LAPLACE TRANSFORMS OF INTEGRALS

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LAPLACE TRANSFORMS OF INTEGRALS:

If $L\{f(t)\} = \emptyset(s)$, then $L\int_0^t f(u)du = \frac{1}{s}\emptyset(s)$

Proof: By Definition of Laplace transform,

$$L\left\{\int_0^t f(u)du\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)du\right\}dt$$

Integrating by parts

$$= \left\{\int_0^t f(u)du.\left(\frac{-e^{-st}}{s}\right)\right\}\Big|_0^\infty - \int_0^\infty \left\{\left(\frac{-e^{-st}}{s}\right)\frac{d}{dt}\int_0^t f(u)du\right\}$$

But $\frac{d}{dt}\int_0^t f(u)du = f(t)$

$$\therefore L\left\{\int_0^t f(u)du\right\} = 0 + \int_0^\infty \frac{1}{s}.e^{-st}f(t)dt$$

$$= \frac{1}{s}.L\{f(t)\}$$

$$= \frac{1}{s}\emptyset(s) \quad \text{Since } \emptyset(s) = L\{f(t)\}$$

$$\therefore L\int_0^t f(u)du = \frac{1}{s}L\{f(t)\}$$

Corollary: The above result can be generalized as follows

$$L\left\{\int_0^t\int_0^t \dots \int_0^t f(u)(du)^n\right\} = \frac{1}{s^n}L[f(t)]$$

Ex :- Find the laplace Transform of $\int_0^t \sin 2u\, du$

Solution :- $L[\sin 2t] = \frac{2}{s^2+4} = \phi(s)$ (say)

$$\therefore L\left[\int_0^t \sin 2u\, du\right] = \frac{1}{s}\phi(s) = \frac{2}{s(s^2+4)}$$

Ex-2 Find Laplace Transform of $\int_0^t u e^{-3u}\cos^2 2u\, du$

Solution $\cos^2 2u = \frac{1+\cos 4u}{2}$

$$\therefore L[\cos^2 2u] = \frac{1}{2}L[1+\cos 4u] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

$$\therefore L[u\cos^2 2u] = -\frac{d}{ds}\,\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$ (multiplication by u)

$$= -\frac{1}{2}\left[\frac{-1}{s^2} + \frac{(s^2+16)\cdot(1) - s(2s)}{(s^2+16)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{-1}{s^2} + \frac{16-s^2}{(s^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-16}{(s^2+16)^2}\right]$$

$$\therefore L[e^{-3u}u\cos^2 2u] = \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{[(s+3)^2+16]^2}\right]$$ [First shifting theorem]

$$\therefore L\left[e^{-3u}\, u \cos^2 2u\right] = \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{\left[(s+3)^2+16\right]^2}\right]$$ [First shifting theorem]

$$= \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right] = \phi(s) \quad (\text{say})$$

$$\therefore L\left[\int_0^t e^{-3u}\, u \cos^2 2u \, du\right] = \frac{1}{s}\phi(s)$$ [Laplace Transform of integral]

$$= \frac{1}{2s}\left[\frac{1}{(s+3)^2} + \frac{s^2+6s-7}{(s^2+6s+25)^2}\right]$$

Ex-3 Find Laplace Transform of $t\int_0^t e^{-4u} \sin 3u \, du$

Solution :- $$L[\sin 3u] = \frac{3}{s^2+9}$$

$$L\left[e^{-4u} \sin 3u\right] = \frac{3}{(s+4)^2+9}$$ [By First shifting theorem]

$$\therefore L\left[\int_0^t e^{-4u} \sin 3u \, du\right] = \frac{1}{s}\cdot\frac{3}{(s+4)^2+9}$$ [Laplace Transform of integral]

$$\therefore L\left[t\int_0^t e^{-4u} \sin 3u \, du\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^3+8s^2+25s}\right]$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

Ex-4 Find Laplace Transform of $e^{-t}\int_0^t e^{u} \cosh u \, du$

Solution :- $$L[\cosh u] = \frac{s}{s^2-1}$$

$$L\left[e^{u} \cosh u\right] = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$$ [First shifting theorem]

$$L\left[\int_0^t e^{u} \cosh u \, du\right] = \frac{1}{s}\cdot\frac{s-1}{s(s-2)} = \frac{s-1}{s^2(s-2)}$$ [Laplace of integral]

$$L\left[e^{-t}\int_0^t e^{u} \cosh u \, du\right] = \frac{(s+1)-1}{(s+1)^2\left[(s+1)-2\right]} = \frac{s}{(s+1)^2(s-1)}$$

[First shifting theorem]

[First shifting theorem]

~~Ex-5~~ Find Laplace Transform of $\cosh t \int_0^t e^u \cosh u \, du$

~~Solution~~ $L[\cosh u] = \frac{s}{s^2-1}$

$$L[e^u \cosh u] = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$$

$$L\left[\int_0^t e^u \cosh u \, du\right] = \frac{1}{s}\cdot\frac{s-1}{s(s-2)} = \frac{s-1}{s^2(s-2)}$$

$$L\left[\cosh t \int_0^t e^u \cosh u \, du\right] = \frac{1}{2} L\left[e^t \int_0^t e^u \cosh u \, du + e^{-t}\int_0^t e^u \cosh u \, du\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2((s-1)-2)} + \frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right]$$

Ex-6 Find $L[\operatorname{erf}\sqrt{t}]$

Solution :- we have $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

Hence, $\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du$

Now put $u^2 = v$

$\therefore u = \sqrt{v} \implies du = \frac{1}{2\sqrt{v}} dv$

u	0	$\sqrt{t}$
v	0	v

$$\therefore \operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^t e^{-v} \cdot \frac{1}{2\sqrt{v}} dv = \frac{1}{\sqrt{\pi}} \int_0^t e^{-v} v^{-1/2} dv$$

Now, $L[v^{-1/2}] = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

$$= \frac{\quad}{s^{1/2}} = \frac{\quad}{\sqrt{s}}$$

$$\therefore L\left[e^{-v} v^{-1/2}\right] = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$\therefore L\left[\int_0^t e^{-v} v^{-1/2} dv\right] = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore L\left[\operatorname{erf}\sqrt{t}\right] = \frac{1}{\sqrt{\pi}} L\left[\int_0^t e^{-v} v^{-1/2} dv\right] = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore L\left[\operatorname{erf}\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$$

Ex-7 Find $L\left[\int_0^t \int_0^t t \cdot \sin t \, (dt)^2\right]$

Solution : $L[\sin t] = \dfrac{1}{s^2+1}$

$$L[t \sin t] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\left[\frac{-2s}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$$

$$L\left[\int_0^t\int_0^t t \sin t \,(dt)^2\right] = \frac{1}{s^2} \cdot \frac{2s}{(s^2+1)^2} = \frac{2}{s(s^2+1)^2}$$