Normal Distribution

NORMAL DISTRIBUTION

- Normal Distribution is one of the most important and commonly used continuous distribution. It was first developed by DeMoivre but it is credited to Gauss who refereed to it first in 1809. A large number of continuous variates follow this distribution, hence, the ‘normal’.

DEFINITION

- A continuous random variable X is said to follow normal distribution with parameter m (called mean) and σ^2 (Called variance), if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \quad -\infty < x < \infty,$$

$$-\infty < m < \infty, \ \sigma^2 > 0$$

REMARKS

1. A continuous random variate X following normal distribution with mean m & standard deviation σ is referred to as $\boldsymbol{X \sim N(m, \sigma)}$
2. If X is a normal variate with parameter m, σ then $Z = \frac{X-m}{\sigma}$ is also a normal variate with mean $=\ 0$ and standard deviation $= 1.$
   It is called **Standard Normal Variate. (S.N.V.)**

IMPORTANCE OF NORMAL DISTRIBUTION

(i) The variables such as height, weight, intelligence etc. follow normal distribution.

(ii) Many other distributions occuring in practice such as Binomial, Poisson etc. Can be approximated by normal distribution.

(iii) Many of the distributions of samples statictic e.g Sample mean, Samples variance tend to normal distribution for large samples

(iv) Normal distribution has wide applications in Statistical Quality Control.

(v) Errors in measurements of Physical quantities follow normal distribution.

(vi) It is also useful in psychological and educational research.

MODE OF NORMAL DISTRIBUTION

By definition, mode is the value of X for which the frequency is maximum i.e $f(x)$ is maximum.

Hence, for mode $f'(x) = 0$

Now, $f(x) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

Differentiating $f(x)$ w.r.t x,

$$f'(x) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} . \left(-\frac{1}{2}\right) . 2. \left(\frac{x-m}{\sigma}\right) \frac{1}{\sigma}$$

$$f'(x) = -f(x).\frac{(x-m)}{\sigma^2} \therefore f''(x) = -f'(x).\left(\frac{x-m}{\sigma^2}\right) - f(x)\frac{1}{\sigma^2}$$

$$= f(x).\frac{(x-m)^2}{\sigma^4} - \frac{f(x)}{\sigma^2}$$

$$= \frac{1}{\sigma^4} f(x)[(x-m)^2 - \sigma^2]$$

when $x = m$, $f'(x) = 0$

and $f''(x)$ is negative

$\therefore Mode = m$

MEDIAN OF NORMAL DISTRIBUTION

By definition if M is the median, then

$$\int_{-\infty}^{M} f(x)dx = \frac{1}{2}$$

$$\int_{-\infty}^{M} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx + \int_{m}^{M} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = \frac{1}{2}$$

................(i)

Put $\frac{x-m}{\sigma} = z$ in $I_1 = \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$

$$\therefore I_1 = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}t^2} dt$$

$$\therefore I_1 = \frac{1}{\sqrt{2\pi}} . \sqrt{\frac{\pi}{2}} = \frac{1}{2}$$

$$\left[\because \int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \quad \therefore \int_0^\infty e^{-t^2/2} dt = \sqrt{\frac{\pi}{2}} \right]$$

From (i) $$\frac{1}{2}+\int_m^M \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx=\frac{1}{2}$$

$$\therefore \int_m^M \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx=0$$

$\therefore m=M$ $\qquad\qquad$ $\therefore Median=m$

Remark: Thus mean, median and mode of the normal distribution are equal to m

$$\therefore Mean=Median=Mode=m$$

LINEAR COMBINATION (ADDITIVE PROPERTY)

1. Let $X_i, i = 1,2,3, \ldots \ldots n$ be n independent normal variates with mean m_i and variance σ_i^2.

Let their linear combination be denoted by Y

i.e. $Y = a_1 X_1 + a_2 X_2 + \ldots \ldots + a_n x_n$

Then Y is also a normal variate with mean m and variance σ^2

where $m = a_1 m_1 + a_2 m_2 + \ldots a_n m_n$

$$and\ \sigma^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \ldots . + a_n^2 \sigma_n^2$$

