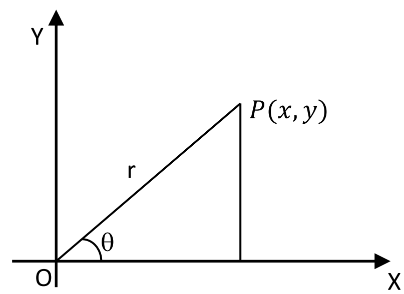
COMPLEX VARIABLE

Introduction:

An ordered pair of real numbers (x, y) connected by an expression $x + i\,y$ and denoted by 'z' is called a complex number. $z = x + i\,y$ is a complex number, where $i = \sqrt{-1}$ is called an imaginary unit.
The real numbers x and y are called real and imaginary parts of z and written as $R(z)$ and $I(z)$ respectively.



In the Argand's diagram, the complex number z is represented by the point $P(x, y)$. If (r, θ) are the polar co – ordinate of P, then $x = r\cos\theta$ and $y = r\sin\theta$ and therefore every non zero complex number z can be expressed as $z = r(\cos\theta + i\sin\theta)$,
$or\ z = r\,e^{i\theta}$ which is the polar form of the complex number.

Then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $|z|$

$\theta = tan^{-1}(y/x)$ is called the amplitude (or argument) of z and is denoted by amp z (or arg z)

If $z = x + i\,y$, then the complex number $x - i\,y$ is called the conjugate of the complex number z and is denoted by $\bar{z}$ i.e $\bar{z} = x - i\,y$

In the polar form, $\bar{z} = r(\cos\theta - i\sin\theta)\ or\ \bar{z} = r\,e^{-i\,\theta}$

Clearly, $|z| = |\bar{z}| = r$, $|z|^2 = z\,\bar{z}$, $amp\ \bar{z} = -tan^{-1}(y/x) = -\theta$

$$x = \frac{z+\bar{z}}{2},\quad y = \frac{z-\bar{z}}{2i}$$

Hence, if x and y are real variables, then $z = x + i\,y$ is called a complex variable.

Definition of a Complex Function:

If by a rule or set of rules we can find one or more complex numbers w for every $z(= x + iy)$ in a given domain, we say that w is a function of z and denote it as $w = f(z)$

Since, both z and w are complex quantities the function is called a **complex function.**

If for a given z there corresponds one and only one w then the function is called **single valued function**, otherwise function is called **multiple valued function**.

Example: (1) $w = z^2$ is a single valued function. **(2)** $w = \sqrt[6]{z}$ is a multiple valued function.

We shall consider single valued functions only.

Note: Whenever we speak of function we shall, unless otherwise stated, assume single – valued function.

Since, $z = x + iy$, $w = f(z)$ can be put in the form $w = u(x, y) + iv(x, y)$ where, u and v are functions of x and y. Thus, we can write $w = u(x, y) + iv(x, y)$

Example: If $w = z^2 + 2z + 3$ then $w = (x + iy)^2 + 2(x + iy) + 3$

$$= x^2 + 2ixy - y^2 + 2x + 2iy + 3$$
$$= (x^2 - y^2 + 2x + 3) + i(2xy + 2y)$$
$$= u(x, y) + iv(x, y)$$

Differentiability of a Function $f(z)$:

Definition: Let $w = f(z)$ be a single valued function of z defined in domain D. f(z) is said to be differentiable at any point z if $\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ Is unique as $\delta z \to 0$ along any path of the domain D

Analytic Functions:

If a single valued function $w = f(z)$ is defined and differentiable at each point of a domain D then it is called **analytic** or **regular** or **holomorphic** function of z in the domain D.

A function is said to be analytic at a point if it has a derivative at that point and in some neighbourhood of that point. If a function ceases to be analytic at a point of the domain then the point is called a **singular point.**

Cauchy – Riemann Equations in Cartesian Coordinates:

Theorem: The necessary and sufficient conditions for a continuous one valued function $w = f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in a region R and

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $(i.e\ u_x = v_y\ and\ u_y = -v_x)$ at each point of R.

The conditions **(ii)** are known as **Cauchy – Riemann equations** or briefly **C – R equations**.

NOTE: **(1)** The Cauchy – Riemann equations are only necessary conditions for a function to be analytic. This means even if Cauchy – Riemann equations are satisfied the function need not be analytic at that point

(2) When $f(z)$ is analytic, its derivative is given by any one of the following expressions.

(i) $f'(z) = u_x + iv_x$ **(ii)** $f'(z) = v_y + iv_x$ **(iii)** $f'(z) = u_x - iu_y$ **(iv)** $f'(z) = v_y - iu_y$

(3) If $f(z)$is analytic then it can be differentiated in usual manner.

e.g **(1)** If $f(z) = z^2\ then\ f'(z) = 2z$ **(2)** If $f(z) = \sin z\ then\ f'(z) = \cos z$

(4) If $f(z) = f(x + iy) = u + iv\ and\ f(z)$ is analytic then the functions u and v of real variables x and y are called **conjugate functions**.

Derivatives of Elementary Functions:

1.	$\frac{d}{dz}(c) = 0$	**2.**	$\frac{d}{dz}(z^n) = n\,z^{n-1}$	**3.**	$\frac{d}{dz}(e^z) = e^z$
4.	$\frac{d}{dz}(a^z) = a^z \log a$	**5.**	$\frac{d}{dz}(\sin z) = \cos z$	**6.**	$\frac{d}{dz}(\cos z) = -\sin z$
7.	$\frac{d}{dz}(\tan z) = sec^2 z$	**8.**	$\frac{d}{dz}(\cot z) = -cosec^2 z$	**9.**	$\frac{d}{dz}(\sec z) = \sec z tan\, z$
10.	$\frac{d}{dz}(\csc z) = -\csc z \cot z$	**11.**	$\frac{d}{dz}(\log z) = \frac{1}{z}$	**12.**	$\frac{d}{dz}(log_a z) = \frac{1}{zlog_e a}$
13.	$\frac{d}{dz}(sin^{-1}z) = \frac{1}{\sqrt{1-z^2}}$	**14.**	$\frac{d}{dz}(cos^{-1}z) = -\frac{1}{\sqrt{1-z^2}}$	**15.**	$\frac{d}{dz}(tan^{-1}z) = \frac{1}{1+z^2}$
16.	$\frac{d}{dz}(cot^{-1}z) = -\frac{1}{1+z^2}$	**17.**	$\frac{d}{dz}(sec^{-1}z) = \frac{1}{z\sqrt{z^2-1}}$	**18.**	$\frac{d}{dz}(cosec^{-1}z) = \frac{-1}{z\sqrt{z^2-1}}$
19.	$\frac{d}{dz}(\sin hz) = \cos hz$	**20.**	$\frac{d}{dz}(\cos hz) = \sin hz$	**21.**	$\frac{d}{dz}(\tan hz) = sech^2 z$
22.	$\frac{d}{dz}(\cot hz) = -cosech^2 z$	**23.**	$\frac{d}{dz}(\sec hz)$ $= -\sec hz \tan hz$	**24.**	$\frac{d}{dz}(cosec\ hz)$ $= -cosec\ hz \cot h\, z$
25.	$\frac{d}{dz}(sinh^{-1}z) = \frac{1}{\sqrt{1+z^2}}$	**26.**	$\frac{d}{dz}(\cos h^{-1} z) = \frac{1}{\sqrt{z^2-1}}$	**27.**	$\frac{d}{dz}(tanh^{-1}z) = \frac{1}{1-z^2}$
28.	$\frac{d}{dz}(cot\ h^{-1}z) = \frac{1}{z^2-1}$	**29.**	$\frac{d}{dz}(sech^{-1}z) = \frac{-1}{z\sqrt{1-z^2}}$	**30.**	$\frac{d}{dz}(cosec\ h^{-1}z) = \frac{-1}{z\sqrt{z^2+1}}$

Cauchy – Riemann Equations In Polar Coordinates:

Let (r, θ) be the polar coordinates of a point whose Cartesian coordinates are (x, y).

$\therefore\ x = r\cos\theta, y = r\sin\theta, \quad z = x + iy = r\,(\cos\theta + isin\,\theta) = \ r\,e^{i\theta}$

Let $f(z) = u + iv$ be the given function.

