EFFECT OF MULTIPLICATION BY t

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EFFECT OF MULTIPLICATION BY t :

If $L|f(t)| = \emptyset(s), then\ L|t^n f(t)| = (-1)^n \frac{d^n}{ds^n}\emptyset(s)$

Proof: We shall prove this property by the **method of induction**.

Step 1: Let $\emptyset(s) = L|f(t)| = \int_0^\infty e^{-st} f(t)dt$

Differentiating both sides w.r.t s and applying the rule of differentiation under integral sign

$\emptyset'(s) = \int_0^\infty \frac{\partial}{\partial s}|e^{-st} f(t)dt| = -\int_0^\infty e^{-st} t\, f(t)dt = -L[t\, f(t)]$

$\therefore L|t\, f(t)| = (-1)\frac{d}{ds}\emptyset(s)$

Thus, the rule is true for n = 1.

Step 2: Now we assume the rule is true for $n = m$ and prove that it is true for $n = m + 1$

i.e we assume that $L|t^m f(t)| = (-1)^m \frac{d^m}{ds^m}\emptyset(s)$

$\therefore (-1)^m \frac{d^m}{ds^m}\emptyset(s) = L|t^m f(t)| = \int_0^\infty e^{-st} t^m f(t)dt$

Differentiating both sides w.r.t. s and applying the rule of differentiation under the integral sign.

$$(-1)^m \frac{d^{m+1}}{ds^{m+1}}\emptyset(s) = \int_0^\infty \frac{\partial}{\partial s}|e^{-st}.t^m f(t)dt|$$

$$= -\int_0^\infty e^{-st}.t^{m+1} f(t)dt$$

$$= -L|t^{m+1} f(t)|$$

$$\therefore L|t^{m+1} f(t)| = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\emptyset(s)$$

Thus, if the property is true form $n = m$ then it is true for $n = m + 1$.

It is true for any value of n.

Note: In particular, if $L\ f(t) = \emptyset(s)$, then $L|t\ f(t)| = -\emptyset'(s), L|t^2 f(t)| = \emptyset''(s)$

Ex :- Find $L[t\, e^{-t}\cosh 2t]$

Solution :- $e^{-t}\cosh 2t = e^{-t}\left[\frac{e^{2t}+e^{-2t}}{2}\right] = \frac{e^{t}+e^{-3t}}{2}$

$$\therefore L(e^{-t}\cosh 2t) = \frac{1}{2}\left[L(e^{t}) + L(e^{-3t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right]$$

$$\therefore L[t\, e^{-t}\cosh 2t] = -\frac{d}{ds}\cdot\frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2}\right]$$

Ex :- Find $L[(1+te^{-t})^3]$

Solution :- $L[(1+te^{-t})^3] = L[1+3te^{-t} + 3t^2e^{-2t} + t^3e^{-3t}]$

$$= L[1] + 3L[te^{-t}] + 3L[t^2e^{-2t}] + L[t^3e^{-3t}]$$

$$= \frac{1}{s} - 3\frac{d}{ds}[L(e^{-t})] + 3\frac{d^2}{ds^2}[L(e^{-2t})] - \frac{d^3}{ds^3}[L(e^{-3t})]$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left[L\left(e^{-t}\right)\right] + 3\frac{d^2}{ds^2}\left[L\left(e^{-2t}\right)\right] - \frac{d^3}{ds^3}\left[L\left(e^{-3t}\right)\right]$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left[\frac{1}{s+1}\right] + 3\frac{d^2}{ds^2}\left[\frac{1}{s+2}\right] - \frac{d^3}{ds^3}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Ex :- Find $L\left[t\, e^{-4t} \sin 3t\right]$

Solution

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L[t \sin 3t] = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$ (using multiplication by t property)

$$= \frac{6s}{(s^2+9)^2}$$

$$L\left[e^{-4t}\, t \sin 3t\right] = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2}$$ [using First shifting]

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

Ex :- Find $L\left[t^5 \cosh t\right]$

Solution **First method**

$$L\left[t^5 \cosh t\right] = L\left[t^5\left(\frac{e^t+e^{-t}}{2}\right)\right]$$

$$= \frac{1}{2} L\left[t^5 e^t + t^5 e^{-t}\right]$$

$$= \frac{1}{2}\left\{-\frac{d^5}{ds^5} L\left(e^t\right) - \frac{d^5}{ds^5} L\left(e^{-t}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{d^5}{ds^5}\left(\frac{1}{s-1}\right) - \frac{d^5}{ds^5}\left(\frac{1}{s+1}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right] = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

