LOGIC

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Logic is the study of the logic **relationships** between objects

and

forms the basis of all mathematical reasoning and all automated reasoning

PROPOSITION

Definition: A ***proposition*** or ***statement*** is a declarative sentence which is either true or false, but not both.

Are the following sentences propositions?

Examples :

(i) There are seven days in a week.

(ii) $2 + 2 = 5$

(iii) The earth is flat.

(iv) The equation $x^2 + x + 1 = 0$ has no real root.

(v) It will rain tomorrow.

(vi) Four is even.

(vii) $43 \geq 21$

(viii) $4 \in \{ 1, 3, 5 \}$

NOT PROPOSITIONS

1. x + 3 = 5
2. Bring that book !
3. When is your interview?
4. What a beautiful painting !
5. This statement is false.
6. C++ is the best language.
7. When is the pretest?
8. Do your homework.

Opinions, exclamatory, interrogative (Qns.) and imperative (commanding) statements are *NOT Propositions*

PROPOSITIONAL LOGIC

Propositional Logic – the area of logic that deals with propositions

Propositional Variables – variables that represent propositions: p, q, r, s

E.g. Proposition p – "Today is Friday."

Truth values – T, F

LOGICAL OPERATIONS

Logical operators are used to form new propositions from two or more existing propositions. The logical operators are also called connectives

- **Negation** (e.g., $\neg a$ or $!a$ or $\bar{a}$)
- **AND** or logical **Conjunction** (denoted $\wedge$)
- **OR** or logical **Disjunction** (denoted $\vee$)
- **XOR** or exclusive or (denoted $\oplus$)
- **Imply** on (denoted $\Rightarrow$ or $\rightarrow$)
- **Bi-conditional** (denoted $\Leftrightarrow$ or $\leftrightarrow$) IFF

We define the meaning (semantics) of the logical connectives using truth tables

LOGICAL CONNECTIVE: NEGATION

~p, the negation of a proposition *p*, is also a proposition

p	**~p**
0	1
1	0

Examples:

p: Today is Monday

Solution: ~p: It is not the case that *today is Monday*.
In simple English, “Today is not Monday.” or “It is not Monday today.”

p:At least 10 inches of rain fell today in Mumbai

Solution: ~p: It is not the case that *at least 10 inches of rain fell today in Mumbai*.
In simple English, “Less than 10 inches of rain fell today in Mumbai.”

LOGICAL CONNECTIVE: LOGICAL CONJUNCTION (AND)

If p and q are statements the compound statement '**p** and **q**' is called as the **conjunction** of p and q; and is denoted by p Λ q

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Example 1 :

Let P : The sun is shining.

q : The birds are singing.

Then p ∧ q is the statement : The sun is shining and the birds are singing.

Example 2 :

Let P : 2 is a prime number.

q : John is an intelligent boy.

Then p ∧ q is the statement : ***2 is a prime number and*** *John* ***is an intelligent boy.***

LOGICAL CONNECTIVE: LOGICAL DISJUNCTION (OR)

If p and q are statements, then the compound statement 'p or q' is called as the ***disjunction*** of p and q , and is denoted p v q

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1:

Let p : There is an error in the program.

q : The data is wrong.

Then ***$p \vee q$:*** There is an error in the program or the data is wrong.

LOGICAL CONNECTIVE: LOGICAL DISJUNCTION (OR)

Let p : I will read a book q : I will go to sleep

Then $p \vee q$ $\Rightarrow$ I will read a book or go to sleep.

In everyday language as this example demonstrates, the connective "or" is used in the 'exclusive sense', i.e. either one or the other activity can happen, but not both.

In logic, the symbol '$\vee$' is used in the inclusive sense only; where we wish to speicfy that "exclusive or" is to be used, we use a new symbol "$\bar{v}$".

Hence in the above example, the more correct notation is $p\bar{v}$ q.