$\therefore f(z) = u + iv = f(r\,e^{i\theta})$ ……………(i)

Differentiating (i) partially w.r.t r, $\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(r\,e^{i\theta})\cdot e^{i\theta}$ ……………….(ii)

Differentiating (i) partially w.r.t θ, $\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(r\,e^{i\theta})\cdot r\,e^{i\theta}\cdot i \quad = i\,r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$ by (ii)

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts

$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} and \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$ **Or** $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ and \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

i. e $u_r = \frac{1}{r}v_\theta\ and\ v_r = -\frac{1}{r}u_\theta$

Note: From (ii) We get an important result $f'\left(r\,e^{i\theta}\right) = \ e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$\therefore f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

SOME SOLVED EXAMPLES:

1. If $f(z) and \overline{f(z)}$ are both analytic, prove that $f(z)$is constant.

Solution: Let $f(z) = u + iv$ then $\overline{f(z)} = u - iv = u + i(-v)$

Since, $f(z)$ is analytic

$u_x = v_y$ and $u_y = -v_x$, $C - R$ equations

Since, $\overline{f(z)}$ is analytic

$u_x = (-v_y)$ and $u_y = -(-v_x)$, $C - R$ equations

Adding $u_x = v_y$ and $u_x = -v_y$, we get, $u_x = 0$

Adding $u_y = -v_x$ and $u_y = v_x$, we get, $u_y = 0$

Since, $u_x = 0$ and $u_y = 0, u = a$ constant

Similarly by subtraction we can prove that $v_x = 0$ and $v_y = 0$ $\quad \therefore v = a$ constant

Hence, $f(z) = u + iv = a$ constant

2. If $f(z)$ is an analytic function, show that $\frac{\partial f}{\partial \bar{z}} = 0$

Solution: Since, $z = x + iy$, $\bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

Let $f(z) = u + iv$

$$\begin{aligned}
\therefore \frac{\partial f}{\partial \bar{z}} &= \frac{\partial}{\partial \bar{z}}(u + iv) \\
&= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) \\
&= \left[\frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right] + i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2} + \frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right] \\
&= \frac{1}{2}u_x - \frac{1}{2i}u_y + \frac{i}{2}v_x - \frac{1}{2}v_y \\
&= \frac{1}{2}u_x + \frac{i}{2}u_y + \frac{i}{2}v_x - \frac{1}{2}v_y
\end{aligned}$$

But, since $f(z)$ is analytic,

$u_x = v_y$ and $u_y = -v_x$

$\therefore \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}v_y - \frac{i}{2}v_x + \frac{i}{2}v_x - \frac{1}{2}v_y = 0$

3. If $f(z)$ is an analytic function and $|f(z)|$ is constant, prove that $f(z)$ is constant.

Or A regular function of constant magnitude is constant

Solution: Let $f(z) = u + iv$ but $|f(z)| = C$

$\therefore u^2 + v^2 = C^2$

Differentiating it partially w.r.t. x, $uu_x + vv_x = 0$

Differentiating it partially w.r.t. y, $uu_y + vv_y = 0$

Since, $f(z)$ is analytic $u_x = v_y$ and $u_y = -v_x$

$\therefore uu_x - vu_y = 0$ and $uu_y + vu_x = 0$

Eliminating u_y, we get, $(u^2 + v^2)u_x = 0$

$\therefore C^2 u_x = 0 \qquad \therefore u_x = 0$

Similarly, we can show that $u_y = 0, v_x = 0, v_y = 0$

Since, $f(z)$ is analytic

$f'(Z) = u_x + iv_x = 0 \qquad \therefore f(z) =$ constant

4. If $f(z)$ is analytic and if the amplitude of $f(z)$ is constant, prove that $f(z)$ is constant.

Solution: Let $f(z) = u + iv$. Since its amplitude $= \tan^{-1}(v/u)$ is constant c say, we have

$\tan^{-1}\frac{v}{u} = c \qquad \therefore \frac{v}{u} = \tan c$

Differentiating this w.r.t. x and y

$\frac{uv_x - vu_x}{u^2} = 0$ and $\frac{uv_y - vu_y}{u^2} = 0$

$\therefore uv_x - vu_x = 0$ and $uv_y - vu_y = 0$

Since $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$

$\therefore -uu_y - vu_x = 0$ .... (1)

and $uu_x - vu_y = 0$ .... (2)

Multiply the first by u and second by v and add

$\therefore (-u^2 - v^2)u_y = 0 \qquad \therefore u_y = 0$

Multiply the first by v and second by v and subtract

$\therefore (-v^2 - u^2)u_x = 0 \qquad \therefore u_x = 0$

But $u_x = v_y$ and $u_y = -v_x$

$\therefore v_y = 0$ and $v_x = 0$

Since, all four partial derivatives of u, v are zero, u and v are constants

$\therefore f(z)$ is constant

5. If $f(z) = u + iv$ is an analytic function and u = constant then $f(z)$ is constant.

Solution: If u is constant $u_x = 0, u_y = 0$

But $f'(z) = u_x + iv_x$

$= u_x - iu_y$ (By $C - R$ equations)

$= 0$ (By data)

$\therefore f(z)$ is constant

6. Show that the following functions are analytic and find their derivatives.

(i) e^z **(ii)** z^3 **(iii)** ze^z **(iv)** $\sin z$ **(v)** $\sin hz$.

Solution: **(i)** $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$\therefore u = e^x \cos y, v = e^x \sin y$

$u_x = e^x \cos y \, , u_y = -e^x \sin y$

$v_x = e^x \sin y \, , v_y = e^x \cos y$

$\therefore u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x, v_y are continuous and Caunchy-Riemann equations are satisfied

Hence, e^z is analytic

Now, $f'(z) \quad = u_x + iv_x$

$= e^x \cos y + ie^x \sin y$

$= e^x(\cos y + i \sin y) = e^x \cdot e^{iy}$

$= e^{x+iy} = e^z$

(ii) $f(z) = z^3 = (x+iy)^3$

$\therefore f(z) = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore u = x^3 - 3xy^2, \;\; v = 3x^2y - y^3$

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy, \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore f(z) = z^3$ is analytic and can be differentiated as usual

$\therefore f'(z) = 3z^2$

(iii) $f(z) = ze^z = (x+iy)e^{x+iy}$

$\therefore f(z) = (x+iy)e^x(\cos y + i \sin y)$

$\therefore u = e^x(x \cos y - y \sin y), v = e^x(x \sin y + y \cos y)$

$\frac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y$

$\frac{\partial u}{\partial y} = e^x(-x \sin y - y \cos y - \sin y)$

$\frac{\partial v}{\partial x} = e^x(x \sin y + y \cos y) + e^x \sin y$

$\frac{\partial v}{\partial y} = e^x(x \cos y + \cos y - y \sin y)$

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore f(z) = ze^z$ is analytic and can be differentiated as usual

$\therefore f(z) = ze^z + e^z = e^z(z+1)$

(iv) $f(z) = \sin z = \sin(x+iy)$

$= \sin x \cosh y + i \cos x \sinh y$

$\therefore u = \sin x \cosh y \, , v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \, , \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y \, , \frac{\partial v}{\partial y} = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore f(z) = \sin z$ is analytic and can be differentiated as usual

$\therefore f(z) = \cos z$

(v) $f(z) = \sin hz = \sinh(x+iy)$

$= \sinh x \cosh iy + \cosh x \sinh iy$

$= \sinh x \cos y + i \cosh x \sinh y$

$\therefore u = \sinh x \cos y\,, v = \cosh x \sin y$

$u_x = \cosh x \cos y\,, u_y = -\sinh x \sin y$

$v_x = \sinh x \sin y\,,\; v_y = \cosh x \cos y$

$\therefore u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x, v_y are continuous and Caunchy-Riemann equations are satisfied

Hence, $\sin hz$ is analytic

Now, $f'(z) \quad = u_x + iv_x$

$= \cosh x \cos y + i \sinh x \sin y$

$= \cosh x \cosh iy + \sinh x \sinh iy$

$= \cosh(x+iy) = \cosh z$

7. If $f(z)$ is equal to **(a)** $\bar{z}$ **(b)** $2x + ixy^2$, show that $f'(z)$ does not exist

Solution: **(a)** $f(z) = \bar{z} = x - iy \quad \therefore u = x, v = -y$

$\therefore u_x = 1, u_y = 0;\; v_y = -1, v_x = 0$

Since, $u_x \neq v_y$ Cauchy – Riemann equations are not satisfied and $f'(z)$ does not exist

Alternatively

$$f'(z) = \lim_{\delta z \to 0} \frac{\overline{z+\delta z} - \bar{z}}{\delta z}$$

$$\therefore f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\overline{(x+\imath y+\delta x+\imath\delta y)} - \overline{(x+\imath y)}}{\delta x + i\delta y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{x-iy+\delta x-i\delta y-x+iy}{\delta x+i\delta y}$$

$$\therefore f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta x - i\delta y}{\delta x + i\delta y}$$

If $\delta y = 0$, the required limit $= \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = 1$

If $\delta x = 0$, the required limit $= \lim_{\delta y \to 0} -\frac{\delta y}{\delta y} = -1$

(b) $f(z) = 2x + ixy^2 \quad \therefore u = 2x, v = xy^2$

$u_x = 2, u_y = 0, v_x = y^2, v_y = 2xy$

Since, $u_x \neq v_y$ and $u_y \neq v_x$,

Cauchy – Riemann equations are not satisfied and hence, $f'(z)$ does not exist

8. Show that $f(z) = z\bar{z} = |z|^2$ satisfies Cauchy – Riemann equations at $z = 0$ and yet is not analytic anywhere

Solution: $f(z) = |z|^2 = x^2 + y^2 \quad \therefore u = x^2 + y^2, v = 0$

$\therefore \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial y} = 0$

Hence, $u_x = v_y = 0$ and $u_y = -v_x = 0$ when $x = 0$ and $y = 0$

The partial derivatives u_x, u_y, v_x, v_y are also continuous everywhere

Thus, $f'(z) = |z|^2$ is differentiable only at $z = 0$ but no other point. There is no neighbourhood of $z = 0$ in which the conditions of analyticity are satisfied. Hence, $f(z)$ is not analytic anywhere

9. Show that $w = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ is an analytic function and find $\frac{dw}{dz}$ in terms of z.

