DIFFERENTIATION

1. DEFINITION

(a) Let us consider a function $y = f(x)$ defined in a certain interval. It has a definite value for each value of the independent variable x in this interval.

Now, the ratio of the increment of the function to the increment in the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now, as $\Delta x \to 0$, $\Delta y \to 0$ and $\frac{\Delta y}{\Delta x} \to$ finite quantity, then derivative $f(x)$ exists and is denoted by y' or $f'(x)$ or $\frac{dy}{dx}$

Thus, $f'(x) = \lim_{x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$

(if it exits)

for the limit to exist,

$$\lim_{h \to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}\frac{f(x-h) - f(x)}{-h}$$

(Right Hand derivative) (Left Hand derivative)

(b) The derivative of a given function f at a point x = a of its domain is defined as :

$\underset{h \to 0}{\text{Limit}}\frac{f(a+h) - f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

Note that alternatively, we can define

$f'(a) = \underset{x \to a}{\text{Limit}}\frac{f(x) - f(a)}{x - a}$, provided the limit exists.

This method is called first principle of finding the derivative of f(x).

2. DERIVATIVE OF STANDARD FUNCTION

(i) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$; $x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \cdot \ln a \ (a > 0)$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a|x|) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(x) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

(xi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(\text{constant}) = 0$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$

(xv) $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$, $\quad x\in R$

(xvi) $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$, $\quad x\in R$

(xvii) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$, $\quad |x|>1$

(xviii) $\frac{d}{dx}\left(\text{cosec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$, $\quad |x|>1$

(xix) **Results :**

If the inverse functions f & g are defined by $y=f(x)$ & $x=g(y)$. Then $g(f(x))=x$.

$\Rightarrow$ $g'(f(x)).f'(x)=1$.

This result can also be written as, if $\frac{dy}{dx}$ exists & $\frac{dy}{dx}\neq 0$, then

$$\frac{dx}{dy}=1/\left(\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx}.\frac{dx}{dy}=1 \text{ or } \frac{dy}{dx}=1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy}\neq 0\right]$$

3. THEOREMS ON DERIVATIVES

If u and v are derivable functions of x, then,

(i) Term by term differentiation : $\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$

(ii) Multiplication by a constant $\frac{d}{dx}(K\,u)=K\frac{du}{dx}$, where K is any constant

(iii) **"Product Rule"** $\frac{d}{dx}(u.v)=u\frac{dv}{dx}+v\frac{du}{dx}$ known as

In general,

(a) If $u_1, u_2, u_3, u_4, \ldots, u_n$ are the functions of x, then

$$\frac{d}{dx}(u_1.u_2.u_3.u_4\ldots.u_n)$$

$$=\left(\frac{du_1}{dx}\right)(u_2\,u_3\,u_4\ldots u_n)+\left(\frac{du_2}{dx}\right)(u_1\,u_3\,u_4\ldots u_n)$$

$$+\left(\frac{du_3}{dx}\right)(u_1\,u_2\,u_4\ldots u_n)+\left(\frac{du_4}{dx}\right)(u_1\,u_2\,u_3\,u_5\ldots u_n)$$

$$+\ldots+\left(\frac{du_n}{dx}\right)(u_1\,u_2\,u_3\ldots u_{n-1})$$

(iv) **"Quotient Rule"** $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\left(\frac{du}{dx}\right)-u\left(\frac{dv}{dx}\right)}{v^2}$ where $v\neq 0$ known as

(b) **Chain Rule :** If $y=f(u)$, $u=g(w)$, $w=h(x)$

then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dw}.\frac{dw}{dx}$

or $\frac{dy}{dx}=f'(u).g'(\;\;).h'(x)$

Note...

In general if $y=f(u)$ then $\frac{dy}{dx}=f'(u).\frac{du}{dx}$.

4. METHODS OF DIFFERENTIATION

4.1 Derivative by using Trigonometrical Substitution

Using trigonometrical transformations before differentiation shorten the work considerably. Some important results are given below :

(i) $\sin 2x=2\sin x\cos x=\frac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x=2\cos^2 x-1=1-2\sin^2 x=\frac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$, $\tan^2 x=\frac{1-\cos 2x}{1+\cos 2x}$

(iv) $\sin 3x=3\sin x-4\sin^3 x$

(v) $\cos 3x=4\cos^3 x-3\cos x$

(vi) $\tan 3x=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$

(vii) $\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}$

(viii) $\tan\left(\frac{\pi}{4}-x\right)=\frac{1-\tan x}{1+\tan x}$

(ix) $\sqrt{(1\pm\sin x)}=\left|\cos\frac{x}{2}\pm\sin\frac{x}{2}\right|$

(x) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right)$

(xi) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right\}$

(xii) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left\{xy\mp\sqrt{1-x^2}\ \sqrt{1-y^2}\right\}$

(xiii) $\sin^{-1}x+\cos^{-1}x=\tan^{-1}x+\cot^{-1}x=\sec^{-1}x+\mathrm{cosec}^{-1}x=\pi/2$

(xiv) $\sin^{-1}x=\mathrm{cosec}^{-1}(1/x)$; $\cos^{-1}x=\sec^{-1}(1/x)$; $\tan^{-1}x=\cot^{-1}(1/x)$

Note...

