Fourier Series in $(0,2\pi)$

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Formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Examples :-

① Find a Fourier series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence, deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$

Solution :- Let $$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{in } (0, 2\pi) \qquad (1)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right]$$

$$\therefore \boxed{a_0 = \frac{8\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\cancelto{0}{\frac{\sin nx}{n}}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\cancelto{0}{\frac{-\sin nx}{n^3}}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2x\,\frac{\cos nx}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[4\pi\,\frac{\cos 2n\pi}{n^2} - 0\right]$$

$$\boxed{a_n = \frac{4}{n^2}}$$

2π 2π

Now $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\cancel{-\frac{\sin nx}{n^2}}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$\to 0$

$$= \frac{1}{\pi}\left[\left(\frac{-4\pi^2\cos 2n\pi}{n} - 0\right) + 2\left(\frac{\cos 2n\pi}{n^3} - \frac{\cos 0}{n^3}\right)\right]$$

Now $\cos 2n\pi = 1$, $\cos 0 = 1$

$\therefore \boxed{b_n = \frac{-4\pi}{n}}$

Substituting these values in (1)

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\therefore x^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \cdots\right] - 4\pi\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots\right]$$

Now put $x = \pi$

$$\therefore \pi^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2}\cdots\right]$$

$$\frac{-\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2}\cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Hence proved.

Example-2 :- Obtain the Fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ and $f(x+2\pi) = f(x)$

Also Deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ (iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

<u>Solution</u> :- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\,dx = \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^3}{-3} - \frac{\pi^3}{-3}\right] = \frac{1}{4\pi}\left[\frac{2\pi^3}{3}\right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{\cancelto{0}{\sin nx}}{n}\right) - (-2(\pi-x))\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\cancelto{0}{\sin nx}}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-2(\pi-x)\left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi\frac{\cos 2n\pi}{n^2} - (-2\pi)\frac{\cos 0}{n^2}\right] = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$\therefore \boxed{a_n = \frac{1}{n^2}}$$

Now, $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - (-2(\pi-x))\left(\frac{-\cancelto{0}{\sin nx}}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\pi^2\frac{\cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3}\right) - \left(-\pi^2\frac{\cos 0}{n} + \frac{2\cos 0}{n^3}\right)\right]$$

$$b_n = \frac{1}{4\pi}\left[\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$\therefore \boxed{b_n = 0}$

Substituting these values in (1)

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots \quad \text{—— (2)}$$

(i) Now we put $x = 0$ in (2)

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{—— (3)}$$

(ii) Again, put $x = \pi$ in (2)

$$0 = \frac{\pi^2}{12} + \frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \cdots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{—— (4)}$$

(iii) To get the result (iii), add (3) & (4)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

(iv) To derive the last result, we use parseval's Identity

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum(a_n^2 + b_n^2) \quad \text{—— (5)}$$

Now $$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^4}{16} dx = \frac{1}{16\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{2\pi}$$

$$= \frac{-1}{80\pi}\left[-\pi^5 - \pi^5\right] = \frac{\pi^4}{40}$$

Substiting in (5)

$$\frac{\pi^4}{40} = \frac{\pi^4}{72} + \sum \frac{1}{n^4}$$

$$\therefore \sum \frac{1}{n^4} = \frac{\pi^4}{40} - \frac{\pi^4}{72} = \frac{9\pi^4 - 5\pi^4}{360} = \frac{4\pi^4}{360} = \frac{\pi^4}{90}$$

$$\therefore \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Hence proved.

Example-3 Find the Fourier series for $f(x) = e^x$ in $(0, 2\pi)$

Solution :- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ —— (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_0^{2\pi} = \frac{1}{\pi}\left[e^{2\pi} - 1\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\,dx$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx + n\,\underbrace{\sin nx}_{\to 0}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(\cos 2n\pi) - \frac{e^0}{1+n^2}(\cos 0)\right]$$

$$\boxed{a_n = \frac{1}{\pi(1+n^2)}\left(e^{2\pi} - 1\right)}$$

(as $\cos 2n\pi = \cos 0 = 1$)

