1. Replication of FIG. 3.

From the article, the FIG. 3, is extracted and placed below.

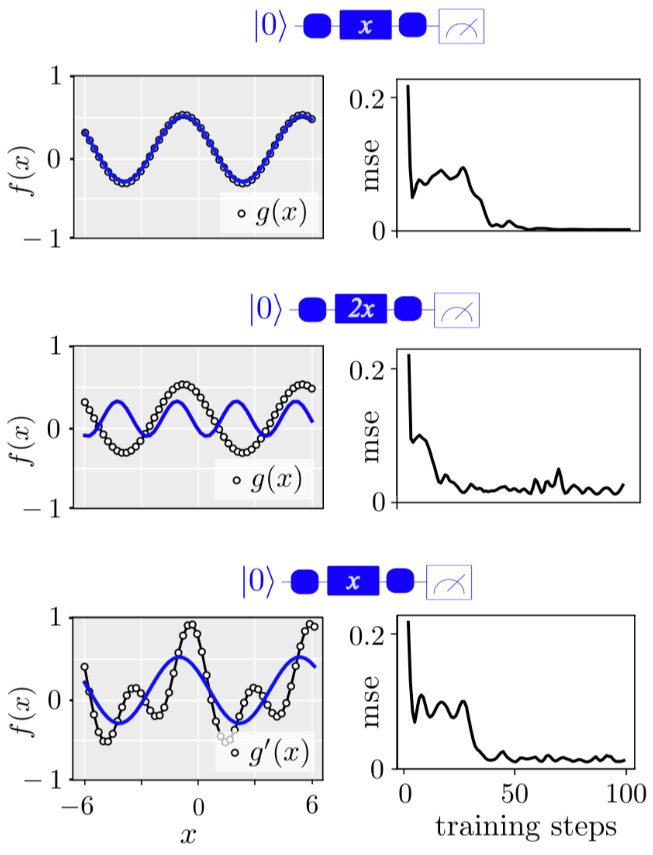


Figure 1 - Quantum Data encoding model for a multi-qubit system

The figure caption is quoted below.

> FIG. 3. A parametrised quantum model is trained with data samples (white circles) to fit a target function g(x) = ′ ∑ n=−1 1 c n e −nix or g (x) = ∑ n=−2 2 c n e −nix with coefficients c 0 = 0.1, c 1 = c 2 = 0.15 − 0.15i. The variational circuit is of the form f(x) = ⟨ 0 | U † (x)σ z U(x) | 0 ⟩ where | 0 ⟩ is a single qubit, and U = W (2) R x (x)W (1) . The W (round blue symbols) are implemented as general rotation gates parametrised by three learnable weights each, and Rx (square blue symbols) is a single Pauli-X rotation. The left panels show the quantum model function f(x) and target function g(x), g ′ (x), while the right panels show the mean squared error between the data sampled from g and f during a typical training run. Feeding in the input x as is (top row), the quantum model easily fits the target of degree 1. Rescaling the inputs x → 2x causes a frequency mismatch, and the model cannot learn the target any more (middle row). However, even with the correct scaling, the variational circuit cannot fit the target function of degree 2 (bottom row). The experiments in this paper were all performed using the PennyLane software library [36].

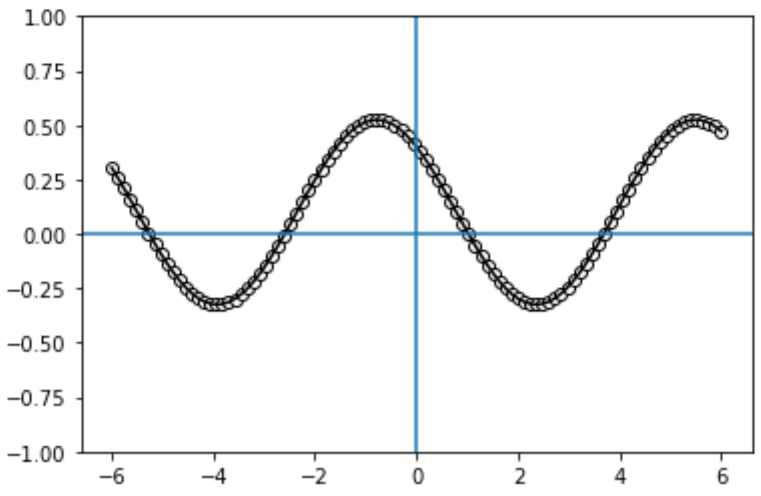
CODE: Target Function

In [28]:

```
import warnings
warnings.filterwarnings('ignore')
warnings.simplefilter('ignore')
```