Solution: Since, $u = \frac{x}{x^2+y^2}, \frac{\partial u}{\partial x} = u_x = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = u_y = -\frac{x\cdot 2y}{(x^2+y^2)^2}$

$v = -\frac{y}{x^2+y^2} \quad \therefore \frac{\partial v}{\partial x} = v_x = +\frac{y\cdot 2x}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial y} = v_y = -\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\therefore u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x and v_y are continuous except at $z = x + iy = 0$ i.e., $(x = 0, y = 0), w$ is analytic everywhere except at $z = 0$

$$\therefore \frac{dw}{dz} = u_x + iv_x = \frac{-x^2+y^2}{(x^2+y^2)^2} + i\cdot\frac{2xy}{(x^2+y^2)^2}$$

$$= -\frac{(x^2-2ixy-y^2)}{(x^2+y^2)^2} = -\frac{(x^2-2ixy+i^2y^2)}{(x^2-i^2y^2)^2} = -\frac{(x-iy)^2}{[(x-iy)(x+iy)]^2}$$

$$= -\frac{(x-iy)^2}{(x-iy)^2(x+iy)^2} = -\frac{1}{(x+iy)^2} = -\frac{1}{z^2}$$

(Or to find $\frac{dw}{dz}$ in terms of z, put $x = z, y = 0$ in (i) $\therefore \frac{dw}{dz} = -\frac{1}{z^2}$)

10. Find k such that $\frac{1}{2}log(x^2+y^2) + itan^{-1}\frac{kx}{y}$ is analytic.

Solution: Let $f(z) = \frac{1}{2}log(x^2+y^2) + itan^{-1}\frac{kx}{y}$

$\therefore u = \frac{1}{2}log(x^2+y^2), v = tan^{-1}\frac{kx}{y}$

$\therefore u_x = \frac{x}{x^2+y^2}, u_y = \frac{y}{x^2+y^2}$

$v_x = \frac{1}{1+\frac{k^2x^2}{y^2}}\cdot\frac{k}{y} = \frac{ky}{k^2x^2+y^2}$

$v_y = \frac{1}{1+\frac{k^2x^2}{y^2}}\cdot\left(-\frac{kx}{y^2}\right) = -\frac{kx}{k^2x^2+y^2}$

Since, the function is analytic $C - R$ equations are satisfied

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore \frac{x}{x^2+y^2} = -\frac{kx}{k^2x^2+y^2}, \quad \frac{y}{x^2+y^2} = \frac{-ky}{k^2x^2+y^2}$

which are satisfied when $k = -1$

11. Find the constants a, b, c, d if $f(z) = (x^2 + 2axy + by^2) + i(cx^2 + 2dxy + y^2)$ is analytic

Solution: We have $f(z) = u + iv$

and $u = x^2 + 2axy + by^2; v = cx^2 + 2dxy + y^2$

$\therefore u_x = 2x + 2ay, u_y = 2ax + 2by$

$v_x = 2cx + 2dy, v_y = 2dx + 2y$

Since, $f(z)$ is analytic, Cauchy – Riemann equations are satisfied

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore 2x + 2ay = 2dx + 2y$ and $2ax + 2by = -2cx - 2dy$

Equating the coefficient of x and y, we get,

$a = 1, d = 1$ and $a = -c, b = -d$

$\therefore a = 1, b = -1, c = -1, d = 1$

12. Find the values of z for which the following functions are not analytic.

(i) $z = e^{-v}(\cos u + i \sin u)$ **(ii)** $z = \sin hu \cos v + i \cos hu \sin v$

Solution: **(i)** We have $z = e^{-v}(\cos u + i \sin u) = e^{-v}e^{iu}$

$\therefore z = e^{-v+iu} = e^{i^2v+iu} = e^{i(u+iv)} = e^{iw}$ where $w = u + iv$

$\therefore iw = \log z$ $\therefore w = \frac{1}{i}\log z$

$\therefore \frac{dw}{dz} = \frac{1}{i} \cdot \frac{1}{z}$

$\therefore w$ is not analytic at $z = 0$

(ii) We have $z = \sinh u \cos v + i \cosh u \sin v$

$= \sinh u \cosh iv + \cosh u \sinh iv$

$= \sinh(u + iv)$ .... $[\because \sinh ix = i \sin x \ and \ \cosh ix = \cos x]$

$= \sinh w$ where $w = u + iv$

$\therefore w = \sinh^{-1} z = \log\left(z + \sqrt{z^2 + 1}\right)$

$\therefore \frac{dw}{dz} = \frac{1}{z+\sqrt{z^2+1}}\left(1 + \frac{z}{\sqrt{z^2+1}}\right) = \frac{1}{\sqrt{z^2+1}}$

$\therefore w$ is not analytic when $\sqrt{z^2 + 1} = 0$ i.e., $z^2 = -1$, i.e., $z = \pm i$

13. Find p if $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.

Solution: Let $w = f(z) = u + iv = r^2 \cos 2\theta + ir^2 \sin p\theta$

$\therefore u = r^2 \cos 2\theta, \frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \frac{\partial u}{\partial r} = 2r \cos 2\theta$

$v = r^2 \sin p\theta, \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta, \frac{\partial v}{\partial r} = 2r \sin p\theta$

Since, $f(z)$ is analytic

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

The first relation gives, $2r\cos 2\theta = \frac{1}{r}\cdot pr^2\cos p\theta \qquad \therefore p = 2$

And the second relation also gives, $2r\sin p\theta = -\frac{1}{r}(-2r^2\sin 2\theta) \qquad \therefore p = 2$ Hence $p = 2$

14. If $w = z^n \; find \; \frac{dw}{dz}$

Solution: Let $z = re^{i\theta} \qquad \therefore z^n = r^n e^{in\theta}$

$\therefore z^n = r^n(\cos n\theta + i\sin n\theta) = u + iv$

$\therefore u = r^n\cos n\theta\,, v = r^n\sin n\theta$

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta\,, \frac{\partial u}{\partial \theta} = -r^n\cdot n\cdot\sin n\theta$

$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta\,, \frac{\partial v}{\partial \theta} = nr^n\cos n\theta$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Also partial derivatives are continuous. Hence, w is analytic

$$\begin{aligned}\therefore \frac{dw}{dz} &= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\\ &= e^{-i\theta}(nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta)\\ &= nr^{n-1}\cdot e^{-i\theta}\cdot e^{in\theta} = nr^{n-1}\cdot e^{i(n-1)\theta}\\ &= n\left(re^{i\theta}\right)^{n-1} = nz^{n-1}\end{aligned}$$

15. Using Cauchy – Riemann equations in polar form prove that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Solution: We know that Cauchy – Riemann equations in polar form are

$u_r = \frac{1}{r}v_\theta$ .... (i)

and $u_\theta = -rv_r$ .... (ii)

Differentiating (i) w.r.t. r, we get,

$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial r}$ .... (iii)

Differentiating (ii) w.r.t. θ, we get,

$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial\theta\partial r}$ .... (iv)

Now, using (iii) and (iv), we get,

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial r}\right) + \frac{1}{r}\cdot\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{1}{r^2}\left(\frac{r\partial^2 v}{\partial\theta\partial r}\right)\\ &= -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial\theta\partial r} = 0\end{aligned}$$

Note: The equation $\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$ is called Laplace's equation in **Cartesian Form** and the equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ is called Laplace's equation in **Polar Form**.

Harmonic Functions:

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies Laplace's equation $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ is called a **Harmonic Function.**

Theorem: The real and imaginary parts u, v of an analytic function $f(z) = u + iv$ are harmonic functions.

Proof: Since, $f(z)$ is an analytic function in some region of the z – plane

∴ u, v satisfy C – R equations.

$\therefore u_x = v_y \; and \; u_y = -v_x$ ………………(i)

Differentiating the first w.r.t x and second w.r.t y, we get

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \quad and \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}$

Assuming $\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$ and adding the above results we get, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Similarly differentiating the equations in (i) with respect to y and x respectively,

we can show that the result $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ $\quad \therefore u\,, v$ are harmonic functions.

