BINOMIAL DISTRIBUTION

BINOMIAL DISTRIBUTION

- This was discovered by James Bernoulli's in 1700 and it expresses probabilities of events of dichotomous (dicho + tomy = Two parts) nature i.e, which results in only two ways, success or failure.

DEFINITION

- A random variable is said to follow the Binomial distribution if the probability of x is given by

$$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x},$$

$$x = 0,1,2,3 \ldots n \ and \ q = 1 - p$$

The two constants n and p are called the parameters of the distribution.

REMARKS:

- The distribution is called “Binomial Distribution” because the probabilities

$$^{n}C_{x}p^{x}q^{n-x}, x = 01,2, \dots, n$$

are the successive terms of the expansion of the binomial expression $(q + p)^{n}$

- If X is a binomial variate with parameters $n \; and \; p$, it is denoted as $b(X, n, p)$

- The sum of the probabilities is 1.

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} {}^{n}C_x p^x q^{n-x}$$
$$= (p+q)^n = 1$$

- Let the experiment of n trials be repeated N times. Then we except x successes to occur

$N.\left({}^{n}C_x p^x q^{n-x}\right)$ times.

This is called **frequency function.**

WHEN DO WE GET BINOMIAL DISTRIBUTION

- We get a binomial distribution when the following conditions are satisfied:

(i) A trial is repeated n times where n is a finite number.

(ii) Each trial results only in two ways success or failure.

(iii) These possibilities are mutually exclusive, exhaustive but not necessarily equally likely.

(iv) If p and q are the probabilities of success and failure then $p + q = 1$

(v) The events are independent, i.e the probability p of success in each trial remains constant in all trials.

USES

Binomial distribution is used in problems involving

(i) The tossing of a coin – heads or tails,

(ii) The results of an examination success or failure,

(iii) The result of an election – success or failure,

(iv) The results of inspection of an article defective or non defective,

(v) Habit of a person smoker or non smoker etc.

MEAN , VARIANCE AND MODE

- $Mean = np$
- $variance = npq$
- If $(n + 1)p$ is an integer, say, k then there are two modes $k\ and\ k - 1$.
- If $(n + 1)p$ is not an integer then the mode is the integral part of $(n + 1)p$.

ADDITIVE PROPERTY OF BINOMIAL DISTRIBUTION

- If X_1 is a Binomial variate with parameter $n_1 \; and \; p_1$ and X_2 is another Binomial variate with parameter $n_2 \; and \; p_2$ then $X_1 + X_2$ in general is not a Binomial variate.
- If $X_1 \; and \; X_2$ are two Binomial variates with parameters n_1, p and n_2, p then $X_1 + X_2$ is a Binomial variate with parameters

$(n_1 + n_2), p$

RECURRENCE RELATION FOR THE PROBABILITY OF BINOMIAL DISTRIBUTION

- $$\frac{p(x+1)}{p(x)} = \frac{{}^{n}C_{x+1}p^{x+1}q^{n-x-1}}{{}^{n}C_{x}p^{x}q^{n-x}}$$

$$= \frac{n(n-1)\ldots..(n-x)}{(x+1)!} \cdot \frac{x!}{n(n-1)\ldots.(n-x+1)} \cdot \frac{p}{q}$$

$$= \frac{n-x}{x+1} \cdot \frac{p}{q}$$

$$\therefore p(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} p(x)$$

- Further since the expected frequency of x

  i.e $f(x) = Np(x)$,

  we have $f(x+1) = N.p(x+1)$

$$= N.\left(\frac{n-x}{x+1} \cdot \frac{p}{q} p(x)\right)$$

$$= \left(\frac{n-x}{x+1} . \frac{p}{q}\right). N.p(x)$$

$$\therefore f(x+1) = \frac{n-x}{x+1} . \frac{p}{q} . f(x)$$

EX 1. The mean and variance of a Binomial variate are 3 and 1.2 . Find 'n', 'p' and $\mathrm{P}(X < 4)$.