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

LOGICAL CONNECTIVE: CONDITIONAL (IF….THEN)

Conditional (If …then)

If p and q are statements, the compound statement 'If p then q', denoted by $p \to q$ is called a ***Conditional Statement*** or implication

p is called the **antecedent (precursor)** or **hypothesis**, while q is called the **consequent**.

Let p : Peter works hard.

q : Peter will pass the exam.

Then $p \to q$: ***If Peter works hard, then he will pass the exam.***

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

Let p be the statement "Maria learns discrete mathematics." and q the statement "Maria will find a good job."

Express the statement $p \rightarrow q$ as a statement in English.

Solution: Any of the following -

"If Maria learns discrete mathematics, then she will find a good job.

"Maria will find a good job when she learns discrete mathematics."

CONVERSE, INVERSE AND CONTRAPOSITIVE PROPOSITIONS

If $p \to q$ is the conditional statement then

1. $q \to p$ is called its **Converse.**
2. $\sim p \to \sim q$ is called its **Inverse.**
3. $\sim q \to \sim p$ is called its **Contrapositive**.

LOGICAL CONNECTIVE: CONDITIONAL (IF….THEN)

State the converse, inverse and contrapositive of the following.

(i) If it is cold then he wears hat.

Let p: It is cold , q: He wears hat.

Converse $(q \rightarrow p)$ **:**

If he wears hat then it is cold.

Contrapositive $(\sim q \rightarrow \sim p)$ **:**

If he does not wear hat, then it is not cold.

Inverse $(\sim p \rightarrow \sim q)$ **:**

If it is not cold then he does not wear hat

(ii) If integer is multiple of 2, then it is even.

Converse $(q \rightarrow p)$ **:**

If integer is even, then it is multiple of 2.

Inverse $(\sim p \rightarrow \sim q)$:

If integer is not multiple of 2 then it is not even.

Contrapositive $(\sim q \rightarrow \sim p)$ **:**

If integer is not even, then it is not multiple of 2.

BI-CONDITIONAL (IF AND ONLY IF)

If p and q are statements, the compound statement 'p if and only if q', denoted by , is called a ***biconditional statement***.

Often "if and only if" is shortened as"iff"

$p \leftrightarrow q$ is also read as "If p then q, and conversely".

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Examples :

(i) An integer is even if and only if it is divisible by 2.

(ii) A right angled triangle is isosceles if and only if the other two angles are each equal to forty-five degrees.

(iii) Two lines are parallel if and only if they have the same slope

PROPOSITIONAL OR STATEMENT FORM

Using the logical connectives, we can construct or form an expression, involving the statement variables. The following are examples of statement forms :

(i) $\sim (p \vee q) \rightarrow p$

(ii) $(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

(iii) $((p \wedge q) \vee (p \wedge \sim r)) \rightarrow ((p \wedge r) \vee q)$

PRECEDENCE OF LOGICAL OPERATORS

- As in arithmetic, an ordering is imposed on the use of logical operators in compound propositions
- However, it is preferable to use parentheses to disambiguate operators and facilitate readability

$\neg p \vee q \wedge \neg r \equiv (\neg p) \vee (q \wedge (\neg r))$

- To avoid unnecessary parenthesis, the following precedences hold:

1. Negation ($\neg$)
2. Conjunction ($\wedge$)
3. Disjunction ($\vee$)
4. Implication ($\rightarrow$)
5. Biconditional ($\leftrightarrow$)

STATEMENT FORM

Ex. 1 : Let

p denotes Raju is rich,

q denotes Raju is happy.

Write each of the following in symbolic form.

1. *Raju is poor but happy*
2. *Raju is neither rich nor happy*
3. *Raju is either rich or unhappy*
4. *Raju is poor or else he is rich and unhappy*

Soln. :

1. $\sim p \wedge q$
2. $\sim p \wedge \sim q$
3. $p \vee \sim q$
4. $\sim p \vee (p \wedge \sim q)$

STATEMENT FORM

Ex. 2 : Express the proposition 'Either my program runs and it contains no bugs, or my program contains bugs' in symbolic form.