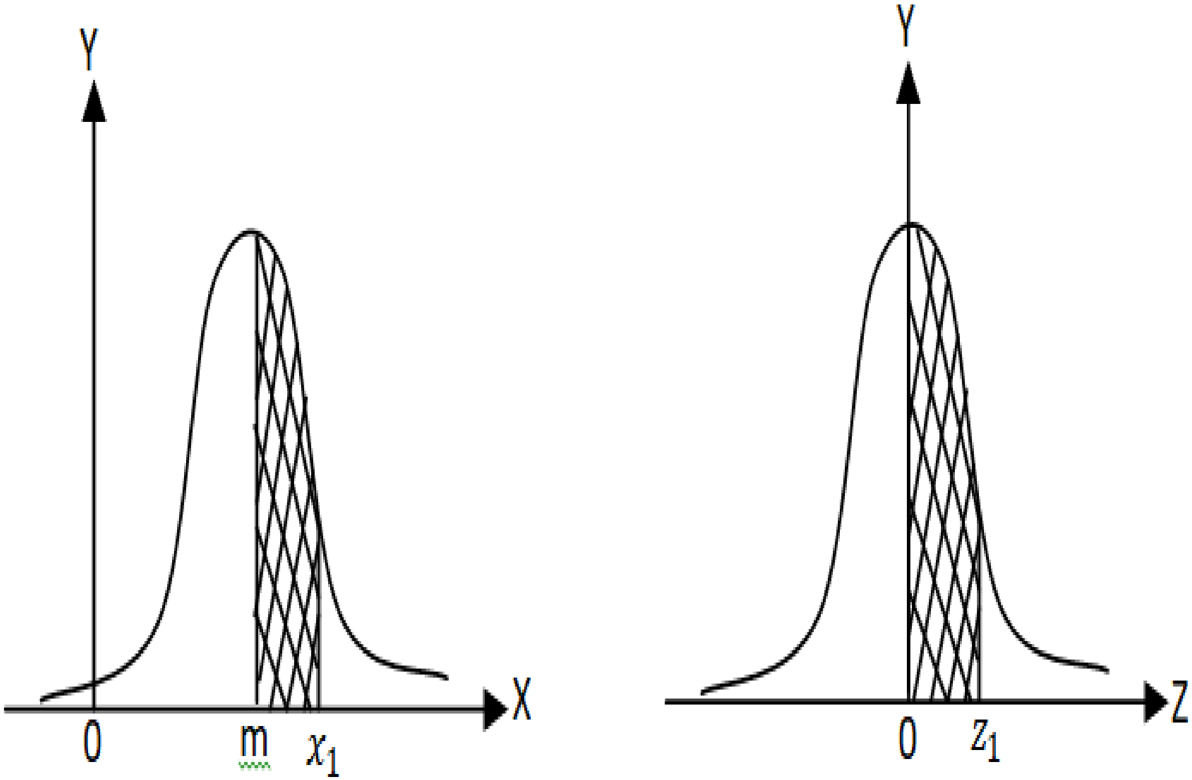
2. $Y = X_1 + X_2$ is normal variate with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$
3. $Y = X_1 - X_2$ is also a normal variate with mean $m_1 - m_2$ and variance $\sigma_1^2 + \sigma_2^2$
4. Comparing Normal Distribution with Poisson Distribution we find that sum of two Normal or Poisson Variates is a Normal or Poisson variate, But although difference of two normal variates is a normal variate, the difference of two Poisson variates is not a Poisson variate.

AREA PROPERTY

1. " If X is a normal variate with mean m and variance σ^2 $and\ Z$ is standard normal variate (with mean zero and variance one) then the area under the normal curve of X between $X = m$ and $X = x_1$ is equal to the area under the S.N Curve of Z between

$Z = 0\ to\ Z = z_1$ (say, corresponding to x_1)

AREA TABLE


Y
X
0
m
x_1
Y
Z
0
z_1

2. In some problems we know that X is a normal variate with mean m and standard deviation σ. We are required to find the value of $X = x_1$ corresponding to a given probability. Suppose, we want to find of $X = x_1$ such that $P(X > x_1) = \alpha$.

In this case also we consider $S.N.V$.

$$Z = \frac{x-m}{\sigma}$$

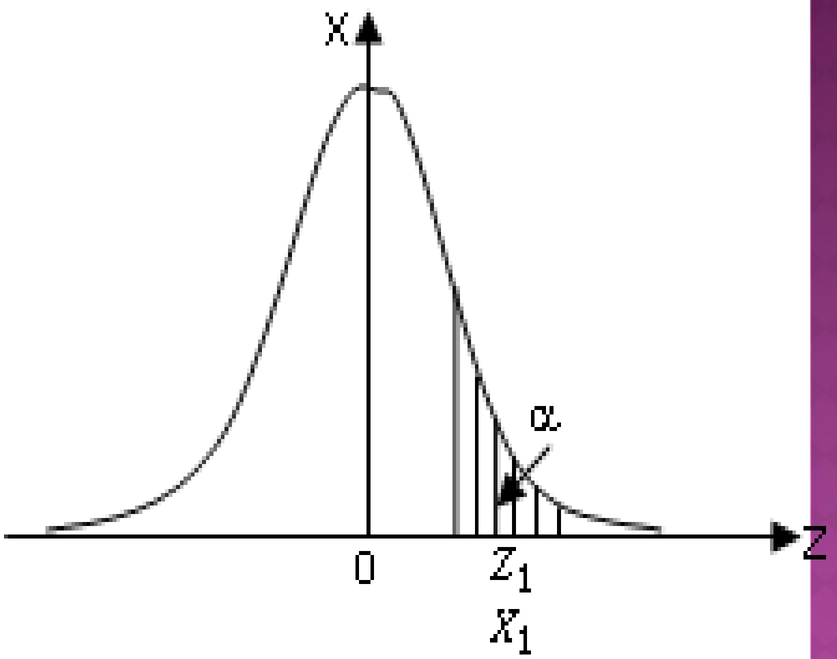
We now consult the area table and

find the value of $Z = z_1$ for which area to the right is α i.e area between $z = 0 \ to \ Z = z_1$ is $(0.5 - \alpha)$.