Note: **(1)** In other words the above theorem states that if $f(z) = u + iv$ is analytic, then its real and imaginary parts u, v satisfy Laplace equation $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$

(2) The above theorem states that $f(z) = u + iv$ is analytic then u and v satisfy Laplace's equation i.e u and v are harmonic functions.

But, the converse is not true. If u and v are any two functions satisfying Laplace's equation then $u + iv$ need not to be analytic.

FIND ANALYTIC FUNCTION WHOSE REAL OR IMAGINARY PART IS GIVEN

Method 1: Let $f(z) = u + iv$ and let u be given,

since, u is given we can find $u_x \; and \; u_y$

As $f(z)$ is analytic, by C – R equations $u_x = v_y \; and \; u_y = -v_x$

$\therefore f'(z) = u_x + iv_x = u_x - iu_y = \Phi(z)$ say

Hence, by mere integration $f(z)$ can be obtained.

Note: The method can be used only when we are able to express $u_x - iu_y$ as a function of z, say $\Phi(z)$

Method 2: Milne – Thompson's Method

Since, $z = x + iy, \bar{z} = x - iy$

$\therefore x = \frac{z+\bar{z}}{2}, \; y = \frac{z-\bar{z}}{2i}$

$\therefore f(z) = u(x, y) + iv(x, y) = u\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$

This can be regarded as an identity in two independent variables, z and $\bar{z}$.

We can, therefore, put $\bar{z} = z$ and get $f(z) = u(z,0) + iv(z,0)$

Thus, $f(z)$ can be obtained in terms of z by putting $x = z$ and $y = 0$ in $f(z) = u(x,y) + iv(x,y)$ when $f(z)$ is analytic.

Now, $f'(z) = u_x + iv_x = u_x - iu_y$ [$\because$ C – R equations]

Let $u_x = \Phi_1(x,y)$ and $u_y = \Phi_2(x,y)$

$\therefore f'(z) = \Phi_1(x,y) - i\Phi_2(x,y) = \Phi_1(z,0) - i\Phi_2(z,0)$

Integrating, we get $\boldsymbol{f(z) = \int \Phi_1(z,0)dz - i\int \Phi_2(z,0)dz + c}$

Similarly if v given arguing on the above lines we can show that

$\boldsymbol{f(z) = \int \Psi_1(z,0)dz + i\int \Psi_2(z,0)dz + c}$ where $v_y = \Psi_1(x,y), v_x = \Psi_2(x,y)$

SOME SOLVED EXAMPLES:

1. Construct an analytic function whose real part is $x^4 - 6x^2y^2 + y^4$

Solution: **Method 1:**

Let $u = x^4 - 6x^2y^2 + y^4$ and let $f(z) = u + iv$ be the required function

$\therefore u_x = 4x^3 - 12xy^2;\ u_y = -12x^2y + 4y^3$

$$f'(z) = u_x - iu_y$$
$$= 4x^3 - 12xy^2 + 12ix^2y - 4iy^3$$
$$= 4[x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3]$$
$$= 4(x+iy)^3 = 4z^3$$

$\therefore f(z) = \int f'(z)\,dz = \int 4z^3\,dz = z^4 + c$

Method 2: Milne-Thompson Method:

$\Phi_1 = u_x = 4x^3 - 12xy^2;\ \Phi_2 = u_y = -12x^2y + 4y^3$

$\therefore f'(z) = \Phi_1(z,0) - i\Phi_2(z,0)$

$\therefore f'(z) = 4z^3 - i(0)$ [Putting $x = z, y = 0$ in Φ_1 and Φ_2]

$\therefore f(z) = \int 4z^3\,dz = z^4 + c$

2. Construct an analytic function whose real part is $(x-1)^3 - 3xy^2 + 3y^2$

Solution: Let $u = (x-1)^3 - 3xy^2 + 3y^2$

$\therefore u_x = 3(x-1)^2 - 3y^2, u_y = -6xy + 6y$

$\therefore \emptyset_1(x,y) = u_x = 3(x-1)^2 - 3y^2, \emptyset_2(x,y) = u_y = -6xy + 6y$

By Milne Thompson Method

$\therefore f'(z) = \emptyset_1(z,0) - i\emptyset_2(z,0) = 3(z-1)^2 - i0 = 3(z-1)^2$

$\therefore f(z) = \int f'(z)\,dz = \int 3(z-1)^2\,dz = (z-1)^3 + c$

which is the required analytic function

3. Construct an analytic function whose real part is $x^2 + y^2 - 5x + y + 2$

Solution: Let $u = x^2 + y^2 - 5x + y + 2$

$u_x = 2x - 5, u_y = 2y + 1$

$\therefore \emptyset_1(x, y) = u_x = 2x - 5, \emptyset_2(x, y) = u_y = 2y + 1$

By Milne Thompson Method

$\therefore f'(z) = \emptyset_1(z, 0) - i\emptyset_2(z, 0) = (2z - 5) - i[2(0) + 1] = 2z - 5 - i$

$\therefore f(z) = \int f'(z)\, dz = \int (2z - 5 - i)\, dz + c = z^2 - 5z - iz + c$ is the required analytic function

4. Construct an analytic function whose real part is $e^x \cos y$.

Solution: Let $u = e^x \cos y$

$\therefore u_x = e^x \cos y$ and $u_y = -e^x \sin y$

$\therefore \Phi_1 = u_x = e^x \cos y\,, \Phi_2 = u_y = -e^x \sin y$

By Milne-Thompson method

$\therefore f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0) = e^z - i(0)$

$\therefore f(z) = \int e^z\, dz = e^z + c$

which is the required analytic function

5. Construct an analytic function whose real part is $e^{-x}(x \sin y - y \cos y)$

Solution: Let $u = e^{-x}(x \sin y - y \cos y)$

$\therefore u_x = \emptyset_1(x, y) = e^{-x}(\sin y) + (x \sin y - y \cos y)(-e^{-x}) = e^{-x}(\sin y - x \sin y + y \cos y)$

$\therefore u_y = \emptyset_2(x, y) = e^{-x}(x \cos y + y \sin y - \cos y)$

By Milne-Thompson method

$\therefore f'(z) = \emptyset_1(z, 0) - i\emptyset_2(z, 0) = e^{-z}(0) - i \cdot e^{-z}(z - 1) = -ie^{-z}(z - 1)$

$\therefore f(z) \quad = \int f'(z)\, dz = \int -ie^{-z}(z - 1)\, dz = -i \int e^{-z}(z - 1)\, dz$

$= -i[(z - 1)(-e^{-z}) - \int (1)(-e^{-z})\, dz] = -i[(-ze^{-z} + e^{-z} - e^{-z})]$

$= ize^{-z} + c$ is the required analytic function

6. Construct an analytic function whose real part is $e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$

Solution: Let $u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$

$\therefore u_x = -e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\} + e^{-x}\{2x \cos y + 2y \sin y\}$

$u_y = e^{-x}[-(x^2 - y^2)\sin y - 2y \cos y + 2x \sin y + 2xy \cos y]$

$\therefore \Phi_1 = u_x$ and $\Phi_2 = u_y$

By Milne-Thompson method

$\therefore f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0) = e^{-z}[-z^2 + 2z]$

$\therefore f(z) \quad = \int e^{-z}(-z^2 + 2z)\, dz$

$= (-z^2 + 2z)(-e^{-z}) - \int (-e^{-z})(-2z + 2)\, dz$

$= e^{-z}(z^2 - 2z) + \int e^{-z}(2 - 2z)\, dz$

$= e^{-z}(z^2 - 2z) + (2 - 2z)(-e^{-z}) - \int (-e^{-z})(-2)\, dz$

$$= e^{-z}(z^2 - 2z) - e^{-z}(2 - 2z) + 2e^{-z}$$
$$= z^2 e^{-z} + c$$

7. Construct an analytic function whose real part is $\frac{\sin 2x}{\cos h\, 2y + \cos 2x}$

Solution: Let $u = \frac{\sin 2x}{\cos h\, 2y + \cos 2x}$

$\therefore \Phi_1 = u_x = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) + \sin 2x \cdot 2\sin 2x}{(\cosh 2y + \cos 2x)^2} = \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$

$\Phi_2 = u_y = \frac{-\sin 2x \cdot 2\sinh(2y)}{(\cosh 2y + \cos 2x)^2}$

By Milne-Thompson method

$\therefore f'(z) = \Phi_1(z,0) - i\Phi_2(z,0) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} - 0 = \frac{2}{1 + \cos 2z} = \sec^2 z$

