Introduction to Laplace Transform

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Introduction:

In Mathematics and with many applications in physics, Engineering and throughout the sciences, the Laplace Transform is a widely used integral transform. The Laplace Transform is a powerful tool formulated to solve a wide variety of initial value problems.

A particular type of definite integral as an operator is called **Laplace Transform.** Laplace transform changes a function of one variable denoted by t into a function of another variable denoted by s.

Definition:

If f (t) is a function of t satisfying certain conditions, then the definite integral $\emptyset(s) = \int_0^\infty e^{-st}.f(t)dt$

When it exists, is called Laplace Transform of $f(t)$ and is written as $L[f(t)]$.

Thus, $L[f(t)] = \int_0^\infty e^{-st}.f(t)dt$

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS:

1. $L\{1\} = \frac{1}{s}, s > 0$

Proof: $L\{1\} = \int_0^\infty e^{-st}\{1\}\,dt = \left|\frac{e^{-st}}{-s}\right|_0^\infty = -\frac{1}{s}|0-1| = \frac{1}{s}$

2. $L\{e^{at}\} = \frac{1}{s-a}, s > 0$

Proof: $L\{e^{at}\} = \int_0^\infty e^{-st}\{e^{at}\}\,dt = \int_0^\infty e^{-(st-at)}\,dt = \left|\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = -\frac{1}{s-a}|0-1| = \frac{1}{s-a}$

3. $L\{t^n\} = \frac{\lfloor n+1}{s^{n+1}}$

$= \frac{n!}{s^{n+1}}$, where $n = 0, 1, 2, 3, \dots.$

Proof: $L\{t^n\} = \int_0^\infty e^{-st}\{t^n\}\,dt = \int_0^\infty e^{-x}\cdot\left(\frac{x}{s}\right)^n \frac{dx}{s}$, on putting $st = x$

$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\cdot x^n\,dx$ .... {by definition of Gamma function}

$= \frac{\lfloor n+1}{s^{n+1}}$, for $n > -1$ and $s > 0$

If n is a positive integer, then $\lfloor n+1 = n!$

$\therefore L\{t^n\} = \frac{n!}{s^{n+1}}$

4. $L\{\cos at\} = \frac{s}{s^2+a^2}, s > 0$

Proof: $L\{\cos at\} = \int_0^\infty e^{-st}\{\cos at\}\,dt$

$= \left|\frac{e^{-st}}{s^2+a^2}\{-s\cos at + a\sin at\}\right|_0^\infty \quad \because \int e^{ax}\cdot\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}|a\cos bx + b\sin bx|$

$= \frac{1}{s^2+a^2}\left|\frac{-s\cos at + a\sin at}{e^{st}}\right|_0^\infty$

$= \frac{1}{s^2+a^2}|0-(-s)| = \frac{s}{s^2+a^2}$ $\left\{\begin{array}{c}\because \text{for the upper limit when } t\to\infty, \cos at \\ \text{and } \sin at \text{ are considered as some real} \\ \text{numbers lying between } -1 \text{ and } 1, \\ \therefore \frac{-s\cos at + a\sin at}{e^{st}} \to 0\end{array}\right\}$

5. $L\{\sin at\} = \frac{a}{s^2+a^2}, s > 0$

Proof: $L\{\sin at\} = \int_0^\infty e^{-st}\{\sin at\}\,dt$

$= \left|\frac{e^{-st}}{s^2+a^2}\{-s\sin at - a\cos at\}\right|_0^\infty \quad \because \int e^{ax}\cdot\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}|a\sin bx - b\cos bx|$

$= \frac{1}{s^2+a^2}\left|\frac{-sa\sin at - a\cos at}{e^{st}}\right|_0^\infty$

$= \frac{1}{s^2+a^2}|0-(-a)| = \frac{a}{s^2+a^2}$ $\left\{\begin{array}{c}\text{here again, for the upper limit when } t\to\infty, \\ -1\le\cos at\le 1, -1\le\sin at\le 1, \\ \therefore \frac{-s\sin at - a\cos at}{e^{st}} \to 0\end{array}\right\}$

Another Method:

Assuming the formula (2) holds for complex numbers, we have

$L\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \left(\frac{s}{s^2+a^2}\right) + i\left(\frac{a}{s^2+a^2}\right)$ .... (1)

