VECTOR CALCULUS

CURVES IN SPACE:

If $\bar{r}$ is a position of a point $\mathrm{P(x, y, z)}$ on a curve $\mathrm{x = f_1(t), y = f_2(t), z = f_3(t)}$, then $\mathrm{\bar{r} = xi + yj + zk}$ where $\mathrm{x = f_1(t), y = f_2(t), z = f_3(t)}$, is the vector equation of the curve in space.

TANGENT VECTOR: If $\mathrm{\bar{r}} = \bar{f}\ (t)$ is a given curve then $\frac{d\bar{r}}{dt} = \dot{\bar{r}}$ is a vector, tangent to the curve at P (t).

VELOCITY AND ACCELERATION:

Let the position vector $\mathrm{\bar{r}}$ of a particle moving along a curve be given by $\mathrm{\bar{r}\ (t) = x\ (t)i + y\ (t)j + z(t)k}$ where x (t), y (t), z (t) are function of t

Velocity is the rate of change of displacement with respect to time, **the velocity $\bar{\mathbf{v}}$** is given by $\mathrm{\bar{v}} = \frac{\mathrm{d\bar{r}}}{\mathrm{dt}} = \dot{\bar{r}}$

This velocity vector $\mathrm{\bar{v}}$ is in the direction of the tangent to the path of the particle at time t and its magnitude is the speed v.

Since acceleration is the rate of change of velocity with respect to time, **the acceleration $\bar{\mathbf{A}}$** is given by $\mathrm{\bar{A}} = \frac{\mathrm{d\bar{v}}}{\mathrm{dt}} = \frac{\mathrm{d^2\bar{r}}}{\mathrm{dt^2}} = \ddot{\mathrm{\bar{r}}}$

STANDARD RESULTS:

If $\bar{a}, \bar{b}, \bar{c}$ are differentiable vector function of a scalar variable t and Φ is a scalar function of t then

(i) $\frac{d}{dt}\left(\bar{a} \pm \bar{b}\right) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$ **(ii)** $\frac{d}{dt}\left(\bar{a} \cdot \bar{b}\right) = \bar{a} \cdot \frac{d\bar{b}}{dt} + \bar{b} \cdot \frac{d\bar{a}}{dt}$

(iii) $\frac{d}{dt}\left(\bar{a} \times \bar{b}\right) = \bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \times \bar{b}$ **(iv)** $\frac{d}{dt}(\Phi\bar{a}) = \Phi\frac{d\bar{a}}{dt} + \bar{a}\frac{d\Phi}{dt}$

(v) $\frac{d}{dt}\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] = \left[\frac{d\bar{a}}{dt}\ \ \bar{b}\ \ \bar{c}\right] + \left[\bar{a}\ \ \frac{d\bar{b}}{dt}\ \ \bar{c}\right] + \left[\bar{a}\ \ \bar{b}\ \ \frac{d\bar{c}}{dt}\right]$

(vi) $\frac{d}{dt}\left[\bar{a} \times \left(\bar{b} \times \bar{c}\right)\right] = \frac{d\bar{a}}{dt} \times \left(\bar{b} \times \bar{c}\right) + \bar{a} \times \left(\frac{d\bar{b}}{dt} \times \bar{c}\right) + \bar{a} \times \left(\bar{b} \times \frac{d\bar{c}}{dt}\right)$

SOME SOLVED EXAMPLES:

1. If $\bar{a} = 4t^2i + 2tj - t^3k, \bar{b} = \sin t\, i + \cos t\, j$, find **(i)** $\frac{d}{dt}\left(\bar{a} \cdot \bar{b}\right)$ **(ii)** $\frac{d}{dt}\left(\bar{a} \times \bar{b}\right)$

Solution: **(i)**
$$\begin{aligned}\frac{d}{dt}\left(\bar{a} \cdot \bar{b}\right) &= \bar{a} \cdot \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \cdot \bar{b} \\ &= (4t^2i + 2tj - t^3k) \cdot (\cos t\, i - \sin t\, j + 0k) \\ &\qquad + (8ti + 2j - 3t^2k) \cdot (\sin t\, i + \cos t\, j + 0k) \\ &= 4t^2 \cos t - 2t \sin t + 8t \sin t + 2 \cos t \\ &= 4t^2 \cos t + 6t \sin t + 2 \cos t\end{aligned}$$

(ii)
$$\begin{aligned}\frac{d}{dt}\left(\bar{a} \times \bar{b}\right) &= \bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{b}}{dt} \times \bar{b} \\ &= (4t^2i + 2tj - t^3k) \times (\cos t\, i - \sin t\, j + 0k)\end{aligned}$$

$$+(8ti+2j-3t^3k)\cdot(\sin t\, i+\cos t\, j+0k)$$

$$=\begin{vmatrix} i & j & k \\ 4t^2 & 2t & -t^3 \\ \cos t & -\sin t & 0 \end{vmatrix}+\begin{vmatrix} i & j & k \\ 8t & 2 & -3t^2 \\ \sin t & \cos t & 0 \end{vmatrix}$$

$$=(-t^3\sin t)i-(t^3\cos t)j+(-4t^2\sin t-2t\cos t)k$$

$$+(3t^2\cos t)i-(3t^2\sin t)+(8t\cos t-2\sin t)k$$

$$=(-t^3\sin t+3t^2\cos t)i-(t^3\cos t+3t^2\sin t)j$$

$$+(-4t^2\sin t+6t\cos t-2\sin t)k$$

2. If $\overline{r}=\overline{a}\sinh t+\overline{b}\cosh t$, where $\overline{a},\overline{b}$ are constants, prove that **(i)** $\frac{d^2\overline{r}}{dt^2}=\overline{r}$ **(ii)** $\frac{d\overline{r}}{dt}\times\frac{d^2\overline{r}}{dt^2}=$ constant

Solution: We have $\frac{d\overline{r}}{dt}=\bar{a}\cosh t+\bar{b}\sinh t$

$$\frac{d^2\overline{r}}{dt^2}=\overline{a}\sinh t+\overline{b}\cosh t=\overline{r}$$

Further $\frac{d\overline{r}}{dt}\times\frac{d^2\overline{r}}{dt^2}=\left(\bar{a}\cosh t+\bar{b}\sinh t\right)\times\left(\overline{a}\sinh t+\overline{b}\cosh t\right)$

$$=\bar{a}\times\bar{a}\sinh t\cosh t+\bar{a}\times\bar{b}\cosh^2 t+\bar{b}\times\bar{a}\sinh^2 t+\bar{b}\times\bar{b}\sinh t$$

But $\bar{a}\times\bar{a}=0,\bar{b}\times\bar{b}=0$ and $\bar{b}\times\bar{a}=-\bar{a}\times\bar{b}$

$\therefore\frac{d\overline{r}}{dt}\times\frac{d^2\overline{r}}{dt^2}=\bar{a}\times\bar{b}\left(\cosh^2 t-\sinh^2 t\right)=\bar{a}\times\bar{b}$, a constant

3. If $\frac{d\overline{a}}{dt}=\overline{u}\times\overline{a}$ and $\frac{d\overline{b}}{dt}=\overline{u}\times\overline{b}$, prove that $\frac{d}{dt}\left(\overline{a}\times\overline{b}\right)=\overline{u}\times\left(\overline{a}\times\overline{b}\right)$

Solution: We have $\frac{d}{dt}\left[\bar{a}\times\bar{b}\right]=\bar{a}\times\frac{d\bar{b}}{dt}+\frac{d\bar{a}}{dt}\times\bar{b}$

$$=\bar{a}\times\left(\bar{u}\times\bar{b}\right)+(\bar{u}\times\bar{a})\times\bar{b}$$

$$=\left(\bar{a}\cdot\bar{b}\right)\bar{u}-(\bar{a}\cdot\bar{u})\bar{b}+\left(\bar{u}\cdot\bar{b}\right)\bar{a}-\left(\bar{a}\cdot\bar{b}\right)\bar{u}$$

$$=\left(\bar{u}\cdot\bar{b}\right)\bar{a}-(\bar{u}\cdot\bar{a})\bar{b}$$

$$=\bar{u}\times\left(\bar{a}\times\bar{b}\right)$$

4. Verify the result $\frac{d}{dt}\left(\overline{a}\times\overline{b}\right)=\overline{a}\times\frac{d\overline{b}}{dt}+\frac{d\overline{a}}{dt}\times\overline{b}$ for $\overline{a}=5t^2i+t\,j-t^3k,\overline{b}=\sin t\, i-\cos t\, j$

Solution: $\overline{a}\times\overline{b}=\begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}=(-t^3\cos t)i-(t^3\sin t)j-(5t^2\cos t+t\sin t)k$

$\therefore\frac{d}{dt}\left(\overline{a}\times\overline{b}\right)=(-3t^2\cos t+t^3\sin t)i-(3t^2\sin t+t^3\cos t)j$

$$-(10t\cos t-5t^2\sin t+\sin t+t\cos t)k \qquad \text{............ (1)}$$

Now, $\frac{d\bar{a}}{dt}=10ti+j-3t^2k,\frac{d\bar{b}}{dt}=\cos ti+\sin tj+0k$

$a\times\frac{d\bar{b}}{dt}=\begin{vmatrix} i & j & k \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix}=(t^3\sin t)i-(t^3\cos t)j+(5t^2\sin t-t\cos t)k$

$$\frac{d\bar{a}}{dt}\times\bar{b}=\begin{vmatrix} i & j & k \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}=(-3t^2\cos t)i-(3t^2\sin t)j-(10t\cos t+\sin t)k$$

$$\therefore\ \bar{a}\times\frac{d\bar{b}}{dt}+\frac{d\bar{a}}{dt}\times\bar{b}=(t^3\sin t-3t^2\cos t)i-(t^3\cos t+3t^2\sin t)j$$

$$-(-5t^2\sin t+t\cos t+10t\cos t+\sin t)k \qquad \text{............ (2)}$$

From (1) and (2), we get, $\frac{d}{dt}\left(\bar{a}\times\bar{b}\right)=\frac{d\bar{a}}{dt}\times\bar{b}+\bar{a}\times\frac{d\bar{b}}{dt}$

POINT FUNCTION:

A variable quantity, which depends for its value upon the coordinates of the point of a region, say (x, y, z) in space, is called a Point Function. There are two types of point function:

SCALAR POINT FUNCTION:

If to each point P of a region R in space there corresponds a definite scalar (x, y, z) , then Φ is called a scalar point function in region R. The region R so defined is called a scalar field.

Examples: Density of a body, temperature of a body at any instant

VECTOR POINT FUNCTIONS:

If to each point P of a region R in space there corresponds a definite vector F(x, y, z) , then F is called a vector point function in region R. The region R so defined is called a vector field.

Examples: The velocity of a moving fluid at any instant, Gravitational force are vector point function.

VECTOR OPERATOR DEL ∇ (OR NABLA):

The vector differential operator del is written as ∇ is defined by $\nabla = \mathrm{i}\frac{\partial}{\partial \mathrm{x}}+\mathrm{j}\frac{\partial}{\partial \mathrm{y}}+\mathrm{k}\frac{\partial}{\partial \mathrm{z}}$

GRADIENT:

If $\Phi(x, y, z)$is a scalar point function then the gradient of Φ written as $\nabla\Phi$ or gradΦ is defind by

$$\mathrm{grad}\ \Phi\ =\nabla\Phi=\mathrm{i}\frac{\partial\Phi}{\partial \mathrm{x}}+\mathrm{j}\frac{\partial\Phi}{\partial \mathrm{y}}+\mathrm{k}\frac{\partial\Phi}{\partial \mathrm{z}}$$

Remark: We also denote grad Φ as, $\mathrm{grad}\ \Phi\ =\nabla\Phi=\sum\mathrm{i}\frac{\partial\Phi}{\partial \mathrm{x}}=\mathrm{i}\frac{\partial\Phi}{\partial \mathrm{x}}+\mathrm{j}\frac{\partial\Phi}{\partial \mathrm{y}}+\mathrm{k}\frac{\partial\Phi}{\partial \mathrm{z}}$

Note: **(1)** grad Φ is a vector point function

(2) If Φ is a constant then $\frac{\partial\Phi}{\partial \mathrm{x}}=\frac{\partial\Phi}{\partial \mathrm{y}}=\frac{\partial\Phi}{\partial \mathrm{z}}=0 \qquad \therefore\ \mathrm{grad}\ \Phi=\bar{0}$.

Results: **(1)** $\nabla(\Phi \pm \Psi) = \nabla\Phi \pm \nabla\Psi$

(2) $\nabla(\Phi\Psi) = \Phi(\nabla\Psi) + (\nabla\Phi)\Psi$

(3) $\nabla f(u) = i\frac{\partial f(u)}{\partial x} + j\frac{\partial f(u)}{\partial y} + k\frac{\partial f(u)}{\partial z} = f'(u)\nabla u$

DIVERGENCE:

Let $\bar{F}(x, y, z) = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ be a vector point function defined in a certain region of space, where the components f_1, f_2, f_3 are functions of x, y, z then the divergence of $\bar{F}$ written as $\nabla \cdot \bar{F}$ or $div\bar{F}$ is defined by

$$div\bar{F} = \nabla \cdot \bar{F} = (\frac{\partial}{\partial x}\bar{\imath} + \frac{\partial}{\partial y}\bar{\jmath} + \frac{\partial}{\partial z}\bar{k}) \cdot (f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note: $\nabla \cdot \bar{F}$ is a scalar point function

CURL :

Let $\bar{F}(x, y, z) = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ be a vector point function defined in a certain region of space then the curl of $\bar{F}$, written as $\nabla \times \bar{F}$ or curl $\bar{F}$ is defined by

$$\mathrm{Curl}\bar{F} = \nabla \times \bar{F} = \left(\frac{\partial}{\partial x}\bar{\imath} + \frac{\partial}{\partial y}\bar{\jmath} + \frac{\partial}{\partial z}\bar{k}\right) \times \left(f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}\right) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note: $\nabla \times \bar{F}$ is a vector point function

1.	**Gradient of Scalar Point Function is Vector Point Function**
2.	**Divergence of Vector Point Function is Scalar Point Function**
3.	**Curl of Vector Point Function is Vector Point Function**

SOME SOLVED EXAMPLES:

1. Prove that $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$ and hence, find f(r) if $\nabla f(r) = 2r^4\bar{r}$

Solution: We have $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

Here, $\Phi = f(r)$ and f is a function of r and r is function of (x, y, z)

$\therefore \nabla f(r) = i\frac{df}{dr}\frac{\partial r}{\partial x} + j\frac{df}{dr}\frac{\partial r}{\partial y} + k\frac{df}{dr}\frac{\partial r}{\partial z}$

But $r^2 = x^2 + y^2 + z^2$ $\quad\therefore 2r = \frac{\partial r}{\partial x} = 2x$ $\quad\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \nabla f(r) = \frac{f'(r)}{r}(xi + yj + zk) = \frac{f'(r)}{r}\bar{r}$

Comparing this with the given expression. i.e. comparing

$\nabla f(r) = f'(r)\frac{\bar{r}}{r}$ with $\nabla f(r) = 2r^4\bar{r} = 2r^5\frac{\bar{r}}{r}$

we see that $f'(r) = 2r^5$

Here, by integration

$\therefore f(r) = \frac{2r^6}{6} + c = \frac{r^6}{3} + c$

2. Prove that $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

Solution: Here, $f(r) = \frac{1}{r}$ $\quad \therefore f'(r) = -\frac{1}{r^2}$

But $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

$\therefore \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\bar{r}}{r} = -\frac{\bar{r}}{r^3}$

Alternatively we have

$\nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$

Here, $\Phi = \frac{1}{r}$ and $r^2 = x^2 + y^2 + z^2$

$\therefore \frac{d\Phi}{dr} = -\frac{1}{r^2}$ and

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial y} = \frac{z}{r}$

$\therefore \nabla\Phi = \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k = -\frac{1}{r^2}\cdot\frac{x}{r}i - \frac{1}{r^2}\cdot\frac{y}{r}j - \frac{1}{r^2}\cdot\frac{z}{r}k = -\frac{1}{r^3}(xi + yj + zk) = -\frac{\bar{r}}{r^3}$

3. Prove that $\nabla r^n = nr^{n-2}\bar{r}$

Solution: We know, from 3(a), $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$

Here, $f(r) = r^n$ $\quad \therefore f'(r) = nr^{n-1}$

$\therefore \nabla f(r) = nr^{n-1}\frac{\bar{r}}{r} = nr^{n-2}\bar{r}$

Alternatively we have

$\nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$

Here $\Phi = r^n$ and $r^2 = x^2 + y^2 + z^2$

$\therefore \frac{d\Phi}{dr} = nr^{n-1}$ and $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\begin{aligned}\therefore \nabla\Phi &= \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial x}i + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial y}j + \frac{d\Phi}{dr}\cdot\frac{\partial r}{\partial z}k \\ &= nr^{n-1}\left(\frac{x}{r}\right)i + nr^{n-1}\left(\frac{y}{r}\right)j + nr^{n-1}\left(\frac{z}{r}\right)k \\ &= nr^{n-2}(xi + yj + zk) \\ &= nr^{n-2}\bar{r}\end{aligned}$$

4. Find $\nabla\left(e^{r^2}\right)$

Solution: Here, $f(r) = e^{r^2}$ $\quad \therefore f'(r) = e^{r^2} \cdot 2r$

$\therefore \nabla f(r) = f'(r)\frac{\bar{r}}{r} = e^{r^2} \cdot 2r \cdot \frac{\bar{r}}{r} = 2e^{r^2}\bar{r}$

Alternatively we have

$\nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$

Here $\Phi = e^{r^2} \cdot 2r$ and $r^2 = x^2 + y^2 + z^2$

$\therefore \frac{d\Phi}{dr} = e^{r^2} \cdot 2r$ and $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \nabla\Phi = \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial x} i + \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial y} j + \frac{d\Phi}{dr} \cdot \frac{\partial r}{\partial z} k$