From this value of z_1 we get X using

$$z_1 = \frac{x_1 - m}{\sigma}$$

$$i.e \ x_1 = m + z_1\sigma$$



3. Since standard normal curve is symmetrical about the y – axis it is enough to find the areas to the right. The areas to the left of y – axis at equal distance will be equal

4. The total area under the curve is unity. Hence, because of symmetry the area under S.N.V to the right of the y – axis is 0.5

5. To find the probability that X – will be lie between $x_1\ and\ x_2\ (x_1 < x_2)$, we find the corresponding values of S.N.V. Z (from $Z = \frac{X-m}{\sigma}$) say z_1 and z_2 and find the area from z_1 and z_2 under the S.N. curve.

The required probability is this area.

$P(x_1 \leq X \leq x_2) = P(z_1 \leq Z \leq z_2)$

$=$ area between $Z = z_1$ and $Z = z_2$ under the S.N. Curve

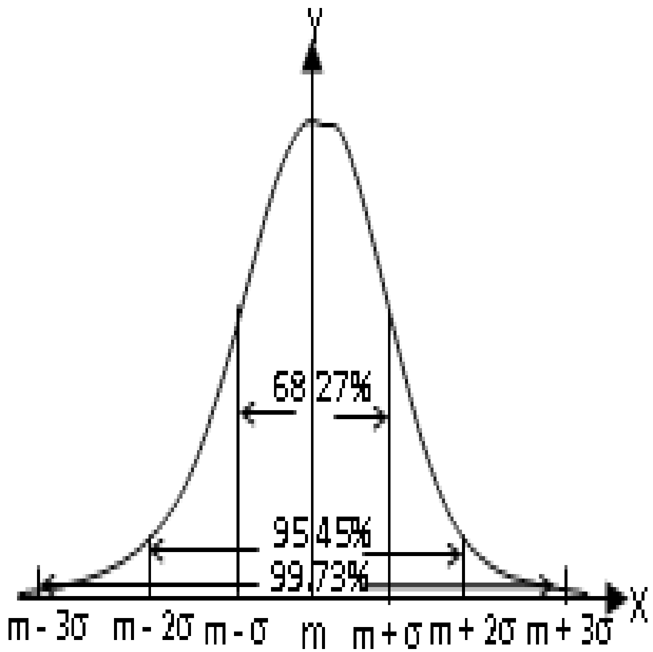
6. The area under the normal curve is distributed as follows

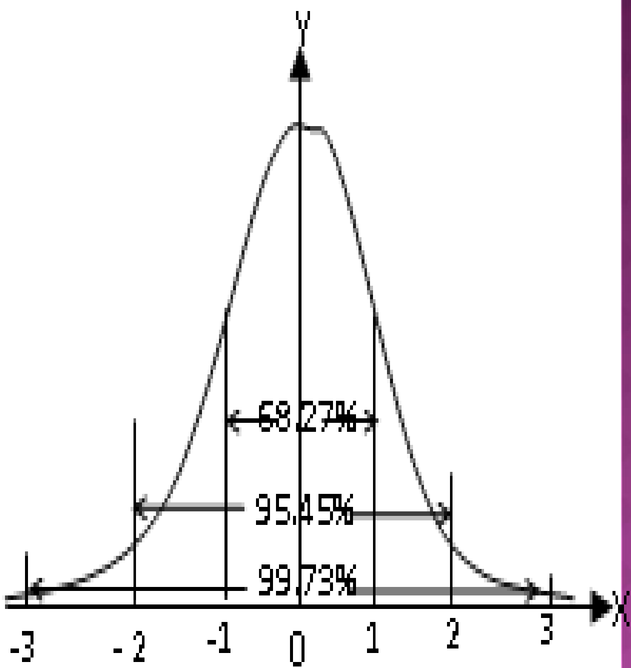
(a) The area between $x = m - \sigma \ and$ $x = m + \sigma \ is$ 68.27%

(b) The area between $x = m - 2\sigma \ and$ $x = m + 2\sigma \ is$ 95.45%

(c) The area between $x = m - 3\sigma \ and$ $x = m + 3\sigma \ is$ 99.73%

These areas under the normal curve and standard normal curve are shown below





QUARTILE DEVIATION

- $Q_1 = m - \frac{2}{3}\sigma$
- $Q_3 = m + \frac{2}{3}\sigma$
- $\therefore Quartile\ Deviation = \frac{Q_3 - Q_1}{2} = \frac{2}{3}\sigma$
- **The Quartile Deviation of a Normal Distribution is (2/3) S.D**
- **The Mean Deviation of a Normal Distribution is (4/5) S.D**

NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

It can be proved, that if X is a Binomial variate with parameter n and p (i.e mean = np and $S.D = \sqrt{npq}$ where $\mathrm{q} = 1 - p$) then $Z = \frac{X-np}{\sqrt{npq}}$

is a standard Normal Variate if $n \to \infty$

(i.e is large) and neither p nor q is small.

