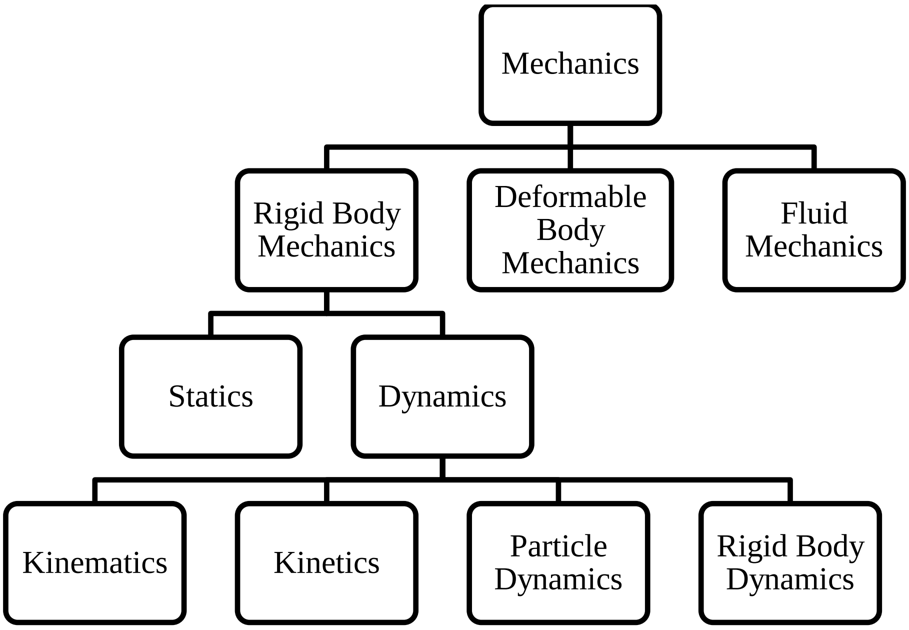
Module 1 – System of Forces

Module Section 1.1 – System of Coplanar Forces

Mechanics is defined as the branch of physics which deals with the study of resultant effect of action of forces on bodies, which may be at rest or in motion.

Classification of Mechanics:



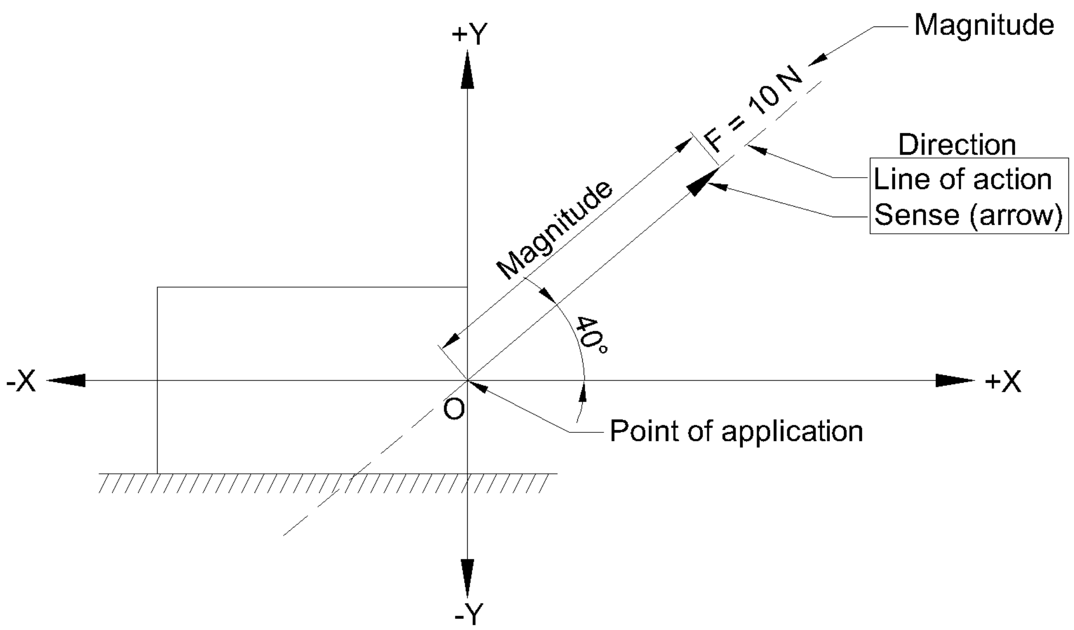
- Rigid Body Mechanics: In this, bodies are assumed to be perfectly rigid, i.e., there is no deformation of bodies under any load acting on them.
- Statics: It is the study of effect of force system acting on a particle or rigid body which is at rest.
- Dynamics: It is the study of effect of force system acting on a particle or rigid body which is in motion. Dynamics can be classified into two categories: "Kinematics and Kinetics", depending on whether forces acting on the body are considered; or "Particle Dynamics and Rigid Body Dynamics", depending on whether the body is considered to be an ideal particle or a rigid body.
    - Kinematics is concerned only with the study of motion of the body without consideration of the forces causing the motion.
    - Kinetics relates the forces acting on the body to the motion of the body.
    - Particle dynamics is the motion analysis of a body considering the body as an idealized particle.
    - Rigid body dynamics is the motion analysis involving the shape and size of the body.

Force: A force is defined as an external agency or action which changes or tends to change the state of rest or of uniform motion of a body upon which it acts.

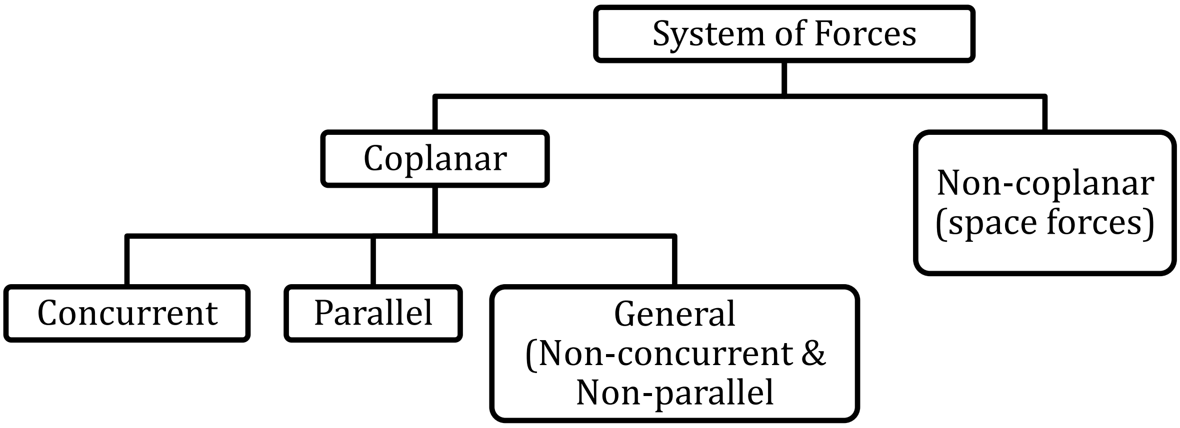
- It is a result of the action of one body on another. It could be due to direct action like physically lifting something or by remote action like gravitational force.
- It imparts motion or affects the motion of a body.
- It can accelerate, decelerate or stop a moving body depending on its direction.
- It may rotate a body.
- It may maintain the equilibrium condition of a body.
- It may be of push type or pull type.

Force is a vector quantity and is quantified by following factors:

a) Magnitude: It is the quantity of force measured in Newtons.
b) Direction (Line of action and sense): It is the orientation of line of action indicated by an angle with respect to the reference axes, along with a sense that indicates on which side the force acts.
c) Point of application (aka Location): It is the exact point at which the force acts and can be indicated by the distance from the origin.



System of Forces: When a number of forces act simultaneously on a body then they are said to form a system of forces or force system.

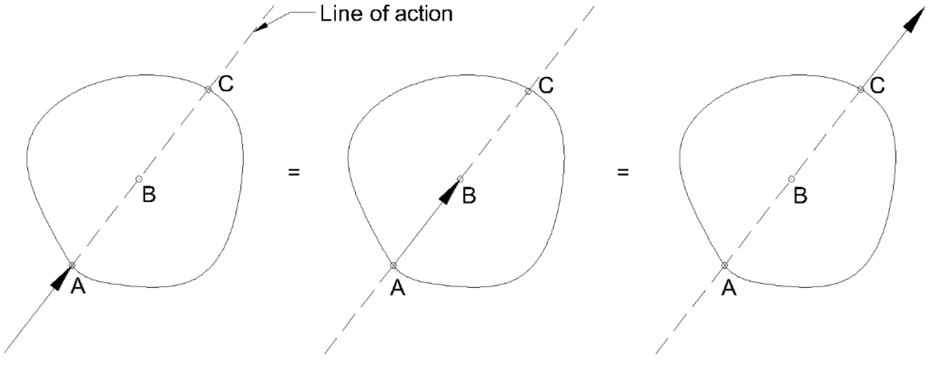


In Coplanar System of Forces, all forces lie in one plane; while in Non-coplanar System of Forces, all the forces in the system do not lie in a single plane.

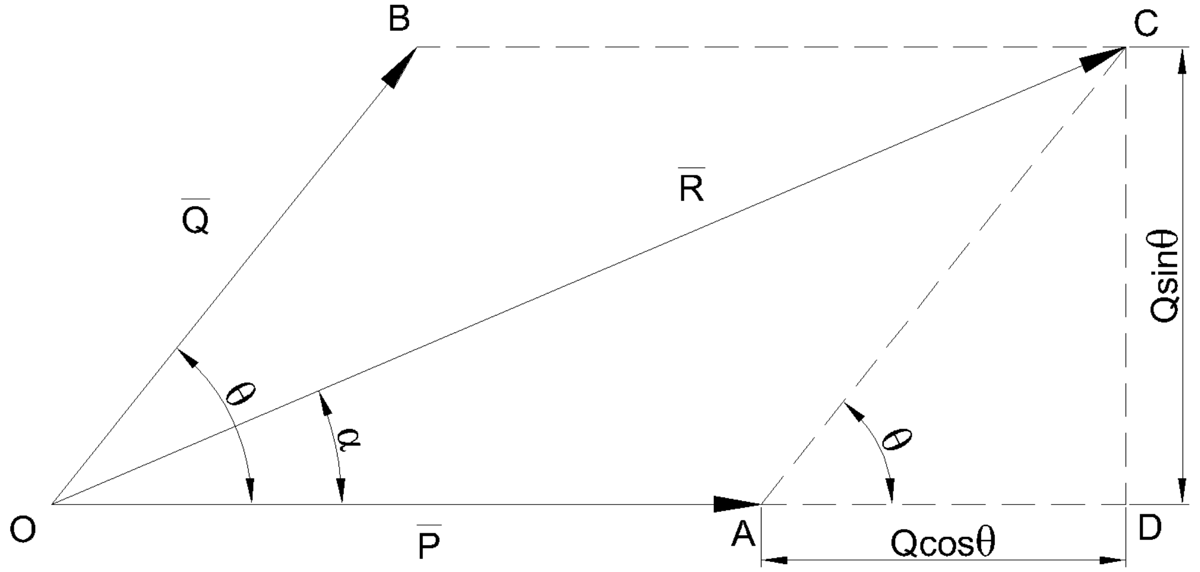
Both systems can be subdivided into:

a) Concurrent Force System: In this, the lines of action of all the forces in the system pass through the same point.
e.g., lamp hanging from string, electric pole supporting heavy cables, forces on a tripod, etc.
b) Parallel Force System: In this, the lines of action of all the forces in the system are parallel to each other.
e.g., weighing scale, things places on a table, people sitting on a bench, etc.
c) General Force System: In this, the lines of action of all the forces in the system are neither concurrent nor parallel to each other.
e.g., a moving vehicle has engine power, friction due to road, wind resistance, weight of vehicle and passengers, etc. acting in various directions.

Principle of Transmissibility of Force: A force being a sliding vector will not affect the state of a rigid body (whether at rest or in motion) if the force acts from a different point along its line of action. E.g., in a train, the engine can be located at the front pulling the other cars with it or at the back pushing them forward.



Law of Parallelogram for Vectors: If two vectors acting simultaneously at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the vectors.

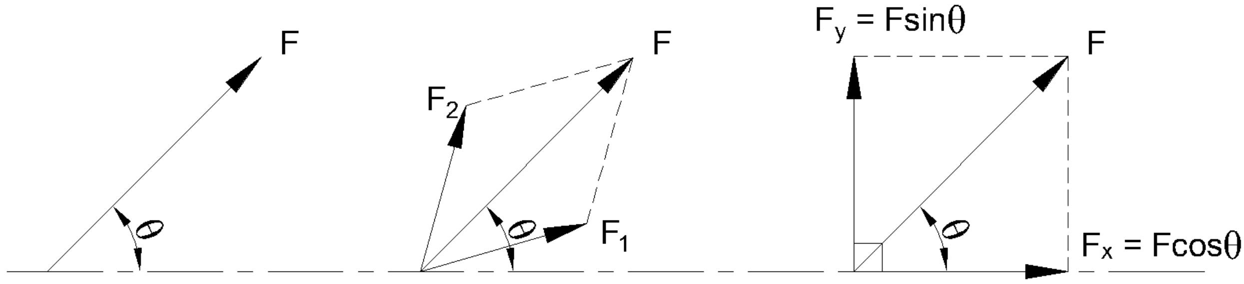


In ΔOCD,

$$OC^2 = OD^2 + CD^2 = (OA + AD)^2 + CD^2$$
$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$
$$R^2 = P^2 + 2PQ\cos\theta + Q^2\cos^2\theta + Q^2\sin^2\theta$$
$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$
$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA + AD}$$
$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

Resolution of Forces: It is the process of breaking a force into components, such that the components combined together would have the same effect as the original force. There are various ways to resolve a force, but the most beneficial for calculations is the resolution of force into its rectangular (or perpendicular) components.



