JOINT PROBABILITY DISTRIBUTION

Nandini Rai

Two Dimensional Discrete Random Variable

In many trials we study two characteristics of an outcome and we need a pair of numbers to describe the outcome. For example,

we may select a student at random from a group and measure his height (h) and weight (w).

We may select a worker from a factory and note his age(x) and salary (y).

We may select a couple at random and note the age of the husband (x) and the age of the wife (y).

We may select a student at random and note the marks in Mathematics (x) and marks in Statistics (y).

In such cases we need a pair of numbers to record the outcome. Such experiments are called **two dimensional.**

Two Dimensional Discrete Random Variable

- We consider two function which associate a pair of real numbers to an outcome of an experiment of this type. These two functions taken jointly are called a **two dimensional discrete random variable**.
- If X and Y assume finite or countable infinite values, then (X, Y) is called a two dimensional **discrete random variable.**

Two Dimensional Discrete Random Variable

Definition: Let (X, Y) be a two dimensional discrete random variable. With each possible outcome $\left(x_i, y_j\right)$ we associate a number p_{ij} representing the probability of the event that $X = x_i \; and \; Y = y_j$ and satisfying the conditions:

(i) $p\left(x_i,\ y_j\right) \geq 0 \;\; for\; all\; i, j$

(ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p\left(x_i, y_j\right) = 1$

The function p defined for all $\left(x_i,\ y_j\right)$ is called the joint **probability mass function** of (X, Y).

The set of values of $\left(x_i,\ y_j\right)$ together with their probabilities p_{ij} is called the **probability distribution** of (X, Y)

Two Dimensional Discrete Random Variable

- Probability Distribution of (X, Y) is given by

X/Y	y_1	y_2	………	………..	y_m
x_1	p_{11}	p_{12}	………	………	p_{1m}
x_2	p_{21}	p_{22}	………	………	p_{2m}
x_3	p_{31}	p_{32}	………	………	p_{3m}
x_n	p_{n1}	p_{n2}	………	………	p_{nm}

Marginal Probability Distribution

- A two dimensional probability distribution gives us the probability that X will take a particular value and Y will take a particular value.
- But we may also be interested in the probability that X will take a particular value irrespective of the values of Y.
- This is called **marginal probability distribution of X**.
- Or we may be interested in the probability that Y will take a particular value **irrespective of the values of X**.
- This is called **marginal probability distribution Y**
- Marginal probabilities distributions in the case of discrete two dimensional distributions are easily obtained by summing probabilities vertically or horizontally.

Marginal Probability Distribution

- In general, consider the following probability distribution of (X,Y).

X/Y	y_1	y_2	………	………..	y_m	Total
x_1	p_{11}	p_{12}	………	………	p_{1m}	p_1
x_2	p_{21}	p_{22}	………	………	p_{2m}	p_2
x_3	p_{31}	p_{32}	………	………	p_{3m}	p_3
x_n	p_{n1}	p_{n2}	………	………	p_{nm}	p_n
Total	p_1'	p_2'	………	………	p_m'	1

Marginal Probability Distribution

- If we find the probabilities $p_1, p_2, \ldots p_i \ldots . p_n$ $of\ x_1, x_2 \ldots . x_i \ldots . . x_n$ only, irrespective of the values taken by Y, the probabilities distribution so obtained is called the marginal probability distribution of X.

Thus, the marginal probability distribution of x is:

X	$x_1, x_2, \ldots . x_i, \ldots . . x_n$	**Sum**
$P(X)$	$p_1, p_2, \ldots . p_i, \ldots . . p_n$	1

Marginal Probability Distribution

- Similarly, if we find the probabilities $p_1', p_2', \ldots . p_j' \ldots \ldots p_m{}'$ of $y_1, y_2, \ldots . y_j \ldots . . y_m$ irrespective of the values taken by X, the probability distribution so obtained is called the marginal probability distribution of Y.
- Thus, the marginal probability distribution of Y is:

Y	$y_1, y_2, \ldots . y_i, \ldots . . y_n$	**Sum**
$P(Y)$	$p_1{}', p_2{}', \ldots . p_i{}', \ldots . . p_m{}'$	1

Conditional Probability Distribution

- $P(X = x_i | Y = y_i) = \frac{P(X=x_i, Y=y_i)}{P(Y=y_i)}$

Example-1

The joint probability distribution of X and Y is given by

$P(X = x, Y = y) = \frac{x+3y}{24}, x = 1,2; y = 1,2$

Find the joint p.m.f s of X and Y , Also Find the Marginal Probability distributions of X and Y.

