Relations, Digraphs

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RELATIONS, DIGRAPHS (07)

- 3.1 Relations, Paths and Digraphs
- 3.2 Properties and types of binary relations
- 3.3 Manipulation of relations, Closures, Warshall‟s algorithm
- 3.4 Equivalence relations

Introduction

Definition: A binary relation from a set A to a set B is a subset $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B\}$

- Let A and B be nonempty sets. A relation R from A 'to' B is a subset of $A \times B$.
- If $R \subseteq A \times B$ and $(a, b) \in R$, we say that a 'is related to' b by R, and we also write a R b.
- If a is not related to b by R, we write $a \not R b$.
- Frequently, A and B are equal. In this case, we often say that $R \subseteq A \times A$ 'is a relation on' A, instead of a relation from A to A.

Examples

Ex. 1 :

Let $A = \{1, 2, 3\}$ and $B = \{r, s\}$

Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B.

Ex. 2 :

Let $A = \{1, 2, 3, 4, 5\}$.

Define the following relation R (less than) on A :

a R b if and only if $a < b$.

Then $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

DEFINITIONS

- Let { $A1, A2, \ldots, An$ } be a finite collection of sets. A subset R of $A1 \times A2 \times \ldots \times An$ is called an **n-ary relation** on A1, A2, …, An.
- If $R=\phi$, then R is called **void** or **empty relation**.
- If $R=A1 \times A2 \times \ldots \times An$, then R is called the **universal** relation.
- If Ai =A for all i, then R is called an 'n - ary relation on A'.
- If n=1, 2 or 3, then R is called a **unary**, **binary** or **ternary** relation respectively.
- Among the relations, binary relations are the most important being widely used in various applications.

Set Arising from Relations

Domain of Relation R :

Let R $\subseteq$ A $\times$ B be a relation from A to B. The **domain** of R, denoted by **Dom (R)**, is the set of elements in A that are related to some element in B. In other words, Dom (R), a subset of A, is the set of all first elements in the pairs that make up R.

Range of relation R :

Similarly, we define the **range** of R, denoted by **Ran (R)**, to be the set of elements in B that are second elements of pairs in R, that is, all elements in B that are related to some element in A.

Examples

Ex. 1 :

Let A={ 1, 2, 3 } , B= { r, s }

and R={(1, r), (2, s), (3, r)}

Dom (R)={ 1, 2, 3 } = A

Ran (R)={ r, s } = B

Ex. 2 :

Let A={ 1, 2, 3, 4, 5 } , B={ 1, 2, 3, 4 , 5 }

a R b, if and only if a < b

R={(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

Dom (R)={1, 2, 3, 4 }

Ran (R)={2, 3, 4, 5}

Representation of Relation

Graphical, Tabular and Matrix forms :

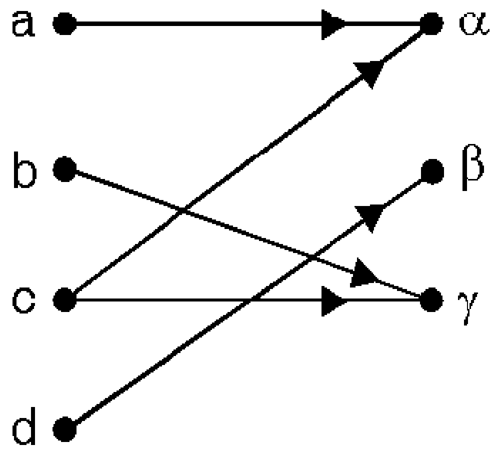
For Example :

Let $A=\{a, b, c, d\}$, $B=\{\alpha, \beta, \gamma\}$

and R is a relation from A to B.

$R=\{(a, \alpha), (b, \gamma), (c, \alpha), (c, \gamma), (d, \beta)\}$

	α	β	γ
a	√		
b			√
c	√		√
d		√	



$$M_R=\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$$

Representation of Relation

Diagraph

If A is a finite set and R is a relation on A, we can also represent R pictorially as follows :

(i) Draw a small circle for each element of A and label the circle with the corresponding element of A. These circles are called **vertices**.

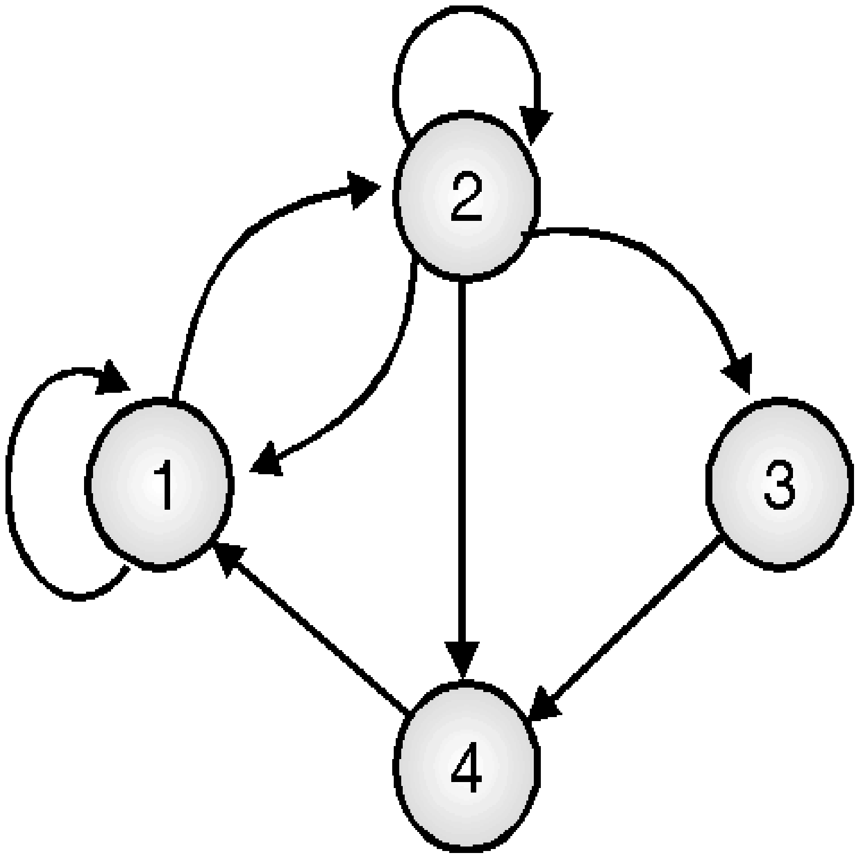
(ii) Draw an arrow, called an **edge**, from vertex ai to vertex aj if and only if ai R aj.

