Scalar Product:

If $\bar{a}$ and $\bar{b}$ are two vectors and θ is the angle between them then the scalar quantity $|\bar{a}||\bar{b}|\cos\theta$ is called the **scalar product** or the dot product of $\bar{a}$ and $\bar{b}$ and is denoted by $\bar{a}\cdot\bar{b}$ and hence $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

- $i\cdot i = j\cdot j = k\cdot k = 1$, $i\cdot j = j\cdot k = k\cdot j = 0$, $\bar{a}\cdot\bar{a} = a^2$
- If $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{b} = b_1 i + b_2 j + b_3 k$ then $\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ **NOTE:**$(\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a})$
- Two vectors $\bar{a},\bar{b},(\bar{a}\neq\bar{0},\ \bar{b}\neq\bar{0})$ are perpendicular if $\bar{a}\cdot\bar{b} = 0$

Vector Product:

Let $\bar{a},\bar{b}$ be two vectors and θ be the angle between them $(0\le\theta\le\pi)$. The **vector product of** $\bar{a}$ and $\bar{b}$ is defined as a vector $ab\sin\theta\ \hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane of $\bar{a},\bar{b}$ and $\bar{a},\bar{b},\hat{n}$ form a right handed screw system.

The vector product of $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a}\times\bar{b}$ and read as $\bar{a}$ cross $\bar{b}$.

Thus, $\bar{a}\times\bar{b} = (ab\sin\theta)\,\hat{n}$

- $\bar{a}\times\bar{b}$ is a vector perpendicular to both vectors $\bar{a}$ and $\bar{b}$.
- Unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$ = $\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$
- **i)** $\bar{a}\times\bar{a} = 0$ **ii)** $\bar{b}\times\bar{a} = -(\bar{a}\times\bar{b})$

  iii) $i\times j = k,\ j\times k = i,\ k\times i = j$ **iv)** $i\times i = j\times j = k\times k = 0$

  v) $j\times i = -k,\ \ k\times j = -i,\ \ i\times k = -j.$
- If $\bar{a} = a_1 i + a_2 j + a_3 k$ and $\bar{b} = b_1 i + b_2 j + b_3 k$ then $\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- The area of parallelogram of sides $\bar{a}$ and $\bar{b} = |\bar{a}\times\bar{b}|$
- The area of parallelogram of diagonals $\bar{a}$ and $\bar{b} = \frac{1}{2}|\bar{a}\times\bar{b}|$
- The area of the triangle whose sides are co initial vectors, $\bar{a},\bar{b}$ is $\Delta = \frac{1}{2}|\bar{a}\times\bar{b}|$
- The vector $\bar{a}\times\bar{b}$ is called the vector area of the parallelogram
- The vector $\frac{1}{2}\bar{a}\times\bar{b}$ is called the vector area of the triangle.

Scalar Triple Product:

If $\bar{a},\bar{b},\bar{c}$ are three vectors then the scalar product $\bar{a}\cdot(\bar{b}\times\bar{c})$ is called **scalar triple product** of the vectors $\bar{a},\bar{b}$ and $\bar{c}$. It is denoted by$[\bar{a}\,\bar{b}\,\bar{c}\,]$. Thus, $[\bar{a}\,\bar{b}\,\bar{c}\,] = \bar{a}\cdot(\bar{b}\times\bar{c})$

- If $\bar{a} = a_1 i + a_2 j + a_3 k,\quad \bar{b} = b_1 i + b_2 j + b_3 k,\quad \bar{c} = c_1 i + c_2 j + c_3 k$ then

$$[\bar{a}\,\bar{b}\,\bar{c}\,] = \bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- $[\bar{a}\,\bar{b}\,\bar{c}] = [\bar{c}\,\bar{a}\,\bar{b}] = [\bar{b}\,\bar{c}\,\bar{a}]$ changing the order of vectors cyclically does not change the value of the product
- $[\bar{a}\,\bar{b}\,\bar{c}] = -[\bar{a}\,\bar{c}\,\bar{b}] = -[\bar{b}\,\bar{a}\,\bar{c}] = -[\bar{c}\,\bar{b}\,\bar{a}]$

  Interchanging the positions of two vector change the sign of the product
- $[\bar{a}\,\bar{a}\,\bar{b}] = [\bar{a}\,\bar{b}\,\bar{a}] = [\bar{b}\,\bar{a}\,\bar{a}] = 0$. If two vectors are same the value of the product is zero.

- $[k\bar{a}\,\bar{b}\,\bar{c}] = [\bar{a}\ \ k\bar{b}\,\bar{c}] = [\bar{a}\,\bar{b}\,k\bar{c}] = k[\bar{a}\,\bar{b}\,\bar{c}]$

  Multiplying any vector by a scalar k multiplies the product by k.
- $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$ Dot and cross can be interchanged.
- If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors in space with same initial point then the volume of the parallelepiped formed by them is given by $V = [\bar{a}\,\bar{b}\,\bar{c}]$
- If $\bar{a},\bar{b},\bar{c}$ are three non-zero vectors in space with same initial point, they will be coplanar if and only if the volume of the parallelepiped form by them is zero. i.e. if $[\bar{a}\bar{b}\bar{c}] = 0$
- If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors in space with same initial point then the volume of the tetrahedron formed by them is given by $V = \frac{1}{6}[\bar{a}\,\bar{b}\,\bar{c}]$

Vector Triple Product:

If $\bar{a},\bar{b},\bar{c}$ are three vectors then the vector or cross product of $\bar{a}\times\bar{b}$ with $\bar{c}$ is called the **vector triple product** of the three vectors $\bar{a},\bar{b}$ and $\bar{c}$ and is written $(\bar{a}\times\bar{b})\times\bar{c}$

The vector triple product is given by $(\bar{\boldsymbol{a}}\times\bar{\boldsymbol{b}})\times\bar{\boldsymbol{c}} = (\bar{\boldsymbol{a}}\cdot\bar{\boldsymbol{c}})\bar{\boldsymbol{b}} - (\bar{\boldsymbol{b}}\cdot\bar{\boldsymbol{c}})\,\bar{\boldsymbol{a}}$