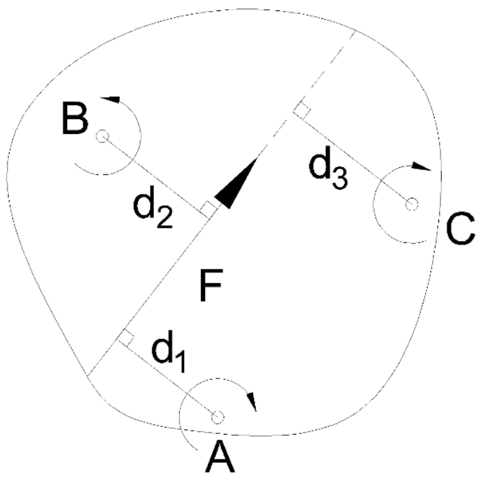
Note: In rectangular resolution, the component of force adjacent to the given angle is taken as Fcosθ, and not always the x-direction component.

Moment of a Force: The rotational effect of a force is known as moment. Basically, the tendency of a force to rotate a rigid body about an axis or a point is measured by the moment of the force about that axis or point. E.g., door being opened or closed about a hinge, tightening or loosening of nut with a spanner, etc.

The point about which we calculate the moment is called the moment centre, and the rotational effect of the same force will vary from one moment centre to another.

The moment is measured by multiplying the magnitude of force and the perpendicular distance from the moment centre. This perpendicular distance is called as moment arm.

Moments of force F about moment centres A, B, C, with moment arms d_1, d_2, d_3:



$$M_A^F = F \times d_1\ (\circlearrowright)$$
$$M_B^F = F \times d_2\ (\circlearrowleft)$$
$$M_C^F = F \times d_3\ (\circlearrowright)$$

Assuming anti-clockwise as positive, written as $\circlearrowleft$ +ve. We can also write the moments as,

$$M_A^F = -F \times d_1$$
$$M_B^F = +F \times d_2$$
$$M_C^F = -F \times d_3$$

Note: To find the direction of rotation, consider the moment centre as a fixed point or a hinge and based on the direction of the force, the body will tend to rotate in clockwise or anti-clockwise. That is the direction of the moment.

Varignon's Theorem: The algebraic sum of the moments of a system of coplanar forces about any point in the plane is equal to the moment of the resultant force of the system about the same point.

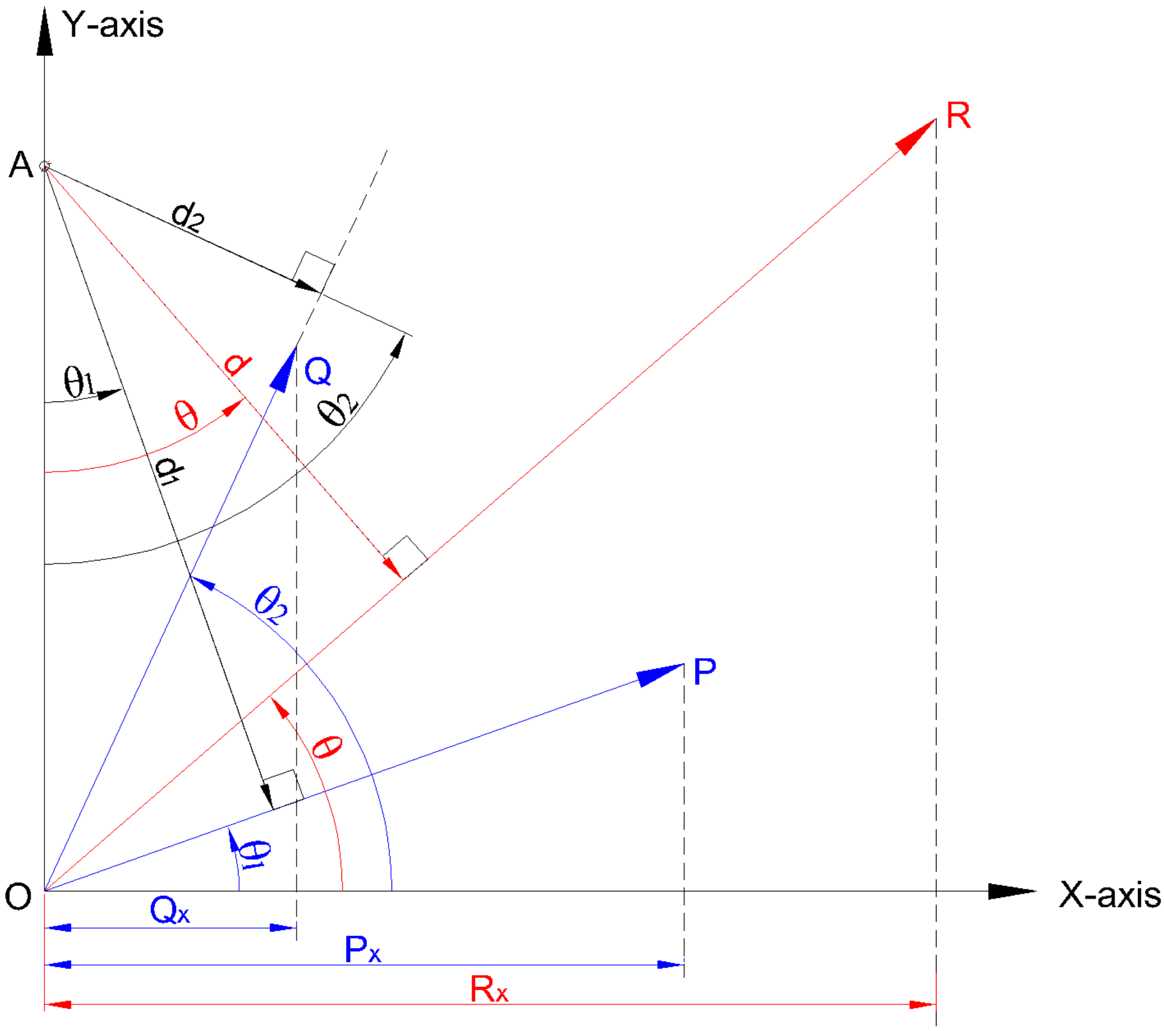
(Note: While the statement and proof is for coplanar forces, the theorem is applicable to a system of non-coplanar forces as well.)

Proof:

Let P & Q be 2 concurrent forces acting at O (origin), making angles θ_1 & θ_2 with x-axis. Let their resultant be R making angle θ with x-axis.

Let A be a point on the y-axis about which moments are to be taken. Let d_1, d_2, & d be the moment arms of P, Q & R respectively from moment centre A.

Let the components of the forces in x-direction be denoted by adding a 'subscript x'.



Moment due to force P about A,

$$M_A^P = +P \times d_1$$
$$M_A^P = P \times OA \cos\theta_1$$
$$M_A^P = P\cos\theta_1 \times OA$$
$$M_A^P = P_x OA$$

Similarly, moment due to Q about A, $M_A^Q = +Q \times d_2 = Q_x OA$

and, moment due to R about A, $M_A^R = +R \times d = R_x OA$

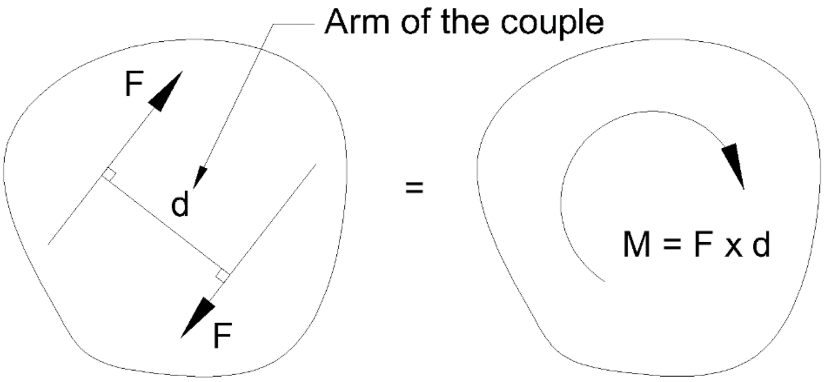
Now, the sum of moments of forces P & Q about A is given by,

$$\sum M_A^F = M_A^P + M_A^Q$$
$$\sum M_A^F = P_x OA + Q_x OA$$
$$\sum M_A^F = (P_x + Q_x)OA$$
$$\sum M_A^F = R_x OA$$
$$\sum M_A^F = M_A^R$$

This means that the sum of moments of the two forces about a point is equal to the moment of resultant force about that point.

Hence, the theorem is proved.

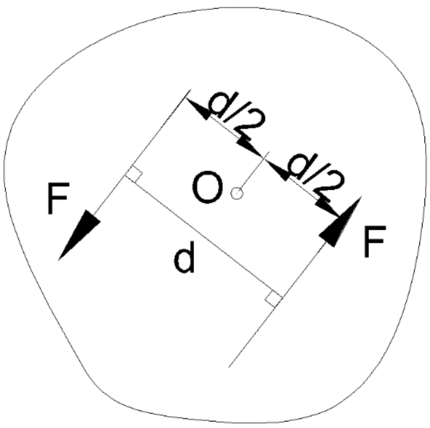
Couple: Two non-collinear parallel forces of equal magnitude but in opposite directions to each other form a couple. It causes rotation of the body. E.g., steering wheel of a vehicle, key rotation to lock or unlock, tap opening or closing, etc.

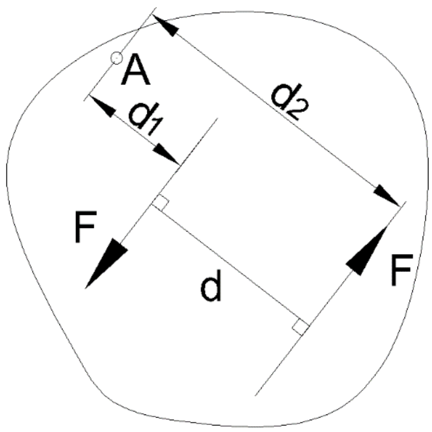


Properties of a couple:

1. Moment of a couple is equal to the product of one of the forces and the arm of the couple.
2. It tends to rotate the body about an axis perpendicular to the plane of the forces involved. It can only rotate and not translate the body.
3. The resultant force if a couple is zero.
4. Moment of a couple can be added algebraically as scalar quantity with proper sign convention.
5. A couple can be replaced by a couple only and not by a single force.
6. A couple is a free vector and does not have a moment centre, like moment of a force.

Couple is a free vector:



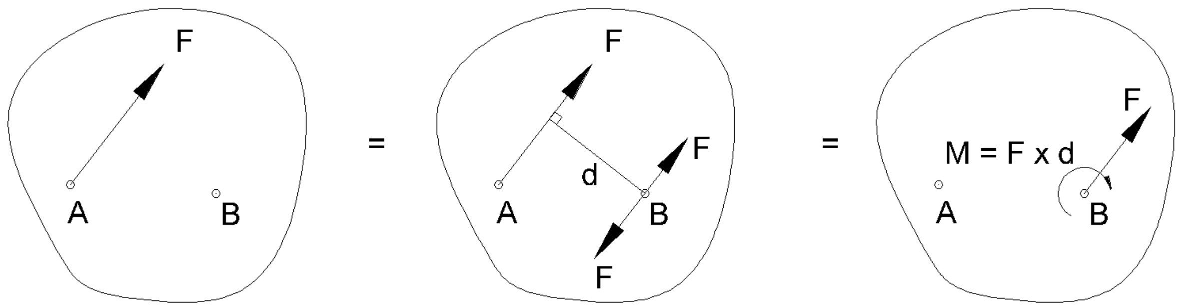


$$M_O^F = +F\times\frac{d}{2} + F\times\frac{d}{2}$$
$$M_O^F = F\times d$$

$$M_A^F = -F\times d_1 + F\times d_2$$
$$M_A^F = F(d_2 - d_1)$$
$$M_A^F = F\times d$$

Shift a force to a new parallel position:

To shift a force to a new parallel position, a couple is required to be added to the system. In below figure, force F acting at A is to be shifted to B. First step is to add two collinear forces of equal magnitude F & -F at B. The force F at A and force –F at B are parallel and form a couple with couple arm d. Thus, we have a single force F at B and a couple $M = F \times d$ in the system.



Force originally at A

Two collinear equal but opposite forces added at B

Force shifted to B with a couple due to two forces.

Composition of Forces or Resultant of Forces: Composition means to combine the forces acting in a system into a single force, which has the same effect as the number of forces acting together. This single force is known as the resultant of the system.

Types of Resultant:

a) Resultant Force: After composition of a number of forces results in a single force, it is called as a resultant force indicated by a magnitude and angle.
b) Resultant Couple: If a resultant force is zero, but the resultant moment is not zero, such a system reduces to a couple. It is possible in a Parallel system or General system, but not in a Concurrent system.
c) Resultant Force-Couple: When a resultant force is shifted to a new parallel position without change in direction, it introduces a couple in the system. This resultant consists of a single force and a single couple.