Solution:

X/Y	1	2	Total
1	1/6	7/24	11/24
2	5/24	1/3	13/24
Total	3/8	5/8	1

Example-1

- The Marginal Probability Distribution of X & Y is given by

X	1	2
P(X)	11/24	13/24

Y	1	2
P(Y)	3/8	5/8

Example-2

- The joint probability distribution of X_1 *and* X_2 is given by $P(X_1 = x_1, X_2 = x_2) = \frac{1}{27}(x_1 + 2x_2)\,, x_1 = 0\,, 1, 2\,, x_2 = 0, 1, 2$ Find the joint p.m.f s X_1 *and* X_2, Also Find the Marginal Probability distributions of X_1 *and* X_2

Solution:

X_1/X_2	**0**	**1**	**2**	**Total**
0	0	2/27	4/27	2/9
1	1/27	1/9	5/27	1/3
2	2/27	4/27	2/9	4/9
Total	1/9	1/3	5/9	1

Example-2

- The Marginal Probability Distribution of X_1 & X_2 is given by

X_1	0	1	2
P(X_1)	2/9	1/3	4/9

X_2	0	1	2
P(X_2)	1/9	1/3	5/9

Example-3

3 balls are drawn at random without replacement from a box containing 2 white, 4 black and 3 red balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find

(i) Joint probability distribution of X and Y

(ii) $P(X \leq 1)$,
$P(X \leq 1, Y \leq 2), P(Y \leq 2 | X \leq 1)\ , P(X + Y \leq 2)$

(iii) Marginal Probability distribution of X

(iv) Marginal Probability distribution of Y

(v) Conditional distribution of X given Y=1

Also check whether X & Y are independent variables.

Example-3

(i) $P(X=0,Y=0) = P(all\ 3\ balls\ are\ black)$

$$= \frac{{}^4C_3}{{}^9C_3} = \frac{1}{21}$$

(ii) $P(X=0,Y=1) = P(1\ red\ ball\ \&2\ black\ balls)$

$$= \frac{{}^3C_1 \times {}^4C_2}{{}^9C_3} = \frac{3}{14}$$

(iii) $P(X=0,Y=2) = P(2\ red\ balls\ \&1\ black\ ball)$

$$= \frac{{}^3C_2 \times {}^4C_1}{{}^9C_3} = \frac{1}{7}$$

(iv) $P(X=0,Y=3) = P(all\ 3\ balls\ are\ red)$

$$= \frac{{}^3C_3}{{}^9C_3} = \frac{1}{84}$$

And so on …

Example-3

The joint probability distribution of X and Y is given by

X/Y	0	1	2	3	total
0	1/21	3/14	1/7	1/84	35/84
1	1/7	2/7	1/14	0	42/84
2	1/21	1/28	0	0	7/84
total	5/21	15/28	3/14	1/84	1

Example-3

- (ii) $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{35}{84} + \frac{42}{84} = \frac{77}{84}$
- $P(X \leq 1, Y \leq 2)$
- $= P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2) + P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$$= \frac{1}{21} + \frac{3}{14} + \frac{1}{7} + \frac{1}{7} + \frac{2}{7} + \frac{1}{14} = \frac{19}{21}$$

Example-3

$$P(Y \leq 2 | X \leq 1) = \frac{P(Y \leq 2, X \leq 1)}{P(X \leq 1)}$$

$$= \frac{19/21}{77/84} = \frac{76}{77}$$

$$\begin{aligned} P(X + Y \leq 2) \\ &= P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2) \\ &\quad + P(X = 1, Y = 0) + P(X = 1, Y = 1) \\ &\quad + P(X = 2, Y = 0) \\ &= \frac{1}{21} + \frac{3}{14} + \frac{1}{7} + \frac{1}{7} + \frac{2}{7} + \frac{1}{21} = \frac{37}{42} \end{aligned}$$