Soln. :

Let	p	:	*My program runs.*
	q	:	*My program contains bugs.*

Solution:

The proposition can be written in symbolic form as,

$(p \wedge \sim q) \vee q.$

STATEMENT FORM

Ex. 3 : Using the following statements :

p : I will study discrete structures.

q : I will go to a movie.

r : I am in a good mood.

Write the following statements in symbolic form :

(i) If I am not in a good mood, then I will go to a movie.

(ii)I will not go to a movie and I will study discrete structures.

(iii)I will go to a movie only if I will not study discrete structures.

(iv)I will not study discrete structures, then I am not in a good mood.

Solution:

(i) $\sim r \rightarrow q$

(ii) $\sim q \land p$

(iii) $\sim p \rightarrow q$

(iv) $\sim p \rightarrow \sim r$

STATEMENT FORM

Ex. 4 Translate the following statement into symbolic form :

If the utility cost goes up or the request for additional funding is desired, then a new computer will be purchased if and only if we can show that the current computing facilities are indeed not adequate.

Soln. : p :The utility cost goes up

q: The request for additional funding is desired

r: A new computer will be purchased

s: We can show that the current computing facilities are indeed adquate.

Then $(p \vee q) \rightarrow (r \leftrightarrow \sim s)$ **is the required symbolic form.**

STATEMENT FORM

Ex. 5 : Let 'a' be the proposition 'high speed driving is dangerous' and 'b' be the proposition 'Rajesh was a wise man.' Write down the meaning of the following proposition.

1. $a \wedge b$
2. $\sim a \wedge b$
3. $\sim (a \wedge b)$
4. $(a \wedge b) \vee (\sim a \wedge \sim b)$

Soln. :

1. **$a \wedge b$: High speed driving is dangerous and Rajesh was a wise man.**
2. **$\sim a \wedge b$: High speed driving is not dangerous and Rajesh is a wise man.**
3. **$\sim (a \wedge b)$: Either high speed driving is not dangerous or Rajesh is not a wise man.**
4. **$(a \wedge b) \vee (\sim a \wedge \sim b)$: High speed driving is dangerous and Rajesh was a wise man or neither high speed driving is dangerous nor Rajesh is a wise man.**

STATEMENT FORM

Ex. 6: *How can this English sentence be translated into a logical expression?*

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Soln: Let q : You can ride the roller coaster.

r : You are under 4 feet tall.

s : You are older than 16 years old.

The sentence can be translated into:

$$(r \land \neg s) \rightarrow \neg q$$

TRUTH TABLES OF COMPOUND PROPOSITIONS

We can use connectives to build up complicated compound propositions involving any number of propositional variables, then use truth tables to determine the truth value of these compound propositions.

Example: **Construct the truth table of the compound proposition**

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

p	q	¬q	p ν ¬q	p Λ q	(p ν ¬q) → (p Λ q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

TRUTH TABLES OF COMPOUND PROPOSITIONS

Construct the truth table for the following compound proposition

$((p \wedge q) \vee \neg q)$

p	q	$p \wedge q$	$\neg q$	$((p \wedge q) \vee \neg q)$
0	0	0	1	1
0	1	0	0	0
1	0	0	1	1
1	1	1	0	1

TRUTH TABLES OF COMPOUND PROPOSITIONS

Construct the truth table for the formula : $\sim [\, p \wedge (p \vee \sim q\,)\,]$

p	q	$\sim q$	$p \vee \sim q$	$p \wedge (p \vee \sim q)$	$\sim [\, p \wedge (p \vee \sim q)\,]$
T	T	F	T	T	F
T	F	T	T	T	F
F	T	F	F	F	T
F	F	T	T	F	T

TAUTOLOGY AND CONTRADICTION

A statement form is called a Tautology if it always assumes the truth value "T" irrespective of the truth values assigned to its variables.

A statement form is called a Contradiction if it is always assumes the truth value "F" irrespective of the truth values assigned to its variables.

A statement form which is neither a tautology nor a contradiction is called a Contingency.

