Problems based on Basic formulae

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1. Find the Laplace transforms of the functions if $f(t) = \begin{cases} t \text{ when } 0 < t < 4 \\ 5 \text{ when } t > 4 \end{cases}$

Solution: By Definition

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^4 e^{-st}(t)\, dt + \int_4^\infty e^{-st}(5)\, dt$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left(\frac{e^{-st}}{-s}\right)_4^\infty$$

(on solving the first integral by parts)

$$= \left[4\left(\frac{e^{-4s}}{-s}\right) - \left(\frac{e^{-4s}-1}{s^2}\right)\right] - \frac{5}{s}\left[0 - e^{-4s}\right]$$

$$L[f(t)] = \boxed{\frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2}\right)e^{-4s}}$$

2. Find the Laplace transform of $f(t) = \begin{cases} \cos t \text{ when } 0 < t < \pi \\ \sin t \text{ when } t > \pi \end{cases}$

Solution:

By Definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$

using the formulae $\int e^{ax} \cos bx\, dx = \frac{1}{(a^2+b^2)} e^{ax}(a\cos bx + b\sin bx)$

$\int e^{ax} \sin bx\, dx = \frac{1}{(a^2+b^2)} e^{ax}(a\sin bx - b\cos bx)$

$$\therefore L[f(t)] = \frac{1}{s^2+1}\left[e^{-st}(-s\cos t + \sin t)\right]_0^\pi + \frac{1}{s^2+1}\left[e^{-st}(-s\sin t - \cos t)\right]_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}(-s\cos\pi + \sin\pi) - (-s\cos 0 + \sin 0)\right]$$

$$+ \frac{1}{s^2+1}\left[e^{-\infty}(-s\sin\infty - \cos\infty) - e^{-s\pi}(-s\sin\pi - \cos\pi)\right]$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}(s) + s\right] + \frac{1}{s^2+1}\left[-e^{-s\pi}\right]$$

$$\therefore \boxed{L[f(t)] = \frac{1}{s^2+1}\left[s + (s-1)e^{-s\pi}\right]}$$

Find **(i)** $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$,

(ii) $L\{\sin^3 t\}$, **(iii)** $L\{\cos^3 t\}$,

(iv) $L\{(t^2+1)^2\}$, **(v)** $L\{\sin(\omega t + \alpha)\}$, ω and α being

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Solution (i) Using the linearity property

$$L[3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t]$$

$$= 3L[t^4] - 2L[t^3] + 4L[e^{-3t}] - 2L[\sin 5t] + 3L[\cos 2t]$$

$$= 3\frac{4!}{s^5} - 2\frac{3!}{s^4} + 4\cdot\frac{1}{s+3} - 2\cdot\frac{5}{s^2+25} + 3\cdot\frac{s}{s^2+4}$$

$$L[f(t)] = \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$$

(ii) $L[\sin^3 t] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$ $\left(\because \sin^3 t = \frac{3\sin t - \sin 3t}{4}\right)$

$$= \frac{3}{4}L[\sin t] - \frac{1}{4}L[\sin 3t]$$

$$= \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9} = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$$= \frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right] = \frac{6}{(s^2+1)(s^2+9)}$$

(iii) $L[\cos^3 t] = L\left[\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right] = \frac{3}{4}L[\cos t] + \frac{1}{4}L[\cos 3t]$

$$= \frac{3}{4}\left[\frac{s}{s^2+1}\right] + \frac{1}{4}\left[\frac{s}{s^2+9}\right] = \frac{1}{4}\left[\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right]$$

$$= \frac{1}{4}\left[\frac{3(s^3+9s) + (s^3+s)}{(s^2+1)(s^2+9)}\right] = \frac{1}{4}\left[\frac{4s^3+28s}{(s^2+1)(s^2+9)}\right]$$

$$\boxed{L[\cos^3 t] = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}}$$

(iv) $L[(t^2+1)^2] = L[t^4+2t^2+1] = L[t^4] + 2L[t^2] + L[1]$

$$= \frac{4!}{s^5} + 2\cdot\frac{2!}{s^3} + \frac{1}{s} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

(v) $\sin(\omega t+\alpha) = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$

$$L[\sin(\omega t+\alpha)] = L[\sin\omega t\cos\alpha] + L[\cos\omega t \sin\alpha]$$

$$= \cos\alpha \, L[\sin\omega t] + \sin\alpha \, L[\cos\omega t]$$

$$= \cos\alpha \left[\frac{\omega}{s^2+\omega^2}\right] + \sin\alpha\left[\frac{s}{s^2+\omega^2}\right]$$

$$L[\sin(\omega t+\alpha)] = \frac{\omega\cos\alpha + s\sin\alpha}{s^2+\omega^2}$$

Q. Evaluate $L[\sin 2t \cdot \sin 3t]$

Solution :- $L[\sin 2t \sin 3t] = \frac{1}{2} L[\cos t - \cos 5t]$

$\left\{ \text{using} \quad \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \right\}$

$$= \frac{1}{2}[L(\cos t) - L(\cos 5t)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

Q. Evaluate $L[\cos t \cos 2t \cos 3t]$

Solution :- using $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

$$L[\cos t(\cos 2t \cos 3t)] = L\left[\cos t \cdot \frac{1}{2}(\cos 5t + \cos t)\right]$$

$$= \frac{1}{2} L[\cos t \cos 5t + \cos^2 t]$$

$$= \frac{1}{2} L\left[\frac{1}{2}(\cos 6t + \cos 4t) + \frac{1}{2}(1+\cos 2t)\right]$$

$$= \frac{1}{4}[L(\cos 6t) + L(\cos 4t) + L(\cos 2t) + L(1)]$$

$$L[\cos t \cos 2t \cos 3t] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right]$$