The vector triple product $\bar{a}\times(\bar{b}\times\bar{c})$ is given by $\bar{\boldsymbol{a}}\times(\bar{\boldsymbol{b}}\times\bar{\mathbf{c}}) = (\bar{\boldsymbol{a}}\cdot\bar{\boldsymbol{c}})\bar{\boldsymbol{b}} - (\bar{\boldsymbol{a}}\cdot\bar{\mathbf{b}})\,\bar{\mathbf{c}}$

In general, $\bar{a}\times(\bar{b}\times\bar{c}) \neq (\bar{a}\times\bar{b})\times\bar{c}$

Scalar Product of four vectors:

If $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are any four vectors then the product $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$ is called the **scalar product of four vectors** $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$

Lagrange's identity: The product of four vectors $\bar{a},\bar{b}$, $\bar{c},\bar{d}$ by $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$

Proof: Let $\bar{c}\times\bar{d} = \bar{m}$

$$(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d}) = (\bar{a}\times\bar{b}).\bar{m} = \bar{a}.(\bar{b}\times\bar{m})$$
$$= \bar{a}.[\bar{b}\times(\bar{c}\times\bar{d})]$$
$$= \bar{a}.[(\bar{b}.\bar{d})\bar{c} - (\bar{b}.\bar{c})\bar{d}]$$
$$= (\bar{a}.\bar{c})(\bar{b}.\bar{d}) - (\bar{a}.\bar{d})(\bar{b}.\bar{c})$$
$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$$

Note: If we put $\bar{c} = \bar{a}$ & $\bar{d} = \bar{b}$ in Lagrange's identity then

$$(\bar{a}\times\bar{b})\cdot(\bar{a}\times\bar{b}) = (\bar{a}\cdot\bar{a})(\bar{b}\cdot\bar{b}) - (\bar{a}\cdot\bar{b})(\bar{a}\cdot\bar{b})$$
$$(\bar{a}\times\bar{b})^2 = a^2b^2 - (\bar{a}\cdot\bar{b})^2$$

Vector Product of Four Vectors:

If $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are any four vectors then the product $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})$ is called the **vector product of the four vectors** $\bar{a},\bar{b},\bar{c}$ and $\bar{d}$

- **Geometrical Meaning:**

Let $\bar{p} = (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})$. $\bar{p}$ is a vector perpendicular to both vectors $(\bar{a}\times\bar{b})$ and $(\bar{c}\times\bar{d})$. hence, $\bar{p}$ represent a vector parallel to line of intersection of two planes – one of which is parallel to the plane containing

$\bar{a}$ and $\bar{b}$ and the other plane is parallel to the plane containing $\bar{c}$ and $\bar{d}$

➢ **Expansion of $(\bar{\boldsymbol{a}} \times \bar{\boldsymbol{b}}) \times (\bar{\boldsymbol{c}} \times \bar{\boldsymbol{d}})$**

The vector product of four vectors $\bar{a}, \bar{b}, \bar{c}\ \bar{d}$ can be expressed

(i) in terms of vector $\bar{a}$ and $\bar{b}$ as $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$

(ii) in terms of vector $\bar{c}$ and $\bar{d}$ as $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$

Proof: **(i)** Let $(\bar{c} \times \bar{d}) = \bar{m}$

$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = (\bar{a} \times \bar{b}) \times \bar{m}$$
$$= (\bar{a}.\bar{m})\bar{b} - (\bar{b}.\bar{m})\bar{a}$$
$$= [\bar{a}\,.(\bar{c} \times \bar{d})]\bar{b} - [\bar{b}.(\bar{c} \times \bar{d})]\bar{a}$$
$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$$

(ii) Let $\bar{a} \times \bar{b} = \bar{n}$

$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = \bar{n} \times (\bar{c} \times \bar{d})$$
$$= (\bar{n}.\bar{d})\bar{c} - (\bar{n}.\bar{c})\bar{d}$$
$$= [(\bar{a} \times \bar{b}).\bar{d}]\bar{c} - [(\bar{a} \times \bar{b}).\bar{c}]\bar{d}$$
$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$$

Examples:

1. Find the area of the parallelogram whose diagonals are given by $3i + j - 2k, i - 3j + 4k$

Solution: If $\bar{a}$ and $\bar{b}$ are the sides of the parallelogram then the diagonals are $\bar{a} + \bar{b}$ and $\bar{b} - \bar{a}$

$\therefore \bar{a} + \bar{b} = 3i + j - 2k$ and $\bar{b} - \bar{a} = i - 3j + 4k$

$$\therefore \bar{a} = (2i + 4j - 6k)/2 = i + 2j - 3k$$
$$\bar{b} = (4i - 2j + 2k)/2 = 2i - j + k$$

The area of parallelogram of sides $\bar{a}$ and $\bar{b} = |\bar{a} \times \bar{b}|$

$\therefore$ Area of parallelogram $= \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix} = -i - 7j - 5k$

$\therefore$ Magnitude of the area $= \sqrt{75} = 5\sqrt{3}$ sq units

Note: We can use formula, the area of parallelogram of diagonals $\bar{a}$ and $\bar{b} = \frac{1}{2}|\bar{a} \times \bar{b}|$

2. If $\bar{a} + \bar{b} + \bar{c} = 0$, prove that $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$

Solution: Since $\bar{a} \times \bar{b} \times \bar{c} = \bar{0}$, $\bar{a} \times (\bar{a} + \bar{b} + \bar{c}) = \bar{0}$

$\therefore \bar{a} \times \bar{a} + \bar{a} \times \bar{b} + \bar{a} \times \bar{c} = \bar{0}$

$\therefore \bar{a} \times \bar{b} = -\bar{a} \times \bar{c}$ $\quad (\because \bar{a} \times \bar{a} = 0)$

$\therefore \bar{a} \times \bar{b} = \bar{c} \times \bar{a}$

Similarly we get the other results $\quad \therefore \bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$

3. If $\bar{a} = 0$, and $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$, $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}$ then prove that $\bar{b} = \bar{c}$

