Fourier Series in $(0, 2l)$

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The Formula for Fourier series in $(0, 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Example (1) Find the Fourier expansion of $f(x) = 2x - x^2$ $0 \le x \le 3$ whose period is 3. Also plot the graph of the function.

Solution :- Here $2l = 3$ $\quad \therefore l = 3/2$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{3}\right) \quad \text{---} (1)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^{3}(2x - x^2)\,dx = \left(x^2 - \frac{x^3}{3}\right)_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx = \frac{2}{3}\int_0^{3}(2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{(2x - x^2)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - (2 - 2x)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right.$$

$$\left. + (-2)\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)\right\}_0^3$$

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$$a_n = \frac{2}{3}\left[\frac{9}{4n^2\pi^2}(2 - 2x)\cos\left(\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}\left[-4\cos(2n\pi) - 2\cos 0\right] = \frac{-9}{n^2\pi^2}$$

$$= \frac{3}{2n^2\pi^2}\left[-4\cos(2n\pi) - 2\cos 0\right] = \frac{-9}{n^2\pi^2}$$

Now $$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{(2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - (2-2x)\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right) + (-2)\left(\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right)\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\frac{3}{2n\pi}\left[-3(-\cos 2n\pi)\right] - 2\left(\frac{27}{8n^3\pi^3}\right)\left[\cos 2n\pi - \cos 0\right]\right\}$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

Substituting these values in 1

$$f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\right)\cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$$

$$\therefore f(x) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

For the graph we see that $y = 2x - x^2$ is a parabola

Now $y - 1 = 2x - x^2 - 1 = -(x^2 - 2x + 1) = -(x-1)^2$

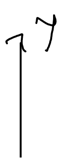
$\therefore Y = -X^2$ where $Y = y-1$ & $X = x-1$

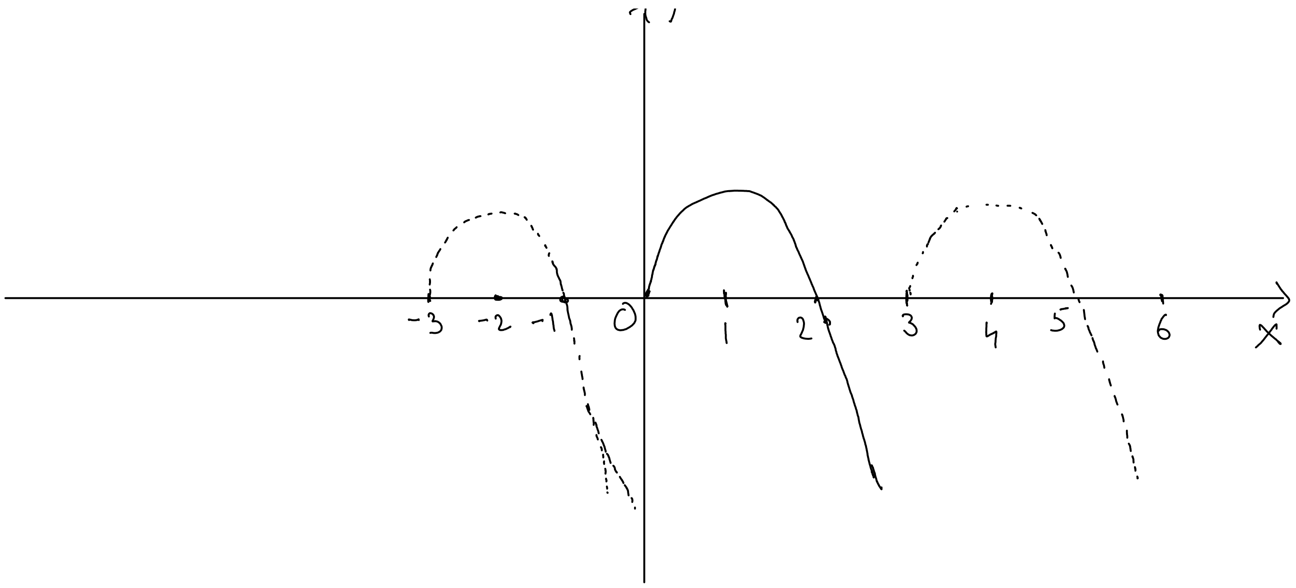
it opens downwards

when $x=0, y=0$, when $x=2, y=0$, when $x=3, y=-3$

Since $f(x)$ is periodic with period 3, the graph repeats at $x = 3, 6, \dots$

Thus, we get, the following graph.





