COMPLEX INTEGRATION

LINE INTEGRAL:

The **line integral** of $f(z)$ along C is denoted by $\int_C f(z)dz$

If C is a closed curve then the integral is called the **contour integral** and is denoted by $\oint_C f(z)dz$

In the case of real variables, the path of integration of $\int_a^b f(x)dx$ is always along the real axis from $x = a$ to $x = b$. But in the case of complex variables the path of the definite integral $\int_a^b f(z)dz$ may be any curve joining the points $z = a \, and \, z = b$. Generally, the value of this integration depends upon the path. However, as we shall see later the value remains the same in some special cases.

Evaluation of line Integral:

The evaluation of a line integral is reduced to the evaluation of two real line integrals as follows:

Since $z = x + iy,\ dz = dx + i\,dy$

If $f(z) = u + iv$ then $\int_c f(z)dz = \int_c (u + iv)(dx + i\,dy)$

$$\int_c f(z)dz = \int_c (udx - vdy) + i\int_c (v\,dx + udy)$$

Thus, the integral on the l.h.s is converted into two integrals on r.h.s.

Note: We shall often need the curves represented by

(i) $|z| = r$ which represents a circle with centre at the origin and radius r

(ii) $|z - z_0| = r$ which represents a circle with centre at (x_0, y_0) and radius r

(iii) $z = r\,e^{i\theta}, 0 \le \theta \le 2\pi$ which represents the circle with centre at the origin and radius r, in polar form.

(iv) $|z - c| + |z + c| = k$ is an ellipse with foci at $A(c, 0)$ and $B(-c, 0)$ and major axis equal to k and minor axis is $2\sqrt{\frac{k^2}{4} - c^2}$

Note: (i) Whenever $f(z)$ is analytic $\int_c f(z)dz$ depends only upon the end points & not on path

(ii) Whenever $|z| = r$ given then put $z = re^{i\,\theta}$

(iii) Whenever $|z - z_0| = r$ given then put $z = z_0 + re^{i\,\theta}$

SOME SOLVED EXAMPLES:

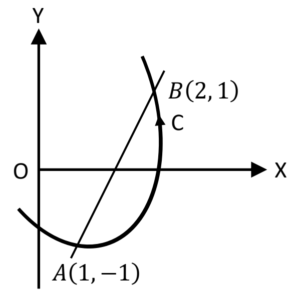
1. Evaluate $\int_{1-i}^{2+i}(2x+1+iy)\,dz$

(a) along the straight line between the given limits, **(b)** along the curve $x=t+1, y=2t^2-1$

Solution: **(a)** Here the path is along the line, between the limits $A(1-i)$ and $B(2+i)$,

Using the formula $\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$, we have the equation of line AB as

$y=2x-3, \ \therefore dy=2dx$



Hence, $I = \int_{1-i}^{2+i}(2x+1+iy)(dx+idy)$

$$= \int_{x=1}^{2}[2x+1+i(2x-3)](dx+2idx)$$

$$= (1+2i)\int_{x=1}^{2}[2x+1+i(2x-3)]dx$$

$$= (1+2i)[x^2+x+i(x^2-3x)]_1^2 = 4+8i$$

(b) Here C is a curve whose parametric equation is given by $x=t+1, y=2t^2-1$, which is passing through the points $A(1-i)$ and $B(2+i)$ (as shown in the figure above)

$\therefore dx=dt$ and $dy=4tdt$,

lower limit: $x=1, y=-1 \Rightarrow t=0$; upper limit: $x=2, y=1 \Rightarrow t=1$

Hence, $I = \int_{1-i}^{2+i}(2x+1+iy)(dx+idy)$

$$= \int_{t=0}^{1}[2(t+1)+1+i(2t^2-1)](dt+i4tdt)$$

$$= \int_{t=0}^{1}\{[(2t+3)-4(2t^3-t)+i(2t^2-1)+4(2t^2+3t)]\}\,dt$$

$$= \int_{t=0}^{1}[(8t^3+6t+3)+i(10t^2+12t-1)]\,dt$$

$$= \left[(-2t^4+3t^2+3t)+i\left(\frac{10t^3}{3}+6t^2-1\right)\right]_0^1 = 4+\left(\frac{25}{3}\right)i$$

2. Evaluate $\int_0^{2+i}(\bar{z})^2\,dz$ along the paths:

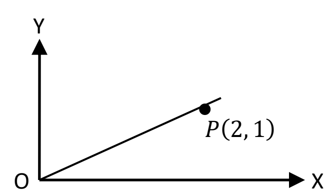
(a) the line $2y=x$, **(b)** the real axis to 2 and then vertically to $2+i$

(c) the imaginary axis to i and then horizontally to $2+i$,

(d) the parabola $2y^2=x$

Solution: **(a)** Here C is the straight line OP in the figure whose equation is $2y=x$

Now, $2y=x$ or $x=2y$ then $dx=2dy$

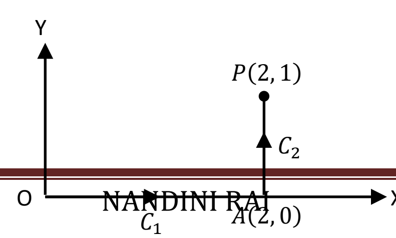


Hence, $I=\int_0^{2+i}(\bar{z})^2\,dz = \int_0^{2+i}(x-iy)^2(dx+idy)$

$$= \int_0^1(2y-iy)^2(2dy+idy) = (2-i)^2(2+i)\int_0^1 y^2\,dy$$

$$= 5(2-i)\left[\frac{y^3}{3}\right]_0^1 = \frac{5}{3}(2-i)$$

(b) Here C consists of the lines $OA=C_1$ and $AP=C_2$



$\therefore I \quad = \int_C (x-iy)^2(dx+idy)$

$= \int_{C_1}(x-iy)^2(dx+idy) + \int_{C_2}(x-iy)^2(dx+idy)$

$C_1: y=0$ so that $dy=0, x$ varies from 0 to 2 and

$C_2: x=2$ so that $dx=0, y$ varies from 0 to 1

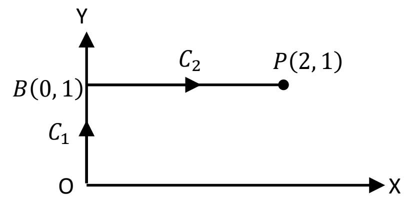
$\therefore \int_{C_1}(x-iy)^2(dx+idy) = \int_{x=0}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$ .... (1)

and $\int_{C_2}(x-iy)^2(dx+idy) \;\; = \int_{y=0}^{1}(2-iy)^2\,idy$

$= i\left[\frac{(2-iy)^3}{-3i}\right]_0^1 = \frac{(2-i)^3-8}{-3} = 2+\frac{11i}{3}$ .... (2)

Hence, from (1) and (2), we have, $I = \frac{8}{3} + \left(2+\frac{11i}{3}\right) = \frac{14+11i}{3}$

(c) Here C consists of the lines $OB = C_1$ and $BP = C_2$



$\therefore I \quad = \int_C (x-iy)^2(dx+idy)$

$= \int_{C_1}(x-iy)^2(dx+idy) + \int_{C_2}(x-iy)^2(dx+idy)$

$C_1: x=0$ so that $dx=0, y$ varies from 0 to 1 and

$C_2: y=1$ so that $dy=0, x$ varies from 0 to 2

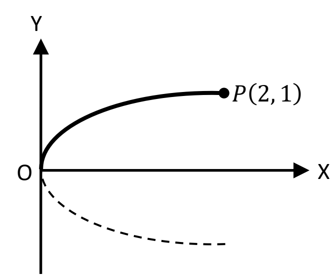
$\therefore \int_{C_1}(x-iy)^2(dx+idy) = \int_{y=0}^{1} -y^2\,idy = -\left[\frac{iy^3}{3}\right]_0^1 = -\frac{i}{3}$ .... (1)

and $\int_{C_2}(x-iy)^2(dx+idy) \;\; = \int_{x=0}^{2}(x-i)^2\,dx$

$= \left[\frac{(x-i)^3}{3}\right]_0^2 = \frac{(2-i)^3+i}{3} = \frac{2-10i}{3}$ .... (2)