TAUTOLOGY AND CONTRADICTION

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	T

p	q	$\sim p$	$(\sim p \vee q)$	$p \wedge (\sim p \vee q)$	$\sim q$	$(p \wedge (\sim p \vee q)) \wedge \sim q$
T	T	F	T	T	F	F
T	F	F	F	F	T	F
F	T	T	T	F	F	F
F	F	T	T	F	T	F

LAWS OF LOGIC

1. **Idempotent laws :**

   $p \vee p \equiv p$

   $p \wedge p \equiv p$

2. **Commutative laws :**

   $p \vee q \equiv q \vee p$

   $p \wedge q \equiv q \wedge p$

3. **Associative laws :**

   $(p \vee q) \vee r \equiv p \vee (q \vee r)$

   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

4. **Distributive laws :**

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. **Identity laws :**

   $p \vee \text{False} \equiv p$

   $p \vee \text{True} \equiv \text{True}$

   $p \wedge \text{False} \equiv \text{False}$

   $p \wedge \text{True} \equiv p$

6. **Absorption law :**

   $p \wedge (p \vee q) \equiv p$

   $p \vee (p \wedge q) \equiv p$

7. **Implication law :**

   $p \rightarrow q \equiv \sim p \vee q$

8. **Complement laws / Inverse law :**

   $p \vee \sim p \equiv \text{True}$

   $p \wedge \sim p \equiv \text{False}$

   $\sim (\sim p) \equiv p$

   $\sim \text{True} \equiv \text{False}$ and $\sim \text{False} \equiv \text{True}$

9. **De Morgan's law :**

   $\sim (p \vee q) \equiv \sim p \wedge \sim q$

   $\sim (p \wedge q) \equiv \sim p \vee \sim q$

EXAMPLES

Show that (i) $a \vee (\bar{a} \wedge b) = a \vee b$

(ii) $a \wedge (\bar{a} \vee b) = a \wedge b$

(i) $a \vee (\bar{a} \wedge b) \equiv (a \vee \bar{a}) \wedge (a \vee b)$ - Distributive law

$\equiv 1 \wedge (a \vee b)$ - Complement law

$\equiv a \vee b$ - Identity law

Hence proved

(ii) $a \wedge (\bar{a} \vee b) \equiv (a \wedge \bar{a}) \vee (a \wedge b)$ - Distributive law

$\equiv 0 \vee (a \wedge b)$ - Complement law

$\equiv a \wedge b$ - Identity law

Hence proved.

Ex. : Use the laws of logic to show that

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ **is a tautology.**

Soln. : $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$\equiv \sim [(\sim p \vee q) \wedge \sim q] \vee \sim p$ Implication law

$\equiv \sim [\sim q \wedge (\sim p \vee q)] \vee \sim p$ Commutative law

$\equiv \sim [(\sim q \wedge \sim p) \vee (\sim q \wedge q)] \vee \sim p$ Distributive law

$\equiv \sim [(\sim q \wedge \sim p) \vee (q \wedge \sim q)] \vee \sim p$ Commutative law

$\equiv \sim [(\sim q \wedge \sim p) \vee F] \vee \sim p$ Complement law

$\equiv \sim (\sim q \wedge \sim p) \vee \sim p$ Identity law

$\equiv (\sim \sim q \vee \sim \sim p) \vee \sim p$ De-morgan's law

$\equiv (q \vee p) \vee \sim p$ Complement law

$\equiv q \vee (p \vee \sim p)$ Associative law

$\equiv q \vee T$ - Inverse law

$\equiv T$ - Identity law

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Ex. :Show that the following statement is tautological.