Solution: $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$

$\therefore \bar{a} \cdot \bar{b} - \bar{a} \cdot \bar{c} = 0$

$\therefore \bar{a} \cdot (\bar{b} - \bar{c}) = 0$

But $\bar{a} \neq 0$ $\quad \therefore \bar{b} - \bar{c} = 0$ or $\bar{a} \perp (\bar{b} - \bar{c})$ .... (1)

Also $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}$

$\therefore \bar{a} \times \bar{b} - \bar{a} \times \bar{c} = 0$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) = 0$

But $\bar{a} \neq 0$ $\quad \therefore \bar{b} - \bar{c} = 0$ or $\bar{a} \parallel (\bar{b} - \bar{c})$ .... (2)

From (1) and (2) $\bar{b} = \bar{c}$

4. Prove that the points $(2, 1, 1), (0, 1, -3), (3, 2, -1)$ and $(7, 2, 7)$ are coplanar

Solution: Let the four points be $A(2i + j + k), B(j - 3k), C(3i + 2j - k), D(7i + 2j + 7k)$. Let O be the origin

Then $\overline{AB} = \overline{OB} - \overline{OA} = -2i + 0j - 4k$

$\overline{AC} = \overline{OC} - \overline{OA} = i + j - 2k$

$\overline{AD} = \overline{OD} - \overline{OA} = 5i + j + 6k$

Now, $[\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} -2 & 0 & -4 \\ 1 & 1 & -2 \\ 5 & 1 & 6 \end{vmatrix} = 0$

Hence, A, B, C, D are coplanar

5. Find the volume of parallelepiped whose conterminal edges are $2i - 3j + 4k, i + 2j - 2k, 3i - j + k$

Solution: The volume of parallelepiped is the scalar triple product of the given vectors

$\therefore V = \bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -2 \\ 3 & -1 & 1 \end{vmatrix} = 2(2 - 2) + 3(1 + 6) + 4(-1 - 6) = -7 = 7$ units

6. A parallelopiped has concurrent edges OA, OB, OC of lengths a, b, c along the lines $x/1 = y/2 = z/3$; $x/2 = y/1 = z/3$; $x/3 = y/1 = z/2$ respectively. Find the volume of the parallelepiped

Solution: Unit vector along the $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is $\frac{i+2j+3k}{\sqrt{14}}$

$\therefore \bar{a} = (i + 2j + 3k)\frac{a}{\sqrt{14}}$

Similarly $\bar{b} = (2i + j + 3k)\frac{b}{\sqrt{14}}$, $\bar{c} = (3i + j + 2k)\frac{c}{\sqrt{14}}$

Volume of parallelepiped $= [\bar{a}\ \bar{b}\ \bar{c}] = \frac{abc}{14\sqrt{14}}\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = \frac{3abc}{7\sqrt{14}}$

7. Find the volume of the tetrahedron formed by $(1, 1, 3), (4, 3, 4), (5, 2, 7)$ and $(6, 4, 8)$

Solution: Let the points be A, B, C, D respectively and O be the origin

Then $\overline{OA} = i + j + 3k, \overline{OB} = 4i + 3j + 4k, \overline{OC} = 5i + 2j + 7k, \overline{OD} = 6i + 4j + 8k,$

$\therefore \overline{AB} = \overline{OB} - \overline{OA} = 3i + 2j + k$

$\overline{AC} = \overline{OC} - \overline{OA} = 4i + j + 4k$

$\overline{AD} = \overline{OD} - \overline{OA} = 5i + 3j + 5k$

$\therefore$ volume of the tetrahedron $ABCD = \frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}] = \frac{1}{6}\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 4 \\ 5 & 3 & 5 \end{vmatrix} = \frac{7}{3}$

8. Prove that $(\bar{p}-\bar{q}).[(\bar{q}-\bar{r})\times(\bar{r}-\bar{p})]=0$

Solution:

LHS $=(\bar{p}-\bar{q}).[(\bar{q}-\bar{r})\times(\bar{r}-\bar{p})]$

$=(\bar{p}-\bar{q})\cdot[\bar{q}\times\bar{r}-\bar{q}\times\bar{p}-\bar{r}\times\bar{r}+\bar{r}\times\bar{p}]$

$=\bar{p}\cdot(\bar{q}\times\bar{r})-\bar{p}\cdot(\bar{q}\times\bar{p})-\bar{p}\cdot(\bar{r}\times\bar{r})$

$+\bar{p}\cdot(\bar{r}\times\bar{p})-\bar{q}\cdot(\bar{q}\times\bar{r})+\bar{q}\cdot(\bar{q}\times\bar{p})+\bar{q}\cdot(\bar{r}\times\bar{r})-\bar{q}\cdot(\bar{r}\times\bar{p})$

But $\bar{p}\cdot(\bar{r}\times\bar{r})=0,\ \bar{p}\cdot(\bar{q}\times\bar{p})=0,$

$\bar{p}\cdot(\bar{r}\times\bar{p})=0,\ \bar{q}\cdot(\bar{q}\times\bar{r})=0,\ \bar{q}\cdot(\bar{q}\times\bar{p})=0,\quad \bar{p}\cdot(\bar{r}\times\bar{p})=0,\ \bar{q}\cdot(\bar{q}\times\bar{r})=0,\ \bar{q}\cdot(\bar{q}\times\bar{p})=0$

$\therefore$ LHS $=\bar{p}\cdot(\bar{q}\times\bar{r})-\bar{q}\cdot(\bar{r}\times\bar{p})=[p\ q\ r]-[q\ r\ p]=[p\ q\ r]-[p\ q\ r]=0$

9. Prove that $[\bar{p}+\bar{q}\quad \bar{q}+\bar{r}\quad \bar{r}+\bar{p}]=(\bar{p}+\bar{q}).[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})]=2[\bar{p}\quad \bar{q}\quad \bar{r}]$

Solution: LHS $=(\bar{p}+\bar{q}).[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})]$

$=(\bar{p}+\bar{q})\cdot[\bar{q}\times\bar{r}+\bar{q}\times\bar{p}+\bar{r}\times\bar{r}+\bar{r}\times\bar{p}]$