But $e^{iat} = \cos at + i\sin at$, hence

$L\{e^{iat}\} = L\{\cos at + i\sin at\} = \int_0^\infty e^{-st}\{\cos at + i\sin at\}\,dt$

$= \int_0^\infty e^{-st}\{\cos at\}\,dt + i\int_0^\infty e^{-st}\{\sin at\}\,dt = L\{\cos at\} + iL\{\sin at\}$ .... (2)

From (1) and (2), on equating real and imaginary parts, we have

$L\{\cos at\} = \frac{s}{s^2+a^2}$ and $L\{\sin at\} = \frac{a}{s^2+a^2}$

6. $L\{\cosh at\} = \frac{s}{s^2-a^2}, s > |a|$

Proof: $L(\cosh at) \quad = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \int_0^\infty e^{-st}\left\{\frac{e^{at}+e^{-at}}{2}\right\}dt = \frac{1}{2}\int_0^\infty e^{-st}\{e^{at}\}dt + \frac{1}{2}\int_0^\infty e^{-st}\{e^{-at}\}dt$$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$ .... {by using the formula (2)}

$= \frac{s}{s^2-a^2}$ for $s > |a|$

7. $L\{\sinh at\} = \frac{a}{s^2-a^2}, s > |a|$

Proof: $L(\sinh at) \quad = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \int_0^\infty e^{-st}\left\{\frac{e^{at}-e^{-at}}{2}\right\}dt = \frac{1}{2}\int_0^\infty e^{-st}\{e^{at}\}dt - \frac{1}{2}\int_0^\infty e^{-st}\{e^{-at}\}dt$$

$$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$$

$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$ .... {by using the formula (2)}

$= \frac{a}{s^2-a^2}$ for $s > |a|$

1.	$L(1) = \frac{1}{s}$	**2.**	$L(e^{at}) = \frac{1}{s-a}, L(e^{-at}) = \frac{1}{s+a}$, $L(c^{at}) = \frac{1}{s - a\log c}$
3.	$L(t^n) = \frac{\lfloor n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}$ if $n \in N$	**4.**	$L(\cos at) = \frac{s}{s^2+a^2}$
5.	$L(\sin at) = \frac{a}{s^2+a^2}$	**6.**	$L(\cos h\ at) = \frac{s}{s^2-a^2}$
7.	$L(\sin h\ at) = \frac{a}{s^2-a^2}$		

1. LINEARITY PROPERTY:

If k_1 and k_2 are constants then, $L|k_1 f_1(t) + k_2 f_2(t)| = k_1 L|f_1(t)| + k_2 L|f_2(t)|$

2. CHANGE OF SCALE PROPERTY:

If $L|f(t)| = \emptyset(s)$, then $L|f(at)| = \frac{1}{a}\emptyset\left(\frac{s}{a}\right)$

Proof: By definition $L|f(at)| = \int_0^\infty e^{-st} f(at)dt.$ Now, put u = at

$$= \int_0^\infty e^{-s(u/a)} f(u).\frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)du = \frac{1}{a}\emptyset\left(\frac{s}{a}\right)$$

e.g If $Lf(t) = \frac{2s}{s^2+4}$, then $L|f(2t)| = \frac{1}{2}\left|\frac{2(s/2)}{(s/2)^2+4}\right| = \frac{2s}{s^2+16}$

ERROR FUNCTION:

Definition: Error function of x is defined as $\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$ and is denoted by $erf(x)$

We write $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$ ....(1)

This function or integral is also called Error function Integral or probability integral and accounted in many branches of Mathematics, Physics or Engineering

COMPLEMENTARY ERROR FUNCTION:

Definition: Complementary Error Function of x defined as $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du$ and is denoted by $erf_c(x)$

We write $erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du$ ....(2)

Alternative Definition of Error Function:

In integral of (1), if we put $u^2 = t$ $\quad \therefore 2udu = dt$ $\quad \therefore du = \frac{dt}{2\sqrt{t}}$

As $u \to 0, t \to 0$ and $u \to x, t \to x^2$

$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^{x^2} e^{-t}\frac{dt}{2\sqrt{t}} = \frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}t^{-1/2}dt$

$\therefore erf(x) = \frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}t^{-1/2}dt$ ....(3)

This is also considered as definition of Error Function of x and either (1) or (3) can be used for $erf(x)$ according to the need of the problem.

NOTE: $\mathrm{erf}(x) + \mathrm{erf_c}(x) = 1$