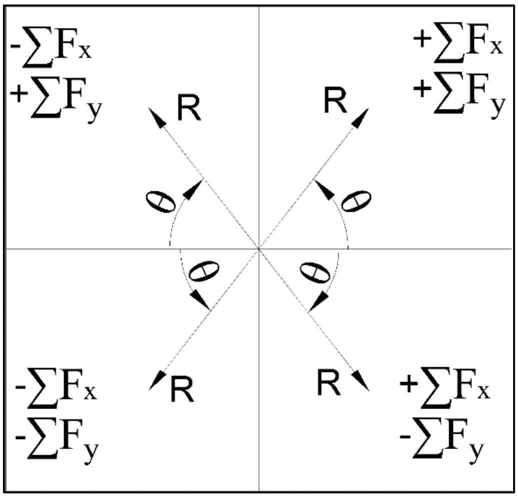
Types of Problems on Resultant of Forces:

1. Resultant of Concurrent Force System using Parallelogram Law of Vectors: Since force is a vector, two concurrent forces can be combined into a resultant using the law of parallelogram. This can be applied to multiple forces, two at a time, but that becomes cumbersome.
2. Resultant of Concurrent Force System using Method of Resolution:
   Step 1: Resolve all forces into their components along horizontal x-direction and vertical y-direction.
   Step 2: Find the algebraic sum of all components in x-direction to get $\sum F_x$, using $\rightarrow$ +ve; and all components in y-direction to get $\sum F_y$, using $\uparrow$ +ve.

Step 3: Find the magnitude if the resultant force, R and its direction considering the angle θ it makes with x-axis.

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2} \;\; \& \;\; \tan\theta = \frac{\sum F_y}{\sum F_x}$$

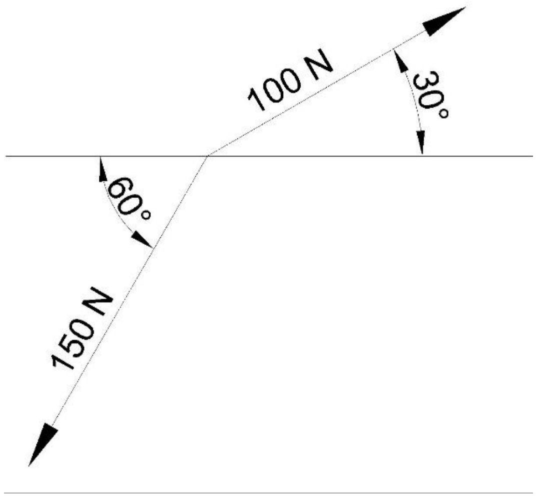
Step 4: Draw a simple diagram of resultant with proper direction considering the signs of $\sum F_x$ & $\sum F_y$ as shown below.



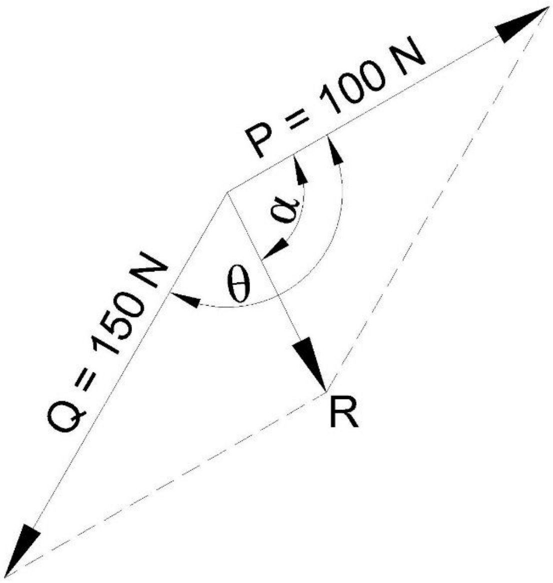
3. <u>Resultant of Parallel Force System using Varignon's Theorem</u>:
   Step 1: Since all forces are in one direction only, they can simply be added taking proper sign conventions. $R = \sum F$
   Step 2: Location of the resultant force (point of application) is found using Varignon's theorem. We assume the resultant to be acting at some perpendicular distance d to the right or left of the reference point.
   $$\sum M_A^F = M_A^R = R \times d$$
   If value of d is positive, the assumption of right or left is correct. If value of d is negative, resultant lies to the opposite side to what was assumed.
4. <u>Resultant of General Force System (Non-coplanar & Non-parallel)</u>:
   The steps for this are basically a combination of both the above systems.
   Step 1: Resolve all forces into their components along horizontal x-direction and vertical y-direction.
   Step 2: Find the algebraic sum of all components in x-direction to get $\sum F_x$, using → +ve; and all components in y-direction to get $\sum F_y$, using ↑ +ve.
   Step 3: Find the magnitude if the resultant force, R and its direction considering the angle θ it makes with x-axis.
   $$R = \sqrt{(\sum F_x)^2 + \left(\sum F_y\right)^2} \;\; \& \;\; \tan\theta = \frac{\sum F_y}{\sum F_x}$$
   Step 4: Find the perpendicular distance d where the resultant acts using Varignon's theorem.
   $$\sum M_A^F = M_A^R = R \times d$$
   Step 5: Draw a simple diagram of resultant with proper direction and distance labelled from reference point.

Numericals:

N1: Find the resultant of the given two forces.



Soln: Let, $P = 100\text{ N}, Q = 150\text{ N}, \theta = 150°$



Using Law of Parallelogram, the resultant force,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$R = \sqrt{100^2 + 150^2 + 2 \times 100 \times 150 \times \cos 150°}$$

$$R = 80.74\text{ N}$$

And, the angle between P & R,

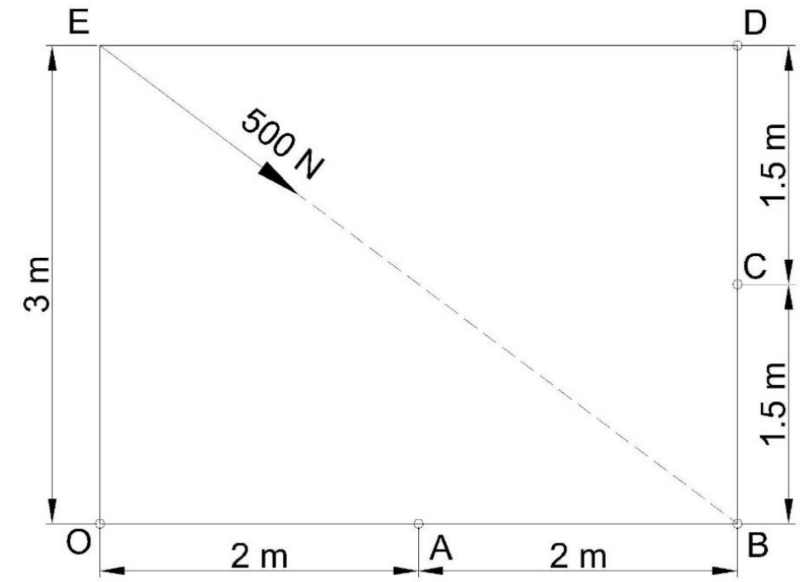
$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

$$\tan\alpha = \frac{150 \times \sin 150°}{100 + 150 \times \cos 150°}$$

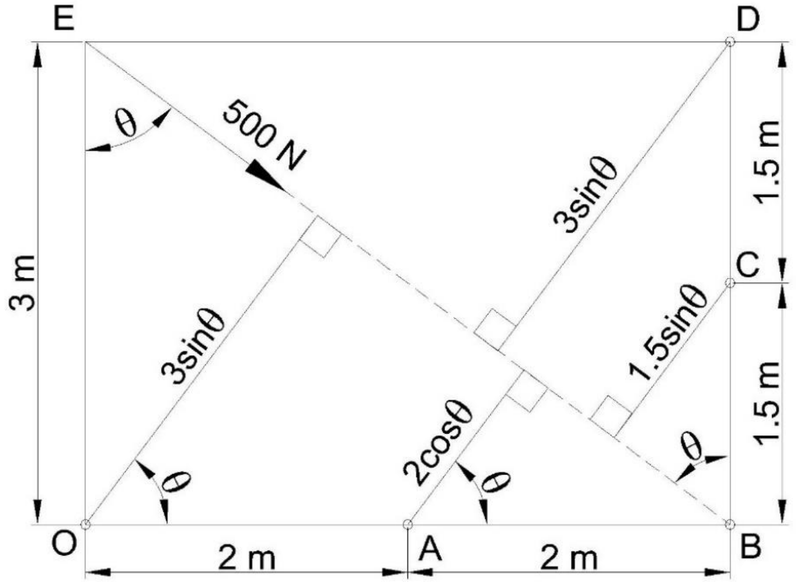
$$\tan\alpha = |-2.51|$$

$$\therefore \alpha = 68.26°$$

N2: Find the moment of the 500 N force about the points O, A, B, C & D.



Soln: Method 1: Perpendicular distances to the line of action



$$\theta = \tan^{-1}\frac{OB}{OE}$$

$$\theta = \tan^{-1}\frac{4}{3}$$

$$\therefore \sin\theta = \frac{4}{5} = 0.8$$

$$\& \cos\theta = \frac{3}{5} = 0.6$$

Moment about O, $M_O = -500 \times 3\sin\theta = -500 \times 3 \times 0.8 = -1200$ Nm

$$M_O = 1200 \text{ Nm } (\circlearrowright)$$

Moment about A, $M_A = -500 \times 2\cos\theta = -500 \times 2 \times 0.6 = -600$ Nm

$$M_A = 600 \text{ Nm } (\circlearrowright)$$

Moment about B, $M_B = 0$, as the point lies on the line of action of force.

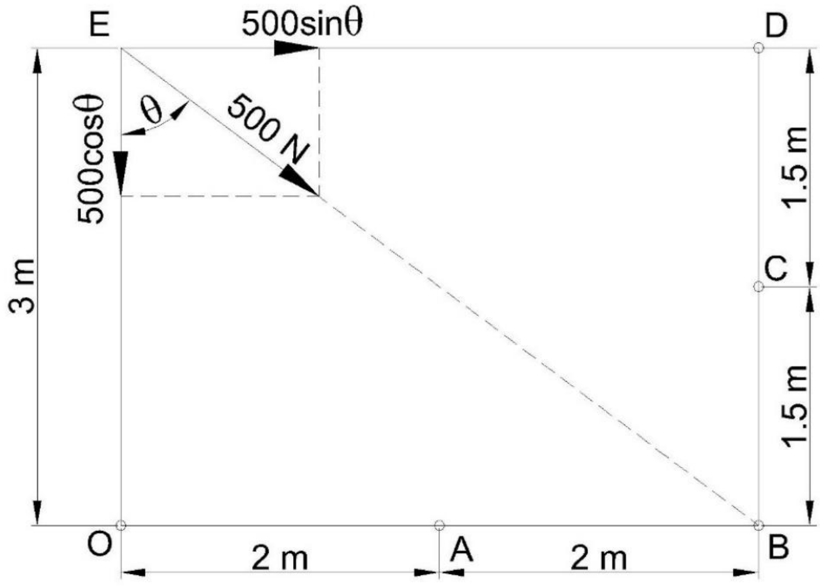
Moment about C, $M_C = +500 \times 1.5\sin\theta = +500 \times 1.5 \times 0.8 = +600$ Nm

$$M_C = 600 \text{ Nm } (\circlearrowleft)$$

Moment about D, $M_D = +500 \times 3\sin\theta = +500 \times 3 \times 0.8 = +1200$ Nm

$$M_D = 1200 \text{ Nm } (\circlearrowleft)$$

Method 2: Resolution of force into components along known distances



Moment about O,

$$\sum M_O = -500\sin\theta \times 3 + 500\cos\theta \times 0$$
$$\sum M_O = -500 \times 0.8 \times 3 + 0 = -1200 \text{ Nm}$$
$$\sum M_O = 1200 \text{ Nm } (\circlearrowright)$$

Moment about A,

$$\sum M_A = -500\sin\theta \times 3 + 500\cos\theta \times 2$$
$$\sum M_A = -500 \times 0.8 \times 3 + 500 \times 0.6 \times 2 = -600 \text{ Nm}$$
$$\sum M_A = 600 \text{ Nm } (\circlearrowright)$$

Moment about B,

$$\sum M_B = -500\sin\theta \times 3 + 500\cos\theta \times 4$$
$$\sum M_B = -500 \times 0.8 \times 3 + 500 \times 0.6 \times 4$$
$$\sum M_B = 0$$

Moment about C,

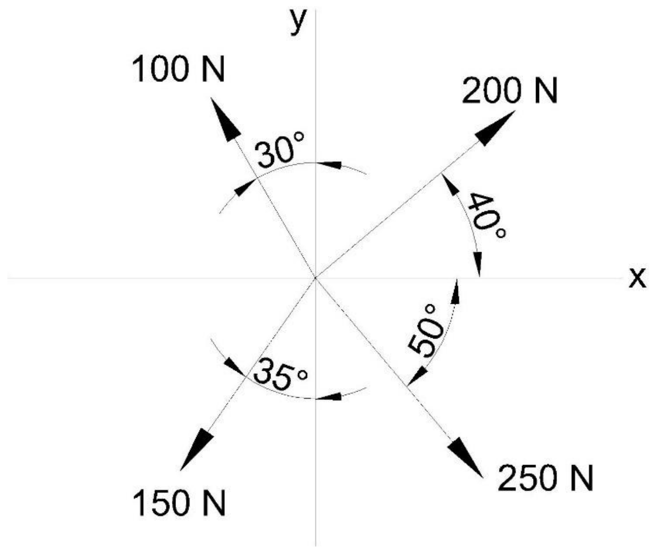
$$\sum M_C = -500\sin\theta \times 1.5 + 500\cos\theta \times 4$$
$$\sum M_C = -500 \times 0.8 \times 1.5 + 500 \times 0.6 \times 4 = +600 \text{ Nm}$$
$$\sum M_C = 600 \text{ Nm } (\circlearrowleft)$$