$$(p \wedge (p \rightarrow q)) \rightarrow q.$$

Soln. : $(p \wedge (p \rightarrow q)) \rightarrow q$

$\equiv \sim (p \wedge (\sim p \vee q)) \vee q$	**Implication Law**
$\equiv (\sim p \vee \sim (\sim p \vee q)) \vee q$	
$\equiv (\sim p \vee (\sim\sim p \wedge \sim q)) \vee q$	**De-Morgan's Law**
$\equiv (\sim p \vee (p \wedge \sim q)) \vee q$	**Complement Law**
$\equiv ((\sim p \vee p) \wedge (\sim p \vee \sim q)) \vee q$	**Distributive Law**
$\equiv (T \wedge (\sim p \vee \sim q)) \vee q$	**Complement law**
$\equiv (\sim p \vee \sim q) \vee q$	**Indentity Law**
$\equiv \sim p \vee (\sim q \vee q)$	**Associative Law**
$\equiv \sim p \vee \text{True}$	**Complement law**
$\equiv \text{True}$	**Identity law**

Ex. : Prove:

$[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \leftrightarrow p \wedge q$

i) L.H.S.

$\equiv (\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)$

$\equiv \sim(\sim p \vee \sim q) \vee (p \wedge q \wedge r)$ - Implication law

$\equiv (p \wedge q) \vee (p \wedge q \wedge r)$ - De morgan's law

Let $p \wedge q$ be x

$\equiv x \vee (x \wedge r)$

$\equiv x$ - Absorption law

$\equiv p \wedge q$

$\equiv$ R.H.S.

Ex. :Prove the distributive law.

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	**T**	T	T	**T**
T	T	F	F	**T**	T	T	**T**
T	F	T	F	**T**	T	T	**T**
T	F	F	F	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	T	F	F	**F**	T	F	**F**
F	F	T	F	**F**	F	T	**F**
F	F	F	F	**F**	F	F	**F**

QUANTIFIER

Consider the following sentences :

(i) "x is tall and handsome."

(ii) "$x + 3 = 5$."

(iii) "$x + y \geq 10$."

These sentences are **NOT** propositions, since they do not have any truth value.

However, if values are assigned to the variables, each of them becomes proposition, which is either true or false.

For example, the above sentences can be converted into:-

(i) "He is tall and handsome".

(ii) "$2 + 3 = 5$" (true statement).

(iii) "$2 + 5 \geq 10$" (false statement).

Examples (i) and (ii) are 'one -place predicate',
while Example (iii) is a '2 - place predicate.'

PREDICATE

An assertion that contains <u>one or more variables</u> is called a **Predicate**; its truth value is predicated after assigning truth values to its variables.

If we want to specify the variables in a predicate, we denote the predicate by $P\ (x_1, x_2, \ldots, x_n)$. Each variable x_i is also called as an **argument.**

A predicate P containing n variables $x_1, x_2, \ldots, x_n$ is called an **n - place predicate.**

For example,

(i) "x is a city in India" is denoted by P(x).

(ii) "x is the father of y" is denoted by P(x,y)

(iii) "$x + y \geq z$" is denoted by P(x, y, z)

The values which the variables may assume constitute a collection or set called as the **universe of discourse.**

BINDING

A predicate becomes a proposition only when all its variables are bound.

Consider the following examples :

(i) $P(x) : x + 3 = 5$

Let the universe of discourse be the set of all integers.

Binding x by putting $x = -1$, we get a false proposition.

Binding x by putting $x = 2$, we get a true proposition.

(ii) $P(x, y) : x + y = 10$

Let the universe of discourse be the set of natural numbers.

Putting $x = 1$, we get the one-place predicate $P(1,y) : 1 + y = 10$.

Further setting $y = 9$, we obtain the proposition $P(1,9)$ which is true proposition.

If we set $y = 10$, $P(1,10)$ is a false proposition.

In each case, we have bound both the variables (x by 1, y by 9 and y by 10).

UNIVERSAL QUANTIFIER

If P(x) is a predicate with the individual variable x as an argument, then the assertion **"For all x, P(x)"** which is interpreted as **"For all values of x, the assertion P(x) is true,"** is a statement in which the variable x is said to be **universally quantified**.

We denote the phrase "For all" by ∀, called the **universal quantifier**. The meaning of ∀ is "for all" or "for every" or "for each".