REMARKS

1. Normal distribution can be used in place of binomial distribution when np and nq are both greater than 15.
2. Normal Distribution can also be obtained from Poission distribution when the parameter $m \to \infty$.

EX 1. Find k and the mean and standard deviation of the normal distribution given by

(i) $y = k\, e^{-\left(\frac{x^2}{18} - x + \frac{9}{2}\right)}$

(ii) $y = k\, e^{-\left(\frac{x^2}{6} - x + \frac{3}{2}\right)}$

Solution: $f(x) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

EX 2. Write down the equation of the curve of the normal distribution with mean 50 and standard deviation 9. What is the first quartile of the distribution ?

Solution: $f(x) = \frac{1}{\sqrt{2\pi}.\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

$$f(x) = \frac{1}{\sqrt{2\pi}.9} e^{-\frac{1}{2}\left(\frac{x-50}{9}\right)^2}$$

$$Q_1 = m - \frac{2}{3}\sigma$$

$$Q_1 = 50 - \frac{2}{3}9 = 44$$

EX 3. Write down the equation of the normal curve with mean 10 and variance 36. What is the quartile deviation of the distribution ?

EX 4. For a normal distribution the mean is 50 and the standard deviation is 15.

Find (i) Q_1 and Q_3, (ii) mean deviation & the interquartile range

Solution:(i) For a normal distribution

$Q_1 = m - \frac{2}{3}\sigma = 50 - \frac{2}{3}15 = 40$

Again $Q_3 = m + \frac{2}{3}\sigma = 50 + \frac{2}{3}15 = 60$

(ii) The mean deviation of the normal distribution is,

$$MD = \frac{4}{5}\sigma = \frac{4}{5} \times 15 = 12$$

$\therefore$ Interquartile range $= Q_3 - Q_1 = 60 - 40 = 20$

EX 5. Find the probability that a random variable having the standard normal distribution will take on a value between 0.87 and 1.28

Solution:

$$P(0.87 < Z < 1.28)$$

$=$ area between $z = 0.87$ and $z = 1.28$

$$= (\text{area from } z = 0 \text{ to } z = 1.28)$$
$$-(\text{area from } z = 0 \text{ to } z = 0.87)$$

$$= 0.3997 - 0.3078 = 0.0919$$

EX 6. Find the probability that a random variable having standard normal distribution will take a value between -0.34 and 0.62

Solution:

$P(-0.34 < Z < 0.62)$= Area from $z = -0.34$ to 0.62

= Area from $z = 0$ to $z = 0.34$ + Area from $z = 0$ to $z = 0.62$

$$\therefore P(-0.34 < Z < 0.62)$$
$$= 0.13331 + 0.2324$$
$$= 0.3655$$

EX 7. For a normal variate X with mean 25 and standard deviation 10, find the area between
(i) $X = 25, X = 35$, **(ii)** $X = 15, X = 35$
and also the area such that,
(iii) $X \geq 15$, **(iv)** $X \geq 35$

Solution: S.N.V. $Z = \frac{X-m}{\sigma} = \frac{X-25}{10}$

(i) When $x = 25, z = 0$, and when $x = 35, z = 1$

$\therefore$ Area(between $x = 25$ and $x = 35$)

$= \text{area (between } z = 0 \text{ and } z = 1)$

$= 0.3413$

(ii) When $x = 15, z = -1$ and when $x = 35, z = 1$

$\therefore$ Area between ($z = 15$ and $z = 35$)

$= \text{ area between } (z = -1 \text{ and } z = 1)$

$= 2 \text{ area between } (z = 0 \text{ and } z = 1)$

$= 2(0.3413) = 0.6826$

(iii) When $X \geq 15, z \geq -1$

$\therefore$ Area to the right of $(x = 15)$

$=$ area to the right of $(z = -1)$

$=$

(area between $z = -1$ to $z = 0$) + (area to the right of $z = 0$)

$= 0.3413 + 0.5 = 0.8413$

(iv) When $X \geq 35, z \geq 1$

$\therefore$ Area to the right of $(x = 35)$

$=$ area to the right of $(z = 1)$

$=$ (area to the right of $z = 0$) $-$ (area between $z = 0$ and $z = 1$)

$= 0.5 - 0.3413 = 0.1587$

EX 8. If X is normally distributed with mean 10 and standard deviation 2, find $P(-3 \leq X \leq 12)$ and $P(|X| \geq 5)$

