KUHN – TUCKER CONDITIONS

NLPP with inequality Constraint

OPTIMIZATION WITH INEQUALITY CONSTRAINTS

Consider the following NLPP with $\boldsymbol{n}$ **−variables and one inequality constraint.**

Maximize $z = f(x_1, x_2, x_3, \dots \dots, x_n)$

Subject to $g(x_1, x_2, x_3, \dots \dots, x_n) \leq b$ and

$x_1, x_2, x_3, \dots \dots, x_n \geq 0$

where b is a constant

These problems can be solved by the **Kuhn-Tucker method**

We will not get into the proof of this method as it is not required

OPTIMIZATION WITH INEQUALITY CONSTRAINTS

The necessary conditions **for maximization** are:

(1) $$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0,$$

(2) $$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, \dots \dots .. ,$$

(3) $$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0,$$

(4) $$\lambda h(x_1, x_2, \dots \dots, x_n) = 0,$$

(5) $$h(x_1, x_2, \dots \dots, x_n) \leq 0,$$

(6) $$\lambda \geq 0$$

These are the Kuhn-Tucker conditions

EXAMPLE-1

Solve the following NLPP

Maximize $z = 4x_1 - {x_1}^3 + 2x_2$

Subject to $x_1 + x_2 \leq 1, \quad x_1, x_2 \geq 0$

Solution: We have

$$f(x_1, x_2) = 4x_1 - {x_1}^3 + 2x_2, \qquad h(x_1, x_2) = x_1 + x_2 - 1$$

Now, the Kuhn-Tucker conditions for maximization are:

(a) $\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0 = 4 - 3{x_1}^2 - \lambda$ .... (1)

(b) $\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 = 2 - \lambda$ .... (2)

(c) $\lambda \geq 0$ .... (3)

(d) $h(x_1, x_2) \leq 0$ i.e. $x_1 + x_2 - 1 \leq 0$ .... (4)

(e) $\lambda h(x_1, x_2) = 0 = \lambda(x_1 + x_2 - 1)$ .... (5)

From the equation (2), we get $\lambda = 2$

From the equation (5), we get $x_1 + x_2 = 1$

From the equation (1), we get $x_1 = \sqrt{2/3}$

From the equation (5), we get $x_2 = 1 - \sqrt{2/3}$

These values satisfy all the necessary conditions

$\therefore$ the optimum solution is $x_1 = \sqrt{2/3}$ and $x_2 = 1 - \sqrt{2/3}$

$$\therefore z_{max} = 4\left(\sqrt{\frac{2}{3}}\right) - \left(\sqrt{\frac{2}{3}}\right)^3 + 2\left(1 - \sqrt{\frac{2}{3}}\right)$$
$$= 3.0887$$

EXAMPLE-2

Solve the following NLPP

Maximize $z = 10x_1 + 4x_2 - 2{x_1}^2 - {x_2}^2$

Subject to $2x_1 + x_2 \leq 5, x_1, x_2 \geq 0$

Solution: We have

$f(x_1, x_2) = 10x_1 + 4x_2 - 2{x_1}^2 - {x_2}^2$ and

$$h(x_1, x_2) = 2x_1 + x_2 - 5$$

Now, the Kuhn-Tucker conditions for maximization are:

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0,$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, h(x_1, x_2) \leq 0, \lambda h(x_1, x_2) = 0, \lambda \geq 0$$

$\therefore$ We get $10 - 4x_1 - 2\lambda = 0$ .... (1)

$4 - 2x_2 - \lambda = 0$ .... (2)

$2x_1 + x_2 - 5 \leq 0$ .... (3)

$\lambda(2x_1 + x_2 - 5) = 0$ .... (4)

$\lambda \geq 0$ .... (5)

From equation (4) we can either have $\lambda = 0$ or $2x_1 + x_2 - 5 = 0$

Case 1: If $\lambda = 0$,

then we get from (1) and (2) $x_1 = \frac{5}{2}$ and $x_2 = 2$

Putting these values in (3) to check whether the condition is satisfied or not, we get

$2 \times \frac{5}{2} + 2 - 5 = 2 \nleq 0$

Thus, these values do not satisfy (3).

