χ^2- TEST

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NON PARAMETRIC TESTS

- So far we have dealt with the problems of estimation of a parameter or testing an hypothesis about a parameter.
- Such tests which deals with the parameter of the population are called parametric tests.
- On the other hand tests which do not deal with the parameter are called non parametric tests.
- One such test, which we are going to study is χ^2- test (pronounced as 'ki' square test – 'ki' as in kite)

DEFINITION OF χ^2

- Suppose we are given a die and we want to know whether it is biased or unbiased.
- OR suppose in a cholera epidemic we inoculate a group and we want to know whether inoculation is effective in preventing the attack of cholera.
- In such situations Chi square test is used to test the hypothesis
- **e.g** the die is not biased or the inoculation is not effective.

- χ^2 is calculated on the assumption of **status quo** i,e there is no change.
- To test such a hypothesis we toss the die for say, 138 times and observe how many times we get 1, 2, 3, 4, 5, 6.
- We can calculate expected frequencies of 1, 2, 3, 4, 5, 6 in 138 tosses. Then we find the value of Chi – square from the following.

- The statistic χ^2 first used by Karl Pearson is defined by
- $\chi^2 = \sum\left(\frac{(O-E)^2}{E}\right)$
- Where, O = observed frequency, E = expected frequency.

We calculate expected frequencies on certain assumption such as

(i) the coin or a die is unbiased,

(ii) there is no association between the attributes.

(iii) the accident occur evenly on all days.

(iv) digits occur evenly on all pages,

(v) the events occur in the given ratio,

(vi) the events occur according to the given distribution (Binomial, Poisson, Normal).

This is called testing goodness of fit.

We now compare the calculated value of χ^2 with the table value for the given degrees of freedom and at a specified level of significance.

If the **calculated value of χ^2 is greater than the table value** we conclude that the difference between the observed values and expected frequencies is significant and the **hypothesis is rejected**.

If on the other hand the **calculated value of χ^2 is less than the table value**, we conclude that the difference between the observed values and the expected frequencies is not significant and the **hypothesis is accepted.**

Note:

The value of χ^2 will be zero if the observed and expected frequencies coincide.

The value of χ^2 is always positive.

As observed frequencies depart from expected frequencies χ^2 goes on increasing.

CONDITIONS FOR χ^2 TEST

- N, the total number of observations must be sufficiently large. Preferably N should be greater than 50.
- Frequency of every cell must be greater than 10. If any frequency is less than 10 it is combined with Neighboring frequencies so that the combined frequency is greater than 10 and the degrees of freedom are reduced accordingly.
- The number of classes n must not be too small not too large. Preferably we should have $4 \leq n \leq 16$.

- It may be noted that the χ^2 test depends upon
- **The observed frequencies**
- **The expected frequencies,**
- **The number of observations only.**
- It does not make any assumption regarding the nature of parent population.

USES OF χ^2 TEST

- **1. To determine association between two or more attributes:**
- χ^2- test is widely used to test whether there is association between two or more attributes.
- **For example**, χ^2- test can be used to determine whether there is association between the colour of mother's eye and daughter's eye, between inoculation and prevention of a diseases.
- In such cases we proceed on the null hypothesis that there is no association between the attributes. If the calculated value of χ^2 at a certain level of significance is less than the table value, the hypothesis is accepted otherwise rejected.

- In the same way χ^2 test is also used to test if a characteristic is dependent upon another characteristic.
- **For example**, χ^2 - test can be used to test whether the performance of workers in a factory is dependent on the training or to test whether performance of students in a particular subject is dependent on the performance in another subject.
- Using χ^2- distribution in this way to test the independence of one attribute on another is called **test of independence**

TO TEST THE GOODNESS OF FIT

- χ^2 - test is very commonly known as χ^2 - test of goodness of fit because it enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal fit the observed frequencies.
- In such cases we proceed on the null hypothesis that the theory supports the observations i.e. the fit is good.

