Stochastic Processes and Queueing Theory

Assignment 2

February 18, 2014

Notations:

1. $X_1,\ X_2, \ldots$ i.i.d non negative random variables with distribution F. $S_n = \sum_{k=1}^{n} X_k$.

2. $N(t)$ is the number of arrivals till time t excluding the one at zero. $m(t) = E[N(t)],\ \mu = E[X_1]$.

Question 1. Show that $P(X_{N(t)+1} \geq x) \geq \bar{F}(x)$ (where $\bar{F}(.)$ is the complementary cdf). Thus also show that $E[(X_{N(t)+1})^m] \geq E[X^m]$ for any positive integer m. Compute $P(X_{N(t)+1} \geq x)$ for $X_i \sim exp(\lambda)$.

Solution 1. a) To show that $P(X_{N(t)+1} \geq x) \geq \bar{F}(x)$,

$$\begin{aligned}
P(X_{N(t)+1} < x) &= P(X_{N(t)+1} < x, S_1 \leq t) + P(X_{N(t)+1} < x, S_1 > t) \\
&= \int_{u=0}^{t} P(X_{N(t)+1} < x | S_1 = u) dF_{S_1}(u) + \int_{u=t}^{\infty} P(X_1 < x | S_1 = u) dF_{S_1}(u) \\
&= \int_{u=0}^{t} P(X_{N(t-u)+1} < x) dF_{S_1}(u) + \int_{u=t}^{\infty} I_{\{x \geq u\}} dF_{S_1}(u)
\end{aligned}$$

If $x > t$,

$$\begin{aligned}
P(X_{N(t)+1} < x) &= \int_{u=0}^{t} P(X_{N(t-u)+1} < x) dF_{S_1}(u) + \int_{u=t}^{x} dF_{S_1}(u) \\
.P(X_{N(t)+1} \geq x) &= 1 - P(X_{N(t)+1} < x) \\
&= \int_{u=0}^{t} dF_{S_1}(u) + \int_{u=t}^{x} dF_{S_1}(u) + \int_{u=x}^{\infty} dF_{S_1}(u) \\
&\quad - \int_{u=0}^{t} P(X_{N(t-u)+1} < x) dF_{S_1}(u) - \int_{u=t}^{x} dF_{S_1}(u)
\end{aligned}$$

$$\begin{aligned}
&= \int_{u=0}^{t} dF_{S_1}(u) + \int_{u=x}^{\infty} dF_{S_1}(u) - \int_{u=0}^{t} P(X_{N(t-u)+1} < x) dF_{S_1}(u) \\
&= \int_{u=0}^{t} (1 - P(X_{N(t-u)+1} < x)) dF_{S_1}(u) + \int_{u=x}^{\infty} dF_{S_1}(u) \\
&\geq \int_{u=x}^{\infty} dF_{S_1}(u) = P(X_1 \geq x).
\end{aligned}$$

If $x \leq t$,

$$\begin{aligned}
P(X_{N(t)+1} < x) &= \int_{u=0}^{t} P(X_{N(t-u)+1} < x) dF_{S_1}(u) + \int_{u=t}^{\infty} I_{\{x \geq u\}} dF_{S_1}(u) \\
&= \int_{u=0}^{t} P(X_{N(t-u)+1} < x) dF_{S_1}(u) \\
&\geq \int_{u=0}^{x} P(X_{N(t-u)+1} < x) dF_{S_1}(u)
\end{aligned}$$

$$\begin{aligned}
P(X_{N(t)+1} \geq x) &= 1 - P(X_{N(t)+1} < x) \\
&= \int_{u=0}^{x} dF_{S_1}(u) + \int_{u=x}^{\infty} dF_{S_1}(u) - \int_{u=0}^{x} P(X_{N(t-u)+1} < x) dF_{S_1}(u) \\
&= \int_{u=0}^{x} (1 - P(X_{N(t-u)+1} < x)) dF_{S_1}(u) + \int_{u=x}^{\infty} dF_{S_1}(u) \\
&\geq \int_{u=x}^{\infty} dF_{S_1}(u) = P(X_1 \geq x).
\end{aligned}$$

b) To show part b) we use the following expectation formula for non negative random variables.

$$\begin{aligned}
E[X^m] &= E[(\int_{t=0}^{X} mt^{(m-1)} dt)] \\
&= \int_{x=0}^{\infty} \int_{t=0}^{x} mt^{(m-1)} dt dF(x) \\
&= \int_{t=0}^{\infty} \int_{x=t}^{\infty} dF(x) mt^{(m-1)} dt \\
&= \int_{t=0}^{\infty} P(X > t) mt^{(m-1)} dt.
\end{aligned}$$

