# Heaviside's Unit Step Function

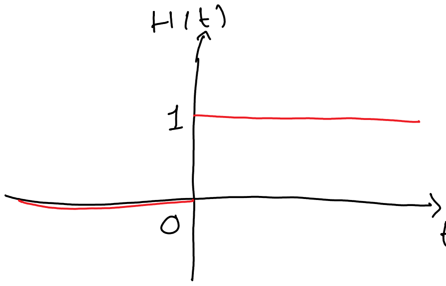
10 July 2023
15:44

The Function takes only two values 0 and 1.
when $t$ is negative, the value of the function is zero and when $t$ is positive its value is 1

It is denoted by $H(t)$ [or $U(t)$], H for Heaviside ( U for unit)

Thus, the value of $H(t)$ is one to the right of the origin and is zero to the left of the origin.

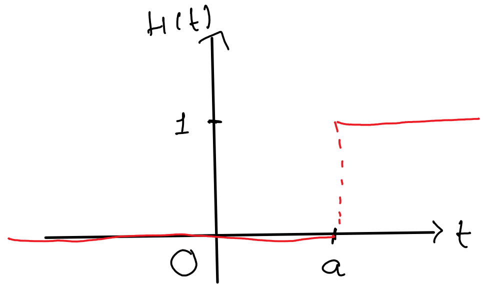
obviously, it is a discontinuous function.




we define it as $H(t) = \begin{cases} 0 & , t < 1 \\ 1 & , t \geq 1 \end{cases}$

The function takes a jump of unit magnitude and remains there thereafter.
The graph of the function is shown in the figure above.

## Displaced Unit Step Function




If the origin is shifted to a point $t = a$
i.e. if the function is zero before $t = a$ and takes a jump of unit magnitude at $t = a$ and remains there thereafter, the function is called displaced unit step function

It is defined by

$$H(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases}$$

. The graph of the function as shown above.

---

## Laplace Transform of H(t)

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

By definition of Laplace Transform,

$$L[H(t)] = \int^{\infty} e^{-st} H(t)\, dt = \int^{\infty} e^{-st} (1)\, dt = \left( \frac{e^{-st}}{-s} \right)^{\infty}$$

$$L[H(t)] = \int_0^\infty e^{-st} H(t)\,dt = \int_0^\infty e^{-st}(1)\,dt = \left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$= \frac{1}{s}$$

$$\therefore L[H(t)] = \frac{1}{s}$$

$$\Rightarrow L^{-1}\left[\frac{1}{s}\right] = H(t)$$

---

Laplace Transform of $H(t-a)$

Since $H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

By definition of Laplace Transform

$$L[H(t-a)] = \int_0^\infty e^{-st} H(t-a)\,dt = \int_a^\infty e^{-st}(1)\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = \frac{1}{s} e^{-as}$$

$$\therefore L[H(t-a)] = \frac{1}{s} e^{-as}$$

$$\Rightarrow L^{-1}\left[\frac{1}{s} e^{-as}\right] = H(t-a)$$

---

Laplace Transform of $f(t)H(t-a)$

Since $f(t)H(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$

By definition of Laplace Transform, we get

$$L[f(t)H(t-a)] = \int_0^\infty e^{-st} f(t)H(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t)\,dt$$

put $t-a=u \quad \therefore dt = du$

$$\therefore L[f(t)H(t-a)] = \int_0^\infty e^{-s(a+u)} f(a+u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su} f(a+u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t+a)\,dt$$

$$L[f(t)H(t-a)] = e^{-as} L[f(t+a)]$$

$$\boxed{L[f(t)H(t-a)] = e^{-as} L[f(t+a)]}$$

put $a=0$

$$\boxed{L[f(t)H(t)] = L[f(t)]}$$

---

Laplace Transform of $f(t-a)H(t-a)$

$$f(t-a)H(t-a) = \begin{cases} 0 & t<a \\ f(t-a) & t \geq a \end{cases}$$ by definition of $H(t-a)$

By definition of Laplace Transform

$$L[f(t-a)H(t-a)] = \int_0^\infty f(t-a)H(t-a)e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)e^{-st}\,dt$$

put $t-a=u \quad \therefore dt = du$

$$= \int_0^\infty f(u)\,e^{-s(a+u)}\,du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\, dt$$

$$\boxed{L[f(t-a) H(t-a)] = e^{-as} L[f(t)]}$$

$$\therefore \boxed{L[f(t-a) H(t-a)] = e^{-as} \phi(s)} \quad \text{where } \phi(s) = L[f(t)]$$

$$\therefore \boxed{L^{-1}[e^{-as} \phi(s)] = f(t-a) H(t-a)} \quad \text{where } f(t) = L^{-1}[\phi(s)]$$

---

## A useful Result

If a given function is a step function having many steps then it is convenient to express it as a unit step function as explained below and then write down its laplace Transform

1. suppose $f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & t > a \end{cases}$

then $\boxed{f(t) = f_1(t)[H(t) - H(t-a)] + f_2(t) H(t-a)}$

2. Suppose $f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & a < t < b \\ f_3(t) & b < t < c \\ f_4(t) & t > c \end{cases}$