Moment about D,

$$\sum M_D = -500\sin\theta \times 0 + 500\cos\theta \times 4$$
$$\sum M_D = 0 + 500 \times 0.6 \times 4 = +1200 \text{ Nm}$$
$$\sum M_D = 1200 \text{ Nm } (\circlearrowleft)$$

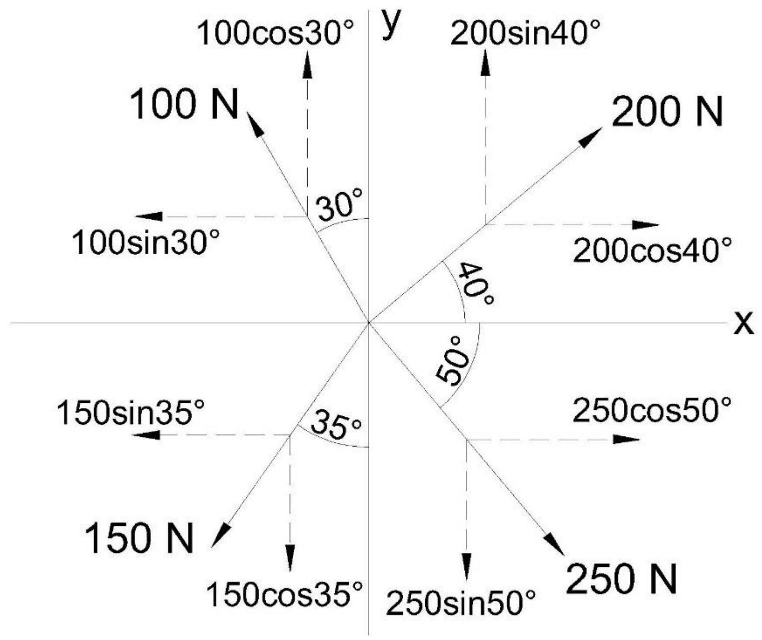
NOTE: Method 1 can become complicated as many perpendicular distances need to be calculated, hence Method 2 is recommended.

N3: Find the resultant of the following four forces as shown in the figure.



Soln: This is a concurrent force system.

Resolving all forces in their horizontal and vertical components:



$$\therefore \sum F_x = 200\cos 40° - 100\sin 30° - 150\sin 35° + 250\cos 50°$$
$$\Rightarrow \sum F_x = +177.869\text{ N or }177.869\text{ N }(\rightarrow)$$

$$\therefore \sum F_y = 200\sin 40° + 100\cos 30° - 150\cos 35° - 250\sin 50°$$
$$\Rightarrow \sum F_y = -99.224\text{ N or }99.224\text{ N }(\downarrow)$$

Hence, the magnitude of the resultant force,

$$\therefore R = \sqrt{(\sum F_x)^2 + \left(\sum F_y\right)^2}$$
$$\Rightarrow R = \sqrt{(177.869)^2 + (-99.224)^2}$$
$$\Rightarrow R = 203.673\text{ N}$$

Also, the angle it makes with the x-axis,

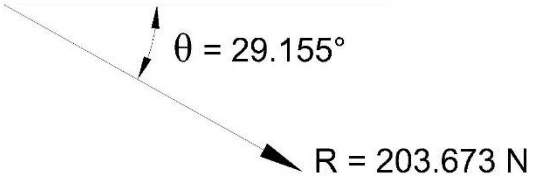
$$\therefore \tan\theta = \frac{\sum F_y}{\sum F_x} = \frac{99.224}{177.869}$$

$$\theta = 29.155°$$

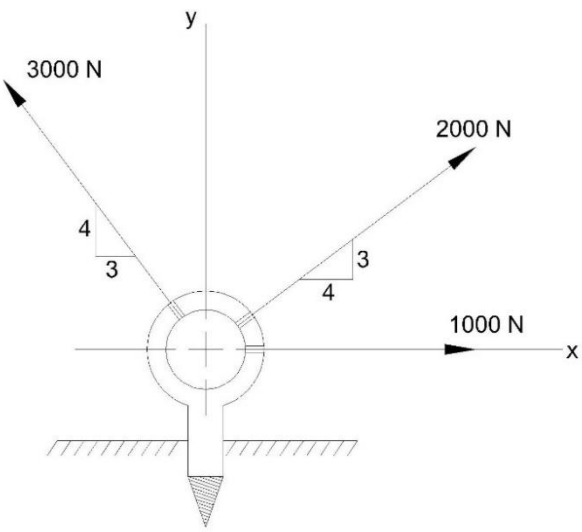
This gives the direction of the resultant.

Since, $\sum F_x$ is in → direction and $\sum F_y$ is in ↓ direction, the resultant R lies in the 2nd quadrant (↘). Hence, the resultant of the given forces is given by,

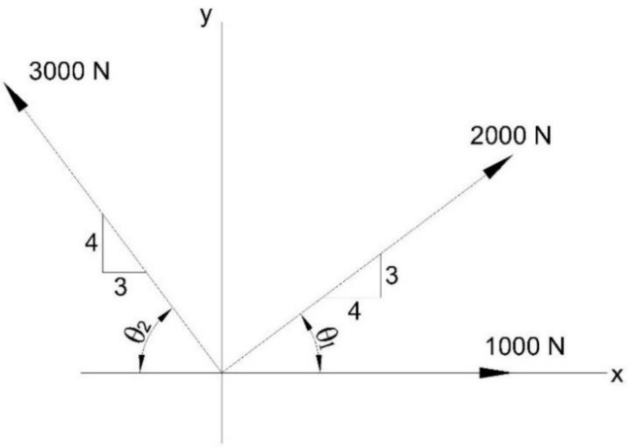
$$\boxed{\begin{array}{c} R = 203.673\ \text{N} \\ \text{at } \theta = 29.155°\ (\searrow) \\ \text{acting at origin.} \end{array}}$$



<u>N4</u>: An eye bolt is being pulled from the ground by three forces as shown. Determine the resultant force acting on the eye bolt.



Soln: This is a concurrent force system, all forces acting at the centre of eye bolt.



$$\theta_1 = \tan^{-1}\frac{3}{4}\ \&\ \theta_2 = \tan^{-1}\frac{4}{3}$$

$$\therefore \sin\theta_1 = \frac{3}{5} = 0.6$$

$$\&\ \sin\theta_2 = \frac{4}{5} = 0.8$$

$$\therefore \cos\theta_1 = \frac{4}{5} = 0.8$$

$$\&\ \cos\theta_2 = \frac{3}{5} = 0.6$$

Resolving the forces in x & y directions and add accordingly, we get,

$$\therefore \sum F_x = 1000 + 2000\cos\theta_1 - 3000\cos\theta_2$$
$$\Rightarrow \sum F_x = 1000 + 2000 \times 0.8 - 3000 \times 0.6$$
$$\Rightarrow \sum F_x = +800\text{ N or }800\text{ N }(\rightarrow)$$

$$\therefore \sum F_y = 2000\sin\theta_1 + 3000\sin\theta_2$$
$$\Rightarrow \sum F_y = 2000 \times 0.6 + 3000 \times 0.8$$
$$\Rightarrow \sum F_y = +3600\text{ N or }3600\text{ N }(\uparrow)$$

Magnitude of the resultant,

$$R = \sqrt{(\sum F_x)^2 + \left(\sum F_y\right)^2}$$
$$R = \sqrt{(800)^2 + (3600)^2}$$
$$R = 3687.82\text{ N}$$

Inclination of the resultant,

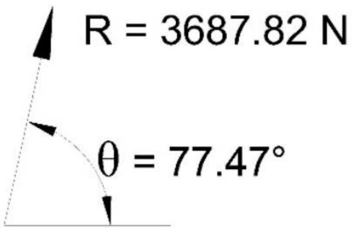
$$\theta = \tan^{-1}\frac{\sum F_y}{\sum F_x} = \tan^{-1}\frac{3600}{800}$$
$$\theta = 77.47°$$

Direction of the resultant,

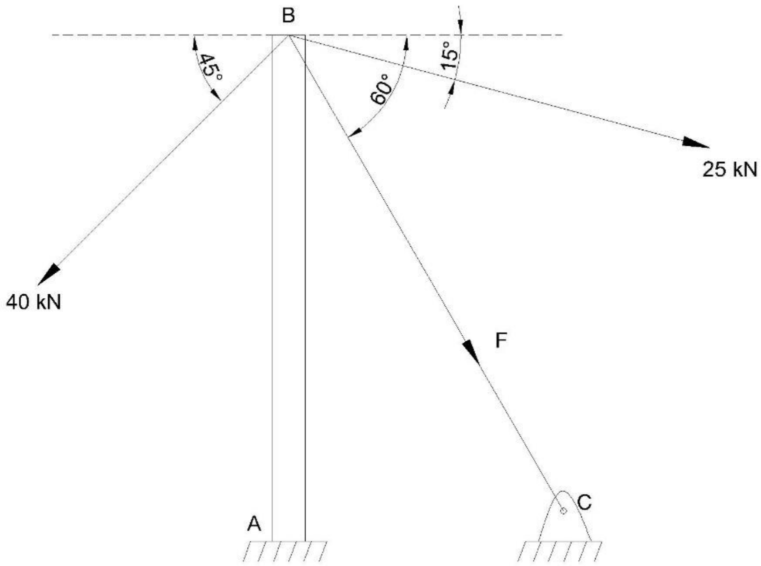
$\because \sum F_x$ is positive ($\rightarrow$) and $\sum F_y$ is positive ($\uparrow$)

$\therefore$ R is in the 1st quadrant ($\nearrow$)

$$\boxed{\therefore\ R = 3687.82\text{ N at }\theta = 77.47°}$$



N5: Determine the force F in the cable BC if the resultant of the 3 concurrent forces acting at B is vertical. Also determine the resultant.



This is a concurrent force system of 3 forces acting at B.

It is given that the resultant of the force system is vertical, it means that the algebraic sum of all the force components in the horizontal direction is zero.

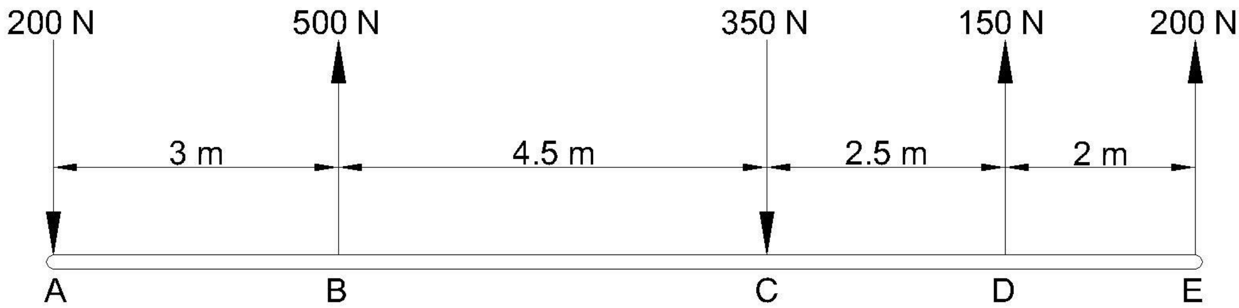
$$\therefore \sum F_x = 0$$
$$\Rightarrow 25\cos 15° - 40\cos 45° + F\cos 60° = 0$$
$$\boxed{\therefore F = 8.27\ \mathrm{kN}}$$

Also, since resultant is vertical,

$$\therefore R = \sum F_y$$
$$\Rightarrow R = -25\sin 15° - 40\sin 45° - 8.27\sin 60°$$
$$\Rightarrow \therefore R = -41.92\ \mathrm{kN}$$
$$\boxed{\therefore R = 41.92\ \mathrm{kN}\ (\downarrow)}$$

<u>N6</u>: Figure shows four parallel forces acting on a beam ABCDE.

i) Determine the resultant of the system and its location from A.
ii) Replace the system by a single force and couple acting at a point B.
iii) Replace the system by a single force and couple acting at a point D.



