DUALITY

FORMULATION OF DUAL PROBLEM

To obtain dual of the problem, we proceed as follows:

1. We write the LPP as canonical form

2.The maximization problem in the primal becomes minimization problem in the dual and vice versa

3. The "less than or equal to" type of constraints in the primal become "greater than or equal to" constraints in the dual and vice versa

4.The coefficients $c_1, c_2, \dots \dots, c_n$ in the objective function in the primal become the right hand side constants of the constraints of the dual

5. The right hand side constants $b_1, b_2, \dots \dots .., b_m$ of the constraints in the primal become the coefficients in the objective function in the dual and vice versa

FORMULATION OF DUAL PROBLEM

5. If the primal has n decision variables and m constraints then the dual has m decision variables and n constraints and vice versa

6. The transpose of the body matrix (coefficient matrix) in the primal is the body matrix (coefficient matrix) in the dual and vice versa

7. The variables in both the primal and dual are non-negative

Following the above rules we can obtain the dual of the problem

Maximize $z = c_1x_1 + c_2x_2 + \ \ldots.\ldots..+c_nx_n$

Subject to $\quad a_{11}x_1 + a_{12}x_2 + \ \ldots\ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ \ldots\ldots + a_{2n}x_n \leq b_2$

....

....

$a_{m1}x_1 + a_{m2}x_2 + \ \ldots\ldots + a_{mn}x_n \leq b_m$

$x_1, x_2, \ldots\ldots, x_n \geq 0$

as Minimize $w = b_1y_1 + b_2y_2 + \cdots + b_my_m$

Subject to $\quad a_{11}y_1 + a_{21}y_1 + \ \ldots\ldots + a_{m1}y_m \geq c_1$

$a_{12}y_1 + a_{22}y_2 + \ \ldots\ldots + a_{m2}y_m \geq c_2$

....

....

$a_{1n}y_1 + a_{2n}y_2 + \ \ldots\ldots + a_{mn}y_m \geq c_n$

$y_1, y_2, \ldots\ldots, y_m \geq 0$

The above pair of programmes can be written as

Primal	Dual
Maximize $Z = \sum_{j=1}^{n} c_j x_j$	Minimize $W = \sum_{i=1}^{m} b_i y_i$
Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, 2, 3, \ldots m$	Subject to $\sum_{i=1}^{n} a_{ij} y_i \geq c_j$, $j = 1, 2, 3, \ldots n$
Where $x_j \geq 0, j = 1, 2, 3, \ldots n$	Where $y_i \geq 0,\ i = 1, 2, 3, \ldots m$

Construct the dual of the following problem

Minimize $z = x_2 + 3x_3$

Subject to $2x_1 + x_2 \leq 3, x_1 + 2x_2 + 6x_3 \geq 5,$

$$-x_1 + x_2 + 2x_3 = 2, x_1, x_2, x_3 \geq 0$$

Solution: Since the objective function is of "minimization" type, the constraints must be of greater than or equal to" type.

Hence, we multiply the first constraints by (-1) and get $-\ 2x_1, -x_2 \geq 3$

Since, the third constraints is of "equal to" type we write it as

$-x_1 + x_2 + 2x_3 \geq 2$ and $-x_1 + x_2 + 2x_3 \leq 2$ i.e. $-\ (-x_1 + x_2 + 2x_3) \geq -2$

i.e. $x_1 - x_2 - 2x_3 \geq -2$

Thus, the given problem becomes

Minimize $z = 0x_1 + x_2 + 3x_3$

Subject to $\quad -2x_1 - x_2 + 0x_3 \geq -3,$

$$x_1 + 2x_2 + 6x_3 \geq 5, -x_1 + x_2 + 2x_3 \geq 2, x_1 - x_2 - 2x_3 \geq -2,$$

$$x_1, x_2, x_3 \geq 0$$

Since the last given equality type constraint is now expressed in the form of two constraints, we write y_3 as $y_3' - y_3''$

$\therefore$ The dual of the problem is

Maximize $\quad w = -3y_1 + 5y_2 + 2y_3' - 2y_3''$

Subject to $\quad -2y_1 + y_2 - y_3' + y_3'' \leq 0,$

$$-y_1 + 2y_2 + y_3' - y_3'' \leq 1,$$

$$0y_1 + 6y_2 + 2y_3' - 2y_3'' \leq 3$$

Replacing $y_3' - y_3''$ by y_3 which is now unrestricted, the dual becomes

Maximize $\quad w = -3y_1 + 5y_2 + 2y_3$

Subject to $\quad -2y_1 + y_2 - y_3 \leq 0,$

$$-y_1 + 2y_2 + y_3 \leq 1, 6y_2 + 2y_3 \leq 3,$$

$y_1, y_2 \geq 0, y_3$ is unrestricted

Construct the dual of the following LPP

Maximize $z = 3x_1 + 17x_2 + 9x_3$

Subject to $x_1 - x_2 + x_3 \geq 3, -3x_1 + 2x_3 \leq 1,$

$$2x_1 + x_2 - 5x_3 = 1, x_1, x_2, x_3 \geq 0$$

Solution: Since the problem is of maximization type, all the constraints must be expressed in less than or equal to form $(\leq)$.