Then
$$\boxed{\begin{aligned} f(t) = {} & f_1(t)[H(t) - H(t-a)] + f_2(t)[H(t-a) - H(t-b)] \\ & + f_3(t)[H(t-b) - H(t-c)] + f_4(t) H(t-c) \end{aligned}}$$

$$+ f_3(t)[H(t-b) - H(t-c)] + f_4(t) H(t-c)$$

This can also be written as

$$\boxed{\begin{aligned} f(t) = {} & f_1(t) H(t) + [f_2(t) - f_1(t)] H(t-a) \\ & + [f_3(t) - f_2(t)] H(t-b) + [f_4(t) - f_3(t)] H(t-c) \end{aligned}}$$

---

## Examples

1) Find Laplace Transform of $(1+2t-t^2+t^3) H(t-1)$

Solution :- we use the formula $L[f(t)H(t-a)] = e^{-as} L[f(t+a)]$

Here $f(t) = 1+2t-t^2+t^3$ & $a = 1$

$$\therefore L[(1+2t-t^2+t^3) H(t-1)] = e^{-s} L[f(t+1)]$$

$$\text{Now } f(t+1) = 1+2(t+1) - (t+1)^2 + (t+1)^3$$

$$= 1+2t+2 - (t^2+2t+1) + (t^3+3t^2+3t+1)$$

$$= 3+3t+2t^2+t^3$$

$$\therefore L[(1+2t-t^2+t^3) H(t-1)] = e^{-s} L[3+3t+2t^2+t^3]$$

$$= e^{-s}\left[\frac{3}{s} + \frac{3}{s^2} + 2\cdot\frac{2!}{s^3} + \frac{3!}{s^4}\right]$$

$$= e^{-s}\left[\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right]$$

---

2) Find $L[e^{-t}\sin t \, H(t-\pi)]$

Solution :- we use the formula

$$L[f(t) H(t-a)] = e^{-as} L[f(t+a)]$$

Here $f(t) = e^{-t}\sin t$ , $a = \pi$

$\therefore f(t+a) = f(t+\pi) = e^{-(t+\pi)} \sin(t+\pi)$

$= -e^{-\pi}\left[e^{-t}\sin t\right]$

$\therefore L\left[e^{-t}\sin t\, H(t-\pi)\right] = e^{-\pi s}\cdot\left(-e^{-\pi} L\left[e^{-t}\sin t\right]\right)$

$= -e^{-\pi(s+1)} \cdot \dfrac{1}{(s+1)^2+1}$

$L\left[e^{-t}\sin t\, H(t-\pi)\right] = \dfrac{-e^{-\pi(s+1)}}{s^2+2s+2}$

---

③ Using Laplace Transform evaluate

$$\int_0^\infty e^{-2t}(1+t+t^2)\,H(t-3)\,dt$$

Solution :- $\int_0^\infty e^{-2t}(1+t+t^2)\,H(t-3)\,dt = L\left[(1+t+t^2)\,H(t-3)\right]$ at $s=2$

Now using $L\left[f(t)H(t-a)\right] = e^{-as} L\left[f(t+a)\right]$

Here $f(t) = 1+t+t^2$ & $a = 3$

$f(t+a) = f(t+3) = 1+(t+3)+(t+3)^2$

$= 1+(t+3)+(t^2+6t+9)$

$f(t+3) = 13+7t+t^2$

$L\left[f(t+3)\right] = L\left[13+7t+t^2\right]$

$= \dfrac{13}{s} + \dfrac{7}{s^2} + \dfrac{2!}{s^3}$

$= 13 \quad . \quad 7 \quad . \quad 2$

$$= \frac{13}{s} + \frac{7}{s^2} + \frac{2}{s^3}$$

$$\therefore L\left[(1+t+t^2)H(t-3)\right] = e^{-3s}\left[\frac{13}{s} + \frac{7}{s^2} + \frac{2}{s^3}\right]$$

Now
$$\int_0^\infty e^{-2t}(1+t+t^2)\,H(t-3)\,dt = e^{-3s}\left[\frac{13}{s} + \frac{7}{s^2} + \frac{2}{s^3}\right]_{\text{at } s=2}$$

$$= e^{-6}\left[\frac{13}{2} + \frac{7}{4} + \frac{1}{4}\right] = e^{-6}\left[\frac{13}{2} + 2\right]$$

$$= e^{-6}\left[\frac{17}{2}\right] = \frac{17}{2e^6}$$

---

(4) Express the following functions as Heaviside's Unit step function and find their Laplace Transform

(i) $f(t) = \begin{cases} 2t & 0 < t < 1 \\ 3t^2 & t > 1 \end{cases}$ (ii) $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$

