Introduction

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Definition of a Periodic Function:

When at equal intervals of abscissa 'x', the value of each ordinate $f(x)$ repeats itself, i.e $f(x) = f(x+T)$, for all x, then $y = f(x)$ is called a periodic function having period T.

eg. $\sin x$ and $\cos x$ are periodic of period 2π, since $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \ldots..$
and $\cos x = \cos(x+2\pi) = \cos(x+4\pi) = \ldots..$

Definition:

Let $f(x)$ be a periodic function of period 2π, defined in the interval $(c, c+2\pi)$, satisfying Dirichlet's Conditions as **(i)** $f(x)$ and its integrals are finite and single valued,

(ii) $f(x)$ has discontinuities finite in number ,

(iii) $f(x)$ has finite number of maxima and minima ,

then $f(x)$ can be expanded as an infinite trigonometric series as: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

which is called **Fourier Series**, where $a_0, a_n, b_n (n = 1,2,3, \ldots)$ are called Fourier coefficients or Fourier constants.

EULER'S FORMULAE:

The Fourier series for the function $f(x)$ in the interval $c < x < c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\cos nx\, dx$, $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\sin nx\, dx$ ......(I)

These values of a_0, a_n, b_n represented by (I) are known as Euler's Formulae.

TO ESTABLISH EULER'S FORMULAE:

The following definite integrals will be required to establish Euler's formulae:

1. $\int_c^{c+2\pi} \cos mx\, dx = \left|\frac{\sin mx}{m}\right|_c^{c+2\pi} = 0, \{for\ all\ m \neq 0\}$

2. $\int_c^{c+2\pi} \sin mx\, dx = -\left|\frac{\cos mx}{m}\right|_c^{c+2\pi} = 0, \{for\ all\ m \neq 0\}$

3. $\int_c^{c+2\pi} \sin mx\, cosnx\, dx = \frac{1}{2}\int_c^{c+2\pi}(\sin(m+n)x + \sin(m-n)x)\, dx = 0$

4. $\int_c^{c+2\pi} \cos mx.\cos nx\, dx$

$$= \begin{cases} \frac{1}{2}\int_c^{c+2\pi}(\cos(m+n)x + \cos(m-n)x)\, dx = 0, & \{for\ m \neq n, by\ virtue\ of\ (1)\} \\ \int_c^{c+2\pi} cos^2 mx\, dx = \frac{1}{2}\int_c^{c+2\pi}[1+\cos 2mx]\, dx = \pi, & \{for\ m = n\} \end{cases}$$

5. $\int_c^{c+2\pi} \sin mx.\sin nx\, dx$

$$= \begin{cases} \frac{1}{2}\int_c^{c+2\pi}(\cos(m-n)x - \cos(m+n)x)\, dx = 0, & \{for\ m \neq n, by\ virtue\ of\ (1)\} \\ \int_c^{c+2\pi} sin^2 mx\, dx = \frac{1}{2}\int_c^{c+2\pi}[1-\cos 2mx]\, dx = \pi, & \{for\ m = n\} \end{cases}$$

Theorem: Establish Euler's formulae represented by (I)

Proof: Let $f(x)$ be represented in the interval $(c, c+2\pi)$ by the Fourier series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx) + \sum_{n=1}^{\infty}(b_n \sin nx)$ ....(*)

To find a_0, integrating both sides of (*) w.r.t x from c to $+2\pi$, we have

$$\int_c^{c+2\pi} f(x)dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(a_n \cos nx)\, dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(b_n \sin nx)\, dx$$

$= \frac{a_0}{2}|c + 2\pi - c| + 0 + 0$ {by the integrals (1) and (2) in the above section }

$= a_0.\pi$

Hence $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

To find a_n, multiplying both sides of (*) by cos nx and then integrating w.r.t x from c to $c+2\pi$, we have

$$\int_c^{c+2\pi} f(x)\cos nx\,dx$$
$$= \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}((a_n\cos nx))\cos nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}((b_n\sin nx))\cos nx\,dx$$

$= 0 + a_n.\pi + 0$ { by the integrals (1), (3) and (4) in the above section }

$= a_n.\pi$

Hence $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\cos nx\,dx$

To find b_n, multiply both sides of (*) by sin nx and then integrating w.r.t x from c to $c+2\pi$, we have

$$\int_c^{c+2\pi} f(x).\sin nx\,dx$$
$$= \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}((a_n\cos nx))\sin nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}((b_n\sin nx))\sin nx\,dx$$

