LAPLACE TRANSFORMS OF DERIVATIVES

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$L(f'(t)) = -f(0) + sL(f(t))$

Proof: By Definition of Laplace transform, $L(f'(t)) = \int_0^\infty e^{-st} f'(t)dt$.

Integrating by parts

$L(f'(t)) = |e^{-st} f(t)|_0^\infty - \int_0^\infty (-s) e^{-st} f(t)dt$

$\therefore L(f'(t)) = -f(0) + s\int_0^\infty e^{-st} f(t)dt$

$\therefore L(f'(t)) = -f(0) + sL(f(t))$ ....(1)

Applying (1) again

$L(f''(t)) = -f'(0) + s[L(f'(t))]$

$\quad = -f'(0) + s[-f(0) + sLf(t)]$

$\therefore L(f''(t)) = -f'(0) - sf(0) + s^2L(f(t))$

Further, $L\,(f'''(t)) = -f''(0) - sf'(0) - s^2f(0) + s^3L(f(t))$

In general $L(\, f^n(t)) = -f^{n-1}(0) - s.f^{n-2}(0) - s^2.f^{n-3}(0) + \;\dots\dots\dots\dots + s^nL(f(t))$

If $f(0) = f'(0) = f''(0) = \cdots . = 0$ we get $Lf'(t) = s.Lf(t)$,

$Lf''(t) = s^2Lf(t)$,

$Lf'''(t) = s^3Lf(t), \dots\dots\dots Lf^n(t) = s^nLf(t)$

These results are going to be highly useful to solve differential equations.

Ex :- Given $f(t) = t+1,\ 0 \le t \le 2$ and $f(t) = 3,\ t > 2$

Find $L[f(t)]$, $L[f'(t)]$ and $L[f''(t)]$

Solution :- By definition : $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^2 e^{-st}(t+1)\,dt + \int_2^\infty e^{-st}(3)\,dt$

$= \left[(t+1)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^2 + 3\left(\frac{e^{-st}}{-s}\right)_2^\infty$

$= 3\left(\frac{e^{-2s}}{-s}\right) - \left(\frac{e^{-2s}}{s^2}\right) - (1)\left(\frac{-1}{s}\right) + \frac{1}{s^2} + 3\left(0 - \frac{e^{-2s}}{-s}\right)$

$L[f(t)] = \frac{1}{s} + \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$

Now, $L[f'(t)] = -f(0) + sL[f(t)]$

But by data $f(0) = 1$

$\therefore L[f'(t)] = -1 + s\left[\frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)\right] = -1 + 1 + \frac{1}{s}\left(1 - e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-2s}\right)$

Also, $L[f''(t)] = s^2L[f(t)] - s[f(0)] - f'(0)$

$\because f(t) = t+1$
$f'(t) = 1$
$\therefore f(0) = 1 \quad f'(0) = 1$

Also, $L[f''(t)] = s^2 L[f'(t)] - s[f(0)] - f'(0)$

$$= s^2\left[\frac{1}{s} + \frac{1}{s^2}(1-e^{-2s})\right] - s - 1$$

$$= s + (1-e^{-2s}) - s - 1 = -e^{-2s}$$

$\left[\begin{array}{l} f'(t) = 1 \\ \therefore f(0) = 1,\ f'(0) = 1 \end{array}\right]$

Ex: Find $L\left[\frac{d}{dt}\left(\frac{\sin 3t}{t}\right)\right]$

Solution :- $L(\sin 3t) = \dfrac{3}{s^2+9}$

$$\therefore L\left(\frac{\sin 3t}{t}\right) = \int_s^\infty \frac{3}{s^2+9}\,ds = \tan^{-1}\left(\frac{s}{3}\right)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \cot^{-1}\left(\frac{s}{3}\right)$$

$$\therefore L[f'(t)] = -f(0) - sL[f(t)]$$

$$f(0) = \lim_{t\to 0}\frac{\sin 3t}{t} = 3\cdot\lim_{t\to 0}\frac{\sin 3t}{3t} = 3\cdot(1) = 3$$

$$\therefore L[f'(t)] = -3 - s\cot^{-1}\left(\frac{s}{3}\right)$$