Université Libre de Bruxelles

Reconfiguration problems

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Introduction

Introductory problem : POWER SUPPLY problem

Let C be a set of customer with fixed demands, P be a set of power stations with fixed capacity and $G = (V, E)$ be a bipartite graph where $V = \{C \cup P\}$ with weights on the vertices.

Introduction

Introductory problem : POWER SUPPLY problem

Let C be a set of customer with fixed demands, P be a set of power stations with fixed capacity and $G = (V, E)$ be a bipartite graph where $V = \{C \cup P\}$ with weights on the vertices.

Can G be partitioned into subtrees, such that each subtree contains exactly one power supply P s.t the sum of the demands of the C vertices (customers) in each subtree is no more than the capacity of the P vertex (power station) in it?

Introduction

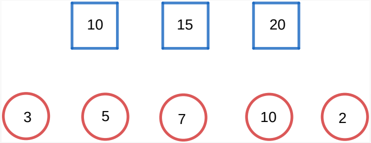

Figure: Input graph G where the blue vertices are the power stations and red vertices are the customers.

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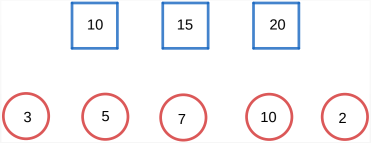

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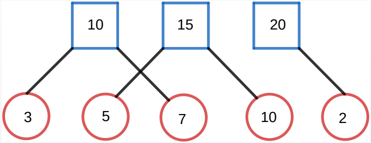

Figure: A feasible solution to the POWER SUPPLY problem.

Theorem (Ito et al.)

The POWER SUPPLY problem is NP-complete [6].

POWER SUPPLY RECONFIGURATION problem

Suppose now that we are given two feasible solutions s_0 and s_t of the POWER SUPPLY problem and are asked:

Can one solution be transformed into the other by moving only one customer at a time and always remaining feasible?

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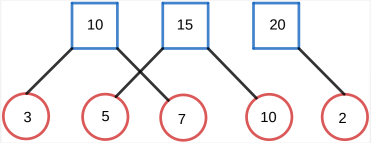


Figure: Feasible solution s_0.

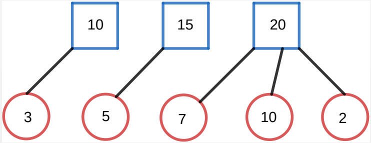
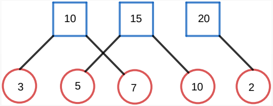


Figure: Feasible solution s_t.

POWER SUPPLY RECONFIGURATION problem



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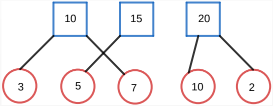

Figure: Intermediate feasible solution s_i where customer 10 is moved.

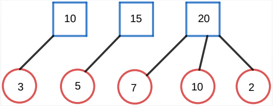

Figure: Target feasible solution s_t where customer 7 is moved from previous intermediate solution.

Theorem (Ito et al.)

POWER SUPPLY RECONFIGURATION problem is PSPACE-complete [6].

RECONFIGURATION problems

Definition

Reconfiguration problems are computational problems in which we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible.

RECONFIGURATION problems

Graph-theoretic perspective

- Reconfiguration graph where :
    1. The vertex set consists of all possible configurations (solutions).
    2. Two nodes are connected if the corresponding configurations can each be obtained from the other by the application of a single transformation rule, a *reconfiguration step*.
- Any path or walk in the Reconfiguration graph = *Reconfiguration sequence*.

RECONFIGURATION problems

Main themes

1. SATISFIABILITY RECONFIGURATION problems.
2. SLIDING TOKENS problems.
3. SUBSET SUM RECONFIGURATION problems.

SATISFIABILITY RECONFIGURATION

satisfiability problem

The satisfiability problem, also called SAT is to test whether a CNF formula is satisfiable. An example of a CNF formula is $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$.