$= 0 + 0 + b_n.\pi$ { by the integrals (2) , (3) and (5) in the above section }

$= b_n.\pi$

Hence $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\sin nx\,dx$

Fourier Series for f(x) [Even / Odd / Neither Even nor Odd] in different interval

INTER -VAL	FOURIER SERIES	a_0	a_n	b_n
$(0,2\pi)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx) + (b_n\sin nx)$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$	$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx$	$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx\,dx$
$(0,2l)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{l})) + (b_n\sin(\frac{n\pi x}{l}))$	$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$	$a_n = \frac{1}{l}\int_0^{2l} f(x).\cos(\frac{n\pi x}{l})\,dx$	$b_n = \frac{1}{l}\int_0^{2l} f(x).\sin(\frac{n\pi x}{l})\,dx$
$(-\pi,\pi)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx) + (b_n\sin nx)$	$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$	$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$	$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\,dx$
$(-l,l)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{l})) + (b_n\sin(\frac{n\pi x}{l}))$	$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x).\cos(\frac{n\pi x}{l})\,dx$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin(\frac{n\pi x}{l})\,dx$
Even function in $(-\pi,\pi)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx)$	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$	$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx$	$b_n = 0$
Even function in $(-l,l)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{l}))$	$a_0 = \frac{2}{l}\int_0^{l} f(x)dx$	$a_n = \frac{2}{l}\int_0^{l} f(x).\cos(\frac{n\pi x}{l})\,dx$	$b_n = 0$
Odd function in $(-\pi,\pi)$	$f(x) = \sum_{n=1}^{\infty}(b_n\sin nx)$	$a_0 = 0$	$a_n = 0$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\,dx$
Odd function in $(-l,l)$	$f(x) = \sum_{n=1}^{\infty}(b_n\sin(\frac{n\pi x}{l}))$	$a_0 = 0$	$a_n = 0$	$b_n = \frac{2}{l}\int_0^{l} f(x).\sin(\frac{n\pi x}{l})\,dx$

SOME IMPORTANT RESULTS:

1. Bernoulli's generalized formula of integration by parts:
   $\int uv = uv_1 - u'v_2 + u''v_3 -$……………… until derivatives of u vanish,
   where $u', u'', \ldots\ldots\ldots\ldots.$ are the successive derivatives of u and $v_1, v_2, \ldots\ldots\ldots\ldots$ are the successive integrations of v.
2. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

ax ax

3. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

4. Property of definite integrals:

$\int_{-a}^{a} f(x)dx = 0,$ when $f(x)$ is odd function

$= 2\int_{0}^{a} f(x)dx$ when $f(x)$ is even function

PARSEVAL'S IDENTITY:

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$ is the Fourier series in $(0,2c)$ then prove that

$$\frac{1}{c}\int_0^{2c}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\} \quad \text{............(I)}$$

The formula represented by (I) is known **as Parseval's Identity.**

Proof: The Fourier for $f(x)$ in $(0,2c)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$ ....(II)

where
$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_0^{2c} f(x)dx \\ a_n &= \frac{1}{c}\int_0^{2c} f(x).\cos\frac{n\pi x}{c}dx \\ b_n &= \frac{1}{c}\int_0^{2c} f(x).\sin\frac{n\pi x}{c}dx \end{aligned}\right\} \quad \text{...........(III)}$$

Multiplying both sides of (II) by $f(x)$ and integrating term by term w.r.t x from 0 to $2c$.

$$\int_0^{2c}[f(x)]^2\,dx = \frac{a_0}{2}\int_0^{2c} f(x)dx + \sum_{n=1}^{\infty}\left(a_n\int_0^{2c} f(x).\cos\frac{n\pi x}{c}dx + b_n\int_0^{2c} f(x).\sin\frac{n\pi x}{c}dx\right)$$

$\therefore \int_0^{2c}[f(x)]^2dx = c.\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$ {Using the result (III)}

$\frac{1}{c}\int_0^{2c}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$ which is the required Parseval's identity.

Intervals	Parseval's identity
$(0,2c)$	$\frac{1}{c}\int_0^{2c}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$
$(0,2\pi)$	$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$
$(-c,c)$	$\frac{1}{c}\int_{-c}^{c}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$
$(-\pi,\pi)$	$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right\}$
half – range cosine series in $(0,c)$	$\frac{2}{c}\int_0^{c}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2)\right\}$
half – range cosine series in $(0,\pi)$	$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2)\right\}$
half – range sine series in $(0,c)$	$\frac{2}{c}\int_0^{c}[f(x)]^2dx = \sum_{n=1}^{\infty}(b_n^2)$
half – range sine series in $(0,\pi)$,	$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2dx = \sum_{n=1}^{\infty}(b_n^2)$