If P(x) is true for every possible value of x, then **∀ x P(x) is true;** otherwise **∀ x P(x) is false.**

For Example :

Let P(x) be the predicate x ≥ 0 ; where x is any positive integer.

Then the proposition ∀ x P(x) is true.

EXISTENTIAL QUANTIFIER

Suppose for the predicate P(x), $\forall x\, P(x)$ is false, but there exists at least one value of x for which P(x) is true, then we say that in this proposition, x is bound by existential quantification.

We denote the words "there exists" by the symbol $\exists$.

Then the notation $\exists\, x\, P(x)$ means "there exists a value of x (in the universe of discourse) for which P(x) is true".

QUANTIFIER

Ex. : If $M(x)$ is "x is man" $C(x)$ is "x is clever"

Then translate the following statements into English.

(i) $\exists x \,(M(x) \rightarrow C(x))$

(ii) $\forall x \,(M(x) \wedge C(x))$

Soln. :

(i) There exists a man who is clever.

(ii) For all men x is man and x is clever.

QUANTIFIER

Ex. : Write the following two propositions in symbols.

(i) 'For every number x there is a number y such that $y = x + 1$.'

(ii) 'There is a number y such that, for every number x, $y = x + 1$.'

Soln. :

Let $P(x,y)$ denote the predicate '$y = x + 1$'.

(i) $\forall\, x\ \exists\, y\ P(x, y)$

(ii) $\exists\, y\ \forall\, x\ \mathbf{P(x,y)}$

QUANTIFIER

Ex. : Transcribe the following into logical notation. Let the universe of discourse be the real numbers.

(i)For any value of x, x^2 is non-negative.

(ii)For every value of x, there is some value of y such that $x \cdot y = 1$.

(iii)There are positive values of x and y such that $x \cdot y > 0$.

(iv)There is a value of x such that if y is positive, then $x + y$ is negative.

Soln. :

(i) $\forall x\, [x^2 \geq 0]$

(ii) $\forall x\, \exists y\, [x \cdot y = 1]$

(iii) $\exists x\, \exists y\, [\, (x > 0) \wedge (y > 0) \wedge (x \cdot y > 0)]$

(iv) $\exists x\, \forall y\, [\, (y > 0) \rightarrow (x + y < 0)\,]$.

QUANTIFIER

Ex. : For the universe of all integers, let $P(x)$, $Q(x)$, $R(x)$, $S(x)$ and $T(x)$ be the following statements :

$P(x) : x > 0.$ $Q(x)$: x is even.

$R(x)$: x is a perfect square. $S(x)$: x is divisible by 4.

$T(x)$: x is divisible by 5.

Write the following statements in symbolic form :

(i) Atleast one integer is even.

(ii) There exists a positive integer that is even.

(iii) If x is even, then x is not divisible by 5.

(iv) There exists an even integer divisible by 5.

(v) If x is even and x is a perfect square, then x is divisible by 4.

Soln. :

(i) $\exists\, x\, Q(x)$

(ii) $\exists\, x\, [\, P(x) \wedge Q(x)\,]$

(iii) $\forall\, x\, [\, Q(x) \rightarrow \sim T(x)\,]$

(iv) $\exists\, x\, [\, Q(x) \wedge T(x)\,]$

(v) $\forall\, x\, [\, (Q(x) \wedge R(x)) \rightarrow S(x)\,]$

NORMAL FORMS

When the statements consist of more variables or are of complex form, then the method of employing truth table is not efficient.

Thus we will have to reduce the statement to so called '**Normal forms**'.

1. Disjunctive Normal form denoted by DNF.
2. Conjunctive Normal form denoted by CNF.

DISJUNCTIVE NORMAL FORM (DNF)

An expression of 'n' variables $x_1, x_2, \ldots x_n$ is said to be a **'minterm'** if it is of the form.

$$x_1 \wedge x_2 \wedge x_3 \wedge \ldots \wedge x_n$$

An expression is said to be in **'disjunctive normal form'** if it is a **join** of minterms.