ANS: given $np = 3$ and $npq = 1.2$

$$\therefore q = \frac{npq}{np} = \frac{1.2}{3} = 0.4$$

$$\therefore p = 1 - q = 0.6$$

$$np = 3 \Rightarrow n \times 0.6 = 3$$

$$\therefore n = 5$$

$$P(X = x) = {}^5C_x 0.6^x 0.4^{5-x}, \ x = 0,1,2,3 \dots 5$$

$$\mathrm{P}(X < 4) = 1 - P(X \geq 4) = 1 - P(X = 4) - P(X = 5)$$

$$= 1 - {}^5C_4 0.6^4 0.4^1 - {}^5C_5 0.6^5 0.4^0$$

$$= \ 0.66304$$

EX 2. Find the Binomial distribution if the mean is 4 and variance is 3. Find also its mode.

Solution:

We have mean $= np = 4$ and variance $= npq = 4$

$$\therefore \frac{np}{npq} = \frac{4}{3} \qquad \therefore \frac{1}{q} = \frac{4}{3} \qquad \therefore q = \frac{3}{4}$$

$$\therefore p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore np = 4 \qquad \therefore n \cdot \frac{1}{4} = 4 \qquad \therefore n = 16$$

$$\therefore P(X = x) = {}^{n}C_x \, p^x \, q^{n-x} = {}^{16}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x}$$

$$(n+1)p = (16+1) \cdot \frac{1}{4} = \frac{17}{4}$$

$\therefore$ Mode $=$ integral part of $\frac{17}{4} = 4$

EX 3. A Binomial variate X satisfies the relation $9P(X=4)=P(X=2)$ when $n=6$. Find the value of the parameter $'p'$.

Solution:

We have $P(X=x)={}^{n}C_x\, p^x\, q^{n-x}={}^{6}C_x\, p^x\, q^{6-x}$

Since, $9\,P(X=4)=\ P(X=2)$

$\therefore 9\,{}^{6}C_4\, p^4\, q^{6-4}={}^{6}C_2\, p^2\, q^{6-2}$

$\therefore 9\ {}^{6}C_4\, p^4\, q^2={}^{6}C_2\, p^2\, q^4$

$\because {}^{6}C_4={}^{6}C_2,$

$9p^2=\ q^2$

$\therefore 9\,p^2=(1-p)^2$

$\therefore 9p^2=1-2p+p^2$

$\therefore 8p^2+2p-1=0$

$\therefore (4p-1)(2p+1)=0$

$\therefore p=-\frac{1}{2}$ or $\frac{1}{4}$

Since p cannot be negative, $p=1/4$

EX 4. If 10 fair coins are tossed simultaneously, what is the chance of getting atleast 7 heads?

Solution: Probability of x successes in a trial $= {}^{n}C_x\, p^x\, q^{n-x}$

Here $n = 10, p = \frac{1}{2}, q = \frac{1}{2}$

(i) $x = 7, P(7\text{ heads}) = {}^{10}C_7 \left(\frac{1}{2}\right)^{10-7} \left(\frac{1}{2}\right)^{7} = 120 \times \left(\frac{1}{2}\right)^{10}$

(ii) $x = 8, P(8\text{ heads}) = {}^{10}C_8 \left(\frac{1}{2}\right)^{10-8} \left(\frac{1}{2}\right)^{8} = 45 \times \left(\frac{1}{2}\right)^{10}$

(iii) $x = 9, P(9\text{ heads}) = {}^{10}C_9 \left(\frac{1}{2}\right)^{10-9} \left(\frac{1}{2}\right)^{9} = 10 \times \left(\frac{1}{2}\right)^{10}$

(iv) $x = 10, P(10\text{ heads}) = {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{0} = 1 \times \left(\frac{1}{2}\right)^{10}$

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= [120 + 45 + 10 + 1]\frac{1}{2^{10}} = \frac{176}{1024} = \frac{11}{64}$$

EX 5. The odds in favour of X's winning a game against Y are 4:3. Find probability of Y's winning 3 games out of 7 played.

