GREEN'S THEOREM IN THE PLANE:

(RELATION BETWEEN ~~DOUBLE INTEGRAL AND LINE INTEGRAL~~)

— ① write P & Q

② $\frac{\partial Q}{\partial x}$ & $\frac{\partial P}{\partial y}$ → solve

Statement:

If R is a closed region of the xy −plane bounded by a simple close curve C and if P and Q are continuous functions of x and y having continuous partial derivatives in R, then

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$\int e^{-x} \sin y\, dx + e^{-x} \cos y\, dy = \int P\, dx + Q\, dy$$

1. Evaluate by Green's Theorem $\oint_C (e^{-x} \sin y\, dx + e^{-x} \cos y\, dy)$ where C is the rectangle whose vertices are $(0,0), (\pi,0), (\pi,\pi/2), (0,\pi/2)$.

$P = e^{-x} \sin y$ $\qquad$ $Q = e^{-x} \cos y$

$\frac{\partial P}{\partial y} = e^{-x} \cos y$ $\qquad$ $\frac{\partial Q}{\partial x} = -e^{-x} \cos y$

By Green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C e^{-x} \sin y\, dx + e^{-x} \cos y\, dy = \iint_R -2e^{-x} \cos y\, dx\, dy$$

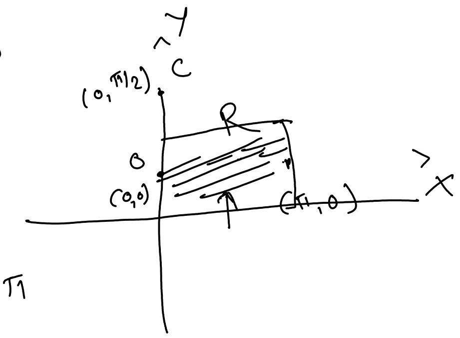
In R taking a vertical strip

$y \rightarrow$ x axis to BC

$y = 0$ to $y = \frac{\pi}{2}$

$x \rightarrow$ y axis to AB

$x = 0$ to $x = \pi$



$$= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} -2e^{-x} \cos y\, dx\, dy$$

$$= \left(\int_{x=0}^{\pi} -2e^{-x} dx\right)\left(\quad\right) \quad \Big| \quad = \int_0^{\pi} -2e^{-} \sin y^{\pi/2}\, dx$$

$$= \left(\frac{-2e^{-x}}{-1}\right)_0^{\pi} \left(\sin y\ \pi/2\right. \quad \Big| \quad = \int_0^{\pi}$$

$$= 2\left(e^{-\pi} - e^{0}\right)\left(\sin\frac{\pi}{2} - \sin 0\right) \quad \Big| \quad = \int^{\pi} -2e^{-x} dx$$

$$= 2\left(e^{-\pi} - e^{0}\right)\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2\left(e^{-\pi} - 1\right)(1-0)$$

$$= 2\left(e^{-\pi} - 1\right)$$

$$= \int_0^{\pi} -2e^{-x}\,dx$$

$$= \left(\frac{-2e^{-x}}{-1}\right)_0^{\pi}$$

$$= 2\left(e^{-\pi} - 1\right)$$

2. Evaluate the Green's theorem $\int_C F \cdot dr$ where $F = -xy(xi - yj)$ and C is $r = a(1 + \cos\theta)$

$= -x^2 y\, i + x y^2\, j$

Soln:-

$$\int_C \bar{F}\cdot d\bar{r} = \int_C -x^2 y\,dx + x y^2\,dy$$

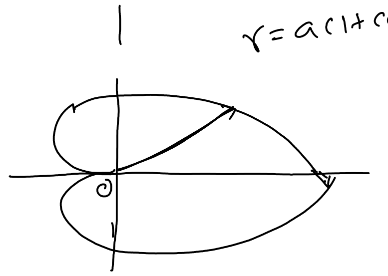
$$\int_C p\,dx + q\,dy$$

Now $p = -x^2 y \qquad q = x y^2$

$$\frac{\partial p}{\partial y} = -x^2 \qquad \frac{\partial q}{\partial x} = y^2 \qquad \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = y^2 - (-x^2) = x^2 + y^2$$

By Green's theorem $\int p\,dx + q\,dy = \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$

$r = a(1+\cos\theta)$



To evaluate the double integral, we put $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

$$x^2 + y^2 = r^2$$

$$\int -x^2 y\,dx + x y^2\,dy = \iint r^2\, r\,dr\,d\theta$$

$r \to 0$ to $a(1+\cos\theta)$

$\theta \to 0$ to π (2 times)