Now, $E[(X_{N(t)+1})^m] = \int_{u=0}^{\infty} P(X_{N(t)+1} > u) mu^{(m-1)} du \geq \int_{t=0}^{\infty} P(X_1 > u) mu^{(m-1)} du = E[X_1^m]$.

c)

$$\begin{aligned}
P(X_{N(t)+1} \geq x) &= \int_{u=0}^{\infty} P(X_{N(t)+1} \geq x, A(t) > u) du \\
&= \int_{u=0}^{\infty} P(X_{N(t)+1} \geq x | A(t) > u) dF_{A(t)}(u) \\
&= \int_{u=0}^{x} P(X_{N(t)+1} \geq x | A(t) > u) dF_{A(t)}(u) + \int_{u=x}^{\infty} P(X_{N(t)+1} \geq x | A(t) > u) dF_{A(t)}(u) \\
&= \int_{u=0}^{x} P(X_1 \geq x | X_1 > u) dF_{A(t)}(u) + \int_{u=x}^{\infty} 1 dF_{A(t)}(u)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{u=0}^{x} \frac{P(X_1 \geq x)}{P(X_1 > u)} dF_{A(t)}(u) + e^{-\lambda x} \\
&= e^{-\lambda x} \int_{u=0}^{x} e^{\lambda u} dF_{A(t)}(u) + e^{-\lambda x} \\
&= \lambda x e^{-\lambda x} + e^{-\lambda x}
\end{aligned}
$$

As $A(t) \sim exp(\lambda)$. Try proving this from renewal argument.

Question 2. Prove the renewal equation: $m(t) = m(t) * F(t)$.

Solution 2.

$$
\begin{aligned}
m(t) &= E[N(t)] = E[N(t) \times 1] = E[N(t)(I_{\{S_1 \leq t\}} + I_{\{S_1 > t\}})] \\
&= E[N(t) I_{\{S_1 \leq t\}}] + E[N(t) I_{\{S_1 > t\}}] \\
&= E[N(t) I_{\{S_1 \leq t\}}] = \int_{u=0}^{t} E[N(t) | S_1 = u] dF_{S_1}(u) \\
&= \int_{u=0}^{t} E[1 + N(t-u)] dF_{S_1}(u) = F(t) + m(t) * F(t).
\end{aligned}
$$

Question 3. If F is uniform on $(0,1)$ then show that $m(t) = e^t - 1,\ 0 \leq t \leq 1$.

Solution 3. If $F \sim$ Uniform$(0,1)$, $dF_{S_1}(u) = du$

$$
m(t) = \int_{u=0}^{t} m(t-u) dF_{S_1}(u) = \int_{u=0}^{t} m(t-u) du = \int_{u=0}^{t} m(u) du.
$$

Together with the initial condition, $m(0) = E[N(0)] = 0$, we solve for the above differential equation to get $m(t) = e^t - 1,\ t \in [0,1]$.

Question 4. Consider single server bank in which potential customers arrive at Poisson rate λ. Customer enters the bank only of the server s free when the customer arrives. Let G denote the service distribution.
a). What fraction of time the server is busy?
b). At what rate customer enters the bank?
c). At fraction of potential customers enter the bank ?

Solution 4. The question can be solved by using Renewal Reward Theorem (RRT)(is a direct consequence of the Renewal theorem.) a). Consider a renewal period. It consists of idle and busy periods. Let the fraction of time the server is busy be the reward during the cycle. From RRT, the answer is $\frac{\int x^2 dG(x)}{(\lambda^{-1}) + \int x^2 dG(x)}$.
b).Let L_k denote the fraction of lost customers in jth cycle. From RRT, rate at which customers not entering the system is $\lambda_l = \frac{\lambda \int x^2 dG(x)]}{(\lambda^{-1}) + \int x^2 dG(x)}$. From this we get the required rate$=\lambda - \lambda_l$.
c). The reward for this case, during a cycle (in terms of customers, not time)

is 1. The total arrival is $L_k + 1$ during cycle k. From RRT, the required rate is $\frac{1}{1+\lambda \int x^2 dG(x)}$.

Question 5. Find the renewal equation for $E[A(t)]$. Then also find $\lim_{t\to\infty} E[A(t)]$.