Hence, from (1) and (2), we have $I = \frac{-i}{3} + \frac{(2-10i)}{3} = \frac{2-11i}{3}$

(d) Here, C is the parabolic arc OP,



The equation of the parabola is given by $x=2y^2$, then $dx=4ydy$

$\therefore I \quad = \int_C (x-iy)^2(dx+idy)$

$= \int_{y=0}^{1}(2y^2-iy)^2(4ydy+idy)$

$= \int_{y=0}^{1}(4y^4-4iy^3-y^2)(4y+i)\,dy$

$= \int_{y=0}^{1}[(16y^5) - i(12y^4+y^2)]\,dy$

$= \left[\left(\frac{16y^6}{6}\right) - i\left(\frac{12y^5}{5}+\frac{y^3}{3}\right)\right]_0^1 = \left(\frac{8}{3}\right) - i\left(\frac{12}{5}+\frac{1}{3}\right) = \left(\frac{8}{3}\right) - i\left(\frac{41}{15}\right)$

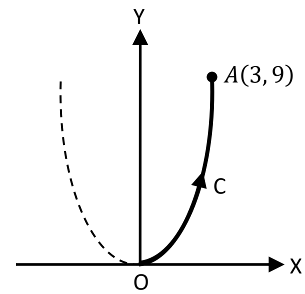
3. Evaluate $\int_C (x+y)dx + x^2 y dy$

(a) along $y = x^2$ from $(0,0)$ to $(3,9)$ **(b)** along $y = x$ between the same limits

Solution: **(a)** $C: y = x^2$ is the upward parabola

$\therefore dy = 2xdx$

x varies from 0 to 3,

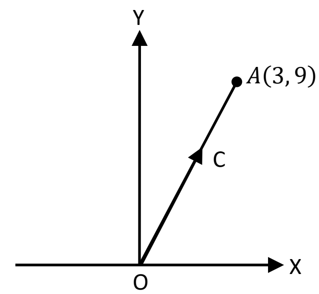


Hence $I = \int_C (x+y)dx + x^2 y dy$

$= \int_{x=0}^{3} (x + x^2)\, dx + (x^4)(2xdx)$

$= \int_0^3 (x + x^2 + 2x^5)\, dx = \left[\frac{x^2}{2} + \frac{x^3}{3} + 2\frac{x^6}{6}\right]_0^3 = \frac{9}{2} + 9 + 243 = \frac{513}{2}$

(b) $C: y = 3x \quad \therefore dy = 3dx$



$\therefore I = \int_C (x+y)dx + x^2 y dy$

$= \int_{x=0}^{3} (4x + 9x^3)\, dx$

$= \left[2x^2 + 9\frac{x^4}{4}\right]_0^3 = 18 + \frac{729}{4} = \frac{801}{4}$

4. Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along the parts:

(a) $z = t^2 + it$, **(b)** the lines from 0 to $2i$ and then to $4 + 2i$,

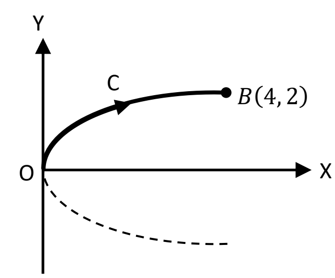
(c) the lines from 0 to i and then to $4 + 2i$

Solution: **(a)** Here the path C is a curve whose parametric equation is given by $x = t^2, y = t$ which is passing through the points $(0,0)$ and $(4,2)$ $\therefore dx = 2tdt$ and $dy = dt$

Lower limit: when $x = 0, y = 0 \Rightarrow t = 0$

Upper limit: when $x = 4, y = 2 \Rightarrow t = 2$

The curve is $y^2 = x$ which is parabola as shown in Figure



Hence, $I = \int_{z=0}^{4+2i} \bar{z}\, dz = \int_{(0,0)}^{(4,2)} (x - iy)(dx + idy)$

$= \int_{t=0}^{2} (t^2 - it)(2tdt + idt)$

$= \int_{t=0}^{2} (2t^3 - 2it^2 + it^2 + t)dt$

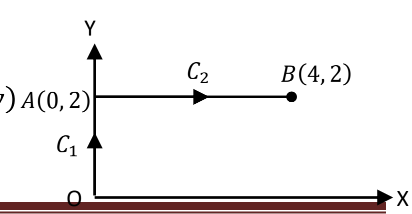
$= \int_{t=0}^{2} (2t^3 - it^2 + t)dt$

$= \left[\frac{2t^4}{4} - i\frac{t^3}{3} + \frac{t^2}{2}\right]_0^2 = \frac{16}{2} - i\frac{8}{3} + \frac{4}{2} = 10 - i\frac{8}{3}$

(b) The path consists of two lines $OA = C_1$ and $AB = C_2$, as shown in the figure

Hence, $I = \int_{z=0}^{4+2i} \bar{z}\, dz$

$= \int_{C_1} (x - iy)(dx + idy) + \int_{C_2} (x - iy)(dx + idy)$



$C_1: x = 0$ so that $dx = 0, y$ varies from 0 to 2 and $C_2 : y = 2$ so that $dy = 0, x$ varies from 0 to 4

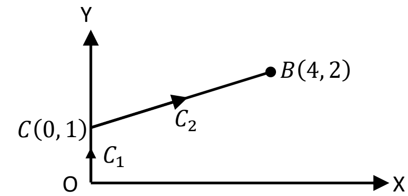
$\therefore \int_{C_1} (x - iy)(dx + idy) = \int_{y=0}^{2} (-iy)(idy) = \left[\frac{y^2}{2}\right]_0^2 = 2$ .... (1)

and $\int_{C_2} (x - iy)(dx + idy) \quad = \int_{x=0}^{4} (x - 2i)\, dx = \left[\frac{x^2}{2} - 2ix\right]_0^4$

$= \frac{16}{2} - 8i = 8 - 8i$ .... (2)

Thus from (1) and (2), we get, $I = 2 + (8 - 8i) = 10 - 8i$

(c) Here, again, the path consists of two lines $OC = C_1$ and $CB = C_2$, as shown in the figure



Hence, $\quad I \quad = \int_{z=0}^{4+2i} \bar{z}\, dz$

$= \int_{C_1} (x - iy)(dx + idy) + \int_{C_2} (x - iy)(dx + idy)$

Along $C_1 : x = 0$ so that $dx = 0, y$ varies from 0 to 1

$\int_{C_1} (x - iy)(dx + idy) = \int_{y=0}^{1} (-iy)(idy) = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$ .... (3)

Along C_2: The lines CB passing through the points $C(0, 1)$ and $B(4, 2)$

$\therefore$ Equation of the line CB is $\frac{y-1}{1-2} = \frac{x-0}{0-4} \Rightarrow 4(y - 1) = x$ or $x = 4(y - 1), dx = 4dy$

y varies from 1 to 2

$\therefore \int_{C_2} (x - iy)(dx + idy) \quad = \int_{y=1}^{2} [4(y - 1) - iy][4dy + idy] = \int_1^2 (4y - 4 - iy)(4 + i)\, dy$

$= (4 + i)\left[\frac{4y^2}{2} - 4y - \frac{iy^2}{2}\right]_1^2$

$= (4 + i)\left[(8 - 8 - 2i) - \left(2 - 4 - \frac{i}{2}\right)\right]$

$= (4 + i)\left(2 - \frac{3}{2}i\right) = \frac{19}{2} - 4i$ .... (4)

Thus from (3) and (4), we get, $I = \frac{1}{2} + \frac{19}{2} - 4i = 10 - 4i$

5. Evaluate $\int_0^{1+i} (x - y + ix^2)\, dz$

(a) along the straight line $z = 0$ to $z = 1 + i$

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$

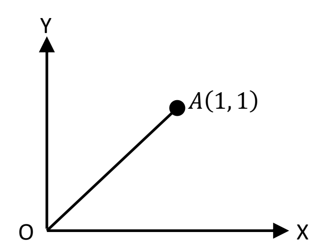
Solution: **(a)** Along the line segment OA