Theorem (Cook-Levin)

SAT is NP-complete [2].

SATISFIABILITY RECONFIGURATION

SAT RECONFIGURATION problems

The solutions (satisfying assignments) of a given n-variable CNF φ induce a subgraph $G(\varphi)$ of the n-dimensional hypercube, introducing two decision problems :

1. Connectivity problem : Given a CNF formula φ, is $G(\varphi)$ connected?
2. st-Connectivity problem : Given a CNF formula φ and two solutions s_0 and s_t of φ, is there a path from s_0 to s_t in $G(\varphi)$?

SATISFIABILITY RECONFIGURATION

Connectivity $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$.

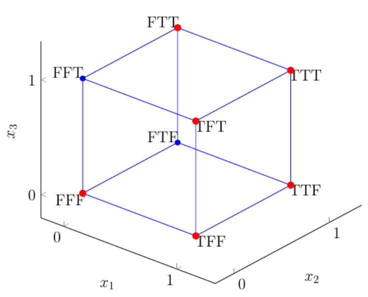

Figure: Reconfiguration graph of φ.

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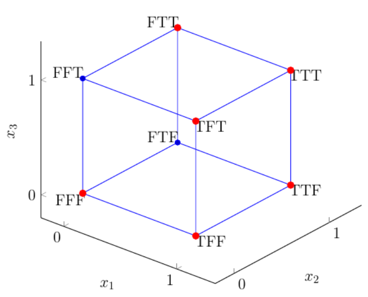

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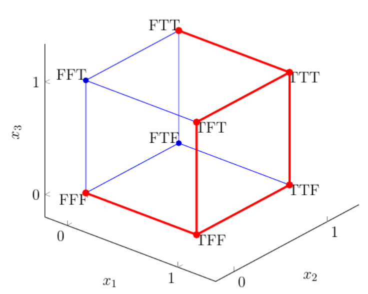

Figure: $G(\varphi)$.

SATISFIABILITY RECONFIGURATION

st-Connectivity $\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$.

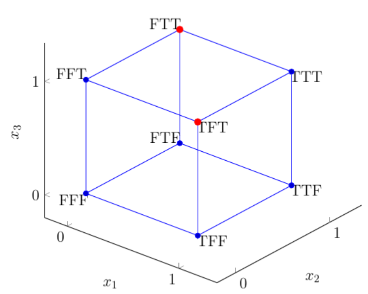

Figure: Reconfiguration graph of φ with two satisfying assignments s_0 and s_t.

SATISFIABILITY RECONFIGURATION

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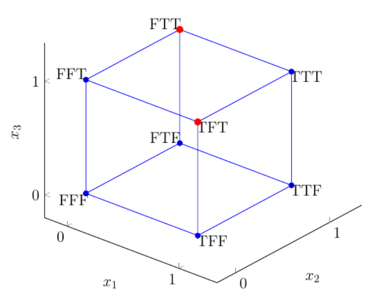

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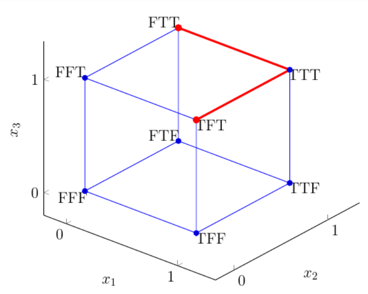


Figure: Reconfiguration sequence transforming s_0 to s_t.

BOOLEAN SATISFIABILITY RECONFIGURATION

Theorem (Gopalan et al.)

The Connectivity problem is PSPACE-complete [3].

Theorem (Gopalan et al.)

The st-Connectivity problem is PSPACE-complete [3].

SLIDING TOKENS RECONFIGURATION

The SLIDING TOKENS problem

Input Instance: Two independent sets I_1 and I_2 of a graph $G = (V, E)$ s.t $|I_1| = |I_2|$ with a token placed on each vertex in I_1.