Solution:- (i) $f(t)$ can be expressed as

$$f(t) = 2t\left[H(t) - H(t-1)\right] + 3t^2 H(t-1)$$

$$= 2t\,H(t) + (3t^2 - 2t)\,H(t-1)$$

$$L[f(t)] = L[2t\,H(t)] + L\left[(3t^2-2t)\,H(t-1)\right]$$

$$= L[2t] + e^{-s} L\left[3(t+1)^2 - 2(t+1)\right]$$

$$= \frac{2}{s^2} + e^{-s} L\left[3(t^2+2t+1) - 2(t+1)\right]$$

$$= \frac{2}{s^2} + e^{-s} L\left[3t^2 + 4t + 1\right]$$

$$= \frac{2}{s^2} + e^{-s}\left[3\cdot\frac{2}{s^3} + 4\cdot\frac{1}{s^2} + \frac{1}{s}\right]$$

$$= \frac{2}{s^2} + e^{-s}\left[3\cdot\frac{2}{s^3} + 4\cdot\frac{1}{s^2} + \frac{1}{s}\right]$$

$$L[f(t)] = \frac{2}{s^2} + e^{-s}\left[\frac{6}{s^3} + \frac{4}{s^2} + \frac{1}{s}\right]$$

---

Sol^n (ii)
$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ \sin 2t & \pi < t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$$

$f(t)$ can be expressed as follows

$$f(t) = \sin t\,[H(t) - H(t-\pi)] + \sin 2t\,[H(t-\pi) - H(t-2\pi)]$$
$$+ \sin 3t\, H(t-2\pi)$$
$$= \sin t\, H(t) + [\sin 2t - \sin t]\, H(t-\pi)$$
$$+ [\sin 3t - \sin 2t]\, H(t-2\pi)$$

$$L[f(t)] = L[\sin t\, H(t)] + L[(\sin 2t - \sin t)\, H(t-\pi)]$$
$$+ L[(\sin 3t - \sin 2t)\, H(t-2\pi)]$$
$$= L[\sin t] + e^{-\pi s} L[\sin 2(t+\pi) - \sin(t+\pi)]$$
$$+ e^{-2\pi s} L[\sin 3(t+2\pi) - \sin 2(t+2\pi)]$$
$$= L[\sin t] + e^{-\pi s} L[\sin 2t + \sin t] + e^{-2\pi s} L[\sin 3t - \sin 2t]$$

$$L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

---

Ex:- Find $L^{-1}\left[\frac{e^{-3s}}{\ldots}\right]$

Ex :- Find $L^{-1}\left[\frac{e^{-3s}}{(s+4)^3}\right]$

Solution :- we use the formula

$$L^{-1}\left[e^{-as}\phi(s)\right] = f(t-a)H(t-a)$$

where $f(t) = L^{-1}\left[\phi(s)\right]$

Now, Here $\phi(s) = \frac{1}{(s+4)^3}$ and $a = 3$

Now $f(t) = L^{-1}\left[\phi(s)\right] = L^{-1}\left[\frac{1}{(s+4)^3}\right] = e^{-4t} L^{-1}\left[\frac{1}{s^3}\right]$

$$f(t) = e^{-4t} \cdot \frac{t^2}{2}$$

$$\therefore L^{-1}\left[\frac{e^{-3s}}{(s+4)^3}\right] = f(t-3)H(t-3)$$

$$= e^{-4(t-3)} \frac{(t-3)^2}{2} H(t-3)$$

---

Ex :- Find $L^{-1}\left[\frac{s e^{-\pi s}}{s^2+2s+2}\right]$

Solution :- $L^{-1}\left[e^{-as}\phi(s)\right] = f(t-a)H(t-a)$

Here $a = \pi$ & $\phi(s) = \frac{s}{s^2+2s+2}$

Also, $f(t) = L^{-1}\left[\phi(s)\right] = L^{-1}\left[\frac{s}{s^2+2s+2}\right]$

$$= L^{-1}\left(\frac{(s+1)-1}{(s+1)^2+1}\right) = e^{-t} L^{-1}\left[\frac{s-1}{s^2+1}\right]$$

$$f(t) = e^{-t}\left[\cos t - \sin t\right]$$

$$\therefore f(t-a) = f(t-\pi) = e^{-(t-\pi)}\left[\cos(t-\pi) - \sin(t-\pi)\right]$$

$$= e^{-(t-\pi)}\left[-\cos t + \sin t\right]$$

$$\therefore L^{-1}\left[\frac{s e^{-\pi s}}{s^2+2s+1}\right] = f(t-\pi)\, H(t-\pi)$$

$$= e^{-(t-\pi)}\left[\sin t - \cos t\right] H(t-\pi)$$

---

**Ex :-** Find $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2(s^2+1)}\right]$

Solution :- $L^{-1}\left[e^{-as}\phi(s)\right] = f(t-a)H(t-a)$

Here $a = \pi$, $\phi(s) = \dfrac{1}{s^2(s^2+1)}$

$$\therefore f(t) = L^{-1}\left[\phi(s)\right] = L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

$$= t - \sin t$$

$$\therefore f(t-a) = f(t-\pi) = (t-\pi) - \sin(t-\pi)$$

$$= (t-\pi) + \sin t$$

$$\therefore L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right] = \left[(t-\pi) + \sin t\right] H(t-\pi)$$