Equation of OA joining the points $(0, 0)$ and $(1, 1)$ is

$\frac{y-0}{0-1} = \frac{x-0}{0-1}$ i.e., $y = x \quad \therefore dy = dx$

Hence, $\quad I \quad = \int_0^{1+i} (x - y + ix^2)(dx + idy)$

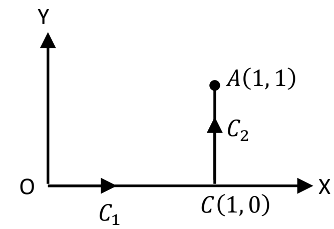
$= \int_{x=0}^{1} (x - x + ix^2)(1 + i)\, dx$



$$= (1+i)\int_0^1 ix^2\,dx = (1+i)i\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}(i-1)$$

(b) The path consist of two lines $OC = C_1$ and $CA = C_2$

Along $C_1: y = 0$, $\therefore dy = 0, x$ varies from 0 to 1



$\therefore \int_{C_1}(x - y + ix^2)(dx + idy)$

$= \int_0^1 (x + ix^2)\,dx = \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 = \frac{1}{2} + i\frac{1}{3}$ .... (1)

Along $C_2: x = 1$, $\therefore dx = 0, y$ varies from 0 to 1

$\therefore \int_{C_2}(x - y + ix^2)(dx + idy) = \int_{y=0}^1 (1 - y + i)\,(idy) = i\left[y - \frac{y^2}{2} + iy\right]_0^1$

$= i\left[1 - \frac{1}{2} + i\right] = \frac{i}{2} - 1$ .... (2)

Thus from (1) and (2), we get

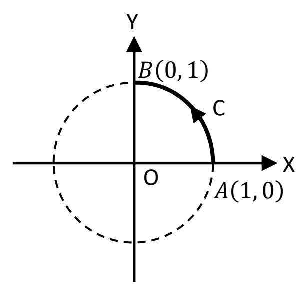
$\therefore I \quad = \int_0^{1+i}(x - y + ix^2)\,dz = \int_{C_1}(x - y + ix^2)(dx + idy) + \int_{C_2}(x - y + ix^2)(dx + idy)$

$= \left(\frac{1}{2} + \frac{i}{3}\right) + \left(\frac{i}{2} - 1\right) = -\frac{1}{2} + \frac{5}{6}i$

6. Evaluate the line integral $\int_C (3y^2dx + 2ydy)$ where C is the circle $x^2 + y^2 = 1$, counter clockwise from $(1, 0)$ to $(0, 1)$

Solution: Equation of the circle $x^2 + y^2 = 1$, can be written in parametric form as $x = \cos\theta$ and $y = \sin\theta$

θ varies from A to B along the curve i.e. θ varies from 0 to $\frac{\pi}{2}$



Hence, $\quad I \quad = \int_C (3y^2dx + 2ydy)$

$= \int_{\theta=0}^{\frac{\pi}{2}}[(3\sin^2\theta)(-\sin\theta\,d\theta) + 2(\sin\theta)(\cos\theta\,d\theta)]$

$= \int_{\theta=0}^{\frac{\pi}{2}}(-3\sin^2\theta + 2\cos\theta)(\sin\theta\,d\theta)$

$= \int_{\theta=0}^{\frac{\pi}{2}}[-3(1 - \cos^2\theta) + 2\cos\theta](\sin\theta\,d\theta)$ .... (1)

Put $\cos\theta = t, \therefore -\sin\theta\,d\theta = dt$, when $\theta = 0, t = 1$ and when $\theta = \frac{\pi}{2}, t = 0$

$\therefore$ From the equation (1) we get,

$\therefore I \quad = \int_1^0[-3(1 - t^2) + 2t](-dt) = \int_0^1(-3 + 3t^2 + 2t)\,dt$

$= \left[-3t + 3\frac{t^3}{3} + 2\frac{t^2}{2}\right]_0^1 = [-3 + 1 + 1] = -1$

7. If C is a circle $|z - a| = r$, evaluate

(a) $\int_C (z - a)^{-1}\,dz$ **(b)** $\int_C (z - a)^{-(n+1)}\,dz$

(c) $\int_C (z - a)^n\,dz \quad (n \neq 1), n$ being any integer

Solution: **(a)** Equation of the circle is $(z-a)=re^{i\theta}$, so that θ varies from 0 to 2π as z describes the contour C. Then $dz=re^{i\theta}id\theta$ and the given integer becomes

$I=\int_0^{2\pi}\left(re^{i\theta}\right)^{-1}re^{i\theta}id\theta=[i\theta]_0^{2\pi}=2\pi i$

(b) $I=\int_C (z-a)^{-(n+1)}\,dz=\int_0^{2\pi}\left(re^{i\theta}\right)^{-(n+1)}re^{i\theta}id\theta=ir^{-n}\int_0^{2\pi}e^{-in\theta}\,d\theta$ .... (1)

Case I: Let $n=0$, then $I=i\int_0^{2\pi}d\theta=[i\theta]_0^{2\pi}=2\pi i$ .... {as in (a)}

Case II: Let $n\neq 0$, then (1) $\Rightarrow$

$$I=ir^{-n}\int_0^{2\pi}(\cos n\theta-i\sin n\theta)d\theta=ir^{-n}\left[\frac{\sin n\theta+i\cos n\theta}{n}\right]_0^{2\pi}=0$$

(c) Here again let $(z-a)=re^{i\theta}$, $\quad\therefore dz=re^{i\theta}id\theta, 0\leq\theta\leq 2\pi$

$$\begin{aligned}\therefore I &=\int_C (z-a)^n\,dz\\ &=\int_0^{2\pi}\left(re^{i\theta}\right)^n\left(ire^{i\theta}d\theta\right)=ir^{n+1}\int_0^{2\pi}e^{-i(n+1)}\,d\theta\\ &=ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}=\frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi}-1\right]\\ &=\frac{r^{n+1}}{n+1}[\cos(n+1)\,2\pi+i\sin(n+1)\,2\pi-1]=\frac{r^{n+1}}{n+1}[1+i0-1]=0\end{aligned}$$

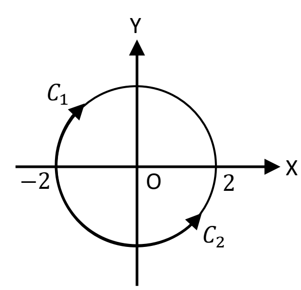
8. Evaluate $\int_{-2}^{2}\frac{2z-3}{z}dz$, when the path C is

(a) the upper half of the circle $|z|=2$, **(b)** the lower half of the same circle,

(c) entire circle in anticlockwise sense

Solution: **(a)** Equation of the circle with the centre at the origin and radius $=2$ is $z=2e^{i\theta}$, $\therefore dz=2ie^{i\theta}d\theta$

Here, the path C_1 is the upper half of the circle from $z=-2$ to $z=2$ i.e., θ varies from π to 0



$$\begin{aligned}\therefore I &=\int_{C_1}\frac{2z-3}{z}dz=\int_{\theta=\pi}^{0}\frac{4e^{i\theta}-3}{2e^{i\theta}}2ie^{i\theta}\,d\theta\\ &=i\int_{\theta=\pi}^{0}\left(4e^{i\theta}-3\right)d\theta=i\left[\frac{4e^{i\theta}}{i}-3\theta\right]_\pi^0\\ &=i\left[\frac{4}{i}-0-\frac{4e^{i\pi}}{i}+3\pi\right]=i\left[\frac{4}{i}-\frac{4}{i}(\cos\pi+i\sin\pi)+3\pi\right]\\ &=i\left[\frac{4}{i}+\frac{4}{i}+3\pi\right]=8+3\pi i\end{aligned}$$