Soln: This is a parallel system of forces; hence we can simply find the algebraic sum for the resultant.

i) Resultant and location from A

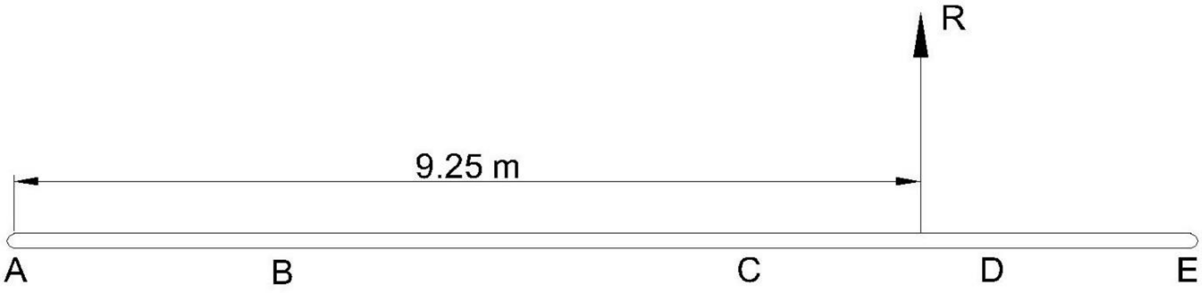
$$R = \sum F \uparrow +ve$$
$$R = -200 + 500 - 350 + 150 + 200 = 300\ \mathrm{N}$$
$$R = 300\ \mathrm{N}\ (\uparrow)$$

For the location of the resultant, let us assume that it is at a distance d to the right side of A. Using Varignon's theorem,

$$\sum M_A^F = M_A^R \ \circlearrowleft +ve$$
$$+500 \times 3 - 350 \times 7.5 + 150 \times 10 + 200 \times 12 = +300 \times d$$
$$\therefore d = 9.25\ \mathrm{m}$$

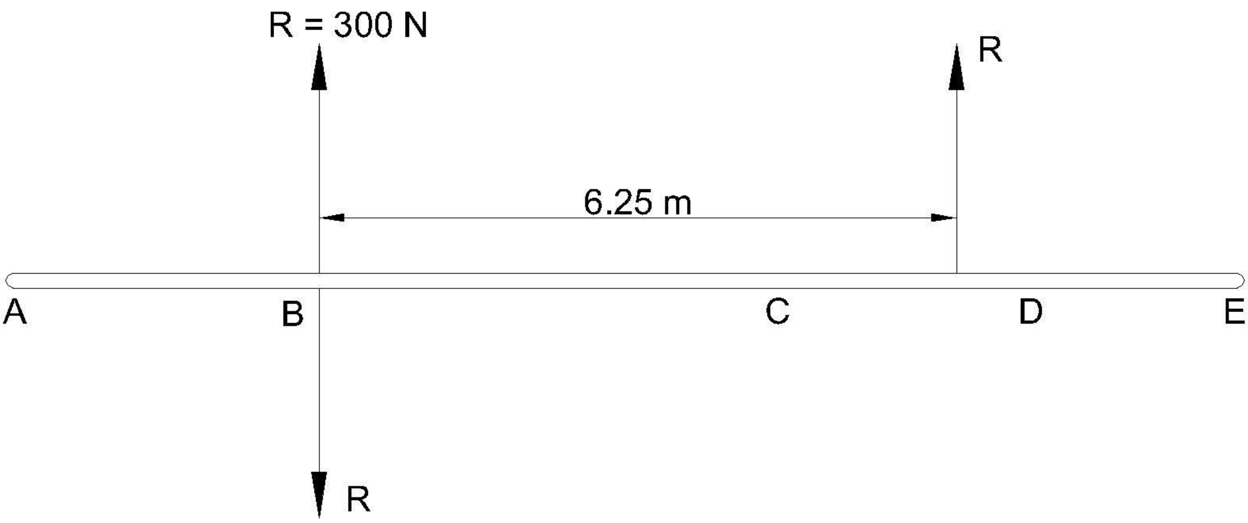
Since, d is +ve, our assumption of resultant being on the right side is correct.

$R = 300$ N (↑) acting at a distance of $d = 9.25$ m to the right of A.



ii) Single force and couple at B

Method 1: To shift a force to a new parallel position, a couple is required to be added to the system. Add two collinear forces of equal magnitude R & -R at B. The resultant R at B is as required and the remaining forces form a couple.



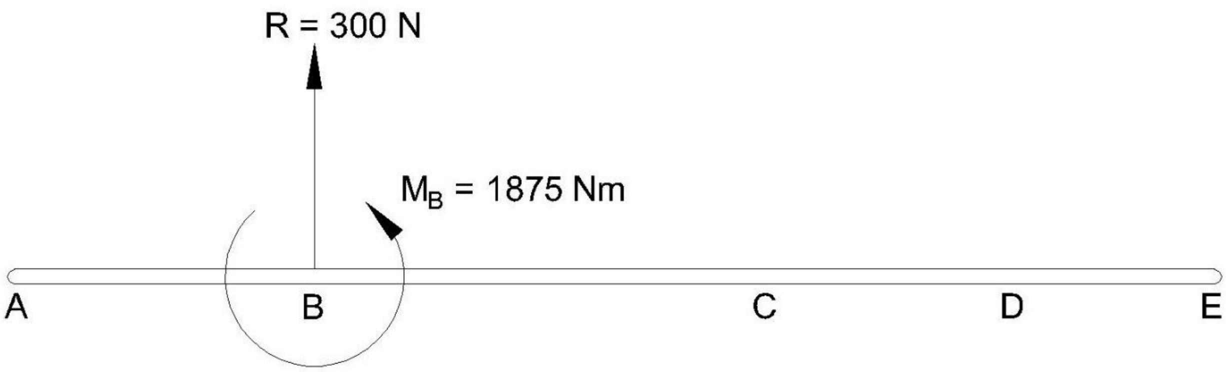
The single force acting at B is $R = 300$ N (↑)

And couple at B would be, $M_B = +R \times 6.25$ (↺ +ve)

$$M_B = +300 \times 6.25 = 1875 \text{ Nm}$$

$$\therefore M_B = 1875 \text{ Nm } ↺$$

Hence, the system can be replaced by a single force $R = 300$ N (↑) and a couple of 1875 Nm ↺ at B.



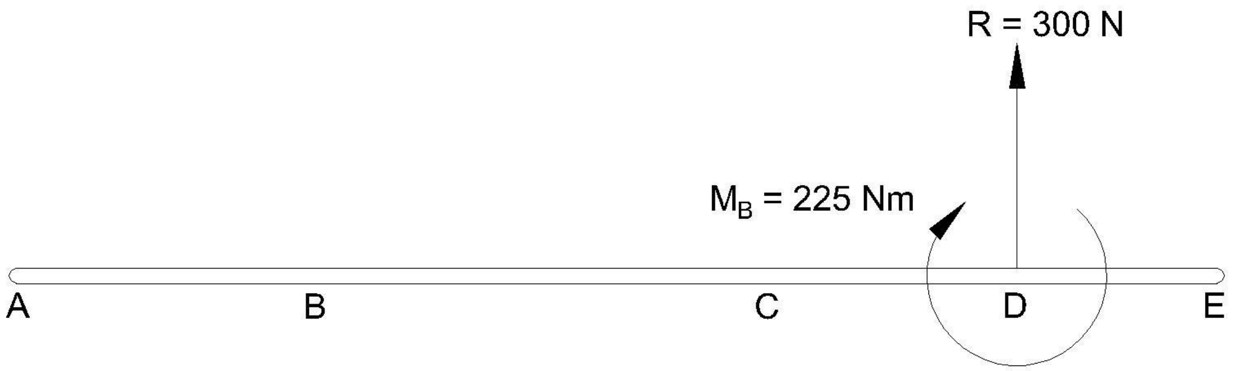
iii) Single force and couple at D

<u>Method 2</u>: The single force acting at D would be the same as $R = 300$ N (↑)

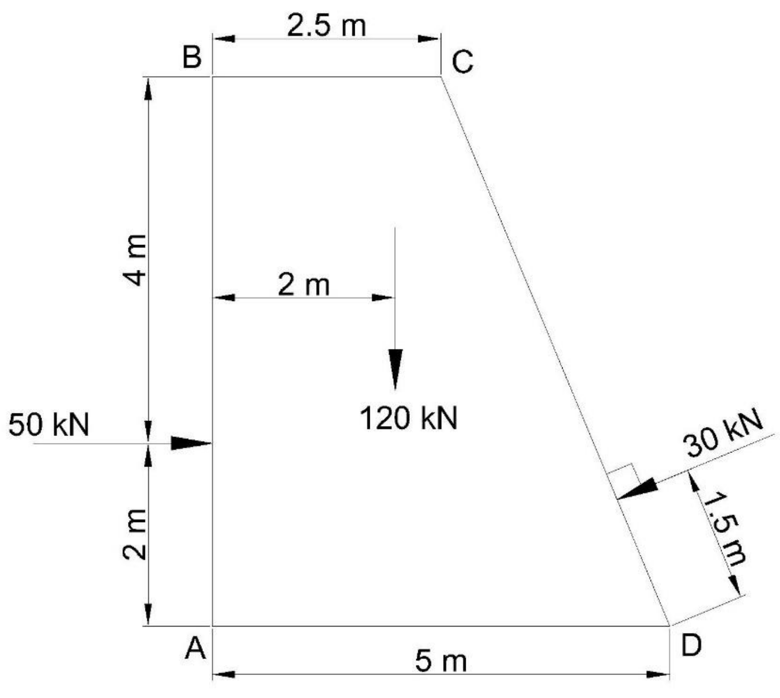
To find couple at D, take moments of all forces about D

$$\sum M_D^F \ \circlearrowleft +ve$$
$$\sum M_D^F = +200 \times 10 - 500 \times 7 + 350 \times 2.5 + 200 \times 2 = -225 \text{ Nm}$$
$$\therefore \sum M_D^F = 225 \text{ Nm} \circlearrowright$$

Hence, the system can be replaced by a single force $R = 300$ N (↑) and a couple of 225 Nm ↻ at D.

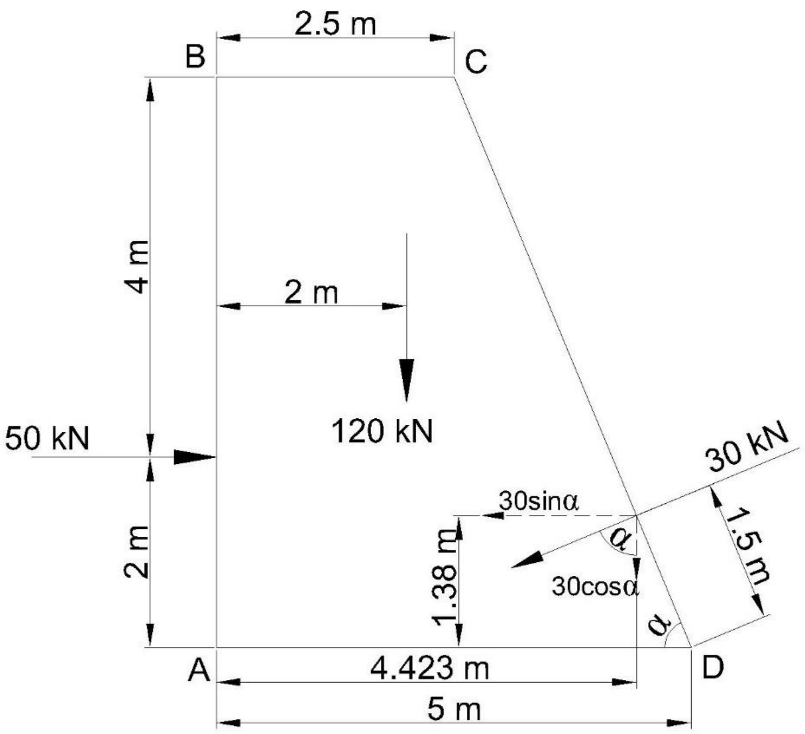


N7: A dam is subjected to three forces 50 kN on the upstream face AB, 30 kN force on the downstream inclined face and its own weight of 120 kN as shown. Determine the single force and locate its point of intersection with the base AD assuming all the forces to lie in a single plane.



Soln: This is a general force system of three coplanar forces acting on the dam.

Finding the angle of the inclined face and resolving the downstream force, we get,



$$\alpha = \tan^{-1}\frac{6}{2.5} = 67.4°$$

So, horizontal component of 30 kN is 30sinα, and vertical component is 30cosα.

Distance between 30sinα and horizontal base AD is

$$1.5\sin\alpha = 1.38 \text{ m}$$

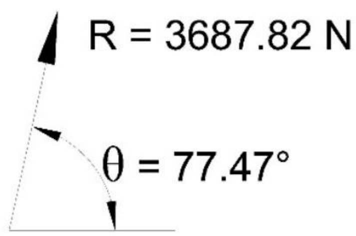
Distance between 30cosα and vertical face AB is

$$l(\text{AD}) - 1.5\cos\alpha$$
$$= 5 - 0.577 = 4.423\ m$$

Adding all the components of the forces in x & y directions, we get,

$$\therefore \sum F_x = +50 - 30\sin 67.4° = +22.3 \text{ kN}$$
$$\Rightarrow \sum F_x = 22.3 \text{ kN } (\rightarrow)$$

$$\therefore \sum F_y = -120 - 30\cos 67.4° = -131.5 \text{ kN}$$
$$\Rightarrow \sum F_y = 131.5 \text{ kN } (\downarrow)$$



Magnitude of the resultant,

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$$
$$R = \sqrt{(22.3)^2 + (131.5)^2}$$
$$R = 133.4 \text{ kN}$$

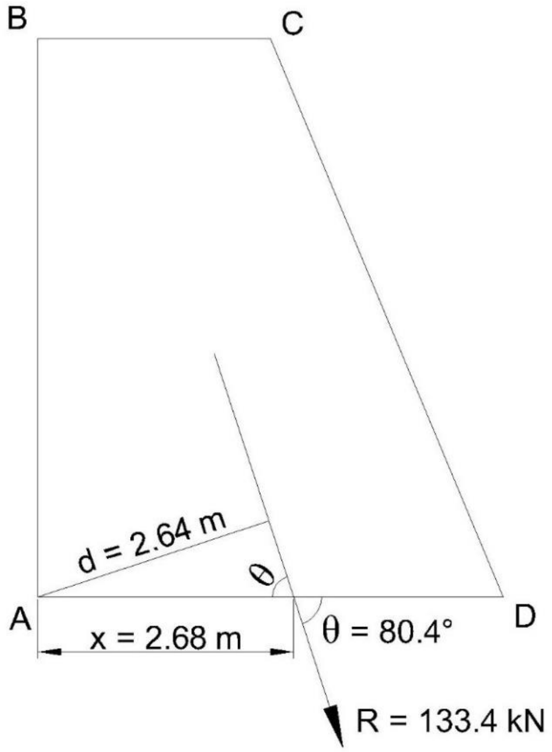
Direction of the resultant,

$$\theta = \tan^{-1}\frac{\sum F_y}{\sum F_x}$$
$$\theta = \tan^{-1}\frac{131.5}{22.3}$$
$$\theta = 80.4°$$

$\because \sum F_x$ is positive ($\rightarrow$) and $\sum F_y$ is negative ($\downarrow$), $\therefore$ R is in the 4th quadrant ($\searrow$).