$= e^{r^2} \cdot 2r\left(\frac{x}{r}\right)i + e^{r^2} \cdot 2r\left(\frac{y}{r}\right)j + e^{r^2} \cdot 2r\left(\frac{z}{r}\right)k$

$= 2e^{r^2}r \cdot \frac{(xi+yj+zk)}{r}$

$= 2e^{r^2} \cdot \bar{r}$

5. Find $\Phi(r)$ such that $\nabla\Phi = -\frac{\bar{r}}{r^5}$ and $\Phi(2) = 3$

Solution: We have $\quad \nabla\Phi = -\bar{r}(x^2 + y^2 + z^2)^{-5/2}$ $\quad \left[\because r = \sqrt{x^2 + y^2 + z^2}\right]$

$= -(x^2 + y^2 + z^2)^{-5/2}(xi + yj + zk)$ .... (1)

But $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$ .... (2)

Comparing (1) and (2), we get,

$\frac{\partial\Phi}{\partial x} = -x(x^2 + y^2 + z^2)^{-5/2}, \frac{\partial\Phi}{\partial y} = -y(x^2 + y^2 + z^2)^{-5/2}, \frac{\partial\Phi}{\partial z} = -z(x^2 + y^2 + z^2)^{-5/2}$

But $d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = -(x^2 + y^2 + z^2)^{-5/2}(xdx + ydy + zdz)$

Now put $x^2 + y^2 + z^2 = t$

$\therefore 2(xdx + ydy + zdz) = dt$

$\therefore \; d\Phi = -t^{-5/2} \cdot \frac{dt}{2}$

Integrating, $\Phi = -\frac{1}{2}\frac{t^{-3/2}}{(-3/2)} + c = \frac{t^{-3/2}}{3} + c$

$= \frac{1}{3}(x^2 + y^2 + z^2)^{-3/2} + c$

$= \frac{1}{3} \cdot \frac{1}{r^3} + c$

But by data $\Phi(r) = 3$ when $r = 2$

$\therefore 3 = \frac{1}{3} \cdot \frac{1}{8} + c$ $\quad \therefore c = \frac{71}{24}$

$\therefore \Phi = \frac{1}{3} \cdot \frac{1}{r^3} + \frac{71}{24} = \frac{1}{3}\left(\frac{1}{r^3} + \frac{71}{8}\right)$

6. If $u = x + y + z, v = x + y,\ w = -2xz - 2yz - z^2$, show that $\nabla u \,.\, [\nabla v \times \nabla w] = 0$

Solution: $\nabla u = i + j + k, \nabla v = i + j, \nabla w = -2zi - 2zj - (2x + 2y + 2z)k$

$$\nabla u \,.\, [\nabla v \times \nabla w] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -2z & -2z & -2x - 2y - 2z \end{vmatrix} = 0$$

(Since first two columns are identicarl)

7. If $\Phi = x^2 + y^2 + z^2,\ \Psi = x^2y^2 + y^2z^2 + z^2x^2$, find $\nabla[\nabla\Phi \cdot \nabla\Psi]$

Solution: $\nabla\Phi = 2xi + 2yj + 2zk$

$$\nabla\Psi = (2xy^2 + 2xz^2)i + \left(2yx^2 + 2yz^2\right)j + \left(2zx^2 + 2zy^2\right)k$$

$$\therefore \nabla\Phi \cdot \nabla\Psi = 4x^2(y^2 + z^2) + 4y^2(x^2 + z^2) + 4z^2(x^2 + y^2) = 8(x^2y^2 + y^2z^2 + z^2x^2)$$

$$\therefore \nabla(\nabla\Phi \cdot \nabla\Psi) = 16x(y^2 + z^2)i + 16y(z^2 + x^2)j + 16z(x^2 + y^2)k$$

8. If $\Phi = x^3 + y^3 + z^3 - 3xyz$, find **(a)** $\overline{r}\,.\nabla\Phi$ **(b)** div $\overline{F}$ **(c)** curl $\overline{F}$ where $\overline{F} = \nabla\Phi$.

Solution: **(a)** $\nabla\Phi = i\dfrac{\partial\Phi}{\partial x} + j\dfrac{\partial\Phi}{\partial y} + k\dfrac{\partial\Phi}{\partial z}$

$$\therefore \overline{F} = i\left(3x^2 - 3yz\right) + j(3y^2 - 3xz) + k(3z^2 - 3xy)$$

But $\bar{r} = xi + yj + zk$

$$\therefore \bar{r} \cdot \nabla\Phi = x\left(3x^2 - 3yz\right) + y(3y^2 - 3xz) + z(3z^2 - 3xy) = 3\left(x^3 + y^3 + z^3 - 3xyz\right) = 3\Phi$$

(b) div $\overline{F} = \nabla \cdot (\nabla\Phi)$

$$= \frac{\partial}{\partial x}\left(3x^2 - 3yz\right) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6x + 6y + 6z = 6(x + y + z)$$

(c) Curl $\overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3\left(x^2 - yz\right) & 3(y^2 - xz) & 3(z^2 - xy) \end{vmatrix}$

$$= i(-3x + 3x) + j(3y - 3y) + k(-3z + 3z) = 0$$

9. If $\bar{\mathrm{f}} = (x + y + 1)i + j - (x + y)k$, prove that $\overline{f}$. Curl $\overline{f} = 0$

Solution: Curl $\overline{f} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + y + 1 & 1 & -x - y \end{vmatrix} = i(-1) - j(-1) + k(-1) = -i + j - k$

$$\therefore \overline{f} \cdot \mathrm{curl}\overline{f} = [(x + y + 1)i + j - (x + y)k] \cdot [-i + j - k]$$

$$= -(x + y + 1) + 1 + (x + y) = 0$$

10. If $\overline{r}_1$ and $\overline{r}_2$ are vectors joining the fixed points $A\ (x_1, y_1, z_1)$ and B (x_2, y_2, z_2) to a variable point P (x, y, z) then prove that **(a)** $\nabla \cdot (\overline{r}_1 \times \overline{r}_2) = 0$, **(b)** $\nabla \times (\overline{r}_1 \times \overline{r}_2) = 2(\overline{r}_1 - \overline{r}_2)$.

Solution: **(a)** We have $\overline{r}_1 = (x - x_1)i + (y - y_1)j + (z - z_1)k$

and $\overline{r}_2 = (x - x_2)i + (y - y_2)j + (z - z_2)k$

$$\text{Now, } \overline{r}_1 \times \overline{r}_2 = \begin{vmatrix} i & j & k \\ x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \end{vmatrix}$$

$$= i[(y - y_1)(z - z_2) - (y - y_2)(z - z_1)] + j[\dots\dots] + k[\dots\dots]$$

$$= i\Phi_1 + j\Phi_2 + k\Phi_3 \text{ say}$$

$$\therefore \nabla \cdot (\overline{r}_1 \times \overline{r}_2) = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_3}{\partial z}$$

But $\frac{\partial \Phi_1}{\partial x} = 0, \ \frac{\partial \Phi_2}{\partial y} = 0, \ \frac{\partial \Phi_3}{\partial z} = 0$

$$\therefore \nabla \cdot (\overline{r}_1 \times \overline{r}_2) = 0$$

(b) $$\nabla \times (\overline{r}_1 \times \overline{r}_2) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y - y_1)(z - z_2) - (y - y_2)(z - z_1) & \dots\dots\dots & \dots\dots\dots\dots \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\{(x - x_1)(y - y_2) - (x - x_2)(y - y_1)\}\right.$$

$$\left. - \frac{\partial}{\partial z}\{(z - z_1)(x - x_2) - (z - z_2)(x - x_1)\}\right] + j[\dots\dots] + k[\dots\dots]$$

$$= i[\{(x - x_1) - (x - x_2)\} - \{(x - x_2) - (x - x_1)\}] + j[\dots] + k[\dots]$$

$$= i[2(x - x_1) - 2(x - x_2)] + j[\dots\dots] + k[\dots\dots]$$

$$= 2[i(x - x_1) + j(y - y_1) + k(z - z_1)] - 2[i(x - x_2) + j(y - y_2) + k(z - z_2)]$$

$$= 2(\overline{r}_1) - 2(\overline{r}_2)$$

$$= 2(\overline{r}_1 - \overline{r}_2)$$

11. Prove that $\nabla . \left(\nabla \times \overline{F}\right) = 0$ where $\overline{F}$ is a vector point function

Solution: Let $\overline{F} = F_1 i + F_2 j + F_3 k$

$$\text{Then } \nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + j\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\therefore \nabla \cdot \left(\nabla \times \overline{F}\right) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y \partial z} = 0$$

12. If $u\overline{F} = \nabla v$ where u and v are scalar fields and $\overline{F}$ is a vector field, prove that $\overline{F} \cdot \text{curl}\, \overline{F} = 0$.

Solution: Let $\overline{F} = F_1 i + F_2 j + F_3 k$. By data $u\overline{F} = \nabla v$

$$\therefore uF_1 i + uF_2 j + uF_3 k = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}$$

$$\therefore uF_1 = \frac{\partial v}{\partial x}, \quad uF_2 = \frac{\partial v}{\partial y}, \quad uF_3 = \frac{\partial v}{\partial z}$$

$$\therefore F_1 = \frac{1}{u}\frac{\partial v}{\partial x}, \quad F_2 = \frac{1}{u}\frac{\partial v}{\partial y}, \quad F_3 = \frac{1}{u}\frac{\partial v}{\partial z}$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= i\left[\left\{-\frac{1}{u^2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{1}{u}\frac{\partial^2 v}{\partial y \partial z}\right\} - \left\{-\frac{1}{u^2}\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} + \frac{1}{u}\frac{\partial^2 v}{\partial y \partial z}\right\}\right] + j[\dots\dots] + k[\dots\dots]$$

$$= i\,\frac{1}{u^2}\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + j[\dots\dots] + k[\dots\dots]$$

$$\therefore \bar{F}\,\cdot\text{curl}\,\bar{F} = \frac{1}{u^3}\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + [\dots\dots] + [\dots\dots] = 0$$

(The terms are cancelled because of symmetry)

OR **If $\overline{\boldsymbol{F}} = \boldsymbol{\Phi}\nabla \boldsymbol{f}$ where $\boldsymbol{\Phi}$ & f are scalar point functions and $\overline{\boldsymbol{F}}$ is a vector point function, prove that $\overline{\boldsymbol{F}}\,.\left(\boldsymbol{\nabla} \times \overline{\boldsymbol{F}}\right) = \mathbf{0}$**

13. If Φ_1 and Φ_2 are scalar functions, then prove that, $\nabla \times (\Phi_1 \nabla \Phi_2) = \nabla \Phi_1 \times \nabla \Phi_2$.

Solution: We have $\nabla \Phi_2 = i\frac{\partial \Phi_2}{\partial x} + j\frac{\partial \Phi_2}{\partial y} + k\frac{\partial \Phi_2}{\partial z}$

$$\therefore \Phi_1 \nabla \Phi_2 = i\Phi_1\frac{\partial \Phi_2}{\partial x} + j\Phi_1\frac{\partial \Phi_2}{\partial y} + k\Phi_1\frac{\partial \Phi_2}{\partial z}$$

$$\therefore \nabla \times (\Phi_1 \nabla \Phi_2) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \Phi_1\frac{\partial \Phi_2}{\partial x} & \Phi_1\frac{\partial \Phi_2}{\partial y} & \Phi_1\frac{\partial \Phi_2}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial \Phi_1}{\partial y}\cdot\frac{\partial \Phi_2}{\partial z} + \Phi_1\frac{\partial^2 \Phi_2}{\partial y \partial z} - \frac{\partial \Phi_1}{\partial z}\cdot\frac{\partial \Phi_2}{\partial y} - \Phi_1\frac{\partial^2 \Phi_2}{\partial y \partial z}\right] + j[\dots\dots] + k[\dots\dots]$$

$$= i\left[\frac{\partial \Phi_1}{\partial y}\cdot\frac{\partial \Phi_2}{\partial z} - \frac{\partial \Phi_1}{\partial z}\cdot\frac{\partial \Phi_2}{\partial y}\right] + j[\dots\dots] + k[\dots\dots]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial y} & \frac{\partial \Phi_1}{\partial z} \\ \frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_2}{\partial y} & \frac{\partial \Phi_2}{\partial z} \end{vmatrix} = \nabla \Phi_1 \times \nabla \Phi_2$$

IDENTIFIES INVOLVING GRAD, DIV, CURL:

1. $grad\ (\Phi \pm \Psi) = grad\ \Phi\ \pm\ grad\ \Psi$

$\nabla(\Phi \pm \Psi) = \nabla\Phi \pm \nabla\Psi$

2. $div\left(\bar{f} \pm \bar{g}\right) = div\,\bar{f}\ \pm\ div\,\bar{g}$

$\nabla\cdot\left(\bar{\mathrm{f}} \pm \bar{\mathrm{g}}\right) = \nabla\cdot\bar{\mathrm{f}}\ \pm\ \nabla\cdot\bar{\mathrm{g}}$

3. $Curl\left(\bar{f} \pm \bar{g}\right) = Curl\,\bar{f}\ \pm Curl\,\bar{g}$

$\nabla \times \left(\bar{\mathrm{f}} \pm \bar{\mathrm{g}}\right) = \nabla \times \bar{\mathrm{f}}\ \pm\ \nabla \times \bar{\mathrm{g}}$

4. $grad\ (\Phi\Psi) =\ \Phi\ grad\Psi + \Psi\ grad\Phi$

$\nabla(\Phi\Psi) =\ \Phi\ \nabla\Psi + \Psi\ \nabla\Phi$ where $\Phi$and Ψ are scalar functions

5. $grad\left(\bar{f}\cdot\bar{g}\right)=\bar{f}\times(curl\,\bar{g})+\bar{g}\times\left(curl\bar{f}\right)+\left(\bar{f}\cdot\nabla\right)\bar{g}+(\bar{g}\cdot\nabla)\bar{f}$

$\nabla\left(\bar{f}\cdot\bar{g}\right)=\bar{f}\times(\nabla\times\bar{g})+\bar{g}\times\left(\nabla\times\bar{f}\right)+\left(\bar{f}\cdot\nabla\right)\bar{g}+(\bar{g}\cdot\nabla)\bar{f}$

6. $div\left(\Phi\bar{f}\right)=\Phi div\bar{f}+\bar{f}\cdot grad\,\Phi$

$\nabla\cdot\left(\Phi\bar{f}\right)=\Phi\left(\nabla\cdot\bar{f}\right)+\bar{f}\cdot(\nabla\Phi)$

7. $div\left(\bar{f}\times\bar{g}\right)=\bar{g}\cdot curl\bar{f}-\bar{f}\cdot curl\bar{g}$

$\nabla\cdot\left(\bar{f}\times\bar{g}\right)=\bar{g}\cdot\left(\nabla\times\bar{f}\right)-\bar{f}\cdot(\nabla\times\bar{g})$

8. $curl\left(\bar{f}\times\bar{g}\right)=\bar{f}div\,\bar{g}-\bar{g}\,div\,\bar{f}+(\bar{g}\cdot\nabla)\bar{f}-\left(\bar{f}\cdot\nabla\right)\bar{g}$

$\nabla\times\left(\bar{f}\times\bar{g}\right)=\bar{f}(\nabla\cdot\bar{g})-\bar{g}\left(\nabla\cdot\bar{f}\right)+(\bar{g}\cdot\nabla)\bar{f}-\left(\bar{f}\cdot\nabla\right)\bar{g}$

9. $curl\left(\Phi\bar{f}\right)=\Phi\left(curl\bar{f}\right)+(grad\,\Phi)\times\bar{f}$

$\nabla\times\left(\Phi\bar{f}\right)=\Phi\left(\nabla\times\bar{f}\right)+(\nabla\,\Phi)\times\bar{f}$

Sigma Notation:

$\nabla=\sum i\frac{\partial}{\partial x}$

$\nabla\Phi=\sum i\frac{\partial\Phi}{\partial x}$

$\nabla\cdot\bar{f}=\sum\bar{i}\cdot\frac{\partial\bar{f}}{\partial x}$

$\nabla\times\bar{f}=\sum\bar{i}\times\frac{\partial\bar{f}}{\partial x}$

Differentiation Formula:

$\frac{d}{dx}\left(\bar{f}\pm\bar{g}\right)=\frac{d\bar{f}}{dx}\pm\frac{d\bar{g}}{\partial x}$

$\frac{d}{dx}\left(\bar{f}\cdot\bar{g}\right)=\bar{f}\cdot\frac{d\bar{g}}{dx}+\frac{d\bar{f}}{\partial x}\cdot\bar{g}$

$\frac{d}{dx}\left(\bar{f}\times\bar{g}\right)=\bar{f}\times\frac{d\bar{g}}{dx}+\frac{d\bar{f}}{\partial x}\times\bar{g}$

$\frac{d}{dx}\left(\Phi\bar{f}\right)=\Phi\frac{d\bar{f}}{dx}+\frac{d\Phi}{\partial x}\bar{f}$

SOME SOLVED EXAMPLES:

1. If $\bar{a}$ is a constant vector and $\bar{r}=xi+yj+zk$, prove that,

(i) $div\,\bar{a}=0$ **(ii)** $curl\,\bar{a}=\bar{0}$ **(iii)** $grad\,r=\frac{1}{r}\,\bar{r}$ **(iv)** $div\,\bar{r}=3$ **(v)** $curl\,\bar{r}=\bar{0}$

Solution: **(i)** Let $\bar{a}=a_1i+a_2j+a_3k$ where a_1,a_2,a_3 are constants

$\therefore\nabla\cdot\bar{a}=\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}=0$

(ii) $\nabla\times\bar{a}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_1 & a_2 & a_3\end{vmatrix}=i\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)+j\left(\frac{\partial a_1}{\partial z}-\frac{\partial a_3}{\partial x}\right)+k\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial z}\right)=\bar{0}$

(iii) grad $r=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}=\frac{x}{r}i+\frac{y}{r}j+\frac{z}{r}k=\frac{1}{r}(xi+yj+zk)=\frac{1}{r}\,\bar{r}$

Aliter: $grad\,f(r)=\frac{f'(r)\bar{r}}{r}$ put $f(r)=r$ $\therefore f'(r)=1$

$\therefore grad\,f(r)=\frac{(1)\bar{r}}{r}=\frac{\bar{r}}{r}$

(iv) div $\bar{r}=\left(\frac{\partial}{\partial x}x\right)+\left(\frac{\partial}{\partial y}y\right)+\left(\frac{\partial}{\partial x}z\right)=1+1+1=3$

(v) curl $\overline{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \overline{0}$

2. If $\overline{a}$ and $\overline{b}$ are constant vectors and $\overline{r} = xi + yj + zk$, prove that, $\nabla(\overline{a} \,.\, \overline{r}) = \overline{a}$

Solution: Let $\overline{a} = a_1 i + a_2 j + a_3 k$ and $\overline{b} = b_1 i + b_2 j + b_3 k$

$\overline{a} \,.\, \overline{r} = (a_1 i + a_2 j + a_3 k) \cdot (xi + yj + zk) = a_1 x + a_2 y + a_3 z$

$\therefore \nabla(\overline{a} \,.\, \overline{r}) = i \frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z) + j \frac{\partial}{\partial y}(a_1 x + a_2 y + a_3 z) + k \frac{\partial}{\partial z}(a_1 x + a_2 y + a_3 z)$

$= a_1 i + a_2 j + a_3 k = \overline{a}$

3. If $\overline{a}$ and $\overline{b}$ are constant vectors and $\overline{r} = xi + yj + zk$, prove that, $div\ (\overline{a} \times \overline{r}) = 0$.