The resulting pictorial representation of R is called a **directed graph** or **digraph** of R.

DIAGRAPH

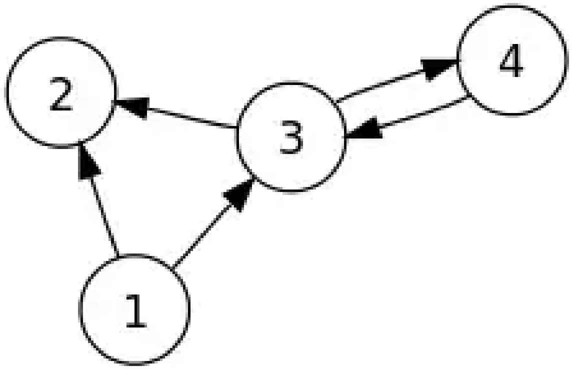
Ex. 1 : Let A = {1, 2, 3, 4}, Let R is a relation from A to A.

R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)}



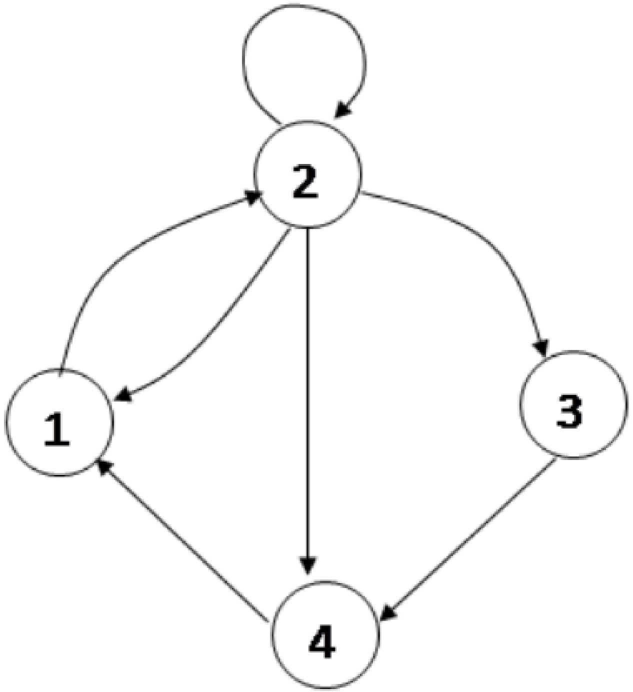
Degree of Vertex in a Directed Graph

- A directed graph, each vertex has an in-egree and an out-degree.
- In-degree of a Graph-Number of edges which are coming into the vertex V.
- Out-degree of a Graph-Number of edges which are going out from the vertex V



VERTEX	1	2	3	4
In Degree	0	2	2	1
Out-degree	2	0	2	1

Find out in degree and out degree



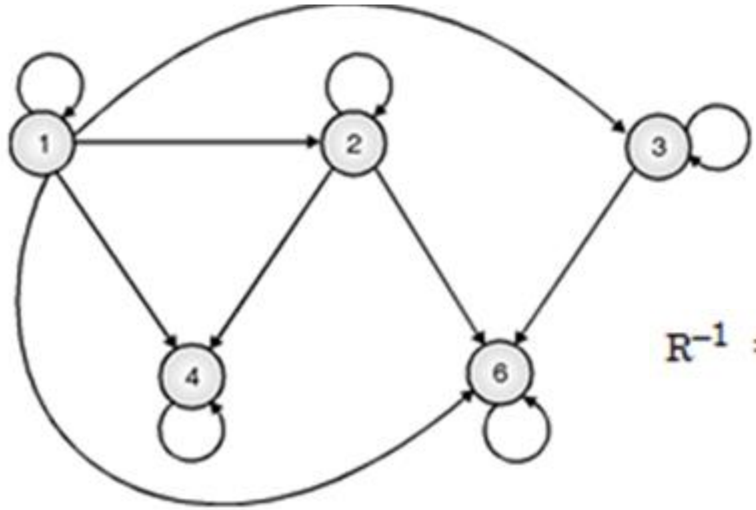
VERTEX	1	2	3	4
In Degree	2	2	1	2
Out-degree	1	4	1	1

Example

Let A = {1, 2, 3, 4, 6} and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find Matrix of R. Find inverse relation of R.

Soln.: R = {(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (6,6)} (4,4)

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



R^{-1} = {(1,1), (2,1) (3,1) (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (6,6) (4,4)

Obtain $M_{R^{-1}}$

EXAMPLE

Let A = {1, 2, 3, 4, 6} = B, a R b if and only if a is a multiple of b. Find R and draw the digraph of R. Find Matrix of R. Find each of the following :

(i) R(3) (ii) R(6) (iii) R({2, 4, 6})

Solution:

R={(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4,1), (4, 2), (4, 4), (6, 1), (6, 2), (6, 3), (6, 6)}

Dom (R) ={1, 2, 3, 4, 6}

Ran (R) ={1, 2, 3, 4, 6}

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

(i) R (3) ={1, 3, 6},

Since (3, 1) ∈ R, (3, 3) ∈ R and (6, 3) ∈ R

(ii) R (6) ={1, 2, 3, 6},

Since (6, 1) ∈ R, (6, 2) ∈ R, (6, 3) ∈ R and (6,6) ∈ R.

(iii) R ({2, 4, 6}) = {1, 2, 4, 3, 6}

Since (2, 1) ∈ R, (4, 2) ∈ R, (6,1) ∈ R, (6,2) ∈ R, (6, 3) ∈ R, (6, 6) ∈ R, (4,4) ∈ R

Paths in Relations and Digraphs

Suppose that R is a relation on a set A. A **path of length n** in R from a to b is a finite sequence π : a, x_1, x_2, … , x_{n-1}, b, beginning with a and ending with b, such that

a R x_1, x_1 R x_2, ………………., x_{n-1} R b

Note that a path of length n involves n + 1 elements of A, although they are not necessarily distinct.