TO TEST THE GOODNESS OF FIT

- **For example,** suppose we toss 3 fair coins 200 times and observe the frequencies of 0,1,2,3 heads. We can also calculate the expected frequencies by using Binomial Distribution.
- χ^2-test can be used to ascertain whether Binomial Distribution fits well. If the calculated value of χ^2 at a certain level of significance is less than the table value, the fit is supposed to be good otherwise the fit is supposed to be poor.

- **To test the discrepancies between observed frequencies and expected frequencies**
- χ^2-test can also be used to ascertain whether the difference between observed frequencies and the expected frequencies is purely due to chance or whether due to inadequacy in the theory applied.

- **To test equality of several proportions:**
- χ^2- test can also be used to test whether the proportions p_1, p_2, p_3, p_4 in different populations are equal
- i.e. χ^2test can also be used to test the null hypothesis that $p_1 = p_2 = p_3 = p_4$.

- **To test the hypothesis about $\boldsymbol{\sigma^2}$**: χ^2 is also used to test the population variance.

EXAMPLE-1

- The following table gives the number of accidents in a city during a week. Find whether the accidents are uniformly distributed over a week.

Day:	Sun.	Mon	Tue.	Wed.	Thu.	Fri.	Sat.	Total
No. of Accidents:	13	15	9	11	12	10	14	84

SOLUTION

- The null hypothesis H_0: Accidents are equally distributed over all days of a week
- Alternative Hypothesis H_a: Accidents do not occur equally
- Calculation of test statistic: If the accidents occur equally on all days of week, there will be $(84)/7 = 12$ accidents per day

SOLUTION

- $\therefore \chi^2 = \sum \frac{(O-E)^2}{E}$
- $= \frac{(13-12)^2}{12} + \frac{(15-12)^2}{12} + \frac{(9-12)^2}{12} + \frac{(11-12)^2}{12} + \frac{(12-12)^2}{12} + \frac{(10-12)^2}{12} + \frac{(14-12)^2}{12}$
- $= 2.33$

- Level of significance: $\alpha = 0.05,$
- Degrees of freedom $= n - 1 = 7 - 1 = 6$
- Critical value: For 6 degrees of freedom at 5% level of significance table value of χ^2 is 12.59
- Decision: Since the calculated value of χ^2 is less than the table value.
- The hypothesis is accepted
- $\therefore$ The accidents occur equally on all working days

EXAMPLE-2

- A die was thrown 132 times and the following frequencies were observed.

Test the hypothesis that the die is unbiased.

No. obtained	1	2	3	4	5	6	Total
Frequency	15	20	25	15	29	28	132

SOLUTION

- Null hypothesis H_0: The die is unbiased
- Alternative Hypothesis H_a: The die is not unbiased
- Calculation of test statistic: On the hypothesis that the die is unbiased we should expect the frequency of each number to be $132/6 = 22$
- $\chi^2 = \sum \frac{(O-E)^2}{E}$
- $= 8.91$

- Level of significance: $\alpha = 0.05$
- Number of degree of freedom $= n - 1 = 5$
- Critical value: For 5 dof. at 5% level of significance the table value of χ^2 is
- 11.07
- Decision: Since the calculated value of $\chi^2 = 8.91$ is less than the table value of $\chi^2 = 11.07,$
- The hypothesis is accepted
- $\therefore$ The die is unbiased

EXAMPLE-3

- 300 digits were chosen at random from a table of random numbers. The frequency of digits was as follows. Using χ^2- test examine the hypothesis that the digits were distributed in equal numbers in the table.