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\,dr\,d\theta$$

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\, dr\, d\theta$$

$$= 2\int_0^{\pi}\left(\frac{r^4}{4}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^4 (1+\cos\theta)^4\, d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi}\left(\frac{2\cos^2\theta}{2}\right)^4 d\theta$$

$$= \frac{2^4 a^4}{2}\int_0^{\pi}\cos^8\frac{\theta}{2}\, d\theta$$

put $\frac{\theta}{2} = t$ $\quad d\theta = 2\,dt$

$t \to 0$ to $\pi/2$

$$\cos^8 t\,(2\,dt)$$

$$= 16a^4 \int_0^{\pi/2} \cos^8 t\, dt$$

$$= 16a^4 \cdot \frac{1}{2} B\left(\frac{8+1}{2}, \frac{0+1}{2}\right) = 8a^4\, B\left(\frac{9}{2}, \frac{1}{2}\right)$$

$$= 8a^4 \frac{\Gamma\frac{9}{2}\,\Gamma\frac{1}{2}}{\quad} \qquad \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\;\Gamma\frac{1}{2}}{4!}$$

$$= \frac{35\pi a^4}{16}$$

3. Verify Green's theorem for $F = x^2 i - xyj$ and C is the triangle having vertices $A(0,2), B(2,0), C(4,2)$

Soln:- $\int_C \bar{F}\cdot d\bar{r} = \int_C x^2\,dx - xy\,dy = \int_C P\,dx + Q\,dy$

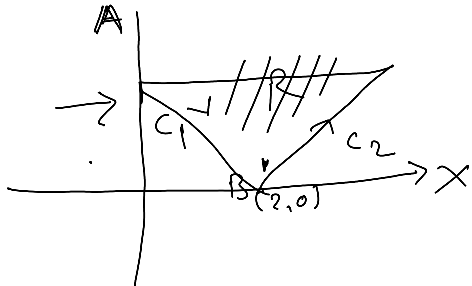
$P = x^2 \qquad Q = -xy$

$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = -y \qquad \therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y$

$\frac{\partial P}{\partial y} = 0$

By Green's thm, $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$x^2 dx - xy\,dy = \iint_R -y\,dx\,dy$



(a) $RHS = \iint_R -y\,dx\,dy$

taking a horizontal strip

$x \to$ AB to BC

$x = 2-y$ to $2+y$

$y \to$ X axis to AC

$y=0$ to $y=2$

eqn of AB

$x+y=2$

eqn of BC

$x-y=2$

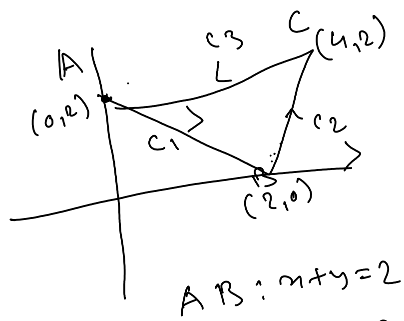
$$RHS = \int_0^2 \int_{2-y}^{2+y} -y\,dx\,dy$$

$$= \int_0^2 -y\,(x)_{2-y}$$

$$= \int_0^2 -y\,[2y]\,dy = -2\int_0^2 y^2\,dy = -2\left(\frac{y^3}{3}\right)_0^2$$

$$RHS = \frac{-16}{3}$$

(b) $LHS = x^2 dx - xy\,dy$



AB : $x+y=2$

BC : $x-y=2$

$C = C_1 + C_2 + C_3$

along C_1 : $x+y=2$

$y = 2-x$

$dy = -dx$

$x \to 0$ to 2

$2 \quad \ldots \quad (2-x)(-dx)$

$x \to 0$ to 2

$$\int_{C_1} x^2\,dx - xy\,dy = \int_0^2 x^2\,dx - x(2-x)(-dx)$$

$$= \int_0^2 2x\,dx = \left(x^2\right)_0^2 = 4-0 = 4$$

Now along C_2 : $BC \to x-y=2$

$y = x-2$

$dy = dx$

$x \to 2$ to 4

$$\int_{C_2} x^2\,dx - xy\,dy = \int_2^4 x^2\,dx - x(x-2)\,dx$$

$$= \int_2^4 2x\,dx = \left(x^2\right)_2^4 = 4^2-2^2 = 12$$

along C_3 : $CA \to y=2$ $\quad x \to 4$ to 0

$dy = 0$

$$\int_{C_3} x^2\,dx - xy\,dy = \int_4^0 x^2\,dx = \left(\frac{x^3}{3}\right)_4^0 = 0 - \frac{4^3}{3} = \frac{-64}{3}$$