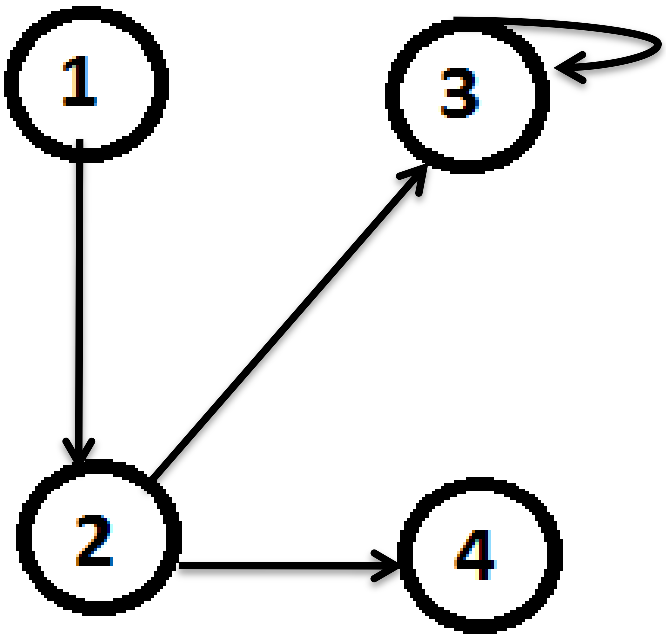
The **length** of a path is the number of edges in the path, where the vertices need not all be distinct.

A path that begins and ends at the same vertex is called a **cycle**.

Paths in Relations and Digraphs

R = { (1 , 2) ,(2 , 3) , (2 , 4), (3 , 3) } is a relation on A = {1,2,3,4}

$R^1 = R = \{(1,2),(2,3),(2,4), (3,3)\}$



$R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$

1 R^2 3 Since: **1 R 2 and 2 R 3**

1 R^2 4 Since: **1 R 2 and 2 R 4**

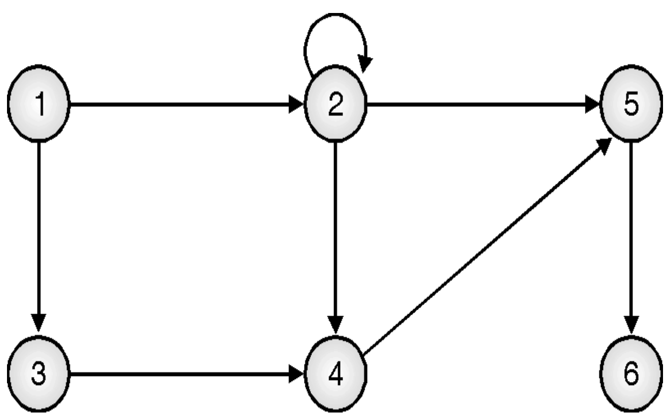
$R^3 = \{ (1 , 3) , (2 , 3), (3 , 3) \}$

$R^4 = \{ (1 , 3) , (2 , 3), (3 , 3)\}$

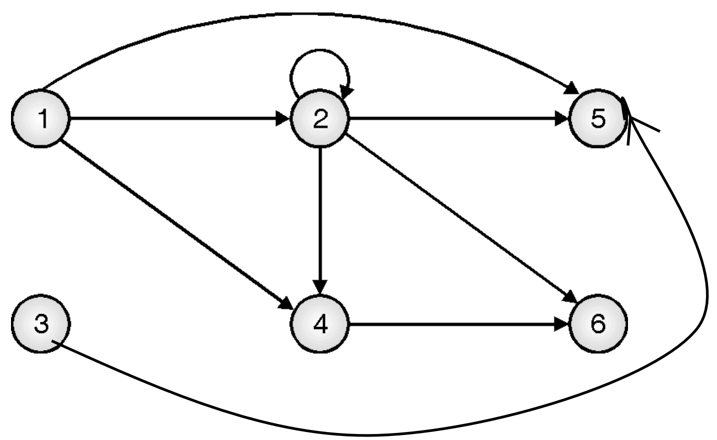
Obtain R^∞

Paths in Relations and Digraphs

Let A = {1, 2, 3, 4, 5, 6}. Let R be the relation whose digraph is shown in Fig. Find R^2 and draw digraph of the relation R^2.



$1\ R^2\ 2$	Since	$1\ R\ 2$	and	$2\ R\ 2$
$1\ R^2\ 4$	Since	$1\ R\ 2$	and	$2\ R\ 4$
$1\ R^2\ 5$	Since	$1\ R\ 2$	and	$2\ R\ 5$
$2\ R^2\ 2$	Since	$2\ R\ 2$	and	$2\ R\ 2$
$2\ R^2\ 4$	Since	$2\ R\ 2$	and	$2\ R\ 4$
$2\ R^2\ 5$	Since	$2\ R\ 2$	and	$2\ R\ 5$
$2\ R^2\ 6$	Since	$2\ R\ 5$	and	$5\ R\ 6$
$3\ R^2\ 5$	Since	$3\ R\ 4$	and	$5\ R\ 5$
$4\ R^2\ 6$	Since	$4\ R\ 5$	and	$5\ R\ 6$

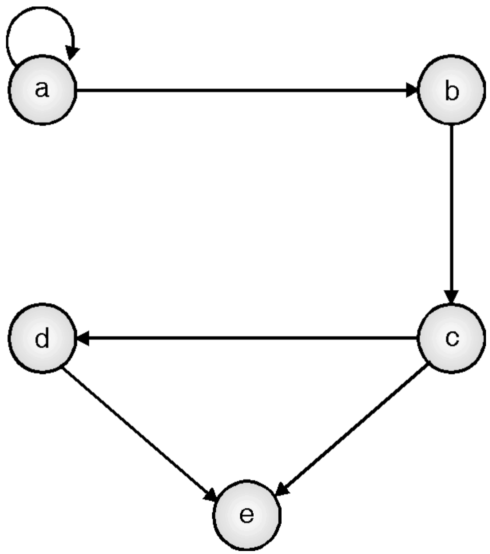


Paths in Relations and Digraphs

- Let A = {a, b, c, d, e}

and R = {(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)}

Compute (i) R^2 (ii) R^∞



a R^2 a	Since	a R a	and	a R a
a R^2 b	Since	a R a	and	a R b
a R^2 c	Since	a R b	and	b R c
b R^2e	Since	b R c	and	c R e
b R^2 d	Since	b R c	and	c R d
c R^2 e	Since	c R d	and	d R e

R^2={(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)}
R ∞ ={(a, a), (a, b), (a, c), (a, d) (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e)}.

