EFFECT OF DIVISION BY t

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If $L|f(t)| = \emptyset(s)$, then $L\left|\frac{1}{t}f(t)\right| = \int_s^\infty \emptyset(s)ds$

Proof: By definition of Laplace transform $\emptyset(s) = \int_0^\infty e^{-st}f(t)dt$

Integrating both sides w.r.t s between the limits $s\ to\ \infty$ and changing the order of integration on r.h.s

$$\int_s^\infty \emptyset(s)ds = \int_0^\infty \left|\int_s^\infty e^{-st}f(t)ds\right|dt$$

$$= \int_0^\infty \left|\frac{e^{-st}}{-t}f(t)\right|_s^\infty dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}dt$$

$$= L\left|\frac{1}{t}f(t)\right|$$

Ex 1 : Find the Laplace Transform of $\frac{1}{t}\left(e^{-at} - e^{-bt}\right)$

Solution : By standard formulae

$$L\left(e^{-at} - e^{-bt}\right) = \frac{1}{s+a} - \frac{1}{s+b}$$

Now, By effect of division by t

$$L\left[\frac{1}{t}\left(e^{-at} - e^{-bt}\right)\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$= \Big[\log(s+a) - \log(s+b)\Big]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)\Big|_s^\infty$$

$$= \log\left(\frac{1+a/s}{1+b/s}\right)\Big|_s^\infty$$

$$= \log(1) - \log\left(\frac{1+a/s}{1+b/s}\right)$$

$$L\left[\frac{1}{t}\left(e^{-at} - e^{-bt}\right)\right] = \log\left(\frac{s+b}{s+a}\right)$$

Ex - 2 Find $L\left[\frac{\sin^2 2t}{t}\right]$

Solution :- $L\left[\sin^2 2t\right] = L\left[\frac{1-\cos 4t}{2}\right] = \frac{1}{2}\left[L(1) - L\left(\cos 4t\right)\right]$

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$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

Now By effect of division by t

$$L\left[\frac{\sin^2 2t}{t}\right] = \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right] ds$$

$$= \frac{1}{2}\left\{\int_s^\infty \frac{1}{s} ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+16} ds\right\}$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+16}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{\sqrt{1+16/s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{1}{\sqrt{1+16/s^2}}\right)\right]$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{2}\log\left(\frac{\sqrt{s^2+16}}{s}\right)$$

Ex-3 Find $L\left[\frac{e^{-2t}\sin 2t \cosh t}{t}\right]$

Solution :- we have, $e^{-2t}\sin 2t \cosh t = e^{-2t}\sin 2t\left(\frac{e^t+e^{-t}}{2}\right)$

$$= \frac{1}{2}\left[e^{-t}\sin 2t + e^{-3t}\sin 2t\right]$$

But $L\left[\sin 2t\right] = \frac{2}{s^2+4}$

By shifting theorem,

$L\left[e^{-t}\sin 2t\right] = \frac{2}{(s+1)^2+4}$ and $L\left[e^{-3t}\sin 2t\right] = \frac{2}{(s+3)^2+4}$

$$\therefore L\left[e^{-2t}\sin 2t \cosh t\right] = \frac{1}{2}\left[\frac{2}{(s+1)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

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$$= \frac{1}{(s+1)^2+2^2} + \frac{1}{(s+3)^2+2^2}$$

Now using the effect of division by t

$$L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right] = \int_s^{\infty} \frac{1}{(s+1)^2+2^2} + \frac{1}{(s+3)^2+2^2}\, ds$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s+1}{2}\right)\right] + \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right] = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)$$

Ex-4 Find Laplace Transform of $\dfrac{\sin t\sin 5t}{t}$

solution we have $\sin t\sin 5t = \frac{1}{2}\left[\cos(4t) - \cos(6t)\right]$

$$\therefore L\left[\sin t\sin 5t\right] = \frac{1}{2} L\left[\cos 4t - \cos 6t\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+16} - \frac{s}{s^2+36}\right]$$

Now using effect of division by t

$$L\left[\frac{\sin t\sin 5t}{t}\right] = \frac{1}{2}\int_s^{\infty}\left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right) ds$$

$$= \frac{1}{4}\int_s^{\infty}\left(\frac{2s}{s^2+16} - \frac{2s}{s^2+36}\right) ds$$

$$= \frac{1}{4}\left[\log(s^2+16) - \log(s^2+36)\right]_s^{\infty}$$

$$= \frac{1}{4}\left[\log(s^2+16) - \log(s^2+36)\right]_s$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty = \frac{1}{4}\left[\log\left(\frac{1+16/s^2}{1+36/s^2}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log(1) - \log\left(\frac{1+16/s^2}{1+36/s^2}\right)\right]$$

$$L\left[\frac{\sin t \sin 5t}{t}\right] = \frac{1}{4}\log\left(\frac{s^2+36}{s^2+16}\right)$$

Ex-5 Find $L\left[\frac{\sin at}{t}\right]$. Does $L\left[\frac{\cos at}{t}\right]$ exist ?

Solution :- $L[\sin at] = \frac{a}{s^2+a^2}$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Now $L[\cos at] = \frac{s}{s^2+a^2}$

$$\therefore L\left[\frac{\cos at}{t}\right] = \int_s^\infty \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2}\,ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

Since $\log(s^2+a^2)$ is infinite when $s \to \infty$,
$L\left[\frac{\cos at}{t}\right]$ does not exist.