For Example : $(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$

A statement which consists of a disjunction of fundamental conjunctions is called **disjunctive normal form**.

EXAMPLES OF DNF

i. $(p \wedge q) \vee \sim q$
ii. $(\sim p \wedge q) \vee (p \wedge q) \vee q$
iii. $(p \wedge q \wedge r) \vee (p \wedge \sim r) \vee (q \wedge r)$
iv. $(p \wedge \sim q) \vee (p \wedge r)$
v. $((p \wedge q \wedge r) \vee \sim r)$

(**Note :** $\vee$ = join)

DNF

Ex. : Obtain the DNF of the form: $\mathbf{p \wedge (p \rightarrow q)}$

Soln.:

$$
\begin{aligned}
\mathbf{p \wedge (p \rightarrow q)} &\equiv p \wedge (\sim p \vee q) \\
&\equiv (p \wedge \sim p) \vee (p \wedge q) \\
&\equiv F \vee (p \wedge q) \\
&\equiv (p \wedge q)
\end{aligned}
$$

CONJUNCTIVE NORMAL FORM (CNF)

An expression of 'n' variables $x_1, x_2, \ldots, x_n$ is said to be a 'maxterm', if it is of the form

$$x_1 \vee x_2 \vee x_3 \vee \ldots \vee x_n$$

An expression is said to be in **conjunctive normal form (CNF)** if it is a meet of maxterms.

For Ex. : $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

Note that a CNF is a tautology if and only if every fundamental disjunction contained in it is a tautology.

Ex. : (i) $p \wedge q$

(ii) $\sim p \wedge (p \vee q)$

(iii) $(p \vee q \vee r) \wedge (\sim p \vee r)$

CNF

Ex. : Obtain the CNF of the form $(\sim p \to r) \wedge (p \leftrightarrow q)$

Soln.: $(\sim p \to r) \wedge (p \leftrightarrow q)$

$\equiv \quad (\sim p \to r) \wedge (p \leftrightarrow q))$

$\equiv \quad (\sim p \to r) \wedge ((p \to q) \wedge (q \to p))$

$\equiv \quad (\sim (\sim p) \vee r) \wedge ((\sim p \vee q) \wedge (\sim q \vee p))$

$\equiv \quad (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

CNF

Ex. : Obtain the CNF of the form $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

Soln.:

$\equiv$ $(p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r))$ Distributive law

$\equiv$ $((p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)) \wedge ((q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r))$

$\equiv$ $(p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \wedge (q \vee r)$

DNF USING TRUTH TABLE

Ex. :Find the DNF of the form $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

p	q	r	$\sim p$	$\sim p \rightarrow r$	$p \leftrightarrow q$	$(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

Consider the rows of p, q, r in which T appears in the last column.

Then the required DNF is

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$$

CNF, DNF

Ex. : Obtain the CNF and DNF of $\sim (p \vee q) \leftrightarrow (p \wedge q)$

Soln.: $\sim (p \vee q) \leftrightarrow (p \wedge q)$

$$\equiv (\sim (p \vee q) \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow \sim (p \vee q))$$

$$\equiv (\sim\sim (p \vee q) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee \sim (p \vee q))$$

$$\equiv ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$$

$$\equiv (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q))$$

$$\equiv (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$$

$$\equiv (p \vee q) \wedge (\sim p \vee \sim q) \quad \Leftarrow \textbf{CNF}$$

Further,

$$(p \vee q) \wedge (\sim p \vee \sim q)$$

$$\equiv((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$$

$$\equiv(p \wedge \sim p) \vee (q \wedge \sim p) \vee ((p \wedge \sim q) \vee (q \wedge \sim q)$$

$$\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F$$

$$\equiv \mathbf{(q \wedge \sim p) \vee (p \wedge \sim q)} \quad \Leftarrow \textbf{DNF}$$

MATHEMATICAL INDUCTION

In Mathematics, we are often required to generalize a particular solution.