Boolean product

The 'Boolean product' of A and B, denoted $A \odot B$ is the $m \times n$ Boolean matrix.

$$C = [C_{ij}] \text{ defined by}$$

$$C_{ij} = \begin{cases} 1 & \text{if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some } k,\ 1 \le k \le P \\ 0 & \text{otherwise} \end{cases}$$

$M_R{}^2 = M_R \odot M_R$

$M_R{}^n = M_R \odot M_R \odot \ldots\ldots\ldots\ldots \odot M_R$ (n factors)

Paths in Relations and Digraphs

Let A = {a, b, c, d, e} and

R = {(a, a), (a, b), (b, c), (c,e), (c, d), (d, e)}

$M_R{}^2 = M_R \odot M_R$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

M_R (i.k)	M_R (k.j)	$M_R{}^2 = M_R \odot M_R$
(a.a)	(a.a)	(a.a)
(a.b)	(b.c)	(a.c)
(b.c)	(c.d)(c.e)	(b.d)(b.e)
(c.d)	(d.e)	(c.e)
(c.e)		-
(d.e)		-
(a.a)	(a.b)	(a.b)

Properties/Types of Relations

- Reflexive
- Symmetric
- Transitive
- Antisymmetric
- Asymmetric

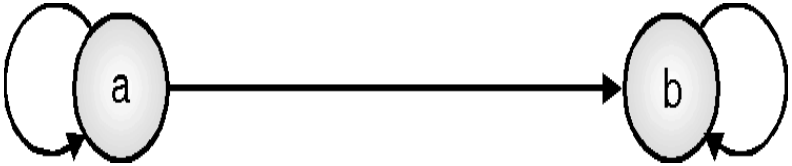
Properties: Reflexivity

A relation R on a set A is **reflexive** if for 'every' element $a \in A$, a R a, i.e. $(a, a) \in R$.

R is not a reflexive relation if for 'some' element $a \in A$, $(a, a) \notin$ **R**

Ex. 1 : Let A = {a, b} and let R = {(a, a), (a, b), (b, b)}.

Then R is reflexive.



Ex. 2 : Let A = {1,2} and let R = {(1,1), (1, 2)}.

R is not reflexive since $(2,2) \notin R$.

PROPERTIES: SYMMETRY

A relation R on a set A is **symmetric** if whenever a R b, then b R a. It then follows that R is not symmetric if we have some a and b $\in$ A with a R b, but b $\not R$ a.

Ex. 1 : A = { 1 , 2, 3} , Is R symmetric ?

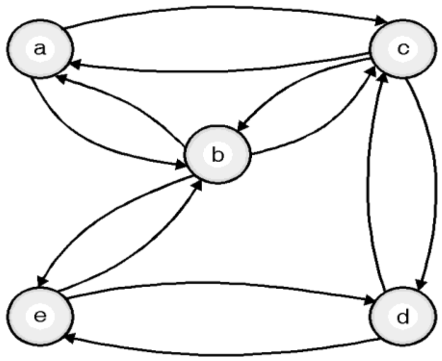
R = { (1,2) , (2,1) , (2,3), (3,2) , (1,1)) }

Ex. 2 : A = { 1 , 2 , 3 , 4 } , Is R symmetric ?

R= { (1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1) , (4,2), (4,3) }

Ex. 3 : A ={a, b, c, d, e}

R={(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e), (c, d), (d, c)}



Properties: Asymmetric relation

A relation R on a set A is **asymmetric** if whenever $(a,b) \in R$. then $(b, a) \notin R$. It then follows that R is not asymmetric if we have some a and $b \in A$ with both $(a,b) \in R$ and $(b, a) \in R$

Examples :

1.Let A = IR , the set of real numbers and let R be the relation '<'. If $a < b$, then $b \nless a$ (b is not less than a), so '<' is asymmetric.

2.Let A = {1, 2, 3, 4} and let

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ Reflexive - Yes, Symmetric- Yes, Asymmetric - No

$R_2 = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$

Then, R_2 is not asymmetric, since $(2, 2) \in R$.

3.Let $A = Z^+$, the set of positive integers, and let

$R=\{(a, b) \in A \times A \mid a \text{ divides } b\}$.

If a = b = 3, say then a R b and b R a, so R is not asymmetric.

Antisymmetric Relations

A relation R on a set A is **antisymmetric** if whenever a R b and b R a, then a = b.

- If a = b then, a R b **and** b R a is Antisymmetric
- If a ≠ b then (a,b) **∉ R or** (b,a) **∉ R** is Antisymmetric
- (If a ≠ b then both a R b and b R a is **NOT** Antisymmetric)

The contrapositive of this definition is that R is antisymmetric if whenever a ≠ b, then (a,b) **∊ R** or (b,a) **∉R**.

It follows that R is not antisymmetric if we have a and b in A, a ≠ b, and both a R b and b R a.

Symmetry versus Antisymmetry

- In a symmetric relation $a\mathcal{R}b \Leftrightarrow b\mathcal{R}a$

- In an antisymmetric relation, if we have $a\mathcal{R}b$ and $b\mathcal{R}a$ hold only when $a = b$

- Example: A={1,2,3}

R1 ={(1,2), (2,2), (2,1)}

------> Symmetric- Yes

------> Antisymmetric- No

R2={(1,2), (2,2), (1,3)}

------> Symmetric- No

------> Antisymmetric- Yes

Symmetry versus Antisymmetry

- An antisymmetric relation is not necessarily a reflexive relation

  $A=\{1,2,3\}$

  $R_1 = \{(1,1),(2,2), (3,3)\}$

  $R_2=\{(1,1),(2,2)\}$

Type of Relation	R_1	R_2
Antisymmetric	Yes	Yes
Reflexive	Yes	No
Symmetric	Yes	Yes

Symmetry versus Antisymmetry

- A relation that is not symmetric is not necessarily asymmetric

  A={1,2,3}

  R={(1,2),(2,2)}

Type of Relation	R
Symmetric	No
Asymmetric	No

Examples

Ex. : Let $A = Z$, the set of integers, and let $R = \{(a, b) \in A \times A \mid a < b\}$ Is R symmetric, asymmetric, or antisymmetric ?

