Lagrangian Method

NLPP with equality Constraint

OPTIMIZATION WITH EQUALITY CONSTRAINTS

A general non-linear programming problem in which the object function is non linear but the constraints are linear and in the form of equalities takes the following form

Optimize $z = f(x_1, x_2, \dots \dots, x_n)$

Subject to $g_1(x_1, x_2, \dots \dots, x_n) = b_1,$

$$g_2(x_1, x_2, \dots \dots, x_n) = b_2,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .$$

$$g_m(x_1, x_2, \dots \dots, x_n) = b_m,$$

$$x_1, x_2, \dots \dots, x_n \geq 0$$

OPTIMIZATION WITH EQUALITY CONSTRAINTS

where, $f(x_1, x_2, \dots \dots, x_n)$ is a non linear function and $g_1(x_1, x_2, \dots \dots, x_n), g_2(x_1, x_2, \dots \dots, x_n), \dots ., g_m(x_1, x_2, \dots \dots, x_n)$ are linear functions and $m < n$

The problem of this type is solved by forming what is called Lagrangian Function with Lagrange's Multiplier λ

NON-LINEAR PROGRAMMING PROBLEM WITH 2 VARIABLES AND 1 EQUALITY CONSTRAINT

Optimize $\boldsymbol{z = f(x_1, x_2)}$
Subject to $\boldsymbol{g(x_1, x_2) = b_2;\ x_1, x_2 \geq 0}$

We first express the constraints with RHS equal to zero

Then the problem becomes,

Optimize $z = f(x_1, x_2)$

Subject to $h(x_1, x_2) = g(x_1, x_2) - b = 0$

$$x_1, x_2 \geq 0$$

We now construct a new function called Lagrangian function, using the Lagrangian multiplier λ

$$\therefore L(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda h(x_1, x_2) \quad \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

The necessary conditions for maxima or minima subjected to the condition $h(x_1, x_2) = 0$ are

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0 \qquad \ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Now from (1) we get,

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}; \ \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}; \ \frac{\partial L}{\partial \lambda} = -h \ldots\ldots\ldots \text{(3)}$$

Using (2) we get from (3) the following three necessary conditions,

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0;$$
$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0;$$
$$h(x_1, x_2) = 0$$

Now, we should just solve these 3 conditions to get x_1, x_2 and λ.

Thus, the point of maxima or minima can be obtained

- To determine whether the point obtained above is maxima or a minima, we consider the following determinant .
- $\Delta_3 =$

$$\begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \left(\frac{\partial^2 f}{\partial {x_1}^2} - \lambda \frac{\partial^2 h}{\partial {x_1}^2}\right) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2}\right) \\ \frac{\partial h}{\partial x_2} & \left(\frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1}\right) & \left(\frac{\partial^2 f}{\partial {x_2}^2} - \lambda \frac{\partial^2 h}{\partial {x_2}^2}\right) \end{vmatrix}$$

If Δ_3 is positive, then X_0 is a maxima and if Δ_3 is negative, then X_0 is a minima

EXAMPLE-1

Using the method of Lagrange's multipliers, solve the following NLPP

Optimize $z = 6{x_1}^2 + 5{x_2}^2$

Subject to $x_1 + 5x_2 = 7,$

$$x_1, x_2 \geq 0$$

$\lambda =$ Lagrange's multiplier

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2)$$

$$f(x_1, x_2) = 6{x_1}^2 + 5{x_2}^2$$

$$h(x_1, x_2) = x_1 + 5x_2 - 7$$

$$\therefore L(x_1, x_2, \lambda) = (6{x_1}^2 + 5{x_2}^2) - \lambda(x_1 + 5x_2 - 7)$$

$$\frac{\partial L}{\partial x_1} = 0 = 12x_1 - \lambda \qquad \text{..............} \text{ (a)}$$

$$\frac{\partial L}{\partial x_2} = 0 = 10x_2 - 5\lambda \qquad \text{..............} \text{ (b)}$$

$$\frac{\partial L}{\partial \lambda} = 0 = x_1 + 5x_2 - 7 \qquad \text{..............} \text{ (c)}$$