Ex Prove that $L[\sin^5 t] = \dfrac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

Solution :- First we simplify $\sin^5 t$

Let $x = \cos t + i\sin t \Rightarrow \frac{1}{x} = \cos t - i\sin t$

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$\therefore \sin t = \frac{1}{2i}\left(x - \frac{1}{x}\right)$

Also $x^n = \cos nt + i\sin nt$ & $\frac{1}{x^n} = \cos nt - i\sin nt$

$\therefore x^n - \frac{1}{x^n} = 2i\sin nt$

$$\therefore \sin^5 t = \left(\frac{1}{2i}\right)^5\left(x - \frac{1}{x}\right)^5$$

$$= \frac{1}{32i}\left(x^5 - 5x^4\cdot\frac{1}{x} + 10x^3\cdot\frac{1}{x^2} - 10x^2\cdot\frac{1}{x^3} + 5x\cdot\frac{1}{x^4} - \frac{1}{x^5}\right)$$

$$= \frac{1}{32i}\left[\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)\right]$$

$$= \frac{1}{32i}\left[2i\sin 5t - 5(2i\sin 3t) + 10(2i\sin t)\right]$$

$$\sin^5 t = \frac{1}{16}\left[\sin 5t - 5\sin 3t + 10\sin t\right]$$

$$\therefore L[\sin^5 t] = \frac{1}{16}\left[L(\sin 5t) - 5L[\sin 3t] + 10L(\sin t)\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+25} - 5\cdot\frac{3}{s^2+9} + 10\cdot\frac{1}{s^2+1}\right]$$

$$= \frac{5}{16}\left[\frac{1}{s^2+25} - \frac{3}{s^2+9} + \frac{2}{s^2+1}\right]$$

On simplifying we get

$$L[\sin^5 t] = \frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$$

Similarly we can find $L[\cos^5 t]$, $L[\sinh^5 t]$, $L[\cosh^5 t]$

or any power of $\sin t, \cos t, \sinh t, \cosh t$

Ex If $f(t) = (\sin 2t - \cos 2t)^2$ then find $L[f(t)]$
Hence find $L[f(2t)]$

Solution $f(t) = (\sin 2t - \cos 2t)^2$

$$= \sin^2 2t - 2\sin 2t\cos 2t + \cos^2 2t$$

$$= 1 - \sin 4t$$

$$\therefore L[f(t)] = L[1 - \sin 4t] = L[1] - L[\sin 4t]$$

$$= \frac{1}{s} - \frac{4}{s^2+16}$$

$$= \frac{s^2+16-4s}{s(s^2+16)}$$

Now using change of scale property

If $L[f(t)] = \phi(s)$ then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

$$\therefore L[f(2t)] = \frac{1}{2}\left[\frac{\left(\frac{s}{2}\right)^2 - 4\left(\frac{s}{2}\right)+16}{\left(\frac{s}{2}\right)\left(\left(\frac{s}{2}\right)^2+16\right)}\right]$$

$$L[f(2t)] = \frac{s^2-8s+64}{s(s^2+64)}$$

Ex :- If $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$. Find $L[f(2t)]$

Solution :- using change of scale property

If $L[f(t)] = \phi(s)$ then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Now $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$

$$\therefore L[f(2t)] = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right) = \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Ex : If $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$ find $L[\operatorname{erf} 3\sqrt{t}]$

Solution using change of scale property

$$L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \phi(s)$$

$$\therefore L[\operatorname{erf} 3\sqrt{t}] = L[\operatorname{erf}\sqrt{9t}]$$

$$= \frac{1}{9}\phi\left(\frac{s}{9}\right) = \frac{1}{9}\cdot\frac{1}{\frac{s}{9}\sqrt{\frac{s}{9}+1}} = \frac{3}{s\sqrt{s+9}}$$

Miscellaneous example

Find Laplace transform of $\sin\sqrt{t}$. Hence find $L[\sin 2\sqrt{t}]$

Solution :- Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

we have $\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots\cdots$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots\cdots$$

$$\therefore L[\sin\sqrt{t}] = L[t^{1/2}] - \frac{1}{3!}L[t^{3/2}] + \frac{1}{5!}L[t^{5/2}] - \frac{1}{7!}L[t^{7/2}] + \cdots\cdots$$

Now $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$, $\Gamma(n) = (n-1)\Gamma(n-1)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\therefore L[\sin\sqrt{t}] = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma(9/2)}{s^{9/2}} + \cdots\cdots$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} - \frac{1}{3!}\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{5/2}} + \frac{1}{5!}\frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{7/2}} \cdots\cdots$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{2s^{3/2}}\left[1 - \left(\frac{1}{2^2\cdot s}\right) + \frac{1}{2!}\left(\frac{1}{2^2\cdot s}\right)^2 \cdots\cdots\right]$$

$$\therefore L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4s} \qquad \left[e^{-x} = 1 - x + \frac{x^2}{2!}\cdots\cdots\right]$$

Now using change of scale property

$$L[\sin 2\sqrt{t}] = L[\sin\sqrt{4t}] = \frac{1}{4}\cdot\frac{\sqrt{\pi}}{2(s/4)^{3/2}}\,e^{-1/4\left(\frac{s}{4}\right)} = \frac{\sqrt{\pi}}{s^{3/2}}\,e^{-1/s}$$

Similarly we can find $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$