Soln.:

Symmetry : If $a < b$,then it is not true that $b < a$, so R is **not symmetric.**

Asymmetry : If $a < b$, then b a (b is not less than a), so R is **asymmetric.**

Antisymmetry : If $a \neq b$, then either $a< b$ or $b < a$, so that R is **antisymmetric.**

EXAMPLES

Ex. : Let A = {1, 2, 3} and let R = {(1, 2), (2, 1), (2, 3)}. Is R symmetric, asymmetric, or antisymmetric?

Soln.:

Symmetry : R is not symmetric either since (2, 3) $\in$ R

but (3, 2) $\notin$ R

Asymmetry : R is also not asymmetric since both (1, 2) and (2, 1) $\in$ R.

Antisymmetry : R is not antisymmetric since (1, 2) and (2, 1) $\in$ R.

Examples

Ex. : Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 2), (2, 2), (3, 4), (4,1)\}$. Is R symmetric, asymmetric, or antisymmetric?

Soln.:

Symmetry : R is not symmetric, since $(1, 2) \in R$, but $(2, 1) \notin R$.

Asymmetry : R is not asymmetric, since $(2, 2) \in R$.

Antisymmetry : R is antisymmetric, since if $a \neq b$, either

$(a, b) \notin R$ or $(b, a) \notin R$.

Properties: Transitivity

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$.

$\forall a,b,c \in A\ ((aRb) \wedge (bRc)) \Rightarrow aRc$

Example

Let A = Z+, the set of positive integers, and let

$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$ Is R transitive?

Soln.: a divides b , aR b and b divides c, bRc

a divides c, aRc . Thus R is transitive.

Special cases

1) Let A = { 1 , 2 , 3 , 4 }

R= { (1 , 2) , (1 , 3) , (4 , 2) }

Is R transitive?

YES

2) R = { }

3)A relation that is symmetric and anti-symmetric

R = {(1,1),(2,2)} on the set A = {1,2,3}

EXAMPLE

Give examples of relations R on A = {1, 2, 3} having the stated property.

(i) R is transitive but not symmetric.

(ii) R is symmetric but not transitive.

(iii) R is both symmetric and anti-symmetric.

(iv) R is neither symmetric nor anti–symmetric.

Solution:

i. R={(1, 2), (2, 3), (1, 3)}

ii. R={(1, 2), (2, 1)}

iii. R={(1,1), (2,2)}

iv. R={(1,2), (2,3),(3,2)}

EXAMPLE

Define a relation on the set {a, b, c, d} that is

(i) transitive, reflexive and symmetric,

(ii) symmetric and transitive but not reflexive

Solution:

(i) Transitive, reflexive and symmetric,

A={ a, b, c, d }

R={(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (a, c), (c, a), (a, d), (d, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c)}

(ii) Symmetric and transitive but not reflexive

A={ a, b, c, d }

R={(a, b), (b, a), (c, d), (d, c), (a, a), (c, c)}

Irreflexive Relations

A relation R on a set A is **irreflexive** if a not related to a, i.e. $(a,a) \notin R$ for every $a \in A$. Thus R is irreflexive if no element is related to itself.

Examples

1. Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$.

   R is not reflexive $(1,1)$ $(2,2) \notin R$

   Then R is irreflexive since $(1, 1)$ $(2, 2) \notin R$.

2. Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 2)\}$.

   Then R is not irreflexive since $(2, 2) \in R$.

   Note: R is not reflexive either; since $(1, 1) \notin R$.

Identity Relation

Identity relation I on set A is reflexive, transitive and symmetric.

Example:

A= {1, 2, 3}

R= {(1, 1), (2, 2), (3, 3)}

Void Relation

It is given by R: A →B such that $R = \emptyset$ ($\subseteq A \times B$) is a null relation.

Void Relation $R = \emptyset$ is symmetric and transitive but not reflexive.

Universal Relation

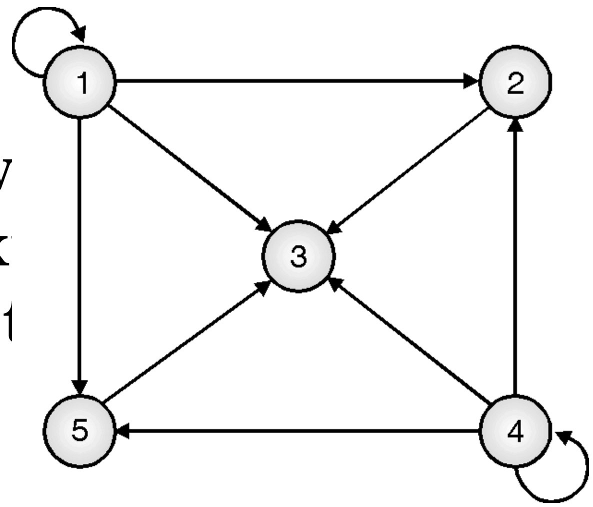
A relation R: A →B such that R = A x B (⊆ A x B) is a universal relation.

Universal Relation from A →B is reflexive, symmetric and transitive.

EXAMPLE

Let $A = \{1, 2, 3, 4, 5\}$

Determine whether the relation R w
digraph is given is reflexive, irreflex
symmetric, asymmetric, antisymmet
transitive.



R is not reflexive (See slide No. 23)

R is not irreflexive (See slide No. 35)

R is not symmetric (See slide No. 24)

R is not asymmetric (See slide No. 25)

R is antisymmetric (See slide No. 26)

R is transitive (See slide No. 31)

Exercise : Properties of Relations

State whether R satisfies property of reflexive , irreflexive , symmetry, asymmetry , antisymmetry , transitivity for A={1,2,3,4}

R={(1,1),(1,2),(2,1),(2,2),(3,3),(4,3),(3,4),(4,4)}

R= {(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)}

R={(1,1),(1,2),(2,1),(2,2),(3,3),(3,4)}

R={(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)}

R={(1,1),(2,2),(3,3),(4,4)}

EQUIVALENCE RELATION

A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive.

Let A = { a , b , c } and

R= { (a,a),(b,b),(b,c),(c,b),(c,c)}

is an equivalence relation since it is reflexive, symmetric, and transitive.

Determine whether R is an Equivalence relation

Let A = {a, b, c} and let , $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Determine whether R is an equivalence relation.