In [29]:

```
import matplotlib.pyplot as plt
import pennylane as qml
from pennylane import numpy as np

np.random.seed(42)

def square_loss(targets, predictions):
    loss = 0
    for t, p in zip(targets, predictions):
        loss += (t - p) ** 2
    loss = loss / len(targets)
    return 0.5*loss

data_points = 100  # number of datas
degree = 1  # degree of the target function

def target_function(x, degree):

    coeffs = [0.15 + 0.15j]*degree  # coefficients of non-zero frequencies
    coeff_0 = 0.1  # coefficient of zero frequency
    scale_target = 1.  # scale_target of the data

    res = 0.0 + 0.0j
    for idx, coeff in enumerate(coeffs):
        exponent = np.complex128((idx+1) * 1j * scale_target * x)
        conj_coeff = np.conjugate(coeff)
        res += coeff * np.exp(exponent) + conj_coeff * np.exp(-exponent)
    return np.real(res + coeff_0)

x = np.linspace(-6, 6, data_points, requires_grad=False)
target_y = np.array([target_function(x_, degree) for x_ in x], requires_grad=

plt.plot(x, target_y, color='black')
plt.scatter(x, target_y, facecolor='white', edgecolor='black')
plt.ylim(-1, 1)
plt.axvline(0.0)
plt.axhline(0.0)
plt.show()
```



My first code block went smooth! The import and variable declaration is fine and all, yet there might be a small detail asking for a sharp attention. It's the fifth variable `scale_target` declared to 1. If this is anything but 1 then the loss is huge and the trainable model all of the sudden becomes untrainable. This is what authors precisely meant is the account between the expressivity and the data encoding strategy.

Better yet, define a `scale_train` and set it to 1. Now as long as the difference between `scale_target` and `scale_train_model` remains 0 the model is trainable otherwise if not.

The second row of the FIG. 3. is the ouput for `scale_target = 1` and `scale_train_model = 2`. I have coded for such case in following section 1.2.

Finally, let's make a trainable model to train it!

Trainable model randomly instantiated

In [30]:
```
scale_train_model = 1
dev = qml.device('default.qubit', wires=1)

@qml.qnode(dev)
def quantum_model(weights, x):

    for (idx, theta) in enumerate(weights):

        ''' This is trainable circuit block. theta_tensor is the tensor conta

        The Rot method in qml is the general rotation operator which takes in
        qml.Rot(theta[0], theta[1], theta[2], wires=0)

        ''' This is encoding gate. It's a roation gate, if x = pi then this i
        if (idx == len(weights) -1):
            continue

        qml.RX(scale_train_model*x, wires=0)

    return qml.expval(qml.PauliZ(wires=0))
```

The 'weights' is a tensor. In this single-qubit case, it a 1×3 row matrix. In a n-qubit model it's $n \times 3$ matrix. The three sticks around because 'Rot' method in the class 'qml' takes in exact three parameters. Moreover, the quantum model returns an expectation value for the Pauli Z-gate. Since Hadamard gate is never applied throught the entire model the Pauli Z-gate has no effect like that of an Identity operator/matrix. Well almost! Except for an additional phase of π when operated on the state/qubit $|1>.

$$\sigma_z|1>= e^{i\pi}|1>$$

Yet, this difference won't impact the measurement since the phase term is cancelled out by its conjugate during the measurement operation. So, for all the intend of measurement Pauli Z-gate has no effect! As simple as:

