Theoretical description

Before running over the solution which is in subsequent section. Let's go through preambles.

Section 1: Qubit

A qubit is a two-state system that can be realized in a state of superposition composed by two orthonormal basis states of $|0\rangle$ and $|1\rangle$. Expressed as following.

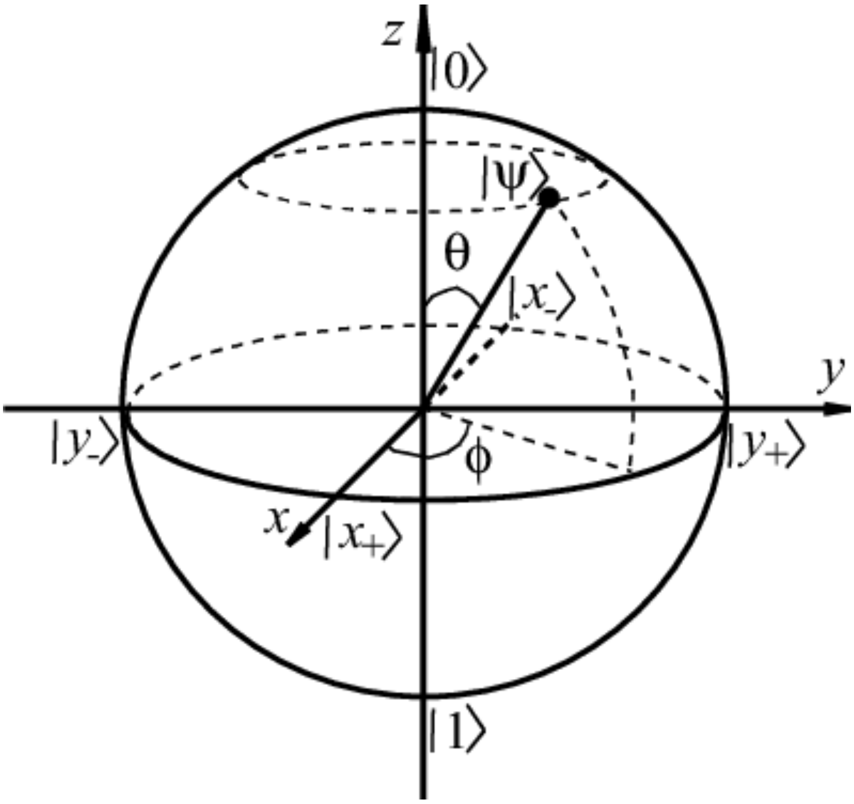
$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Here, α and β are the probability amplitudes along with the two basis states respectively. Whose square of norms are the probabilities for the qubit to collapse over the same. Following the normalization of the qubit $|\Psi\rangle$.

$$|\alpha|^2 + |\beta|^2 = 1$$

Section 2: Bloch sphere

Bloch Sphere is a unit sphere. The radius of the sphere equals to the total probability of the residing qubit. All the points on its surface are traced by the vector representation of the qubit. The convention is to associate the positive z-axis to $|0\rangle$ and the other one to $|1\rangle$. This puts an angle of π between these two orthonormal states. Immediately, it seems to be a counterintuitive approach. However, together with all of these cases, the idea of probability amplitudes (α and β) are expressible in polar coordinates makes us ready to choose the Hopf coordinates for the representation of the Bloch Sphere as illustrated below.



Source: Cong, Shuang. (2008). The analysis of two-level Quantum system states and control in the Bloch ball. 618-622. 10.1109/CHICC.2008.4604907.

In the desired co-ordinate, θ is called polar angle and ϕ is called azimuthal angle with range $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ The probability amplitude for the two basis state of the qubit becomes:

$$\alpha = cos\left(\frac{\theta}{2}\right)e^{i\phi_1}$$
$$\beta = sin\left(\frac{\theta}{2}\right)e^{i\phi_2}$$
$$\alpha^2 + \beta^2 = 1$$

Consequently, the qubit takes the form as shown below.

$$|\Psi\rangle = cos\left(\frac{\theta}{2}\right)e^{i\phi_1}|0\rangle + sin\left(\frac{\theta}{2}\right)e^{i\phi_2}|1\rangle$$
$$|\Psi\rangle = e^{i\phi_1}\left(cos\left(\frac{\theta}{2}\right)|0\rangle + sin\left(\frac{\theta}{2}\right)e^{i(\phi_2-\phi_1)}|1\rangle\right)$$

Global phase does not have much of a significance. So most of the times, arbitary choice like $e^{i\phi_1} = 1$ and $e^{i(\phi_1-\phi_1)} = e^{i\phi}$ eliminates it. Eventually the qubit finalizes to following form.

$$|\Psi\rangle = cos\left(\frac{\theta}{2}\right)|0\rangle + sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$$

Aliter:

With an non-zero global phase, the azimuthal angle ϕ_1 can be replaced by ϕ. So that the other one can be expressed with a phase denoted by λ so, ϕ_2 = $\phi + \lambda$. Hence the qubit takes the form as shown below.

$$|\Psi\rangle = cos\left(\frac{\theta}{2}\right)e^{i\phi}|0\rangle + sin\left(\frac{\theta}{2}\right)e^{i(\phi+\lambda)}|1\rangle$$

Section 3: Hadamard gate

Hadamard gate (H-gate) performs a rotation operation on a single qubit thus it's termed as a single-qubit gate. The qubit, after rotation, moves away from the poles of the Bloch sphere and create a superposition of $|0\rangle$ and $|1\rangle$. The probability for the qubit to collapse into either of the states is equal. The matrix form of H-gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad =: |x_+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad =: |x_-\rangle$$

When H-gate is applied to $|0\rangle$ the resulting state is along the $|x_+\rangle$ axis and when applied to $|1\rangle$ it's along the $|x_-\rangle$ axis. If the H-gate is applied for the second time we observe an interesting result. It reverts the superposition into the state of origin. In simple expressions we write:

$$H|x_+\rangle = |0\rangle$$
$$H|x_-\rangle = |1\rangle$$

Section 4: SWAP gate

Unlike the H-gate, SWAP Gate is a two-qubit gate. It acts upon the composite state of qubits. The composite state is achieved by the tensor product between two or more states. In the case of a composite state composed of two qubits, the gate swaps the position of two qubits.

For brevity, let's take two qubits spanning over the same vector space with basis vectors $|0\rangle$ & $|1\rangle$

$$|\Psi\rangle_1 = a_1|0\rangle + b_1|1\rangle$$
$$|\Psi\rangle_2 = a_2|0\rangle + b_2|1\rangle$$

$$|\Psi\rangle_1 \otimes |\Psi\rangle_2 = a_1 a_2 |0\rangle|0\rangle + a_1 b_2 |0\rangle|1\rangle + b_1 a_2 |1\rangle|0\rangle + b_1 b_2 |1\rangle|1\rangle$$

This expression is simplified as:

$$|\Psi\rangle_1 \otimes |\Psi\rangle_2 = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

The SWAP gate exchanges the two qubits. It transforms the basis vectors as

$$|00\rangle \rightarrow |00\rangle, \quad |01\rangle \rightarrow |10\rangle, \quad |10\rangle \rightarrow |01\rangle, \quad |11\rangle \rightarrow |11\rangle$$

Controlled SWAP Gate (Fredkin Gate)

However, in the case of a composite state composed of three qubits a controlled SWAP gate (Fredkin Gate) is implemented. This is a three-qubit gate We take the first qubit as a control qubit and the other two qubits are the targets. The targets are swapped only if the reference qubit is in $|1\rangle$ state.