APPLICATION OF LT TO SOLVE DE

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For the use of Laplace Transform to solve Differential Equations we need the following results.

If Laplace transform of y ie $L[y]$ is denoted by $\bar{y}$ then

$$\boxed{\begin{aligned} L[y'] &= s\bar{y} - y(0) \\ L[y''] &= s^2\bar{y} - sy(0) - y'(0) \\ L[y'''] &= s^3\bar{y} - s^2y(0) - sy'(0) - y''(0) \end{aligned}}$$

Examples

① $(3D+2)y = e^{3t}$, $y(0)=1$

Soln: Given DE is

$$3y' + 2y = e^{3t}$$

$$\therefore 3L[y'] + 2L[y] = L[e^{3t}]$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s+3}$$

$$(3s+2)\bar{y} - 3 = \frac{1}{s+3}$$

$$(3s+2)\bar{y} = \frac{1}{s+3} + 3 = \frac{s+4}{s+3}$$

$$\therefore \boxed{\bar{y} = \frac{s+4}{(s+3)(3s+2)}}$$

Now we apply inverse Laplace to find y

Let $\frac{s+4}{(s+3)(3s+2)} = \frac{A}{s+3} + \frac{B}{3s+2}$

$$\therefore s+4 = A(3s+2) + B(s+3)$$

$$= (3A+B)s + (2A+3B)$$

$$\Rightarrow 3A+B=1 \quad \& \quad 2A+3B=4$$

$$\Rightarrow A = \frac{-1}{7} \quad \& \quad B = \frac{10}{7}$$

$$\therefore \bar{y} = \frac{10}{7} \cdot \frac{1}{3s+2} - \frac{1}{7} \cdot \frac{1}{s+3}$$

$$= \frac{10}{7\times 3} \cdot \frac{1}{s+\frac{2}{3}} - \frac{1}{7} \cdot \frac{1}{s+3}$$

$$\therefore y = \frac{10}{21} L^{-1}\left[\frac{1}{s+\frac{2}{3}}\right] - \frac{1}{7} L^{-1}\left[\frac{1}{s+3}\right]$$

$$\boxed{y = \frac{10}{21} e^{-\frac{2}{3}t} - \frac{1}{7} e^{-3t}}$$

Example -2

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1 \quad \text{where } y(0)=0,\ y'(0)=1$$

Solution : Given DE is

$$y'' + 4y' + 8y = 1$$

$$\therefore L[y''] + 4L[y'] + 8L[y] = L[1]$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 4[s\bar{y} - y(0)] + 8\bar{y} = \frac{1}{s}$$

$$[s^2\bar{y} - 1] + 4[s\bar{y}] + 8\bar{y} = \frac{1}{s}$$

$$(s^2+4s+8)\bar{y} = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\therefore \bar{y} = \frac{s+1}{s(s^2+4s+8)}$$

Now we find Laplace inverse of $\bar{y}$ to get y

Let $\dfrac{s+1}{s(s^2+4s+8)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4s+8}$

$\therefore s+1 = A(s^2+4s+8) + (Bs+C)s$

$s+1 = (A+B)s^2 + (4A+C)s + 8A$

$\Rightarrow A+B=0, \quad 4A+C=1, \quad 8A=1$

$\Rightarrow \boxed{A=\frac{1}{8}} \qquad C = 1-\frac{1}{2} \qquad \therefore \boxed{C=\frac{1}{2}}$

and $B=-A \qquad \therefore \boxed{B=-\frac{1}{8}}$

$$\therefore \bar{y} = \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2+4s+8} + \frac{1}{2}\cdot\frac{1}{s^2+4s+8}$$

$$= \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{(s+2)-2}{(s+2)^2+4} + \frac{1}{2}\cdot\frac{1}{(s+2)^2+4}$$

$$= \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s+2}{(s+2)^2+4} + \frac{3}{4}\cdot\frac{1}{(s+2)^2+4}$$

$$y = \frac{1}{8}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{8}L^{-1}\left[\frac{s+2}{(s+2)^2+4}\right] + \frac{3}{4}L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= \frac{1}{8} - \frac{1}{8}e^{-2t}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{3}{4}e^{-2t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{4}\cdot e^{-2t}\cdot\frac{1}{2}\sin 2t$$

$$\boxed{y = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t}$$

Example-3

$$\frac{d^2y}{dt^2} + 9y = \cos 2t, \quad y(0)=1 \text{ and } y\left(\frac{\pi}{2}\right) = -1$$

(This type of problem is called Boundary Value problem)

Solution given DE is

$$y'' + 9y = \cos 2t$$

$$L[y''] + 9L[y] = L[\cos 2t]$$

$$[s^2\bar{y} - s y(0) - y'(0)] + 9\bar{y} = \frac{s}{s^2+4}$$

Assume that $y'(0) = \alpha$

$$[s^2\bar{y} - s - \alpha] + 9\bar{y} = \frac{s}{s^2+4}$$

$$(s^2+9)\bar{y} = \frac{s}{s^2+4} + s+\alpha$$

$$\therefore \bar{y} = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{\alpha}{s^2+9}$$

$$= \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{\alpha}{s^2+9}$$ (use partial Fraction)

$$\bar{y} = \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9} + \frac{\alpha}{s^2+9}$$

$$\therefore y = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{4}{5}L^{-1}\left[\frac{s}{s^2+9}\right] + \alpha L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{\alpha}{3}\sin 3t$$

Now to find α, we use the given condition

$$y\left(\frac{\pi}{2}\right) = -1$$

$$\therefore y\left(\frac{\pi}{2}\right) = -1 = \frac{1}{5}\cos 2\left(\frac{\pi}{2}\right) + \frac{4}{5}\cos 3\left(\frac{\pi}{2}\right) + \frac{\alpha}{3}\sin 3\left(\frac{\pi}{2}\right)$$

$$\therefore\ y\left(\frac{\pi}{2}\right) = -1 = \frac{1}{5}\cos 2\left(\frac{\pi}{2}\right) + \frac{\ }{5}\cos 3\left(\frac{\pi}{2}\right) \quad 3 \quad 2)$$

$$-1 = -\frac{1}{5} + 0 - \frac{\alpha}{3}$$

$$\therefore\ -\frac{4}{5} = -\frac{\alpha}{3}$$

$$\therefore\ \alpha = \frac{12}{5}$$

$$\therefore\ \boxed{y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t}$$

Example $\quad y + \int_0^t y\,dt = 1 - e^{-t}$

Solution : $\quad y + \int_0^t y\,dt = 1 - e^{-t}$

$$L[y] + L\left[\int_0^t y\,dt\right] = L(1) - L(e^{-t})$$

$$\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$$

$$\left(\frac{s+1}{s}\right)\bar{y} = \frac{1}{s(s+1)}$$

$$\therefore\ \bar{y} = \frac{1}{(s+1)^2}$$

$$\therefore\ y = L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2}\right] = te^{-t}$$

$$\therefore\ \boxed{y = te^{-t}}$$