HW

EX 9. If X is a normal variate with mean 10 and standard deviation 4, find

(i) $P(|X-14|<1)$, (ii) $P(5 \leq X \leq 18)$,(iii) $P(X \leq 12)$

Solution: We have $Z=\frac{x-m}{\sigma}=\frac{x-10}{4}$

(i) When $x=13, z=\frac{13-10}{4}=\frac{3}{4}=0.75$

When $x=15, z=\frac{15-10}{4}=\frac{5}{4}=1.25$

$$\therefore P(|X-14|<1)=P(-1<X-14<1)$$
$$=P(13<X<15)$$
$$=P(0.75<z<1.25)$$

$=$
$(\text{Area from } z=0 \text{ to } z=1.25)-$
$(\text{Area from } z=0 \text{ to } z=0.75)$

$$=0.1210$$

(ii) When $x = 5, z = \frac{5-10}{4} = -1.25$

When $x = 18, z = \frac{18-10}{4} = 2$

$\therefore P(5 \leq X \leq 18) = P(-1.25 \leq Z \leq 2)$

$= \text{area between } z = -1.25 \text{ and } z = 2$

$= (\text{area between z} = 0 \text{ and z} = 1.25)$

$+ (\text{area between z} = 0 \text{ and z} = 2)$

$= 0.3944 + 0.4772 = 0.8716$

(iii) When $x = 12, z = \frac{12-10}{4} = 0.5$

$\therefore P(X \leq 12) = P(Z \leq 0.5)$

$= \text{area upto } z \leq 0.5$

$= (\text{area from} - \infty \text{ to } z = 0) + (\text{area from } z = 0 \text{ to } z = 0.5)$

$= 0.5 + 0.1015 = 0.6915$

EX 10. If Z is a standard normal variate, find c such that **(i)** $P(-c < Z < c) = 0.95$,

(ii) $P(|Z| > c) = 0.01$

If X is a normal variate with the mean120 and standard deviation 10, find c such that

(i) $P(X > c) = 0.02$ **(ii)** $P(X < c) = 0.05$

EX 12. If X is a normal variate with mean 25 standard deviation 5, find the value

(i) of $X = x_1$, such that $P(X \geq x_1) = 0.32$,

(ii) of $X = x_2$, such that $P(X \leq x_2) = 0.73$,

(iii) of $X = x_3$, such that $P(X \leq x_{3.}) = 0.24$

Solution:(i) Since 0.32 is less than 0.5.

We have to find

$Z = z_1$ = corresponding area $= 0.5 - 0.32 = 0.18$

Now, from the table we find that corresponding to

$Z = 0.47$ the area under S.N.V. is 0.18

$\therefore Z = \frac{X-m}{\sigma}$ gives $0.47 = \frac{X-25}{5}$

$$\therefore X = 25 + 5 \times 0.47 = 25 + 2.35 = 27.35$$

$\therefore P(X \geq 27.35) = 0.32$

(ii) Since 0.73 is greater than 0.5, we have to find $Z = z_1 =$ corresponding to area $= 0.73 - 0.5 = 0.23$

Now, from the table we find that corresponding to $Z = 0.61$ the area under S.N.V. is 0.23

$\therefore Z = \frac{X-m}{\sigma}$ gives $0.61 = \frac{X-25}{5}$

$\therefore X = 25 + 5 \times 0.61 = 28.05$

$\therefore P(X \leq 28.05) = 0.73$

(iii) Since 0.24 is less than 0.5, we have to find $Z = z_1 =$ corresponding to area $= 0.5 - 0.24 = 0.26$

Now, from the table, we find that corresponding to $Z = 0.71$, the area under S.N.V. is 0.26

Since, we want X less than the desired value, we must take Z_1 on the left hand area i.e. $Z_1 = -0.71$

$\therefore Z = \frac{X-m}{\sigma}$ gives $-0.71 = \frac{X-25}{5}$

$\therefore X = 25 - 5 \times 0.71 = 25 - 3.55 = 21.45$

$\therefore P(X \leq 21.45) = 0.24$

EX 13. The first and third quartiles of a normal distribution are respectively 92 and 128. Find the mean and the standard deviation.

Solution:

We have $Q_1 = m - \frac{2}{3}\sigma$ and $Q_3 = m + \frac{2}{3}\sigma$

$\therefore 92 = m - \frac{2}{3}\sigma$ and $128 = m + \frac{2}{3}\sigma$

Adding $220 = 2m$

$\therefore m = 110$

Then, $92 = 110 - \frac{2}{3}\sigma$

$\therefore \frac{2}{3}\sigma = 18$

$\therefore \sigma = 27$

EX 14. For a normal distribution the first quartile is 46 and the variance is 144. Find **(i)** mode,

(ii) limits of central 50% items,

(iii) mean deviation.

EX 15. The mean and the standard deviation of a normal distribution are 70 and 15. Find the quartile deviation and mean deviation.