Example-3

- (iii) Marginal Probability distribution of X

X	0	1	2	Total
P(X)	35/84	42/84	7/84	1

Example-3

- (iv) Marginal Probability distribution of Y

Y	0	1	2	3	Total
P(Y)	5/21	15/28	3/14	1/84	1

Example-3

- (v) Conditional probability of X given Y=1

$$= P(X = 0|Y = 1) + P(X = 1|Y = 1) + P(X = 2|Y = 1)$$

$$= \frac{P(X = 0, Y = 1)}{P(Y = 1)} + \frac{P(X = 1, Y = 1)}{P(Y = 1)} + \frac{P(X = 2, Y = 1)}{P(Y = 1)}$$

$$= \frac{3/14}{15/28} + \frac{2/7}{15/28} + \frac{1/28}{15/28} = 1$$

(vi) Conditional probability of Y if X=2

$$= P(Y = 0|X = 2) + P(Y = 1|X = 2)$$

$$= \frac{P(Y = 0, X = 2)}{P(X = 2)} + \frac{P(Y = 1, X = 2)}{P(X = 2)}$$

$$= \frac{1/21}{7/84} + \frac{1/28}{7/84} = 1$$

Example-3

- (vii) Check whether X & Y are independent variables
- X and Y are said to be independent if
- $P(X, Y) = P(X) \times P(Y) \ \ \forall X \,\&Y$

Here consider $X = 0, Y = 1$

$$P(X = 0, Y = 1) = \frac{3}{14}$$

And $P(X = 0) \times P(Y = 1) = \frac{35}{84} \times \frac{15}{28} = \frac{25}{108}$

Hence X & Y are not independent variables.

Two dimensional continuous Probability Distribution

Let (X, Y) be a two dimensional continuous variable and let $f_{XY}(x, y)$ be function of (x, y) such that

(i) $f_{XY}(x, y) \geq 0$,

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx\, dy = 1$

(iii) $\int_{a}^{b} \int_{c}^{d} f_{XY}(x, y) dx\, dy$ represents the probability that $P(a \leq X < b, c \leq Y \leq d)$

then $f_{XY}(x, y)$ is called two dimensional probability density function.

Marginal Probability Distribution

(a) The marginal probability density function of X is obtained by integrating two dimensional p.d.f $f_{XY}(x,y)$ w.r.t y from $-\infty \; to \; \infty$

Thus, the marginal p..d.f of X is $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$

where $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$

(b) The marginal probability density function of Y is obtained by integrating two dimensional p.d.f $f_{XY}(x,y)$ w.r.t x from $-\infty \; to \; \infty$.

Thus, the marginal p.d.f of Y is $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$

where $f_Y(y) \geq 0 \; and \; \int_{-\infty}^{\infty} f_Y(y)dy = 1$

Example-4

The joint probability density of two random variables is given by $f_{XY}(x,y) = \begin{cases} 15e^{-3x-5y}, & x > 0, y > 0 \\ 0, & eleswhere \end{cases}$

(a) Find the probability that

(i) $1 < X < 2 \; and \; 0.2 < Y < 0.3$ **(ii)** $X < 2 \; and \; Y > 0.2$

(b) Find marginal probability distributions of X and Y.

Example-4

(i)$P(1 < x < 2, 0.2 < y < 0.3)$

$= \int_{1}^{2} \int_{0.2}^{0.3} 15e^{-3x-5y}\, dxdy$

$= 15 \int_{1}^{2} e^{-3x}\, dx \int_{0.2}^{0.3} e^{-5y}\, dy$

$= 0.00684$

(ii) $P(x < 2, y > 0.2)$

$= \int_{0}^{2} \int_{0.2}^{\infty} 15e^{-3x-5y}\, dxdy$

$= 15 \int_{0}^{2} e^{-3x}\, dx \int_{0.2}^{\infty} e^{-5y}\, dy$

$= 0.367$

Example-4

(iii) Marginal Probability distribution of X

$$f_X(x) = \int_0^{\infty} 15e^{-3x}(e^{-5y})dy$$

$$= -3e^{-3x}(0-1)$$

$$\therefore f_X(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0 & , otherwise \end{cases}$$