Solving (a), (b) and (c) we get

$$x_1 = \frac{7}{31}, x_2 = \frac{42}{31}, \lambda = \frac{84}{31}$$

Here, $X_0(x_1, x_2) = X_0\left(\frac{7}{31}, \frac{42}{31}\right)$

$$\Delta_3 = \begin{vmatrix} 0 & \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} \\ \dfrac{\partial h}{\partial x_1} & \left(\dfrac{\partial^2 f}{\partial {x_1}^2} - \lambda \dfrac{\partial^2 h}{\partial {x_1}^2}\right) & \left(\dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_2}\right) \\ \dfrac{\partial h}{\partial x_2} & \left(\dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_1}\right) & \left(\dfrac{\partial^2 f}{\partial {x_2}^2} - \lambda \dfrac{\partial^2 h}{\partial {x_2}^2}\right) \end{vmatrix}$$

- $h(x_1, x_2) = x_1 + 5x_2 - 7$
- $\therefore \frac{\partial h}{\partial x_1} = 1; \frac{\partial^2 h}{\partial {x_1}^2} = 0$ and $\frac{\partial h}{\partial x_2} = 5; \frac{\partial^2 h}{\partial {x_2}^2} = 0$
- $f(x_1, x_2) = 6{x_1}^2 + 5{x_2}^2$
- $\therefore \frac{\partial f}{\partial x_1} = 12x_1; \frac{\partial^2 f}{\partial {x_1}^2} = 12$ and $\frac{\partial f}{\partial x_2} = 10x_2; \frac{\partial^2 f}{\partial {x_2}^2} = 10$
- $\frac{\partial^2 f}{\partial {x_1}^2} - \lambda \frac{\partial^2 h}{\partial {x_1}^2} = 12; \qquad \frac{\partial^2 f}{\partial {x_2}^2} - \lambda \frac{\partial^2 h}{\partial {x_2}^2} = 10$
- $\frac{\partial^2 h}{\partial x_1 \partial x_2} = 0; \qquad \frac{\partial^2 h}{\partial x_2 \partial x_1} = 0$
- $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0; \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$
- $\frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} = 0$ and $\frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} = 0$

We have $\Delta_3 = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{vmatrix}$

$= -310$

Since Δ_3 is negative, X_0 is a minima

$\therefore z_{min} = 6{x_1}^2 + 5{x_2}^2$

$= 6\left(\frac{7}{31}\right)^2 + 5\left(\frac{42}{31}\right)^2$

$= 9.48$

EXAMPLE-2

Using the method of Lagrange's multipliers, solve the following NLPP

Optimize $z = 4x_1 + 8x_2 - {x_1}^2 - {x_2}^2$

Subject to $x_1 + x_2 = 4,$

$$x_1, x_2 \geq 0$$

$$L(x_1, x_2, \lambda) = (4x_1 + 8x_2 - {x_1}^2 - {x_2}^2) - \lambda(x_1 + x_2 - 4)$$

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0,$$

$$4 - 2x_1 - \lambda = 0, 8 - 2x_2 - \lambda = 0, x_1 + x_2 = 4$$

Hence, X_0 is $(1, 3)$

Now,

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \left(\frac{\partial^2 f}{\partial {x_1}^2} - \lambda \frac{\partial^2 h}{\partial {x_1}^2}\right) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2}\right) \\ \frac{\partial h}{\partial x_2} & \left(\frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1}\right) & \left(\frac{\partial^2 f}{\partial {x_2}^2} - \lambda \frac{\partial^2 h}{\partial {x_2}^2}\right) \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 4$$

Since, Δ_3 is positive, X_0 is a maxima

Hence, $x_1 = 1, x_2 = 3, z_{max} = 18$

In case of more than 2 variables

If the signs of the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots..$ are alternatively positive and negative, i.e. $\Delta_3 > 0, \Delta_4 < 0, \Delta_5 > 0, \ldots.$ etc. then the point X_0 is a maxima

If all the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots\ldots, \Delta_{n+1}$ are negative i.e. $\Delta_3 < 0, \Delta_4 < 0, \ldots\ldots, \Delta_{n+1} < 0 \ldots..$ etc then the point X_0 is a minima