Solution: $\overline{a} \times \overline{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)i + (a_3 x - a_1 z)j + (a_1 y - a_2 x)k$

$div\ (\overline{a} \times \overline{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) = 0$

Aliter:

Assuming the result that $\nabla \cdot (\overline{a} \times \overline{r})$ can be looked upon as a scalar triple product treating ∇ as a vector we have

$div(\overline{a} \times \overline{r}) = \nabla \cdot (\overline{a} \times \overline{r}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x) = 0$

Aliter:

$\text{div}\ (\overline{a} \times \overline{r}) = \overline{r} \cdot \text{curl}\ \overline{a} - \overline{a} \cdot \text{curl}\ \overline{r}$

$= \overline{r} \cdot \overline{0} - \overline{a} \cdot \overline{0}$ $\quad [\because \text{curl}\ \overline{a} = \overline{0}, \text{curl}\ \overline{r} = \overline{0}]$

$= \overline{0}$

4. If $\overline{a}$ and $\overline{b}$ are constant vectors and $\overline{r} = xi + yj + zk$, prove that, $div\ (\overline{a} \,.\, \overline{r})\overline{a} = a^2$

Solution: $\overline{a} \cdot \overline{r} = a_1 x + a_2 y + a_3 z$

$\therefore (\overline{a} \cdot \overline{r})\overline{a} = (a_1 x + a_2 y + a_3 z)a_1 i + (a_1 x + a_2 y + a_3 z)a_2 j + (a_1 x + a_2 y + a_3 z)a_3 k$

$\therefore$ div $(\overline{a} \cdot \overline{r})\overline{a} = \frac{\partial}{\partial x}[a_1(a_1 x + a_2 y + a_3 z)] + \frac{\partial}{\partial y}[a_2(a_1 x + a_2 y + a_3 z)]$

$+ \frac{\partial}{\partial z}[a_3(a_1 x + a_2 y + a_3 z)]$

$= {a_1}^2 + {a_2}^2 + {a_3}^2 = a^2$

Aliter:

div $[(\overline{a} \,.\, \overline{r})\overline{a}] = \sum i \cdot \frac{\partial}{\partial x}[(\overline{a} \,.\, \overline{r})\overline{a}]$

$$= \sum i \cdot \left\{\frac{\partial}{\partial x}(\bar{a}\,.\,\bar{r})\right\}\bar{a} + \sum i \cdot (\bar{a}\,.\,\bar{r})\frac{\partial}{\partial x}\bar{a}$$

$$= \sum i \cdot \left\{\frac{\partial}{\partial x}(\bar{a}\,.\,\bar{r})\right\}\bar{a} \qquad \left[\because \frac{\partial \bar{a}}{\partial x} = 0\right]$$

$$= \sum i \cdot \left[\frac{\partial \bar{a}}{\partial x}\cdot \bar{r} + \bar{a}\cdot\frac{\partial \bar{r}}{\partial x}\right]\bar{a}$$

$$= \sum i \cdot \left[\bar{a}\cdot\frac{\partial \bar{r}}{\partial x}\right]\bar{a} \qquad \left[\because \frac{\partial \bar{a}}{\partial x} = 0\right]$$

But $\frac{\partial \bar{r}}{\partial x} = \frac{\partial}{\partial x}(xi + yj + zk) = i$

$\therefore$ div $[(\bar{a}\ \cdot \bar{r})\bar{a}] = \sum i \cdot [\bar{a}\cdot i]\,\bar{a} = \sum i \cdot a_1\bar{a} = \sum a_1{}^2 = a_1{}^2 + a_2{}^2 + a_3{}^2 = a^2$

Aliter:

$$\text{div}\,[(\bar{a}\cdot\bar{r})\bar{a}] = (\bar{a}\cdot\bar{r})\,\text{div}\,\bar{a} + \bar{a}\cdot\ \text{grad}(\bar{a}\cdot\bar{r})$$

$$= (\bar{a}\cdot\bar{r})0 + \bar{a}\cdot\bar{a} \qquad [\because\ \text{div}\,\bar{a} = 0\ \&\ \text{grad}(\bar{a}\cdot\bar{r}) = \bar{a}]$$

$$= \bar{a}\cdot\bar{a}$$

$$= a^2$$

Aliter:

Let $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{r} = xi + yj + zk$, then

$\bar{a}\cdot\bar{r} = a_1x + a_2y + a_3z = \Phi$, say

$\therefore \nabla\cdot\{(\bar{a}\cdot\bar{r})\bar{a}\} = (\bar{a}\cdot\bar{r})\nabla\cdot\bar{a} + \bar{a}\cdot\nabla(\bar{a}\cdot\bar{r})$

$$= (a_1x + a_2y + a_3z)\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) + (a_1 i + a_2 j + a_3 k)\cdot\left(\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y} + \frac{\partial\Phi}{\partial z}\right)$$

$$= (a_1x + a_2y + a_3z)\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) + (a_1 i + a_2 j + a_3 k)\cdot(a_1 i + a_2 j + a_3 k)$$

$$= 0 + a_1{}^2 + a_2{}^2 + a_3{}^2 = a^2$$

5. If $\bar{a}$ and $\bar{b}$ are constant vectors and $\bar{r} = xi + yj + zk$, prove that, $div\ (\bar{a}\times(\bar{r}\times\bar{a})) = 2a^2$

Solution: $\bar{r}\times\bar{a} = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = (a_3y - a_2z)i + (a_1z - a_3x)j + (a_2x - a_1y)k$

$$\therefore \bar{a}\times(\bar{r}\times\bar{a}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$$= [a_2(a_2x - a_1y) - a_3(a_1z - a_3x)]i + [\ldots\ldots.]j + [\ldots\ldots.]k$$

$\therefore$ div $[\bar{a}\times(\bar{r}\times\bar{a})] = \frac{\partial}{\partial x}[a_2(a_2x - a_1y) - a_3(a_1z - a_3x)] + \frac{\partial}{\partial y}[\ldots\ldots] + \frac{\partial}{\partial z}[\ldots\ldots]$

$$= (a_2{}^2 + a_3{}^2) + (a_3{}^2 + a_1{}^2) + (a_1{}^2 + a_2{}^2)$$

$$= 2(a_1{}^2 + a_2{}^2 + a_3{}^2) = 2a^2$$

Aliter:

div $[\bar{a}\times(\bar{r}\times\bar{a})]$ $=$ div $[(\bar{a}\cdot\bar{a})\bar{r} - (\bar{a}\cdot\bar{r})\bar{a}]$

$=$ div $[a^2\bar{r} - (a_1x + a_2y + a_3z)\bar{a}]$

$$= \sum i\cdot\frac{\partial}{\partial x}[(a^2\bar{r}) - (a_1x + a_2y + a_3z)\bar{a}]$$

$$= \sum i\cdot[a^2 - a_1\bar{a}]$$

$= \sum a^2 - \sum a_1{}^2$

$= a^2 + a^2 + a^2 - a_1{}^2 - a_2{}^2 - a_3{}^2$

$= 3a^2 - a^2 = 2a^2$

Aliter:

$\text{div}\,[\bar{a} \times (\bar{r} \times \bar{a})] = \text{div}\,[(\bar{a} \cdot \bar{a})\bar{r} - (\bar{a} \cdot \bar{r})\bar{a}]$

$= \text{div}\,[a^2\bar{r} - (\bar{a} \cdot \bar{r})\bar{a}]$

$= \text{div}\,(a^2\bar{r}) - \text{div}\,[(\bar{a} \cdot \bar{r})\bar{a}]$

Now $\text{div}\,(a^2\bar{r}) = a^2 \text{div}\,\bar{r} + \bar{r} \cdot\ \text{grad}(a^2)$

$= a^2 3 + \bar{r} \times \bar{0}$

$= 3a^2$

$\text{div}\,[(\bar{a} \cdot \bar{r})\bar{a}] = (\bar{a} \cdot \bar{r})\ \text{div}\,\bar{a} + \bar{a} \cdot\ \text{grad}(\bar{a} \cdot \bar{r})$

$= (\bar{a} \cdot \bar{r})0 + \bar{a} \cdot \bar{a}$ $\qquad [\because \text{grad}(\bar{a} \cdot \bar{r}) = \bar{a}]$

$= \bar{a} \cdot \bar{a}$

$= a^2$

$\therefore \text{div}\,[\bar{a} \times (\bar{r} \times \bar{a})] = 3a^2 - a^2 = 2a^2$

6. If $\overline{a}$ and $\overline{b}$ are constant vectors and $\overline{r} = xi + yj + zk$, prove that, $curl(\overline{a} \times \overline{r}) = 2\overline{a}$

Solution: $\overline{a} \times \overline{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z - a_3y)i + (a_3x - a_1z)j + (a_1y - a_2x)k$

$\therefore$ Curl $(\overline{a} \times \overline{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$

$= i(a_1 + a_1) - j(-a_2 - a_2) + k(a_3 + a_3)$

$= 2a_1i + 2a_2j + 2a_3k$

$= 2\overline{a}$

Aliter:

$\text{curl}\,(\bar{a} \times \bar{r}) = \bar{a}(\text{div}\,\bar{r}) - \bar{r}(\text{div}\,\bar{a}) + (\bar{r} \cdot \nabla)\bar{a} - (\bar{a} \cdot \nabla)\bar{r}$

$= \bar{a}(3) - \bar{r}(0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\bar{a} - \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)$

$= 3\bar{a} - \bar{0} + \bar{0} - \left(a_1\bar{\imath} + a_2\bar{\jmath} + a_3\bar{k}\right)$

$= 3\bar{a} - \bar{a}$

$= 2\bar{a}$

7. If $\overline{a}$ and $\overline{b}$ are constant vectors and $\overline{r} = xi + yj + zk$, prove that, $\nabla . \left(\frac{\overline{a} \times \overline{r}}{r}\right) = 0$

Solution: $\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r}\right)=\nabla\cdot\left(\bar{a}\times\frac{\bar{r}}{r}\right)=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_1 & a_2 & a_3\\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r}\end{vmatrix}$

$$=\frac{\partial}{\partial x}\left(\frac{a_2z-a_3y}{r}\right)+\frac{\partial}{\partial y}\left(\frac{a_3x-a_1z}{r}\right)+\frac{\partial}{\partial z}\left(\frac{a_1y-a_2x}{r}\right)$$

$$=-\frac{1}{r^2}(a_2z-a_3y)\frac{\partial r}{\partial x}+\cdots+\cdots$$

$$=-(a_2z-a_3y)\frac{x}{r^3}+\cdots+\cdots$$

$$=-\frac{1}{r^3}[(a_2z-a_3y)x+(a_3x-a_1z)y+(a_1y-a_2x)z]$$

$$=0$$

Aliter:

$$\operatorname{div}\left(\frac{\bar{a}\times\bar{r}}{r}\right)=\operatorname{div}\left(\bar{a}\times\frac{\bar{r}}{r}\right)$$

$$=\frac{\bar{r}}{r}\cdot\operatorname{curl}\bar{a}-\bar{a}\cdot\operatorname{curl}\frac{\bar{r}}{r}$$

$$=\frac{\bar{r}}{r}\cdot\bar{0}-\bar{a}\cdot\left(\frac{1}{r}\operatorname{curl}\bar{r}+\left[\operatorname{grad}\left(\frac{1}{r}\right)\right]\times\bar{r}\right)\qquad\because\operatorname{curl}\bar{a}=\bar{0}$$

$$=\bar{0}-\bar{a}\cdot\left[\frac{1}{r}\bar{0}+\left(\frac{\left(\frac{-1}{r^2}\right)\bar{r}}{r}\right)\times\bar{r}\right]\qquad\because\operatorname{curl}\bar{r}=\bar{0}\ \&\ grad\, f(r)=\frac{f'(r)\bar{r}}{r}$$

$$=\bar{a}\cdot\frac{1}{r^3}(\bar{r}\times\bar{r})$$

$$=\bar{a}\cdot\frac{1}{r^3}(\bar{0})=\bar{a}\cdot\bar{0}=0\qquad\because\bar{r}\times\bar{r}=\bar{0}$$

Aliter:

$$\operatorname{div}\left(\frac{\bar{a}\times\bar{r}}{r}\right)=\frac{1}{r}\operatorname{div}(\bar{a}\times\bar{r})+(\bar{a}\times\bar{r})\cdot\operatorname{grad}\left(\frac{1}{r}\right)$$

$$=\frac{1}{r}0+(\bar{a}\times\bar{r})\cdot\left[\frac{\left(-\frac{1}{r^2}\right)\bar{r}}{r}\right]$$

$$\because\operatorname{div}(\bar{a}\times\bar{r})=0\ \&\ grad\, f(r)=\frac{f'(r)\bar{r}}{r}$$

$$=-\frac{1}{r^3}(\bar{a}\times\bar{r})\cdot\bar{r}$$

$=-\frac{1}{r^3}0$ by box product property

$$=0$$

10. If $\bar{a}$ and $\bar{b}$ are constant vectors and $\bar{r}=xi+yj+zk$, prove that, $\nabla\times[(\bar{r}\times\bar{a})\times\bar{b}]=\bar{b}\times\bar{a}$

Solution: By vector triple product we have $(\bar{r}\times\bar{a})\times\bar{b}=(\bar{r}\cdot\bar{b})\bar{a}-(\bar{a}\cdot\bar{b})\bar{r}$

$\therefore\nabla\times[(\bar{r}\times\bar{a})\times\bar{b}]=\nabla\times[(\bar{r}\cdot\bar{b})\bar{a}]-\nabla\times[(\bar{a}\cdot\bar{b})\bar{r}]$ .... (1)

Now let $\bar{a}=a_1i+a_2j+a_3k,\ \ \bar{b}=b_1i+b_2j+b_3k$

$\therefore(\bar{r}\cdot\bar{b})\bar{a}=(b_1x+b_2y+b_3z)(a_1i+a_2j+a_3k)=f_1i+f_2j+f_3k$ say

$$\therefore \nabla \times (\bar{r} \cdot \bar{b})\bar{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + j(\dots\dots) + k(\dots\dots)$$

But $f_1 = (b_1x + b_2y + b_3z)a_1$, $f_2 = (b_1x + b_2y + b_3z)a_2$, $f_3 = (b_1x + b_2y + b_3z)a_3$

$\therefore \frac{\partial f_3}{\partial y} = b_2a_3$, $\frac{\partial f_2}{\partial z} = b_3a_2$ and so on

$$\therefore \nabla \times (\bar{r} \cdot \bar{b})\bar{a} = i(b_2a_3 - b_3a_2) + j(\dots\dots) + k(\dots\dots) = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \bar{b} \times \bar{a}$$

Further, $\nabla \times [(\bar{a} \cdot \bar{b})\bar{r}] = (\bar{a} \cdot \bar{b})\nabla \times \bar{r} = \bar{0}$

Hence, the result

Aliter:

By vector triple product we have $(\bar{r} \times \bar{a}) \times \bar{b} = (\bar{r} \cdot \bar{b})\bar{a} - (\bar{a} \cdot \bar{b})\bar{r}$

$$\therefore \nabla \times [(\bar{r} \times \bar{a}) \times \bar{b}] = \nabla \times [(\bar{r} \cdot \bar{b})\bar{a}] - \nabla \times [(\bar{a} \cdot \bar{b})\bar{r}]$$

$$= [\nabla(\bar{r} \cdot \bar{b}) \times \bar{a} + (\bar{r} \cdot \bar{b})\nabla \times \bar{a}] - [\nabla(\bar{a} \cdot \bar{b}) \times \bar{r} + (\bar{a} \cdot \bar{b})\nabla \times \bar{r}]$$

$\because \bar{a}$ is constant, $\nabla \times \bar{a} = 0$.