Hence, $\lambda = 0$ does not yield a feasible solution and we reject these values

Case 2: If $\lambda \neq 0$, then $2x_1 + x_2 - 5 = 0$ ………… (6)

We now solve equations (1), (2) and (6) to get values of x_1 and x_2

After solving, we get $x_1 = \frac{11}{6}$ and $x_2 = \frac{4}{3}$

From the equation (2),

we get $\lambda = 4 - 2x_2 = 4 - \frac{8}{3} = \frac{4}{3}$

These values satisfy all the conditions.

$2 \times \frac{11}{6} + \frac{4}{3} - 5 = 0$ which satisfy the condition

$$2x_1 + x_2 - 5 \leq 0$$

$\therefore$ Optimal solution is $x_1 = \frac{11}{6}$ and $x_2 = \frac{4}{3}$

$$\therefore z_{max} = 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}$$

EXAMPLE-3

Use the Kuhn-Tucker conditions to solve the following NLPP

Maximize $z = 8x_1 + 10x_2 - {x_1}^2 - {x_2}^2$

Subject to $3x_1 + 2x_2 \leq 6, x_1, x_2 \geq 0$

Solution: We rewrite the problem as

$f(x_1, x_2) = 8x_1 + 10x_2 - {x_1}^2 - {x_2}^2$ and

$h(x_1, x_2) = 3x_1 + 2x_2 - 6$

Now, the Kuhn-Tucker conditions are:

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0,$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, h(x_1, x_2) \leq 0, \lambda h(x_1, x_2) = 0, \lambda \geq 0$$

$\therefore$ We get $8 - 2x_1 - 3\lambda = 0$ .... (1)

$10 - 2x_2 - 2\lambda = 0$ .... (2)

$3x_1 + 2x_2 - 6 \leq 0$ .... (3)

$\lambda(3x_1 + 2x_2 - 6) = 0$ .... (4)

$x_1, x_2, \lambda \geq 0$ .... (5)

From equation (4) we can either have $\lambda = 0$ or $3x_1 + 2x_2 - 6 = 0$

Case 1: If $\lambda = 0$,

then from equations (1) and (2), we get

$8 - 2x_1 = 0$ and $10 - 2x_2 = 0$

$\therefore x_1 = 4$ and $x_2 = 5$

But then for $x_1 = 4, x_2 = 5$, the condition in equation (3) is not satisfied

Hence, $\lambda = 0$ does not yield a feasible solution and we reject these values

Case 2: If $\lambda \neq 0,$ then $3x_1 + 2x_2 - 6 = 0$……………. (6)

To find x_1 and x_2 we obtain one more relation between x_1, x_2 by eliminating λ from (1) and (2)

Now, multiplying equation (1) by 2, (2) by 3 and subtracting

$\therefore 16 - 4x_1 - 30 + 6x_2 = 0$ $\qquad$ $\therefore -2x_1 + 3x_2 - 7 = 0$……………. (7)

Multiplying equation (6) by 3, (7) by 2 and subtracting, we get

$$\therefore 9x_1 - 18 + 4x_1 + 14 = 0 \qquad \therefore 13x_1 = 4 \qquad \therefore x_1 = \frac{4}{13}$$

$\therefore$ From the equation (7), we get,

$$\therefore -\frac{8}{13} + 3x_2 - 7 = 0 \qquad \therefore 3x_2 = \frac{99}{13} \qquad \therefore x_2 = \frac{33}{13}$$

Now, from (1), $4 \times \frac{4}{13} + 12 \times \frac{33}{13} = 2\lambda$

$\therefore 2\lambda = \frac{412}{33}$ $\qquad$ $\therefore \lambda > 0$

These values satisfy all the necessary conditions

$\therefore$ Optimal solution is $x_1 = \frac{4}{13}$ and $x_2 = \frac{33}{13}$

$$\therefore z_{max} = 8\left(\frac{4}{13}\right) + 10\left(\frac{33}{13}\right) - \frac{16}{169} - \frac{1089}{169} = \frac{3601}{169} = \frac{277}{13} = 21.3$$