$=\bar{p}\cdot(\bar{q}\times\bar{r})+\bar{p}\cdot(\bar{q}\times\bar{p})+\bar{p}\cdot(\bar{r}\times\bar{r})$

$+\bar{p}\cdot(\bar{r}\times\bar{p})+\bar{q}\cdot(\bar{q}\times\bar{r})+\bar{q}\cdot(\bar{q}\times\bar{p})+\bar{q}\cdot(\bar{r}\times\bar{r})+\bar{q}\cdot(\bar{r}\times\bar{p})$

But $\bar{p}\cdot(\bar{r}\times\bar{r})=0,\ \bar{p}\cdot(\bar{q}\times\bar{p})=0,$

$\bar{p}\cdot(\bar{r}\times\bar{p})=0,\ \bar{q}\cdot(\bar{q}\times\bar{r})=0,\ \bar{q}\cdot(\bar{q}\times\bar{p})=0,$

$\bar{p}\cdot(\bar{r}\times\bar{p})=0,\ \bar{q}\cdot(\bar{q}\times\bar{r})=0,\ \bar{q}\cdot(\bar{q}\times\bar{p})=0$

LHS $=\bar{p}\cdot(\bar{q}\times\bar{r})+\bar{q}\cdot(\bar{r}\times\bar{p})=[\bar{p}\quad \bar{q}\quad \bar{r}]+[\bar{q}\quad \bar{r}\quad \bar{p}]=[\bar{p}\quad \bar{q}\quad \bar{r}]+[\bar{p}\quad \bar{q}\quad \bar{r}]=2[\bar{p}\quad \bar{q}\quad \bar{r}]$

10. Prove that $\left[(\bar{a}+\bar{b}+\bar{c})\times(\bar{b}+\bar{c})\right].\bar{c}=\bar{a}.(\bar{b}\times\bar{c})$

Solution: LHS $=\left[\bar{a}\times(\bar{b}+\bar{c})+\bar{b}\times(\bar{b}+\bar{c})+\bar{c}\times(\bar{b}+\bar{c})\right]\cdot\bar{c}$

$=\left[(\bar{a}\times\bar{b})+(\bar{a}\times\bar{c})+(\bar{b}\times\bar{b})+(\bar{b}\times\bar{c})+(\bar{c}\times\bar{b})+(\bar{c}\times\bar{c})\right]\cdot\bar{c}$

$=(\bar{a}\times\bar{b})\cdot\bar{c}+(\bar{a}\times\bar{c})\cdot\bar{c}+(\bar{b}\times\bar{b})\cdot\bar{c}+(\bar{b}\times\bar{c})\cdot\bar{c}+(\bar{c}\times\bar{b})\cdot\bar{c}+(\bar{c}\times\bar{c})\cdot\bar{c}$

$=(\bar{a}\times\bar{b})\cdot\bar{c}$ other terms being zero

$=\bar{a}\cdot(\bar{b}\times\bar{c})$

11. If $\bar{a},\bar{b},\bar{c}$ are the position vectors of the points A, B, C, prove that the vector $\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}$ is perpendicular to the plane of the triangle ABC.

Solution: We have $\overline{AB}=\bar{b}-\bar{a}$

$\therefore(\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a})\cdot(\bar{b}-\bar{a})$

$=(\bar{a}\times\bar{b})\cdot\bar{b}+(\bar{b}\times\bar{c})\cdot\bar{b}+(\bar{c}\times\bar{a})\cdot\bar{b}-(\bar{a}\times\bar{b})\cdot\bar{a}-(\bar{b}\times\bar{c})\cdot\bar{a}-(\bar{c}\times\bar{a})\cdot\bar{a}$

$=0+0+(\bar{c}\times\bar{a})\cdot\bar{b}-0-(\bar{b}\times\bar{c})\cdot\bar{a}-0=0$

$\therefore$ The given vector is perpendicular to $\overline{AB}$.

Similarly we can show that it is perpendicular to $\overline{BC}$ and $\overline{CA}$

12. If $\bar{l}, \bar{m}, \bar{n}$ are three non – coplanar vectors, prove that $[\bar{l}\ \bar{m}\ \bar{n}](\bar{a} \times \bar{b}) = \begin{vmatrix} \bar{l}.\bar{a} & \bar{l}.\bar{b} & \bar{l} \\ \bar{m}.\bar{a} & \bar{m}.\bar{b} & \bar{m} \\ \bar{n}.\bar{a} & \bar{n}.\bar{b} & \bar{n} \end{vmatrix}$

Solution: Let $\bar{l} = l_1 i + l_2 j + l_3 k,\ \bar{m} = m_1 i + m_2 j + m_3 k$, and $\bar{n} = n_1 i + n_2 j + n_3 k,\ \ \bar{a} = a_1 i + a_2 j + a_3 k$, $\bar{b} = b_1 i + b_2 j + b_3 k$. Then,

$$[\bar{l}\ \bar{m}\ \bar{n}](\bar{a} \times \bar{b}) = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ (Two determinates are multiplied by row by row

$$= \begin{vmatrix} l_1 i + l_2 j + l_3 k & l_1 a_1 + l_2 a_2 + l_3 a_3 & l_1 b_1 + l_2 b_2 + l_3 b_3 \\ m_1 i + m_2 j + m_3 k & m_1 a_1 + m_2 a_2 + m_3 a_3 & m_1 b_1 + m_2 b_2 + m_3 b_3 \\ n_1 i + n_2 j + n_3 k & n_1 a_1 + n_2 a_2 + n_3 a_3 & n_1 b_1 + n_2 b_2 + n_3 b_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{l} & \bar{l} \cdot \bar{a} & \bar{l} \cdot \bar{b} \\ \bar{m} & \bar{m} \cdot \bar{a} & \bar{m} \cdot \bar{b} \\ \bar{n} & \bar{n} \cdot \bar{a} & \bar{n} \cdot \bar{b} \end{vmatrix} = \begin{vmatrix} \bar{l}.\bar{a} & \bar{l}.\bar{b} & \bar{l} \\ \bar{m}.\bar{a} & \bar{m}.\bar{b} & \bar{m} \\ \bar{n}.\bar{a} & \bar{n}.\bar{b} & \bar{n} \end{vmatrix}$$