{Since $\cos\pi=-1$ and $\sin\pi=0$}

(b) Here, the path is C_2, which is the lower half of the same circle from -2 to 2, i.e., θ varies from π to 2π

$$\therefore I=\int_{C_2}\frac{2z-3}{z}dz=\int_{\theta=\pi}^{2\pi}\frac{4e^{i\theta}-3}{2e^{i\theta}}2ie^{i\theta}\,d\theta$$

$$= i\int_{\theta=\pi}^{2\pi}\left(4e^{i\theta}-3\right)d\theta = i\left[\frac{4e^{i\theta}}{i}-3\theta\right]_{\pi}^{2\pi}$$

$$= i\left[\frac{4e^{i2\pi}}{i}-6\pi-\frac{4e^{i\pi}}{i}+3\pi\right] = i\left[\frac{4}{i}(\cos 2\pi + i\sin 2\pi)-\frac{4}{i}(\cos\pi + i\sin\pi)-3\pi\right]$$

$$= 8-3\pi i$$

(c) Here, the path is C is around in the anticlockwise direction, i.e. $C=-C_1+C_2, \theta$ varies from 0 to 2π

$$\therefore I \quad = \int_{C=-C_1+C_2}\frac{2z-3}{z}dz = \int_{-C_1}\frac{2z-3}{z}dz + \int_{C_2}\frac{2z-3}{z}dz$$

$$= -\int_{C_1}\frac{2z-3}{z}dz + \int_{C_2}\frac{2z-3}{z}dz$$

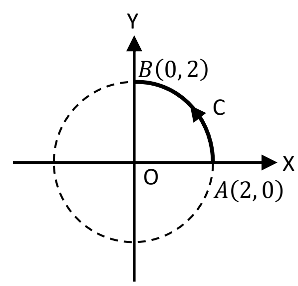
$$= -[8+3\pi i]+[8-3\pi i] = -6\pi i \qquad \text{\{From (a) and (b)\}}$$

9. Evaluate $\int_C (z^2+3z)\,dz$, along the circle $|z|=2$ from $(2,0)$ to $(0,2)$

Solution: C is the circle $|z|=2$ from $A(2,0)$ to $B(0,2)$

The equation of the circle is $z=2e^{i\theta}, dz=2ie^{i\theta}d\theta$ and $0\le\theta\le\frac{\pi}{2}$

The given integral becomes



$$\therefore I \quad = \int_C (z^2+3z)\,dz$$

$$= \int_0^{\pi/2}\left(4e^{2i\theta}+6e^{i\theta}\right)2ie^{i\theta}d\theta$$

$$= \int_0^{\pi/2}\left(8ie^{3i\theta}+12ie^{2i\theta}\right)d\theta$$

$$= \left[\frac{8ie^{3i\theta}}{3i}+\frac{12ie^{2i\theta}}{2i}\right]_0^{\pi/2}$$

$$= \left(\frac{8}{3}e^{3\pi/2i}+6e^{\pi i}\right)-\left(\frac{8}{3}+6\right)$$

$$= \frac{8}{3}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)+6(\cos\pi+i\sin\pi)-\frac{8}{3}-6$$

$$= \frac{8}{3}(0-i)+6(-1+0i)-\frac{8}{3}-6$$

$$= -\frac{8i}{3}-6-\frac{8}{3}-6$$

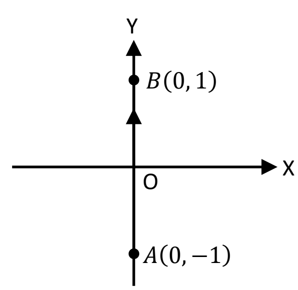
$$= -\frac{44}{3}-\frac{8i}{3}$$

10. Evaluate $\int_C |z|\,dz$, where C is

(a) straight line from $z=-i$ to $z=i$

(b) right half of the unit circle $|z|=1$ from, $z=-i$ to $z=i$,

(c) the circle $|z-1|=1$ described in the positive sense



Solution: **(a)** The path is a straight line from $z=-i$ to $z=i$,

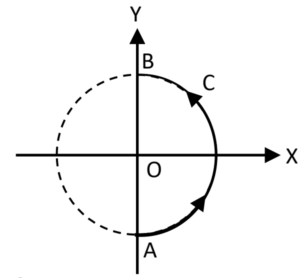
i.e. the path along y –axis, between the limits

$A(0,-1)$ and $B(0,1)$. The equation of y −axis is $x=0$, then $dx=0, y$ varies from -1 to 1

$$\therefore I = \int_C |z|\,dz = \int_{y=-1}^{1}\sqrt{x^2+y^2}(dx+idy)$$

$$= \int_{-1}^{1} yi\,dy = i\left[\frac{y^2}{2}\right]_{-1}^{1} = 0$$

(b) Here path C is a circle whose equation is $z=e^{i\theta}$, so that θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (as shown in the Figure). Then $dz = e^{i\theta}id\theta$ and $|z|=1$



$$\therefore I = \int_C |z|\,dz = \int_{-\pi/2}^{\pi/2} e^{i\theta}id\theta$$

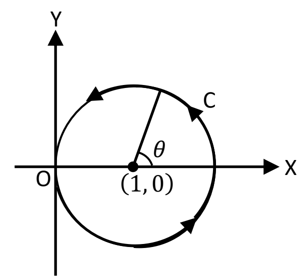
$$= i\int_{-\pi/2}^{\pi/2}(\cos\theta + i\sin\theta)d\theta = i[\sin\theta - i\cos\theta]_{-\pi/2}^{\pi/2} = 2i$$

(c) Equation of the circle is $(z-1)=e^{i\theta}$, so that θ varies from 0 to 2π as z describes the contour C, then $dz = e^{i\theta}id\theta$

Now, $(z-1)=e^{i\theta}$,

i.e. $z = 1+e^{i\theta} = 1+\cos\theta + i\sin\theta$

and therefore the given integrand is



$$\therefore |z| = \sqrt{(1+\cos\theta)^2+\sin^2\theta} = \sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta} = \sqrt{2(1+\cos\theta)}$$

$$= \sqrt{2\left[2\cos^2\left(\frac{\theta}{2}\right)\right]} = 2\cos\left(\frac{\theta}{2}\right)$$

Here $\quad I = \int_C |z|\,dz = \int_0^{2\pi} 2\cos\left(\frac{\theta}{2}\right)e^{i\theta}id\theta$

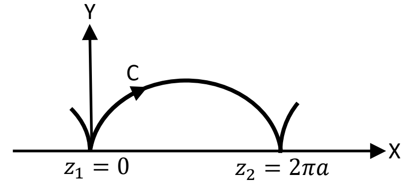
$$= 2i\left[\frac{e^{i\theta}}{-1+\left(\frac{1}{4}\right)}\left\{i\cos\left(\frac{\theta}{2}\right)+\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\right\}\right]_0^{2\pi} \quad \text{..........}\left\{\begin{matrix}\text{Using } \int(e^{ax}\cos bx)\,dx \\ = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\end{matrix}\right\}$$

$$= 2i\left(\frac{-4}{3}\right)\left[e^{2\pi i}\left(i\cos\pi + \frac{1}{2}\sin\pi\right) - e^0\left(i\cos 0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \left(\frac{-8i}{3}\right)(-2i) = \frac{-16}{3} \quad \left\{\begin{matrix}\text{Since } e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 \\ \cos\pi = -1, \cos 0 = 1\end{matrix}\right\}$$

11. Evaluate $\int_C (4z^3+9z^2-8z+5)\,dz$, where C is the arc of the cycloid $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$ between $(0,0)$ and $(2\pi a, 0)$

Solution: Since $f(z) = 4z^3+9z^2-8z+5$ is analytic everywhere, by the corollary, the integral is independent of the path of integration and only depends on the end points $z_1 = 0+i0$ and $z_2 = 2\pi a + i0$



$$\therefore \int_C (4z^3+9z^2-8z+5)\,dz = \int_0^{2\pi a}(4z^3+9z^2-8z+5)\,dz$$

$$= [z^4 + 3z^3 - 4z^2 + 5z]_0^{2\pi a}$$
$$= 2\pi a[8\pi^3 a^3 + 12\pi^2 a^2 - 8\pi a + 5]$$

