Cauchy's Stress Principle and Equilibrium Equations

Prateeksha Sharma

Abstract

Cauchy's Stress Principle is a concept in continuum mechanics which states that the state of stress at a point within a material can be completely described by a second-order tensor $\boldsymbol{\sigma}$, which depends linearly on the unit vector $\mathbf{n}$. This article defines the principle, derives the equilibrium equations, and explains their physical meaning.

1 Introduction

In continuum mechanics, stress is the internal force transmitted within a material. While forces act on surfaces, the Cauchy stress tensor defines their intensity and direction at a point. Understanding this principle allows us to connect the physical idea of internal forces with a rigorous mathematical framework.

2 Cauchy's Stress Principle

Consider an arbitrary volume element within a deformable body. The internal forces acting across any surface with unit normal vector $\mathbf{n}$ are described by the *traction vector* $\mathbf{t}(\mathbf{n})$. Cauchy's stress principle states:

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\,\mathbf{n} \tag{1}$$

where $\boldsymbol{\sigma}$ is the *Cauchy stress tensor*, a 3×3 second-order tensor whose components σ_{ij} represent the force in the i-direction acting on a face with outward normal in the j-direction.

2.1 Physical Meaning

The traction vector $\mathbf{t}(\mathbf{n})$ depends linearly on $\mathbf{n}$, and the stress tensor encapsulates all possible tractions on all possible orientations through a single point.

3 Symmetry of the Stress Tensor

By conservation of angular momentum for an infinitesimal element, it can be shown that:

$$\sigma_{ij} = \sigma_{ji} \tag{2}$$

This reduces the number of independent components from 9 to 6.

4 Derivation of the Equilibrium Equations

Consider a differential cube of size $\Delta x \times \Delta y \times \Delta z$ with density ρ and subjected to body forces $\mathbf{b}$ (force per unit mass). Applying Newton's second law in the absence of acceleration (static equilibrium), we have:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \tag{3}$$

where $i = 1, 2, 3$ and Einstein summation convention is used over repeated indices. These are the *static equilibrium equations.*

5 Example: 1D Rod in Tension

For a prismatic bar under axial load P, the only non-zero stress component is σ_{xx}, which is constant along the length if the load is uniformly distributed and the bar is in static equilibrium.

6 Applications

- Stress analysis in structural members.
- Basis for finite element formulation in solid mechanics.
- Derivation of more advanced balance laws in continuum mechanics.

7 Conclusion

Cauchy's Stress Principle provides the bridge between physical forces acting inside a body and the mathematics needed to describe them. The equilibrium equations play a central role, forming the basis for both analytical and numerical approaches in solving mechanics problems.

References

References

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[2] Hjelmstad, K. D. (2005). *Fundamentals of Structural Mechanics* (2nd ed.). Springer.