$$LHS = \int_C x^2\,dx - xy\,dy$$

$$= \int_{C_1} x^2\,dx - xy\,dy + \int_{C_2} x^2\,dx - xy\,dy + \int_{C_3} x^2\,dx - xy\,dy$$

$$= 4+12-\frac{64}{3} = 16 - \frac{64}{3} = \frac{-16}{3}$$

$\therefore$ $LHS = RHS = \frac{-16}{3}$

Hence Green's theorem is verified.

4. Verify Green's theorem for $\int_C \left(\frac{1}{y}dx + \frac{1}{x}dy\right)$ where C is the boundry of the region defined by $x = 1, x = 4$, $y = 1$ and $y = \sqrt{x}$

Sol^n :- $\int_C \frac{1}{y}dx + \frac{1}{x}dy = \int P\,dx + Q\,dy$

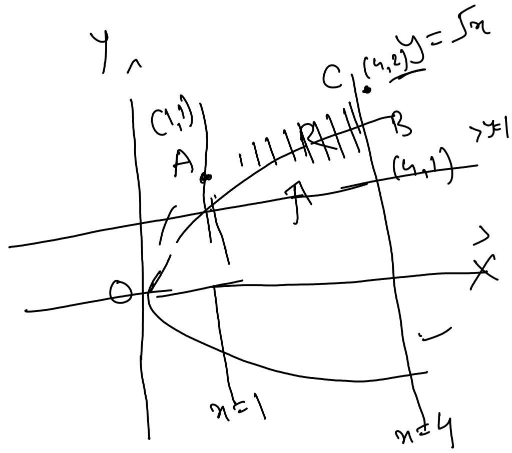
$P = \frac{1}{y}$ $\qquad Q = \frac{1}{x}$

$\frac{\partial P}{\partial y} = \frac{-1}{y^2}$ $\qquad \frac{\partial Q}{\partial x} = \frac{-1}{x^2}$ $\qquad \therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-1}{x^2} + \frac{1}{y^2}$

By Green's theorem $\int P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$\int_C \frac{1}{y}\,dx + \frac{1}{x}\,dy = \iint_R \left(-\frac{1}{x^2}+\frac{1}{y^2}\right) dx\,dy$$

(a) $$\text{RHS} = \iint_R \left(\frac{-1}{x^2}+\frac{1}{y^2}\right) dx\,dy$$



Taking a vertical strip

$y \to$ AB to parabola

$y=1$ to $y=\sqrt{x}$

$x \to$ A to BC

$x=1$ to $x=4$

$$\text{RHS} = \int_1^4 \int_1^{\sqrt{x}} \left(-\frac{1}{x^2}+\frac{1}{y^2}\right) dy\,dx$$

$$= \int_1^4 \left(-\frac{y}{x^2}-\frac{1}{y}\right)_1^{\sqrt{x}} dx = \int_1^4 \left(-\frac{\sqrt{x}}{x^2}-\frac{1}{\sqrt{x}}\right) - \left(\frac{-1}{x^2}-1\right) dx$$

$$= \int_1^4 \left(\frac{-1}{x^{3/2}} - \frac{1}{\sqrt{x}} + \frac{1}{x^2} + 1\right) dx$$

$$= \int_1^4 \left(-x^{-3/2} - x^{-1/2} + x^{-2} + 1\right) dx$$

$$= \left(\frac{-x^{-1/2}}{-1/2} - \frac{x^{1/2}}{1/2} + \frac{x^{-1}}{-1} + x\right)_1^4$$

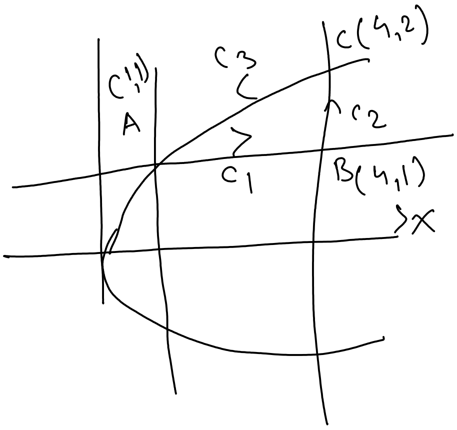
$$= \left(2(4)^{-1/2} - 2(4)^{1/2} - (4)^{-1} + 4\right) - \left(2(1)^{-1/2} - 2(1)^{1/2} - (1) + 1\right)$$