Hence, we multiply the first constraints by -1 and get

$$-x_1 + x_2 - x_3 \leq 3$$

Since, the third constraint is in the form of an equality, we write it as $2x_1 + x_2 - 5x_3 \geq 1$ and $2x_1 + x_2 - 5x_3 \leq 1$

$\therefore -2x_1 - x_2 + 5x_3 \leq -1$ and $2x_1 + x_2 - 5x_3 \leq 1$

Thus, the given problem becomes,

Maximize $\qquad z = 3x_1 + 17x_2 + 9x_3$

Subject to $\qquad -x_1 + x_2 - x_3 \leq -3,$

$$-3x_1 + 0x_2 + 2x_3 \leq 1, 2x_1 + x_2 - 5x_3 \leq 1,$$

$$-2x_1 - x_2 + 5x_3 \leq -1, x_1, x_2, x_3 \geq 0$$

Since the last constraint in the primal is an equality, y_3 must be unrestricted.

Let $y_1, y_2, y_3', y_3"$ be the associated non-negative variables of the dual. Then the dual is

Minimize $\quad w = -3y_1 + y_2 + y_3' - y_3"$

Subject to $\quad -y_1 - 3y_2 + 2y_3' - 2y_3" \geq 3$

$y_1 + 0y_2 + y_3' - y_3" \geq 17$

$-y_1 + 2y_2 - 5(y_3' - y_3") \geq 19$

Putting $y_3' - y_3" = y_3$ where y_3 is unrestricted, the required dual is

Minimize $w = -3y_1 + y_2 + y_3$

Subject to $-y_1 - 3y_2 + 2y_3 \geq 3, y_1 + y_3 \geq 17,$

$-y_1 + 2y_2 - 5y_3 \geq 19, y_1, y_2 \geq 0, y_3$ is unrestricted

Obtain the dual from the following primal

Minimize $z = x_1 - 3x_2 - 2x_3$

Subject to $3x_1 - x_2 + 2x_3 \leq 7, 2x_1 - 4x_2 \geq 12,$

$-4x_1 + 3x_2 + 8x_3 = 10, x_1, x_2 \geq 0, x_3$ is unrestricted

Solution: First we express the given problem in standard form

Minimize $z = x_1 - 3x_2 - 2x_3$

Subject to $-3x_1 + x_2 - 2x_3 \geq -7, 2x_1 - 4x_2 + 0x_3 \geq 12,$

$-4x_1 + 3x_2 + 8x_3 \geq 10$ and $-4x_1 + 3x_2 + 8x_3 \leq 10$

i.e. $4x_1 - 3x_2 - 8x_3 \geq -10$

Since, x_3 is unrestricted, we put $x_3 = x_3' - x_3''$

$\therefore$ Minimize $z = x_1 - 3x_2 - 2x_3' + 2x_3''$

Subject to

$$-3x_1 + x_2 - 2x_3' + 2x_3'' \geq -7,$$
$$2x_1 - 4x_2 + 0x_3' - 0x_3'' \geq 12,$$
$$-4x_1 + 3x_2 + 8x_3' - 8x_3'' \geq 10,$$
$$4x_1 - 3x_2 - 8x_3' + 8x_3'' \geq -10,$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

If $y_1, y_2, y_3', y_3"$ are dual variables and w is the function of the dual then dual of the given problem will be

$$
\begin{aligned}
\text{Maximize} \quad & w = -7y_1 + 12y_2 + 10y_3' - 10y_3" \\
\text{Subject to} \quad & -3y_1 + 2y_2 - 4y_3' + 4y_3" \leq 1, \\
& y_1 - 4y_2 + 3y_3' - 3y_3" \leq -3, \\
& -2y_1 + 8y_3' - 8y_3" \leq -2, \\
& 2y_1 - 8y_3' + 8y_3" \leq 2,
\end{aligned}
$$

Putting $y_3' - y_3" = y_3$, we get

Maximize $w = -7y_1 + 12y_2 + 10y_3$

Subject to $-3y_1 + 2y_2 - 4y_3 \leq 1,$

$$y_1 - 4y_2 + 3y_3 \leq -3,$$

$$-2y_1 + 8y_3 = -2,$$

$y_1, y_2 \geq 0, y_3$ unrestricted