EVALUATION OF PARTICULAR INTEGRAL

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Laplace Transforms can be very useful for solving integrals.

Ex-1 :- Evaluate $\int_0^\infty e^{-2t} \sin^3 t \, dt$

Solution :- $L[\sin^3 t] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$

$$= \frac{3}{4} L(\sin t) - \frac{1}{4} L(\sin 3t)$$

$$= \frac{3}{4} \cdot \frac{1}{s^2+1} - \frac{1}{4} \cdot \frac{3}{s^2+9} = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

This means that $\int_0^\infty e^{-st} \sin^3 t \, dt = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$

Now put $s = 2$

$$\therefore \int_0^\infty e^{-2t} \sin^3 t \, dt = \frac{3}{4}\left[\frac{1}{5} - \frac{1}{13}\right] = \frac{3}{4}\left[\frac{8}{65}\right] = \frac{6}{65}$$

Ex-2 If $\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\, dt = \frac{1}{4}$, then find α

Solution :- $\sin(t+\alpha)\cos(t-\alpha) = \frac{1}{2}\left[\sin(2t) + \sin(2\alpha)\right]$

$$\therefore L[\sin(t+\alpha)\cos(t-\alpha)] = \frac{1}{2}\left[L(\sin 2t) + L(\sin(2\alpha))\right]$$

$$= \frac{1}{2} \cdot \frac{2}{s^2+4} + \sin 2\alpha \left(\frac{1}{s}\right)$$

$$= \frac{1}{s^2+4} + \frac{1}{2}\sin 2\alpha \left(\frac{1}{s}\right)$$

This means that $\int_0^\infty e^{-st} \sin(t+\alpha)\cos(t-\alpha)\, dt = \frac{1}{s^2+4} + \frac{\sin 2\alpha}{2}\left(\frac{1}{s}\right)$

put $s = 2$

$$\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\, dt = \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

$$\int_0 e \sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{1}{8} + \frac{1}{4}\sin \ldots$$

but it is given that this value is $\frac{1}{4}$

$$\therefore \frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{1}{4} \Rightarrow \frac{1}{4}\sin 2\alpha = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

$$\Rightarrow \sin 2\alpha = \frac{1}{2}$$

$$\Rightarrow 2\alpha = \frac{\pi}{6} \Rightarrow \boxed{\alpha = \frac{\pi}{12}}$$

Ex-3 Evaluate $\int_0^\infty e^{-3t}\, t \sin 3t \, dt$

Solution :- $L(\sin 3t) = \frac{3}{s^2+9}$

$$L(t \sin 3t) = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right] = \frac{6s}{(s^2+9)^2}$$

This means that $\int_0^\infty e^{-st}\, t \sin 3t \, dt = \frac{6s}{(s^2+9)^2}$

put $s = 3$

$$\int_0^\infty e^{-3t}\, t \sin 3t \, dt = \frac{18}{(18)^2} = \frac{1}{18}$$

Ex-4 Evaluate $\int_0^\infty \frac{e^{-2t} - e^{-3t}}{t}\, dt$

Solution $L\left[e^{-2t} - e^{-3t}\right] = \frac{1}{s+2} - \frac{1}{s+3}$

$$\therefore L\left[\frac{e^{-2t} - e^{-3t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+2} - \frac{1}{s+3}\right) ds = \log\left(\frac{s+2}{s+3}\right)\Big/_s^\infty$$

$$= \log\left(\frac{s+3}{s+2}\right)$$

By definition of Laplace Transform,

$$\int_0^\infty e^{-st}\left(\frac{e^{-2t} - e^{-3t}}{t}\right) dt = \log\left(\frac{s+3}{s+2}\right)$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-2t}-e^{-3t}}{t}\right)dt = \log\left(\frac{s+3}{s+2}\right)$$

put $s=0$

$$\int_0^\infty \frac{e^{-2t}-e^{-3t}}{t}\,dt = \log\left(\frac{3}{2}\right)$$

Ex-5 Prove that $\displaystyle\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)dt = \frac{1}{4}\log 5$

Solution $\displaystyle \sin^2 t = \frac{1-\cos 2t}{2}$

$$\therefore L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds\right]$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$\Rightarrow \int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t}\right)dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

put $s=1$

$$\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)dt = \frac{1}{4}\log(5)$$ Hence proved.

