Examples Using Partial Fraction Method

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Ex-1 Find $L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}\right]$

Solution Let $\dfrac{5s^2-15s-11}{(s+1)(s-2)^2} = \dfrac{a}{s+1} + \dfrac{b}{s-2} + \dfrac{c}{(s-2)^2}$

$\therefore\ 5s^2-15s-11 = a(s-2)^2 + b(s+1)(s-2) + c(s+1)$

put $s=2$, $20-30-11 = c(3) \Rightarrow -21 = 3c \Rightarrow c=-7$

put $s=-1$, $5+15-11 = 9a \Rightarrow 9a = 9 \Rightarrow a=1$

put $s=0$, $-11 = 4a-2b+c$

$\Rightarrow -11 = 4-2b-7$

$\Rightarrow -11 = -2b-3 \Rightarrow -2b=-8 \Rightarrow b=4$

$\therefore \dfrac{5s^2-15s-11}{(s+1)(s-2)^2} = \dfrac{1}{s+1} + \dfrac{4}{s-2} - \dfrac{7}{(s-2)^2}$

$\therefore L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}\right] = L^{-1}\left[\dfrac{1}{s+1}\right] + 4L^{-1}\left[\dfrac{1}{s-2}\right] - 7L^{-1}\left[\dfrac{1}{(s-2)^2}\right]$

$= e^{-t} + 4e^{2t} - 7e^{2t}L^{-1}\left[\dfrac{1}{s^2}\right]$

$= e^{-t} + 4e^{2t} - 7e^{2t}\cdot t$

Ex 2:- Find $L^{-1}\left[\dfrac{s+29}{(s+4)(s^2+9)}\right]$

Solution Let $\dfrac{s+29}{(s+4)(s^2+9)} = \dfrac{a}{s+4} + \dfrac{bs+c}{s^2+9}$

$\therefore\ s+29 = a(s^2+9) + (bs+c)(s+4)$

$s+29 = (a+b)s^2 + (4b+c)s + (9a+4c)$

Comparing the coefficients of similar powers of s

$$\Rightarrow \quad a+b=0, \quad 4b+c=1, \quad 9a+4c=29$$

$$\begin{array}{r} 4a+4b=0 \\ \underset{-}{c}+\underset{-}{4b}=\underset{-}{1} \\ \hline 4a-c=-1 \end{array}$$

$$\begin{array}{r} 16a-4c=-4 \\ 9a+4c=29 \\ \hline 25a \quad =25 \end{array} \quad \Rightarrow \boxed{a=1}$$

$$\therefore \boxed{b=-1} \qquad \text{also } c=1-4b=1+4 \Rightarrow \boxed{c=5}$$

$$\therefore \frac{s+29}{(s+4)(s^2+9)} = \frac{1}{s+4} + \frac{(-s+5)}{s^2+9}$$

$$L^{-1}\left[\frac{s+29}{(s+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{s+4}\right] - L^{-1}\left[\frac{s}{s^2+9}\right] + 5L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t$$

Ex 3 Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Solution :- For the purpose of doing partial fraction, it will be convenient to put $s^2=x$

$$\text{let } \frac{x}{(x+a^2)(x+b^2)} = \frac{A}{x+a^2} + \frac{B}{x+b^2}$$

$$\therefore x = A(x+b^2) + B(x+a^2)$$

put $x=-a^2$, $\quad -a^2 = A(-a^2+b^2) \Rightarrow A = \dfrac{a^2}{a^2-b^2}$

put $x=-b^2$, $\quad -b^2 = B(-b^2+a^2) \Rightarrow B = \dfrac{-b^2}{a^2-b^2}$

$$\therefore \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{\ldots}\right] = \frac{1}{\ldots}\left[L^{-1}\left(\frac{a^2}{\ldots}\right) - L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[L^{-1}\left(\frac{a^2}{s^2+a^2}\right) - L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[a^2\cdot\frac{1}{a}\sin at - b^2\cdot\frac{1}{b}\sin bt\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Ex 4 :- Find $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]$

Solution :- $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = L^{-1}\left[\frac{(s+1)^2+2}{[(s+1)^2+4][(s+1)^2+1]}\right]$

$$= e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]$$

As above example, Assume $s^2 = x$

Let $\frac{x+2}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1}$

$\therefore x+2 = A(x+1) + B(x+4)$

put $x=-1$, $1 = B(3)$ $\therefore B = \frac{1}{3}$

put $x=-4$, $-2 = -3A$ $\therefore A = \frac{2}{3}$

$$\therefore L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = e^{-t}L^{-1}\left[\frac{2}{3}\cdot\frac{1}{s^2+4} + \frac{1}{3}\,\frac{1}{s^2+1}\right]$$

$$= e^{-t}\left\{\frac{2}{3}L^{-1}\left(\frac{1}{s^2+4}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s^2+1}\right)\right\}$$

$$= e^{-t}\left\{\frac{2}{3}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}\sin t\right\}$$

$$= \frac{e^{-t}}{3}\left(\sin 2t + \sin t \right)$$

Ex-5 Find $L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right]$

Solution : First we consider,

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$\therefore \frac{s}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{b^2-a^2}\left\{ L^{-1}\left(\frac{s}{s^2+a^2}\right) - L^{-1}\left(\frac{s}{s^2+b^2}\right)\right\}$$

$$= \frac{1}{b^2-a^2}\left\{ \cos at - \cos bt \right\}$$