Example-2 Find Fourier series for $f(x) = \begin{cases} x & 0<x<1 \\ 1-x & 1<x<2 \end{cases}$

with $f(x+2) = f(x)$

solution Here $2l = 2 \quad \therefore l = 1$

let $f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{———} \quad (1)$$

Now $a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx = \frac{1}{1}\int_0^2 f(x)\, dx = \left[\int_0^1 x\, dx + \int_1^2 (1-x)\, dx\right]$

$$\therefore a_0 = \left(\frac{x^2}{2}\right)_0^1 + \left(x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{1}{2} + \left(2 - \frac{2^2}{2}\right) - \left(1 - \frac{1}{2}\right) = \frac{1}{2} + (0) - \frac{1}{2}$$

$$\therefore \boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\, dx$$

$$= \int_0^1 x\cos(n\pi x)\, dx + \int_1^2 (1-x)\cos(n\pi x)\, dx$$

$$= \left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+ \left[(1-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$+\left[(1-x)\left(\cancelto{0}{\frac{\sin n\pi x}{n\pi}}\right)-(-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$=\frac{1}{n^2\pi^2}\left[\cos n\pi-\cos 0\right]-\frac{1}{n^2\pi^2}\left[\cos 2n\pi-\cos n\pi\right]$$

$$=\frac{1}{n^2\pi^2}\left[(-1)^n-1-1+(-1)^n\right]=\frac{2\left[(-1)^n-1\right]}{n^2\pi^2}$$

$$\therefore a_n=\begin{cases}\dfrac{-4}{n^2\pi^2} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$$

Now $\quad b_n=\dfrac{1}{l}\displaystyle\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx=\frac{1}{1}\int_0^{2} f(x)\sin n\pi x\,dx$

$$=\int_0^1 x\sin(n\pi x)\,dx+\int_1^2(1-x)\sin(n\pi x)\,dx$$

$$=\left\{x\left[-\frac{\cos(n\pi x)}{n\pi}\right]-1\left[-\cancelto{0}{\frac{\sin(n\pi x)}{n^2\pi^2}}\right]\right\}_0^1$$

$$+\left\{(1-x)\left[\frac{-\cos(n\pi x)}{n\pi}\right]-(-1)\left[\cancelto{0}{\frac{-\sin(n\pi x)}{n^2\pi^2}}\right]\right\}_1^2$$

$$\therefore b_n=\left(\frac{-\cos n\pi}{n\pi}-0\right)+\left[(-1)\left(\frac{-\cos 2n\pi}{n\pi}\right)-0\right]$$

$$=\frac{-(-1)^n}{n\pi}+\frac{1}{n\pi}=\frac{1-(-1)^n}{n\pi}$$

$$\therefore b_n=\begin{cases}\dfrac{2}{n\pi} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$$

Hence the Fourier series for $f(x)$ is given by

$$f(x)=0+\sum\frac{2\left[(-1)^n-1\right]}{n^2\pi^2}\cos(\pi n x)+\sum\frac{1-(-1)^n}{n\pi}\sin(n\pi x)$$

$$=\frac{-4}{\pi^2}\cos(\pi x)-\frac{4}{3^2\pi^2}\cos(3\pi x)-\frac{4}{5^2\pi^2}\cos(5\pi x)-\cdots$$

$$+\frac{2}{\pi}\sin(\pi x)+\frac{2}{3\pi}\sin(3\pi x)+\frac{2}{5\pi}\sin(5\pi x)+\cdots$$

$$+ \frac{2}{\pi} \sin(\pi x) + \frac{2}{3\pi} \sin(3\pi x) + \frac{2}{5\pi} \sin(5\pi x) + \cdots$$

$$\therefore f(x) = \frac{-4}{\pi^2} \left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots \right]$$

$$+ \frac{2}{\pi} \left[\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots \right]$$