13. Prove that $[\bar{a}\ \bar{b}\ \bar{c}]^2 = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix}$

Solution: $[\bar{a}\ \bar{b}\ \bar{c}]^2 = [\bar{a}\ \bar{b}\ \bar{c}][\bar{a}\ \bar{b}\ \bar{c}]$

Let $\bar{a} = a_1 i + a_2 j + a_3 k,\ \bar{b} = b_1 i + b_2 j + b_3 k,\ \bar{c} = c_1 i + c_2 j + c_3 k$

By definition of scalar triple product

$$\therefore [\bar{a}\ \bar{b}\ \bar{c}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Now, the product of two determinants is obtained by row $\times$ row multiplication

$$\therefore [\bar{a}\ \bar{b}\ \bar{c}]^2 = \begin{vmatrix} a_1 a_1 + a_2 a_2 + a_3 a_3 & a_1 b_1 + a_2 b_2 + a_3 b_3 & a_1 c_1 + a_2 c_2 + a_3 c_3 \\ b_1 a_1 + b_2 a_2 + b_3 a_3 & b_1 b_1 + b_2 b_2 + b_3 b_3 & b_1 c_1 + b_2 c_2 + b_3 c_3 \\ c_1 a_1 + c_2 a_2 + c_3 a_3 & c_1 b_1 + c_2 b_2 + c_3 b_3 & c_1 c_1 + c_2 c_2 + c_3 c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{a} \cdot \bar{a} & \bar{a} \cdot \bar{b} & \bar{a} \cdot \bar{c} \\ \bar{b} \cdot \bar{a} & \bar{b} \cdot \bar{b} & \bar{b} \cdot \bar{c} \\ \bar{c} \cdot \bar{a} & \bar{c} \cdot \bar{b} & \bar{c} \cdot \bar{c} \end{vmatrix}$$

14. If $\bar{a} = 3i - 2j + 2k, \bar{b} = 6i + 4j - 2k, \bar{c} = 3i + 2j + 4k$, find $\bar{a} \times (\bar{b} \times \bar{c})$

Solution: $\bar{b} \times \bar{c} = \begin{vmatrix} i & j & k \\ 6 & 4 & -2 \\ 3 & 2 & 4 \end{vmatrix} = 20i - 30j$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} i & j & k \\ 3 & -2 & 2 \\ 20 & -30 & 0 \end{vmatrix} = 60i + 40j - 50k = 10(6i + 4j - 5k)$$

Alternatively, we have,

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

But $\bar{a} \cdot \bar{c} = (3i - 2j + 2k)(3i + 2j + 4k) = 13$

$\bar{a} \cdot \bar{b} = (3i - 2j + 2k)(6i + 4j - 2k) = 6 \therefore (\bar{a} \cdot \bar{c})\bar{b} = 13(6i + 4j - 2k) = 78i + 52j - 26k$

$$(\bar{a} \cdot \bar{b})\bar{c} = 6(3i + 2j + 4k) = 18i + 12j + 24k$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = 60i + 40j - 50k = 10(6i + 4j - 5k)$$

15. Find the scalars p, q if $(\overline{a} \times \overline{b}) \times \overline{c} = \overline{a} \times (\overline{b} \times \overline{c})$ where $\overline{a} = 2i + j + pk,\ \overline{b} = i - j,\ \overline{c} = 4i + qj + 2k$.

Solution: We have

$(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{b} \cdot \overline{c})\overline{a}$ .... (1) $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}$ .... (2)

Now $\overline{a} \cdot \overline{b} = (2i + j + pk) \cdot (i - j) = 2 - 1 = 1$

$\overline{b} \cdot \overline{c} = (i - j) \cdot (4i + qj + 2k) = 4 - q$

$\overline{c} \cdot \overline{a} = (4i + qj + 2k) \cdot (2i + j + pk) = 8 + q + 2p$ (1) and (2) $(\overline{a} \cdot \overline{c})\overline{b} - (\overline{b} \cdot \overline{c})\overline{a} = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}$

$\therefore (\overline{b} \cdot \overline{c})\overline{a} = (\overline{a} \cdot \overline{b})\overline{c}$

$\therefore (4 - q) \cdot (2i + j + pk) = 4i + qj + 2k$

Equating coefficient of i, j, k we get $2(4 - q) = 4,\ 4 - q = q,\ (4 - q)p = 2$

Hence, $p = 1$ and $q = 2$

16. Prove that, $i \times (\overline{a} \times i) + j \times (\overline{a} \times j) + k \times (\overline{a} \times k) = 2\overline{a}$.

Solution: By definition of vector triple product $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c})\overline{b} - (\overline{a} \cdot \overline{b})\overline{c}$

$\therefore i \times (\overline{a} \times i) = (i \cdot i)\overline{a} - (i \cdot \overline{a})i$

$j \times (\overline{a} \times j) = (j \cdot j)\overline{a} - (j \cdot \overline{a})j$

$k \times (\overline{a} \times k) = (k \cdot k)\overline{a} - (k \cdot \overline{a})k$

But $i \cdot i = j \cdot j = k \cdot k = 1$

If $\overline{a} = a_1 i + a_2 j + a_3 k$

$\therefore (i \cdot \overline{a})i = a_1 i,\ \ (j \cdot \overline{a})j = a_2 j,\ \ (k \cdot \overline{a})k = a_3 k$

$\therefore$ LHS $= 3\overline{a} - (a_1 i + a_2 j + a_3 k) = 3\overline{a} - \overline{a} = 2\overline{a}$

17. If $\overline{a}, \overline{b}, \overline{c}$ are coplanar vectors, prove that $\overline{a} \times \overline{b}, \overline{b} \times \overline{c}$ and $\overline{c} \times \overline{a}$ are also coplanar vectors.

