A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Solve the Differential equation to express v in terms of x.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dv}{dx} + P(x)v = Q(x)$$

$$xv\frac{dv}{dx} + v^2 = gx$$

$$v\frac{dv}{dx} + \frac{v^2}{x} = g.$$

put $v^2 = y$

$$\Rightarrow 2v\frac{dv}{dx} = \frac{dy}{dx}$$

$$\frac{1}{2}\frac{dy}{dx} + \frac{y}{x} = g \Rightarrow \frac{dy}{dx} + \frac{2y}{x} = 2g$$

$$P = \frac{2}{x} \qquad Q = 2g$$

Soln is

$$y e^{\int P dx} = \int e^{\int P dx} Q\, dx + c \quad \text{—— (1)}$$

$$e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Soln is from (1)

$$y x^2 = \int x^2 2g \, dx + c$$

$$v^2 x^2 = 2g\frac{x^3}{3} + c$$

$$v^2 = \frac{2gx}{3} + \frac{c}{x^2}$$

In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$. Find the current i at time t if at $t = 0, i = 0$ and L, R, E are constants.

$y\, e^{\int P\,dx}$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$y = i$

$x = t$

$$\frac{di}{dt} + P(t)\,i = Q(t)$$

$$L\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

$$P = \frac{R}{L} \qquad Q = \frac{E}{L}$$

Sol^n is

$$i\, e^{\int P\,dt} = \int e^{\int P\,dt}\, Q\, dt + C$$

$$e^{\int P\,dt} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

$$i\, e^{\frac{R}{L}t} = \int e^{\frac{R}{L}t}\frac{E}{L}\, dt + C$$

$$= \frac{E}{\cancel{L}}\,\frac{e^{\frac{R}{L}t}}{\frac{R}{\cancel{L}}} + C = \frac{E}{R}e^{\frac{R}{L}t} + C$$

$$i = \frac{E}{R} + C e^{-\frac{R}{L}t} \quad \text{———} \quad (1)$$

given at $t=0$ $i=0$

$$0 = \frac{E}{R} + C e^{0} \Rightarrow C = -\frac{E}{R}$$

from (1) $$i = \frac{E}{R} - \frac{E}{R}e^{-R/L\,t} = \frac{E}{R}\left(1 - e^{-R/L\,t}\right).$$

In a circuit containing inductance L resistance R and voltage E, the current i is given by $E = Ri + L\frac{di}{dt}$ .If $L = 640h, R = 250\ \Omega$ and $E = 500$volts and $i = 0$ when $t = 0$, find the time that elapses before the current reaches 90% of its maximum value.

$$L\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

from earlier problem Soln is

$$i = \frac{E}{R}\left(1 - e^{-R/L\, t}\right). \quad \text{--- } (1)$$

Consider max. value of current is I which we'll get at $t \to \infty$

i.e. here $i = I$ at $t \to \infty$

from (1) $I = \frac{E}{R}\left(1 - e^{-\infty}\right)$ $\qquad e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$$I = \frac{E}{R}$$

given $i = 90\%$ of I (max value of current).

$$= \frac{90}{100} I = \frac{9}{10}\frac{E}{R}$$

from (1)

$$i = \frac{E}{R}\left(1 - e^{-R/L\, t}\right).$$

$$\frac{9}{10}\frac{\cancel{E}}{\cancel{R}} = \frac{\cancel{E}}{\cancel{R}}\left(1 - e^{-R/L\, t}\right)$$

$$e^{-R/L\, t} = 1 - \frac{9}{10} = \frac{1}{10}$$

$$\frac{1}{e^{R/L\, t}} = \frac{1}{10} \Rightarrow e^{R/L\, t} = 10$$

$$\frac{R}{L}t = \ln(10)$$

$$t = \frac{L}{R}\ln(10) = \frac{640}{250}\ln(10)$$

$$= 5.89 \text{ sec.}$$

The charge q on the plate of a condenser of capacity C charged through a resistance R by the steady voltage V satisfies the differential equation $R\frac{dq}{dt}+\frac{q}{C}=V$. If $q=0$ at $t=0$, show that $q=CV(1-e^{-t/RC})$. Find also the current flowing into the plate.

$$\frac{dq}{dt}+P(t)q=Q(t)$$

$$R\frac{dq}{dt}+\frac{q}{C}=V$$

$$\frac{dq}{dt}+\frac{1}{RC}q=\frac{V}{R}$$

$$P=\frac{1}{RC} \qquad Q=\frac{V}{R}$$

$$q e^{\int P\,dt}=\int e^{\int P\,dt}Q\,dt+c \quad \text{—— (1)}$$

$$e^{\int P\,dt}=e^{\int \frac{1}{RC}dt}=e^{\frac{t}{RC}}$$

from (1)

$$q e^{t/RC}=\int e^{t/RC}\frac{V}{R}dt+C_1$$

$$=\frac{V}{R}\frac{e^{t/RC}}{1/RC}+C_1$$

$$q e^{t/RC}=\frac{V}{\cancel{R}}\times\cancel{R}C\, e^{t/RC}+C_1$$

$$q=VC+C_1e^{-t/RC}$$

at $t=0$ $q=0$

$$0=VC+C_1 \Rightarrow C_1=-VC$$

$$q=VC-VCe^{-t/RC}$$

$$=VC(1-e^{-t/RC})$$

$$i=\frac{dq}{dt}=VC\left(0-e^{-t/RC}\left(-\frac{1}{RC}\right)\right)$$

$$=V\cancel{C}\frac{e^{-t/RC}}{R\cancel{C}}=\frac{V}{R}e^{-t/RC}$$