BAYE’S THEOREM

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If $E_1, E_2, E_3, \dots E_n$ are n mutually exclusive and exhaustive events from the sample space S, A is any other event from S and if probability of occurrence of E_i's and probability of occurrence of A given that $E_i, i = 1, 2, 3, \dots, n$ has occurred are known, then probabilities of occurrence of E_i's given that A has occurred are given by

$$P(E_i/A) = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)}; i = 1, 2, 3, \dots, n$$

BAYE'S THEOREM

Note: Three types of probabilities occur in the above theorem $P(E_i), P(A/E_i), P(E_i/A)$.

(i) The probabilities $P(E_i), i = 1, 2, 3, \ldots, n$ are called **priori** or simply **prior** probabilities

and $P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n) = 1$

(ii) The probabilities $P(A/E_i)$ is the probability of event A, given each and **every prior** probability.

(iii) The probabilities $P(E_i/A)$ are called **posterior** probabilities.

EXAMPLE-1

The chances of X, Y, Z becoming managers of a certain company are 4:2:3. The probabilities that the bonus scheme will be introduced if X, Y, Z become managers are 0.3,0.5 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that X is appointed as the manager?

Let event E_1: Person X becomes manager

event E_2: Person Y becomes manager

And event E_3: Person Z becomes manager

$\therefore P(E_1) = \frac{4}{9}$ $\quad P(E_2) = \frac{2}{9}$ $\quad P(E_3) = \frac{3}{9}$ (Note that E_1, E_2 and E_3 are mutually exclusive and exhaustive events)

Let event A: Bonus is introduced

$\therefore P(A/E_1) = P$(Bonus is introduced under the condition that person X becomes manager) $= 0.3$

$P(A/E_2) = P$(Bonus is introduced under the condition that person Y becomes manager) $= 0.5$

$P(A/E_3) = P$(Bonus is introduced under the condition that person Z becomes manager) $= 0.8$

$\therefore$ required probability $= P$(Person X becomes manager under the condition that bonus Scheme is introduced)

$$= P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)} = \frac{\left(\frac{2}{15}\right)}{\left(\frac{23}{45}\right)} = \frac{6}{23}$$

EXAMPLE-2

- The members of a consulting firm hire cars from three rental agencies, 60 percent from agency 1, 30 percent from agency 2 and 10 percent from agency 3. 9 percent of the cars from agency 1 need repairs, 20 percent of the car from agency 2 need repairs and 6 percent of the cars from agency 3 need repairs. If a rental car delivered to the consulting firms needs repairs, what is theprobability that it came from rental agency 2?
- **Solution:** Let $E_1 \equiv$ events that the car comes from rental agencies 1
- $E_2 \equiv$ events that the car comes from rental agencies 2
- $E_3 \equiv$ events that the car comes from rental agencies 3
- $A \equiv$ event that the car needs repairs
- $P(E_1) = 0.6, P(E_2) = 0.3, P(E_3) = 0.1$
- $P(A/E_1) = 0.09, P(A/E_2) = 0.2$ and $P(A/E_3) = 0.06$
- $P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$
- $= \frac{(0.30)(0.20)}{(0.60)(0.09)+(0.30)(0.20)+(0.10)(0.06)}$
- $= \frac{0.060}{0.120} = 0.5$

EXAMPLE-3

In certain area 25% of women were black, 75% were white. The literacy rates for black women were 48% and for white women was 83%. What proportion of the literate women was black?

Solution: Let $E_1 \equiv$ event that women were black

$E_2 \equiv$ event that women were white

$A \equiv$ event that women were literate

$$\therefore P(E_1) = 25\% = \frac{25}{100} = \frac{1}{4}$$

$$P(E_2) = 75\% = \frac{75}{100} = \frac{3}{4}$$

Since the literacy rates for black women were 48% and for white women were 83%,

$P(A/E_1) = 48\% = \frac{48}{100}$ and $P(A/E_2) = 83\% = \frac{83}{100}$

By Baye's theorem, the proportion of the literate women who were black is given by

$$P(E_1/A) = \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\left(\frac{1}{4}\right)\left(\frac{48}{100}\right)}{\left(\frac{1}{4}\right)\left(\frac{48}{100}\right)+\left(\frac{3}{4}\right)\left(\frac{83}{100}\right)}$$

$$= \frac{48}{48+249} = \frac{48}{297} = \frac{16}{99}$$

EXAMPLE-4

If E_1 and E_2 are equally likely, mutually exclusive and exhaustive events and $P(A/E_1) = 0.2, P(A/E_2) = 0.3$. Find $P(E_1/A)$.