ANS: 0.293

EX 6. If X is the random variable showing the number of boys in a family with 4 children, construct a table showing the probability distribution of X

EX 7. In a multiple choice examination there are 20 questions. Each question has 4 alternative answers following it-and the student must select one correct answer. 4 marks are given for correct answer and 1 mark is deducted for wrong answer. A student must secure at least 50% of maximum possible marks to pass the examination. Suppose a student has not studied at all, so that he answers the questions by guessing only. What is the probability that he will pass the examination?

Solution: Since there are 20 questions and each carries 4 marks, maximum marks are 80.
If the student solves 12 questions correctly and 8 questions wrongly, he gets more than 40 marks
Now, probability of getting a correct answer is $p = 1/4$, of wrong answer $q = 3/4$

$$P(X = x) = {}^{n}C_x \, p^x \, q^{n-x}$$

$$n = 20, p = 1/4, q = 3/4$$

$\therefore$ Probability of passing $= P(X = 12, 13, \ldots., 20)$

$$= \sum_{x=12}^{20} {}^{20}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{20-x}$$

EX 8. In a Binomial distribution consisting of 5 independent trials, probability of 1 and 2 successes are $0.4096 \quad and \; 0.2048$ respectively. Find the parameter 'p' of the distribution.

HINT : similar to EX 3

We have $P(X = x) = {}^{n}C_{x}\, p^{x}\, q^{n-x} = {}^{5}C_{x}\, p^{x}\, q^{5-x}$

Since, $P(X = 1) = 2P(X = 2)$

Solve further….

EX 9. If hens of a certain breed lay eggs on 5 days a week on an average; find on how many days during a season of 100 days, a poultry keeper with 5 hens of this breed, will expect to receive at least 4 eggs?

Solution: Probability of an hen laying an egg, $p = 5/7$ and probability of not laying an egg, $q = 1 - p = 2/7$

$$P(X = x) = {}^{n}C_x\, p^x\, q^{n-x} \text{ and } n = 5, p = \frac{5}{7}, q = \frac{2}{7}$$

$$\therefore P(X \geq 4) = P(X = 4) + P(X = 5)$$

$$= {}^{5}C_4 \left(\frac{5}{7}\right)^4 \left(\frac{2}{7}\right)^1 + {}^{5}C_5 \left(\frac{5}{7}\right)^5 \left(\frac{2}{7}\right)^0$$

$$= 0.5578$$

$$\text{Expectation} = Np = 100 \times 0.5578 = 55.78 = 56$$

EX 10. A communication system consists of n components, each of which functions independently with probability p. The total system will be able to function effectively if at least one – half of its components are functioning. For what value of p is a 5 – component system more likely to operate effectively than a 3 – component system ?

Solution: Here, we have a Binomial distribution with parameters n and p

$$\therefore P(X = x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}, x = 0, 1, 2, \dots\dots\dots, n$$

$$P(\text{5 component system will work effectively})$$
$$= P(X = 3, \text{or } 4, \text{or } 5)$$
$$= P(X = 3) + P(X = 4) + P(X = 5)$$
$$= \sum_{x=3}^{5} {}^{3}C_{x}\, p^{x}\, q^{5-x} \qquad (\because n = 5)$$

$$P(\text{3 component system will work effectively}) = P(X = 2 \text{ or } 3)$$
$$= \sum_{x=2}^{3} {}^{3}C_{x}\, p^{x}\, q^{3-x} \qquad (\because n = 3)$$