Digit	0	1	2	3	4	5	6	7	8	9	Total
Frequency	28	29	33	31	26	35	32	30	31	25	300

SOLUTION

- Null hypothesis H_0: The digits are distributed equally
- Alternative Hypothesis H_a: The digits are not distributed equally
- Calculation of test statistic:
- On the basis of the hypothesis, the frequency of each digit $= (\text{total})/10 = 30$
- $\therefore \chi^2 = \sum \frac{(O-E)^2}{E}$
- $= 2.87$

- Level of significance: $\alpha = 0.05$
- degrees of freedom $= 9$
- Critical value: For 9 d.f. at 5% level of significance, the table value of χ^2 is
- 16.92
- Decision: Since the calculated value of $\chi^2 = 2.87$ is less than the table value of χ^2,
- the hypothesis is accepted
- $\therefore$ Digits are equally distributed in the table.

EXAMPLE-4

- Theory predicts that the proportion of beans in the four groups A, B, C, D should be $9:3:3:1$. In an experiment among 1600 beans the numbers in the four groups were $882, 313, 287$ and 118. Does the Experimental results support the theory?

SOLUTION

- Null hypothesis H_0: The proportion of the beans in the four groups A, B, C, D is the given proportion $9:3:3:1$
- Alternative Hypothesis H_a: The proportion is not as given above
- Calculation of test statistic: On the basis of this hypothesis, since the sum is $9+3+3+1=16$, the number of beans in the four groups will be

SOLUTION

- $A = \frac{9}{16} \times 1600 = 900,$
- $B = \frac{3}{16} \times 1600 = 300,$
- $C = \frac{3}{16} \times 1600 = 300,$
- $D = \frac{1}{16} \times 1600 = 100$
- $\therefore \chi^2 = \sum \frac{(O-E)^2}{E} =$
- 4.72

- Level of significance: $\alpha = 0.05$
- Number of degree of freedom $= 3$
- Critical value: For 3 degree of freedom at 5% level of significance, the table value of χ^2 is
- 7.81
- Decision: Since the calculated value of $\chi^2 = 4.72$ is less than the table value of $\chi^2 = 7.81,$
- the null hypothesis is accepted
- $\therefore$ The proportion $9:3:3:1$ is correct

EXAMPLE-5

- Investigate the association between the darkness of eye colour in father and son from the following data.

		Colour of father's eyes		
		Dark	Not Dark	Total
Colour of son's eyes	Dark	48	90	138
	Not Dark	80	782	862
	Total	128	872	1000

SOLUTION

- Null hypothesis H_0: There is no association between the darkness of eye colour in father and son
- Alternative Hypothesis H_a: There is an association

SOLUTION

- Calculation of test statistic: On the basis of this hypothesis the expected frequency of dark eyed sons with dark eyed fathers $= \frac{A \times B}{N}$
- where, A $=$ number of dark eyed fathers (total of first column)
- B $=$ number of dark eyed sons (total of first row)
- N $=$ total number of observations
- $\therefore$ Expected frequency $= \frac{128 \times 138}{1000} = 18$

- Having obtained the expected frequency in the first cell, since the totals remain the same, the figures in other cells can be easily obtained as,

		Colour of father's eyes		
		Dark	Not Dark	Total
Colour of son's eyes	Dark	18	120	138
	Not Dark	110	752	862
	Total	128	872	1000

- **Calculation of $(\boldsymbol{O - E})^2/\boldsymbol{E}$**

O	E	$(O-E)^2$	$(O-E)^2/E$
48	18		
80	110		
90	120		
782	752		
		Total	

- $\chi^2 =$
- 66.88
- Level of significance: $\alpha = 0.05$
- Degrees of freedom $= (r-1)(c-1) = 1$
- Critical value: For 1 d.f. at 5% level of significant the table value of χ^2 is
- 3.84

- Decision: Since the calculated value of $\chi^2 = 66.88$ is much greater than the table value of $\chi^2 = 3.84$,
- the hypothesis is rejected
- $\therefore$ There is an association between darkness of eye colour of fathers and sons

EXAMPLE-6

- The figures given below are (a) the observed frequencies of a distribution, (b) the frequencies of the normal distribution, having the same mean, standard deviation and the total frequency as in (a)

(a) $1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1$

(b) $2, 15, 66, 210, 484, 799, 943, 799, 484, 210, 66, 15, 2$

Apply χ^2 test of goodness of fit.