$$<1|\sigma_z^\dagger\sigma_z|1>=<1|e^{-i\pi}e^{i\pi}|1>$$

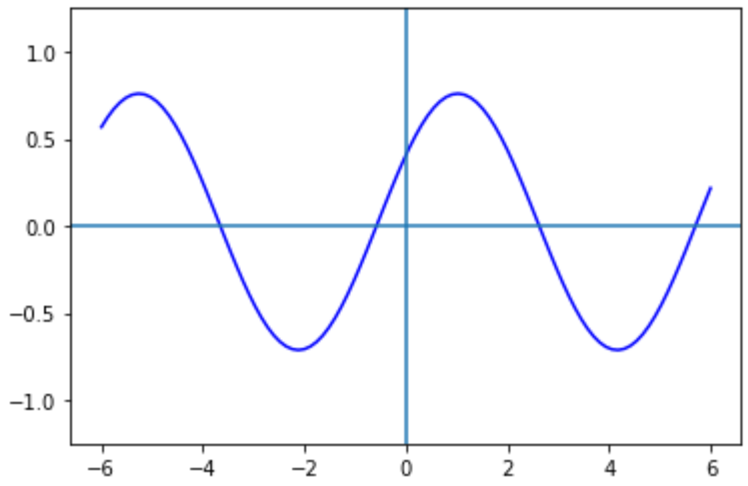
$$\therefore <1|\sigma_z^\dagger\sigma_z|1>=<1|1>$$

I must mention there're nice places to play with qubits [8] [9]. The websites provide a visual and dataful experience.

In the next code snippet, I have built a random trainable model. After it's visualized it's time to begin the long awaited training!

In [31]:

```
# number of times the encoding gets repeated (here equal to the number of lay
r = 1

# some random initial weights
weights = 2 * np.pi * np.random.random(size=(r+1, 3), requires_grad=True)

x = np.linspace(-6, 6, data_points, requires_grad=False)
random_quantum_model_y = [quantum_model(weights, x_) for x_ in x]

plt.plot(x, random_quantum_model_y, color='blue')
plt.ylim(-1.25, 1.25)
plt.axvline(0.0)
plt.axhline(0.0)
plt.show()
```



In [32]:
```
print(qml.draw(quantum_model)(weights, x[-1]))
```

```
0: ──Rot(2.35, 5.97, 4.6)──RX(6)──Rot(3.76, 0.98, 0.98)──┤ ⟨Z⟩
```

1.1 The first row

To obtain this result, the scale for target and training model must be 1. In the code snippet below they are the variables `scale_target = 1` and `scale_train_model = 1`.

The degree of truncated Fourier series in 1. There's one qubit in the model, so, the number of encoding gate is 1 too.

1.1.1 Optimization/Learning for the parameteric circuit

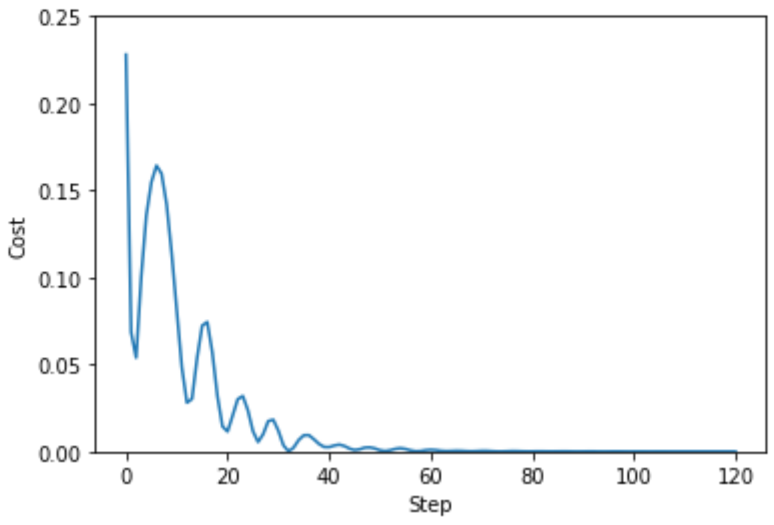
In [33]:
```
def cost(weights, x, y):
    predictions = [quantum_model(weights, x_) for x_ in x]
    return square_loss(y, predictions)

cost_ = [cost(weights, x, target_y)]

def optimizer_func(weights):
    # max_steps = 150
    # opt = qml.AdamOptimizer(stepsize=0.25)
    # batch_size = 30

    max_steps = 120
    opt = qml.AdamOptimizer(stepsize=0.4)
    batch_size = 40


    for step in range(max_steps):

        batch_index = np.random.randint(0, len(x), (batch_size,))
        x_batch = x[batch_index]
        y_batch = target_y[batch_index]

        # Update the weights by one optimizer step
        weights, _, _ = opt.step(cost, weights, x_batch, y_batch)

        # Save, and possibly print, the current cost
        c = cost(weights, x, target_y)
        cost_.append(c)

        if (step + 1) % 15 == 0:
            print("Cost at step {0:3}: {1}".format(step + 1, c))

    return (weights, cost_)

(weights_scale_1_1, cost_1_1 )= optimizer_func(weights)
```

```
Cost at step  15: 0.0722666231955169
Cost at step  30: 0.011944659602031369
Cost at step  45: 0.0008478812775901242
Cost at step  60: 0.0009749460834098915
Cost at step  75: 0.00012814578078917135
Cost at step  90: 3.678618857270698e-05
Cost at step 105: 1.664770666812511e-05
Cost at step 120: 5.637304497008881e-07
```