Soln.: R = {(a, a), (b, b), (b, c), (c, b), (c, c)}

R is reflexive since (a, a), (b, b), (c, c) ∈ R

R is symmetric since (b, c) ∈ R → (c, b) ∈ R

R is transitive since,

(b,b)	and	(b, c) ∈ R	implies	(b, c) ∈ R,
(b, c)	and	(c, b) ∈ R	implies	(b, b) ∈ R,
(c, c)	and	(c, b) ∈ R	implies	(c, b) ∈ R,
(c, b)	and	(b, b) ∈ R	implies	(c, b) ∈ R,
(c, b)	and	(b, c) ∈ R	implies	(c, c) ∈ R,
(b, c)	and	(c, c) ∈ R	implies	(b, c) ∈ R,

Hence R is an equivalence relation.

Determine whether R is an Equivalence relation

Let $A = Z$, the set of integers, and let R be defined by a R b if and only if $a \leq b$. Is R an equivalence relation?

- Since $a \leq a$, R is reflexive.
- If $a \leq b$, it need not follow that $b \leq a$, so R is not symmetric.
- Incidentally, R is transitive, since $a \leq b$ and $b \leq c$ imply that $a \leq c$.
- We see that R is **not** an equivalence relation.

Determine whether R is an Equivalence relation

Let A = {1, 2, 3, 4} and

Let R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)}

Determine whether the relation R on the set A is an equivalence relation.

Soln.:

R is reflexive since (1, 1), (2, 2), (3, 3), (4, 4) $\in$ R. R is not symmetric since, (4, 1) $\in$ R but (1, 4) $\notin$ R.

R is not transitive since,

(2, 1), (1, 3) $\in$ R but (2, 3) $\notin$ R

Hence given relation R is not an equivalence relation.

Let R be a binary relation on the set of all positive integers such that,

R = {(a, b) | a – b is an odd positive integer}

Is R reflexive ? Symmetric ? Transitive ? An equivalence relation ?

Soln.: R is not Reflexive since

$$a - a = 0 \neq \text{odd positive integer}$$

$$\therefore \quad a \not R a$$

R is not symmetric also, since, as if a R b then

$$a - b = 2n + 1 \quad \text{where } n = \text{integer number}$$

if b R a then $b - a = -2n - 1$ where n = integer number

$$\therefore \quad b - a \neq \text{odd positive integer}$$

R is not transitive since,

Let a R b and b R c

i.e. $a - b = 2n_1 + 1$

$b - c = 2n_2 + 1$ ... i.e. odd positive integer

$$a - c = (2n_1 + 1) + (2n_2 + 1) = 2(n_1 + n_2 + 1)$$

$$\neq \text{odd positive integer}$$

Hence R is not transitive.

Therefore R is not an equivalence relation.

Equivalence Class and Partitions

Let $A = \{ 1 , 2 , 3 , 4 \}$ and consider the partition

$P = \{ \{ 1 , 2 , 3 \} , \{ 4\} \}$ of A.

Find the equivalence relation R on A determined by P

Soln: " Each element in a block is related to every other element in the same block and only to those elements"

$R= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,4)\}$

Problems

Find the equivalence relation on A by P

1) Let A ={ a , b , c , d } and P = {{a , b } , { c }, { d } }

R={(a,a),(a,b),(b,b),(b,a),(c,c),(d,d)}

2) Let A={1,2,3,4,5} and P={{ 1 ,2 },{ 3 },{ 4, 5}}

R={(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,5),(5,5),(5,4)}

3) If {{1,3,5},{2,4}} is a partition on the set A={1,2,3,4,5},determine the corresponding equivalence relation

R={(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(2,2),(2,4),(4,2),(4,4)}

Equivalence Class

Let $A = \{1,2,3,4,5,6\}$ and let R be the equivalence relation on A defined by $R=\{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$

Find the equivalence classes of R and find the partition of A induced by R

$R=\{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$

Equivalence Classes: $R(1)$, $R(2)$, $R(3)$, $R(4)$, $R(5)$, $R(6)$.

$R(1) = \{1,5\}$

$R(2) = \{2,3,6\}$

$R(3) = \{2,3,6\}$

$R(4) = \{4\}$

$R(5) = \{1,5\}$

$R(6) = \{2,3,6\}$

Therefore, the partition of A induced by R i.e

$A | R = \{\{1,5\},\{2,3,6\},\{4\}\}$

Rank = R = Number of distinct equivalence classes **= 3**

Problems: Find Equivalence Classes, partition and rank

1. Let $A=\{1,2,3\}$ and let $R=\{(1,1),(2,2),(1,3),(3,1),(3,3)\}$.

   Find $A|R$.

Ans: $R(1)=\{1,3\}$ $R(2)=\{2\}$ $R(3)=\{1,3\}$

$A|R=\{\{1,3\},\{2\}\}$, Rank=2

2. Let $A=\{1,2,3,4\}$,and let $R=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$

   Determine $A|R$.

Ans: $R(1)=\{1,2\}$ $R(2)=\{1,2\}$ $R(3)=\{3,4\}$ $R(4)=\{3,4\}$

$A|R=\{\{1,2\},\{3,4\}\}$ Rank=2

3. Let $A=\{1,2,3,4\}$,and let $R=\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(2,3),(3,2),(3,3),(4,4)\}$ Show that R is an equivalence relation and determine the equivalence classes and hence find the rank of R

Ans: $R(1)=\{1,2,3\}$ $R(2)=\{1,2,3\}$ $R(3)=\{1,2,3\}$ $R(4)=\{4\}$

$A|R=\{\{1,2,3\},\{4\}\}$

Rank=2

Combining Relations

- Relations are simply… sets (of ordered pairs); subsets of the Cartesian product of two sets
- Therefore, in order to <u>combine</u> relations to create new relations, it makes sense to use the usual set operations
  - Compliment $\bar{R}$
  - Intersection ($R_1 \cap R_2$)
  - Union ($R_1 \cup R_2$)
  - Set difference ($R_1 \setminus R_2$)
  - Inverse R^{-1}