Ex-6 Prove that $\displaystyle\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right)dt = \frac{3\pi}{4}$

Solution $$\int_0^\infty \frac{\sin 2t + \sin 3t}{te^t}\,dt = \int_0^\infty e^{-t}\left(\frac{\sin 2t+\sin 3t}{t}\right)dt$$

$$= L\left[\frac{\sin 2t + \sin 3t}{t}\right]$$

$$= L\left[\frac{\sin 2t + \sin 3t}{t}\right] \text{ at } s=1$$

$$L[\sin 2t + \sin 3t] = \frac{2}{s^2+4} + \frac{3}{s^2+9}$$

$$L\left[\frac{\sin 2t + \sin 3t}{t}\right] = \int_s^{\infty}\left(\frac{2}{s^2+4} + \frac{3}{s^2+9}\right) ds = \left[\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right) + \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \pi - \tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{3}\right)$$

$$\therefore \int_0^{\infty} e^{-t}\left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \left[\pi - \tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{3}\right)\right]_{s=1}$$

$$= \pi - \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)$$

$$= \pi - \left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \pi - \left[\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\cdot\frac{1}{3}\right)}\right)\right]$$

$$= \pi - \tan^{-1}\left(\frac{5/6}{1-1/6}\right) = \pi - \tan^{-1}(1)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4} \qquad \text{Hence proved}$$

<u>Ex-7</u> Evaluate $\int_0^{\infty} e^{-2t}\left(\int_0^t \frac{e^{-u}\sin u}{u} du\right) dt$

<u>Solution</u> :- $L[\sin u] = \frac{1}{s^2+1}$

$$L\left[\frac{\sin u}{u}\right] = \int_s^{\infty} \frac{1}{s^2+1} ds = \left[\tan^{-1}(s)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s)$$

$$L\left[\frac{e^{-u}\sin u}{u}\right] = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$L\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right] = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right]$$

This means that $\displaystyle\int_0^\infty e^{-st}\left(\int_0^t \frac{e^{-u}\sin u}{u}\,du\right)dt = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right]$

put $s=2$

$$\int_0^\infty e^{-2t}\left(\int_0^t \frac{e^{-u}\sin u}{u}\,du\right)dt = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(3)\right]$$

$$= \frac{1}{2}\cot^{-1}(3)$$

Ex-8 Evaluate $\displaystyle\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\,du\right)dt$

Solution $L[\sinh u \cosh u] = \frac{1}{2} L[\sinh 2u]$

$$= \frac{1}{2}\left[\frac{2}{s^2-4}\right]$$

$$L[u^2 \sinh u \cosh u] = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{2}{s^2-4}\right]$$

$$= \frac{1}{2}\frac{d}{ds}\left[\frac{-4s}{(s^2-4)^2}\right] = -2\frac{d}{ds}\left[\frac{s}{(s^2-4)^2}\right]$$

$$= -2\left[\frac{(s^2-4)^2\cdot(1) - s\cdot 2(s^2-4)(2s)}{(s^2-4)^4}\right]$$

$$= -2\left[\frac{s^2-4-4s^2}{(s^2-4)^3}\right] = \frac{2(4+3s^2)}{(s^2-4)^3}$$

$$L\left[\int_0^t u^2 \sinh u \cosh u\,du\right] = \frac{2(4+3s^2)}{s(s^2-4)^3}$$

This means that $\int_0^\infty e^{-st}\left(\int_0^t u^2 \sinh u \cosh u \, du\right) dt = \frac{2(4+3s^2)}{s(s^2-4)^3}$

put $s=1$

$$\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u \, du\right) dt = \frac{2(4+3)}{(1)(1-4)^3}$$

$$= \frac{14}{-27}$$