$\therefore f(z) = \int \sec^2 z\, dz = \tan z + c$

8. Find an analytic function whose imaginary part is $(x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

Solution: We have $v = (x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

$\therefore v_y = \Psi_1(x,y) = -12x^2y + 4y^3 - 2y + 2x$

$v_x = \Psi_2(x,y) = 4x^3 - 12xy^2 + 2x + 2y$

We use Milne-Thompson method

$\therefore \Psi_1(z,0) = 2z, \Psi_2(z,0) = 4z^3 + 2z$

Now, $f(z) = \int \Psi_1(z,0)\, dz + i\int \Psi_2(z,0)\, dz$

$$= \int 2z\, dz + i\int (4z^3 + 2z)\, dz$$
$$= z^2 + i(z^4 + z^2) + c$$

9. Find an analytic function whose imaginary part is $\cos x \cos h\, y$

Solution: Let $v = \cos x \cosh y$

$\therefore v_y = \Psi_1(x,y) = \cos x \sinh y\,, v_x = \Psi_2(x,y) = -\sin x \cosh y$

By using Milne-Thompson method

$\Psi_1(z,0) = 0, \Psi_2(z,0) = -\sin z$

$f'(z) = \Psi_1(z,0) + i\Psi_2(z,0) = i(-\sin z)$

$\therefore f(z) = \int f'(z)\, dz = \int -i \sin z\, dz = i\cos z + c$ is the required analytic function

10. Find an analytic function whose imaginary part is $\sin h\, x \sin y$

Solution: Let $v = \sin h\, x \sin y$

$\therefore v_y = \Psi_1(x,y) = \sin h\, x \cos y\,, v_x = \Psi_2(x,y) = \cosh x \sin y$

By using Milne-Thompson method

$\Psi_1(z,0) = \sinh z\,, \Psi_2(z,0) = 0$

$\therefore f'(z) = \Psi_1(z,0) + i\Psi_2(z,0) = \sinh z$

$\therefore f(z) = \int f'(z)\, dz = \int \sinh z\, dz = \cosh z + c$ is the required analytic function

11. Find an analytic function whose imaginary part is $e^x(x\sin y + y\cos y)$

Solution: Let $v = e^x(x\sin y + y\cos y)$

$\therefore v_y = \Psi_1(x,y) = e^x(x\cos y - y\sin y + \cos y), v_x = \Psi_2(x,y) = e^x(\sin y + x\sin y + y\cos y)$

By using Milne-Thompson method

$\Psi_1(z,0) = e^z(z+1), \Psi_2(z,0) = 0$

$\therefore f'(z) = \Psi_1(z,0) + i\Psi_2(z,0) = e^z(z+1)$

$\therefore f(z) = \int f'(z)\,dz = \int e^z(z+1)\,dz = (z+1)e^z - \int(1)e^z\,dz = (z+1)e^z - e^z + c = ze^z + c$

is the required analytic function

12. Find an analytic function whose imaginary part is $e^{-x}(y\cos y - x\sin y)$

Solution: We have $v = e^{-x}(y\cos y - x\sin y)$

$\therefore v_y = \Psi_1(x,y) = e^{-x}(\cos y - y\sin y - x\cos y)$

$v_x = \Psi_2(x,y) = -e^{-x}(y\cos y - x\sin y) + e^{-x}(-\sin y)$

$= e^{-x}(-\sin y - y\cos y + x\sin y)$

We use Milne-Thompson method

$\therefore \Psi_1(z,0) = e^{-z}(1-z), \Psi_2(z,0) = 0$

Now, $f(z) = \int \Psi_1(z,0)\,dz + i\int \Psi_2(z,0)\,dz = \int(1-z)e^{-z}\,dz$

$= (1-z)(-e^{-z}) - \int(-e^{-z})(-1)\,dz$

$= -e^{-z} + ze^{-z} + e^{-z} = ze^{-z} + c$

13. Find an analytic function whose imaginary part is $e^{-x}(\,y\sin y + x\cos y)$

Solution: We have $v = e^{-x}(\,y\sin y + x\cos y)$

$\therefore v_y = \Psi_1(x,y) = e^{-x}(\sin y + y\cos y - x\sin y)$

$v_x = \Psi_2(x,y) = -e^{-x}(y\sin y + x\cos y) + e^{-x}(\cos y)$

$= e^{-x}(\cos y - y\sin y - x\cos y)$

We use Milne-Thompson method

$\therefore \Psi_1(z,0) = 0, \Psi_2(z,0) = e^{-z}(1-z)$

Now, $f(z) = \int \Psi_1(z,0)\,dz + i\int \Psi_2(z,0)\,dz = i\int e^{-z}(1-z)\,dz$

$= i[(1-z)(-e^{-z}) - \int -e^{-z}(-1)\,dz]$

$= i[(1-z)(-e^{-z}) + e^{-z}]$

$f(z) = ie^{-z}z + c$

14. Find an analytic function whose imaginary part is $tan^{-1}\frac{y}{x}$

Solution: We have $v = tan^{-1}\frac{y}{x}$

$\therefore v_y = \Psi_1(x,y) = \frac{1}{1+(y^2/x^2)} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

$v_x = \Psi_2(x,y) = \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

We use Milne-Thompson method

$\therefore \Psi_1(z,0) = \frac{z}{z^2} = \frac{1}{z}, \Psi_2(z,0) = 0$

Now, $f(z) = \int \Psi_1(z,0)\,dz + i\int \Psi_2(z,0)\,dz = \int \frac{1}{z}dz = \log z + c$

15. If the imaginary part of the analytic function $w = f(z)$ is $v = x^2 - y^2 + \frac{x}{x^2+y^2}$. Show that the real part $u = -2xy + \frac{y}{x^2+y^2} + c$

Solution: We have $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

$\therefore v_y = \Psi_1(x,y) = -2y - \frac{2xy}{(x^2+y^2)^2}$

$v_x = \Psi_2(x,y) = 2x - \frac{x^2-y^2}{(x^2+y^2)^2}$

We use Milne-Thompson method

$\therefore \Psi_1(z,0) = 0, \Psi_2(z,0) = 2z - \frac{1}{z^2}$

$f'(z) = v_y + iv_x = \Psi_1(z,0) + i\,\Psi_2(z,0)$

Now, $f(z) = \int \Psi_1(z,0)\,dz + i\int \Psi_2(z,0)\,dz$

$= i\int\left(2z - \frac{1}{z^2}\right)dz = i\left(z^2 + \frac{1}{z}\right)$

$= i(x+iy)^2 + i\cdot\frac{1}{x+iy}\cdot\frac{x-iy}{x-iy}$

$= i(x^2 + 2ixy - y^2) + i\frac{(x-iy)}{x^2+y^2}$

$\therefore f(z) = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$

$\therefore u = -2xy + \frac{y}{x^2+y^2} + c$

16. Check whether $u = x + e^{xy} + y + e^{-xy}$ is harmonic

Solution: $u = x + e^{xy} + y + e^{-xy}$; for a function to harmonic, it must satisfy Laplace's equation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u = x + e^{xy} + y + e^{-xy}$

$\therefore \frac{\partial u}{\partial x} = 1 + e^{xy}(y) + e^{-xy}(-y)$

$\frac{\partial^2 u}{\partial x^2} = y^2 e^{xy} + y^2 e^{-xy} y^2$

$\frac{\partial u}{\partial y} = e^{xy}(x) + 1 + e^{-xy}(-x)$

$\frac{\partial^2 u}{\partial y^2} = e^{xy}(x^2) + x^2 e^{-xy}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2+y^2)(e^{xy} + e^{-xy}) \neq 0$

It does not satisfy Laplace's equations $\therefore$ the function u is not harmonic

17. State true or false with proper justification "There does not exist an analytic function whose real part is $x^3 - 3x^2y - y^3$"

Solution: We shall use the theorem to check whether $u = x^3 - 3x^2y - y^3$ is a real part of some analytic function. By the result, $u = x^3 - 3x^2y - y^3$ must satisfy Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if it is real part of some analytic function

Now $\frac{\partial u}{\partial x} = 3x^2 - 6xy, \frac{\partial^2 u}{\partial x^2} = 6x - 6y$

$\frac{\partial u}{\partial y} = -3x^2 - 3y^2, \frac{\partial^2 u}{\partial y^2} = -6y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 12y \neq 0$

$\therefore$ There does not exist an analytic function whose real part is $u = x^3 - 3x^2y - y^3$

18. If $u(x, y)$ is a harmonic function then prove that $f(z) = u_x - i\, u_y$ is an analytic function.

Solution: Since u is harmonic

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ……………… (1)

By data $f(z) = u_x - i\, u_y$

Let $u_x = U$ and $-u_y = V$, so that $f(z) = U + iV$

We have to show that $f(z)$ is analytic

Now, $U_x = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ [By (1)]

and $U_y = \frac{\partial^2 u}{\partial x\, \partial y}$

$V_x = -\frac{\partial^2 u}{\partial y \partial x}$ and $V_y = -\frac{\partial^2 u}{\partial y^2}$