Now for location of resultant, specifically, its point of intersection with base AD:

Let the resultant be at a perpendicular distance ‘d’ m to the right of A. Also, let it cut the base at a distance ‘x’ m from end A.



From Varignon’s theorem,

$$\sum M_A^F = M_A^R \; (\circlearrowleft +\mathrm{ve})$$

$$-(50 \times 2) - (120 \times 2)$$
$$-(30 \cos 67.4° \times 4.423)$$
$$+(30 \sin 67.4° \times 1.38)$$
$$= -(133.4 \times d)$$

$$\therefore d = 2.64 \text{ m}$$

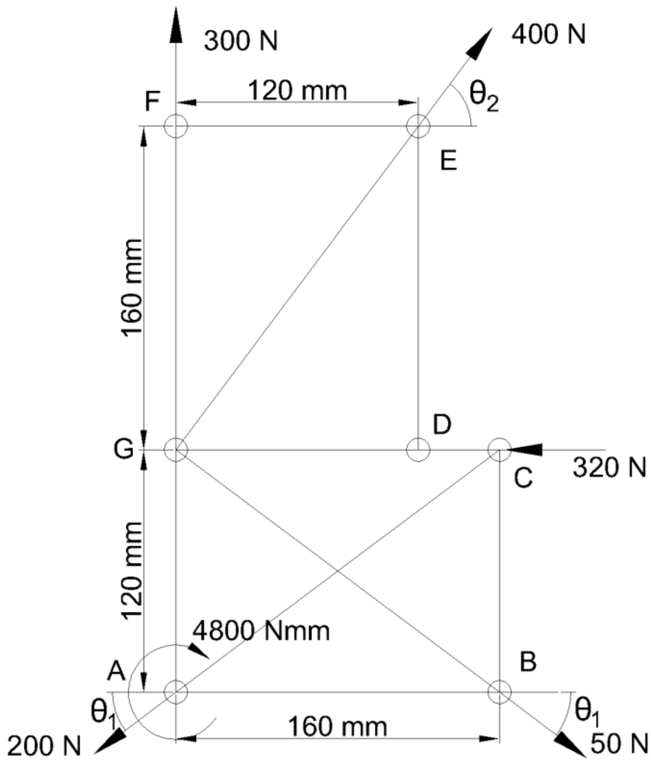
Also,

$$\sin 80.4° = \frac{d}{x} = \frac{2.64}{x}$$

$$\rightarrow x = 2.68 \text{ m}$$

Hence, the resultant is $R = 133.4$ kN at $\theta = 80.4°$ (↘) having its point of intersection with the base AD at $x = 2.68$ m from the right of A.

N8: Find the resultant of a coplanar forces system given in the figure below, located it on AB with due consideration to the applied moment.

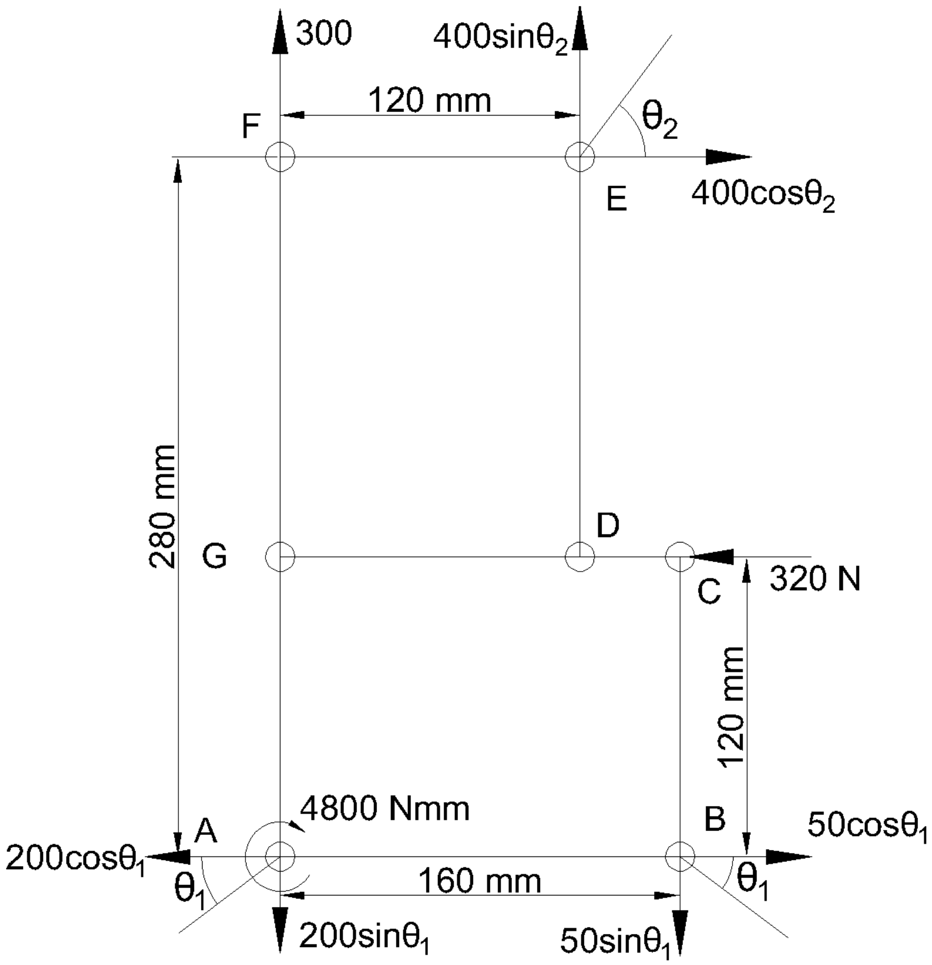


Soln: This is a general force system, since the lines of action of all forces don't meet at a single point.

Calculations of angles:

$$\theta_1 = \tan^{-1}\frac{BC}{BD} = \tan^{-1}\frac{120}{160} = \tan^{-1}\frac{3}{4}$$
$$\therefore \sin\theta_1 = 0.6 \ \& \ \cos\theta_1 = 0.8$$
$$\theta_2 = \tan^{-1}\frac{FG}{EF} = \tan^{-1}\frac{160}{120} = \tan^{-1}\frac{4}{3}$$
$$\therefore \sin\theta_2 = 0.8 \ \& \ \cos\theta_2 = 0.6$$

Resolving the various forces and adding their horizontal and vertical components,



$$\therefore \sum F_x = -200\cos\theta_1 + 50\cos\theta_1 - 320 + 400\cos\theta_2$$
$$\Rightarrow \sum F_x = -200 \times 0.8 + 50 \times 0.8 - 320 + 400 \times 0.6$$
$$\Rightarrow \sum F_x = -200 \text{ N}$$
$$\Rightarrow \sum F_x = 200 \text{ N } (\leftarrow)$$

$$\therefore \sum F_y = -200\sin\theta_1 - 50\sin\theta_1 + 400\sin\theta_2 + 300$$
$$\Rightarrow \sum F_y = -200 \times 0.6 - 50 \times 0.6 + 400 \times 0.8 + 300$$
$$\Rightarrow \sum F_y = +470 \text{ N}$$
$$\Rightarrow \sum F_y = 470 \text{ N } (\uparrow)$$

Magnitude of the resultant,

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$$
$$R = \sqrt{(200)^2 + (470)^2}$$
$$R = 510.78 \text{ N}$$

Direction of the resultant,

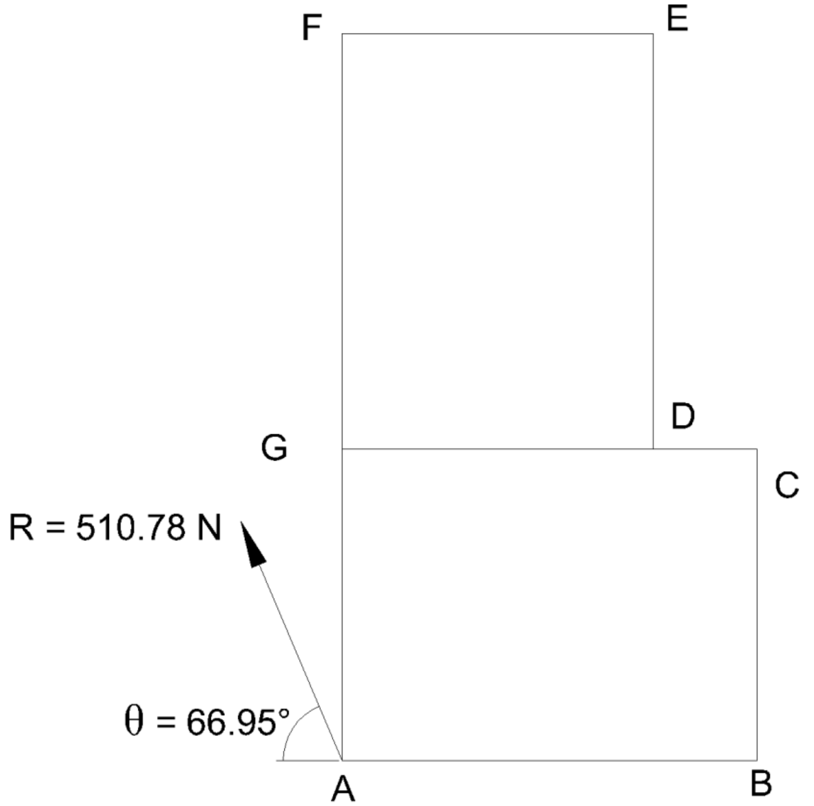
$$\theta = \tan^{-1}\frac{\sum F_y}{\sum F_x}$$
$$\theta = \tan^{-1}\frac{470}{200}$$
$$\theta = 66.95°$$

$\because \sum F_x$ is negative ($\leftarrow$) and $\sum F_y$ is positive ($\uparrow$), $\therefore$ R is in the 2nd quadrant ($\nwarrow$).

For the location of the resultant, let us assume that it is at a distance d to the right side of A. Using Varignon's theorem,

$$\sum M_A^F = M_A^R \circlearrowleft +ve$$
$$-4800 - (50\sin\theta_1 \times 160) - 320 \times 120$$
$$-(400\cos\theta_2 \times 280) + (400\sin\theta_2 \times 120) = +510.78 \times d$$
$$0 = 510.78 \times d$$
$$\therefore d = 0$$

This implies that the resultant acts exactly at point A.



Module Section 1.2 – Forces in Space

Vectors:

1. Basic Vector Operations:

$$\overline{P} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$$
$$\overline{Q} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$$

a) Dot Product

$$\overline{P} \cdot \overline{Q} = x_1x_2 + y_1y_2 + z_1z_2 \quad \text{or}$$
$$\overline{P} \cdot \overline{Q} = |\overline{P}||\overline{Q}| \cos\theta$$

b) Cross Product

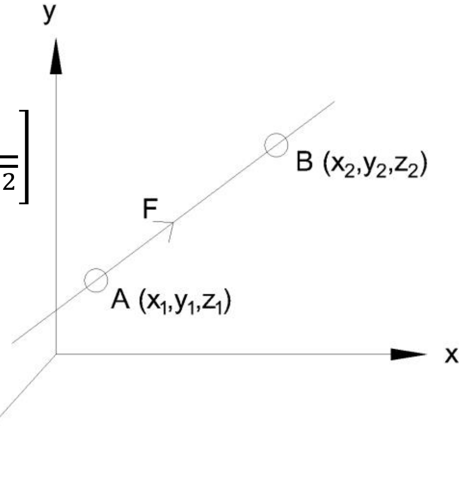
$$\overline{P} \times \overline{Q} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \quad \text{or}$$
$$\overline{P} \times \overline{Q} = |\overline{P}||\overline{Q}| \sin\theta\, \hat{n}$$

(where $\hat{n}$ is the unit vector normal to the plane of $\overline{P}$ & $\overline{Q}$)

2. Force Vector:

$$\overline{F} = (F)(\hat{e}_{AB})$$
$$\overline{F} = (F)\left[\frac{(x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}\right]$$
$$\overline{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

(where $\hat{e}_{AB}$ is the unit vector in the direction of AB)