Solution 5. Let $g(t) = E[A(t)]$.

$$\begin{aligned}
g(t) &= E[A(t)] = E[A(t) \times 1] = E[A(t)(I_{\{S_1 \le t\}} + I_{\{S_1 > t\}})] \\
&= E[A(t)I_{\{S_1 \le t\}}] + E[A(t)I_{\{S_1 > t\}}] \\
&= E[A(t)I_{\{S_1 \le t\}}] + E[tI_{\{S_1 > t\}}] \\
&= \int_{u=0}^{t} E[A(t)|S_1 = u]dF_{S_1}(u) + tP(S_1 > t) \\
&= \int_{u=0}^{t} E[A(t-u)]dF_{S_1}(u) + tP(S_1 > t) = g(t) * F(t) + h(t).
\end{aligned}$$

where $h(t) = tP(S_1 > t)$. We know that the solution to the above renewal equation (which is the solution to part a) of the question) is $g(t) = m(t) * h(t) + h(t)$. Now, we are interested in finding the limit $\lim_{t\to\infty} g(t)$. Recall, Key Renewal Theorem (KRT) which says that, if $h(.)$ is directly Riemann integrable (dRi), then,

$$\lim_{t\to\infty} g(t) * h(t) = \frac{\int_{u=0}^{\infty} h(u)du}{E[X_1]}.$$

Hence, if the distribution function of S_1 is such that $h(t)$ is dRi ((eg. Exponential distribution. A sufficient condition for dRi is $E[X^2] < \infty$. This is easy to see as $\int_{t=0}^{\infty} h(t) = \frac{1}{2}E[X^2]$), then,

$$\begin{aligned}
\lim_{t\to\infty} g(t) * h(t) &= \frac{\int_{u=0}^{\infty} h(u)du}{E[X_1]} \\
&= \frac{1}{E[X_1]} \int_{u=0}^{\infty} uP(S_1 > u)du \\
&= \frac{1}{E[X_1]} \int_{u=0}^{\infty} u \int_{s=u}^{\infty} dF_{S_1}(s)du \\
&= \frac{1}{E[X_1]} \int_{s=0}^{\infty} \int_{u=0}^{s} udF_{S_1}(s)du \\
&= \frac{1}{E[X_1]} \int_{s=0}^{\infty} \int_{u=0}^{s} ududF_{S_1}(s) \\
&= \frac{1}{E[X_1]} \int_{s=0}^{\infty} [\frac{u^2}{2}]_{u=0}^{s} dF_{S_1}(s) \\
&= \frac{1}{E[X_1]} \int_{s=0}^{\infty} s^2 dF_{S_1}(s) = \frac{E[X_1^2]}{2E[X_1]}.
\end{aligned}$$

Also, here we have h(t) to be a non-negative function. Hence (from class notes dated 28-01-2014, Tuesday, second lecture on Renewal process), if $h(t)$ is dRi, $\lim_{t\to\infty} h(t) = 0$. Thus whenever the distribution function F is such that $h(.)$ is dRi, the solution to part b) is $\frac{E[X_1^2]}{2E[X_1]}$.

Question 7. Consider a delayed renewal process $\{N_D(t), t \geq 0\}$ whose first interarrival time has distribution G and the others have distribution F. Let $m_D(t) = E[N_D(t)]$.
a) Prove that, $m_D(t) = G(t) + m(t) * G(t)$, where $m(t) = \sum_{n=1}^{\infty} F^{*n}(t)$.
b) Let $A_D(t)$ denote the age time at time t. Show that if F is non lattice with $\int x^2 dF(x) < \infty$ and $t\bar{G}(t) \to 0$ at $t \to \infty$, then

$$E[A_D(t)] \to \frac{\int_{x=0}^{\infty} x^2 dF(x)}{2\int_{x=0}^{\infty} x dF(x)}$$

Solution 7.

$$\begin{aligned} m_D(t) &= E[N_D(t)] = E[N_D(t) \times 1] = E[N_D(t)(I_{\{S_1 \leq t\}} + I_{\{S_1 > t\}})] \\ &= E[N_D(t) I_{\{S_1 \leq t\}}] + E[N_D(t) I_{\{S_1 > t\}}] \\ &= E[N_D(t) I_{\{S_1 \leq t\}}] = \int_{u=0}^{t} E[N_D(t)|S_1 = u] dG_{S_1}(u) \\ &= \int_{u=0}^{t} E[1 + N(t-u)] dG_{S_1}(u) = G(t) + m(t) * G(t). \end{aligned}$$