EX 16. The marks obtained by students in a college are normally distribution with mean 65 and variance 25. If 3 students are selected at random from this college what is the probability that at least one of them would have scored more than 75 marks ?

EX 17. The weights of 4000 students are found to be normally distributed with mean 50 kgs. and standard deviation 5 kgs. Find the probability that a student selected at random will have weight

(i) less than 45 kgs. **(ii)** between 45 and 60 kgs.

EX 18. The daily sales of a firm are normally distributed with mean Rs. 8000 and variance of Rs. 10,000.

(i) What is the probability that on a certain day the sales will be less than Rs. 8210 ?

(ii) What is % of days on which the sales will be between Rs. 8100 and Rs. 8200 ?

EX 19. Mean and standard deviation of chest measurements of 1200 soldiers are 85cms and 5cms respectively. How many of them are expected to have their chest measurements exceeding 95cms. assuming the measurements to follow the normal distribution ?

EX 20. The height of 22 year old boys is distributed normally with mean 63” and standard deviation 2.5” A boy is eligible if his height is between 62” and 56”. Find the expected number of boys out of 180 who will be ineligible because of excess height.

EX 21. A large number of automobile batteries have average life of 24 months. If 34 percent of them average between 22 and 26 months and 272 of them last longer than 29 months how many were in the group tested ? Assume the distribution to be normal.

Solution: $P(22 < X < 26) = 0.34$

$\therefore P\left(\frac{22-24}{\sigma} < Z < \frac{26-24}{\sigma}\right) = 0.34$

Because of symmetry of the normal distribution

$2 \times P\left(0 < Z < \frac{26-24}{\sigma}\right) = 0.34$

$\therefore P\left(0 < Z < \frac{26-24}{\sigma}\right) = 0.17$

$\therefore \frac{26-24}{\sigma} = 0.44 \qquad \therefore \sigma = \frac{2}{0.44} = \frac{50}{11}$

Now when $x = 29, Z = \frac{X-m}{\sigma} = \frac{29-24}{50/11} = \frac{11}{10} = 1.1$

By data area between $z = 0$ and $z = 1.1$ is 0.364

$\therefore$ The area to the right of $z = 1.1$ i.e. $x = 29$ is $= 0.5 - 0.364 = 0.136$ which is the probability that a battery will last longer than 29 months

But $p = \frac{f}{N}$ and $f = 272, p = 0.136$

$\therefore N = \frac{f}{p} = \frac{272}{0.136} = 2000$

Hence, 2000 batteries where tested

EX 22. In a factory a large number of workers have average daily income of Rs. 120. If 38.3% of them have income between Rs. 100 – 140 and 528 of them get more than Rs. 170, how many workers were interrogated ?

EX 23. Monthly salary X in a big organization is normally distributed with mean Rs. 3000 and standard deviation of Rs. 250. What should be the minimum salary of a worker is this organization, so that the probability that he belongs to top 5% workers ?

Solution: We have $Z = \frac{X-m}{\sigma} = \frac{X-3000}{250}$

We want to find z_1 such that $P(Z > z_1) = \frac{5}{100} = 0.05$

Since $0.5 - 0.05 = 0.45$, and corresponding to 0.45 the entry in the area table is 1.64

$\therefore z_1 = 1.64$

$$\therefore 1.64 = \frac{X - 3000}{250}$$

$$\therefore X = 3000 + 250 \times 1.64 = \text{Rs } 3410$$

EX 24. The heights of 1000 cakes baked with certain mix have a normal distribution with a mean of 5.75 cms. And standard deviation of 0.75 cms. Find the numbers of cakes having heights between 5 cms and 6.25 cms. Also find the maximum height of the flattest 200 cakes.

EX 25. The distribution of monthly income of 3000 primary teachers confirms to a normal curve with mean equal to Rs. 600 and standard deviation equal to Rs. 100. Find

(i) the percentage of teachers having monthly income of more than Rs. 800,

(ii) the number of teachers having monthly income of less than Rs. 400,

(iii) the highest monthly income among the lowest paid 100 teachers, and

(iv) the lowest monthly income of the highest paid 100 teachers.

EX 27. The local authorities in a certain city installed 10,000 electric lamps in the streets of the city. If these lamps have average life of 1000 burning hours with a standard deviation of 200 hours, what number of lamps might be excepted to fail

(i) in the first 800 hours,

(ii) between 800 and 1200 hours ?

After what period of burning hours would you expect that **(i)** 10% of the lamps would fail,

(ii) 10% of the lamps would be still burning ?

EX 28. Assuming that the diameters of 100 brass plugs taken consecutively from a Normal distribution with mean 0.7515 cm. and standard deviation 0.0020cm. how many plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cms ?

EX 29. The diameters of can tops produced by a machine are normally distributed with standard deviation of 0.01 cms. At what mean diameter the machine be set so that not more than 5% of the can tops produced by the machine have diameters exceeding 3 cms ?