Marginal Probability distribution of Y

$$f_Y(y) = \int_0^{\infty} 15e^{-5y}(e^{-3x})dx$$

$$= -5e^{-5y}(0-1)$$

$$\therefore f_Y(y) = \begin{cases} 5e^{-5y} , & y > 0 \\ 0 , & otherwise \end{cases}$$

Example-5

Given $f_{xy}(x,y) = \begin{cases} cx(x-y), & 0 < x < 2\ , -x < y < x \\ 0\ , & otherwise \end{cases}$

(i)Evaluate c (ii) find $f_x(x)$ (iii) find $f_{\frac{y}{x}}\left(\frac{y}{x}\right)$

Solution: by the property of joint probability distribution

$$\int\int f_{xy}(x,y)\, dx\, dy = 1$$

$$\Rightarrow \int_0^2 \int_{-x}^{x} cx(x-y)dxdy = 1$$

$$\Rightarrow c = \frac{1}{8}$$

Example-5

$(ii) f_x(x) = \int_{-x}^{x} f_{xy}(x, y)\ dy$

$= \int_{-x}^{x} \frac{1}{8} x(x - y) dy$

$= \frac{x^3}{4}\ 0 < x < 2$

(iii) $f_{\frac{y}{x}}\left(\frac{y}{x}\right)$ = conditional density function of y given x

$= \frac{f_{xy}(x,y)}{f_x(x)}$

$= \frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}}$

$= \frac{x-y}{2x^2}\quad, -x < y < x$

Example-6

$$f_{xy}(x, y) = xy^2 + \frac{x^2}{8} \;, 0 \leq x \leq 2, 0 \leq y \leq 1$$

Compute (i) $P(x > 1)$ (ii) $P\left(y < \frac{1}{2}\right)$ (iii) $P\left(x > 1, y < \frac{1}{2}\right)$

(iv) $P\left(y < \frac{1}{2} \middle| x > 1\right)$ (v) $P(x < y)$ (vi) $P(x + y \leq 1)$

Solution:

(i) $P(x > 1) = \iint f_{xy}(x, y)\, dx\, dy$

$$= \int_{y=0}^{1} \int_{x=1}^{2} \left(xy^2 + \frac{x^2}{8}\right) dxdy$$

$$= \frac{19}{24}$$

Example-6

(ii) $P\left(y < \frac{1}{2}\right) = \iint f_{xy}(x,y)\, dx\, dy$

$$= \int_{y=0}^{1/2} \int_{x=0}^{2} \left(xy^2 + \frac{x^2}{8}\right) dxdy$$

$$= \frac{1}{4}$$

(iii) $P\left(x > 1, y < \frac{1}{2}\right) = \int_{y=0}^{1/2} \int_{x=1}^{2} \left(xy^2 + \frac{x^2}{8}\right) dxdy$

$$= \frac{5}{24}$$

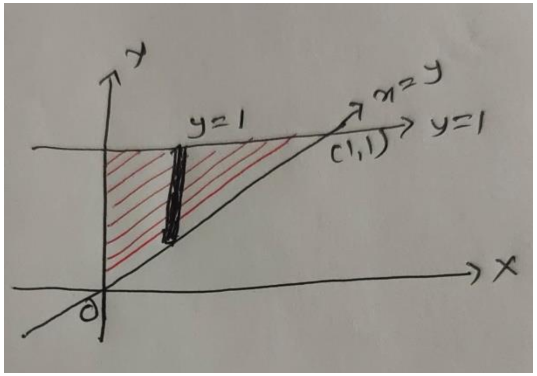
(iv) $P\left(y < \frac{1}{2} \middle| x > 1\right) = \frac{P\left(x>1, y<\frac{1}{2}\right)}{P(x>1)} = \frac{5/24}{19/24} = \frac{5}{19}$

•

Example-6

(v) $P(x < y) = \int_{x=0}^{1} \int_{y=x}^{1} \left(xy^2 + \frac{x^2}{8} \right) dxdy$

$$= \frac{53}{480}$$



(vi) $P(x + y \leq 1) = \int_{x=0}^{1} \int_{y=0}^{1-x} \left(xy^2 + \frac{x^2}{8} \right) dxdy$

$$= \frac{13}{480}$$