Because $\bar{a}, \bar{b}$ are constants, $\bar{a} \cdot \bar{b} =$ a constant

$\therefore \nabla(\bar{a} \cdot \bar{b}) = 0$, $\nabla \times \bar{r} = \bar{0}$ and $\nabla(\bar{r} \cdot \bar{b}) = \bar{b}$

$$\therefore \nabla \times [(\bar{r} \times \bar{a}) \times \bar{b}] = \bar{b} \times \bar{a}$$

Aliter:

$$\nabla \times [(\bar{r} \times \bar{a}) \times \bar{b}] = \text{curl}\,[(\bar{r} \times \bar{a}) \times \bar{b}]$$

$$= \text{curl}\,[(\bar{r} \cdot \bar{b})\bar{a} - (\bar{a} \cdot \bar{b})\bar{r}]$$

$$= \text{curl}[(\bar{r} \cdot \bar{b})\bar{a}] - \text{curl}[(\bar{a} \cdot \bar{b})\bar{r}]$$

$$= [(\bar{r} \cdot \bar{b})\text{curl}\,\bar{a} + \text{grad}\,(\bar{r} \cdot \bar{b}) \times \bar{a}] - [(\bar{a} \cdot \bar{b})\text{curl}\,\bar{r} + \text{grad}\,(\bar{a} \cdot \bar{b}) \times \bar{r}]$$

$$= [\bar{0} + \bar{b} \times \bar{a}] - [\bar{0} + \bar{0}]$$

$$= \bar{b} \times \bar{a} \qquad \left[\begin{matrix} \because \text{curl}\,\bar{a} = 0, \text{curl}\,\bar{r} = 0 \\ \text{grad}\,(\bar{a} \cdot \bar{b}) = 0, \text{grad}\,(\bar{r} \cdot \bar{b}) = \bar{b} \end{matrix}\right]$$

11. If $\bar{a}$ and $\bar{b}$ are constant vectors and $\bar{r} = xi + yj + zk$, prove that, $\nabla \times [\bar{r} \times (\bar{a} \times \bar{r})] = 3\bar{r} \times \bar{a}$

Solution: We have $\bar{r} \times (\bar{a} \times \bar{r}) = (\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}$

But $\bar{r} = xi + yj + zk$, $\bar{a} = a_1i + a_2j + a_3k$

$\therefore \bar{r} \cdot \bar{r} = x^2 + y^2 + z^2$ and $\bar{r} \cdot \bar{a} = a_1x + a_2y + a_3z$

$$\therefore \bar{r} \times (\bar{a} \times \bar{r}) = (x^2 + y^2 + z^2)\bar{a} - (a_1x + a_2y + a_3z)\bar{r}$$

$$= (x^2 + y^2 + z^2)(a_1i + a_2j + a_3k) - (a_1x + a_2y + a_3z)(xi + yj + zk)$$

$$= (a_1x^2 + a_1y^2 + a_1z^2)i + (a_2x^2 + a_2y^2 + a_2z^2)j + (a_3x^2 + a_3y^2 + a_3z^2)k$$
$$+(-a_1x^2 - a_2xy - a_3xz)i + (-a_1xy - a_2y^2 - a_3yz)j$$
$$+(-a_1xz - a_2yz - a_3z^2)k$$

$$= (a_1y^2 - a_1z^2 - a_2xy - a_3xz)i + \left(a_2x^2 + a_2z^2 - a_1xy - a_3yz\right)j$$
$$+\left(a_3x^2 + a_3y^2 - a_1xz - a_2yz\right)k$$
$= f_1i + f_2j + f_3k$ say

$$\therefore \nabla \times [\bar{r} \times (\bar{a} \times \bar{r})] = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + j(\dots\dots) + k(\dots\dots)$$
$$= i[2a_3y - a_2z - 2a_2z + a_3y] + j(\dots\dots) + k(\dots\dots)$$
$$= 3i(a_3y - a_2z) + j(\dots\dots) + k(\dots\dots)$$

But $\bar{r} \times \bar{a} = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = i(a_3y - a_2z) + j(a_1z - a_3x) + k(a_2x - a_1y)$

Hence, $\nabla \times [\bar{r} \times (\bar{a} \times \bar{r})] = 3\bar{r} \times \bar{a}$

Aliter:

$$\bar{r} \times (\bar{a} \times \bar{r}) = (\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}$$
$$= r^2\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}$$
$$\therefore \nabla \times [\bar{r} \times (\bar{a} \times \bar{r})] = \nabla \times (r^2\bar{a}) - \nabla \times [(\bar{r} \cdot \bar{a})\bar{r}]$$
$$= \nabla(r^2) \times \bar{a} + r^2\nabla \times \bar{a} - \nabla(\bar{r} \cdot \bar{a}) \times \bar{r} - (\bar{r} \cdot \bar{a})\nabla \times \bar{r}$$

$\because \nabla r^2 = 2\bar{r}$, Since $\bar{a}$ is constant, $\nabla \times \bar{a} = \bar{0}$, $\nabla(\bar{r} \cdot \bar{a}) = \bar{a}$, $\nabla \times \bar{a} = \bar{0}$

$$\therefore \nabla \times [\bar{r} \times (\bar{a} \times \bar{r})] = 2\bar{r} \times \bar{a} + \bar{0} - \bar{a} \times \bar{r} - \bar{0}$$
$$= 2\bar{r} \times \bar{a} + \bar{r} \times \bar{a}$$
$$= 3\bar{r} \times \bar{a}$$

Aliter:

$$\nabla \times [\bar{r} \times (\bar{a} \times \bar{r})] = \text{curl}\,[(\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}]$$
$$= \text{curl}[r^2\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}]$$
$$= \text{curl}(r^2\bar{a}) - \text{curl}[(\bar{r} \cdot \bar{a})\bar{r}]$$
$$= [r^2(\text{curl}\,\bar{a}) + (\text{grad}\,r^2) \times \bar{a}] - \left[(\bar{r} \cdot \bar{a})(\text{curl}\,\bar{r}) + \left(\text{grad}\,(\bar{r} \cdot \bar{a})\right) \times \bar{r}\right]$$
$$= \left[\bar{r}(\bar{0}) + \left(\frac{2r\bar{r}}{r}\right) \times \bar{a}\right] - [(\bar{r} \cdot \bar{a})(\bar{0}) + \bar{a} \times \bar{r}]$$
$$= 2(\bar{r} \times \bar{a}) - \bar{a} \times \bar{r}$$
$$= 2(\bar{r} \times \bar{a}) + (\bar{r} \times \bar{a})$$
$$= 3(\bar{r} \times \bar{a})$$
$$= 3\bar{r} \times \bar{a} \qquad \left[\because m\bar{a} \times \bar{b} = \bar{a} \times m\bar{b} = m\left(\bar{a} \times \bar{b}\right)\right]$$

13. If $\bar{a}$, $\bar{b}$ are constant vectors and r and $\bar{r}$ have usual meanings, prove that $\bar{a} \,.\, \nabla\left(\bar{b} \,.\, \nabla\frac{1}{r}\right) = \frac{3(\bar{a}\,.\,\bar{r})(\bar{b}\,.\,\bar{r})}{r^5} - \frac{\bar{a}\,.\,\bar{b}}{r^3}$

Solution: $\nabla\frac{1}{r} = -\frac{\bar{r}}{r^3} = -\frac{1}{r^3}(xi + yj + zk)$

$$\therefore \bar{b} \cdot \nabla\left(\frac{1}{r}\right) = (b_1i + b_2j + b_3k) \cdot \left[-\frac{1}{r^3}(xi + yj + zk)\right] = -\frac{1}{r^3}(b_1x + b_2y + b_3z) = \Phi \text{ say}$$

$$\therefore \nabla\left(\overline{b}\,.\nabla\frac{1}{r}\right) = \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z} \qquad \text{............. (1)}$$

Now, $\frac{\partial\Phi}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{b_1x+b_2y+b_3z}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= -\frac{\left(x^2+y^2+z^2\right)^{3/2}\cdot b_1-(b_1x+b_2y+b_3z)\cdot\left(x^2+y^2+z^2\right)^{1/2}x}{(x^2+y^2+z^2)^3}$$

$$= -\frac{b_1}{(x^2+y^2+z^2)^{3/2}} + \frac{3(b_1x+b_2y+b_3z)x}{(x^2+y^2+z^2)^{5/2}}$$

$$= -\frac{b_1}{r^3} + \frac{3(b_1x+b_2y+b_3z)x}{r^5}$$

Similarly, $\frac{\partial\Phi}{\partial y} = -\frac{b_2}{r^3} + \frac{3(b_1x+b_2y+b_3z)y}{r^5}$, $\frac{\partial\Phi}{\partial x} = -\frac{b_3}{r^3} + \frac{3(b_1x+b_2y+b_3z)z}{r^5}$

Hence, from (1)

$$\therefore \nabla\left(\overline{b}\,.\nabla\frac{1}{r}\right) = -\frac{1}{r^3}(b_1i+b_2j+b_3k) + \frac{3}{r^5}(b_1x+b_2y+b_3z)(xi+yj+zk)$$

$$\because \overline{b}\cdot\overline{r} = (b_1i+b_2j+b_3k)\cdot(xi+yj+zk) = (b_1x+b_2y+b_3z)$$

$$\therefore \nabla\left(\overline{b}\,.\nabla\frac{1}{r}\right) = -\frac{\bar{b}}{r^3} + \frac{3(\bar{b}\cdot\bar{r})\bar{r}}{r^5}$$

$$\therefore \overline{a}\,.\nabla\left(\overline{b}\,.\nabla\frac{1}{r}\right) = \frac{3(\overline{a}\,.\,\overline{r})(\overline{b}\,.\,\overline{r})}{r^5} - \frac{\overline{a}\,.\,\overline{b}}{r^3}$$

Aliter:

$$\nabla\left(\frac{1}{r}\right) = \frac{-r^2\bar{r}}{r} = -\frac{\bar{r}}{r^3} \qquad \text{using } \nabla f(r) = \frac{f'(r)\bar{r}}{r}$$

$$\bar{a}\cdot\nabla\left(\frac{1}{r}\right) = \frac{-\bar{a}\cdot\bar{r}}{r^3}$$

$$\nabla\left[\bar{a}\cdot\nabla\left(\frac{1}{r}\right)\right] = \nabla\left(\frac{-\bar{a}\cdot\bar{r}}{r^3}\right)$$

$$= -\left[\nabla\left(\frac{1}{r^3}\right)(\bar{a}\cdot\bar{r}) + \frac{1}{r^3}\nabla(\bar{a}\cdot\bar{r})\right]$$

$$= -\left[\left(\frac{-3r^{-4}\bar{r}}{r}\right)(\bar{a}\cdot\bar{r}) + \frac{1}{r^3}\bar{a}\right]$$

$$\left[\because \nabla(\bar{a}\cdot\bar{r}) = \nabla(a_1x_1+a_2y+a_3z) = \bar{\imath}(a_1)+\bar{\jmath}(a_2)+\bar{k}(a_3) = \bar{a}\right]$$

$$= -\left[\frac{\bar{a}}{r^3} - \frac{3\bar{r}}{r^5}(\bar{a}\cdot\bar{r})\right]$$

$$= -\frac{\bar{a}}{r^3} + \frac{3(\bar{a}\cdot\bar{r})}{r^5}\bar{r}$$

$$\bar{b}\cdot\nabla\left[\bar{a}\cdot\nabla\left(\frac{1}{r}\right)\right] = \frac{-\bar{b}\cdot\bar{a}}{r^3} + \frac{3(\bar{a}\cdot\bar{r})}{r^5}(\bar{b}\cdot\bar{r})$$

14. If $\overline{a}$, $\overline{b}$ are constant vectors and r and $\overline{r}$ have usual meanings, prove that $\nabla\left[\frac{(\overline{a}\,.\,\overline{r})}{r^n}\right] = \frac{\overline{a}}{r^n} - \frac{n(\overline{a}\,.\,\overline{r})\overline{r}}{r^{n+2}}$

Solution: We have $\frac{\overline{a}\,.\,\overline{r}}{r^n} = \frac{(a_1i+a_2j+a_3k)\cdot(xi+yj+zk)}{r^n} = \frac{a_1x+a_2y+a_3z}{r^n}$

Let $\Phi = \frac{\overline{a}\,.\,\overline{r}}{r^n} = \frac{a_1x+a_2y+a_3z}{r^n}$

$$\therefore \frac{\partial\Phi}{\partial x} = \frac{r^na_1-(a_1x+a_2y+a_3z)nr^{n-2}\cdot x}{r^{2n}} = \frac{a_1}{r^n} - \frac{n(a_1x+a_2y+a_3z)x}{r^{n+2}}$$

Similarly, $\frac{\partial\Phi}{\partial y} = \frac{a_2}{r^n} - \frac{n(a_1x+a_2y+a_3z)y}{r^{n+2}}$ and $\frac{\partial\Phi}{\partial z} = \frac{a_3}{r^n} - \frac{n(a_1x+a_2y+a_3z)z}{r^{n+2}}$

$$\therefore \nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$

$$= \frac{1}{r^n}(a_1 i + a_2 j + a_3 k) - \frac{n}{r^{n+2}}[(a_1 x + a_2 y + a_3 z)(xi + yj + zk)]$$

But $\overline{a} \,.\, \overline{r} = (a_1 i + a_2 j + a_3 k)(xi + yj + zk) = a_1 x + a_2 y + a_3 z$

$$\therefore \nabla\Phi = \frac{\overline{a}}{r^n} - \frac{n(\overline{a}\,.\,\overline{r})\overline{r}}{r^{n+2}}$$

16. If $\Phi = (\overline{r} \times \overline{a}) \cdot (\overline{r} \times \overline{b})$, prove that $\nabla\Phi = \overline{b} \times (\overline{r} \times \overline{a}) + \overline{a} \times (\overline{r} \times \overline{b})$ where $\overline{a}, \overline{b}$ are constant vectors.

Solution: Let $\overline{a} = a_1 i + a_2 j + a_3 k, \overline{b} = b_1 i + b_2 j + b_3 k, \overline{r} = xi + yj + zk$

$$\therefore \overline{r} \times \overline{a} = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} \text{ and } \overline{r} \times \overline{b} = \begin{vmatrix} i & j & k \\ x & y & z \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\therefore \overline{r} \times \overline{a} = i(a_3 y - a_2 z) + j(a_1 z - a_3 x) + k(a_2 x - a_1 y)$$

and $\overline{r} \times \overline{b} = i(b_3 y - b_2 z) + j(b_1 z - b_3 x) + k(b_2 x - b_1 y)$

$$\therefore (\overline{r} \times \overline{a}) \cdot (\overline{r} \times \overline{b}) = (a_3 y - a_2 z)(b_3 y - b_2 z) + (a_1 z - a_3 x)(b_1 z - b_3 x)$$
$$+(a_2 x - a_1 y)(b_2 x - b_1 y) \qquad \text{............. (1)}$$

$$\therefore \nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$
$$= [(a_1 z - a_3 x)(-b_3) + (b_1 z - b_3 x)(-a_3) + (a_2 x - a_1 y)b_2 + (b_2 x - b_1 y)a_2]i$$
$$+[\dots\dots]j + [\dots\dots]k$$
$$= [-a_1 b_3 z + a_3 b_3 x - a_3 b_1 z + a_3 b_3 x + a_2 b_2 x - a_1 b_2 y + a_2 b_2 x - a_2 b_1 y]i$$
$$+[\dots\dots]j + [\dots\dots]k$$
$$= [2(a_2 b_2 + a_3 b_3)x - a_1(b_2 y + b_3 z) - b_1(a_2 y + a_3 z)]i + [\dots\dots]j + [\dots\dots]k \qquad \text{........... (2)}$$

But $\quad \overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}$

$$\therefore \overline{b} \times (\overline{r} \times \overline{a}) = (\overline{b} \cdot \overline{a})\overline{r} - (\overline{b} \cdot \overline{r})\overline{a}$$

and $\quad \overline{a} \times (\overline{r} \times \overline{b}) = (\overline{a} \cdot \overline{b})\overline{r} - (\overline{a} \cdot \overline{r})\overline{b}$

$$\therefore \overline{b} \times (\overline{r} \times \overline{a}) + \overline{a} \times (\overline{r} \times \overline{b}) = 2(\overline{a} \cdot \overline{b})\overline{r} - (\overline{b} \cdot \overline{r})\overline{a} - (\overline{a} \cdot \overline{r})\overline{b}$$
$$= 2(a_1 b_1 + a_2 b_2 + a_3 b_3)\overline{r} - (b_1 x + b_2 y + b_3 z)\overline{a}$$
$$-(a_1 x + a_2 y + a_3 z)\overline{b} \qquad \text{........... (3)}$$

Now, we add $2a_1 b_1 x$ to the first term and subtract $a_1 b_1 x$ from the second and third terms of (2)

$$\therefore \nabla\Phi = [2(a_1 b_1 + a_2 b_2 + a_3 b_3)x - (b_1 x + b_2 y + b_3 z)a_1 - (a_1 x + a_2 y + a_3 z)b_1]i$$
$$+[\dots\dots]j + [\dots\dots]k \qquad \text{........... (4)}$$

From (3) and (4) it can be seen that the result is proved

Aliter:

Use formula $\nabla(\bar{f} \cdot \bar{g}) = \bar{f} \times (\nabla \times \bar{g}) + \bar{g} \times (\nabla \times \bar{f}) + (\bar{f} \cdot \nabla)\bar{g} + (\bar{g} \cdot \nabla)\bar{f}$

17. Prove that $\nabla\left\{\nabla \cdot \frac{\overline{r}}{r}\right\} = -\frac{2}{r^3}\overline{r}$

Solution: We have $\overline{r} = xi + yj + zk$

Now, let $\bar{f} = \frac{\bar{r}}{r} = \frac{xi}{\sqrt{x^2+y^2+z^2}} + \frac{yj}{\sqrt{x^2+y^2+z^2}} + \frac{zk}{\sqrt{x^2+y^2+z^2}}$

$\therefore \nabla \cdot \frac{\bar{r}}{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Now, $\frac{\partial f_1}{\partial x} = \frac{\sqrt{x^2+y^2+z^2}\cdot(1) - x\cdot x/\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)} = \frac{r^2-x^2}{(r^2)^{3/2}} = \frac{r^2-x^2}{r^3}$

Similarly, we get two more results

$\therefore \nabla \cdot \frac{\bar{r}}{r} = \frac{(r^2-x^2)+(r^2-y^2)+(r^2-z^2)}{r^3} = \frac{3r^2-(x^2+y^2+z^2)}{r^3} = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2+z^2}}$

$$\begin{aligned}
\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) &= \nabla\left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right) \\
&= 2\left[i \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + j(\dots\dots) + k(\dots\dots)\right] \\
&= 2\left[i \cdot \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot 2x + j(\dots\dots) + k(\dots\dots)\right] \\
&= -2\left[\frac{xi}{r^3} + \frac{yj}{r^3} + \frac{zk}{r^3}\right] = -\frac{2}{r^3}\bar{r}
\end{aligned}$$