EXAMPLE-3

Using the method of Lagrange's multipliers solve

Optimize

$$z = {x_1}^2 + {x_2}^2 + {x_3}^2 - 10x_1 - 6x_2 - 4x_3$$

Subject to

$$x_1 + x_2 + x_3 = 7, x_1, x_2, x_3 \geq 0$$

$$\begin{aligned} L(x_1, x_2, x_3, \lambda) &= ({x_1}^2 + {x_2}^2 + {x_3}^2 - 10x_1 - 6x_2 - 4x_3) \\ &- \lambda(x_1 + x_2 + x_3 - 7) \end{aligned}$$

$$\frac{\partial L}{\partial x_1} = 0 = 2x_1 - 10 - \lambda \qquad \text{……………. (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 = 2x_2 - 6 - \lambda \qquad \text{……………. (2)}$$

$$\frac{\partial L}{\partial x_3} = 0 = 2x_3 - 4 - \lambda \qquad \text{……………. (3)}$$

$$\frac{\partial L}{\partial \lambda} = 0 = -(x_1 + x_2 + x_3 - 7) \qquad \text{……………. (4)}$$

On solving we get

$$(x_1, x_2, x_3) = (4, 2, 1)$$

Now, Δ_4

$$\begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \left(\frac{\partial^2 f}{\partial {x_1}^2} - \lambda \frac{\partial^2 h}{\partial {x_1}^2}\right) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2}\right) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3}\right) \\ \frac{\partial h}{\partial x_2} & \left(\frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1}\right) & \left(\frac{\partial^2 f}{\partial {x_2}^2} - \lambda \frac{\partial^2 h}{\partial {x_2}^2}\right) & \left(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3}\right) \\ \frac{\partial h}{\partial x_3} & \left(\frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1}\right) & \left(\frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2}\right) & \left(\frac{\partial^2 f}{\partial {x_3}^2} - \lambda \frac{\partial^2 h}{\partial {x_3}^2}\right) \end{vmatrix}$$

where $f = {x_1}^2 + {x_2}^2 + {x_3}^2 - 10x_1 - 6x_2 - 4x_3$ and
$h = x_1 + x_2 + x_3 - 7$

We have $$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

By $C_2 - C_4, C_3 - C_4$, we get,

$$\Delta_4 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -2 & -2 & 2 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & -2 \end{vmatrix} = -12$$

Δ_4 is negative.

Hence finding Δ_3 $$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -4$$

Since both Δ_3 and Δ_4 are negative, X_0 is a minima

$$X_0(x_1, x_2, x_3) = X_0(4, 2, 1)$$

$$z_{min} = 16 + 4 + 1 - 40 - 12 - 4 = -35$$

EXAMPLE-4

Using the method of Lagrange's multipliers solve the following NLPP

Optimize

$$z = 12x_1 + 8x_2 + 6x_3 - {x_1}^2 - {x_2}^2 - {x_3}^2 - 23$$

subject to $x_1 + x_2 + x_3 = 10, x_1, x_2, x_3 \geq 0$

$$L(x_1, x_2, x_3, \lambda) = (12x_1 + 8x_2 + 6x_3 - {x_1}^2 - {x_2}^2 - {x_3}^2 - 23) - \lambda(x_1 + x_2 + x_3 - 10)$$

On solving we will get

$$X_0(x_1, x_2, x_3) = X_0(5, 3, 2)$$

$\Delta_4 =$

$$\begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \left(\frac{\partial^2 f}{\partial {x_1}^2} - \lambda \frac{\partial^2 h}{\partial {x_1}^2}\right) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2}\right) & \left(\frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3}\right) \\ \frac{\partial h}{\partial x_2} & \left(\frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1}\right) & \left(\frac{\partial^2 f}{\partial {x_2}^2} - \lambda \frac{\partial^2 h}{\partial {x_2}^2}\right) & \left(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3}\right) \\ \frac{\partial h}{\partial x_3} & \left(\frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1}\right) & \left(\frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2}\right) & \left(\frac{\partial^2 f}{\partial {x_3}^2} - \lambda \frac{\partial^2 h}{\partial {x_3}^2}\right) \end{vmatrix}$$

- $\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix}$

By $C_2 - C_4, C_3 - C_4$, we get,

$$\Delta_4 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 2 & -2 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & 2 \end{vmatrix}$$

- By $C_2 + 2C_1$, $\Delta_4 = (-1)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -2 \\ 1 & 4 & 2 \end{vmatrix}$

$$= (-1)(4+8) = -12$$

- Now, $\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$
- Since Δ_3 is positive and Δ_4 is negative, $X_0(5,3,2)$ is a maxima

$\therefore x_1 = 5, x_2 = 3, x_3 = 2$ and

$$\therefore z_{max} = 35$$