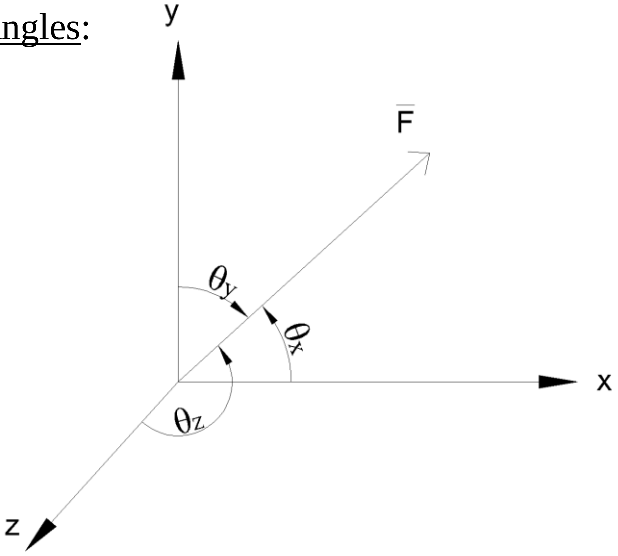
3. Magnitude of Force & Direction Angles:

$$\overline{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$
$$|\overline{F}| \text{ or } F = \sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2}$$

$$F_x = F\cos\theta_x$$
$$F_y = F\cos\theta_y$$
$$F_z = F\cos\theta_z$$

By direction cosine rule,

$$\cos{\theta_x}^2 + \cos{\theta_y}^2 + \cos{\theta_z}^2 = 1$$



4. <u>Moment Vector</u>:

$$\bar{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

$$\bar{r}_{CA} = (x_1 - x_3)\hat{\imath} + (y_1 - y_3)\hat{\jmath} + (z_1 - z_3)\hat{k}$$
$$\bar{r}_{CA} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\bar{M}_C = \bar{r}_{CA} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$
$$\bar{M}_C = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$$

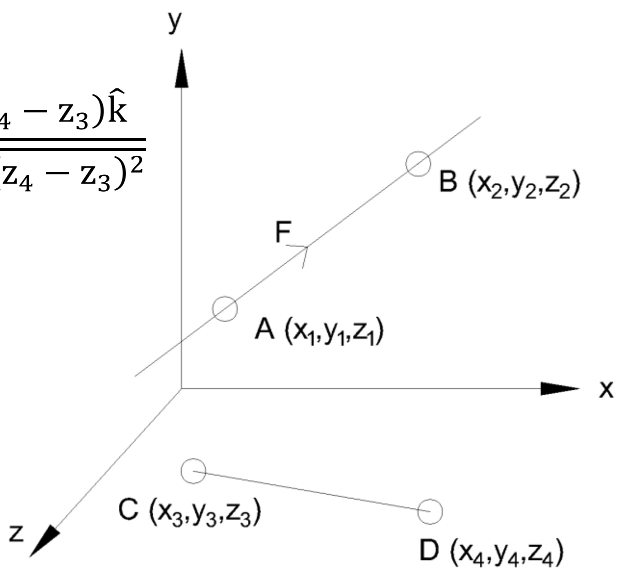
5. <u>Vector Component of a Force along a given line</u>:

$$\bar{F} = (F)(\hat{e}_{AB})$$

$$\hat{e}_{CD} = \frac{(x_4 - x_3)\hat{\imath} + (y_4 - y_3)\hat{\jmath} + (z_4 - z_3)\hat{k}}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$
$$\hat{e}_{CD} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$F_{CD} = \bar{F} \cdot \hat{e}_{CD}$$
$$F_{CD} = F_x x + F_y y + F_z z$$

$$\bar{F}_{CD} = (F_{CD})(\hat{e}_{CD})$$



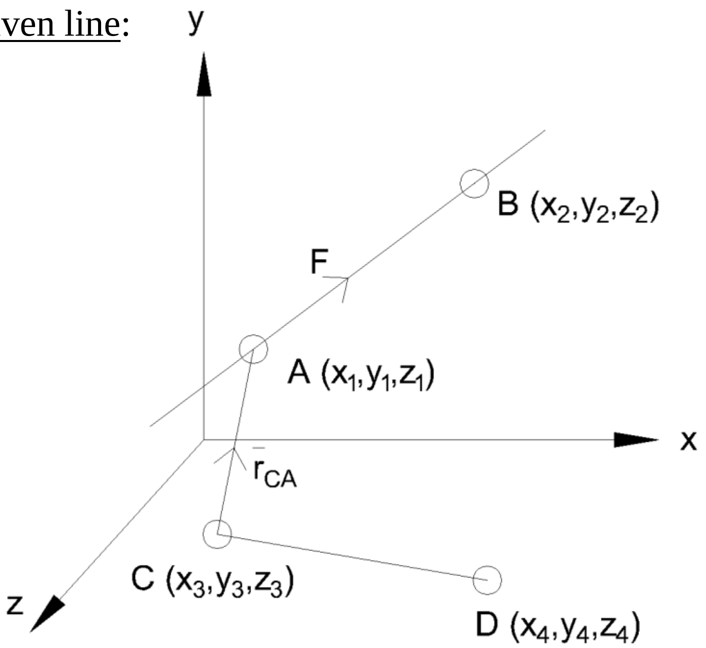
6. <u>Moment of a Force about a given line</u>:

$$\overline{M}_C = \overline{r}_{CA} \times \overline{F}$$
$$\overline{M}_C = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$$

$$\hat{e}_{CD} = x\hat{i} + y\hat{j} + z\hat{k}$$

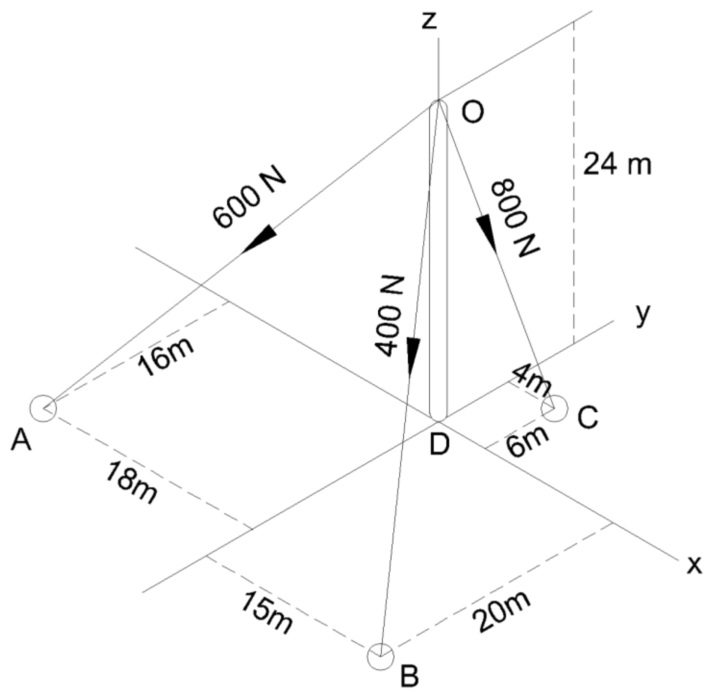
$$M_{CD} = \overline{M}_C \cdot \hat{e}_{CD}$$
$$M_{CD} = M_x x + M_y y + M_z z$$

$$\overline{M}_{CD} = (M_{CD})(\hat{e}_{CD})$$



Numericals:

<u>N1</u>: A tower is being held in place by three cables. If the force of each cable acting on the tower is shown in figure, determine the resultant.

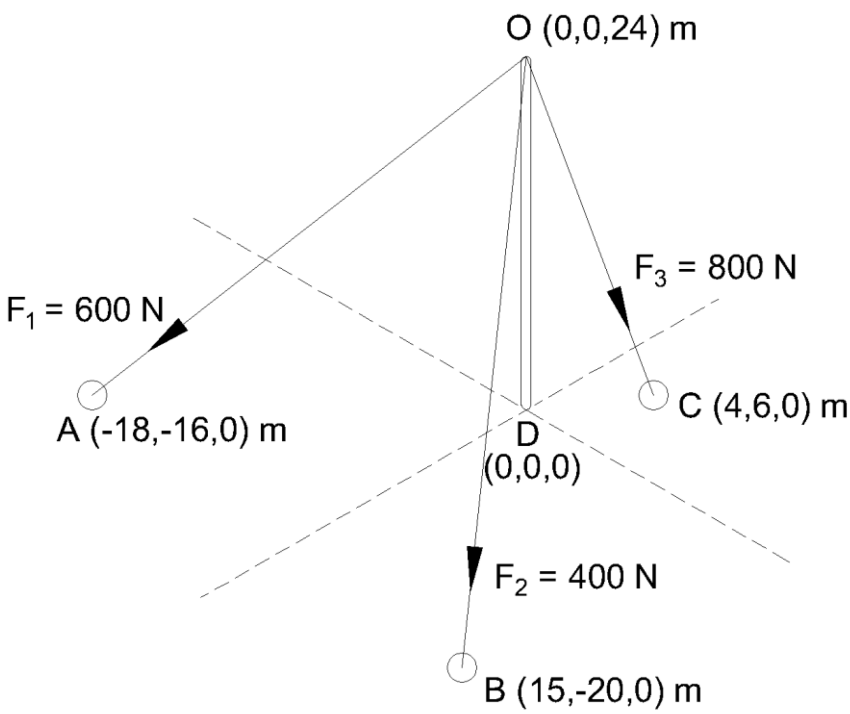


Soln: This is a concurrent space force system of 3 forces acting at O.

Let $\overline{F}_1$, $\overline{F}_2$, and $\overline{F}_3$ be the forces in the cables OA, OB and OC respectively.

$\therefore F_1 = 600$ N, $\quad F_2 = 400$ N, $\quad F_3 = 800$ N

And the co-ordinates of the points based on their distances from origin D are:
D (0,0,0) m
O (0,0,24) m
A (-18,-16,0) m
B (15,-20,0) m
C (4,6,0) m



$$\therefore \bar{F}_1 = (F_1)(\hat{e}_{OA}) = 600\left[\frac{(-18-0)\hat{\imath} + (-16-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(-18)^2 + (-16)^2 + (-24)^2}}\right]$$

$$\bar{F}_1 = \left(-317.6\hat{\imath} - 282.4\hat{\jmath} - 423.5\hat{k}\right) \text{N}$$

$$\therefore \bar{F}_2 = (F_2)(\hat{e}_{OB}) = 400\left[\frac{(15-0)\hat{\imath} + (-20-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(15)^2 + (-20)^2 + (-24)^2}}\right]$$

$$\bar{F}_2 = \left(+173.1\hat{\imath} - 230.8\hat{\jmath} - 277\hat{k}\right) \text{N}$$

$$\therefore \bar{F}_3 = (F_3)(\hat{e}_{OC}) = 800\left[\frac{(4-0)\hat{\imath} + (6-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(4)^2 + (6)^2 + (-24)^2}}\right]$$

$$\bar{F}_3 = \left(+127.7\hat{\imath} + 191.5\hat{\jmath} - 766.2\hat{k}\right) \text{N}$$

Resultant force in vector form is simply given by vector addition of the forces.

$$\begin{aligned} \therefore \bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 &= \left(-317.6\hat{\imath} - 282.4\hat{\jmath} - 423.5\hat{k}\right) \\ &\quad + \left(+173.1\hat{\imath} - 230.8\hat{\jmath} - 277\hat{k}\right) \\ &\quad + \left(+127.7\hat{\imath} + 191.5\hat{\jmath} - 766.2\hat{k}\right) \end{aligned}$$

$$\mathbf{\bar{R} = \left(-16.8\hat{\imath} - 321.7\hat{\jmath} - 1466.7\hat{k}\right) N}$$

N2: The lines of actions of three forces concurrent at origin O pass respectively through point A (-1,2,4), B (3,0,-3), C (2,-2,4). Force F_1 = 40 N passes through A, F_2 = 10 N passes through B, F_3 = 30 N passes through C. Find the magnitude and direction of their resultant.

Soln: In this concurrent space force system, putting the forces in vector form we get,

$$\bar{F}_1 = (F_1)(\hat{e}_{OA}) = 40\left[\frac{-1\hat{\imath} + 2\hat{\jmath} + 4\hat{k}}{\sqrt{1^2 + 2^2 + 4^2}}\right] = \left(-8.729\hat{\imath} + 17.457\hat{\jmath} + 34.915\hat{k}\right) \text{N}$$

$$\bar{F}_2 = (F_2)(\hat{e}_{OB}) = 10\left[\frac{3\hat{\imath} + 0\hat{\jmath} - 3\hat{k}}{\sqrt{3^2 + 0^2 + 3^2}}\right] = \left(+7.071\hat{\imath} + 0\hat{\jmath} - 7.071\hat{k}\right) \text{N}$$

$$\bar{F}_3 = (F_3)(\hat{e}_{OC}) = 30\left[\frac{2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{\sqrt{2^2 + 2^2 + 4^2}}\right] = \left(+12.247\hat{\imath} - 12.247\hat{\jmath} + 24.495\hat{k}\right) \text{N}$$

Resultant of these forces is,

$$\therefore \bar{R} = \left(-8.729\hat{\imath} + 17.457\hat{\jmath} + 34.915\hat{k}\right) + \left(+7.071\hat{\imath} + 0\hat{\jmath} - 7.071\hat{k}\right) + \left(+12.247\hat{\imath} - 12.247\hat{\jmath} + 24.495\hat{k}\right)$$

$$\bar{R} = \left(+10.589\hat{\imath} + 5.27\hat{\jmath} + 52.339\hat{k}\right) \text{N}$$

Magnitude of the resultant force,

$$\mathbf{R} = \sqrt{{R_x}^2 + {R_y}^2 + {R_z}^2} = \sqrt{10.589^2 + 5.27^2 + 52.339^2} = \mathbf{53.66\ N}$$

Direction of the resultant force is given by the angles θ_x, θ_y, and θ_z.