Aliter:

$$\begin{aligned}
\nabla \cdot \frac{\bar{r}}{r} = \text{div}\left(\frac{1}{r}\bar{r}\right) &= \frac{1}{r}(\text{div}\,\bar{r}) + \bar{r}\cdot\left(\text{grad}\,\frac{1}{r}\right) \\
&= \frac{1}{r}(3) + \bar{r}\cdot\left(\frac{-\frac{1}{r^2}\bar{r}}{r}\right) \qquad \because \text{div}\,\bar{r} = 3 \text{ and } grad f(r) = \frac{f'(r)\bar{r}}{r} \\
&= \frac{3}{r} - \frac{1}{r^3}(\bar{r}\cdot\bar{r}) \\
&= \frac{3}{r} - \frac{1}{r^3}(r^2) = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}
\end{aligned}$$

$$\nabla\left(\nabla \cdot \frac{\bar{r}}{r}\right) = \nabla\left(\frac{2}{r}\right) = \frac{2\left(-\frac{1}{r^2}\right)\bar{r}}{r} = \frac{-2}{r^3}\bar{r}$$

18. Prove that $\nabla \cdot \left(r\nabla \frac{1}{r^n}\right) = \frac{n(n-2)}{r^{n+1}}$

Solution: $r^{-n} = (x^2+y^2+z^2)^{-n/2}$

$\therefore \nabla\left(\frac{1}{r^n}\right) = \left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-\frac{n}{2}-1}\cdot(2x)i + (\dots\dots)j + (\dots\dots)k = -nr^{-n-2}(xi+yj+zk)$

$\therefore r\nabla\left(\frac{1}{r^n}\right) = -nr^{-n-1}x\,i - nr^{-n-1}y\,j - nr^{-n-1}z\,k$

$$\begin{aligned}
\therefore \nabla\cdot\left[r\nabla\left(\frac{1}{r^n}\right)\right] &= -n\frac{\partial}{\partial x}\{r^{-n-1}x\} - (\dots\dots) - (\dots\dots) \\
&= -n\left[(-n-1)xr^{-n-2}\frac{\partial r}{\partial x} + r^{-n-1}\cdot 1\right] - (\dots\dots) - (\dots\dots) \\
&= -n\left[(-n-1)\frac{x^2}{r^{n+3}} + \frac{1}{r^{n+1}}\right] - (\dots\dots) - (\dots\dots) \\
&= -n\left[-\frac{(n+1)}{r^{n+3}}(x^2+y^2+z^2) + \frac{3}{r^{n+1}}\right] \\
&= -n\left[-\frac{n+1}{r^{n+1}} + \frac{3}{r^{n+1}}\right] = \frac{n(n-2)}{r^{n+1}}
\end{aligned}$$

Aliter:

$$\nabla\left(\frac{1}{r^n}\right) = \frac{-nr^{-n-1}}{r}\bar{r} = \frac{-n}{r^{n+2}}\bar{r} \qquad \because \mathrm{grad f(r)} = \frac{\mathrm{f'(r)\bar{r}}}{\mathrm{r}}$$

$$r\nabla\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+1}}\bar{r}$$

$$\nabla\cdot\left(r\nabla\frac{1}{r^n}\right) = \mathrm{div}\left(r\nabla\frac{1}{r^n}\right)$$

$$= \mathrm{div}\left(\frac{-n}{r^{n+1}}\bar{r}\right)$$

$$= \frac{-n}{r^{n+1}}\ \mathrm{div}\,\bar{r} + \bar{r}\cdot\mathrm{grad}\left(\frac{-n}{r^{n+1}}\right)$$

$$= \frac{-n}{r^{n+1}}3 + \bar{r}\cdot\left(\frac{-n(-(n+1))r^{-n-1-1}}{r}\bar{r}\right) \qquad \because \mathrm{div}\,\bar{\mathrm{r}} = 3 \text{ and } \mathrm{grad f(r)} = \frac{\mathrm{f'(r)\bar{r}}}{\mathrm{r}}$$

$$= \frac{-3n}{r^{n+1}} + n(n+1)\frac{r^{-n-2}}{r}(\bar{r}\cdot\bar{r})$$

$$= \frac{-3n}{r^{n+1}} + \frac{n(n+1)}{r^{n+3}}(r^2)$$

$$= \frac{-3n}{r^{n+1}} + \frac{n(n+1)}{r^{n+1}}$$

$$= \frac{n}{r^{n+1}}[-3 + (n+1)]$$

$$= \frac{n(n-2)}{r^{n+1}}$$

20. Prove that div grad $r^n = n(n+1)r^{n-2}$

Solution: grad $r^n = nr^{n-2}\bar{r}$ $\qquad \because \mathrm{grad f(r)} = \frac{\mathrm{f'(r)\bar{r}}}{\mathrm{r}}$

$\therefore$ div grad $r^n = \nabla\cdot\{nr^{n-2}(xi + yj + zk)\}$

$$= n\left[\frac{\partial}{\partial x}(r^{n-2}\cdot x) + \frac{\partial}{\partial y}(r^{n-2}\cdot y) + \frac{\partial}{\partial z}(r^{n-2}\cdot z)\right]$$

$$= n\left[r^{n-2} + x\cdot(n-2)\cdot r^{n-3}\frac{\partial r}{\partial x} + \cdots\right]$$

$$= n\left[r^{n-2} + x\cdot(n-2)\cdot r^{n-3}\frac{x}{r} + \cdots + \cdots\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-4}\cdot\left(x^2 + y^2 + z^2\right)\right]$$

$$= n[3r^{n-2} + (n-2)r^{n-2}]$$

$$= n(n+1)r^{n-2}$$

Aliter:

$$\nabla\cdot(\nabla r^n) = \nabla\cdot\left(\frac{nr^{n-1}\bar{r}}{r}\right) \qquad \because \mathrm{grad f(r)} = \frac{\mathrm{f'(r)\bar{r}}}{\mathrm{r}}$$

$$= \nabla\cdot(nr^{n-2}\bar{r})$$

$$= nr^{n-2}(\nabla\cdot\bar{r}) + \bar{r}\cdot\nabla(nr^{n-2})$$

$$= nr^{n-2}\mathrm{div}\,\bar{r} + \bar{r}\cdot\ \mathrm{grad}\ (nr^{n-2})$$

$$= nr^{n-2}3 + \bar{r}\cdot\frac{n(n-2)r^{n-3}\bar{r}}{r} \qquad \because \mathrm{div}\,\bar{\mathrm{r}} = 3 \text{ and } \mathrm{grad f(r)} = \frac{\mathrm{f'(r)\bar{r}}}{\mathrm{r}}$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}(\bar{r}\cdot\bar{r})$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}r^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$
$$= (3+n-2)nr^{n-2}$$
$$= n(n+1)r^{n-2}$$

21. Prove that $\nabla \cdot \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = 0$

Solution: LHS $= \nabla \cdot \left(\frac{\overline{a}\times\overline{r}}{r^n}\right)$

$$= \nabla \cdot [r^{-n}(\overline{a}\times\overline{r})]$$
$$= (\nabla r^{-n}) \cdot (\overline{a}\times\overline{r}) + r^{-n}\big(\nabla \cdot (\overline{a}\times\overline{r})\big)$$
$$= (-nr^{-n-2}\overline{r}) \cdot (\overline{a}\times\overline{r}) + r^{-n}\big(\nabla \cdot (\overline{a}\times\overline{r})\big) \quad \because \text{gradf(r)} = \frac{f'(r)\overline{r}}{r}$$
$$= -nr^{-n-2}\big(\overline{r}\cdot(\overline{a}\times\overline{r})\big) + r^{-n}\,0 \quad \because div\,(\overline{a}\times\overline{r}) = 0$$
$$= 0 + 0 = 0$$

22. Prove that $\nabla \times \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \frac{(2-n)\overline{a}}{r^n} + \frac{n(\overline{a}\,.\,\overline{r})\overline{r}}{r^{n+2}}$

Solution: We have $\frac{\overline{a}\times\overline{r}}{r^n} = \frac{1}{r^n}\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \frac{1}{r^n}(a_2z - a_3y)i + \frac{1}{r^n}(a_3x - a_1z)j + \frac{1}{r^n}(a_1y - a_2x)k$

$$\therefore \nabla \times \frac{(\overline{a}\times\overline{r})}{r^n} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{a_2z-a_3y}{r^n} & \frac{a_3x-a_1z}{r^n} & \frac{a_1y-a_2x}{r^n} \end{vmatrix} = i\left[\frac{\partial}{\partial y}\left(\frac{a_1y-a_2x}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{a_3x-a_1z}{r^n}\right)\right] + j[\dots\dots] + k[\dots\dots]$$

$\because r^2 = x^2 + y^2 + z^2 \qquad \therefore 2r\frac{\partial r}{\partial x} = 2x$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$= i\left[\left\{-nr^{-n-1}\cdot\left(\frac{y}{r}\right)(a_1y - a_2x) + \frac{1}{r^n}(a_1)\right\} - \left\{-nr^{-n-1}\cdot\left(\frac{z}{r}\right)(a_3x - a_1z) + \frac{1}{r^n}(-a_1)\right\}\right]$$
$$+j[\dots\dots\dots] + k[\dots\dots\dots]$$
$$= i\left[-\frac{n}{r^{n+2}}(a_1y^2 - a_2xy) + \frac{a_1}{r^n} + \frac{n}{r^{n+2}}(a_3xz - a_1z^2) + \frac{a_1}{r^n}\right] + j[\dots\dots\dots] + k[\dots\dots\dots]$$
$$= i\left[\frac{2a_1}{r^n} - \frac{n}{r^{n+2}}a_1(y^2+z^2) + \frac{n}{r^{n+2}}(a_2xy + a_3xz)\right] + j[\dots\dots\dots] + k[\dots\dots\dots]$$

Adding $\frac{n}{r^{n+2}}a_1x^2$ to the third term and subtracting it from the second term,

$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}(x^2+y^2+z^2) + \frac{n}{r^{n+2}}(a_1x^2 + a_2xy + a_3xz)\right] + j[\dots\dots\dots] + k[\dots\dots\dots]$$
$$= i\left[\frac{2a_1}{r^n} - \frac{na_1}{r^{n+2}}r^2 + \frac{n}{r^{n+2}}x(a_1x + a_2y + a_3z)\right] + j[\dots\dots\dots] + k[\dots\dots\dots]$$
$$= \frac{(2-n)}{r^n}(a_1i + a_2j + a_3k) + \frac{n}{r^{n+2}}(a_1x + a_2y + a_3z)(xi + yj + zk)$$
$$= \frac{(2-n)}{r^n}\overline{a} + \frac{n}{r^{n+2}}(\overline{a}\,.\,\overline{r})\overline{r}$$

Aliter:

LHS $= \nabla \times \left(\frac{\overline{a}\times\overline{r}}{r^n}\right) = \nabla \times [r^{-n}(\overline{a}\times\overline{r})]$

$$= (\nabla r^{-n}) \times (\overline{a} \times \overline{r}) + r^{-n}\big(\nabla \times (\overline{a} \times \overline{r})\big)$$

$$= (-nr^{-n-2}\overline{r}) \times (\overline{a} \times \overline{r}) + r^{-n}2\overline{a} \qquad \because \text{gradf(r)} = \frac{\text{f}'(\text{r})\overline{\text{r}}}{\text{r}} \ \& \ curl(\overline{a} \times \overline{r}) = 2\overline{a}$$

$$= -nr^{-n-2}\big(\overline{r} \times (\overline{a} \times \overline{r})\big) + 2r^{-n}\overline{a}$$

$$= -nr^{-n-2}[(\overline{r} \cdot \overline{r})\overline{a} - (\overline{r} \cdot \overline{a})\overline{r}] + 2r^{-n}\overline{a} \qquad \because \overline{r} \times (\overline{a} \times \overline{r}) = (\overline{r} \cdot \overline{r})\overline{a} - (\overline{r} \cdot \overline{a})\overline{r}$$

But $\overline{r} \cdot \overline{r} = r^2$

$$\therefore \text{LHS} = -nr^{-n-2}r^2\overline{a} + nr^{-n-2}(\overline{r} \cdot \overline{a})\overline{r} + 2r^{-n}\overline{a}$$

$$= (2-n)r^{-n}\overline{a} + nr^{-n-2}(\overline{r} \cdot \overline{a})\overline{r}$$

$$= \frac{(2-n)\overline{a}}{r^n} + \frac{n(\overline{a}\,.\,\overline{r})}{r^{n+2}}r^2 = \text{RHS}$$

Aliter:

$$\text{LHS} = \nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \text{curl}\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)$$

$$= \frac{1}{r^n}\text{curl}\,(\bar{a} \times \bar{r}) + \left(\text{grad}\,\frac{1}{r^n}\right) \times (\bar{a} \times \bar{r})$$

but $\text{curl}\,(\bar{a} \times \bar{r}) = 2\bar{a}$ & $\text{grad}\left(\frac{1}{r^n}\right) = \frac{-nr^{-n-1}}{r}\bar{r} = \frac{-n}{r^{n+2}}\bar{r}$

$$\text{LHS} = \frac{1}{r^n}2\bar{a} + \left(\frac{-n\bar{r}}{r^{n+2}}\right) \times (\bar{a} \times \bar{r})$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}[\bar{r} \times (\bar{a} \times \bar{r})]$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}[(\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}]$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^{n+2}}[r^2\bar{a} - (\bar{r} \cdot \bar{a})\bar{r}]$$

$$= \frac{2\bar{a}}{r^n} - \frac{n\bar{a}}{r^n} + \frac{n(\bar{r}\cdot\bar{a})\bar{r}}{r^{n+2}}$$

$$= \frac{(2-n)\bar{a}}{r^n} + \frac{n(\bar{r}\cdot\bar{a})\bar{r}}{r^{n+2}}$$

$$= \text{RHS}$$

26. Prove that $\overline{b}\,.\nabla\left(\frac{\overline{a}\,.\,\overline{r}}{r^n}\right) = \frac{\overline{a}\cdot\overline{b}}{r^n} - n\frac{(\overline{a}\,.\,\overline{r})}{r^{n+2}}(\overline{b}\,.\,\overline{r})$

Solution: LHS $= \overline{b} \cdot [\nabla \cdot \{r^{-n}(\overline{a} \cdot \overline{r})\}]$

$$= \overline{b} \cdot [\nabla r^{-n}(\overline{a} \cdot \overline{r}) + r^{-n}\nabla(\overline{a} \cdot \overline{r})]$$

$$= \overline{b} \cdot [(-nr^{-n-2}\overline{r})(\overline{a} \cdot \overline{r}) + r^{-n}\overline{a}] \qquad \because \text{gradf(r)} = \frac{\text{f}'(\text{r})\overline{\text{r}}}{\text{r}} \ \& \ \nabla(\overline{a} \cdot \overline{r}) = \overline{a}$$

$$= -n\frac{(\overline{b}\cdot\overline{r})(\overline{a}\cdot\overline{r})}{r^{n+2}} + \frac{(\overline{b}\cdot\overline{a})}{r^n}$$

$$= \frac{\overline{a}\cdot\overline{b}}{r^n} - n\frac{(\overline{a}\cdot\overline{r})(\overline{b}\cdot\overline{r})}{r^{n+2}}$$

27. Prove that $\nabla \log r = \frac{\overline{r}}{r^2}$ and hence, show that $\nabla \times (\overline{a} \times \nabla \log\, \text{r}) = 2\,\frac{(\overline{a}\,.\,\overline{r})\overline{r}}{r^4}$ where $\overline{a}$ is a constant vector.