Solution: E_1 and E_2 are equally likely, mutually exclusive and exhaustive events

$\therefore P(E_1) = P(E_2), P(E_1 \cap E_2) = 0$ and $P(E_1 \cup E_2) = 1$

But $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\therefore 1 = P(E_1) + P(E_2)$

$\therefore P(E_1) + P(E_1) = 1$ ………… $[\because P(E_1) = P(E_2)]$

$\therefore 2 \cdot P(E_1) = 1$

$\therefore P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$

Also, $P(A/E_1) = 0.2$ and $P(A/E_2) = 0.3$

$\therefore$ by Baye's Theorem

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$= \frac{\left(\frac{1}{2}\right)(0.2)}{\left(\frac{1}{2}\right)(0.2) + \left(\frac{1}{2}\right)(0.3)} = \frac{0.2}{0.2+0.3} = \frac{2}{5} = 0.4$$

EXAMPLE-5

Suppose that a product is produced in 3 factories X, Y & Z. It is known that factory X produces thrice as many times as factory Y, and that factory Y and Z produce the same number of items. Assume that it is known that 3% of the items produced by each of the factories X and Z are defective while 5% of those manufactured by Y are defective. All the items produced in 3 factories are stocked and an item is selected at random

(1) What is the probability that this item is defective

(2) If an item selected at random is found to be defective, what is the probability that it was produced by factory X, Y and Z respectively.

EXAMPLE-5

- **Solution:** Let $E_1 \equiv$ event that the item is produced by factory X
- $E_2 \equiv$ event that the item is produced by factory Y
- $E_3 \equiv$ event that the item is produced by factory Z
- $A \equiv$ event that the item is defective
- Let the number of items produced by Y and Z each be n. then number of items produced by X = 3n
- $P(E_1) = \frac{3n}{3n+n+n} = 0.6, P(E_2) = \frac{n}{5n} = 0.2, P(E_3) = \frac{n}{5n} = 0.2$
- $P(A/E_1) = P(A/E_2) = 0.03$ and $P(A/E_3) = 0.05$

(1) P(item is defective)=$P(A) = \sum_{i=1}^{3} P(A \cap E_i)$

$$= \sum_{i=1}^{3} P(E_i)P(A/E_i)$$

$$= P(E_1)\,P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)\,P(A/E_3)$$

$$= 0.034$$

EXAMPLE-5

Solution: $$P(E_1/A) = \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$$

$$= \frac{9}{17}$$

$$P(E_2/A) = \frac{P(E_2)\cdot P(A/E_2)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$$

$$= \frac{5}{17}$$

$$P(E_3/A) = \frac{P(E_3)\cdot P(A/E_3)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$$

$$= \frac{3}{17}$$

EXAMPLE-6

In 2002 there will be 3 candidates for the position of Principal- X, Y, Z, whose chances of getting the appointment are in the proportion $4: 2: 3$ respectively. The probability that Mr X is selected who would introduce co-education in the college is 0.3. the probability of Y & Z doing the same are respectively 0.5 & 0.8

(i) What is the probability that there will be co-education in the college in 2003

(ii) If there is co-education in the college in 2003, what is the probability that Mr Z is the principal

EXAMPLE-6

Let event E_1: Mr X becomes Principal

event E_2: Mr Y becomes Principal

And event E_3: Mr Z becomes Principal

$\therefore P(E_1) = \frac{4}{9} \quad P(E_2) = \frac{2}{9} \quad P(E_3) = \frac{3}{9}$

(Note that E_1, E_2 and E_3 are mutually exclusive and exhaustive events)

Let event A: Co-education is introduced

$\therefore P(A/E_1) = P$(co-education is introduced under the condition that Mr X becomes principal) $= 0.3$

$P(A/E_2) = P$(co-education is introduced under the condition that Mr Y becomes principal) $= 0.5$

$P(A/E_3) = P$(co-education is introduced under the condition that Mr Z becomes principal) $= 0.8$

EXAMPLE-6

(1) P(Co-education is introduced)=$P(A) = \sum_{i=1}^{3} P(A \cap E_i)$

$$= \sum_{i=1}^{3} P(E_i)P(A/E_i)$$

$$= P(E_1)\, P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)\, P(A/E_3)$$

$$= \frac{46}{90}$$

(2) $P(E_3/A) = \frac{P(E_3)\cdot P(A/E_3)}{P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+P(E_3)\cdot P(A/E_3)}$

$$= \frac{12}{23}$$