$$\begin{aligned} g(t) &= E[A_D(t)] = E[A_D(t) \times 1] = E[A_D(t)(I_{\{S_1 \leq t\}} + I_{\{S_1 > t\}})] \\ &= E[A_D(t) I_{\{S_1 \leq t\}}] + E[A_D(t) I_{\{S_1 > t\}}] \\ &= E[A_D(t) I_{\{S_1 \leq t\}}] + E[t I_{\{S_1 > t\}}] \\ &= \int_{u=0}^{t} E[A_D(t)|S_1 = u] dG_{S_1}(u) + tP(S_1 > t) \\ &= \int_{u=0}^{t} E[A(t-u)] dG_{S_1}(u) + tP(S_1 > t) = \tilde{g}(t) * G(t) + h(t). \end{aligned}$$

where $h(t) = tP(S_1 > t) = t\bar{G}(t)$.

From KRT,

$$\lim_{t\to\infty} \tilde{g}(t) * h(t) = \frac{\int_{u=0}^{\infty} h(u) du}{E[X_2]}.$$

Also from the hypothesis given, $h(t)$ is dRi. Hence,

$$\begin{aligned}
\lim_{t\to\infty} \tilde{g}(t) * h(t) &= \frac{\int_{u=0}^{\infty} h(u)du}{E[X_2]} \\
&= \frac{1}{E[X_2]} \int_{u=0}^{\infty} uP(S_2 > u)du \\
&= \frac{1}{E[X_2]} \int_{u=0}^{\infty} u \int_{s=u}^{\infty} dF_{S_2}(s)du \\
&= \frac{1}{E[X_2]} \int_{s=0}^{\infty} \int_{u=0}^{s} udF_{S_2}(s)du \\
&= \frac{1}{E[X_2]} \int_{s=0}^{\infty} \int_{u=0}^{s} ududF_{S_2}(s) \\
&= \frac{1}{E[X_2]} \int_{s=0}^{\infty} [\frac{u^2}{2}]_{u=0}^{s} dF_{S_2}(s) \\
&= \frac{1}{E[X_2]} \int_{s=0}^{\infty} s^2 dF_{S_2}(s) = \frac{E[X_2^2]}{2E[X_2]}.
\end{aligned}$$

Question 8. Consider a $GI/GI/1$ queue. Interarrival times $\{A_n\}$ are iid and service times $\{S_n\}$ are iid with $E[S_n] < E[A_n] < \infty$. Let $V(t)$ be the virtual service time in the queue at time t $\triangleq$ sum of remaining service time of all customers present in the system at time t. Show that,
a).

$$V \triangleq \lim_{t\to\infty} \frac{1}{t} \int_0^t V(s)ds$$

exists a.s and is a constant.

b). Let D_n the amount of time n^{th} customer waits in the queue. Define

$$W_Q \triangleq \lim_{n\to\infty} \frac{D_1 + D_2 \dots D_n}{n}$$

Show that W_Q exists a.s and is a constant.

c). Show that $V = \lambda E[Y]W_Q + \lambda\frac{E[Y^2]}{2}$ where $\frac{1}{\lambda} = E[A_n]$ and Y has the distribution of service time.

Solution 8. To solve this question, we have to use the regenerative argument or implicitly, Renewal Reward Theorem (RRT). Here, one could view, $R(t) = \int_0^t V(s)$ as the total reward earned till time t. Now, RRT gives

$$\frac{R(t)}{t} \overset{t\to\infty}{\to} \frac{E[\text{Reward earned in a cycle}]}{E[\text{Cycle duration}]}$$

This proves part a).

For part b), to prove that the limit exists, consider the following:

The waiting time D_n of n^{th} customer has the following recursive structure(see the figure below):

$$D_n = \max\{0, D_{(n-1)} + S_n - A_{(n-1)}\}.$$

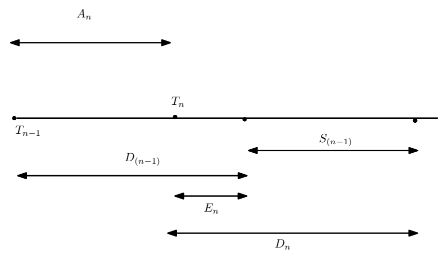

Figure 1: Recursive formula for waiting time.