Examples

Let A ={1,2,3} and B={u,v} and

R1={(1,u),(2,u),(2,v),(3,u)} and

R2={(1,v),(3,u),(3,v)}

R1 U R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}

R 1 ∩ R2 ={(3,u)}

R 1 – R 2 = {(1,u),(2,u),(2,v)}

R 2 – R 1 ={(1,v),(3,v)}

Examples

Let A={ 1 , 2 , 3 , 4 } and B={ a , b , c } and let

R ={(1,a),(1,b),(2,b),(2,c),(3,b),(4,a)} and

S={(1,b),(2,c),(3,b),(4,b)}

Compute $R \cap S$, $R \cup S$, R^{-1}

$R \cap S$={(1,b),(2,c),(3,b)}

$R \cup S$={(1,a),(1,b),(2,b),(2,c),(3,b),(4,a),(4,b)}

R^{-1} ={(a,1),(b,1),(b,2),(c,2),(b,3),(a,4)}

Combining Relations: Example

Let

$A=\{1,2,3,4\}$

$B=\{1,2,3,4\}$

$R_1=\{(1,2),(1,3),(1,4),(2,2),(3,4),(4,1),(4,2)\}$

$R_2=\{(1,1),(1,2),(1,3),(2,3)\}$

$R_1 \cup R_2 =$

$\{(1,2),(1,3),(1,4),(2,2),(3,4),(4,1),(4,2),(1,1),(2,3\}$

$R_1 \cap R_2 =$

$\{(1,2),(1,3)\}$

$R_1 - R_2 =$

$\{(1,4),(2,2),(3,4),(4,1),(4,2)\}$

$R_2 - R_1 =$

$\{(1,1),(2,3)\}$

Composite of Relations

- **Definition**: Let R_1 be a relation from the set A to B and R_2 be a relation from B to C, i.e.

$$R_1 \subseteq A \times B \text{ and } R_2 \subseteq B \times C$$

the composite of R_1 and R_2 is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_{\mathbf{2}} \circ R_{\mathbf{1}}$$

Composite of Relations

Ex: Let $A = \{1,2,3\}$, $B =\{0,1,2\}$ and $C = \{a,b\}$

$R = \{(1,0),(1,2),(3,1),(3,2)\}$

$S = \{(0,b),(1,a),(2,b)\}$

$S \circ R = ?$

Since $(1,0) \in R$ and $(0,b) \in S$, $\therefore$ $(1,b) \in S \circ R$

Since $(1,2) \in R$ and $(2,b) \in S$, $\therefore$ $(1,b) \in S \circ R$

Since $(3,1) \in R$ and $(1,a) \in S$, $\therefore$ $(3,a) \in S \circ R$

Since $(3,2) \in R$ and $(2,b) \in S$, $\therefore$ $(3,b) \in S \circ R$

$$\mathbf{S \circ R = \{ (1 , b) , (3 , a) , (3 , b) \}}$$

PROBLEMS

1) Let A={1,2,3} and let

R={(1,1),(1,3),(2,1),(2,2),(2,3),(3,2)} and

S={(1,1),(2,2),(2,3),(3,1),(3,3)}.

Find SoR and M_{SoR}

SoR={(1,1),(1,3),(2,1),(2,2),(2,3),(3,2),(3,3)}

2) Let A={1,2,3,4}

R={(1,1),(1,2),(2,3),(2,4),(3,4),(4,1),(4,2)}

S={(3,1),(4,4),(2,3),(2,4),(1,1),(1,4)}

Compute SoR, RoS, RoR, SoS

SoR={(1,1),(1,3),(2,1),(2,4),(3,4),(4,1),(4,4),(1,4)}

RoS={(3,1),(3,2),(4,1),(4,2),(2,4),(2,1),(2,2),(1,1),(1,2)}

RoR

SoS

Closures

The 'smallest' relation R_1 on A that contains R and possesses the property we desire. Sometimes R_1 does not exist. If a relation such as R_1 does exist, we call it the 'closure' of R with respect to the property in question.

Reflexive closure

Let R be a relation on a set A, and R is not reflexive (i.e. some pairs of the diagonal relation Δ are not in R).

A relation $R_1 = R \cup \Delta$ is the reflexive closure of the relation R if $R \cup \Delta$ is the smallest relation containing R which is reflexive.

$R_1 = R \cup \Delta$

where Δ is the set of elements of the type (a, a) where $a \in A$.

EXAMPLE

A = {1, 2, 3} and the relation R is given by

R = {(1, 1), (1, 2), (2, 3)} then

R_1 = $R \cup \Delta$ where

Δ = {(1, 1), (2, 2), (3, 3)}

$R \cup \Delta$ = {(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)}

Reflexive closure is,

$$\mathbf{R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}}$$

Symmetric closure

Suppose that R is a relation on A that is not symmetric. Then there must exist pairs (x, y) in R such that (y, x) is not in R. Of course, $(y, x) \in R^{-1}$, so if R is to be symmetric we must add all pairs from R^{-1}; that is we must enlarge R to $R \cup R^{-1}$. Clearly $(R \cup R^{-1})^{-1} = R \cup R^{-1}$, So $R \cup R^{-1}$ is the smallest symmetric relation containing R; that is $R \cup R^{-1}$ is the 'symmetric closure' of R.

EXAMPLE

$A = \{a, b, c, d\}$ and

$R=\{(a, b), (b, c), (a, c), (c, d)\}$ then

$R^{-1}=\{(b, a), (c, b), (c, a), (d, c)\}$

so the symmetric closure of R is

$R \cup R^{-1}=\{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a), (c, d), (d, c)\}$

TRANSITIVE CLOSURE

- Let R be a relation on a set A. Then the 'transitive closure' of a relation R is the smallest transitive relation containing R. The transitive closure of R is just the connectivity relation R^{∞}.