$\therefore U_x = V_y$ and $U_y = -V_x$

$\therefore f(z) = U + iV$ is analytic i.e., $f(z) = u_x - iu_y$ is analytic

19. If u, v are harmonic conjugate functions, show that uv is a harmonic function.

Solution: Let $f(z) = u + iv$ is analytic function

$\therefore u_x = v_y$ and $u_y = -v_x$

And u, v are harmonic

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ ……………. (1)

Now, $\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$

$\therefore \frac{\partial^2}{\partial x^2}(uv) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2} = u\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}$ ……………. (2)

Similarly, we can prove that

$\therefore \frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$

But $u_x = v_y$ and $u_y = -v_x$

$\therefore \frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}$ ……………. (3)

Adding (2) and (3), we get

$$\frac{\partial^2}{\partial x^2}(uv) + \frac{\partial^2}{\partial y^2}(uv) = u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + v\left(\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial y^2}\right) = 0 \text{ [By (1)]}$$

$\therefore uv$ is harmonic

20. If Φ and ψ are function of x and y satisfying Laplace equation and if $u = \Phi_y - \psi_x, v = \Phi_x + \psi_y$ prove that $u + iv$ is analytic (holomorphic)

Solution: Since Φ and ψ satisfy Laplace equation, we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad \text{.............. (1)}$$

and $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ .... (2)

Now, $u_x = \Phi_{yx} - \psi_{xx} = \Phi_{xy} + \psi_{yy}$ [By (2)]

And $u_y = \Phi_{yy} - \psi_{xy} = -(\Phi_{xx} + \psi_{xy})$ [By (1)]

Similarly, $v_x = \Phi_{xx} + \psi_{xy}$ and $v_y = \Phi_{xy} + \psi_{yy}$

Hence, $u_x = v_y$ and $u_y = -v_x$

Hence, $u + iv$ is analytic

21. If Φ and ψ are functions satisfying Laplace equation, then show that $s + it$ is holomorphic (analytic) where $s = \frac{\partial \Phi}{\partial y} - \frac{\partial \psi}{\partial x}$ and $t = \frac{\partial \Phi}{\partial x} + \frac{\partial \psi}{\partial y}$

Solution: Since Φ and ψ satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \text{ and } \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{............... (1)}$$

Now, $\frac{\partial s}{\partial x} = \frac{\partial^2 \Phi}{\partial y \partial x} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2}$ [By (1)] .... (2)

$\frac{\partial s}{\partial y} = \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$ [By (1)] .... (3)

Also, $\frac{\partial t}{\partial x} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$ .... (4)

$\frac{\partial t}{\partial y} = \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2}$ .... (5)

From (2) and (5), we have $\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$

From (3) and (4), we have $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$

Since, $s + it$ satisfies Cauchy-Riemann equations it is analytic

22. Find the imaginary part of the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$ also verify that v is harmonic.

Solution: Let $u = e^{2x}(x\cos 2y - ysin\, 2y)$

$$\therefore \Phi_1 = u_x = e^{2x} \cdot 2(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$$
$$= e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$$

$\Phi_2 = u_y = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$

By Milne-Thompson method

$\therefore f'(z) = \Phi_1(z,0) - i\Phi_2(z,0) = e^{2z}(2z+1) - ie^{2z}(0) = e^{2z}(2z+1)$

$$\therefore f(z) \quad = \int e^{2z}(2z+1)\,dz = (2z+1)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{2}\cdot 2\cdot dz$$

$$= (2z+1)\frac{e^{2z}}{2} - \int e^{2z}\,dz = (2z+1)\frac{e^{2z}}{2} - \frac{e^{2z}}{2}$$

$$= e^{2z}z + c$$

Now, $f(z) = e^{2(x+iy)}\cdot(x+iy) = e^{2x}\cdot e^{2iy}(x+iy) = e^{2x}[\cos 2y + i\sin 2y](x+iy)$

$\therefore v = e^{2x}(y\cos 2y + x sin\, 2y)$

$\therefore \frac{\partial v}{\partial x} = 2e^{2x}(y\cos 2y + x\sin 2y) + e^{2x}(\sin 2y)$

$\frac{\partial^2 v}{\partial x^2} = 4e^{2x}(y\cos 2y + x\sin 2y) + 4e^{2x}(\sin 2y)$

$\frac{\partial v}{\partial y} = e^{2x}(\cos 2y - 2y\sin 2y + 2x\cos 2y)$

$$\frac{\partial^2 v}{\partial y^2} \quad = e^{2x}(-2\sin 2y - 2\sin 2y - 4y\cos 2y - 4x\sin 2y)$$

$$= e^{2x}(-4\sin 2y - 4y\cos 2y - 4x\sin 2y)$$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \therefore v$ is harmonic

23. Show that the following function is harmonic and find the corresponding analytic function $f(z) = u + iv$

$u = \sin x\cos hy + 2\cos x\sin hy + x^2 - y^2 + 4xy$

Solution: We have $\frac{\partial u}{\partial x} = \cos x\cosh y - 2\sin x\sinh y + 2x + 4y$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x\cosh y - 2\cos x\sinh y + 2$$

$$\frac{\partial u}{\partial y} = \sin x\sinh y + 2\cos x\cosh y - 2y + 4x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x\cosh y + 2\cos x\sinh y - 2$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Hence, u satisfies Laplace's equation $\qquad \therefore u$ is a harmonic function

Now $u_x = \Phi_1(x,y) = \cos x\cosh y - 2\sin x\sinh y + 2x + 4y$

$\therefore \Phi_1(z,0) = \cos z + 2z$

$u_y = \Phi_2(x,y) = \sin x\sinh y + 2\cos x\cosh y - 2y + 4x$

$\Phi_2(z,0) = 2\cos z + 4z$

Now, Milne-Thompson Method

$\therefore f'(z) = \Phi_1(z,0) - i\Phi_2(z,0) = (\cos z + 2z) - i(2\cos z + 4z)$

$\therefore f(z) = \int[(\cos z + 2z) - i(2\cos z + 4z)]\,dz = \sin z + z^2 - i(2\sin z + 2z^2) + c$

24. Show that the following functions are harmonic. Also find the corresponding harmonic conjugate function and analytic function.

(i) $u = y^3 - 3x^2y$

Solution: Since, $u = y^3 - 3x^2y$

$u_x = -6xy, u_{xx} = -6y;\ u_y = 3y^2 - 3x^2, u_{yy} = 6y$

$\therefore \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -6y + 6y = 0$

$\therefore u = x^3 - 3x^2y$ is a harmonic function

Since, $u = y^3 - 3x^2y$ by Milne-Thompson method

$u_x = \Phi_1 = -6xy, u_y = \Phi_2 = 3y^2 - 3x^2$

$\therefore f'(z) = \Phi_1(z,0) - i\Phi_2(z,0) = 0 + 3iz^2$

$\therefore f(z) = \int 3iz^2\, dz = iz^3 + c$ as above is required analytic function

Now, $f(z) = i(x+iy)^3 = i(x^3 + 3ix^2y - 3xy^2 - iy^3)$

$\therefore u + iv = -3x^2y + y^3 + i(x^3 - 3xy^2)$

$\therefore v = x^3 - 3xy^2$ is harmonic conjugate

(ii) $v = e^x \sin y$

Solution: We have

$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial^2 v}{\partial x^2} = e^x \sin y$

$\frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial^2 v}{\partial y^2} = -e^x \sin y$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore v$ satisfies Laplace's equation $\qquad \therefore v$ is a harmonic function

Now, we use, Milne-Thompson Method

$v_x = e^x \sin y \quad \therefore\ \psi_2(z,0) = 0$

$v_y = e^x \cos y \quad \therefore\ \psi_1(z,0) = e^z$

$\therefore f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = e^z + 0$

$\therefore f(z) = e^z + c$

Now, $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$\therefore u = e^x \cos y$

(iii) $u = \cos x \cos hy$

Solution: We have

$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$

$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$

$\therefore \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore u$ satisfies Laplace's equation $\qquad \therefore u$ is a harmonic function

Now, we use, Milne-Thompson Method

Now, $u_x = \Phi_1(x,y) = -\sin x \cosh y$

$\therefore \Phi_1(z,0) = -\sin z$

$u_y = \Phi_2(x,y) = \cos x \sinh y$

$\Phi_2(z,0) = 0$

$\therefore f'(z) = \Phi_1(z,0) - i\Phi_2(z,0) = -\sin z$

$\therefore f(z) = \int -\sin z \, dz = \cos z + c$ is the required analytic function

Now, $f(z) = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$

$\therefore u + iv = \cos x \cosh y - \sin x \sinh y$

$\therefore v = -\sin x \sinh y$ is the required harmonic conjugate

(iv) $v = 3x^2y + 6xy - y^3$

Solution: We have

$\frac{\partial v}{\partial x} = 6xy + 6y, \quad \frac{\partial^2 v}{\partial x^2} = 6y$