5 −component system will work more effectively than 3 −component system if

$$\sum_{x=3}^{5} {}^{3}C_x\, p^x\, q^{5-x} \geq \sum_{x=2}^{3} {}^{3}C_x\, p^x\, q^{3-x}$$

$$\therefore ({}^{5}C_3\, p^3\, q^2 + {}^{5}C_4\, p^4\, q + 5c_5 p^5) \geq ({}^{3}C_2\, p^2\, q + {}^{3}C_3\, p^3)$$

$$\therefore (10\, p^3(1-p)^2 + 5\, p^4(1-p) + \ p^5) \geq 1(3\, p^2(1-p) + \ p^3)$$

$$\therefore 10p^3 - 20p^4 + 10p^5 + 5p^4 - 5p^5 + p^5 - 3p^2 + 3p^3 - p^3 \geq 0$$

$$6p^5 - 15p^4 + 12p^3 - 3p^2 \geq 0$$

$$\therefore 3p^2(2p^3 - 5p^2 + 4p - 1) \geq 0$$

$$3p^2(p-1)^2(2p-1) \geq 0$$

$$\therefore 2p - 1 \geq 0 \qquad [\because p^2 \geq 0, (p-1)^2 \geq 0]$$

$\therefore p \geq \frac{1}{2}$ is the required value

EX 11. An irregular six faced die is thrown and the probability that in 20 throws it will give 5 even numbers is twice the probability that it will give 5 odd numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even number?

Solution: Let the probability of getting an even number in a single throw by p then, the probability of getting an odd number is $q = 1 - p$

Now, the probability of getting 5 even numbers in 20 throws $= {}^{20}C_5 q^{15}\, p^5$

And the probability of getting 5 odd numbers in 20 throws $= {}^{20}C_5 q^5\, p^{15}$

By data ${}^{20}C_5 p^5\, q^{15} = 2\,{}^{20}C_5 q^5\, p^{15}$

$\therefore \dfrac{q^{15}}{q^5} = \dfrac{2p^{15}}{p^2}$ $\qquad \therefore q^{10} = 2p^{10}$ $\qquad \therefore \left(\dfrac{q}{p}\right)^{10} = 2$

$$\therefore \left[\frac{q}{p}\right] = 2^{1/10} \qquad \therefore \log\left[\frac{q}{p}\right] = \frac{1}{10}\log 2$$

$$\therefore \log\left[\frac{q}{p}\right] = \frac{1}{10}\log(0.3010) = 0.0301$$

$$\therefore \frac{q}{p} = \text{antilog}\, 0.0301 = 1.072$$

$$\therefore q = p(1.072) = (1-q)(1.072) = 1.072 - 1.072q$$

$$\therefore 2.072q = 1.072 \qquad \therefore q = \frac{1.072}{2.072} = 0.517$$

$\therefore$ Probability of getting no even number in 10 throws $= {}^{10}C_0 q^{10} = q^{10} = (0.517)^{10} = 0.001365$

The expected number $= Np = 10{,}000(0.001365) = 13.65$

EX 12. Out of 800 families with 5 children each how many would you expect to have

(i) 3 Boys and 2 Girls, **(ii)** 5 girls, **(iii)** 5 boys?

Solution: Here $P(\text{Boy}) = p = \frac{1}{2}, P(\text{Girl}) = q = \frac{1}{2}, n = 5,$

(i)$P(3\text{ boys and }2\text{ girl}) = {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16}$

$\therefore$ Expected number of families $= Np = 800 \times \frac{5}{16} = 250$

(ii)$P(5\text{ girls}) = {}^5C_0\left(\frac{1}{2}\right)^5 = \frac{1}{32}$

$\therefore$ Expected number of families $= Np = 800 \times \frac{1}{32} = 25$

(ii)$P(5\text{ boys }) = {}^5C_5\left(\frac{1}{2}\right)^5 = \frac{1}{32}$

$\therefore$ Expected number of families $= Np = 800 \times \frac{1}{32} = 25$

EX 13. Let X, Y be two independent binomial variates with parameters $(n_1 = 6, p = 1/2)$ and $(n_2 = 4, p = 1/2)$ respectively.

