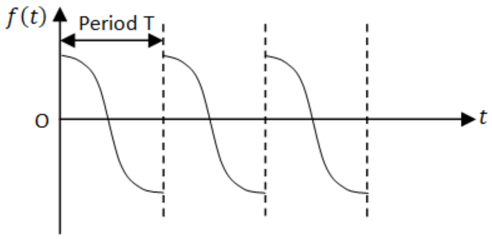
10 July 2023
15:19

PERIODIC FUNCTIONS:

Definition: The function $f(t)$ is said to be periodic function of period T, if $f(t+rT) = f(t), T > 0$ where $r = 0, 1, 2, 3, \ldots\ldots$ as shown in the following figure



For example, $f(t)$ =sint, is a period function of period 2π, as

$$f(t+2r\pi) = \sin(t+2r\pi) = \sin t, \text{ where } r = 0, 1, 2, 3, \ldots\ldots$$

Laplace Transform of Periodic Functions:

If $f(t)$ is a periodic function of period a, show that $L|f(t)| = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$

Proof: Since $f(t)$ is periodic with period a, $f(t) = f(t+a) = f(t+2a) = \ldots\ldots\ldots\ldots$

$\therefore L|f(t)| = \int_0^\infty e^{-st} f(t)dt = \int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt + \ldots\ldots\ldots\ldots\infty$

Now, $\int_a^{2a} e^{-st} f(t)dt = \int_0^a e^{-s(u+a)} f(u+a)du$ (where $t = u + a$)

$= e^{-as} \int_0^a e^{-su} f(u+a)du$

$= e^{-as} \int_0^a e^{-st} f(t+a)dt$ (Changing u to t by dummy variable property)

$= e^{-as} \int_0^a e^{-st} f(t)dt$ $\quad |\because f(t+a) = f(t)|$

Similarly, we can show that $\int_{2a}^{3a} e^{-st} f(t)dt = e^{-2as} \int_0^a e^{-st} f(t)dt$ and so on.

$\therefore L|f(t)| = (1 + e^{-as} + e^{-2as} + \ldots\ldots\ldots\infty) \int_0^a e^{-st} f(t)dt$

$\therefore L|f(t)| = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$ $\qquad \because$ For a G.P. $s_\infty = \frac{a}{1-r}$

Ex-1 :- Find the Laplace Transform of

$$f(t) = \begin{cases} \frac{t}{a}, & 0 < t \le a \\ \frac{1}{a}(2a-t), & a < t < 2a \end{cases} \quad \text{and} \quad f(t) = f(t+2a)$$

Solution :- As $f(t)$ is periodic function with period $2a$,

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left\{ \int_0^a \frac{t}{a} e^{-st}\,dt + \int_a^{2a} \frac{1}{a}(2a-t)\, e^{-st}\,dt \right\}$$

$$= \frac{1}{1-e^{-2as}} \Big\{ \frac{1}{a} \Big[t \Big(e^{-st} \Big) - (1) \Big(e^{-st} \Big) \Big]^a$$

$$= \frac{1}{1-e^{-2as}}\left\{ \frac{1}{a}\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a \right.$$

$$\left. + \frac{1}{a}\left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{ \frac{1}{a}\left[a\left(\frac{e^{-as}}{-s}\right) - \left(\frac{e^{-as}}{s^2}\right) - 0 + \frac{1}{s^2}\right] \right.$$

$$\left. + \frac{1}{a}\left[0 + \frac{e^{-2as}}{s^2} - (a)\left(\frac{e^{-as}}{-s}\right) - \left(\frac{e^{-as}}{s^2}\right)\right] \right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{ \cancel{-\frac{e^{-as}}{s}} - \frac{1}{a}\frac{e^{-as}}{s^2} + \frac{1}{a}\cdot\frac{1}{s^2} + \frac{1}{a}\frac{e^{-2as}}{s^2} + \cancel{\frac{e^{-as}}{s}} - \frac{1}{a}\frac{e^{-as}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{ \frac{1}{as^2}\left[1 - 2e^{-as} + e^{-2as}\right] \right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{ \frac{1}{as^2}\left(1-e^{-as}\right)^2 \right\}$$

$$= \frac{1}{as^2}\cdot\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{as^2}\left(\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right)$$

$$L[f(t)] = \frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$$

Ex-2 :- Find Laplace Transform of

$$f(t) = \begin{cases} \sin 2t & 0 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases} \quad \text{and} \quad f(t) = f(t+\pi)$$

Solution :- $f(t)$ is periodic function of period π

$$\therefore L[f(t)] = \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\int_0^{\pi/2} e^{-st}\sin 2t\, dt\right\}$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\frac{e^{-st}}{s^2+2^2}\left[-s\sin 2t - 2\cos 2t\right]\right\}_0^{\pi/2}$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\frac{e^{-s\pi/2}}{s^2+4}\left[-s(0)-2(-1)\right] - \frac{1}{s^2+4}\left[0-2\right]\right\}$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\frac{2}{s^2+4}\left(1+e^{-s\pi/2}\right)\right\}$$

$$L[f(t)] = \frac{2}{1-e^{-\pi s/2}} \cdot \frac{1}{s^2+4}$$

Ex-3 Find $L[f(t)]$ where $f(t) = \begin{cases} t, & 0<t<1 \\ 0, & 1<t<2 \end{cases}$

and $f(t+2) = f(t)$

Solution :- Since $f(t)$ is periodic with period $a=2$, we have

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\int_0^2 e^{-st} f(t)\, dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 t e^{-st}\, dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^1$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - 0 + \frac{1}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2s})}\left(1 - se^{-s} - e^{-s}\right)$$

$$= \frac{1}{s^2(1-e^{-2s})}\left[1 - se^{-s} - e^{-s}\right]$$