Some standard substitutions :

Expressions	Substitutions
$\sqrt{(a^2-x^2)}$	$x=a\sin\theta$ or $a\cos\theta$
$\sqrt{(a^2+x^2)}$	$x=a\tan\theta$ or $a\cot\theta$
$\sqrt{(x^2-a^2)}$	$x=a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
$\sqrt{\left(\frac{a+x}{a-x}\right)}$ or $\sqrt{\left(\frac{a-x}{a+x}\right)}$	$x=a\cos\theta$ or $a\cos 2\theta$
$\sqrt{(a-x)(x-b)}$ or	$x=a\cos^2\theta+b\sin^2\theta$
$\sqrt{\left(\frac{a-x}{x-b}\right)}$ or $\sqrt{\left(\frac{x-}{a-x}\right)}$	
$\sqrt{(x-a)(x-b)}$ or	$x=a\sec^2\theta-b\tan^2\theta$
$\sqrt{\left(\frac{x-a}{x-b}\right)}$ or $\sqrt{\left(\frac{x-}{x-a}\right)}$	
$\sqrt{(2ax-x^2)}$	$x=a(1-\cos\theta)$

4.2 Logarithmic Differentiation

To find the derivative of :

If $y=\{f_1(x)\}^{f_2(x)}$ or $y=f_1(x).f_2(x).f_3(x)...$

or $$y=\frac{f_1(x).f_2(x).f_3(x)...}{g_1(x).g_2(x).g_3(x)...}$$

then it is convenient to take the logarithm of the function first and then differentiate. This is called derivative of the logarithmic function.

Important Notes (Alternate methods)

1. If $y=\{f(x)\}^{g(x)}=e^{g(x)\ln f(x)}$ ((variable)$^{\text{variable}}$) $\{\because x=e^{\ln x}\}$

$$\therefore \frac{dy}{dx}=e^{g(x)\ln f(x)}.\left\{g(x).\frac{d}{dx}\ln f(x)+\ln f(x).\frac{d}{dx}g(x)\right\}$$

$$=\{f(x)\}^{g(x)}.\left\{g(x).\frac{f'(x)}{f(x)}+\ln f(x).\,g'(x)\right\}$$

2. If $y=\{f(x)\}^{g(x)}$

$\therefore \frac{dy}{dx}=$ Derivative of y treating $f(x)$ as constant + Derivative of y treating $g(x)$ as constant

$$=\{f(x)\}^{g(x)}.\ln f(x).\frac{d}{dx}g(x)+g(x)\{f(x)\}^{g(x)-1}.\frac{d}{dx}f(x)$$

$$=\{f(x)\}^{g(x)}.\ln f(x).g'(x)+g(x).\{f(x)\}^{g(x)-1}.f'(x)$$

4.3 Implict Differentiation : $\phi(x, y)=0$

(i) In order to find dy/dx in the case of implicit function, we differentiate each term w.r.t. x, regarding y as a function of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit function, both x & y are present.

Alternate Method : If $f(x, y)=0$

then $$\frac{dy}{dx}=-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}=-\frac{\text{diff. of f w.r.t. x treating y as constant}}{\text{diff. of f w.r.t. y treating x as constant}}$$

4.4 Parametric Differentiation

If $y = f(t)$ & $x = g(t)$ where t is a Parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad ...(1)$$

Note...

1. $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

2. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} \quad \left(\because \frac{dy}{dx} \text{ in terms of t}\right)$

$$= \frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right) \cdot \frac{1}{f'(t)} \quad \{\text{From}\,(1)\}$$

$$= \frac{f''(t)g'(t) - g''(t)f'(t)}{\{f'(t)\}}$$

4.5 Derivative of a Function w.r.t. another Function

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

4.6 Derivative of Infinite Series

If taking out one or more than one terms from an infinite series, it remains unchanged. Such that

(A) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$

then $y = \sqrt{f(x) + y} \Rightarrow (y^2 - y) = f(x)$

Differentiating both sides w.r.t. x, we get $(2y - 1)\frac{dy}{dx} = f'(x)$

(B) If $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{.......\infty}}}$ then $y = \{f(x)\}^y \Rightarrow y = e^{y \ln f(x)}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{y\{f(x)\}^{y-1} \cdot f'(x)}{1 - \{f(x)\}^y \cdot \ell n\, f(x)} = \frac{y^2 f'(x)}{f(x)\{1 - y\, \ell n\, f(x)\}}$$

5. DERIVATIVE OF ORDER TWO & THREE

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b), is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w.r.t. x. If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ it is also denoted by $f'''(x)$ or y'''.

