FIRST SHIFTING THEOREM AND SECOND SHIFTING THEOREM

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FIRST SHIFTING THEOREM:

If $L[f(t)] = \emptyset(s)$, then $L[e^{-at}f(t)] = \emptyset(s+a)$

Proof: By definition $L[e^{-at}f(t)] = \int_0^\infty e^{-st}\{e^{-at}f(t)\}\,dt$

$$= \int_0^\infty e^{-(s+a)t}f(t)\,dt = \emptyset(s+a)$$

e.g $L(\sin at) = \frac{a}{s^2+a^2}$ $\therefore L[e^{-bt}\sin at] = \frac{a}{(s+b)^2+a^2}$

Cor. Changing sign of a, we get,

If $L[f(t)] = \emptyset(s)$, then $L[e^{at}f(t)] = \emptyset(s-a)$

Ex :- If $L[f(t)] = \frac{s}{s^2+s+4}$, find $L[e^{-3t}f(2t)]$

Solution By change of scale property

$$L[f(2t)] = \frac{1}{2}\cdot\frac{s/2}{(s/2)^2+(\frac{s}{2})+4} = \frac{s}{s^2+2s+16} = \phi(s)$$

Now, using first shifting property

$$L[e^{-3t}f(2t)] = \phi(s+3)$$

$$= \frac{s+3}{(s+3)^2+2(s+3)+16} = \frac{s+3}{s^2+8s+31}$$

Ex :- Find $L[\cosh 2t\cos 2t]$

Solution :- $L[\cosh 2t\cos 2t] = L\left[\frac{1}{2}(e^{2t}+e^{-2t})\cos 2t\right]$

$$= \frac{1}{2}\left[L(e^{2t}\cos 2t) + L(e^{-2t}\cos 2t)\right]$$

But $L[\cos 2t] = \frac{s}{s^2+4}$

$\therefore$ By shifting theorem

$$L[\cosh 2t\cos 2t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4}\right]$$

on simplification

$$= \frac{s^3}{s^4+64}$$

Ex Find $L[(t^2\sinh t)^2]$

Solution :- $(t^2\sinh t)^2 = t^4\left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{t^4}{4}\left[e^{2t}-2+e^{-2t}\right]$

Solution :- $(t^2 \sinh t)^2 = t^4 \left(\frac{e^t - e^{-t}}{2} \right)^2 = \frac{t^4}{4} \left[e^{2t} - 2 + e^{-2t} \right]$

$$\therefore L\left[(t^2 \sinh t)^2 \right] = L\left[\frac{t^4}{4} \left(e^{2t} - 2 + e^{-2t} \right) \right]$$

$$= \frac{1}{4} \left[L\left(e^{2t} t^4\right) - 2L\left(t^4\right) + L\left[e^{-2t} t^4\right) \right]$$

But $L\left[t^4\right] = \frac{4!}{s^5}$

$$\therefore L\left[(t^2 \sinh t)^2 \right] = \frac{1}{4} \left[\frac{4!}{(s-2)^5} - \frac{2 \cdot 4!}{s^5} + \frac{4!}{(s+2)^5} \right]$$

$$= 6 \left[\frac{1}{(s-2)^5} - \frac{2}{s^5} + \frac{1}{(s+2)^5} \right]$$

Ex :- Show that $L\left[\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right) \right] = \frac{\sqrt{3}}{2} \cdot \frac{s}{s^4 + s^2 + 1}$

Solution we have $\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right) = \left(\frac{e^{t/2} - e^{-t/2}}{2} \right) \sin\left(\frac{\sqrt{3}t}{2}\right)$

Now, $L\left[\sin\left(\frac{\sqrt{3}t}{2}\right) \right] = \frac{\sqrt{3}/2}{s^2 + 3/4}$

By First shifting theorem

$$L\left[e^{t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) \right] = \frac{\sqrt{3}/2}{\left(s - \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{\sqrt{3}/2}{s^2 - s + 1}$$

$$L\left[e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right) \right] = \frac{\sqrt{3}/2}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{\sqrt{3}/2}{s^2 + s + 1}$$

$$\therefore L\left[\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right) \right] = \frac{1}{2} \left[\frac{\sqrt{3}/2}{(s^2+1) - s} - \frac{\sqrt{3}/2}{(s^2+1) + s} \right]$$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \left[\frac{s^2 + 1 + s - (s^2 + 1 - s)}{(s^2+1)^2 - s^2} \right]$$

$$L\left[\sinh\left(\frac{t}{2}\right) \sin\left(\frac{\sqrt{3}t}{2}\right) \right] = \frac{\sqrt{3}}{2} \left[\frac{s}{s^4 + s^2 + 1} \right]$$

Ex Find $L\left[e^{-3t} \cosh 5t \sin 4t \right]$

Solution we have

Solution: we have

$$e^{-3t}\cosh 5t \sin 4t = e^{-3t}\left[\frac{e^{5t}+e^{-5t}}{2}\right]\sin 4t$$

$$= \frac{1}{2}\left[e^{2t}\sin 4t + e^{-8t}\sin 4t\right]$$

Now $L[\sin 4t] = \frac{4}{s^2+16}$

∴ By First shifting theorem

$$L[e^{-3t}\cosh 5t \sin 4t] = \frac{1}{2}L\left[e^{2t}\sin 4t + e^{-8t}\sin 4t\right]$$

$$= \frac{1}{2}\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+8)^2+16}\right]$$

$$= \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+80)}$$

SECOND SHIFTING THEOREM:

If $L|f(t)| = \emptyset(s)$ and $g(t) = f(t-a)$ when $t > a$ and $g(t) = 0$ when $t < a$, then $L|g(t)| = e^{-as}\emptyset(s)$

Proof: By definition of Laplace transform

$$L\{g(t)\} = \int_0^\infty e^{-st}g(t)dt$$

$$= \int_0^a e^{-st}g(t)dt + \int_a^\infty e^{-st}g(t)dt$$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}f(t-a)dt$$

$$= \int_a^\infty e^{-st}f(t-a)dt$$

Now put $t - a = p$ ∴ $dt = dp$

$$\therefore L\{g(t)\} = \int_0^\infty e^{-s(a+p)}f(p)dp$$

$$= e^{-as}\int_0^\infty e^{-sp}f(p)dp$$

$$= e^{-as}\int_0^\infty e^{-st}f(t)dt = e^{-as}L|f(t)| = e^{-as}\emptyset(s)$$

Ex :- Using second shifting theorem find,

(i) $L[f(t)]$ where $f(t) = \cos(t-\alpha)$, $t > \alpha$ and $f(t) = 0$, $t < \alpha$

(ii) $L[f(t)]$ where $f(t) = e^{t-k}$, $t > k$ and $f(t) = 0$, $t < k$

Solution: (i) we have $L(\cos t) = \frac{s}{s^2+1}$

Hence by second shifting theorem

$$L[\cos(t-\alpha)] = e^{-\alpha s}\cdot\frac{s}{s^2+1}$$

(ii) we have $L(e^t) = \frac{1}{s-1}$

Hence by second shifting theorem

$$L[e^{t-k}] = e^{-ks}\cdot\frac{1}{\quad}$$

Hence by second shifting theorem

$$L\left(e^{t-k}\right) = e^{-ks} \cdot \frac{1}{s-1}$$