R^*=Transitive closure of R

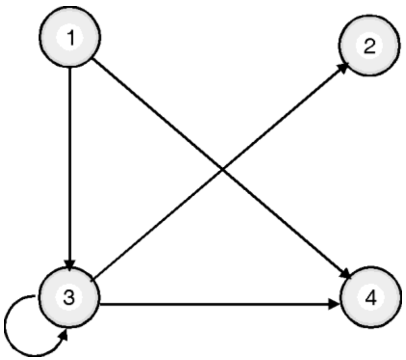
$= R \cup \{(a, c)$, if and only if $(a, b), (b, c) \in R\}$

EXAMPLE

Find the transitive closure **R*** of the relation R on A = {1, 2, 3, 4} defined by the directed graph shown

Soln.:

R = {(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)}



Here transitive closure of R is

$= R \cup \{(a, c) \mid \text{if } (a, b), (b, c) \in R\}$

To find transitive closure

$(1, 3) \in R$ and $(3, 4) \in R$, hence add (1, 4) in R

$(1, 3) \in R$ and $(3, 3) \in R$, hence add (1, 3) in R

$(1, 3) \in R$ and $(3, 2) \in R$, hence add (1, 2) in R

Transitive closure of R = {(1, 2), (1, 3), (1, 4),(3, 2), (3, 3), (3, 4)}

Matrix method

Let A = {1, 2, 3, 4} and let R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the transitive closure of R. The matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^2_{\odot} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (M_R)^3_{\odot} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R^2 ={(1, 1), (1, 3), (2, 2), (2, 4)}

R^3 ={(1, 2), (1, 4), (2, 1), (2, 3)}

R^4 ={(1, 1), (1, 3), (2, 2), (2, 4)}

$R^\infty = R \cup R^2 \cup R^3 \cup R^4$

R^∞ ={(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)}

$$(M_R)^4_{\odot} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{\infty} = M_R \vee (M_R)^2_{\odot} \vee (M_R)^3_{\odot} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

WARSHALL'S ALGORITHM

Ex. 1 : Let A = {1, 2, 3, 4} and let R = {(1, 2), (2, 3), (3, 4), (2, 1)}.

Find transitive closure of R using Warshall's algorithm.

Solution:

$$W_0 = M_R = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

First we find W_1, so that k = 1. W_0 has 1's in location 2 of column 1 i.e. (2, 1) and location 2 of row 1 i.e. (1, 2)

i j

p_1:(2, 1)

i j

q_1:(1, 2)

add (p_i, q_j) i.e. (2, 2) in W_k

Thus W_1 is just W_0 with a new 1 in position (2, 2)

$$W_1 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

Matrix W_1 has 1's at row 1 and 2 of column 2 and columns 1, 2, and 3 of row 2. i.e.

$p_1 : \overset{i\ \ j}{(1, 2)}$ $\quad p_2 : \overset{i\ \ j}{(2, 2)}$

$q_1 : \overset{i\ \ j}{(2, 1)}$ $\quad q_2 : \overset{i\ \ j}{(2, 2)}$ $\quad q_3 : \overset{i\ \ 1}{(2, 3)}$

We must put 1's in positions (p_i, q_j) i.e. (1, 1), (1, 2), (1, 3), (2, 1), (2,2) and (2, 3) of matrix W_1 (if 1's are not already there).

$$W_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$p_1 : \overset{i\ \ j}{(1, 3)}$ $\quad p_2 : \overset{i\ \ j}{(2, 3)}$

$q_1 : \overset{i\ \ j}{(3, 4)}$

(1, 4) and (2, 4)

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, W_3 has 1's in locations 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and $MR_\infty = W_4 = W_3$.

EXAMPLE:

Let $A = \{1,2,3,4,5\}$ $R = \{(1,1), (1,2), (2,1), (2,2)\ (3,3), (3,4), (4,3),(4,4), (5,5)\}$ and $S = \{(1,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$ The reader may verify that both R and S are equivalence relations. The partition $A \mid R$ of A corresponding to R is $\{\{1,2\}, \{3,4\}, \{5\}\}$, and the partition $A \mid S$ of A corresponding to S is $\{\{1\}, \{2\}, \{3\}, \{4,5\}\}$. Find the smallest equivalence relation containing R and S, and compute the partition of A that it produces.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{So } M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We now compute $M_{(R \cup S)^\infty}$ by Warshall's algorithm. First, $W_o = M_{R \cup S}$. We next compute W_1 so k = 1. Since W_o has 1's in locations 1 and 2 of column 1 and in locations 1 and 2 of row 1, we find that no new 1's must be adjoined to W_1 . Thus

$$W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

K=1

$$\begin{array}{llll} & \ \ i\ \ \ j & & \ \ i\ \ \ j \\ p_1 : & (1,\ 1) & p_2 : & (2,\ 1) \\ & \ \ i\ \ \ j & & \ \ i\ \ \ j \\ q_1 : & (1,\ 1) & q_2 : & (1,\ 2) \end{array}$$

To obtain W_1, we must put is in positions (1, 1), (1, 2), (2, 1) and (2, 2). We see that

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_1 = W_0$

We now compute W2, so k = 2. Since W_1 has 1's in locations 1 and 2 : of column 2 and in locations 1 and 2 of row 2, we find that no new 1's must be added to W_1. That is,

$$\begin{array}{llll} & \;\, i \;\; j & & \;\, i \;\; j \\ p_1 : & (1, 2) & p_2 : & (2, 2) \\ & \;\, i \;\; j & & \;\, i \;\; j \\ q_1 : & (2, 1) & q_2 : & (2, 2) \end{array}$$

To obtain W_2, we must put is in positions (1, 1), (1, 2), (2, 1), (2, 2). We see that

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_2 = W_1$

We next compute W_3, so k = 3. Since W_2 has 1's in locations 3 and 4 of column 3 and in locations 3 and 4 of row 3, we find that no new 1's must be added to W_2. That is

$$\begin{array}{llll} & \;\, i \;\; j & & \;\, i \;\; j \\ p_1 : & (3, 3) & p_2 : & (4, 3) \\ & \;\, i \;\; j & & \;\, i \;\; j \\ q_1 : & (3, 3) & q_2 : & (3, 4) \end{array}$$

To obtain W_3, we must put 1's in position (3, 3), (3, 4), (4, 3), (4, 4). We see that

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_3 = W_2$

Things change when we now compute W4. Since W3 has I's in locations 3, 4, and 5 of column 4 and in locations 3, 4 and 5 of column 4, and in locations 3, 4 and 5 of row 4 we must add new 1's to W3 in positions 3, 5, and 5, 3, i.e.

$$
\begin{array}{lll}
 & \ \ i\ \ j & \\
p_1 : (3, 4) & p_2 : (4, 4) & p_3 : (5, 4) \\
 & \ \ i\ \ j & \\
q_1 : (4, 3) & q_2 : (4, 4) & q_3 : (4, 5)
\end{array}
$$

To obtain W4, we must put 1's in positions (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5). We see that,

$$
W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}
$$

You may verify that $W_5 = W_4$ and thus

$(R \cup S)^{\infty}$ ={(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)}