Now $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\,dx$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{\quad}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cancel{\sin nx}^{\to 0} - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{2\pi}(-n\cos 2n\pi) - e^0(-n\cos 0)\right]$$

$$\therefore \boxed{b_n = \frac{n(1-e^{2\pi})}{\pi(1+n^2)}}$$

$\therefore$ Fourier series for $f(x) = e^x$ in $(0, 2\pi)$

$$f(x) = e^x = \frac{(e^{2\pi}-1)}{2\pi} + \sum \frac{(e^{2\pi}-1)}{\pi(1+n^2)}\cos nx + \sum \frac{n(1-e^{2\pi})}{\pi(1+n^2)}\sin nx$$

Example-4: Find Fourier series for $f(x) = \begin{cases} -\pi & 0<x<\pi \\ x-\pi & \pi<x<2\pi \end{cases}$

state the value of the series at $x=\pi$ and hence, show that $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

Solution :- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} -\pi\,dx + \int_{\pi}^{2\pi} x-\pi\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi(x)_0^{\pi} + \left(\frac{x^2}{2} - \pi x\right)_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\pi(\pi) + \left(\frac{4\pi^2}{2} - 2\pi^2 - \frac{\pi^2}{2} + \pi^2\right)\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = \frac{1}{\pi}\left\{\frac{-\pi^2}{2}\right\}$$

$$\therefore \boxed{a_0 = -\frac{\pi}{2}}$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} -\pi\cos nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{ -\pi\cancel{\left(\frac{\sin nx}{n}\right)_0^{\pi}}^{\to 0} + \left[(x-\pi)\cancel{\left(\frac{\sin nx}{n}\right)}^{\to 0} - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi} = \frac{\cos 2n\pi - \cos n\pi}{\pi n^2} = \frac{1-(-1)^n}{\pi n^2}$$

$$a_n = \begin{cases} \frac{2}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} -\pi\sin nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_0^{\pi} + \left[(x-\pi)\left(\frac{-\cos nx}{n}\right) - (1)\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{\to 0}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left(\cos n\pi - \cos 0\right) + \left[\frac{-\pi}{n}\cos 2n\pi - 0\right]\right]$$

$$= \frac{1}{n}\left[\cos n\pi - \cos 0 - \cos 2n\pi\right]$$

$$b_n = \frac{1}{n}\left[(-1)^n - 2\right]$$

$$b_n = \begin{cases} -\frac{3}{n} & \text{if } n \text{ is odd} \\ -\frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ Fourier series for $f(x)$ in $(0, 2\pi)$ is

$$f(x) = -\frac{\pi}{4} + \sum \frac{1-(-1)^n}{\pi n^2}\cos nx + \sum \frac{(-1)^n - 2}{n}\sin nx$$

$$= -\frac{\pi}{4} + \frac{2}{\pi}\cos x + \frac{2}{\pi 3^2}\cos 3x + \frac{2}{\pi 5^2}\cos 5x + \cdots\cdots$$

$$+\left(-\frac{3}{1}\sin x-\frac{1}{2}\sin 2x-\frac{3}{3}\sin 3x-\frac{1}{4}\sin 4x\cdots\cdots\right) \quad -(2)$$

Now, the function is discontinuous at $x=\pi$

$\therefore$ at $x=\pi$, the Fourier series will take a value

$$f(\pi)=\frac{1}{2}\left[\lim_{x\to\pi-}f(x)+\lim_{x\to\pi+}f(x)\right]=\frac{1}{2}\left[-\pi+0\right]=\frac{-\pi}{2}$$

Now put $x=\pi$ in (2)

$$f(\pi)=-\frac{\pi}{4}+\frac{2}{\pi}\cos(\pi)+\frac{2}{\pi 3^2}(\cos 3\pi)+\frac{2}{\pi 5^2}\cos 5\pi+\cdots\cdots$$

$$+\left(-\frac{3}{1}\sin\pi-\frac{1}{2}\sin 2x-\frac{3}{3}\sin 3x\cdots\cdots\right)$$

$$-\frac{\pi}{2}=-\frac{\pi}{4}-\left(\frac{2}{\pi}+\frac{2}{\pi 3^2}+\frac{2}{\pi 5^2}+\cdots\right)$$

$$-\frac{\pi}{2}+\frac{\pi}{4}=-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]=-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

$$\therefore \frac{\pi^2}{8}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

Hence proved.