Question: Is there a reconfiguration sequence from I_1 to I_2 ?

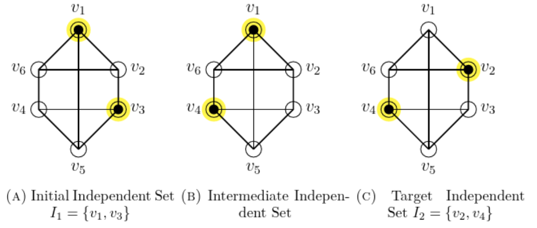


Figure: Reconfiguration sequence from I_1 and I_2.

SLIDING TOKENS RECONFIGURATION

Seen as the reconfiguration version of the Independent Set problem.

Theorem (E.Demaine & R.Hearn)

The SLIDING TOKENS problem is PSPACE-complete [4].

LABELLED SLIDING TOKENS

The LABELLED SLIDING TOKENS problem

Input Instance: Two independent sets I_1 and I_2 of a graph $G = (V, E)$ s.t $|I_1| = |I_2|$ with a labelled token placed on each each vertex in I_1 and each label is unique.

Theorem

The LABELLED SLIDING TOKENS problem is PSPACE-complete.

Proof.

Reduction from the Nondeterministic Constraint Logic. □

Nondeterministic Constraint Logic

Graph formulation

The computational model is a constraint graph $G = (V, E)$ where :

- Each edge is assigned a weight.
- Each vertex has a minimum inflow constraint.
- A configuration = orientation of the edges.

Nondeterministic Constraint Logic

Restricted NCL

The constraint graph $G = (V, E)$

- 3-regular.
- Uses only weights $\in \{1, 2\}$
  Red and blue edges.
- Uses only two types of vertices
  AND and OR vertices.
- The minimum inflow constraint $= 2$.

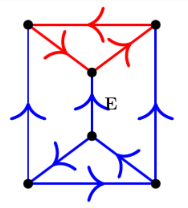


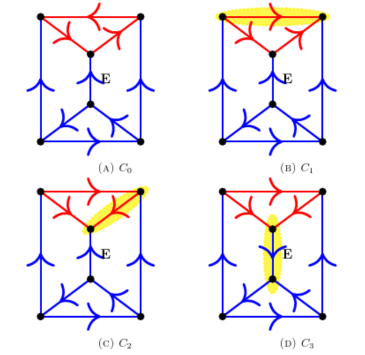
Figure: Restricted NCL instance.

Nondeterministic Constraint Logic

Configuration-to-edge for Restricted NCL.

Theorem (E.Demaine & R.Hearn)

CONFIGURATION-TO-EDGE for restricted NCL is PSPACE-complete [4].



(A) C_0 (B) C_1 (C) C_2 (D) C_3

SUBSET SUM RECONFIGURATION

Subset Sum Problem

Given an integer x and a set of integers $S = \{a_1, a_2, \ldots, a_n\}$, we wish to find a subset $A \subseteq [n]$ such that $\sum_{i \in A} a_i = x$.

SUBSET SUM RECONFIGURATION

Two variants :

1. Add/remove y, keep sum in target range.
   (considered by Ito and Demaine, referred as SUBSET SUM RECONFIGURATION problem (SSR).)

2. Swap y, z and $y + z$, keep target sum.
   (considered by Cardinal et al., referred as k-move SUBSET SUM RECONFIGURATION problem (k-move SSR).)