$R_x = R\cos\theta_x$

$\Rightarrow 10.589 = 53.66\cos\theta_x$

$\Rightarrow \boldsymbol{\theta_x = 78.62°}$

$R_y = R\cos\theta_y$

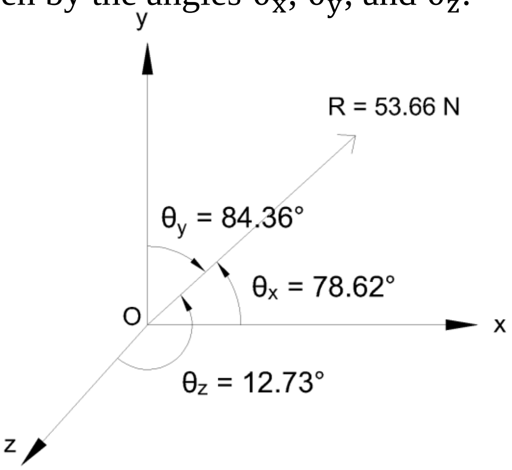
$\Rightarrow 5.27 = 53.66\cos\theta_y$

$\Rightarrow \boldsymbol{\theta_y = 84.36°}$

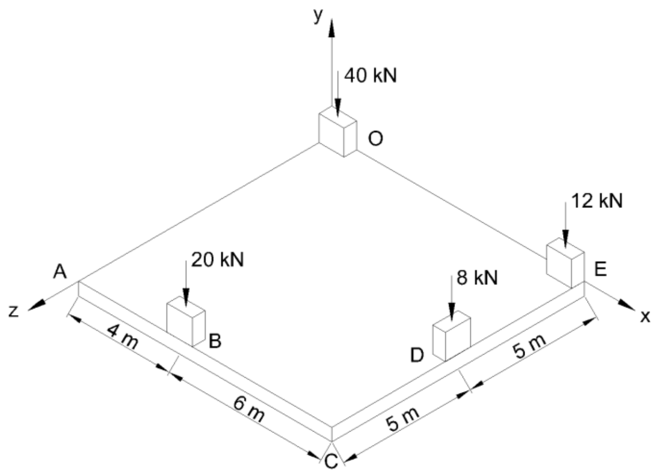
$R_z = R\cos\theta_z$

$\Rightarrow 52.339 = 53.66\cos\theta_z$

$\Rightarrow \boldsymbol{\theta_z = 12.73°}$



N3: A square foundation mat supports the four columns as shown. Determine the magnitude and point of application of the resultant of the four loads.



Soln: This is a parallel space force system with 4 forces. The co-ordinates of the points through which the forces act are as follows,

O (0,0,0), B (4,0,10), D (10,0,5), E (10,0,0)

Let the forces 20 kN, 8 kN, 12 kN & 40 kN be F_1, F_2, F_3 & F_4 respectively.

All the forces are parallel to y axis and in the downward direction; hence all of them will have $-\hat{j}$ in their vector forms.

$\bar{F}_1 = -20\hat{j}\ \text{kN}$; $\quad \bar{F}_2 = -80\hat{j}\ \text{kN}$; $\quad \bar{F}_3 = -12\hat{j}\ \text{kN}$; $\quad \bar{F}_4 = -40\hat{j}\ \text{kN}$

Resultant, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4 = -20\hat{j} - 80\hat{j} - 12\hat{j} - 40\hat{j} = -80\hat{j}\ \text{kN}$

For point of application, we first need to find out the moment of all forces about a point (let's take it from origin). So, the position vectors for each force will be,

$\bar{r}_{OB} = (4-0)\hat{i} + (0-0)\hat{j} + (10-0)\hat{k} = \left(4\hat{i} + 10\hat{k}\right)\ \text{m}$

$\bar{r}_{OD} = \left(10\hat{i} + 5\hat{k}\right)\ \text{m}$; $\quad \bar{r}_{OE} = \left(10\hat{i} + 0\hat{k}\right) = 10\hat{i}\ \text{m}$; $\quad \bar{r}_{OO} = 0\ \text{m}$

Let resultant act at a point P (x,0,z) m. $\therefore \bar{r}_{OB} = \left(x\hat{i} + z\hat{k}\right)\ \text{m}$

Now, the moment vectors of the forces about the origin,

$$\bar{M}_O^{F_1} = \bar{r}_{OB} \times \bar{F}_1 = \left(4\hat{i} + 10\hat{k}\right) \times (-20\hat{j}) = -80(\hat{i} \times \hat{j}) - 200\left(\hat{k} \times \hat{j}\right)$$
$$\bar{M}_O^{F_1} = \left(200\hat{i} - 80\hat{k}\right)\ \text{kNm} \qquad \{\because \hat{i} \times \hat{j} = \hat{k}, \qquad \hat{k} \times \hat{j} = -\hat{i}\}$$

$$\bar{M}_O^{F_2} = \bar{r}_{OD} \times \bar{F}_2 = \left(10\hat{i} + 5\hat{k}\right) \times (-8\hat{j}) = \left(40\hat{i} - 80\hat{k}\right)\ \text{kNm}$$

$$\bar{M}_O^{F_3} = \bar{r}_{OE} \times \bar{F}_3 = (10\hat{i}) \times (-12\hat{j}) = \left(-120\hat{k}\right)\ \text{kNm}$$

$$\bar{M}_O^{F_4} = 0\ \{\because \bar{F}_4 \text{ passes through the origin}\}$$

And the moment of resultant about the origin in terms of x and z,

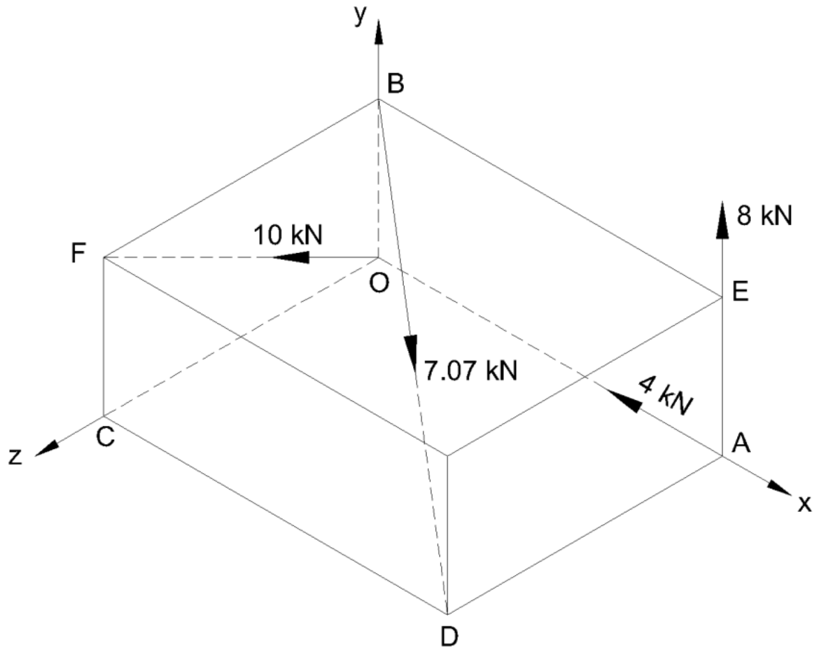
$$\bar{M}_O^{R} = \bar{r}_{OP} \times \bar{R} = \left(x\hat{i} + z\hat{k}\right) \times (-40\hat{j}) = \left[(80z)\hat{i} - (80x)\hat{k}\right]\ \text{kNm}$$

From Varignon's theorem,

$$\sum \bar{M}_O^{F} = \bar{M}_O^{R} \Rightarrow \bar{M}_O^{F_1} + \bar{M}_O^{F_2} + \bar{M}_O^{F_3} + \bar{M}_O^{F_4} = \bar{M}_O^{R}$$
$$\Rightarrow \left(200\hat{i} - 80\hat{k}\right) + \left(40\hat{i} - 80\hat{k}\right) + \left(-120\hat{k}\right) + 0 = (80z)\hat{i} - (80x)\hat{k}$$
$$\Rightarrow 240\hat{i} - 280\hat{k} = (80z)\hat{i} - (80x)\hat{k}$$
$$\Rightarrow 80z = 240\ \& \quad -80x = -280$$
$$\Rightarrow z = 3\ \text{m} \quad \& \quad x = 3.5\ \text{m}$$

Hence, the magnitude of the resultant is $\mathbf{R = 80\ kN}$ and passes through point $\mathbf{P\ (3.5, 0, 3)\ m}$.

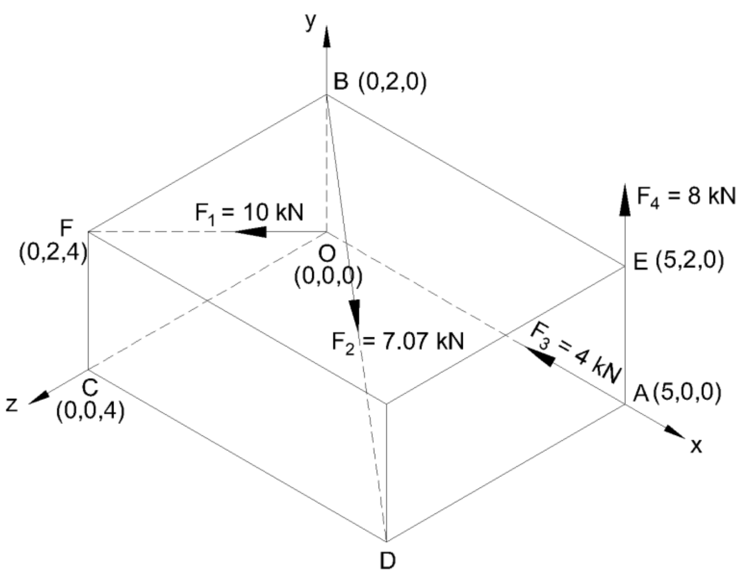
N4: A rectangle parallelepiped carries four forces as shown in the figure. Reduce the force system to a resultant force applied at the origin and moment around the origin. OA = 5 m, OB = 2 m, OC = 4 m.



Soln: The given system is a general space force system of 4 forces.

Let forces 10 kN, 7.07 kN, 4 kN & 8 kN be labelled as F_1, F_2, F_3 & F_4 respectively.

The co-ordinates of the various points through which the forces pass are:



Now, putting the forces in vector form,

$$\bar{F}_1 = (F_1)(\hat{e}_{OF}) = 10\left[\frac{0\hat{\imath} + 2\hat{\jmath} + 4\hat{k}}{\sqrt{0^2 + 2^2 + 4^2}}\right] = \left(0\hat{\imath} + 4.472\hat{\jmath} + 8.944\hat{k}\right)\text{ kN}$$

$$\bar{F}_2 = (F_2)(\hat{e}_{BD}) = 7.07\left[\frac{5\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{\sqrt{5^2 + 2^2 + 4^2}}\right] = \left(5.27\hat{\imath} - 2.108\hat{\jmath} + 4.216\hat{k}\right)\text{ kN}$$

$\bar{F}_3 = (F_3)(\hat{e}_{AO}) = 4\,(-\hat{\imath}) = (-4\hat{\imath})$ kN $\{\because$ it is along $x -$ axes towards origin$\}$
$\bar{F}_4 = (F_4)(\hat{e}_{AE}) = 8\,(\hat{\jmath}) = (8\hat{\jmath})$ kN $\{\because$ it is along $y -$ axes upwards$\}$

The resultant force, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$

$$\bar{R} = \left(0\hat{\imath} + 4.472\hat{\jmath} + 8.944\hat{k}\right) + \left(5.27\hat{\imath} - 2.108\hat{\jmath} + 4.216\hat{k}\right) + (-4\hat{\imath}) + (8\hat{\jmath})$$

$$\bar{\mathbf{R}} = \left(\mathbf{1.27\hat{\imath} + 10.364\hat{\jmath} + 13.16\hat{k}}\right)\ \mathbf{kN}$$

Taking moments of all force about the origin,

$\bar{M}_O^{F_1} = 0$ $\{\because \bar{F}_1$ passes through the origin$\}$

$$\bar{M}_O^{F_2} = \bar{r}_{OB} \times \bar{F}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & 0 \\ 5.27 & -2.108 & 4.216 \end{vmatrix} = \left(8.432\hat{\imath} + 0\hat{\jmath} - 10.54\hat{k}\right) \text{ kNm}$$

$\bar{M}_O^{F_3} = \bar{r}_{OA} \times \bar{F}_3 = 0$ $\{\because \bar{r}_{OA} \& \bar{F}_3$ are along the same directions$\}$

$$\bar{M}_O^{F_4} = \bar{r}_{OA} \times \bar{F}_4 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} = \left(40\hat{k}\right) \text{ kNm}$$

The resultant moment about the origin, $\bar{M}_O = \bar{M}_O^{F_1} + \bar{M}_O^{F_2} + \bar{M}_O^{F_3} + \bar{M}_O^{F_4}$

$$\bar{M}_O = 0 + \left(8.432\hat{\imath} + 0\hat{\jmath} - 10.54\hat{k}\right) + 0 + \left(40\hat{k}\right)$$

$$\bar{\mathbf{M}}_{\mathbf{O}} = \left(\mathbf{8.432\hat{\imath} + 29.46\hat{k}}\right)\ \mathbf{kNm}$$

Hence, the resultant force and moment at origin is,

$$\bar{\mathbf{R}} = \left(\mathbf{1.27\hat{\imath} + 10.364\hat{\jmath} + 13.16\hat{k}}\right)\ \mathbf{kN}$$

$$\bar{\mathbf{M}}_{\mathbf{O}} = \left(\mathbf{8.432\hat{\imath} + 29.46\hat{k}}\right)\ \mathbf{kNm}$$