$$= \left(\frac{2}{2} - (2\times 2) - \frac{1}{4} + 4\right) - \left(2 - 2 - 1 + 1\right)$$

$$= \left(1 - 4 + 4 - \frac{1}{4}\right) = \boxed{\frac{3}{4}}$$

$\boxed{4}$

(b) $LHS = \int_C \left(\frac{1}{y}dx + \frac{1}{x}dy\right)$



$C \to C_1 + C_2 + C_3$

along C_1: $AB \to$ eqn of AB

$y = 1$

$dy = 0$

$x \to 1$ to 4

$$\int_{C_1} \frac{1}{y}dx + \frac{1}{x}dy = \int_1^4 \frac{1}{y}dx = \int_1^4 (1)dx = (x)_1^4 = 3$$

along C_2: $BC \to$ eqn of BC is $x = 4$

$dx = 0$

$y \to 1$ to 2

$$\int_{C_2} \frac{1}{y}dx + \frac{1}{x}dy = \int \frac{1}{x}dy = \int_1^2 \frac{1}{4}dy = \frac{1}{4}(y)_1^2 = \frac{1}{4}$$

along C_3: Curve CA : $y = \sqrt{x}$

$x = y^2 \to dx = 2y\,dy$

$y \to 2$ to 1

$$\frac{1}{y}dx + \frac{1}{x}dy = \int_2^1 \frac{1}{y}(2y\,dy) + \frac{1}{y^2}dy$$

$$= \int_2^1 \left(2 + \frac{1}{y^2}\right)dy$$

$$= \left(2y - \frac{1}{y}\right)_2^1 = (2-1) - \left(4 - \frac{1}{2}\right)$$

$$= 1 - \left(\frac{7}{2}\right) = -\frac{5}{2}$$

$LHS = \int_C \frac{1}{y}dx + \frac{1}{x}dy$

$\int \frac{1}{\,}dx + \frac{1}{\,}dy$

$$\int_C$$

$$= \int_{C_1} \frac{1}{y}dx + \frac{1}{x}dy + \int_{C_2} \frac{1}{y}dx + \frac{1}{x}dy + \int_{C_3} \frac{1}{y}dx + \frac{1}{x}dy$$

$$= 3 + \frac{1}{4} - \frac{5}{2} = \frac{13}{4} - \frac{5}{2} = \boxed{\frac{3}{4}}$$

Hence LHS = RHS = $\frac{3}{4}$

$\therefore$ Green's theorem is verified.

5. Verify Green's Theorem in the plane for $\int_C (xy + y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

Sol^n:- $$\int_C (xy + y^2)\,dx + x^2dy = \int_C P\,dx + Q\,dy$$

$\therefore P = xy + y^2 \qquad Q = x^2$

$\frac{\partial P}{\partial y} = x + 2y \qquad \frac{\partial Q}{\partial x} = 2x$

$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (x + 2y) = x - 2y$$

By Green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C (xy + y^2)\,dx + x^2dy = \iint_R (x - 2y)\,dx\,dy$$

Pt of intersection

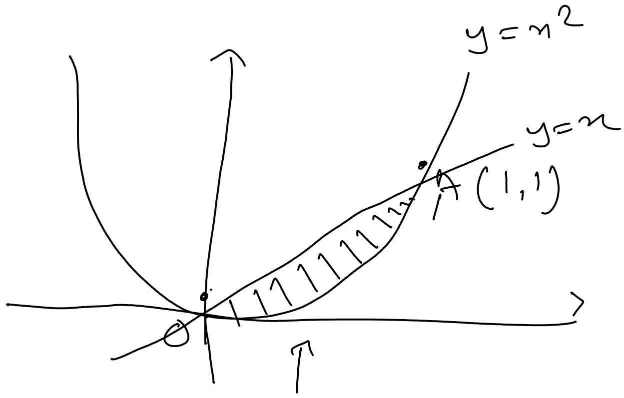
$y = x$ & $y = x^2$

$x = x^2$

$x(1 - x) = 0$

$x = 0$ or $x = 1$

$y = 0$ or $y = 1$



RHS = $\iint (x - 2y)\,dx\,dy$ Taking a vertical strip

$RHS = \iint (x-2y)\,dx\,dy$

$y \to$ parabola to line OA

$y = x^2$ to $y = x$

$x \to 0$ to 1.