$\frac{\partial v}{\partial y} = 3x^2 + 6x - 3y^2, \quad \frac{\partial^2 v}{\partial y^2} = -6y$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0$

$\therefore v$ satisfies Laplace's equation $\therefore v$ is a harmonic function

Now, we use, Milne-Thompson Method

$v_x = 6xy + 6y \quad \therefore \psi_2(z,0) = 0$

$v_y = 3x^2 + 6x - 3y^2 \quad \therefore \psi_1(z,0) = 3z^2 + 6z$

$\therefore f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = (3z^2 + 6z) + 0$

$\therefore f(z) = \int (3z^2 + 6z)\, dz = (z^3 + 3z^2) + c$

$\therefore f(z) = z^3 + 3z^2$

$= (x+iy)^3 + 3(x+iy)^2$

$= (x^3 + 3ix^2y - 3xy^2 - iy^3) + 3(x^2 + 2ixy - y^2)$

$= (x^3 - 3xy^2 + 3x^2 - 3y^2) + i(3x^2y - y^3 + 6xy)$

$\therefore$ harmonic conjugate

$u = x^3 - 3xy^2 + 3x^2 - 3y^2$

(v) $u = 2x(1-y)$

Solution: $\frac{\partial u}{\partial x} = 2(1-y) \quad \frac{\partial^2 u}{\partial x^2} = 0$

$\frac{\partial u}{\partial y} = 2x(-1) \quad \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ it satisfies Laplace equation

$\therefore u$ is a harmonic function

$\therefore u_x = \emptyset_1(x,y) = 2(1-y) \quad \emptyset_1(z,0) = 2$

$u_y = \emptyset_2(x,y) = -2x \quad \emptyset_2(z,0) = -2z$

By Milne-Thompson Method, $f'(z) = \emptyset_1(z,0) - i\emptyset_2(z,0) = 2 - i(-2z) = 2 + i(2z)$

$\therefore f(z) = \int f'(z)\,dz = \int 2 + i(2z)dz = 2z + iz^2 + c$

$\therefore f(z) \quad = 2(x+iy) + i(x+iy)^2 + c = 2x + 2iy + i(x^2 + 2ixy - y^2) + c$

$= i(2y + x^2 - y^2) + (2x - 2xy) + c$

imaginary part $= v = 2y + x^2 - y^2$

(vi) $u = 3x^2y - y^3$

Solution: $\frac{\partial u}{\partial x} = 6xy \qquad \frac{\partial^2 u}{\partial x^2} = 6y$

$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 \qquad \frac{\partial^2 u}{\partial y^2} = -6y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ it satisfies Laplace equation

$\therefore u$ is a harmonic function

$\therefore u_x = \emptyset_1(x,y) = 6xy \qquad \therefore \emptyset_1(z,0) = 0$

$u_y = \emptyset_2(x,y) = 3x^2 - 3y^2 \qquad \emptyset_2(z,0) = 3z^2$

By Milne-Thompson Method,

$\therefore f'(z) = u_x - iu_y = \emptyset_1(z,0) - i\emptyset_2(z,0) = -i(3z^2)$

$\therefore f(z) = \int f'(z)\,dz = \int -i(3z^2)\,dz = -iz^3 + c$

$\therefore f(z) = -iz^3 + c$

$= -i[x+iy]^3 + c$

$= -i[x^3 + 3ix^2y - 3xy^2 - iy^3]$

$= (3x^2y - y^3) - i(x^3 - 3xy^2)$

$\therefore$ Harmonic conjugate is $v = -x^3 + 3xy^2$

(vii) $u = 2a\,xy + b\,(y^2 - x^2)$

Solution: $u = 2axy + b(y^2 - x^2)$

$\frac{\partial u}{\partial x} = 2ay + b(-2x) \qquad \frac{\partial^2 u}{\partial x^2} = -2b$

$\frac{\partial u}{\partial y} = 2ax + 2by \qquad \frac{\partial^2 u}{\partial y^2} = 2b$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2b + 2b = 0$

it satisfies Laplace equation

$\therefore u$ is a harmonic function

Now we use Milne Thompson method

$u_x = 2ay - 2bx \qquad \emptyset_1(z,0) = -2bz$

$u_y = 2ax + 2by \qquad \emptyset_2(z,0) = 2az$

$f'(z) = \emptyset_1(z,0) - i\emptyset_2(z,0) = -2bz - i2az$

$\therefore f(z) = \int f'(z)\,dz = \int -2bz\,dz - \int i2az\,dz = -bz^2 - iaz^2 + c = -z^2(b + ai) + c$

$\therefore f(z) \quad = -(x+iy)^2(b+ai) + c$

$= -(x^2 + 2ixy - y^2)(b + ai) + c$

$= -(x^2b + aix^2 + 2xybi - 2axy - by^2 - ay^2i) + c$

$= (2axy - x^2b + by^2) + (ay^2 - ax^2 - 2xyb)i + c$

$\therefore v = ay^2 - ax^2 - 2xyb$ is the harmonic conjugate

(viii) $u = \frac{1}{2}\log(x^2 + y^2)$

Solution: We have $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

Similarly, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2} = 0$

$\therefore u$ satisfies Laplace's equation $\therefore u$ is a harmonic function

Now, $u_x = \Phi_1(x, y) = \frac{x}{x^2+y^2}$ $\therefore \Phi_1(z, 0) = \frac{z}{z^2+0} = \frac{1}{z}$

$u_y = \Phi_2(x, y) = \frac{y}{x^2+y^2}$ $\therefore \Phi_2(z, 0) = 0$

By Milne-Thompson Method

$\therefore f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0) = \frac{1}{z} - i0 = \frac{1}{z}$

$\therefore f(z) = \int \frac{1}{z}dz = \log z + c = \log(x + iy) + c$

$\therefore u + iv = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x} + c$

$\therefore v = \tan^{-1}\frac{y}{x} + c$ is the corresponding harmonic conjugate

25. Prove that $u = x^2 - y^2, v = -\frac{y}{x^2+y^2}$ both u and v satisfy Laplace's equation, but that $u + iv$ is not an analytic function of z.

Solution: $u_x = 2x, u_{xx} = 2;\ u_y = -2y, u_{yy} = -2$

$v_x = \frac{2xy}{(x^2+y^2)^2}, v_{xx} = 2y\left[\frac{(x^2+y^2)^2\cdot 1 - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}\right]$

$\therefore v_{xx} = \frac{2y(x^2+y^2)[x^2+y^2-4x^2]}{(x^2+y^2)^4} = 2y\frac{(y^2-3x^2)}{(x^2+y^2)^3}$

$v_y = -\left[\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$

$v_{yy} = \frac{(x^2+y^2)^2\cdot 2y - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$

$= 2y(x^2 + y^2)\frac{[x^2+y^2-2y^2+2x^2]}{(x^2+y^2)^4}$

$= 2y\frac{(3x^2-y^2)}{(x^2+y^2)^3}$

$\therefore u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Hence, u, v satisfy Laplace's equations

But Cauchy-Riemann equations are not satisfied as $u_x \neq v_y$ and $u_y \neq -v_x$

Hence, $u + iv$ is not analytic

26. State Laplace's equation in polar form and verity it for $u = r^2 \cos 2\theta$ and also find v and $f(z)$.

Solution: Laplace's equation in polar form is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$\therefore u = r^2 \cos 2\theta$

$\therefore \frac{\partial u}{\partial r} = 2r\cos 2\theta\,,\ \frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta$

$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta \qquad \therefore \frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta$

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$= 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta) = 4\cos 2\theta - 4\cos 2\theta = 0$

$\therefore$ Laplace's equation is satisfied

By Cauchy-Riemann equations in polar form

$u_r = \frac{1}{r}v_\theta \qquad \therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\therefore \frac{\partial v}{\partial \theta} = r(2r\cos 2\theta) = 2r^2\cos 2\theta$

Integrating w.r.t. $\theta, v = r^2 \sin 2\theta + c$

Hence, $f(z) = u + iv = r^2\cos 2\theta + ir^2 \sin 2\theta + c$

$= r^2(\cos 2\theta + i\sin 2\theta) + c$

$= r^2 e^{i2\theta} = \left(re^{i\theta}\right)^2 + c = z^2 + c$

27. Verify Laplace's equation for $u = \left(r + \frac{a^2}{r}\right)\cos\theta$. Also find v and $f(z)$.

Solution: $\because u = \left(r + \frac{a^2}{r}\right)\cos\theta$

$\therefore \frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta,\ \frac{\partial^2 u}{\partial r^2} = \frac{2a^2}{r^3}\cos\theta$

$\frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta\,, \frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{a^2}{r}\right)\cos\theta$