Solution: Since $\overline{a}, \overline{b}, \overline{c}$ are coplanar $[\overline{a}\,\overline{b}\,\overline{c}] = 0$

The vectors $\overline{a} \times \overline{b},\ \overline{b} \times \overline{c},\ \overline{c} \times \overline{a}$ will be coplanar if $[\overline{a} \times \overline{b}\ \ \overline{b} \times \overline{c}\ \ \overline{c} \times \overline{a}] = 0$

LHS $= [\overline{a} \times \overline{b}\ \ \overline{b} \times \overline{c}\ \ \overline{c} \times \overline{a}]$

$= (\overline{a} \times \overline{b}) . [(\overline{b} \times \overline{c}) \times (\overline{c} \times \overline{a})]$

$= (\overline{a} \times \overline{b}) \cdot \{[\overline{b}\ \overline{c}\ \overline{a}]\overline{c} - [\overline{b}\ \overline{c}\ \overline{c}]\overline{a}\}$

$= (\overline{a} \times \overline{b}) \cdot [\overline{b}\ \overline{c}\ \overline{a}]\overline{c}$ $\because [\overline{b}\ \overline{c}\ \overline{c}] = 0$

$= \left((\overline{a} \times \overline{b}) \cdot \overline{c}\right)[\overline{b}\ \overline{c}\ \overline{a}]$

$= [\overline{a}\ \overline{b}\ \overline{c}][\overline{a}\ \overline{b}\ \overline{c}]$

$= 0$ $[\because [\overline{a}\,\overline{b}\,\overline{c}] = 0$ by data$]$

Hence, $\overline{a} \times \overline{b}, \overline{b} \times \overline{c}, \overline{c} \times \overline{a}$ are coplanar

18. If the vectors $\overline{u}, \overline{v}, \overline{w}$ are non – coplanar show that the vectors $\overline{u} \times \overline{v},\ \ \overline{v} \times \overline{w},\ \ \overline{w} \times \overline{u}$ are also non –coplanar. Hence, obtain the scalars l, m, n such that $\overline{u} = l(\overline{v} \times \overline{w}) + m(\overline{w} \times \overline{u}) + n(\overline{u} \times \overline{v})$.

Solution: To prove non-coplanarity of the vectors $\overline{u}\times\overline{v}$, $\overline{v}\times\overline{w}$ and $\overline{w}\times\overline{u}$ we have to prove that their scalar triple product is non-zero.

Hence, consider $[\overline{u}\times\overline{v}\ \ \overline{v}\times\overline{w}\ \ \overline{w}\times\overline{u}] = (\overline{u}\times\overline{v})\cdot[(\overline{v}\times\overline{w})\times(\overline{w}\times\overline{u})]$ $= (\overline{u}\times\overline{v})\cdot\{[\overline{v}\,\overline{w}\,\overline{u}]\overline{w}-[\overline{v}\,\overline{w}\,\overline{w}]\overline{u}\}$

$= (\overline{u}\times\overline{v})\cdot\{[\overline{v}\,\overline{w}\,\overline{u}]\overline{w}-0\}$

$= [\overline{v}\,\overline{w}\,\overline{u}](\overline{u}\times\overline{v})\cdot\overline{w} \quad = [\overline{v}\,\overline{w}\,\overline{u}][\overline{u}\,\overline{v}\,\overline{w}]$

$$= [\overline{u}\,\overline{v}\,\overline{w}][\overline{u}\,\overline{v}\,\overline{w}] = [\overline{u}\,\overline{v}\,\overline{w}]^2 \neq 0$$

Hence, $\overline{u}\times\overline{v}$, $\overline{v}\times\overline{w}$, $\overline{w}\times\overline{u}$ are also non-coplanar

Hence, $\overline{u}$ can be expressed as a linear combination of $\overline{u}\times\overline{v}$, $\overline{v}\times\overline{w}$ and $\overline{w}\times\overline{u}$

$\therefore \overline{u} = l(\overline{v}\times\overline{w}) + m(\overline{w}\times\overline{u}) + n(\overline{u}\times\overline{v})$ where l, m, n are scalars to be determined

Now $\overline{u}\cdot\overline{u} = l(\overline{u}\times\overline{v})\cdot\overline{u} + m(\overline{v}\times\overline{w})\cdot\overline{u} + n(\overline{w}\times\overline{u})\cdot\overline{u}$

$= m(\overline{v}\times\overline{w})\cdot\overline{u} = m[\overline{v}\,\overline{w}\,\overline{u}] = m[\overline{u}\,\overline{v}\,\overline{w}]$

$\therefore m = \frac{\overline{u}\,.\,\overline{u}}{[\overline{u}\ \overline{v}\ \overline{w}]}$

Similarly $l = \frac{\overline{w}\,.\,\overline{w}}{[\overline{u}\ \overline{v}\ \overline{w}]}, n = \frac{\overline{v}\,.\,\overline{v}}{[\overline{u}\ \overline{v}\ \overline{w}]}$

19. Prove that the vectors $\overline{a}\times(\overline{b}\times\overline{c}), \overline{b}\times(\overline{c}\times\overline{a}), \overline{c}\times(\overline{a}\times\overline{b})$ are coplanar.

Solution: We have $\overline{a}\times(\overline{b}\times\overline{c}) = (\overline{a}\cdot\overline{c})\overline{b} - (\overline{a}\cdot\overline{b})\overline{c} = l\overline{b} - m\overline{c}$, say

$\overline{b}\times(\overline{c}\times\overline{a}) = (\overline{b}\cdot\overline{a})\overline{c} - (\overline{b}\cdot\overline{c})\overline{a} = m\overline{c} - n\overline{a}$, say

$\overline{c}\times(\overline{a}\times\overline{b}) = (\overline{c}\cdot\overline{b})\overline{a} - (\overline{c}\cdot\overline{a})\overline{b} = n\overline{a} - l\overline{b}$, say

The given vectors are coplanar if

$[\overline{a}\times(\overline{b}\times\overline{c})\ \ \overline{b}\times(\overline{c}\times\overline{a})\ \ \overline{c}\times(\overline{a}\times\overline{b})] = 0$

Now LHS $= [l\overline{b}-m\overline{c}\quad m\overline{c}-n\overline{a}\quad n\overline{a}-l\overline{b}]$