SOLUTION

- Since the frequencies at the beginning and end are less than 10, we group them and then apply the χ^2 test

O	1	12	66	220	495	792	924	792	495	220	66	12	1
E	2	15	66	210	484	799	943	799	484	210	66	15	2
$(O-E)^2/E$	0.94		0.0	0.48	0.25	0.06	0.38	0.06	0.25	0.48	0	0.94	

- Null hypothesis H_0: The fit is good
- Alternative Hypothesis H_a: The fit is not good
- Calculation of test statistic: $\chi^2 = 3.84$
- Level of significance: $\alpha = 0.05$

- Number of degree of freedom: There are originally 13 classes. Since they are reduced to 11 by grouping twice, the degrees of freedom is reduced by 2. Further, since the mean, the standard deviation and the total frequency of original data are used, there constrains are introduced, reducing the degree of freedom by 3. For calculating the mean and the standard deviation three sums $\sum f_i \sum f_i x_i$ and $\sum f_i {x_i}^2$ are required. Hence, the degree of freedom is reduced by 3. Thus, the d.f. $= 13 - (2) - (3) = 8$

- Critical value: For 8 d.f. at 5% level of significance, the table value of $\chi^2 = 15.51$
- Decision: Since the calculated value of $\chi^2 = 3.84$ is less than the table value of $\chi^2 = 15.51$,
- the hypothesis is accepted
- $\therefore$ The fit is good

YATE'S CORRECTION

- In a 2×2 table the degrees of freedom is $(2 - 1)\,(2 - 1) = 1$. If any of the cell frequency is less than 5, we have to use pooling method. But this will result in χ^2 with zero degrees of freedom. This is meaningless.

YATE'S CORRECTION

- In this case Yates in 1934 suggested to use
$$\chi^2 = \sum\left[\frac{\{|O-E|-0.5\}^2}{E}\right]$$
- He showed that by taking χ^2 as defined above, χ^2 approximation is improved

EXAMPLE-7

- Two batches of 12 animals each are given test of inoculation. One batch was inoculated and the other was not. The number of dead and surviving animals are given in the following table for both cases. Can the inoculation be regarded as effective against the disease at 5% level of significance (Make Yates correction)

	Dead	Surviving	Total
Inoculated	2	10	12
Non- Inoculated	8	4	12
Total	10	14	24

SOLUTION

- Null hypothesis H_0: There is no association between inoculation and death
- Alternative Hypothesis H_a: There is association between inoculation and death
- Calculation of test statistic: On the basis of this hypothesis the number in the first cell $= \frac{A \times B}{N}$
- where, $A =$ total in the first column, $B =$ total in the first row, $N =$ Total no of observations
- $\therefore$ The number in the first cell $= \frac{10 \times 12}{24} = 5$
- apply Yates correction and prepare the table

- Calculation of χ^2

O	E	$\lvert O-E\rvert-0.5$	$\frac{\{\lvert O-E\rvert-0.5\}^2}{E}$
2	$\frac{10\times12}{24}=5$		
10	$12-5=7$		
8	$10-5=5$		
4	$12-5=7$		

- $\chi^2 = 4.29$
- Level of significance: $\alpha = 0.05$
- Degrees of freedom $= (r-1)(c-1) = 1$
- Critical value: For 1 degree of freedom at 5% level of significance, the table value of $\chi^2 =$
- 3.81
- Decision: Since the calculated value of $\chi^2 = 4.29$ is greater than the table value of $\chi^2 = 3.81,$
- the hypothesis is rejected.
- There is association between inoculation and death i.e. inoculation is effective against the disease

EXAMPLE-8

- The following mistakes per page were observed in a book . Fit a Poisson distribution and test the goodness of fit .