12. Show that for every path between the limits, the value of the integral $\int_{-2}^{-2+i}(2+z)^2\,dz$ is the same and find its value

Solution: Here $f(z) = (2+z)^2 = (4 + 4z + z^2)$

$$= 4 + 4(x+iy) + (x+iy)^2 = (4 + 4x + x^2 - y^2) + i(4y + 2xy)$$

$\therefore u = 4 + 4x + x^2 - y^2$ and $v = 4y + 2xy$

$$\left.\begin{aligned} \therefore \frac{\partial u}{\partial x} &= 4 + 2x, \quad \frac{\partial u}{\partial y} = -2y \\ \frac{\partial v}{\partial x} &= 2x, \quad \frac{\partial v}{\partial y} = 4 + 2x \end{aligned}\right\} \qquad \text{............. (1)}$$

Hence from (1), we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i.e. $f(z)$ satisfies $C - R$ equations hence it is analytic function in every region

Thus, by the corollary above, the integral is independent of the path of integration and only depends on the end points, i.e. the value of the integral is same everywhere

$$\therefore \int_{-2}^{-2+i}(2+z)^2\,dz = \left[\frac{(2+z)^3}{3}\right]_{-2}^{-2+i} = \frac{1}{3}\{[2 + (-2+i)]^3 - [2 + (-2)]^3\} = \frac{1}{3}\{i^3 - 0\} = -\frac{i}{3}$$

13. Evaluate

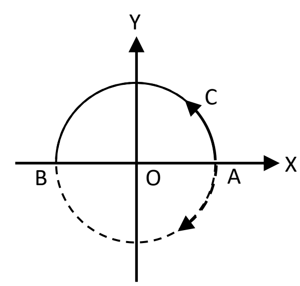
(a) $\int_C (z - z^2)\,dz$, where C is upper half of the unit circle traversing in the positive sense

(b) What is the value of this integral if C is the lower half of the above circle

Solution: **(a)** $|z| = 1$ is the given unit circle

$\therefore$ Equation of the circle is $z = e^{i\theta}$,

$$\therefore dz = ie^{i\theta}d\theta; 0 \le \theta \le \pi$$



$$\therefore I = \int_C (z - z^2)\,dz = \int_0^{\pi}\left(e^{i\theta} - e^{2i\theta}\right) ie^{i\theta}d\theta$$
$$= i\int_0^{\pi}\left(e^{2i\theta} - e^{3i\theta}\right)d\theta = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi}$$
$$I = \left[\left(\frac{e^{2i\pi}}{2} - \frac{e^{3i\pi}}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)\right]$$
$$= \left[\left(\frac{\cos 2\pi + i\sin 2\pi}{2} - \frac{\cos 3\pi + i\sin 3\pi}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)\right]$$
$$= \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{1}{3}\right] = \frac{2}{3}$$

(b) If the path C is along the lower half of the circle, traversing from A to B, the value of the integral is the same, since $f(z) = (z - z^2)$ is analytic function i.e. $I = \frac{2}{3}$

If C is traversing in anticlockwise direction, i.e. from B to A, then $I = -\frac{2}{3}$

SIMPLY & MULTIPLY CONNECTED REGIONS:

If a closed curve does not intersect itself, it is called a simple closed curve or a Jordan curve otherwise called multiple curve. A region R is called a simply connected region if every closed curve in the region encloses points of the region R only.

CAUCHY'S INTEGRAL THEOREM :

If $f(z)$ is an analytic function and if its derivative $f'(z)$ is continuous at each point within and on a simple closed curve C then the integral of $f(z)$ along the closed curve C is zero **i.e** $\oint f(z)dz = 0$

Note: The French mathematician Goursat proved the theorem without assuming that $f'(z)$ is continuous.

Hence, the theorem now becomes, "If $f(z)$ is analytic in and on a closed counter C then $\oint_C f(z)dz = 0.$"

This theorem is known as **Cauchy – Gaursat Theorem**.

Corollary: If $f(z)$ is analytic in R then the line integral of f(z) along any curve in R joining any two points of R is the same if the curve wholly lies in R i.e the line integral is independent of the path joining the two points.

EXTENSION OF CAUCHY'S INTEGRAL THEOREM:

Cauchy's theorem can be applied even if the region is multiply connected.

Theorem: If $f(z)$ is analytic in R between two simple closed curves C_1 *and* C_2 then, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

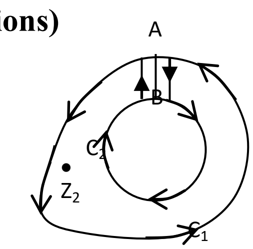
CAUCHY'S INTEGRAL FORMULA (FUNDAMENTAL FORMULA):

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C, then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$ i.e $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

Corollary: Cauchy's Integral Formula for derivatives:

$$\int_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{n-1}(z_0)$$

Theorem (Extension of Cauchy's Integral Formula To Multiply Connected Regions)



If $f(z)$ is analytic in a region R bounded by two closed curves C_1 and C_2 one within the other, if z_0 is any point in the region R then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$

Converse of Cauchy's Theorem (Morera's Theorem):

If $f(z)$ is continuous in a region R and if $\int_C f(z)dz = 0$ for every simple closed curve C which can be drawn in R then $f(z)$ is analytic in R.

SOME SOLVED EXAMPLES:

1. Evaluate $\oint_C \frac{z+6}{z^2-4} dz$, where C is

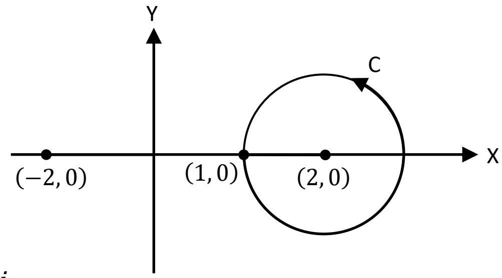
(a) the circle $|z| = 1$; **(b)** the circle $|z-2| = 1$; **(c)** the circle $|z+2| = 1$

Solution: The singular points of $f(z) = \frac{z+6}{z^2-4}$ are given by $z^2 - 4 = 0$, i.e., $z = 2$ and $z = -2$ are the singular points of $f(z)$

(a) C is the circle $|z| = 1$. Both the points $z = 2$ and $z = -2$ lie outside the region of the circle.

$\therefore f(z)$ is analytic on and within C. Hence by Cauchy's theorem $\oint_C f(z)\, dz = 0$

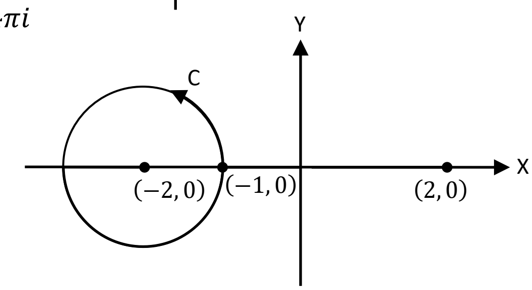
(b) C is the circle $|z-2| = 1$ with the centre at $(2, 0)$ and radius $= 1$. Of the two singular points only $z = 2$ lies within C.



We may, therefore, apply Cauchy's integral formula, by taking, $f(z) = \frac{z+6}{z+2}$ and $a = 2$,

so that the given integral $= 2\pi i\, f(2) = 2\pi i \frac{(2+6)}{(2+2)} = 4\pi i$

(c) Here C is the circle $|z+2| = 1$ with the centre at $(-2, 0)$ and radius $= 1$. Of the two singular points only $z = -2$ lies within C.