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$= \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\cdot\left(1 - \frac{a^2}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{a^2}{r}\right)\cos\theta = 0$

$\therefore$ Laplace's equation is satisfied

By Cauchy-Riemann equations in polar form

$u_r = \frac{1}{r}v_\theta \qquad \therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\therefore \left(1 - \frac{a^2}{r^2}\right)\cos\theta = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$

$\therefore \frac{\partial v}{\partial \theta} = \left(r - \frac{a^2}{r}\right)\cos\theta$

Integrating w.r.t. θ,

$v = \left(r - \frac{a^2}{r}\right)\sin\theta + c$

Hence, $f(z) = u + iv = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$

$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta) + c$

$= z + \frac{a^2}{z} + c$

Alternatively we can express u in terms of x and y and use Cartesian form of Laplace's equation, it may be noted that this method is rather tedius

28. If $u = k(1 + cos\theta)$, find v so that $u + iv$ is analytical.

Solution: Since, $u = k + kcos\theta$

$\frac{\partial u}{\partial r} = 0$ and $\frac{\partial u}{\partial \theta} = -k\sin\theta$

But by $C - R$ equation in polar coordinates

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$\therefore \frac{\partial v}{\partial \theta} = 0, \frac{\partial v}{\partial r} = -\frac{1}{r}(-k\sin\theta)$

Integrating the first equation partially w.r.t. θ,

$v = f(r)$ where $f(r)$ is an arbitrary function

$\therefore \frac{\partial v}{\partial r} = f'(r) = \frac{k\sin\theta}{r}$

$\therefore v = k\sin\theta\log r + c$

Hence, the analytic function is $f(z) = u + iv = k(1 + cos\theta) + ik\sin\theta\log r + c$

29. Find the analytic function $f(z)$ whose real part is $-r^3\sin 3\theta$

Solution: We have $\frac{\partial u}{\partial r} = -3r^2\sin\theta$ and $\frac{\partial u}{\partial \theta} = -3r^3\cos 3\theta$

By Cauchy-Riemann equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore \frac{\partial v}{\partial \theta} = -3r^3\sin 3\theta$

Integrating w.r.t. θ,

$v = r^3\cos 3\theta$

$\therefore f(z) = u + iv$

$= -r^3\sin 3\theta + ir^3\cos 3\theta$

$= ir^3(\cos 3\theta + i\sin 3\theta)$

$= ir^3e^{i3\theta} = iz^3 + c$

ORTHOGONAL CURVES:

Theorem: If $f(z) = u(x,y) + iv(x,y)$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally.

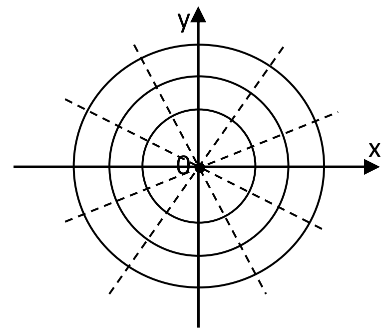
Proof: Let $u = f(x,y) = c_1 \; and \; v = \Phi(x,y) = c_2$

Then $\left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial u/\partial x}{\partial u/\partial y}$ And $\left(\frac{dy}{dx}\right)_{v=c_2} = -\frac{\partial \Phi/\partial x}{\partial \Phi/\partial y} = -\frac{\partial v/\partial x}{\partial v/\partial y}$

Since, $f(z)$ is analytic C – R equations give $u_x = v_y \; and \; u_y = -v_x$

$\therefore \left(\frac{\partial y}{\partial x}\right)_{u=c_1} \times \left(\frac{dy}{dx}\right)_{v=c_2} = \frac{\partial u/\partial x}{\partial u/\partial y} \times \frac{\partial v/\partial x}{\partial v/\partial y} = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1$

Hence, $u = c_1 \; and \; v = c_2$ intersect orthogonally



ORTHOGONAL TRAJECTORIES:

By orthogonal trajectory of a family of curves we mean a curve which cuts every member of the given family at right angles. For example, consider a family to straight lines passing through the origin given by $y = mx$, where m is an arbitrary constant.

It is easy to see that these straight lines are cut by a circle with centre at the origin at right angles at every point of intersection. Its equation is of the form $x^2 + y^2 = a^2$ where a is a parameter.

Thus the family of circles $x^2 + y^2 = a^2$ represents the family of orthogonal trajectories to the family of straight lines given by $y = mx$

Orthogonal trajectories of the family of curves given by $u = c$.

We have seen that if $f(z) = u + iv$ is an analytic function then the curves $u = c_1 \; and \; v = c_2$ intersect orthogonally i.e $v = c_2$ is the family of orthogonal trajectories of the family of curves $u = c_1$

Hence, to find the orthogonal trajectory of $u = c_1$(or $v = c_2$) we find the harmonic conjugate $v = c_2(or \; u = c_1)$ of u (or v)

SOME SOLVED EXAMPLES:

1. Find the orthogonally trajectories of the family of the curve $x^3y - xy^3 = c$

Solution: The orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u

$\because u = x^3y - xy^3$

$\therefore u_x = 3x^2y - y^3$ and $u_y = x^3 - 3xy^2$

$\therefore f'(z) = u_x + iv_x$

$= u_x - iu_y$ (By $C - R$ equations)

$= (3x^2y - y^3) - i(x^3 - 3xy^2)$

By Milne-Thompson's method, we put $x = z, y = 0$

$\therefore f'(z) = -iz^3$

$\therefore f(z) = -\int iz^3 \, dz = -i\frac{z^4}{4} + c$

$= -\frac{i}{4}(x + iy)^4 + c$

$= -\frac{i}{4}(x^4 + 4x^3iy - 6x^2y^2 - 4xiy^3 + y^4) + c$

$\therefore$ Imaginary part $v = -\frac{1}{4}(x^4 - 6x^2y^2 + y^4) + c$

Hence, the required orthogonal trajectories are $x^4 - 6x^2y^2 + y^4 = c'$

2. Find the orthogonally trajectories of the family of the curvs $e^{-x}cosy + xy = \alpha$

Solution: The orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u

$\therefore u = e^{-x}cosy + xy$

$u_x = -e^{-x}cosy + y$ and $u_y = -e^{-x}\sin y + x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$ (By $C - R$ equations)

$= (-e^{-x}cosy + y\ ) - i(-e^{-x}\sin y + x)$

By Milne-Thompson's method, we replace x by z and y by zero

$\therefore f'(z) = -e^{-z} - iz$

By integrating $f(z) = e^{-z} - i\frac{z^2}{2} + c$

$\therefore f(z) = e^{-(x+iy)} - i\frac{(x+iy)^2}{2} + c = e^{-x}(\cos y - i\sin y) - \frac{i}{2}(x^2 + 2ixy - y^2) + c$

$\therefore$ Imaginary part, $v = -e^{-x}\sin y - \frac{1}{2}(x^2 - y^2)$

Hence, the required orthogonal trajectories are $e^{-x}\sin y + \frac{1}{2}(x^2 - y^2) = c_2$

3. Find the orthogonally trajectories of the family of the curvs $2x - x^3 + 3xy^2 = a$

Solution: The orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u

Let $u = \ 2x - x^3 + 3xy^2$

$\therefore u_x = 2 - 3x^2 + 3y^2, u_y = 6xy$

$\therefore f'(z) \quad = u_x + iv_x$

$= u_x - iu_y$ (By $C - R$ equations)

$= 2 - 3x^2 + 3y^2 - i \cdot 6xy$

By Milne-Thompson's method, we put $x = z, y = 0$

$\therefore f'(z) = 2 - 3z^2$

Integrating w.r.t. z, we get,

$\therefore f(z) \quad = 2z - z^3 + c$

$= 2(x + iy) - (x + iy)^3 + c$

$= 2x + 2iy - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$

$\therefore$ Imaginary part $v = 2y - 3x^2y + y^3 + c$

$\therefore$ The required orthogonal trajectories are $2y - 3x^2y + y^3 = c$

4. For the function $f(z) = z^3$, verify that the families of curves $u = c_1$and $v = c_2$ cut orthogonally where $c_1$and $c_2$are constant and $f(z) = u + iv$

Solution: $f(z) = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

$\therefore u_x = 3x^2 - 3y^2, u_y = -6xy$

$v_x = 6xy, v_y = 3x^2 - 3y^2$

$\therefore m_1 = \left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{u_x}{u_y} = -\frac{3(x^2-y^2)}{-6xy}$

$m_2 = \left(\frac{dy}{dx}\right)_{u=c_2} = -\frac{v_x}{v_y} = -\frac{6xy}{3(x^2-y^2)}$

$\therefore m_1 \times m_2 = \frac{3(x^2-y^2)}{6xy} \cdot \left(-\frac{6xy}{3(x^2-y^2)}\right) = -1$

Hence, the families cut orthogonally