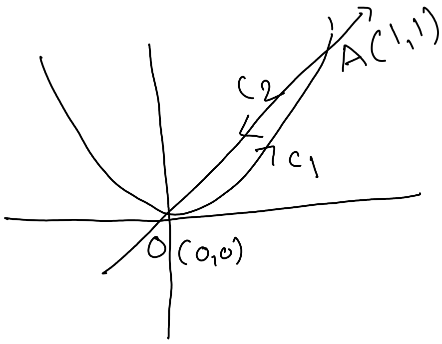
$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left(xy - y^2\right)_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^2) - (x^3 - x^4)\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \boxed{\frac{-1}{20}}$$

(b) $LHS = \int_C (xy + y^2)\,dx + x^2\,dy$



$C \to C_1 + C_2$

along C_1 : parabola $y = x^2$

$dy = 2x\,dx$

$x \to 0$ to 1

$$\int_{C_1} (xy + y^2)\,dx + x^2\,dy = \int_0^1 (x^3 + x^4)\,dx + x^2(2x\,dx)$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \left(\frac{3x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

along C_2 : line AO : $y = x$

$dy = dx$

$x \to 1$ to 0

$$\int (xy + y^2)\,dx + x^2\,dy = \int^0 (x^2 + x^2)\,dx + x^2\,dx$$

$$\int_{C_2} (xy+y^2)\,dx + x^2\,dy = \int_1^0 (x^2+x^2)\,dx + x^2\,dx$$

$$= \int_1^0 3x^2\,dx = \left(x^3\right)_1^0 = 0-1$$

$$= -1$$

$$LHS = \int_C (xy+y^2)\,dx + x^2\,dy = \int_{C_1} (xy+y^2)\,dx + x^2\,dy$$

$$+ \int_{C_2} (xy+y^2)\,dx + x^2\,dy$$

$$= \frac{19}{20} - 1 = \boxed{\frac{-1}{20}}$$

$\therefore$ $LHS = RHS = \frac{-1}{20}$

Hence Green's theorem is verified.

6. Using Green's theorem evaluate $\oint(e^{x^2} - xy)dx - (y^2 - ax)dy$ where C is the circle $x^2 + y^2 = a^2$

$$\int (e^{x^2} - xy)\,dx - (y^2 - ax)\,dy = \int P\,dx + Q\,dy$$

$$P = e^{x^2} - xy \qquad Q = -y^2 + ax$$

$$\frac{\partial P}{\partial y} = -x \qquad \frac{\partial Q}{\partial x} = a$$

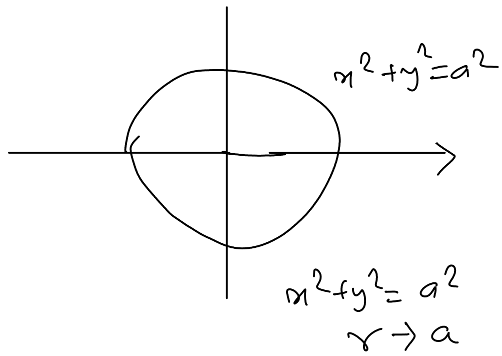
$$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = a - (-x) = a + x$$

By Green's thm

By Green's thm

$$\int_C \left(e^{x^2} - xy\right) dx - \left(y^2 - ax\right) dy = \iint_R (a+x)\, dx\, dy$$

$$\iint_R (a+x)\, dx\, dy$$



Using polar coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta$

$r \to 0$ to a

$\theta \to 0$ to 2π

$$\iint (a+x)\, dx\, dy = \int_0^{2\pi}\int_0^{a} (a + r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{a} \left(ar + r^2\cos\theta\right) dr\, d\theta$$

$$= \int_0^{2\pi} a\left(\frac{r^2}{2}\right)_0^a + \left(\frac{r^3}{3}\right)_0^a \cos\theta\, d\theta$$

$$= \int_0^{2\pi} \frac{a \cdot a^2}{2} + \frac{a^3}{3}\cos\theta\, d\theta$$

$$= a^3 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta$$

$$= a^3 \left[\frac{\theta}{2} + \frac{1}{3}\sin\theta\right]_0^{2\pi}$$

$$= a^3\left(\frac{2\pi}{2} + \frac{1}{3}\sin 2\pi - 0 + \frac{1}{3}\sin 0\right)$$

$$= a^3 \cdot \pi$$