Solution: $\log r = \frac{1}{2}\log(x^2+y^2+z^2)$

$\therefore \frac{\partial}{\partial x}(\log r) = \frac{1}{2}\cdot\frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{x}{r^2}$

Similarly, $\frac{\partial}{\partial y}(\log r) = \frac{y}{r^2}, \quad \frac{\partial}{\partial z}(\log r) = \frac{z}{r^2}$

$\therefore \nabla \log r = i\cdot\frac{x}{r^2} + j\cdot\frac{y}{r^2} + k\cdot\frac{z}{r^2} = \frac{1}{r^2}(xi+yj+zk) = \frac{\bar{r}}{r^2}$

$\nabla \log r = \frac{1}{r}\cdot\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$

Now, $\bar{a}\times\nabla\log r = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2}\end{vmatrix} = \frac{(a_2z-a_3y)}{r^2}i + \frac{(a_3x-a_1z)}{r^2}j + \frac{(a_1y-a_2x)}{r^2}k$

$$\therefore \nabla\times(\bar{a}\times\nabla\log r) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{(a_2z-a_3y)}{r^2} & \frac{(a_3x-a_1z)}{r^2} & \frac{(a_1y-a_2x)}{r^2}\end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}\left[\frac{(a_1y-a_2x)}{r^2}\right] - \frac{\partial}{\partial z}\left[\frac{(a_3x-a_1z)}{r^2}\right]\right\} + j[\dots\dots\dots] + k[\dots\dots\dots]$$

$\because r^2 = x^2+y^2+z^2, \quad 2r\frac{\partial r}{\partial x} = 2x, \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$= i\left[\left\{\frac{r^2(a_1)-(a_1y-a_2x)2r(y/r)}{r^4}\right\} - \right]\left\{\frac{r^2(-a_1)-(a_3x-a_1z)2r(z/r)}{r^4}\right\} + j[\dots.] + k[\dots..]$$

$$= i\left[\frac{a_1}{r^2} - \frac{2}{r^4}(a_1y^2 - a_2xy) + \frac{a_1}{r^2} + \frac{2}{r^4}(a_3xz - a_1z^2)\right] + j[\dots.] + k[\dots..]$$

$$= \frac{2i}{r^4}\left[a_1r^2 - a_1(y^2+z^2) + (a_2xy + a_3xz)\right] + j[\dots.] + k[\dots..]$$

Adding a_1x^2 to the third term and subtracting it from the second term,

$$= \frac{2i}{r^4}\left[a_1r^2 - a_1(x^2+y^2+z^2) + (a_1x^2 + a_2xy + a_3xz)\right] + j[\dots.] + k[\dots..]$$

$$= \frac{2i}{r^4}\left[a_1r^2 - a_1r^2 + x(a_1x + a_2y + a_3z)\right] + j[\dots.] + k[\dots..]$$

$$= \frac{2}{r^4}(a_1x + a_2y + a_3z)(xi + yj + zk)$$

$$= \frac{2}{r^4}(\bar{a}\cdot\bar{r})\bar{r}$$

Aliter:

$\nabla\log r = \frac{\left(\frac{1}{r}\right)\bar{r}}{r} = \frac{\bar{r}}{r^2} \qquad \because \mathrm{grad} f(r) = \frac{f'(r)\bar{r}}{r}$

$\therefore$ LHS $= \nabla\times(\bar{a}\times\nabla\log r)$

$= \nabla\times\left(\bar{a}\times\frac{\bar{r}}{r^2}\right)$

$= \nabla\times\{r^{-2}(\bar{a}\times\bar{r})\}$

$= (\nabla r^{-2})\times(\bar{a}\times\bar{r}) + r^{-2}(\nabla\times(\bar{a}\times\bar{r}))$

$= -2r^{-4}\bar{r}\times(\bar{a}\times\bar{r}) + r^{-2}2\bar{a} \qquad \because \mathrm{grad} f(r) = \frac{f'(r)\bar{r}}{r}$ & $curl(\bar{a}\times\bar{r}) = 2\bar{a}$

$= -2r^{-4}[(\bar{r}\cdot\bar{r})\bar{a} - (\bar{r}\cdot\bar{a})\bar{r}] + 2r^{-2}\bar{a} \qquad \because \bar{r}\times(\bar{a}\times\bar{r}) = (\bar{r}\cdot\bar{r})\bar{a} - (\bar{r}\cdot\bar{a})\bar{r}$

But $\bar{r}\cdot\bar{r} = r^2$

$\therefore$ LHS $= -2r^{-4}r^2\bar{a} + 2r^{-4}(\bar{r}\cdot\bar{a}) + 2r^{-2}\bar{a}$

$= -2r^{-2}\bar{a} + 2r^{-4}(\bar{r}\cdot\bar{a})\bar{r} + 2r^{-2}\bar{a}$

$= \frac{2(\bar{r}\cdot\bar{a})\bar{r}}{r^4} =$ RHS

28. If $\bar{a}$ is a constant unit vector, prove that $\bar{a}\,.\{grad(\bar{f}\,.\,\bar{a}) - curl(\bar{f}\times\bar{a})\} = div\,\bar{f}$

Solution: Let $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{f} = f_1 i + f_2 j + f_3 k$

$\therefore \bar{f}\cdot\bar{a} = a_1 f_1 + a_2 f_2 + a_3 f_3$

$$\bar{f}\times\bar{a} = \begin{vmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = i(f_2 a_3 - f_3 a_2) + j(f_3 a_1 - f_1 a_3) + k(f_1 a_2 - f_2 a_1)$$

$$\text{grad}\left(\bar{f}\cdot\bar{a}\right) = i\left(a_1\frac{\partial f_1}{\partial x} + a_2\frac{\partial f_2}{\partial x} + a_3\frac{\partial f_3}{\partial x}\right) + j(\ldots..) + k(\ldots\ldots.)$$

$$\text{curl}\left(\bar{f}\times\bar{a}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (f_2 a_3 - f_3 a_2) & \ldots\ldots. & \ldots\ldots. \end{vmatrix}$$

$$= i\left(a_2\frac{\partial f_1}{\partial y} - a_1\frac{\partial f_2}{\partial y} - a_1\frac{\partial f_3}{\partial z} + a_3\frac{\partial f_1}{\partial z}\right) + j(\ldots..) + k(\ldots\ldots.)$$

$$\therefore \text{grad}\left(\bar{f}\cdot\bar{a}\right) - \text{curl}\left(\bar{f}\times\bar{a}\right) = i\left[a_1\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + a_2\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) + a_3\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\right]$$
$$+ j(\ldots..) + k(\ldots\ldots.)$$

$$\bar{a}\cdot\{\text{grad}\left(\bar{f}\cdot\bar{a}\right) - \text{curl}\left(\bar{f}\times\bar{a}\right)\} = ({a_1}^2 + {a_2}^2 + {a_3}^2)\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)$$

Other terms getting cancelled $= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \text{div}\,\bar{f}$ $\quad [\because |\bar{a}| = 1]$

Aliter:

$$\text{grad}\left(\bar{f}\cdot\bar{a}\right) = \bar{f}\times\text{curl}\,\bar{a} + \bar{a}\times\text{curl}\,\bar{f} + \left(\bar{f}\cdot\nabla\right)\bar{a} + (\bar{a}\cdot\nabla)\bar{f}$$
$$= \bar{f}\times\bar{0} + \bar{a}\times\text{curl}\,\bar{f} + \bar{0} + (\bar{a}\cdot\nabla)\bar{f}$$
$$= \bar{a}\times\text{curl}\,\bar{f} + (\bar{a}\cdot\nabla)\bar{f}$$

$$\text{curl}\left(\bar{f}\times\bar{a}\right) = \bar{f}\,\text{div}\,\bar{a} - \bar{a}\,\text{div}\,\bar{f} + (\bar{a}\cdot\nabla)\bar{f} - \left(\bar{f}\cdot\nabla\right)\bar{a}$$
$$= \bar{f}\,(0) - \bar{a}\,\text{div}\,\bar{f} + (\bar{a}\cdot\nabla)\bar{f} - \bar{0}$$
$$= -\bar{a}\,\text{div}\,\bar{f} + (\bar{a}\cdot\nabla)\bar{f}$$

$$\text{grad}\left(\bar{f}\cdot\bar{a}\right) - \text{curl}\left(\bar{f}\times\bar{a}\right) = \left(\bar{a}\times\text{curl}\,\bar{f} + (\bar{a}\cdot\nabla)\bar{f}\right) - \left(-\bar{a}\,\text{div}\,\bar{f} + (\bar{a}\cdot\nabla)\bar{f}\right)$$
$$= \bar{a}\times\text{curl}\,\bar{f} + \bar{a}\,\text{div}\,\bar{f}$$

$$\therefore \bar{a}\cdot\left[\text{grad}\left(\bar{f}\cdot\bar{a}\right) - \text{grad}\left(\bar{f}\times\bar{a}\right)\right] = \bar{a}\cdot\left(\bar{a}\times\text{curl}\,\bar{f} + \bar{a}\,\text{div}\,\bar{f}\right)$$
$$= \bar{a}\cdot\bar{a}\times\text{curl}\,\bar{f} + \bar{a}\cdot\bar{a}\times\text{div}\,\bar{f}$$
$$= 0 + a^2\text{div}\,\bar{f}$$
$$= \text{div}\,\bar{f} \qquad [\because |\bar{a}| = 1]$$

29. Prove that $\nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\}=\frac{1}{r^2}\frac{d}{dr}[r^2f(r)]$. Hence or otherwise prove that div $(r^n\bar{r})=(n+3)r^n$

Solution: $\frac{f(r)}{r}\bar{r}=\frac{f(r)}{r}(xi+yj+zk)$

$$\nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\}=\frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right]+\frac{\partial}{\partial y}\left[\frac{f(r)}{r}y\right]+\frac{\partial}{\partial z}\left[\frac{f(r)}{r}z\right]$$

$$=\left[\frac{f'(r)}{r}\times\frac{\partial r}{\partial x}-\frac{f(r)}{r^2}\times\frac{\partial r}{\partial x}+\frac{f(r)}{r}\right]+[\dots\dots]+[\dots\dots]$$

But $r^2=x^2+y^2+z^2$ $\quad\therefore\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$

$$\therefore\nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\}=\left[\frac{f'(r)}{r}\frac{x^2}{r}-\frac{f(r)}{r^2}\frac{x^2}{r}+\frac{f(r)}{r}\right]+[\dots\dots]+[\dots\dots]$$

$$=\left[f'(r)\frac{x^2}{r^2}-\frac{f(r)}{r}\frac{x^2}{r^2}+\frac{f(r)}{r}\right]+[\dots\dots]+[\dots\dots]$$

$$=f'(r)\frac{(x^2+y^2+z^2)}{r^2}-\frac{f(r)}{r}\frac{(x^2+y^2+z^2)}{r^2}+3\frac{f(r)}{r}$$

$$=f'(r)+2\frac{f(r)}{r}\qquad\dots\dots\dots\dots\dots\dots(1)$$

$\left[\because\text{The term }\frac{f(r)}{r}\text{ will come from each bracket}\right]$

Now, $\frac{1}{r^2}\frac{d}{dr}[r^2f(r)]=\frac{1}{r^2}\left[r^2f'(x)+2rf(r)\right]=f'(x)+\frac{2f(r)}{r}$ .....(2)

From (1) and (2) we get the required result

Now, put $\frac{f(r)}{r}=r^n$ i.e. $f(r)=r^{n+1}$

$$\therefore\nabla\cdot\{r^n\bar{r}\}=\frac{1}{r^2}\frac{d}{dr}\{(r^{n+1})r^2\}=\frac{1}{r^2}\frac{d}{dr}\{r^{n+3}\}=\frac{1}{r^2}(n+3)r^{n+2}=(n+3)r^n$$

Aliter:

LHS $=\nabla\cdot\left(\frac{f(r)}{r}\bar{r}\right)=\text{div}\left(\frac{f(r)}{r}\bar{r}\right)$

$=\frac{f(r)}{r}\text{ div }\bar{r}+\bar{r}\left(\text{grad }\frac{f(r)}{r}\right)$

$=\frac{f(r)}{r}3+\bar{r}\cdot\left[\frac{\left(\frac{rf'(r)-f(r)}{r^2}\right)\bar{r}}{r}\right]$ $\qquad\because\text{div }\bar{r}=3\text{ and gradf(r)}=\frac{f'(r)\bar{r}}{r}$

$=\frac{3f(r)}{r}+\left(\frac{rf'(r)-f(r)}{r^3}\right)(\bar{r}\cdot\bar{r})$

$=\frac{3f(r)}{r}+\left(\frac{rf'(r)-f(r)}{r^3}\right)r^2$

$=\frac{3f(r)}{r}+\frac{rf'(r)-f(r)}{r}$

$=\frac{3f(r)}{r}+f'(r)-\frac{f(r)}{r}$

$=\frac{2f(r)}{r}+f'(r)$

RHS $=\frac{1}{r^2}\frac{d}{dr}[r^2f(r)]$

$=\frac{1}{r^2}[2rf(r)+r^2f'(r)]$

$=\frac{2f(r)}{r}+f'(r)$

$$\therefore \text{div}\left(\frac{f(r)}{r}\bar{r}\right) = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$$

put $\frac{f(r)}{r} = r^n \qquad \therefore f(r) = r^{n+1}$

$$\therefore \text{div}(r^n\bar{r}) = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$$

$$= \frac{1}{r^2}\frac{d}{dr}(r^2 r^{n+1})$$

$$= \frac{1}{r^2}\frac{d}{dr}(r^{n+3})$$

$$= \frac{1}{r^2}(n+3)r^{n+2}$$

$$= (n+3)r^n$$

30. Prove that $\nabla\cdot\left[\frac{\log r}{r}\bar{r}\right] = \frac{1}{r}[1 + 2\log r]$

Solution: Putting $f(r) = \log r$ in above example.

$$\nabla\cdot\left[\frac{\log r}{r}\bar{r}\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$$

$$= \frac{1}{r^2}\frac{d}{dr}[r^2\log r]$$

$$= \frac{1}{r^2}\left[r^2\frac{1}{r} + 2r\log r\right]$$

$$= \frac{1}{r^2}[r + 2r\log r]$$

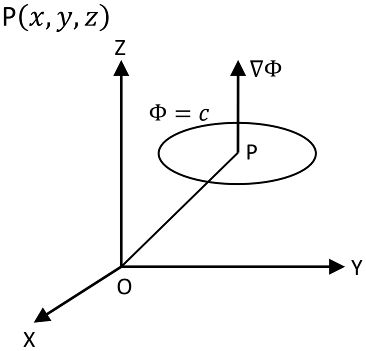
$$= \frac{1}{r}[1 + 2\log r]$$

GEOMETRICAL MEANING OF grad Φ :

Consider a scalar point function Φ and let $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ be the position vector of a point P on the surface $\Phi\,(x,y,z) = c$.

Such a surface for which the value of the function is constant is called a **level surface.**

Then $d\bar{r} = dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k}$ lies in the tangent plane to the surface at P(x,y,z)



Since $\Phi\,(x,y,z) = c$

$$\therefore d\Phi = 0$$

$$\therefore \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0$$

Hence, $\nabla\Phi\cdot\bar{dr} = \left(i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}\right)\cdot(dxi + dyj + dzk)$

$$= \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0$$

$\therefore$ $\nabla\Phi$ is a vector perpendicular to $d\bar{r}$. But since $d\bar{r}$ lies in the tangent plane, $\nabla\Phi$ is a vector perpendicular to the tangent plane to the surface $\Phi\,(x,y,z) = c$.

$\therefore$ $\nabla\Phi$ is a normal vector to the surface $\Phi\,(x,y,z) = c$ in the outward direction.

ANGLE BETWEEN TWO SURFACES:

We know that $\nabla\Phi$ is perpendicular to the tangent plane to the surface $\Phi\,(x, y, z) = c$. Hence, if $\Phi\,(x, y, z) = c_1$ and $\Psi\,(x, y, z) = c_2$ are two surfaces the angle between the two surface is equal to the angle between the normal i.e. the angle between $\nabla\Phi$ and $\nabla\Psi$.

If θ is the angle between them then $\theta = cos^{-1}\left|\frac{\nabla\Phi\cdot\nabla\Psi}{|\nabla\Phi||\nabla\Psi|}\right|$

If the surfaces are orthogonal then $\nabla\Phi\cdot\nabla\Psi = 0$

EXERCISE – IV

1. Find the unit normal to the surface

(i) $2x^2 + 4yz - 5z^2 = -10\ at\ (3, -1, 2)$. **(ii)** $x^2 + y^2 + z^2 = a^2\ at\ \left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

(iii) $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).

2. Find the equation of the tangent plane to the surface,

(i) $x^2y + 2xz^2 = 8$ at $(1, 0, 2)$. **(ii)** $xz^2 + x^2y = z - 1$ at $(1, -3, 2)$.

3. Find the angle between the normal to the surface $x^2y + 2xz = 4$ at $(2, -2, 3)$ and to $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.

4. Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 9$, $z = x^2 + y^2 - 3$ at $(2, -1, 2)$

5. Find the angle between the normal to the surface $\mathrm{xy} = \mathrm{z}^2$ at $\mathrm{P}(1, 1, 1)$ and $\mathrm{Q}(4, 1, 2)$

6. Find the angle between the normal to the surface $x^2y + z = 3$ and $x\log z - y^2 + 4 = 0$ at $(-1, 2, 1)$.

7. Find the angle between the surface $x^2 + y^2 + z^2 = 12$ and $x^2 + y^2 - z = 6$ at $(2, -2, 2)$.

8. Find the angle between the two surface $x^2 + y^2 + az^2 = 6$ and $z = 4 - y^2 + bxy$ at $P(1, 1, 2)$

9. Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and $bx^2y + y^2z + z = 1$ at $(1, 1, 0)$

10. Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at $(1, -1, 2)$.

11. Find the constants a, b such that the surface $ax^2 - bxz + xz = 10$ is orthogonal to the surface $x^2 + y^2 = 4 + xz$ at $(1, 2, 1)$.

12. Find the constants a and b such that the surface $ax^2 - 2byz = (a + 4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4\ at\ (1, -1, 2)$.

13. Find the constants a, b, c if the normal to the surfaces $ax^2 + bxz + z^2y = c$ at $P(-1, 1, 2)$ is parallel to $x^2 - y^2 + 2z = 2$ at $Q(1, 1, 1)$.

14. Find the constants a, b, c if the normal to the surface $ax^2 + yz + bxz^3 = c$ at $P(1, 2, 1)$ is parallel to the normal to the surface $y^2 + xz = 61$ at $(10, 1, 6)$.

15. Find the constants a, b if the angle between the surfaces $x^2 + axz + byz = 2$ & $x^2z + xy + y + 1 = z$

at $(0,1,2)$ $is\ cos^{-1}(1/\sqrt{3})$

ANSWERS

1. **(i)** $\frac{3i+2j-6k}{7}$ **(ii)** $\frac{i+j+k}{\pm\sqrt{3}}$ **(iii)** $(8i+j+8k)/\sqrt{129}$

2. **(i)** $8x+y+8z=24$ **(ii)** $2x-y-3z+1=0$

3. $\cos\theta=11/\sqrt{126}$ **4.** $\cos\theta=8/3\sqrt{21}$ **5.** $\cos\theta=13/\sqrt{198}$

6. $\cos\theta=-5/3\sqrt{34}$ **7.** $\cos\theta=7/\sqrt{33}$ **8.** $a=1, b=-1, \cos\theta=\frac{\sqrt{6}}{\sqrt{11}}$

9. $a=1, b=1, \theta=45°$ **10**. $a=4, b=1$ **11**. $a=0, b=-9$

12. $a=5, b=1$ **13.** $a=10, b=8, c=-2$ **14.** $a=1, b=1, c=4$

15. $a=1, b=1$

SOME SOLVED EXAMPLES:

1. Find the unit normal to the surface $xy^3z^2=4$ at$(-1,-1,2)$

Solution: we know that $\nabla\Phi$ is the vector normal to the surface $\Phi(x,y,z)=c$ at P

Now, $\nabla\Phi = i\frac{\partial}{\partial x}(xy^3z^2)+j\frac{\partial}{\partial y}(xy^3z^2)+k\frac{\partial}{\partial z}(xy^3z^2)$

$= y^3z^2i+3xy^2z^2j+2xy^3zk$

$= -4i-12j-4k$ at $(-1,-1,2)$

Unit vector normal to the surface at $(-1,-1,2)$

$= \frac{\nabla\Phi}{|\nabla\Phi|}=\frac{-4i-12j+4k}{\sqrt{16+144+16}}=-\frac{1}{\sqrt{11}}(i+3j-k)$

2. Find the equation of the tangent plane to the surface $2xz^2-3xy-4x=7$ at $(1,-1,2)$.

Solution: $\nabla\Phi = (2z^2-3y-4)i-3xj+4xzk$

$= 7i-3j+8k$ at $(1,-1,2)$

The normal to the surface at $(1,-1,2)$ is $7i-3j+8k$

∴ The direction rations of the normal to the tangent plane are $7,-3,8$.