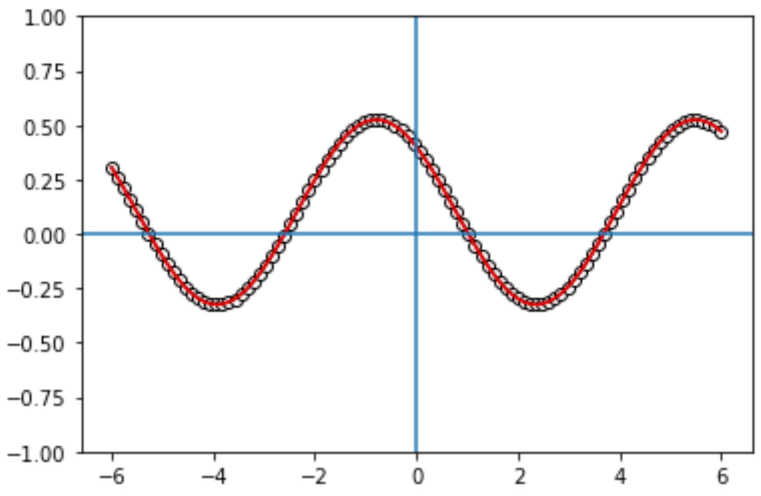
1.1.2 Result

The Loss profile for the training and the graph with both traget model and trained model plotted together is placed below.

In [34]:
```
plt.plot(range(len(cost_1_1)), cost_1_1)
plt.ylabel("Cost")
plt.xlabel("Step")
plt.ylim(0, 0.25)
plt.show()
```



In [35]:
```
predictions = [quantum_model(weights_scale_1_1, x_) for x_ in x]

plt.plot(x, target_y, c='black')
plt.scatter(x, target_y, facecolor='white', edgecolor='black')
plt.plot(x, predictions, c='red')
plt.ylim(-1,1)
plt.axvline(0.0)
plt.axhline(0.0)
plt.show()
```



1.1.3 Conclusion

When the variables `scale_target = 1` and `scale_train_model = 1`, in other words, the scale is identica then the parametric variational model learns with very minimum loss. The quantum model is trainable!

1.2 The second row - Change of scale!

The above result was for scale_target = 1. and scale_train_model = 1. To obtain the second row figure from the FIG. 3 from the paper set the scale_train_model = 2

Finally, trigger the optimizer!

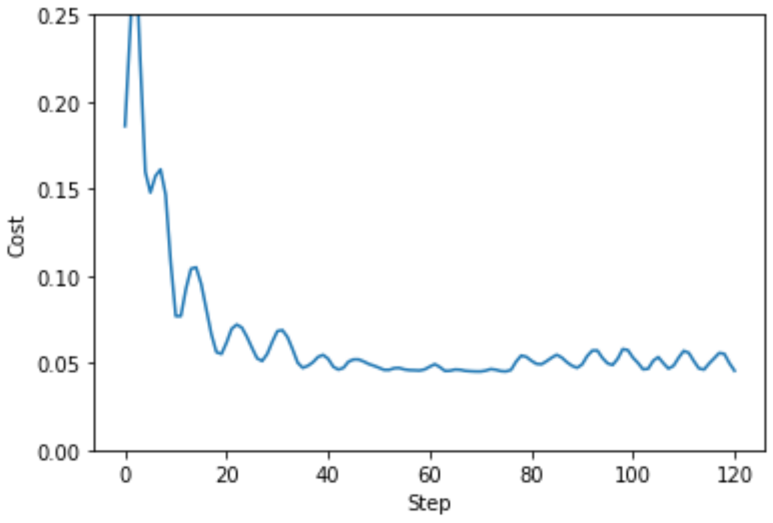
1.2.1 Optimization/Learning for the parameteric circuit

In [36]:
```
scale_train_model = 2

# Reinitialize the (seeded) random initial weights
weights = 2 * np.pi * np.random.random(size=(r+1, 3), requires_grad=True)
cost_ = [cost(weights, x, target_y)]

#  Run the optimizer for scale_target = 1 and scale_train_model = 2
(weights_scale_1_2, cost_1_2)= optimizer_func(weights)
```