Find $P(X + Y = 3)$, $P(X + Y \geq 3)$

Solution: By the additive property of Binomial variates $Z = X + Y$ is a Binomial variate with parameters

$n = n_1 + n_2 = 6 + 4 = 10$ and $p = 1/2$

$$\therefore P(Z) = {}^{n}C_{z}\, p^{z} \cdot q^{n-z}$$

$$\therefore P(Z = 3) = {}^{10}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7 = \frac{15}{128} = 0.1172$$

$$P(Z \geq 3) = 1 - [P(Z = 0) + P(Z = 1) + P(Z = 2)]$$

$$= 1 - \left[{}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9 + {}^{10}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8\right]$$

$$= 1 - \left[({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2) \left(\frac{1}{2}\right)^{10}\right]$$

$$= 0.945$$

EX 14. A lot contains 1% defective items. What should be the number of items in a lot so that the probability of finding at least one defective item in it, is at least 0.95?

EX 15. Five fair coins are tossed 3200 times, find the frequency distribution of number of heads obtained. Also find mean and standard deviation.

Solution: We have $p = \frac{1}{2}, q = \frac{1}{2}, n = 5$

$$\therefore P(X = x) = {}^{n}C_x\, p^x\, q^{n-x} = {}^{5}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

Putting $x = 0, 1, 2, 3,4,5$ we get

$$P(X = 0) = {}^{5}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$P(X = 1) = {}^{5}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 = \frac{5}{32}$$

$$P(X = 2) = {}^{5}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{10}{32}$$

$$P(X = 3) = {}^{5}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{10}{32}$$

$$P(X = 4) = {}^{5}C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 = \frac{5}{32}$$

$$P(X = 5) = {}^{5}C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = \frac{1}{32}$$

The corresponding frequencies are obtained by multiplying these probabilities by 3000

No of Heads	0	1	2	3	4	5
Frequency	100	500	1000	1000	500	100

Now the mean of the binomial distribution $= np = 5 \times \frac{1}{2} = 2.5$

The standard deviation of the Binomial distribution $= \sqrt{npq} = \sqrt{5 \times \frac{1}{2} \times \frac{1}{2}} = \frac{\sqrt{5}}{2}$

Ex 16 Seven coins are tossed and the number of heads obtained is noted. The experiment is repeated 128 times and the following distribution

No. of heads	0	1	2	3	4	5	6	7	Total
Frequency	7	6	19	35	30	23	7	1	128

Fit a Binomial distribution if **(i)** the coins are unbiased, **(ii)** if the nature of the coins is not known.

Solution: To fit a distribution to given data means to find the constants of the distribution which will adequately describe the given situation

(i) When the coins are unbiased

$p = \frac{1}{2}, q = \frac{1}{2}$ and by data $n = 7$

$\therefore P(X = x) = {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$

Putting $x = 0, 1, 2, 3, \ldots \ldots, 7$, we get,

$P(0) = \frac{1}{2^7}, \quad P(1) = \frac{7}{2^7}, \quad P(2) = \frac{21}{2^7}, \ldots \ldots \ldots.$

Expected frequency $= Np$ and $N = 128$

Multiplying the above probabilities by 128 i.e. by 2^7 we get the expected frequencies as

$1, 7, 21, 35, 35, 21, 7, 1$

(ii) When the nature of the coins is not known

We have $\bar{x} = \frac{\sum f_i x_i}{N} = \frac{0\times7+1\times6+2\times19+\cdots+7\times1}{128} = \frac{433}{128} = 3.38$

But $\bar{x} = np$,

$\therefore p = \frac{\bar{x}}{n} = \frac{3.38}{7} = 0.48$ $\therefore q = 1 - p = 0.52$

$\therefore P(X = x) = {}^7C_x(0.48)^x\,(0.52)^{7-x}$

Putting $x = 0, 1, 2, 3, \ldots \ldots, 7$ we get

$P(0) = 0.01, P(1) = 0.066, P(2) = 0.184, \ldots \ldots$

Multiply these probabilities by 128 we get the expected frequencies as $1, 8, 23, 36, 33, 18, 6, 3$

(Last term $= 128 -$ sum of other terms)