Use of differentiation of $\emptyset(s)$

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Use of differentiation of $\emptyset(s)$:

We know that, If $L|f(t)| = \emptyset(s)$ then $L|tf(t)| = -\emptyset'(s)$

Taking inverse Laplace transform, this means, If $L^{-1}|\emptyset(s)| = f(t)$ then $L^{-1}|-\emptyset'(s)| = tf(t)$

$i.e\ L^{-1}|\ \emptyset'(s)| = -tf(t)$

$i.e\ L^{-1}|\ \emptyset'(s)| = -t\ L^{-1}\ \emptyset(s)$

$L^{-1}\ \emptyset(s) = -\frac{1}{t}L^{-1}|\ \emptyset'(s)|$

These results can be profitably used to find $L^{-1}\emptyset(s)$ if we know $L^{-1}\emptyset'(s)$ i.e if $\emptyset'(s)$ comes out to be a standard results.

OR to find $L^{-1}\emptyset'(s)$ if we know $L^{-1}\emptyset(s)$ i.e the given function is the derivative of a standard results

Ex-1 find $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]$

Solution :- using $L^{-1}[\phi(s)] = -\frac{1}{t}L^{-1}[\phi'(s)]$

$$L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\log\left(\frac{s+a}{s+b}\right)\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[\log(s+a) - \log(s+b)\right]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= -\frac{1}{t}\left[e^{-at} - e^{-bt}\right]$$

$$L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = \frac{e^{-bt} - e^{-at}}{t}$$

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② $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$

solution using $L^{-1}[\phi(s)] = -\frac{1}{t}L^{-1}[\phi'(s)]$

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right] = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]\right\}$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{ \frac{2s}{s^2+a^2} - \frac{1}{2}\frac{1}{s+b} \right\}$$

$$= -\frac{1}{t}\left[2\cos at - \frac{1}{2} e^{-bt} \right]$$

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right] = \frac{1}{t}\left[\frac{1}{2}e^{-bt} - 2\cos at\right]$$

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③ $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

<u>Solution</u> using $L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$

$$L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}\frac{1}{s}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\left(\frac{1}{s}\right)^2} \cdot \left(\frac{-1}{s^2}\right)\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = \frac{1}{t}\sin t$$

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④ $L^{-1}\left[\cot^{-1}(s)\right]$

<u>Solution</u> :- using $L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$

$$L^{-1}\left[\cot^{-1}(s)\right] = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\cot^{-1}(s)\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left[\frac{-1}{1+s^2}\right] = \frac{1}{t} L^{-1}\left[\frac{1}{1+s^2}\right] = \frac{1}{t}\sin t$$

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⑤ $L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]$

<u>Solution</u> :- using $L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$

Solution :- using $L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}[\phi'(s)]$

$$L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right] = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\left(\frac{s+a}{b}\right)^2} \cdot \frac{1}{b}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$= -\frac{1}{t} e^{-at} L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= -\frac{1}{t} e^{-at} \sin bt$$

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⑥ $L^{-1}\left[\log\left(\frac{s^2+1}{s(s+1)}\right)\right]$

Solution :- using $L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}[\phi'(s)]$

$$L^{-1}\left\{\log\left[\frac{s^2+1}{s(s+1)}\right]\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log\frac{s^2+1}{s(s+1)}\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\log(s^2+1) - \log s - \log(s+1)\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= -\frac{1}{t}\left[2\cos t - 1 - e^{-t}\right]$$

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⑦ Using convolution theorem, prove that

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+4}\right)\right] = \int_0^t \frac{e^{-4u} - e^{-3u}}{u}\, du$$

Solution :- $\phi(s) = \frac{1}{s}\log\left(\frac{s+3}{s+4}\right) = \phi_1(s) \cdot \phi_2(s)$

where $\phi_1(s) = \log\left(\frac{s+3}{s+4}\right)$ & $\phi_2(s) = \frac{1}{s}$

$\therefore f_1(t) = L^{-1}[\phi_1(s)]$ & $f_2(t) = L^{-1}\left[\frac{1}{s}\right] = 1$

$$= L^{-1}\left[\log\left(\frac{s+3}{s+4}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+3}{s+4}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+3} - \frac{1}{s+4}\right]$$

$$= -\frac{1}{t}\left[e^{-3t} - e^{-4t}\right]$$

$$f_1(t) = \frac{e^{-4t} - e^{-3t}}{t}$$

$\therefore$ By convolution theorem

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+4}\right)\right] = \int_0^t f_1(u) f_2(t-u)\,du$$

$$= \int_0^t \frac{e^{-4u} - e^{-3u}}{u}\,du$$

Ex (HW) Prove using convolution theorem

$$L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{a}{s}\right] = \int_0^t \frac{1}{u}\sin au\,du$$