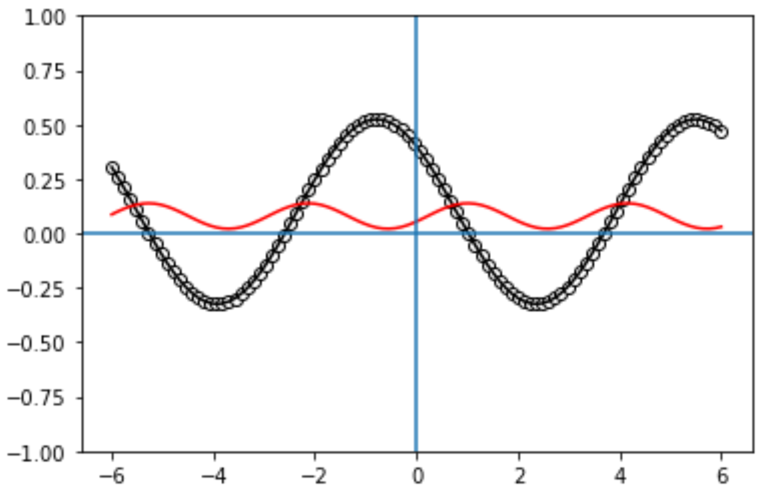
```
Cost at step  15: 0.09542347605236906
Cost at step  30: 0.06831038772178118
Cost at step  45: 0.052006645759923004
Cost at step  60: 0.04787771789091735
Cost at step  75: 0.04515621242201365
Cost at step  90: 0.04914696748837678
Cost at step 105: 0.05338856198069949
Cost at step 120: 0.045558859563352316
```

1.2.2 Result

In [37]:
```
plt.plot(range(len(cost_1_2)), cost_1_2)
plt.ylabel("Cost")
plt.xlabel("Step")
plt.ylim(0, 0.25)
plt.show()
```



In [38]:
```
predictions = [quantum_model(weights_scale_1_2, x_) for x_ in x]

plt.plot(x, target_y, c='black')
plt.scatter(x, target_y, facecolor='white', edgecolor='black')
plt.plot(x, predictions, c='red')
plt.ylim(-1,1)
plt.axvline(0.0)
plt.axhline(0.0)
plt.show()
```



1.2.3 Conclusion

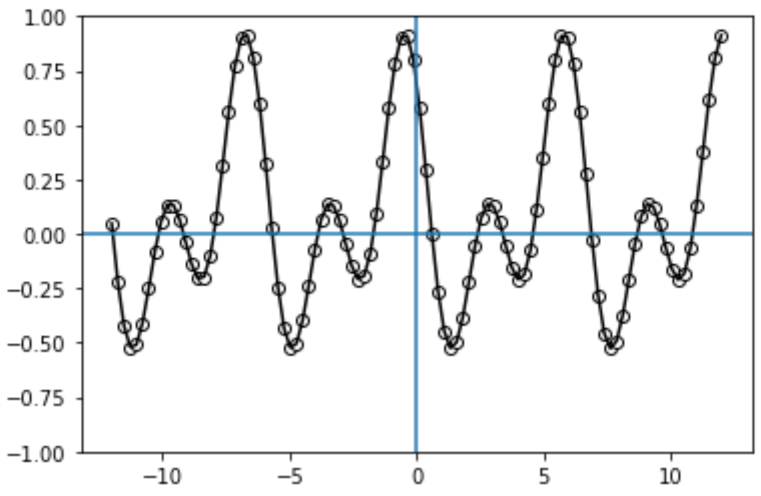
When the variables `scale_target = 1` and `scale_train_model = 2` , in other words, the scale is different then the parametric variational model doesn't learn. The loss does not minimise. The quantum model is untrainable!

1.3 The third row - Change the degree of the Fourier series to 2 and no difference in scale between the trainable model and the target model.

`scale_target = 1.` and `scale_train_model = 1.` To obtain the third row figure from the FIG. 3 from the paper set the `degree = 2` .

Finally, trigger the optimizer!