2nd method

$$L\left[t^5 \cosh t\right] = L\left[t^5\left(\frac{e^t+e^{-t}}{2}\right)\right]$$

$$= \frac{1}{2} L\left[e^{t} t^{5} + e^{-t} t^{5}\right]$$

But $L[t^5] = \frac{5!}{s^6}$ and using first shifting property

$$L\left[t^5 \cosh t\right] = \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right] = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

Ex :- Find $L\left[t\sqrt{1+\sin t}\right]$

Solution :- we have $\sqrt{1+\sin t} = \sqrt{\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right) + 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$

$$= \sqrt{\left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right)^2}$$

$$= \sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)$$

$$\therefore L\left[\sqrt{1+\sin t}\right] = L\left[\sin\left(\frac{t}{2}\right)\right] + L\left[\cos\left(\frac{t}{2}\right)\right]$$

$$= \frac{1/2}{s^2 + (1/2)^2} + \frac{s}{s^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{2}{4s^2+1} + \frac{4s}{4s^2+1} = \frac{2(2s+1)}{(4s^2+1)}$$

Now using multiplication by t property

$$L\left[t\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{(4s^2+1)}\right]$$

$$= -2\left[\frac{(4s^2+1)\cdot 2 - (2s+1)(8s)}{(4s^2+1)^2}\right]$$

$$\therefore L\left[t\sqrt{1+\sin t}\right] = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Ex :- Find $L\left[t e^{3t} \operatorname{erf}\sqrt{t}\right]$

Solution :- $L[\operatorname{erf}\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$

using multiplication by t

$$L[t \operatorname{erf}\sqrt{t}] = -\frac{d}{ds}\left[\frac{1}{s\sqrt{s+1}}\right]$$

$$= -\left[\frac{-1}{s^2(s+1)} \cdot \frac{d}{ds}\left(s\sqrt{s+1}\right)\right]$$

$$= \frac{1}{s^2(s+1)}\left[s \cdot \frac{1}{2\sqrt{s+1}} + \sqrt{s+1}\right]$$

$$= \frac{1}{s^2(s+1)}\left[\frac{s+2(s+1)}{2\sqrt{s+1}}\right]$$

$$L[t \operatorname{erf}\sqrt{t}] = \frac{3s+2}{2s^2(s+1)^{3/2}}$$

Now using First shifting theorem

$$L[e^{3t} t \operatorname{erf}\sqrt{t}] = \frac{3(s-3)+2}{2(s-3)^2(s-3+1)^{3/2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$$

Ex :- Find Laplace Transform of $t\left(\dfrac{\sin t}{e^t}\right)^2$

Solution $f(t) = t\left(\dfrac{\sin t}{e^t}\right)^2 = t\, e^{-2t}\sin^2 t$

$$= t e^{-2t}\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2} t e^{-2t}(1-\cos 2t)$$

Now $L(1-\cos 2t) = L(1) - L(\cos 2t)$

$$= \frac{1}{s} - \frac{s}{s^2+4}$$

$\therefore$ By multiplication by t

$$L[t(1-\cos 2t)] = -\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= -\left[\frac{-1}{s^2} - \frac{(s^2+4)\cdot 1 - s(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-1}{s^2} - \frac{(s^2+4)\cdot 1 - s(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-1}{s^2} - \frac{4-s^2}{(s^2+4)^2}\right] = \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$$

Now using first shifting theorem

$$L\left[e^{-2t}\, t\, \sin^2 t\right] = \frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{\left[(s+2)^2+4\right]^2}$$

$$L\left[e^{-2t}\, t\, \sin^2 t\right] = \frac{1}{(s+2)^2} - \frac{s^2+4s}{(s^2+4s+8)^2}$$

Ex:- Find $L\left[\left(t+e^{-t}+\sin t\right)^2\right]$

Solution :- $(t+e^{-t}+\sin t)^2 = t^2 + e^{-2t} + \sin^2 t + 2te^{-t} + 2t\sin t + 2e^{-t}\sin t$

$$\therefore L\left[\left(t+e^{-t}+\sin t\right)^2\right] = L\left[t^2\right] + L\left[e^{-2t}\right] + L\left[\frac{1-\cos 2t}{2}\right] + 2L\left[te^{-t}\right] + 2L\left[t\sin t\right] + 2L\left[e^{-t}\sin t\right]$$

$$= \frac{2!}{s^3} + \frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) + \frac{2}{(s+1)^2} - 2\frac{d}{ds}\left(\frac{1}{s^2+1}\right) + 2\left(\frac{1}{(s+1)^2+1}\right)$$

[using standard formulae, first shifting and multiplication by t property]

$$= \frac{2}{s^3} + \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] + \frac{2}{(s+1)^2} + \frac{4s}{(s^2+1)^2} + \frac{2}{(s^2+2s+2)}$$