$\therefore$ Applying Cauchy's integral formula, by taking $f(z) = \frac{z+6}{z-2}$ and $a = -2$,

so that the given integral $= 2\pi i\, f(-2) = 2\pi i \frac{(-2+6)}{(-2-2)} = -2\pi i$

2. Evaluate $\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, where $C: |z| = 1$

Solution: The points $z = \frac{\pi}{6}$ lies inside the region of the circle $|z| = 1$. We thus apply the generalization of Cauchy's integral formula, by which

$\oint_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$

We take $f(z) = \sin^2 z$ and $a = \frac{\pi}{6}$

$\therefore f'(z) = 2 \sin z \cos z = \sin 2z\,; f''(z) = 2 \cos 2z\,;$ giving $f''\left(\frac{\pi}{6}\right) = 2 \cos\left(\frac{\pi}{3}\right) = 1$

Hence $I = \oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right) = \pi i$

3. Evaluate $\oint_C \frac{z+1}{z^3-2z^2} dz$, where C is

(a) the circle $|z| = 1$; **(b)** the circle $|z-2-i| = 2$; **(c)** the circle $|z-1-2i| = 2$

Solution: The singular points of the integrand are given by $z^3 - 2z^2 = 0$, i.e., $z^2(z-2) = 0$ which gives $z = 0$ and $z = 2$

(a) C is the circle $|z| = 1$. The point $z = 0$ lies inside the circle. We thus apply the generalization of Cauchy's integral formula.

$I = \oint_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{1!} f'(a)$

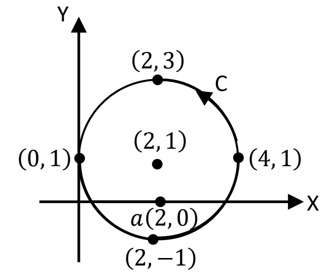
We take $f(z) = \frac{(z+1)}{(z-2)}$ and $a = 0$

$\therefore f'(z) = \frac{(z-2)-(z+1)}{(z-2)^2} = \frac{-3}{(z-2)^2}$

$\Rightarrow f'(0) = \frac{-3}{4}$

Hence $I = \oint_C \frac{z+1}{z^3-2z^2} dz = 2\pi i \left(\frac{-3}{4}\right) = \frac{-3\pi i}{2}$

(b) Here C is the circle $|z - 2 - i| = 2$ with centre at $z = 2 + i$, i.e., at $(2, 1)$ and radius $= 2$. The singular point $z = 2$ lies inside the circle.



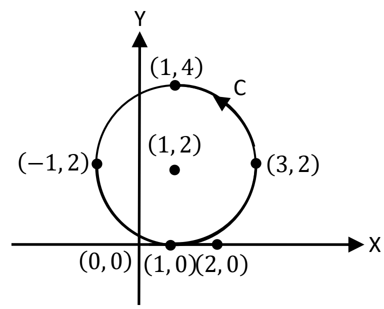
Apply Cauchy's integral formula, by taking

$f(z) = \frac{(z+1)}{z^2}$ and $a = 2$, we have

$I = 2\pi i f(2) = 2\pi i \left(\frac{3}{4}\right) = \frac{3\pi i}{2}$

(c) Here C is the circle $|z - 1 - 2i| = 2$ with centre at $z = 1 + 2i$, i.e. at $(1, 2)$ and radius $= 2$

Both the points $z = 0$ and $z = 2$ lie outside the region of the circle



$\therefore f(z) = \frac{(z+1)}{(z^3-2z^2)}$ is analytic on and within C

Hence, by Cauchy's theorem, $I = \oint_C f(z)\, dz = 0$

4. Evaluate $\oint_C \frac{z+4}{z^2+2z+5} dz$, where C is

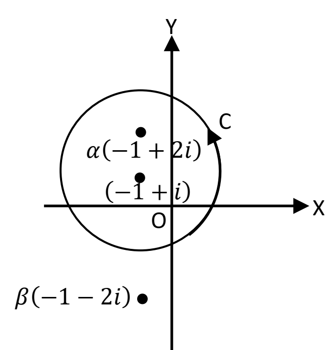
(a) the circle $|z + 1 - i| = 2$; **(b)** the circle $|z + 1 + i| = 2$; **(c)** the circle $|z| = 1$

Solution: The roots of $(z^2 + 2z + 5) = 0$ are $-1 \pm 2i$

(a) $C: |z - (-1 + i)| = 2$ is the circle with centre at $(-1 + i)$, i.e., $(-1, 1)$ and radius $= 2$

$I = \oint_C \frac{z+4}{z^2+2z+5} dz = \oint_C \frac{z+4}{(z-\alpha)(z-\beta)} dz$

where $\alpha = -1 + 2i$ and $\beta = -1 - 2i$, i.e. the points are $(-1, 2)$ and $(-1, -2)$ of which α is lying inside C (see figure)



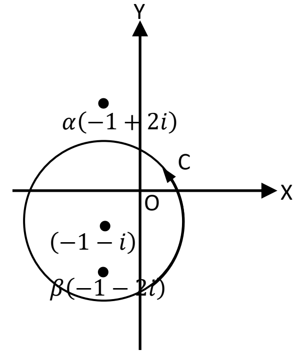
$\therefore$ Taking $f(z) = \frac{z+4}{z-\beta}$; using Cauchy's integral formula,

$I = 2\pi i f(\alpha) = 2\pi i \left(\frac{\alpha+4}{\alpha-\beta}\right) = 2\pi i \frac{(-1+2i+4)}{(-1+2i+1+2i)} = \frac{\pi}{2}(3 + 2i)$

(b) $C: |z - (-1 - i)| = 2$ is the circle with centre at $(-1 - i)$, i.e., the point $(-1,-1)$ and radius $= 2$

Here, $\beta(-1-2i)$ is lying inside C and $\alpha(-1+2i)$

lies outside C (see figure)



$\therefore$ Taking $f(z) = \frac{z+4}{z-\alpha}$;

Using Cauchy's integral formula, we have

$\therefore I \quad = 2\pi i f(\beta)$

$= 2\pi i \left(\frac{\beta+4}{\beta-\alpha}\right)$

$= 2\pi i \frac{(-1-2i+4)}{(-1-2i+1-2i)}$

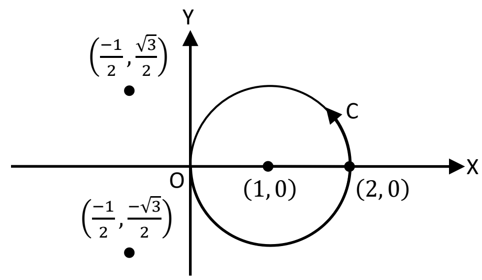
$= -\frac{\pi}{2}(3-2i)$

(c) Since both the points $z = -1+2i$ and $z = -1-2i$ lie outside the circle $|z| = 1$,

$f(z) = \frac{z+4}{z^2+2z+5}$ is analytic in the unit circle. Hence by Cauchy's theorem

$I = \oint_C \frac{z+4}{z^2+2z+5} dz = 0$

5. Evaluate $\oint_C \frac{1}{(z^3-1)^2} dz$, where C is the circle $|z-1| = 1$



Solution: The singular points of the integrand are given by

the roots of $(z^3-1)^2 = 0$

i.e. $(z-1)^2(z^2+z+1)^2 = 0$

i.e. $z = 1$ (twice) and $z = \frac{-1\pm\sqrt{3}i}{2}$ (twice)

Now, the region is bounded by the circle $|z-1| = 1$,

with centre at $(1,0)$ and radius $= 1$

The point $z = 1$, i.e. $(1,0)$ lies inside C

$z = \frac{-1+\sqrt{3}i}{2}$, i.e. the points $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ and $z = \frac{-1-\sqrt{3}i}{2}$, i.e. the point $\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$ both lie outside C

Hence, taking $f(z) = \frac{1}{(z^2+z+1)^2}$ and applying generalized Cauchy's integral formula, we have