Solution: Let X denote the diameter of the can tops. X is normally distributed with mean μ (unknown) and standard deviation $\sigma = 0.05$.

We are given that $\mathrm{P}(X > 3) = 0.05$

$$P(Z > z_1) = 0.05$$

$$\therefore P(0 < z < z_1) = 0.5 - 0.05 = 0.45$$

$$\therefore z_1 = 1.64$$

$$1.64 = \frac{3-m}{0.01}$$

- $$\therefore m = 3 - 1.64 \times 0.01 = 2.984$$

EX 30. Find the mean and the standard deviation of a normal distribution of marks in an examination where 58% of the candidates obtained marks below 75, 4% got above 80 and the rest between 75 and 80

Solution: Let m and σ be the mean and standard deviation of the variate

Since 58% students are below 75, $58 - 50 = 8\%$ students are between 75 and m

Since 4% students are above 80, $50 - 4 = 46\%$ students are between m and 80

The area between z=0 and z= $\frac{75-m}{\sigma}$ is 0.08

$\therefore \frac{75-m}{\sigma} = 0.2$

and Area between Z=0 and Z= $\frac{80-m}{\sigma}$ is 0.46

$\therefore \frac{80-m}{\sigma} = 1.8$

$\therefore 75 - m = 0.2\sigma$ and $80 - m = 1.8\sigma$

Subtracting $-5 = -1.6\sigma$ $\therefore \sigma = \frac{5}{1.6} = 3.125$

$\therefore m = 75 - 0.2\sigma = 75 - 3.125 \times 0.2 = 74.4$ mark

EX 31. The qualifying marks for a certain examination are 35 and to secure distinction one has to score more than 74. If 25% of the students fails, whereas 6.681% obtained distinction, determine the mean and the standard deviation assuming that the distribution of marks is normal.

Hint: $P(X < 35) = 0.25$

And $P(X > 74) = 0.06681$

EX 32. If X_1 and X_2 are two independent normal variates with means 30 and 25 variances 16 and 12 and if $Y = 3X_1 - 2X_2$, find $P(60 \le Y \le 80)$

Solution: Since X_1, X_2 are independent normal variates with means 30 and 25 and variances 16 and 12,

$Y = 3X_1 - 2X_2$ is also a normal variate with mean

$m = a_1\bar{X}_1 + a_2\bar{X}_2 = 3(30) + (-2)(25) = 90 - 50 = 40$

and variance $\sigma^2 = {a_1}^2{\sigma_1}^2 + {a_2}^2{\sigma_2}^2 = 9(16) + 4(12) = 192$

S.N.V. $Z = \frac{Y-m}{\sigma} = \frac{Y-40}{\sqrt{192}}$

When $Y = 60, Z = \frac{20}{\sqrt{192}} = 1.44$

When $Y = 80, Z = \frac{40}{\sqrt{192}} = 2.89$

$\therefore P(60 \le Y \le 80) \quad = P(1.44 \le Z \le 2.89)$

$= \text{area between } Z = 1.44 \text{ and } Z = 2.89$

$= (\text{area from } Z = 0 \text{ to } Z = 2.89)$

$- (\text{area from } Z = 0 \text{ to } Z = 1.44)$

$= 0.4981 - 0.4251 = 0.0730$

EX 33. Two independent random variates X and Y are distributed normally with mean and standard deviations $(52, 3)\ and\ (50, 4)$ respectively. Find the probability that a randomly chosen pair of values of X and Y will differ by 1.7 or more.

Solution: If $U = X - Y$ then by the additive property of normal variates, U is a normal variate with mean $= 52 - 50 = 2$ and standard deviation $\sqrt{9 + 16} = 5$ i.e. $N(2,5)$

$\therefore Z = \frac{U-m}{\sigma} = \frac{U-2}{5}$ is a S.N.V.

Now, $P(X \text{ and } Y \text{ will differ by 1.7 or more}) = P(|X - Y| \geq 1.7)$

$= P(|U| \geq 1.7) = 1 - P(|U| \leq 1.7) = 1 - P(-1.7 \leq U \leq 1.7)$

Now, when $U = -1.7, z = \frac{-1.7-2}{5} = -0.74$

and when $U = 1.7, \frac{1.7-2}{5} = -0.06$

$\therefore P(X \text{ and } Y \text{ will differ by 1.7 or more}) = 1 - P(-0.74 \leq z \leq -0.06)$

$= 1 - (\text{area from } z = 0.06 \text{ to } z = 0.74)$

$=$

$1 - [(\text{area from } z = 0 \text{ to } z = 0.74) - (\text{area from } z = 0 \text{ to } z = 0.06)]$

$= 1 - (0.2704 - 0.0239) = 1 - 0.2465 = 0.7535$

EX 34. If X and Y are two independent random variates $N(3,4)$ and $N(8,5)$, find the probability that a point (X,Y) will lie between the lines $5X+3Y=8$ and $5X+3Y=15$

Solution: Since X is $N(3,4)$ and Y is $N(8,5)$ by additive property of normal distribution $U=5X+3Y$ follows a normal distribution with mean $m=5\times 3+3\times 8=39$ and $\sigma=\sqrt{25\times 16+9\times 25}=25$

$$P(\text{the point }(X,Y)\text{ lies between the lines }5X+3Y=8\text{ and }5X+3Y=15)=P(8\le U\le 15)$$

$\therefore z=\frac{U-39}{25}$ is a S.N.V.