Let $\alpha_n = S_n - A_{(n-1)}$. Also, $\max(0, \max(0, b) + c) = \max(0, c, b + c)$. Now, we can write the recursion as follows:

$$\begin{aligned}
D_n &= \max\{0, D_{(n-1)} + S_n - A_{(n-1)}\} \\
&= \max\{0, \max\{0, D_{(n-2)} + S_{(n-1)} - A_{(n-2)}\} + S_n - A_{(n-1)}\} \\
&= \max\{0, \max\{0, D_{(n-2)} + \alpha_{(n-1)}\} + \alpha_n\} \\
&= \max\{0, D_{(n-2)} + \alpha_{(n-1)} + \alpha_n, \alpha_n\} \\
&= \max\{0, \sum_{k=1}^{n} \alpha_k, \sum_{k=1}^{n-1} \alpha_k, \ldots\}
\end{aligned}$$

Thus, if we denote $\max(0, x)$ as x^+, $\lim_{n\to\infty} D_n$ is equal to $D \triangleq (\sup_{n\geq 1} \sum_{k=1}^{n} \alpha_k)^+$. Now, note that, $E[\alpha_n] = E[S_n - A_{(n-1)}] < 0$ (given). And, α_is are iid with finite mean. Hence, from SLLN,

$$\frac{1}{n} \sum_{k=1}^{n} \alpha_k \to -\epsilon,$$

where, ϵ is some positive quantity. Thus, $\sum_{k=1}^{n} \alpha_k$ converges to $-\infty$ a.s. Hence, the supremum of it should be finite. thus we proves the a.s existence of a finite limit of D_n. Now, part b) asks for the limit of Cesaro sum of the D_n. Thus shows the almost sure existence of W_Q.

to show part c), we need to compute the expression in part a). The renewal instants for the process are at times when $V(t) = 0$. Let C denote the **inter** renewal time. Let $\tilde{N}$ denote the number of arrivals during C duration. Hence,

$E[C] = E[\sum_{k=1}^{\tilde{N}} A_k] = E[\sum_{k=2}^{\tilde{N}+1} A_k] = E[\tilde{N}+1]E[A_k] = \frac{E[\tilde{N}+1]}{\lambda}$. Similarly we are interested in computing the numerator in part a) of the problem. See the figure below. During inter renewal period, we have the following graph for the virtual waiting time $V(t)$. Note that, the renewal happens once $V(t)$ hits t. Thus, total reward during a cycle is the area of the graph which can in turn be found from the areas of triangles and parallelograms. In the figure below, the area is equal to the area of triangles P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2, R_3 and the area of parallelograms P_3, Q_2, Q_3, L_1, L_1, R_2, R_3, L_2. The base and altitude of the triangles are the service times. The base of parallelogram is the service time and height is the waiting time. Also, the trianglea have 45 degree angles owing as the virtual waiting time decreases linearly with t. In fact, for an interarrival time $[T_m, T_{n-1})$ the virtual waiting time $V(t) = \max\{0, W(T_n-)+S_n-(t-T_n)\}$.

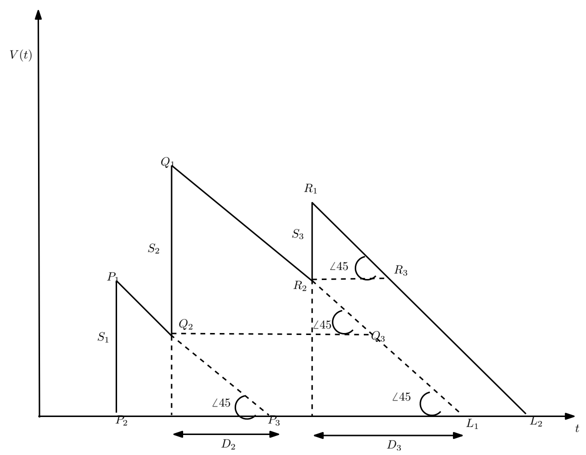

Figure 2: Virtual waiting time.

Thus,

$$
\begin{aligned}
E[\text{Reward during cycle}] &= E[\sum_{k=1}^{\tilde{N}} (\frac{1}{2}S_k^2 + D_k S_k)] \\
&= E[\sum_{k=2}^{\tilde{N}+1} (\frac{1}{2}S_k^2 + D_k S_k)] = E[\tilde{N}+1](E[\frac{S_k^2}{2}] + E[S_k D_k]) \\
&= E[\tilde{N}+1](E[\frac{Y^2}{2}] + E[Y]W_Q).
\end{aligned}
$$

Hence, $V = \lambda E[Y]W_Q + \frac{\lambda}{2}E[Y^2]$.