$I = \oint_C \frac{f(z)}{(z-a)^2} dz = \frac{2\pi i}{2!} f'(a)$, where $a = 1$

$f(z) = \frac{1}{(z^2+z+1)^2} = (z^2+z+1)^{-2}$

$\therefore f'(z) = (-2)(2z+1)(z^2+z+1)^{-3}$

$\therefore f'(1) = (-2)(3)(3)^{-3} = \frac{-6}{27} = \frac{-2}{9}$

Hence $I = 2\pi i \left(\frac{-2}{9}\right) = \frac{-4\pi i}{9}$

6. Evaluate $\int_C \frac{\cos \pi z}{z^2-1} dz$, where C is

(a) an rectangle with vertices at $2 \pm i$ and $-2 \pm i$

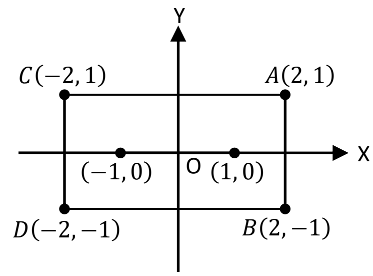
(b) a square with vertices at $\pm i$ and $2 \pm i$

Solution: The singular points of the integrand are given by $z^2 - 1 = 0$, i.e., $z = 1$ and $z = -1$

(a) The region is the rectangle with the vertices at $A(2,1), B(2,-1), C(-2,1), D(-2,-1)$

Both the points $z = 1$ and $z = -1$ lie inside the region. By partial fraction method, we have

$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$



$\therefore I \quad = \int_C \frac{\cos \pi z}{z^2-1} dz$

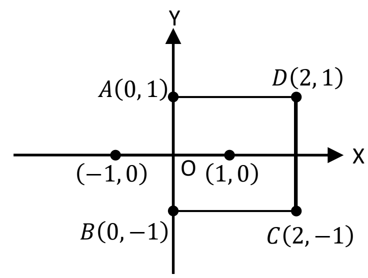
$= \frac{1}{2}\left\{\int_C \frac{\cos \pi z}{z-1} dz - \int_C \frac{\cos \pi z}{z+1} dz\right\}$

{Applying Cauchy's integral formula}

$= \frac{1}{2} 2\pi i\{\cos \pi - \cos(-\pi)\} = \pi i\{(-1) - (-1)\} = 0$

(b) Here the region R is the square with vertices at $A(0,1), B(0,-1), C(2,-1)$ and $D(2,1)$

The point $z = 1$ lies inside R and the point $z = -1$ lies outside R

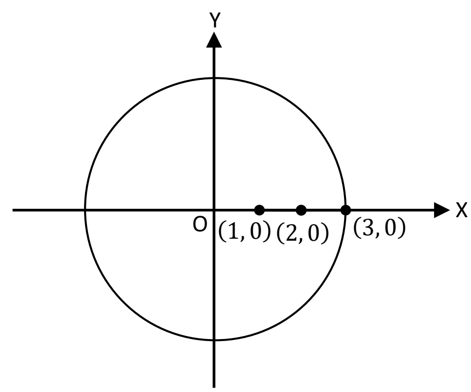


$\therefore f(z) = \frac{\cos \pi z}{z+1}$

Hence, applying Cauchy's integral formula, we have

$I = \int_C \frac{\left(\frac{\cos \pi z}{z+1}\right)}{(z-1)} dz = 2\pi i \left\{\frac{\cos \pi}{2}\right\} = -\pi i$

7. Evaluate $\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$



Solution: The singular points are $z = 1$ and $z = 2$.

Both the points $z = 1$ and $z = 2$ lie within C.

By partial fraction method, we have $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

Hence, $I \quad = \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz$

$= \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{z-2} dz - \int_C \frac{\cos \pi z^2 + \sin \pi z^2}{z-1} dz$

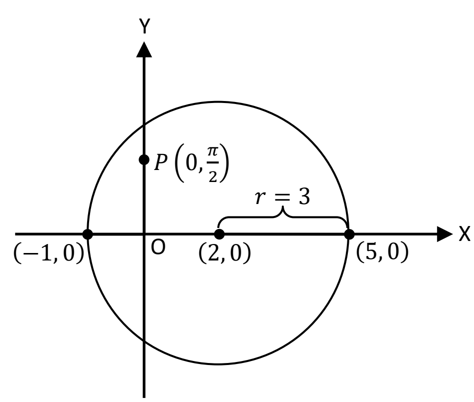
$= 2\pi i(\cos 4\pi + \sin 4\pi) - 2\pi i(\cos \pi + \sin \pi)$

$= 2\pi i(1 + 0) - 2\pi i(-1 + 0) = 4\pi i$

8. Evaluate $\int_C \frac{\sinh z}{z - i\frac{\pi}{2}} dz$, where C is the circle $|z - 2| = 3$

Solution: The circle C: $|z - 2| = 3$, with the centre at $(2,0)$ and radius $= 3$

The singular point $z = i\frac{\pi}{2}$, i.e. the point $P\left(0, \frac{\pi}{2}\right)$ {or $(0, 1.57)$} lies within C



Hence by Cauchy's integral formula, we have

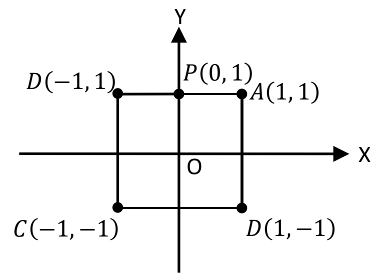
$$\therefore I = \int_C \frac{\sin z}{z - i\frac{\pi}{2}} dz$$

$$= 2\pi i \left[\sinh\left(i\frac{\pi}{2}\right)\right] = 2\pi i \left[i \sin\frac{\pi}{2}\right] \qquad \{\text{Since } \sinh i\theta = i \sin\theta\}$$

$$= -2\pi$$

9. Evaluate $\int_C \frac{z^3-2z+1}{(z-i)^2} dz$, where C is the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

Solution: C is the square $ABCD$ as shown in the figure. The singular point is $z = i$ i.e. the point $P(0,1)$ which lies on the boundary C. Hence applying generalized Cauchy's integral formula, taking $f(z) = z^3 - 2z + 1$, we have



$I = \int_C \frac{z^3-2z+1}{(z-i)^2} dz = \frac{2\pi i}{1!} f'(i)$

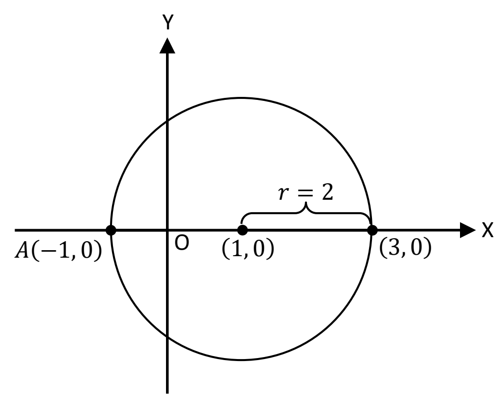
$f(z) = z^3 - 2z + 1, \quad \therefore f'(z) = 3z^2 - 2$

$\therefore f'(i) = -3 - 2 = -5$

Hence, $I = 2\pi i f'(i) = -10\pi i$

10. Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, around the circle $|z-1| = 2$

Solution: $C: |z-1| = 2$, is the circle with centre at $(1,0)$ and radius $r = 2$



The singular point $z = -1$, i.e. the point $A(-1,0)$ lies on the boundary C.

By Cauchy's generalized integral formula, we have

$I = \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$

where $f(z) = e^{2z}, \quad \therefore f'(z) = 2e^{2z};\ f''(z) = 4e^{2z}, f'''(z) = 8e^{2z}$

$\therefore f'''(-1) = 8e^{-2}$

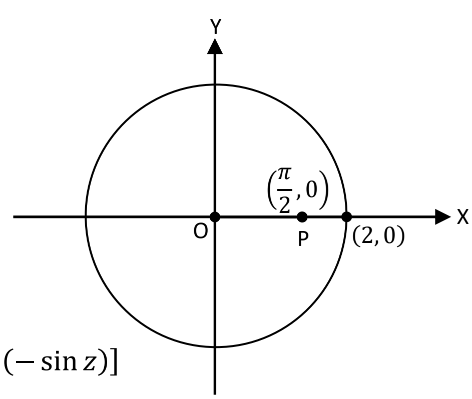
Hence $I = \frac{2\pi i}{6} \times 8e^{-2} = \frac{8\pi i}{3e^2}$

11. Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz$, around the circle $|z| = 2$

Solution: $C: |z| = 2$, is the circle with centre at $(0,0)$ and radius $r = 2$

The singular point $z = \frac{\pi}{2}$, i.e. the point $P\left(\frac{\pi}{2}, 0\right)$ lies within C.