Since it passes through $(1,-1,2)$,

its equation is $7(x-1)-3(y+1)+8(z-2)=0$ i.e. $7x-3y+8z-26=0$

3. Find the angle between the normals to the surface $xy=z^2$ at the points $(1,4,2)$ and $(-3,-3,3)$

Solution: Let $\Phi=xy-z^2$

$\nabla\Phi = i\frac{\partial}{\partial x}(xy-z^2)+j\frac{\partial}{\partial y}(xy-z^2)+k\frac{\partial}{\partial z}(xy-z^2)$

$= yi+xj-2zk$

$\nabla\Phi = 4i+j-4k$ at $(1,4,2)$

and $\nabla\Phi = -3i - 3i - 6k$ at $(-3, -3, 3)$

But these are the normals to the surfaces at the given points

Angle between two vectors $\bar{a}, \bar{b}$ is given by $\bar{a} \cdot \bar{b} = |a||b| \cos\theta$

If θ is the angle between the above vectors

$(4i + j - 4k) \cdot (-3i - 3i - 6k) = |4i + j - 4k| \cdot |-3i - 3i - 6k| \cos\theta$

$\therefore -12 - 3 + 24 = \sqrt{33}\sqrt{54} \cos\theta$

$\therefore \cos\theta = \frac{9}{9\sqrt{22}} = \frac{1}{\sqrt{22}}$

4. Find the angle between the surfaces $x \log z + 1 - y^2 = 0, x^2y + z = 2$ at $(1, 1, 1)$

Solution: We have $\Phi = x \log z + 1 - y^2$

$\therefore \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$= \log z i - 2yj + \frac{x}{z}k$

$= 0i - 2j + k$ at $(1, 1, 1)$

Unit vector at $(1, 1, 1) = \frac{0i - 2j + k}{\sqrt{5}}$

$\Psi = x^2y + z - 2$

$\nabla\Psi = i\frac{\partial\Psi}{\partial x} + j\frac{\partial\Psi}{\partial y} + k\frac{\partial\Psi}{\partial z} = 2xyi + x^2j + k = 2i + j + k$ at $(1, 1, 1)$

Unit vector at $(1, 1, 1) = \frac{2i + j + k}{\sqrt{6}}$

$\cos\theta = \frac{(0 - 2j + k)}{\sqrt{5}} \cdot \frac{(2i + j + k)}{\sqrt{6}} = -\frac{1}{30}$

5. Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$.

Solution: Let $u = ax^2 - byz - (a + 2)x$ and $v = 4x^2y + z^3 - 4$

$\therefore \nabla u = (2ax - a - 2)i + (-bz)j + (-by)k$

$= (a - 2)i - 2bj + bk$ at $(1, -1, 2)$

The direction ratios of the normal to this surface at $(1, -1, 2)$ are $a - 2, -2b, b$

And $\nabla v = 8xyi + 4x^2j + 3z^2k = -8i + 4j + 12k$ at $(1, -1, 2)$

The directions ratios of the normal to this surface at $(1, -1, 2)$ are $-8, 4, 12$ i.e. $-2, 1, 3$

Since the surfaces are orthogonal, normals are perpendicular to each other

$\therefore (a - 2)(-2) + (-2b)(1) + (b)(3) = 0$ i.e. $-2a + b = -4$ ………… (1)

Since $(1, -1, 2)$ lies on the surface $ax^2 - byz - (a + 2)x = 0$, we have $a + 2b - a - 2 = 0$

i.e. $b = 1$ ………… (2)

Then from (1) we get $a = 5/2$. Hence, $a = 5/2$ and $b = 1$

DIRECTIONAL DERIVATIVE:

$\nabla\Phi$ is a vector quantity its component (or resolved part) in the direction of a vector $\bar{a}$ is $\frac{\nabla\Phi\cdot\bar{a}}{|\bar{a}|}$

This component is called the directional derivative of Φ in the direction of $\bar{a}$.

Thus, the directional derivative of Φ in the direction of $\bar{a} = \frac{\nabla\Phi\cdot\bar{a}}{|\bar{a}|}$

Physically the directional derivative is the rate of change of Φ at (x, y, z) in the given direction.

MAXIMUM DIRECTIONAL DERIVATIVE:

Since the resolved part of a vector is maximum in its own direction, the directional derivative is maximum in the direction $\nabla\Phi$. Since $\nabla\Phi$ is normal to the surface, we can also say that $\nabla\Phi$ is maximum in the direction of the normal to the surface and the maximum directional derivative is $|\nabla\Phi|$.

SOME SOLVED EXAMPLES:

1. Find the directional derivative of $\Phi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $i + 2j + 2k$.

Solution: $\nabla\Phi = i\frac{\partial}{\partial x}(xy^2 + yz^3) + j\frac{\partial}{\partial y}(xy^2 + yz^3) + k\frac{\partial}{\partial z}(xy^2 + yz^3)$

$= y^2 i + (2xy + z^3)j + 3yz^2 k$

$= i - 3j - 3k$ at $(2, -1, 1)$

Directional derivative in the direction of $(i + 2j + 2k)$

$= (i - 3j - 3k)\cdot\frac{(i+2j+2k)}{\sqrt{1+4+4}} = \frac{1}{3}(1 - 6 - 6) = -\frac{11}{3}$

2. Find the directional derivative of $\Phi = x^4 + y^4 + z^4$ at point $A(1, -2, 1)$ in the direction of AB where B is $(2, 6, -1)$.

Solution: $\nabla\Phi = i\frac{\partial}{\partial x}(x^4 + y^4 + z^4) + j\frac{\partial}{\partial y}(x^4 + y^4 + z^4) + k\frac{\partial}{\partial z}(x^4 + y^4 + z^4)$

$= 4(x^3 i + y^3 j + z^3 k) = 4(i - 8j + k)$ at $(1, -2, 1)$

$\overline{AB} = \overline{OB} - \overline{OA} = (2 - 1)i + (6 + 2)j + (-1 - 1)k = i + 8j - 2k$

$\therefore$ Directional derivative in the direction of $\overline{AB} = \nabla\Phi\frac{\overline{AB}}{|\overline{AB}|}$

$= 4(i - 8j + k)\cdot\frac{(i+8j-2k)}{\sqrt{1+64+4}} = \frac{4(1-64-2)}{\sqrt{69}} = -\frac{260}{\sqrt{69}}$

3. Find the directional derivative of $\Phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at $(1, 2, 3)$

Solution: $\nabla\Phi = i\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + j\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$

$= 2(xi + yj + zk)$

$= 2(i + 2j + 3k)$ at $(1, 2, 3)$

Given direction $\bar{a} = 3i + 4j + 5k$

Directional derivative in the given direction $= \nabla\Phi \cdot \frac{\bar{a}}{|\bar{a}|} = 2(i + 2j + 3k) \cdot \frac{(3i+4j+5k)}{\sqrt{9+16+25}}$

$$= \frac{2(3+8+15)}{5\sqrt{2}} = \frac{26}{5}\sqrt{2}$$

4. Find the directional derivative of $\Phi = e^{2x}\cos yz$ at $(0,0,0)$ in the direction of the tangent to the curve $x = a\sin t\,,\ y = a\cos t,\ \ z = at$ at $t = \pi/4$.

Solution: $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$= 2e^{2x}\cos yz\, i - ze^{2x}\sin yz\, j - ye^{2x}\sin yz\, k$

$= 2i + 0j + 0k$ at $(0, 0, 0)$

The equation of the curve is $\bar{r} = (a\sin t)i + (a\cos t)j + at\, k$

$\therefore \dot{\bar{r}} = a\cos t\, i - a\sin t\, j + ak$

At $t = \frac{\pi}{4}$, $\dot{\bar{r}} = \frac{a}{\sqrt{2}}i - \frac{a}{\sqrt{2}}j + ak$

$\therefore |\dot{\bar{r}}| = \sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2} = \sqrt{2}a$

$\therefore$ Unit tangent vector $= \frac{\dot{\bar{r}}}{|\dot{\bar{r}}|}$

$\hat{T} = \frac{(a\cos t i - a\sin t j + ak)}{\sqrt{2}a} = \frac{\left(\frac{a}{\sqrt{2}}i - \frac{a}{\sqrt{2}}j + ak\right)}{\sqrt{2}a} = \frac{(i - j + \sqrt{2}k)}{2}$ at $t = \frac{\pi}{4}$

$\therefore$ The directional derivative in the given direction $= \nabla\Phi \cdot \hat{T} = (2i + 0j + 0k) \cdot \frac{(i - j + \sqrt{2}k)}{2} = \frac{2}{2} = 1$

5. Find the directional derivative of $\Phi = \frac{y}{x^2+y^2}$ at $(0,1)$ in the direction making an angle of $30°$ with the positive x – axis .

Solution: $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y}$

$= \left[-\frac{y}{(x^2+y^2)^2} \cdot 2x\right] i + \left[\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right] j$

$= \frac{-2xy}{(x^2+y^2)^2} i + \frac{(x^2-y^2)}{(x^2+y^2)^2} j$

$= 0i - j$ at $(0, 1)$

Unit vector making an angle of $30°$ with the x –axis $= \cos 30°\, i + \sin 30°\, j = \frac{\sqrt{3}}{2}i + \frac{1}{2}j$

$\therefore$ Required directional derivative $= (0i - j) \cdot \left(\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right) = -\frac{1}{2}$

6. In what direction from the point $(2, 1, -1)$ is the directional derivative of $\Phi = x^2yz^3$ maximum? What is its magnitude ?

Solution: $\nabla\Phi \ = \nabla x^2yz^3$

$= i\frac{\partial}{\partial x}(x^2yz^3) + j\frac{\partial}{\partial y}(x^2yz^3) + k\frac{\partial}{\partial z}(x^2yz^3)$

$= 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$

$= -4i - 4j + 12k$ at $(2, 1, -1)$

Directional derivative is maximum in the direction of $\nabla\Phi$.

Hence, directional derivative is maximum in the direction of $\nabla\Phi = -4i - 4j + 12k$

Its magnitude = $|\nabla\Phi| = \sqrt{16 + 16 + 144} = 4\sqrt{11}$

7. Find the maximum directional derivative of $\Phi = (4x - y + 2z)^2$ at (1, 2, 1).

Solution: $\nabla\Phi \quad = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$= 2(4x - y + 2z)4i + 2(4x - y + 2z)(-1)j + 2(4x - y + 2z)\cdot 2k$

$= 2(4x - y + 2z)(4i - j + 2k)$

At $(1, 2, 1)$ $\quad \nabla\Phi = 8(4i - j + 2k)$

maximum directional derivative = $|\nabla\Phi| = 8\sqrt{16 + 1 + 4} = 8\sqrt{21}$

8. Find the constants a, b, c if the directional derivative of $\Phi = ax^2y + by^2z + cz^2x$ at $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

Solution: We know that the maximum magnitude of directional derivative is in the direction of $\nabla\Phi$ itself

Now, $\nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$= (2axy + cz^2)i + \left(ax^2 + 2byz\right)j + (by^2 + 2czx)k$

$= (2a + c)i + (2b + a)j + (2c + b)k$ at $(1, 1, 1)$ .... (1)

But the given direction is $2i - 2j + k$ .... (2)

Since, the two directions are parallel, from (1) and (2) we get,

$\frac{2a+c}{2} = \frac{2b+a}{-2} = \frac{2c+b}{1}$

$\therefore 3a + 2b + c = 0$ and $a + 4b + 4c = 0$ (From the first two and the last two equalities)

$\therefore \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$ say (By Crammer's rule)

$\therefore a = 4k, b = -11k, c = -10k$

But the magnitude of maximum directional derivative i.e. $|\nabla\Phi|$ is given to be 15

$\therefore 15 = \sqrt{[(2a + c)^2 + (2b + a)^2 + (2c + b)^2]}$

$\therefore 15 = \sqrt{[(8k + 10k)^2 + (-22k + 4k)^2 + (20k - 11k)]^2} = \sqrt{[18^2 + 18^2 + 9^2]}\cdot k$

$\therefore 15 = 9\sqrt{(4 + 4 + 1)} = \pm 27k \qquad \therefore k = \pm\frac{5}{9}$

$\therefore a = \pm\frac{20}{9}, \qquad b = \mp\frac{55}{9}, \qquad c = \pm\frac{50}{9}$

9. Find the values of a, b, c if the directional derivative of $\Phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum magnitude 64 in the direction parallel to the z – axis.

Solution: We have $\Phi = axy^2 + byz + cz^2x^3$

$\therefore \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

$= (ay^2 + 3cx^2z^2)i + (2axy + bz)j + (by + 2czx^3)k$

$= (4a + 3c)i + (4a - b)j + (2b - 2c)k$ at $(1, 2, -1)$ ………… (1)

The directional derivative is maximum in the direction of $\nabla\Phi$

i.e. in the direction of $(4a + 3c)i + (4a - b)j + (2b - 2c)k$

But by data directional derivative is maximum in the direction of the z –axis

i.e. in the direction of $0i + 0j + k$

Hence, the two directions are parallel

$\therefore \frac{4a+3c}{0} = \frac{4a-b}{0} = \frac{2b-2c}{1}$

$\therefore 4a + 3c = 0$ and $4a - b = 0$

Hence, from (1), $\therefore \nabla\Phi = (2b - 2c)k$ $\therefore |\nabla\Phi| = |2b - 2c|$ $[\because |k| = 1]$

But directional derivative is maximum in the direction of $\nabla\Phi$ and is given to be 64,

$\therefore 2b - 2c = 64$ $\therefore b - c = 32$

Solving $4a + 3c = 0, 4a - b = 0$ and $b - c = 32$, we get $a = 6, b = 24, c = -8$

PHYSICAL INTERPRETATION OF DIVERGENCE:

Consider a region of space filled with a fluid which moves so that its velocity vector at any point P(x,y,z) is $\bar{F}(x, y, z) = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ where f_1, f_2, f_3 are scalar functions of x, y, z and are the components of velocity parallel to the axes.

Now construct a parallelepiped having centre at P(x, y, z) and edges parallel to the coordinates axes. Since the components of normal to any face is responsible for the flow through that face. Total gain of fluid in the parallelepiped per unit volume per unit time is $\nabla \cdot \bar{F}$ which is $div\bar{F}$.

A vector $\bar{F}$ whose $div\bar{F} = 0$ is called **solenoid** for such a vector there is no loss or gain of fluid. If there is no gain of fluid anywhere then $\nabla \cdot \bar{F}$ = 0. Since fluid is neither created nor destroyed at any point, it is said to have no source or sinks

Examples:

1. If V represents an electric flux then div V is the amount of flux which diverges per unit volume in unit time.

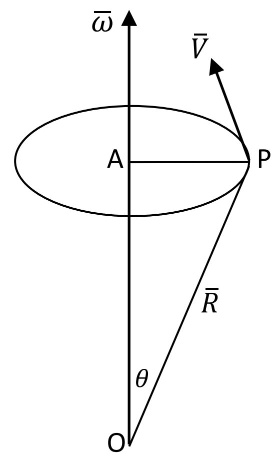
2. If V represents heat flux then div V is the rate at which heat is issuing from a point per unit volume in unit time.

3. If V represents velocity of fluid then div V gives the rate at which fluid is originating at a point per unit volume.

PHYSICAL INTERPRETATION OF CURL:

Consider a motion of a rigid body rotating about a fixed axis OA through the origin. Let Ω be its angular velocity. Let P be a point of the body whose position vector is $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

Let $\angle AOP = \theta$ and PA be perpendicular to OA



If $\bar{n}$ is a unit vector perpendicular to Ω and $\bar{r}$ then

$\Omega \times \bar{r} = (\omega\, r \sin\theta)\, \bar{n}$

$= (\omega\, PA)\, \bar{n}$

$= (Speed\ of\ P)\bar{n}$

= the velocity $\bar{V}$ of p in the direction perpendicular to the plane AOP

$\therefore\ \Omega \times \bar{r} = \bar{V}$

Let $\Omega = \omega_1\bar{\imath} + \omega_2\bar{\jmath} + \omega_3\bar{k}$ then

$$\bar{V} = \Omega \times \bar{r} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\bar{\imath} + (\omega_3 x - \omega_1 z)\bar{\jmath} + (\omega_1 y - \omega_2 x)\bar{k}$$

$$Curl\,\bar{V} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$= (\omega_1 + \omega_1)\bar{\imath} + (\omega_2 + \omega_2)\bar{\jmath} + (\omega_3 + \omega_3)\bar{k}$ since $\omega_1, \omega_2, \omega_3$, are constant

$= 2(\omega_1\bar{\imath} + \omega_2\bar{\jmath} + \omega_3\bar{k})$

$= 2\Omega$

Hence $\Omega = \frac{1}{2}$ curl $\bar{V}$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector. In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Any motion in which the curl of the velocity vector is zero i.e. if $Curl\,\bar{V} = 0$ then $\Omega = 0$ i.e angular velocity is zero then the motion is said to be **irrotational or conservative**, otherwise it is rotational. In view of this interpretation of curl, curl is also called the **rotation of $\bar{F}$** and is sometimes denoted by rot $\bar{F}$

In general, if $\nabla \times \bar{V} = 0$ i.e curl $\bar{V} = \bar{0}$ then we can find scalar field Φ so that $\bar{V} = \nabla\Phi$.

