Dirac Delta Function

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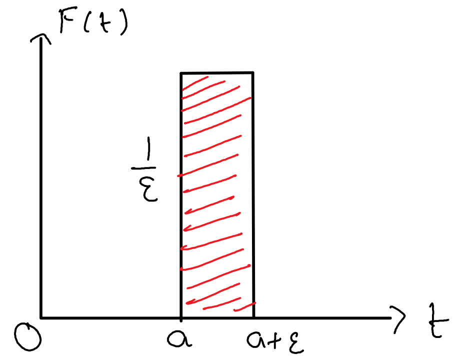
Consider The function $F(t)$ defined by



$$F(t) = \begin{cases} 0 & t < a \\ \frac{1}{\varepsilon} & a \leqslant t \leqslant a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases}$$

The function is represented by the adjoining figure.

Integrating $F(t)$ we get

$$\int_0^\infty F(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\left[t\right]_a^{a+\varepsilon} = \frac{1}{\varepsilon}\left[a+\varepsilon-a\right]$$

$$= \frac{1}{\varepsilon}\left[\varepsilon\right] = 1 \quad \text{for all } \varepsilon$$

As $\varepsilon \to 0$, the function $F(t)$ tends to ∞ at a and is zero everywhere else, but the integral of $F(t)$ is unity

If $F(t)$ represents a force acting for a short time ε at time $t = a$ then the integral $\lim_{\varepsilon \to 0} \int_0^\infty F(t)\,dt$ $(=1)$ represents unit-impulse at $t = a$.

Hence the limiting form of $F(t)$ (as $\varepsilon \to 0$) is known as **Unit impulse Function** or **Dirac-delta function** and is denoted by $\delta(t-a)$

$$\therefore \delta(t-a) = \lim_{\varepsilon \to 0} F(t)$$

when $a = 0$, the unit function is $\boxed{\delta(t) = \lim_{\varepsilon \to 0} F(t)}$

Laplace Transform of Dirac-delta Function

By definition of Laplace Transform ∞ $a+\varepsilon$ e^{-st} $a+\varepsilon$

By definition of Laplace Transform

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st}\, dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}$$

$$= \frac{-1}{\varepsilon s}\left[e^{-s(a+\varepsilon)} - e^{-sa}\right] = \frac{1}{s}\cdot e^{-as}\left[\frac{1-e^{-\varepsilon s}}{\varepsilon}\right]$$

$$\therefore L[\delta(t-a)] = \lim_{\varepsilon\to 0} L[F(t)] = \frac{1}{s}e^{-as}\lim_{\varepsilon\to 0}\left[\frac{1-e^{-\varepsilon s}}{\varepsilon}\right]$$

$$= \frac{1}{s}e^{-as}\lim_{\varepsilon\to 0}\left[\frac{-e^{-\varepsilon s}(-s)}{1}\right] \quad \text{(By L'Hospital's Rule)}$$

$$= \frac{1}{s}e^{-as}(s)$$

$$= e^{-as}$$

$$\therefore \boxed{L[\delta(t-a)] = e^{-as}}$$

Cor: Putting $a=0$, $\boxed{L[\delta(t)] = 1}$

Inverse Laplace Transform

From the above results, we get

$$\boxed{L^{-1}(e^{-as}) = \delta(t-a) \quad \text{and} \quad L^{-1}(1) = \delta(t)}$$

Laplace Transform of $f(t)\,\delta(t-a)$

$$\boxed{L[f(t)\,\delta(t-a)] = e^{-as} f(a)}$$

Taking inverse Laplace Transform

$$\boxed{L^{-1}[e^{-as} f(a)] = f(t)\cdot\delta(t-a)}$$

Examples :-

① Find $L^{-1}\left[\frac{s}{s+1}\right]$

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Soln :- we have $L^{-1}\left[\frac{s}{s+1}\right] = L^{-1}\left[\frac{(s+1)-1}{s+1}\right] = L^{-1}\left[1-\frac{1}{s+1}\right]$

$$= L^{-1}[1] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= \delta(t) - e^{-t}$$

② Find $L^{-1}\left[\frac{s^2+6s+6}{s^2+5s+6}\right]$

Soln :- $L^{-1}\left[\frac{s^2+6s+6}{s^2+5s+6}\right] = L^{-1}\left[\frac{(s^2+5s+6)+s}{s^2+5s+6}\right] = L^{-1}\left[1+\frac{s}{s^2+5s+6}\right]$

$$= L^{-1}(1) + L^{-1}\left(\frac{s}{(s+2)(s+3)}\right)$$

$$= L^{-1}(1) - 2L^{-1}\left[\frac{1}{s+2}\right] + 3L^{-1}\left[\frac{1}{s+3}\right]$$

(by partial Fraction)

$$L^{-1}\left[\frac{s^2+6s+6}{s^2+5s+6}\right] = \delta t - 2e^{-2t} + 3e^{-3t}$$

③ Find $L[\sin 2t \; \delta(t-2)]$

Soln :- Here $f(t) = \sin 2t$, $a = 2$

$$L[f(t)\,\delta(t-a)] = e^{-as} f(a)$$

$$L[\sin 2t \; \delta(t-2)] = e^{-2s}\sin 2(2) = e^{-2s}\sin 4$$

④ Evaluate $\int_0^\infty e^{-t}\left[t^2 H(t-2) - \cosh t \; \delta(t-4)\right] dt$

Soln :- This is a question of particular integral.

① Find $L\left[t^2 H(t-2) - \cosh t\, \delta(t-4)\right]$

② put $s=1$

Now $L\left[t^2 H(t-2)\right] = e^{-2s} L\left[(t+2)^2\right]$ $\left(\text{using } L\left[f(t)H(t-a)\right] = e^{-as} L\left[f(t+a)\right]\right)$

$= e^{-2s} L\left[t^2+4t+4\right]$

$L\left[t^2 H(t-2)\right] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

Also $L\left[\cosh t\, \delta(t-4)\right] = e^{-4s}\cosh(4)$ $\left[\text{using } L\left[f(t)\delta(t-a)\right] = e^{-as} f(a)\right]$

$\therefore L\left[t^2 H(t-2) - \cosh t\, \delta(t-4)\right]$

$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cosh 4$

This means that

$$\int_0^\infty e^{-st}\left[t^2 H(t-2) - \cosh t\, \delta(t-4)\right] dt = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cosh 4$$

put $s=1$

$$\therefore \int_0^\infty e^{-t}\left[t^2 H(t-2) - \cosh t\, \delta(t-4)\right] dt = e^{-2}(2+4+4) - e^{-4}\cosh 4$$

$$= \frac{10}{e^2} - \frac{\cosh 4}{e^4}$$

⑤ Find $L^{-1}\left[e^{-3s}\cos 3\right]$

Soln: we have the formula $L^{-1}\left[e^{-as} f(a)\right] = f(t)\,\delta(t-a)$

$\therefore L^{-1}\left[e^{-3s}\cos 3\right] = \cos t\, \delta(t-3)$