Some Standard Results :

(i) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} \cdot a^n \cdot (ax+b)^{m-n}$, $m \geq n$.

(ii) $\frac{d^n}{dx^n} x^n = n!$

(iii) $\frac{d^n}{dx^n}\left(e^{mx}\right) = m^n \cdot e^{mx}$, $m \in R$

(iv) $\frac{d^n}{dx^n}(\sin(ax+b)) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$, $n \in N$

(v) $\frac{d^n}{dx^n}(\cos(ax+b)) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$, $n \in N$

(vi) $\frac{d^n}{dx^n}\{e^{ax}\sin(bx+c)\} = r^n \cdot e^{ax} \cdot \sin(bx + c + n\phi)$, $n \in N$

where $r = \sqrt{(a^2+b^2)}$, $\phi = \tan^{-1}(b/a)$.

(vii) $\frac{d^n}{dx^n}\{e^{ax} \cdot \cos(bx+c)\} = r^n \cdot e^{ax} \cdot \cos(bx + c + n\phi)$, $n \in N$

where $r = \sqrt{(a^2+b^2)}$, $\phi = \tan^{-1}(b/a)$.

6. DIFFERENTIATION OF DETERMINANTS

If $F(X)=\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

where *f, g, h, ℓ, m, n, u, v, w* are differentiable function of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

7. L' HOSPITAL'S RULE

If $f(x)$ & $g(x)$ are functions of x such that :

(i) $\lim\limits_{x\to a} f(x)=0=\lim\limits_{x\to a} g(x)$ or $\lim\limits_{x\to a} f(x)=\infty=\lim\limits_{x\to a} g(x)$ and

(ii) Both $f(x)$ & $g(x)$ are continuous at x = a and

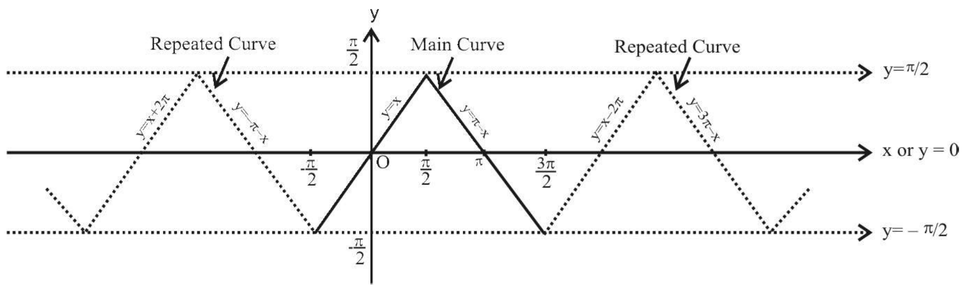
(iii) Both $f(x)$ & $g(x)$ are differentiable at x = a and

(iv) Both $f'(x)$ & $g'(x)$ are continuous at x = a, Then

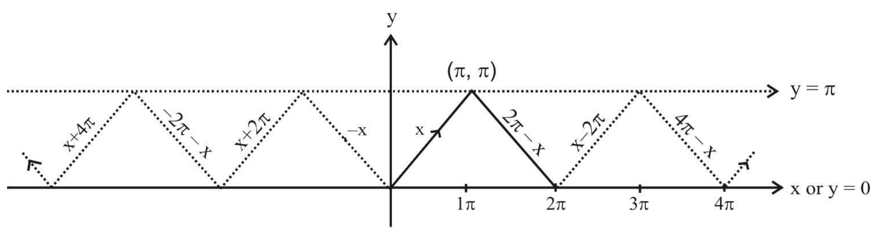
$\underset{x\to a}{\text{Limit}}\dfrac{f(x)}{g(x)}=\underset{x\to a}{\text{Limit}}\dfrac{f'(x)}{g'(x)}=\underset{x\to a}{\text{Limit}}\dfrac{f''(x)}{g''(x)}$ & so on till indeterminant form vanishes..

8. ANALYSIS & GRAPHS OF SOME USEFUL FUNCTION

(i) $y=\sin^{-1}(\sin x)$ $\quad x\in R\,;\ y\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$



(ii) $y=\cos^{-1}(\cos x)$ $\quad x\in R\,;\ y\in[0,\pi]$



(iii) $y=\tan^{-1}(\tan x)$ $\quad x\in R-\left\{x:x=(2n+1)\dfrac{\pi}{2},n\in Z\right\};\ y\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$