NLPP

Non Linear Programming Problems

DEFINITIONS

- An optimization problem in which either the object function and/or some or all constraints are non-linear is called a Non linear Programming Problem. It is abbreviated as **NLPP**
- The variables $x_1, x_2, x_3, \ldots\ldots, x_n$ which enter into a problem are called **decision variables**
- The function which is to be maximized (i.e. z) or minimized is called **objective function**

DEFINITIONS

- The restrictions imposed on the relationships between the variables in the form of equalities or inequalities are called **constraints**
- Any feasible solution which maximizes or minimizes the objective function is called **optimum solution**
- Any solution in which one or more of the variables become zero is called **degenerate solution**

EXAMPLES

- Optimize $z = 3x_1 + 8x_2 + 9x_3$;

Subject to $3x_1 + x_2 + x_3 \leq 1$;

$$x_1^{\,2} + x_2^{\,2} \geq 20;$$

$$x_1, x_2, x_3 \geq 0$$

Here, the object function (z) is linear but one of the constraints is non-linear

EXAMPLES

- Optimize $z = {x_1}^2 + 2{x_2}^2 - 8x_1x_2$

Subject to ${x_1}^2 + x_1x_2 - x_3 = 35;$

$$x_1, x_2 \geq 0$$

Here, both the object function as well as the constraints are non-linear

RECALL

QUADRATIC PROGRAMMING PROBLEMS:

- In this type of problems the object function is the sum of quadratic function and linear functions of the decision variables $x_1, x_2, \dots \dots \dots, x_n$ and there are no constraints on $x_1, x_2, \dots \dots \dots, x_n$.
- The object function looks like
- $z = a_{11}{x_1}^2 + a_{22}{x_2}^2 + \dots \dots \dots + a_{nn}{x_n}^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \dots \dots \dots + a_{23}x_2x_3 + a_{24}x_2x_4 + \dots \dots \dots + c_1x_1 + c_2x_2 + \dots \dots + c_nx_n$

METHOD OF SOLUTION

- We assume that all the first order and second order partial derivatives i.e., $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist for all i and j.
- The points of maxima and minima are obtained by solving the equations
- $\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \dots \dots, \frac{\partial f}{\partial x_n} = 0$ .... (1)

- Suppose by solving these equations we get the point $X_0(x_1, x_2, \ldots \ldots \ldots, x_n)$
- Now we consider the Hessian matrix defined below.

- $$H = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots\dots\dots\dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \dots\dots\dots\dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \dots\dots\dots\dots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \dots\dots\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots & \dots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots\dots\dots\dots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$$

- Hessian Matrix is highly useful for determining the nature of stationary points, i.e. for determining whether the stationary points x_0 is a maxima or a minima

- If all the principal minor determinants of Hessian Matrix at x_0 are positive then X_0 is a minima.
- If the principal minor determinants $D_1, D_3, \ldots.$ are negative and $D_2, D_4, \ldots.$ are positive, X_0 is a maxima.
- In general if Hessian Matrix is indefinite at X_0, X_0 is a saddle point i.e. neither a maxima nor a minima.

EXAMPLE-1

- Find maximum or minimum of the function
  $z = {x_1}^2 + {x_2}^2 + {x_3}^2 - 4x_1 - 8x_2 - 12x_3 + 100$

Here, the objective function is

$$z = f(x_1, x_2, x_3)$$
$$= {x_1}^2 + {x_2}^2 + {x_3}^2 - 4x_1 - 8x_2 - 12x_3 + 100$$

Stationary points are given by

(1) $\frac{\partial f}{\partial x_1} = 0$, (2) $\frac{\partial f}{\partial x_2} = 0$, (3) $\frac{\partial f}{\partial x_3} = 0$

$$\frac{\partial f}{\partial x_1} = 2x_1 - 4 \qquad \therefore 2x_1 - 4 = 0 \Rightarrow x_1 = 2$$

$$\frac{\partial f}{\partial x_2} = 2x_2 - 8 \qquad \therefore 2x_2 - 8 = 0 \Rightarrow x_2 = 4$$

$$\frac{\partial f}{\partial x_3} = 2x_3 - 12 \qquad \therefore 2x_3 - 12 = 0 \Rightarrow x_3 = 6$$

- $\therefore X_0(x_1, x_2, x_3) = X_0(2, 4, 6)$ is the stationary point

- Now, consider the Hessian matrix at $X_0(2,4,6)$
- $$H = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial {x_3}^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The principal minors are: $[2], \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Value of these minors are $2, 4, 8$. Since all these determinants are positive , z is minimum at $X_0(2, 4, 6)$

$$\therefore z_{min} = 44$$

EXAMPLE-2

- Obtain the relative maximum or minimum (if any) of the function
- $z = x_1 + 2x_3 + x_2 x_3 - {x_1}^2 - {x_2}^2 - {x_3}^2$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = 0$$

Now, $\dfrac{\partial f}{\partial x_1} = 1 - 2x_1 \qquad \therefore 1 - 2x_1 = 0 \Rightarrow x_1 = \dfrac{1}{2}$

$\dfrac{\partial f}{\partial x_2} = x_3 - 2x_2 \qquad \therefore x_3 - 2x_2 = 0$

$\dfrac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 \qquad \therefore 2 + x_2 - 2x_3 = 0$

Solving the last two simultaneous equations, we get

$x_2 = 2/3\,, x_3 = 4/3$

$\therefore X_0(1/2\,, 2/3\,, 4/3)$ is the stationary point

SOLUTION

- $X_0(1/2, 2/3, 4/3)$ is the stationary point
- $H = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial {x_3}^2} \end{bmatrix}$
- $= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

- The principal minors are: $[-2], \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

The values of their determinants are $-2, 4, -6$

Since all the values of $\Delta_1, \Delta_2, \Delta_3$ are alternatively negative, positive and negative, $X_0(1/2\,, 2/3\,, 4/3)$ is a maxima

- $\therefore z_{max} = \frac{19}{12}$

SOME MORE EXAMPLES

- Find maximum or minimum of the following functions

1. $z = {x_1}^2 + {x_2}^2 + {x_3}^2 - 6x_1 - 8x_2 - 10x_3$
2. $z = x_1x_2 + 9x_1 + 6x_3 - {x_1}^2 - {x_2}^2 - {x_3}^2$
3. $z = {x_1}^2 + {x_2}^2 + {x_3}^2 - 6x_1 - 10x_2 - 14x_3 + 103$
4. $z = 2x_1 + x_3 + 3x_2x_3 - {x_1}^2 - 3{x_2}^2 - 3{x_3}^2 + 17$
5. $z = {x_1}^2 + {x_2}^2 + {x_3}^2 - 8x_1 - 10x_2 - 12x_3 + 100$
6. $z = 2x_1 + 6x_3 + 9x_2x_3 - 4{x_1}^2 - 9{x_2}^2 - 9{x_3}^2$