When $U=8, z=\frac{8-39}{25}=-1.24$

and when $U=15, z=\frac{15-39}{25}=-0.96$

$\therefore P(\text{the point lies between the two lines})$

$$=\text{area between }z=-0.96\text{ and }z=-1.24$$

$$=\text{area between }z=0.96\text{ and }z=1.24$$

$$=(\text{area from }z=0\text{ to }z=1.24)-(\text{area from }z=0\text{ to }z=0.96)$$

$$=0.3925-0.3315=0.061$$

EX 35. In an examination marks obtained by students in Mathematics, Physics and Chemistry are normally distributed with means 51, 53 and 46 with standard deviation $15, 12, 16$ respectively. Find the probability of securing total marks

(i) 180 or above, **(ii)** 80 or below.

Solution: Hint: Let X_1, X_2 and X_3 be the variates of marks in Mathematics, Physics and Chemistry respectively.

Let $Y = X_1, +X_2 + X_3$

$\therefore Y$ is a normal variate with mean $= 51 + 53 + 46 = 150$ and standard deviation $= \sqrt{15^2 + 12^2 + 16^2} = 25$

(i)Probability of securing total marks 180 or above

$$= P(Y > 180) = P(Z > 1.2)$$

(ii)Probability of securing total marks 180 or above

$$= P(Y < 80) = P(Z < -2.8)$$

EX 36. Determine in two different ways, the probability that by guess work a student can correctly answer 25 to 30 questions in a multiple choice quiz consisting of 80 questions. Assume that in each question with four choices only one is correct and the student has no knowledge

Solution: (a) By using Normal approximation to Binomial Distribution

Mean $= m = np = 80 \times (1/4) = 20$ $\quad$ $\text{S.D.}, \sigma = \sqrt{npq} = \sqrt{80 \times \frac{1}{4} \times \frac{3}{4}} = 3.873$

$$\therefore z = \frac{x - m}{\sigma} = \frac{x - 20}{3.873}$$

Since from discrete (Binomial distribution) we are approximating to continuous (Normal distribution) we extend the range 25 to 30 by half units on either side i.e. we take the range as 24.5 to 30.5

When $x = 24.5,\quad z = \frac{24.5-20}{3.873} = 1.16$ $\quad$ When $x = 30.5,\quad z = \frac{30.5-20}{3.873} = 2.71$

$$\therefore P(24.5 \le X \le 30.5) = P(1.16 \le Z \le 2.71)$$

$$= (\text{area from } z = 0 \text{ to } z = 2.71) - (\text{area from } z = 0 \text{ to } z = 1.16)$$

$$= 0.4966 - 0.6770 = 0.1196$$

(b) By using Binomial Distribution

We have $p = \frac{1}{4}, q = \frac{1}{4}, n = 80$

$$P(X = x) = {}^{n}C_x p^x q^{n-x} = {}^{80}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{80-x}$$

$\therefore$ The required probability $= \sum_{x=25}^{30} {}^{80}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{80-x} = 0.1193$

(**Note** that the two values differ only by 0.0003)

EX 37. Using normal distribution, find the probability of getting 55 heads in the toss of 100 fair coins. (Compare the result with that obtained from Binomial distribution)

Solution: Since the coins are fair, we have $p = 1/2\,, q = 1/n$. By data $n = 100$

$\therefore m = np = 100 \times \frac{1}{2} = 50,\ \sigma = \sqrt{npq} = \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} = 5$

Hence, we have S.N.V. $z = \frac{x-m}{\sigma} = \frac{x-50}{5}$

When $x = 54.5, \quad z = \frac{54.5-50}{5} = 0.9$

When $x = 55.5, \quad z = \frac{55.5-50}{5} = 1.1$

$\therefore P(0.9 < z < 1.1) \quad = \text{area from } z = 0.9 \text{ to } z = 1.1$

$$= (\text{area from } z = 0 \text{ to } z = 1.1) - (\text{area from } z = 0 \text{ to } z = 0.9)$$

$$= 0.3643 - 0.3159 = 0.0484$$

Now, for the second part, we have,

$$P(X = x) = {}^{n}C_x p^x q^{n-x} = {}^{100}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{100-x}$$

$$\therefore\ P(X = 55) = {}^{100}C_{55} \left(\frac{1}{2}\right)^{55} \left(\frac{1}{2}\right)^{45} = 0.04847$$