Applying generalized Cauchy's integral formula, we have



$I = \int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{2}\right)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{2}\right)$

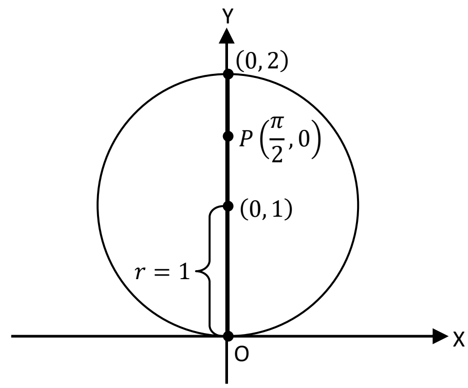
where $f(z) = \sin^6 z$,

$\therefore f'(z) = 6\sin^5 z \cdot \cos z\,;\ f''(z) = 6[5\sin^4 z \cdot \cos^2 z + \sin^5 z\,(-\sin z)]$

$\therefore f''\left(\frac{\pi}{2}\right) = 6\left[5\sin^4\frac{\pi}{2}\cdot\cos^2\frac{\pi}{2} - \sin^6\frac{\pi}{2}\right] = 6[0-1] = -6$

Hence $I = \frac{2\pi i}{2!} \times (-6) = -6\pi i$

12. Evaluate $\int_C \frac{\cosh z}{\left(z - i\frac{\pi}{2}\right)^2} dz$, where C is the circle $|z - i| = 1$



Solution: $C: |z - i| = 1$, is the circle with centre at $(0, 1)$ and radius $r = 1$

The singular point $z = \frac{\pi}{2}i$, i.e. the point $P\left(0, \frac{\pi}{2}\right)$ lies within C.

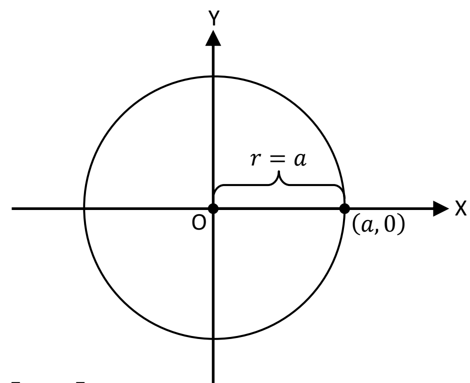
Applying generalized Cauchy's integral formula, we have

$I = \int_C \frac{\cosh z}{\left(z - \frac{\pi}{2}i\right)^2} dz = \frac{2\pi i}{1!} f'\left(\frac{\pi}{2}i\right)$

where $f(z) = \cosh z$, $\therefore f'(z) = \sinh z$; $\therefore f'\left(\frac{\pi}{2}i\right) = \sinh\frac{\pi}{2}i = i$ $\left\{\text{Since} \sinh\frac{\pi}{2}i = i\sin\frac{\pi}{2} = i\right\}$

Hence $I = 2\pi i(i) = -2\pi$

13. Evaluate $\int_C \frac{ze^z}{(z-a)^3} dz$, where C is the circle $|z| = a$



Solution: $C: |z| = a$, is the circle with centre at $(0, 0)$ and radius $r = a$

The singular point $z = a$, lies on the boundary C.

Applying generalized Cauchy's integral formula, we have

$I = \int_C \frac{ze^z}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$

where $f(z) = ze^z$, $\therefore f'(z) = ze^z + e^z$; $f''(z) = ze^z + e^z + e^z$;

$\therefore f''(a) = ae^a + 2e^a = e^a(a+2)$

Hence, $I = \frac{2\pi i}{2} \times e^a(a+2) = \pi i e^a(a+2)$

14. Evaluate $\int_C \frac{1-e^{2z}}{z^4} dz$, where C is $|z| = 1$

Solution: $C: |z| = 1$, is a unit circle with centre at $(0, 0)$ and radius $r = 1$

The singular point $z = 0$, lies within C.

Applying generalized Cauchy's integral formula, we have

$I = \int_C \frac{1-e^{2z}}{z^4} dz = \frac{2\pi i}{3!} f'''(0)$

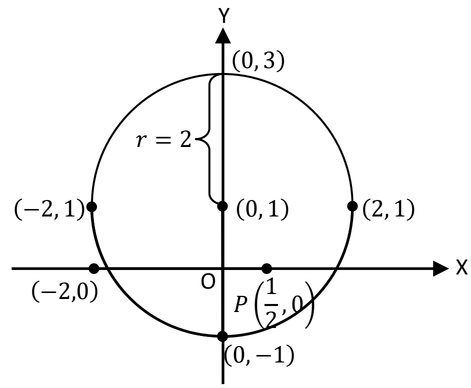
where $f(z) = 1 - e^{2z}$, $\therefore f'(z) = -2e^{2z}$; $f''(z) = -4e^{2z}, f'''(z) = -8e^{2z}$

$\therefore f'''(0) = -8$

Hence $I = \frac{2\pi i}{6} \times (-8) = \frac{-8\pi i}{3}$

15. Evaluate $\oint_C \frac{z+3}{2z^2+3z-2} dz$, where C is the circle $|z-i| = 2$

Solution: C is the circle with centre at $z = i$, i.e. the point is $(0, 1)$, and radius $r = 2$



The singular points of integrand are given by the roots of $2z^2 + 3z - 2 = 0$

i.e. $(2z-1)(z+2) = 0$

i.e. $z = \frac{1}{2}$ and $z = -2$ are the singular points of which $z = \frac{1}{2}$ lies inside C.

$\therefore$ Let $f(z) = \frac{z+3}{z+2}, a = \frac{1}{2}$

By Cauchy's integral formula, we have,

$$\therefore I = \oint_C \frac{f(z)}{(2z-1)} dz = \frac{1}{2}\oint_C \frac{\frac{(z+3)}{(z+2)}}{\left(z-\frac{1}{2}\right)} dz$$

$$= \frac{1}{2}(2\pi i) f\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}(2\pi i)\left(\frac{\frac{1}{2}+3}{\frac{1}{2}+2}\right) = (\pi i)\frac{1}{5} = \frac{7\pi i}{5}$$

16. If $f(t) = \oint_C \frac{4z^2+z+5}{(z-t)} dt$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find the value of

(a) $f(i)$ **(b)** $f'(-1)$ **(c)** $f"(-i)$ **(d)** $f(3)$

Solution: **Case 1:** If the point $z = t$ lies inside the given ellipse, then taking $\emptyset(z) = 4z^2 + z + 5$ and using Cauchy's integral formula, we have $f(t) = 2\pi i\, \emptyset(t)$

Case 2: If the point $z = t$ lies outside the given ellipse, then $\frac{4z^2+z+5}{z-t}$ is analytic in the given region and therefore by Cauchy's theorem, we have $f(t) = 0$

(a) $z = t = i$ lies inside C. $\therefore$ By Case 1 we have,

$f(i) = 2\pi i\, \emptyset(i) = 2\pi i\,(-4 + i + 5) = 2\pi i\,(1 + i)$

(b) $z = t = -1$ lies inside C. $\therefore$ By Case 1 we have,

$f(t) = 2\pi i\, \emptyset(t) = 2\pi i\,(4t^2 + t + 5)$

$\therefore f'(t) = 2\pi i(8t + 1)$. $f'(-1) = -14\pi i$

(c) Again, $z = t = -i$ lies inside C. $\therefore$ By Case 1 we have,

$f(t) = 2\pi i\, \emptyset(t) = 2\pi i\,(4t^2 + t + 5)$

$\therefore f'(t) = 2\pi i(8t + 1)$. $\therefore f''(t) = 2\pi i \times 8 = 16\pi i$

(d) $z = t = 3$ lies outside C. $\therefore$ By Case 2 we have $f(t) = 0$