$= (l\overline{b}-m\overline{c})\cdot[(m\overline{c}-n\overline{a})\times(n\overline{a}-l\overline{b})]$

$$= (l\overline{b}-m\overline{c})\cdot[mn(\overline{c}\times\overline{a}) - ml(\overline{c}\times\overline{b}) - n^2(\overline{a}\times\overline{a}) + nl(\overline{a}\times\overline{b})]$$

$$= lmn\overline{b}\cdot(\overline{c}\times\overline{a}) - ml^2\overline{b}\cdot(\overline{c}\times\overline{b}) - ln^2\overline{b}\cdot(\overline{a}\times\overline{a}) + nl^2\overline{b}\cdot(\overline{a}\times\overline{b}) - m^2n\overline{c}\cdot(\overline{c}\times\overline{a})$$

$$+m^2l\overline{c}\cdot(\overline{c}\times\overline{b}) - mn^2\overline{c}\cdot(\overline{a}\times\overline{a}) - mnl\overline{c}\cdot(\overline{a}\times\overline{b})$$

But $\overline{b}\cdot(\overline{c}\times\overline{b}) = 0, \overline{b}\times(\overline{a}\times\overline{a}) = 0, \overline{b}\cdot(\overline{a}\times\overline{b}) = 0, \overline{c}\cdot(\overline{c}\times\overline{a}) = 0, \overline{c}\cdot(\overline{c}\times\overline{b}) = 0, \overline{c}\cdot(\overline{a}\times\overline{a}) = 0$

(If two vectors are same then the scalar triple product in zero)

$\therefore$ LHS $= lmn\overline{b}\cdot(\overline{c}\times\overline{a}) - lmn\overline{c}\cdot(\overline{a}\times\overline{b}) \qquad = lmn[\overline{b}\cdot(\overline{c}\times\overline{a}) - \overline{c}\cdot(\overline{a}\times\overline{b})]$

$= lmn\{[\overline{b}\,\overline{c}\,\overline{a}] - [\overline{c}\,\overline{a}\,\overline{b}]\} \qquad = lmn\{[\overline{a}\,\overline{b}\,\overline{c}] - [\overline{a}\,\overline{b}\,\overline{c}]\} = 0$

($\because$ changing the order of the vectors cyclically does not change the value of the scalar triple product)

20. If the vector $\overline{x}$ and a scalar λ satisfy the equation $\overline{a}\times\overline{x} = \lambda\overline{a}+\overline{b}\ and\ \overline{a}\,.\,\overline{x} = 1$, find the values of λ and $\overline{x}$ in terms of $\overline{a}, \overline{b}$. Also determine them if $\overline{a} = i - 2j$ and $\overline{b} = 2i + j - 2k$.

Solution: To find λ, we multiply the given equation $\overline{a}\times\overline{x} = \lambda\overline{a}+\overline{b}$ scalarly by $\overline{a}$

$$\therefore \overline{a}\cdot(\overline{a}\times\overline{x}) = \overline{a}\cdot\lambda\overline{a} + \overline{a}\cdot\overline{b}$$

$\therefore \lambda\overline{a}\cdot\overline{a} + \overline{a}\cdot\overline{b} = 0 \qquad [\because \overline{a}\times(\overline{a}\times\overline{x}) = 0]$

$\therefore \lambda = -\frac{\overline{a}\,.\,\overline{b}}{a^2}$ .... (1)

To find $\overline{x}$, we multiply the given equation $\overline{a} \times \overline{x} = \lambda\overline{a} + \overline{b}$ vectorially by $\overline{a}$

$\therefore \overline{a} \times (\overline{a} \times \overline{x}) = \overline{a} \times \lambda\overline{a} + \overline{a} \times \overline{b}$

$\therefore (\overline{a} \cdot \overline{x}) \cdot \overline{a} - (\overline{a} \cdot \overline{a})\overline{x} = \lambda\overline{a} \times \overline{a} + \overline{a} \times \overline{b}$

But $\overline{a} \times \overline{a} = 0$ and $\overline{a} \cdot \overline{x} = 1$ by data

$\therefore \overline{a} - a^2\overline{x} = \overline{a} \times \overline{b}$ $\quad \therefore \overline{x} = \frac{\overline{a} - \overline{a} \times \overline{b}}{a^2}$ .... (2)

For the second part, put $\overline{a} = i - 2j$ and $\overline{b} = 2i + j - 2k$ in (1) and (2),

From (1), $\lambda = -\frac{(i-2j)\cdot(2i+j-2k)}{5} = 0$

(2), $\overline{x} = \frac{[(i-2j)-(i-2j)\times(2i+j-2k)]}{5} = -\frac{(3i+4j+5k)}{5}$

21. If $[\overline{a}\,\overline{b}\,\overline{c}] \neq 0$, prove that a vector $\overline{d}$ can be expressed as $\overline{d} = \frac{[\overline{d}\,\overline{b}\,\overline{c}]\overline{a} + [\overline{d}\,\overline{c}\,\overline{a}]\overline{b} + [\overline{d}\,\overline{a}\,\overline{b}]\overline{c}}{[\overline{a}\,\overline{b}\,\overline{c}]}$

Solution: Since $[\overline{a}\,\overline{b}\,\overline{c}] \neq 0, \overline{a}, \overline{b}, \overline{c}$ are non-coplanar vectors and any vector $\overline{d}$ can be uniquely expressed as a linear combination of $\overline{a}, \overline{b}, \overline{c}$

$\therefore$ Let $\overline{d} = x\overline{a} + y\overline{b} + z\overline{c}$ ....(i)

Taking dot product of $\overline{d}$ with $\overline{b} \times \overline{c}$,

$$\overline{d} \cdot (\overline{b} \times \overline{c}) = x\overline{a} \cdot (\overline{b} \times \overline{c}) + y\overline{b} \cdot (\overline{b} \times \overline{c}) + z\overline{c} \cdot (\overline{b} \times \overline{c}) = x\overline{a} \cdot (\overline{b} \times \overline{c})$$