In [39]:
```
degree = 2

x = np.linspace(-6*degree, 6*degree, data_points, requires_grad=False)
target_y = np.array([target_function(x_, degree) for x_ in x], requires_grad=

plt.plot(x, target_y, color='black')
plt.scatter(x, target_y, facecolor='white', edgecolor='black')
plt.ylim(-1, 1)
plt.axvline(0.0)
plt.axhline(0.0)
plt.show()
```



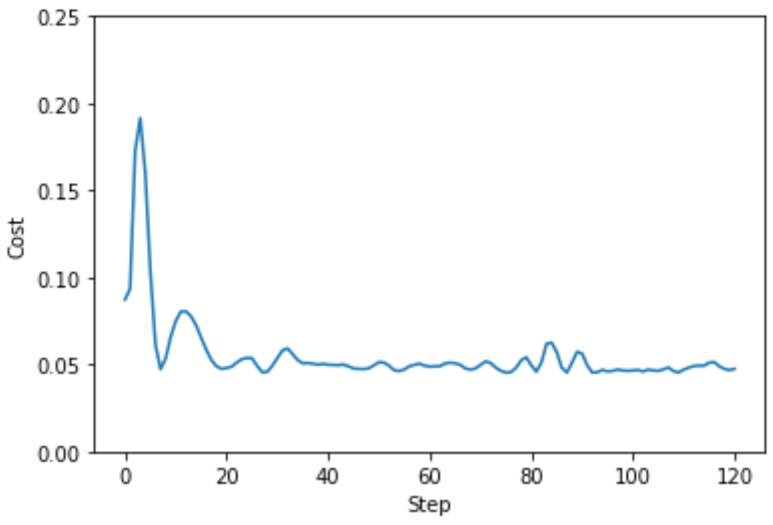
1.3.1 Optimization/Learning for the parameteric circuit

In [40]:
```
scale_train_model = 1

# Reinitialize the (seeded) random initial weights
weights = 2 * np.pi * np.random.random(size=(r+1, 3), requires_grad=True)
cost_ = [cost(weights, x, target_y)]

#  Run the optimizer for scale_target = 1 and scale_train_model = 2
(weights_scale_1_3, cost_1_3)= optimizer_func(weights)
```

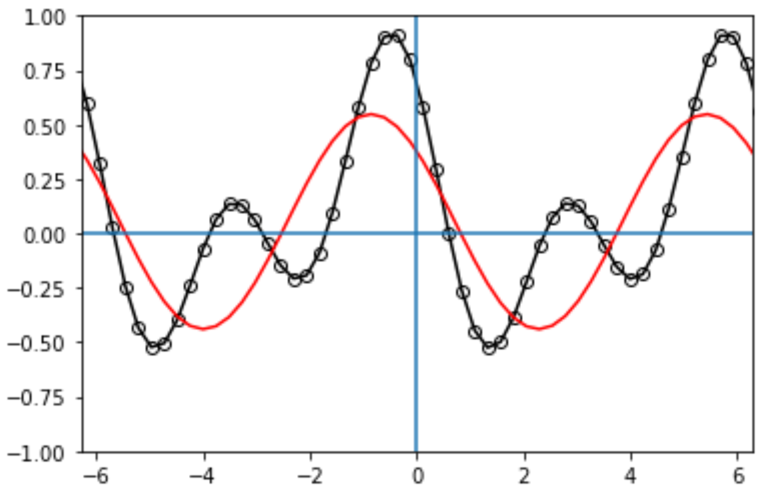
```
Cost at step  15: 0.06535137050511156
Cost at step  30: 0.05368598705922631
Cost at step  45: 0.04769011188688005
Cost at step  60: 0.04879199500721183
Cost at step  75: 0.045332820122448395
Cost at step  90: 0.0562430805997559
Cost at step 105: 0.04632581464547507
Cost at step 120: 0.04746396006390508
```

1.2.2 Result

In [41]:
```
plt.plot(range(len(cost_1_3)), cost_1_3)
plt.ylabel("Cost")
plt.xlabel("Step")
plt.ylim(0, 0.25)
plt.show()
```



In [44]:
```
predictions = [quantum_model(weights_scale_1_3, x_) for x_ in x]

plt.plot(x, target_y, c='black')
plt.scatter(x, target_y, facecolor='white', edgecolor='black')
plt.plot(x, predictions, c='red')
plt.ylim(-1,1)
plt.xlim(-np.pi*degree, np.pi*degree)
plt.axvline(0.0)
plt.axhline(0.0)
plt.show()
```


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−6
−4
−2
0
2
4
6