SUBSET SUM RECONFIGURATION

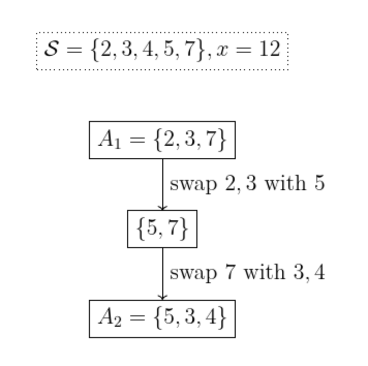
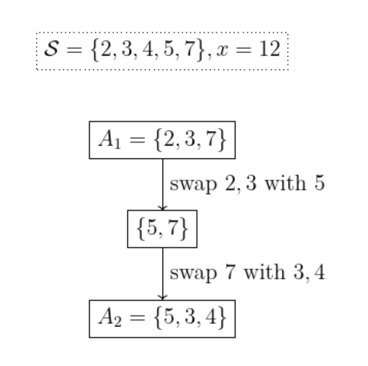


Figure: An instance of the k-move SSR problem where $k = 3$.

SUBSET SUM RECONFIGURATION



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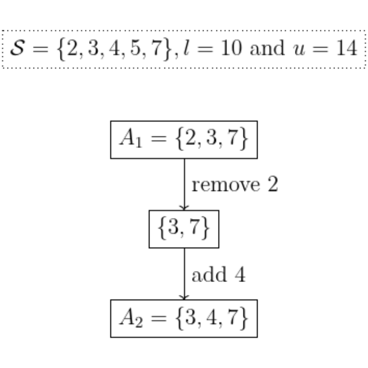

Figure: An instance of the SSR problem.

SUBSET SUM RECONFIGURATION

Theorem (Ito et al.)

The SUBSET SUM RECONFIGURATION problem is NP-hard [5].

Theorem (Cardinal et al.)

The k-move SUBSET SUM RECONFIGURATION Problem is PSPACE-complete for $k = 3$ [1].

Geometric interpretations

Constrained Hypercube Path problem

Given two vertices s, t of the n-hypercube, both contained in a polytope $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ for some $A = (a_{ij}) \in \mathbb{Z}^{d \times n}$ and $b \in \mathbb{Z}^d$, does there exist a path from s to t in the hypercube, all vertices of which lie in P?

Geometric interpretations

SSR problem $\rightarrow$ Constrained hypercube path

- Let $x \subseteq \{0,1\}^n$ Boolean variable indicating if an item is chosen or not.
- The solution space of an instance of the SSR problem is represented by an n-hypercube.
- The solutions to this given instance are the points of the n-hypercube that lie in the polytope $P := \{k \leq \sum_{i=1}^{n} x_i w_i \leq c\}$.
- The SSR problem is equivalent to the Constrained Hypercube path problem where $d = 2$ since it involves exactly two linear constraints.

Geometric interpretation

Example : SSR input instance

- $\mathcal{S} = \{1, 3, 6\}$.
- The lower bound $= 1$.
- The upper bound $= 7$.
- $A_1 = \{6\}$ and $A_2 = \{1, 3\}$.

Geometric interpretation

SSR instance $\rightarrow$ Constrained Hypercube path problem.

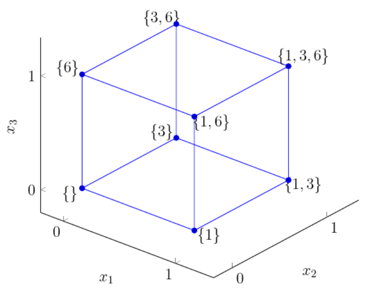

Figure: n-hypercube induced by all possible configurations of the given input SSR instance.

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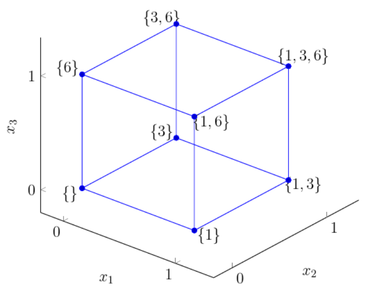

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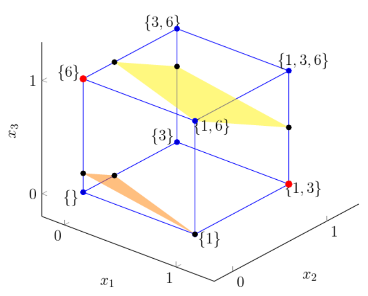

Figure: Polytope defined by the two linear constraints of the SSR problem.