No. of mistakes per page	0	1	2	3	4
No. of pages	211	90	19	5	0

SOLUTION

- Null hypothesis H_0: The mistakes follow Poisson's distribution. The fit is good
- Alternative Hypothesis H_a: The mistakes do not follow Poisson's distribution
- Calculation of test statistic: The expected frequencies by Poisson's distribution are given by

Expected frequency $= Np = N \times \frac{e^{-m}m^x}{x!}$

SOLUTION

where, m = mean of the distribution, x =random variable, N =number of observations

- Here, $m = \frac{\sum fx}{\sum f} = 0.44$
- $x = 0, 1, 2, 3, 4; N = 325$
- Exp. Freq $= \frac{325 \times e^{-0.44}(0.44)^x}{x!}$

No. of mistakes per page	0	1	2	3	4
No. of pages	209.31	92.1	20.26	2.97	0.36

- Calculation of $(O - E)^2/E$

No of mistakes	O	E	$(O-E)^2$	$(O-E)^2/E$
0	211	209.31	2.87	0.014
1	90	92.10	4.41	0.048
2	19	20.26		
3	5	2.97	0.17	0.007
4	0	0.36		
Total				0.069

- $\therefore \chi^2 = \sum \frac{(O-E)^2}{E} = 0.069$
- Level of significance: $\alpha = 0.05$
- Degrees of freedom $5 - 4 = 1$
- (The number of degrees of freedom is 1 for each class. There are 5 classes originally. Hence, the degrees of freedom originally is 5. But we reduced the classes by two, thus reduced the degree of freedom by 2. Further, while calculating the parameter m, we used two sums $\sum f_i$ and $\sum f_i x_i$, thus, reducing the degree of freedom again by 2)

- Critical value: For 1 degree of freedom at 5% level of significance the table value of χ^2 is 3.84
- Decision: Since the calculated value of $\chi^2 = 0.069$ is less than the table value of $\chi^2 = 3.84$, we accept the hypothesis
- $\therefore$ The mistakes follow Poisson's distribution.

EXAMPLE-9

- Samples of three shipments A, B, C of defective items gave the following results. Test whether the proportion of defective items is same in the three shipments at 0.05 level of significance

	Shipment A	Shipment B	Shipment C	Total
Defective	5	8	9	22
Non-defective	35	42	51	128
Total	40	50	60	150

SOLUTION

- Null hypothesis H_0: $p_1 = p_2 = p_3$
- Alternative Hypothesis H_a: $p_1 \neq p_2 \neq p_3$

SOLUTION

- Calculation of test statistic: If the proportion of defective item is same in the three shipment then there will be $\frac{\mathbf{40\times22}}{\mathbf{150}}=\mathbf{5.86}$ defective items in shipment A and the remaining $40-5.86=34.14$ non defective.
- In the same ways defectives in $B=\frac{50\times22}{150}=7.33$ and remaining $50-7.33=42.67$ non defective.
- Defective in $C=\frac{60\times22}{150}=8.8$ and remaining $60-8.8=51.2$ non defective

- Calculation of χ^2

O	E	$(O-E)^2$	$(O-E)^2/E$
5	5.86		
35	34.14		
8	7.33		
42	42.67		
9	8.8		
51	51.2		

- Calculation of χ^2

O	E	$(O-E)^2$	$(O-E)^2/E$
5	5.86	0.7396	0.1262
35	34.14	0.7396	0.0217
8	7.33	0.4489	0.0612
42	42.67	0.4489	0.0105
9	8.8	0.04	0.0045
51	51.2	0.04	0.0008
			0.2249

- $\chi^2 = 0.2249$
- Level of significance: $\alpha = 0.05$
- Degrees of freedom: $= (r-1)(c-1) = 2$
- Critical value: For 2 degrees of freedom at 5% level of significance, the table value of
- $\chi^2 = 5.991$

- Decision: Since the calculated value of $\chi^2 = 0.2249$ is less than the table value $\chi^2 = 5.991$,
- the hypothesis is accepted
- $\therefore$ Proportion of defective items is same in all shipments.