A vector field $\bar{V}$ which can be derived from a scalar field Φ so that $\bar{V} = \nabla\Phi$ is called a **conservation vector field** and Φ is called the **scalar potential**

conversely also, if $\bar{V} = \nabla\Phi$ then $\nabla \times \bar{V} = \bar{0}$ i.e curl $\bar{V} = \bar{0}$

SOME SOLVED EXAMPLES:

1. If $\bar{F} = (x + 3y)i + (y - 2z)j + (az + x)k$ is solenoidal, find the value of a.

Solution: We know that $\bar{F}$ is solenoidal if div $\bar{F} = \nabla \cdot \bar{F} = 0$

Now, $\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(az + x) = 1 + 1 + a = 2 + a$

$\bar{F}$ is solenoidal if $\nabla \cdot \bar{F} = 0$

$\therefore 2 + a = 0 \qquad \therefore a = -2$

2. Find a, b, c if $\overline{F} = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ is irrotational.

Solution: $\overline{F}$ is irrotational if curl $\overline{F} = \overline{0}$

$$\therefore \text{curl}\,\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + bz^3 & 3x^2 - cz & 3xz^2 - y \end{vmatrix}$$

$$= i(-1 + c) - j(3z^2 - 3bz^2) + k(6x - ax) = 0i + 0j + 0k$$

$$\therefore c - 1 = 0,\ 3z^2 - 3bz^2 = 0,\ \ 6x - ax = 0$$

$$\therefore c = 1, b = 1, a = 6$$

3. Prove that $\overline{F} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is solenoidal and determine the constants a, b, c if $\overline{F}$ is irrotational.

Solution: $\overline{F}$ is solenoidal if $\nabla \cdot \overline{F} = 0$

Now, $\nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 - 3 + 2 = 0$

Hence, for all values of $a, b, c, \overline{F}$ is solenoidal

$\overline{F}$ is irrotational if curl $\overline{F} = \overline{0}$

$$\text{Now, curl}\,\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k$$

$$= (c + 1)i + (a - 4)j + (b - 2)k = 0i + 0j + 0k$$

$$\therefore c + 1 = 0,\ a - 4 = 0,\ \ b - 2 = 0$$

$$\therefore a = 4,\ b = 2,\ \ c = -1$$

4. Is $\overline{F} = \frac{\overline{a} \times \overline{r}}{r^n}$ a solenoidal vector? ($\overline{a}$ is a constant vector).

Solution: By data $\overline{F} = \frac{\overline{a} \times \overline{r}}{r^n} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \frac{x}{r^n} & \frac{y}{r^n} & \frac{z}{r^n} \end{vmatrix} = i\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) + j\left(\frac{a_3 x}{r^n} - \frac{a_1 z}{r^n}\right) + k\left(\frac{a_1 y}{r^n} - \frac{a_2 x}{r^n}\right)$

Now, $\nabla \cdot \overline{F} = \frac{\partial}{\partial x}\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) + \frac{\partial}{\partial y}(\dots\dots) + \frac{\partial}{\partial z}(\dots\dots)$

$$\frac{\partial}{\partial x}\left(\frac{a_2 z}{r^n} - \frac{a_3 y}{r^n}\right) = (a_2 z - a_3 y)\frac{\partial}{\partial x} r^{-n}$$

$$= (a_2 z - a_3 y)\left(-nr^{-n-1}\frac{\partial r}{\partial x}\right)$$

$$= (a_2 z - a_3 y)\left(-nr^{-n-1}\frac{x}{r}\right) = (a_2 z - a_3 y)\left(\frac{-nx}{r^{n+2}}\right)$$

.... $\left(\because r^2 = x^2 + y^2 + z^2 \quad \therefore 2r\frac{\partial r}{\partial x} = 2x\right)$

By symmetry, we get two more expressions

$$\therefore \nabla \cdot \overline{F} = r^{n+2}(a_3 xy - a_2 xz + a_1 zy - a_3 xy + a_2 xz - a_1 yz) = 0$$

Hence $\overline{F}$ is solenoidal

Aliter:

$\nabla \cdot \overline{F} = \nabla \cdot \left(\frac{\overline{a}\times\overline{r}}{r^n}\right)$

$= \nabla \cdot [r^{-n}(\overline{a} \times \overline{r})]$

$= (\nabla r^{-n}) \cdot (\overline{a} \times \overline{r}) + r^{-n}\big(\nabla \cdot (\overline{a} \times \overline{r})\big)$

$= (-nr^{-n-2}\overline{r}) \cdot (\overline{a} \times \overline{r}) + r^{-n}\big(\nabla \cdot (\overline{a} \times \overline{r})\big) \quad \because \mathrm{grad f(r)} = \frac{\mathrm{f}'(\mathrm{r})\overline{\mathrm{r}}}{\mathrm{r}}$

$= -\, nr^{-n-2}\big(\overline{r} \cdot (\overline{a} \times \overline{r})\big) + r^{-n}\, 0 \quad \because div\ (\overline{a} \times \overline{r}) = 0$

$= 0 + 0 = 0$

Hence $\overline{F}$ is solenoidal

5. If $\overline{r}$ is the position vector of a point (x, y, z) and r is the modulus of $\overline{r}$ then prove that $r^n\overline{r}$ is an irrotational vector for any value of n but is solenoidal only if $n = -3$.

Solution: **(a)** By definition

$$\mathrm{curl}\ r^n\overline{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] + j(\dots\dots) + k(\dots\dots)$$

$$= i\left[znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right] + j(\dots\dots) + k(\dots\dots)$$

Now, $r^2 = x^2 + y^2 + z^2 \quad \therefore 2r\frac{\partial r}{\partial x} = 2x \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \mathrm{curl}\ r^n\overline{r} = i[nr^{n-2}zy - nr^{n-2}zy] + j(\dots\dots) + k(\dots\dots) = \overline{0}$

Hence, $r^n\overline{r}$ is irrotational

Aliter:

$\mathrm{curl}\ (r^n\overline{r}) \quad = r^n \mathrm{curl}\ \overline{r} + (\mathrm{grad}\ r^n) \times \overline{r}$

$= r^n\overline{0} + \frac{nr^{n-1}\overline{r}}{r} \times \overline{r}$

$= \overline{0} + \left(\frac{nr^{n-1}}{r}\right)(\overline{r} \times \overline{r})$

$= \overline{0} + \overline{0}$

$= \overline{0}$

Hence, $r^n\overline{r}$ is irrotational

(b) $\mathrm{div}\ (r^n\overline{r}) \quad = \nabla \cdot (r^n xi + r^n yj + r^n zk)$

$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= \left[r^n + xnr^{n-1}\frac{\partial r}{\partial x}\right] + [\dots\dots] + [\dots\dots]$

$$= \left[r^n + nxr^{n-1}\frac{x}{r}\right] + \left[r^n + nyr^{n-1}\frac{y}{r}\right] + \left[r^n + nzr^{n-1}\frac{z}{r}\right]$$

$$= 3r^n + nr^{n-1}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) = 3r^n + nr^{n-1}\frac{(x^2+y^2+z^2)}{r}$$

$$= 3r^n + nr^{n-1}\cdot r = (n+3)r^n$$

Hence, div $(r^n\bar{r}) = 0$ if $n = -3$

Aliter:

$$\text{div}\,(r^n\bar{r}) = r^n \text{div}\,\bar{r} + (\text{grad}\, r^n)\cdot\bar{r}$$

$$= r^n 3 + \left(\frac{nr^{n-1}\bar{r}}{r}\right)\cdot\bar{r} \qquad \because div\,\bar{r} = 3\ \&\ grad\, f(\text{r}) = \frac{f'(\text{r})\bar{\text{r}}}{\text{r}}$$

$$= 3r^n + \left(\frac{nr^{n-1}}{r}\right)(\bar{r}\cdot\bar{r})$$

$$= 3r^n + nr^n$$

$$= (3+n)r^n$$

$\therefore$ div $(r^n\bar{r}) = 0 \Rightarrow n = -3$

$\therefore$ $r^n\bar{r}$ solenoidal only if $n = -3$

6. Find f (r), so that the vector $f(r)\bar{r}$ is both solenoidal and irrotational .

Solution: We have $f(r)\bar{r} = f(r)xi + f(r)yj + f(r)zk$

div $[f(r)\bar{r}] = \nabla\cdot f(r)\bar{r}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot[f(r)xi + f(r)yj + f(r)zk]$$

$$= \frac{\partial}{\partial x}[f(r)x] + \frac{\partial}{\partial y}[f(r)y] + \frac{\partial}{\partial z}[f(r)z]$$

Now, $\frac{\partial f(r)}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$

Similarly, $\frac{\partial}{\partial y}f(r) = f'(r)\frac{y}{r}$, $\frac{\partial}{\partial z}f(r) = f'(r)\frac{z}{r}$

$\therefore \frac{\partial}{\partial x}[f(r)x] = x\frac{\partial}{\partial x}f(r) + f(r) = x\frac{f'(r)}{r}\cdot x + f(r)$

$\therefore$ div $[f(r)\bar{r}] = f'(r)\frac{x}{r}\cdot x + f(r) + f'(r)\frac{y}{r}\cdot y + f(r) + f'(r)\frac{z}{r}\cdot z + f(r)$

$$= 3f(r) + f'(r)\cdot\frac{1}{r}(x^2+y^2+z^2)$$

$$= 3f(r) + f'(r)r$$

If $f(r)\bar{r}$ is solenoidal

div $[f(r)\bar{r}] = 3f(r) + f'(r)r = 0$

$\therefore \frac{f'(r)}{f(r)} = -\frac{3}{r}$

Integrating $\log f(r) = -3\log r + \log c$

$\therefore \log f(r) = \log\frac{c}{r^3} \qquad \therefore f(r) = \frac{c}{r^3}$

$\therefore f(r)\bar{r}$ is solenoidal if $f(r) = \frac{c}{r^3}$

Aliter:

$f(r)\bar{r}$ is solenoidal if div $(f(r)\bar{r}) = 0$

$f(r)\,\mathrm{div}\,\bar{r} + \mathrm{grad}\, f(r)\cdot\bar{r} = 0$

$f(r)3 + \frac{f'(r)}{r}\bar{r}\cdot\bar{r} = 0$

$f(r)3 + f'(r)r = 0$

$\frac{f'(r)}{f(r)} + \frac{3}{r} = 0$

Integrating, $\log f(r) + 3\log r = \log c$

$\log[f(r)\cdot r^3] = \log c$

$f(r)\cdot r^3 = c \qquad f(r) = \frac{c}{r^3} \qquad \therefore f(r)\bar{r} = \frac{c}{r^3}\bar{r}$

Now, curl $[f(r)\bar{r}]$ $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$

$= i\left[\frac{\partial}{\partial y}zf(r) - \frac{\partial}{\partial z}yf(r)\right] + j[\dots\dots] + k[\dots\dots]$

$= i\left[zf'(r)\cdot\frac{y}{r} - y\cdot f'(r)\cdot\frac{z}{r}\right] + j[\dots\dots] + k[\dots\dots]$

$= f'(r)i\left[\frac{zy}{r} - \frac{zy}{r}\right] + j[\dots\dots] + k[\dots\dots] = 0$

$\therefore f(r)\bar{r}$ is irrotational for any $f(r)$ and hence for $f(r) = \frac{c}{r^3}$.

Aliter:

$f(r)\bar{r}$ is irrotational if curl $[f(r)\bar{r}] = \bar{0}$

$f(r)\,\mathrm{curl}\,\bar{r} + (\mathrm{grad}\, f(r))\times\bar{r} = \bar{0}$

$f(r)\bar{0} + \frac{f'(r)\bar{r}}{r}\times\bar{r} = \bar{0}$ which is always true

Hence, $f(r)\bar{r} = \frac{c}{r^3}\bar{r}$ is both solenoidal and irrotational

7. Show that $\bar{F} = \frac{\bar{r}}{r^2}$ is irrotational. Find Φ such that $\bar{F} = -\nabla\Phi$ where $\bar{r} = xi + yj + zk$.

Solution: $\mathrm{curl}\,\bar{F} = \mathrm{curl}\frac{\bar{r}}{r^2} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix}$

$= \left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right]i + [\dots\dots]j + [\dots\dots]k$

$= \left(-\frac{2}{r^3}\frac{yz}{r} + \frac{2}{r^3}\frac{zy}{r^2}\right)i + [\dots\dots]j + [\dots\dots]k$

$= 0i + 0j + 0k = \bar{0} \qquad \therefore \bar{F}$ is irrotational

Aliter:

$\mathrm{curl}\,\bar{F} = \mathrm{curl}\frac{\bar{r}}{r^2}$

$= \frac{1}{r^2}(curl\,\bar{r}) + grad\left(\frac{1}{r^2}\right)\times\bar{r}$

$= \frac{1}{r^2}(\bar{0}) + \left(\frac{-2r^{-3}\bar{r}}{r}\right)\times\bar{r} \qquad \because curl\,\bar{r} = \bar{0}\ \&\ \mathrm{grad\ f(r)} = \frac{\mathrm{f'(r)}\bar{\mathrm{r}}}{\mathrm{r}}$

$$= \bar{0} - \frac{2}{r^4}(\bar{r} \times \bar{r})$$

$$= \bar{0} - \frac{2}{r^4}(\bar{0}) = \bar{0}$$

$\therefore \overline{F}$ is irrotational

Now, $\overline{F} = -\nabla\Phi$ gives

$$\frac{\bar{r}}{r^2} = \frac{xi+yi+zk}{r^2} = -\left[\frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k\right]$$

$$\therefore \frac{\partial\Phi}{\partial x} = -\frac{x}{r^2} = -\frac{x}{x^2+y^2+z^2}, \quad \frac{\partial\Phi}{\partial y} = -\frac{y}{r^2} = -\frac{y}{x^2+y^2+z^2}, \quad \frac{\partial\Phi}{\partial z} = -\frac{z}{r^2} = -\frac{z}{x^2+y^2+z^2}$$

But $d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = -\frac{xdx+ydy+zdz}{x^2+y^2+z^2}$

By integrating, $\Phi = -\frac{1}{2}\log(x^2+y^2+z^2)$

8. A vector field is given by $\overline{F} = (x^2+xy^2)i + (y^2+x^2y)j$. Show that $\overline{F}$ is irrotational and find its scalar potential.

Solution: $\operatorname{curl} \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2+x^2y)\right]i - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+xy^2)\right]j + \left[\frac{\partial}{\partial x}(y^2+x^2y) - \frac{\partial}{\partial y}(x^2+xy^2)\right]k$$

$$= 0i + 0j + (2xy - 2xy)k = 0i + 0j + 0k$$

Hence $\overline{F}$ is irrotational

If Φ is the scalar potential then $\overline{F} = \nabla\Phi$

$$\therefore (x^2+xy^2)i + (y^2+x^2y)j + 0k = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$

$$\therefore \frac{\partial\Phi}{\partial x} = x^2+xy^2, \quad \frac{\partial\Phi}{\partial y} = y^2+x^2y, \quad \frac{\partial\Phi}{\partial z} = 0 \qquad \text{.......(1)}$$

But $d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$

$$= [x^2+xy^2]dx + [y^2+x^2y]dy + 0dz$$

$$= x^2dx + y^2dy + (xy^2dx + xy^2dy)$$

By integrating $\Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{2}x^2y^2$

Aliter:

Integrating (1) w.r.t. x, y, z respectively treating the other variables constant, we get,

$$\Phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \Psi_1(y,z), \quad \Phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + \Psi_2(x,z), \quad \Phi = \Psi_3(x,y)$$

Comparing these equations, we find that

$$\Psi_1(y,z) = \frac{y^3}{3}, \quad \Psi_2(x,z) = \frac{x^3}{3}, \quad \Psi_3(x,y) = 0$$

$$\therefore \Phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2}$$

9. Show that $\overline{F} = (y\sin z - sin\ x)i + (x\sin z + 2yz)j + (xy\cos z + y^2)k$. is irrotational and find its scalar potential.

Solution: $\text{curl}\ \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(xy\cos z + y^2) - \frac{\partial}{\partial z}(x\sin z + 2yz)\right]i + \left[\frac{\partial}{\partial z}(y\sin z - \sin x) - \frac{\partial}{\partial x}(xy\cos z + y^2)\right]j$$
$$+ \left[\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right]k$$
$$= [x\cos z + 2y - x\cos z - 2y]i + [y\cos z - y\cos z]j + [\sin z - \sin z]k$$
$$= 0i + 0j + 0k = \overline{0}$$

Hence, $\overline{F}$ is irrotational

If Φ is the scalar potential then $\overline{F} = \nabla\Phi$

$\therefore\ (y\sin z - \sin x)i + (x\sin z + 2yz)j + (xy\cos z + y^2)k = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$

$\therefore\ \frac{\partial\Phi}{\partial x} = y\sin z - \sin x$ .... (1) $\quad \frac{\partial\Phi}{\partial y} = x\sin z + 2yz$ .... (2)

$\frac{\partial\Phi}{\partial z} = xy\cos z + y^2$ .... (3)

But $\quad d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$

$$= (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$
$$= [y\sin z\,dx + x\sin z\,dy + xy\cos z\,dz] + (-\sin x)dx + (2yzdy + y^2dz)$$

By integrating, $\Phi = xy\sin z + \cos x + y^2z$

Aliter:

Integrating (1), (2) and (3) partially w.r.t. x, y, z treating other variables constant, we get,

$\Phi = (xy\sin z + \cos x) + \Psi_1(y,z)$ .... (4)

$\Phi = (xy\sin z + y^2z) + \Psi_2(x,z)$ .... (5)

$\Phi = (xy\sin z + y^2z) + \Psi_3(x,y)$ .... (6)

Comparing (4), (5) and (6) we find that

$\Psi_1(y,z) = y^2z,\ \Psi_2(x,z) = \cos x,\ \ \Psi_3(x,y) = \cos x$

$\Phi = xy\sin z + \cos x + y^2z$