In order to do this, we look for **a pattern** in the particular solution.

Mathematical induction generalizes this pattern of solutions by proving that it is always possible to extend the solution to a group that is one larger than the previous.

The generalization is achieved by using a statement involving a variable natural number.

To the **software engineer**, mathematical induction is an important tool in algorithm verification, to check whether a program statement is loop invariant, that is, whether it is true before and after every pass through a programming loop.

MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving a natural number n.

Basis of induction 1. If $P(n)$ is true for $n = n_0$ and

Induction step 2. Assuming $P(k)$ is true, $(k \geq n_0)$ we prove $P(k + 1)$ is also true, then $P(n)$ is true for all natural numbers $n \geq n_0$

The assumption that $P(n)$ is true for $n = k$ is called as the **Induction hypothesis.**

MATHEMATICAL INDUCTION

Ex. 1 : Prove by induction :

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$ for all natural number values of n.

Soln. : Let P(n) be the statement :

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(i) Basis of induction :

for n = 1,

$$P(1) : 1 = \frac{1(2)}{2} = 1$$

Hence P(1) is true.

(ii) Induction step :

Assume P(k) is true,

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \qquad \ldots(i)$$

(This assumption is called the induction hypothesis)

Prove P(k + 1) is also true.

$$P(k+1) : 1 + 2 + 3 + \ldots + k + (k+1)$$

$$= \frac{(k+1)[(k+1)+1]}{2} \qquad \ldots(ii)$$

$$= \frac{(k+1)(k+2)}{2}$$

Using equation (i)

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also ture. Therefore P(n) is true for all natural number values of n.

Prove that : $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$

Soln. : Let P (n) be the statement :

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1} \qquad \text{(i)}$$

(i) Basis of induction :

For n = 1 $\quad P(1) : \frac{1}{1.4} = \frac{1}{4}$ $\qquad$ Hence P(1) is true.

(ii) Induction step :

Assume P(k) is true, and prove P(k + 1) is also true.

$$P(k) : \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$

$$P(k+1) : \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{k+1}{3(k+1)+1} \qquad \text{...(ii)}$$

Using induction hypothesis (i),

$$L.H.S. of\ Equation\ (ii) = \frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$\dots\left[\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}\right]$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}=\frac{k+1}{3(k+1)+1}$$

$\therefore$ L.H.S = R.H.S. of Equation (ii)

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Show that $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$

Soln. :

Let P(n) : $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$

(i) Basis of induction :

For $n = 1 (1)^3 = (1)^2$ which is true

$\therefore$ P(1) is true.

(ii) Induction step :

Assume P(k) is true i.e.,

$$1^3 + 2^3 + 3^3 + \dots + k^3 = (1 + 2 + \dots + k)^2 \qquad \dots(i)$$

To prove that P(k + 1) is true i.e.:

$$1^3 + 2^3 + 3^3 + \dots + (k+1)^3 = [1 + 2 + \dots + (k+1)]^2$$

$$\begin{aligned}
\text{L.H.S.} &= 1^3 + 2^3 + 3^3 + \dots k^3 + (k+1)^3 \\
&= [1^3 + 2^3 + 3^3 + \dots k^3] + (k+1)^3 \\
&= (1 + 2 + \dots + k)^2 + (k+1)^3 \quad \dots\text{by induction hypothesis, From Equation (i)} \\
&= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \qquad \dots\left[\text{Recall that } 1 + 2 + \dots + k = \frac{k(k+1)}{2}\right] \\
&= \frac{k^2 (k+1)^2}{4} + (k+1)^3 \\
&= (k+1)^2 \left[\frac{k^2}{4} + (k+1)\right] \\
&= (k+1)^2 \left[\frac{k^2 + 4k + 4}{4}\right] \\
&= \frac{(k+1)^2 (k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2 \\
&= [1 + 2 + \dots + (k+1)]^2
\end{aligned}$$

Hence, L.H.S = R.H.S.

Therefore P(n) is true.