Geometric interpretations

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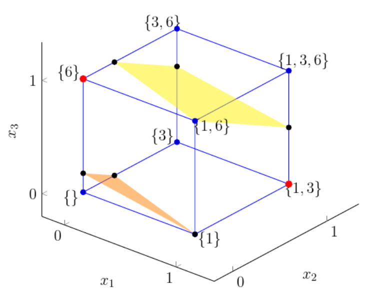

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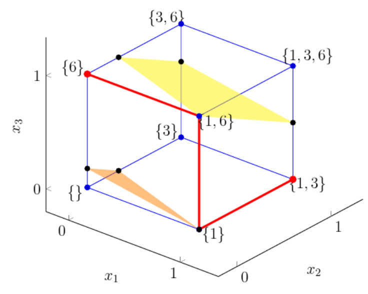


Figure: Reconfiguration sequence S transforming A_1 to A_2 while satisfying the capacity and treshold constraint.

Geometric interpretations

k-move SSR $\rightarrow$ Constrained hypercube path

- Let $x \subseteq \{0,1\}^n$ Boolean variable indicating if an integer is chosen or not.
- The solution space of a k-move SSR instance is represented by $H_n^k[Q]$ which is the kth power of the n-hypercube Q where two vertices are connected iff their symmetric difference is at most k.
- The solutions to this given instance are the points of the $H_n^k[Q]$ that lie in the polytope $P := \{\sum_{i=1}^n x_i a_i = x\}$.
- The k-move SSR is equivalent to the Constrained Hypercube path problem where $d = 1$ since it involves exactly one linear constraint.

Geometric interpretation

Example : 3-move SSR input instance

- $\mathcal{S} = \{2, 3, 5\}$.
- Target sum $x = 5$.
- $A_1 = \{5\}$ and $A_2 = \{2, 3\}$.

Geometric interpretations

3-move SSR instance $\rightarrow$ Constrained Hypercube path problem.

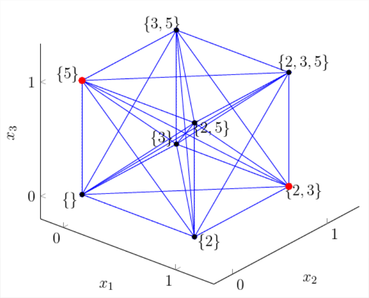


Figure: $H_3^3[Q]$.

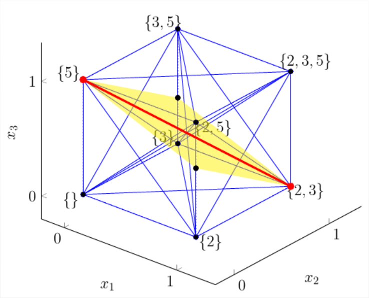
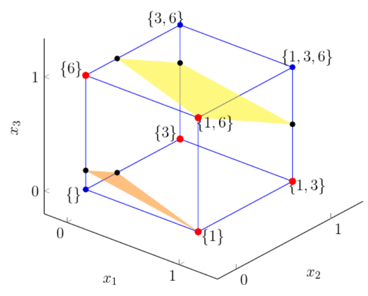


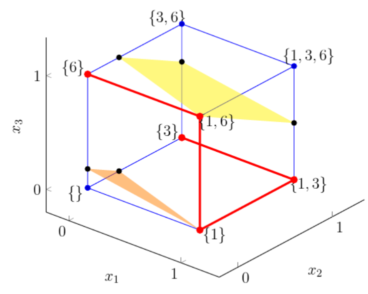
Figure: Reconfiguration sequence S transforming A_1 to A_2 while satisfying the target sum constraint.

Open questions

SSR

Given an SSR instance, is the subgraph induced by all the feasible solutions connected ?





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k-move SSR

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References I

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Thank you for your attention.