$\therefore x = \frac{[\overline{d}\,\overline{b}\,\overline{c}]}{[\overline{a}\,\overline{b}\,\overline{c}]}$

Similarly taking dot product of $\overline{d}$ with $\overline{c} \times \overline{a}$ and $\overline{a} \times \overline{b}$, we get $y = \frac{[\overline{d}\,\overline{c}\,\overline{a}]}{[\overline{a}\,\overline{b}\,\overline{c}]}$, $z = \frac{[\overline{d}\,\overline{a}\,\overline{b}]}{[\overline{a}\,\overline{b}\,\overline{c}]}$

Putting the values of x, y, z we get from (i) $\quad \overline{d} = \frac{[\overline{d}\,\overline{b}\,\overline{c}]\overline{a} + [\overline{d}\,\overline{c}\,\overline{a}]\overline{b} + [\overline{d}\,\overline{a}\,\overline{b}]\overline{c}}{[\overline{a}\,\overline{b}\,\overline{c}]}$

22. By considering the product $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d})$ in two different ways, show that

$[\overline{b}\,\overline{c}\,\overline{d}]\,\overline{a} + [\overline{c}\,\overline{a}\,\overline{d}]\,\overline{b} + [\overline{a}\,\overline{b}\,\overline{d}]\,\overline{c} = [\overline{a}\,\overline{b}\,\overline{c}]\,\overline{d}$ where a, b, c are non coplanar vectors

Solution:

(i) $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = [\overline{a}\,\overline{c}\,\overline{d}]\overline{b} - [\overline{b}\,\overline{c}\,\overline{d}]\overline{a}$

(ii) $(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = [\overline{a}\,\overline{b}\,\overline{d}]\overline{c} - [\overline{a}\,\overline{b}\,\overline{c}]\overline{d}$

$$\therefore [\overline{a}\,\overline{b}\,\overline{d}]\,\overline{c} - [\overline{a}\,\overline{b}\,\overline{c}]\,\overline{d} = [\overline{a}\,\overline{c}\,\overline{d}]\,\overline{b} - [\overline{b}\,\overline{c}\,\overline{d}]\,\overline{a}$$

$$\therefore [\overline{b}\,\overline{c}\,\overline{d}]\,\overline{a} - [\overline{a}\,\overline{c}\,\overline{d}]\,\overline{b} + [\overline{a}\,\overline{b}\,\overline{d}]\,\overline{c} = [\overline{a}\,\overline{b}\,\overline{c}]\,\overline{d}$$

$$\therefore [\overline{b}\,\overline{c}\,\overline{d}]\,\overline{a} + [\overline{c}\,\overline{a}\,\overline{d}]\,\overline{b} + [\overline{a}\,\overline{b}\,\overline{d}]\,\overline{c} = [\overline{a}\,\overline{b}\,\overline{c}]\,\overline{d}$$

23. Prove that $[\overline{d} \times (\overline{a} \times \overline{b})]\,.\,(\overline{a} \times \overline{c}) = [\overline{a}\,\overline{b}\,\overline{c}](\overline{a}.\overline{d})$

Solution: LHS $= [(\overline{d} \cdot \overline{b})\overline{a} - (\overline{d} \cdot \overline{a})\overline{b}] \cdot (\overline{a} \times \overline{c})$

$= (\overline{d} \cdot \overline{b})(\overline{a} \cdot (\overline{a} \times \overline{c})) - (\overline{d} \cdot \overline{b})\left(\overline{b} \cdot (\overline{a} \times \overline{c})\right)$

$= 0 - (\overline{d} \cdot \overline{a})[\overline{b}\,\overline{a}\,\overline{c}] = [\overline{a}\,\overline{b}\,\overline{c}](\overline{a}.\overline{d})$

24. Prove that $\bar{d}\,.\left[\bar{a}\times\left[\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right]\right]=\left(\bar{b}\,.\,\bar{d}\right)\left[\bar{a}\,\bar{c}\,\bar{d}\right]$

Solution: LHS $=\left[\bar{a}\times\left\{\left(\bar{b}\,.\,\bar{d}\right)\bar{c}-\left(\bar{b}\,.\,\bar{c}\right)\bar{d}\right\}\right]\cdot\bar{d}$

$=\left[\left(\bar{b}\,.\,\bar{d}\right)(\bar{a}\times\bar{c})-\left(\bar{b}\,.\,\bar{c}\right)(\bar{a}\times\bar{d})\right]\cdot\bar{d}$

$=\left(\bar{b}\,.\,\bar{d}\right)\left((\bar{a}\times\bar{c})\cdot\bar{d}\right)-\left(\bar{b}\,.\,\bar{c}\right)\left((\bar{a}\times\bar{d})\cdot\bar{d}\right)=\left(\bar{b}\,.\,\bar{d}\right)\left((\bar{a}\times\bar{c})\cdot\bar{d}\right)-0$

$=\left(\bar{b}\,.\,\bar{d}\right)\left[\bar{a}\,\bar{c}\,\bar{d}\right]$

25. Prove that $\bar{a}\times\left[\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right]=\left(\bar{b}\,.\,\bar{d}\right)(\bar{a}\times\bar{c})-\left(\bar{b}\,.\,\bar{c}\right)(\bar{a}\times\bar{d})$

Solution: By vector triple product

$\bar{a}\times\left[\bar{b}\times\left(\bar{c}\times\bar{d}\right)\right]\quad=\bar{a}\times\left[\left(\bar{b}\,.\,\bar{d}\right)\bar{c}-\left(\bar{b}\,.\,\bar{c}\right)\bar{d}\right]$

$=\left(\bar{b}\,.\,\bar{d}\right)(\bar{a}\times\bar{c})-\left(\bar{b}\,.\,\bar{c}\right)(\bar{a}\times\bar{d})$

26. Prove that $\left(\bar{a}\times\bar{b}\right).\left[\left(\bar{b}\times\bar{c}\right)\times(\bar{c}\times\bar{a})\right]=\left[\bar{a}\,.\left(\bar{b}\times\bar{c}\right)\right]^2$

Solution: LHS $=\left(\bar{a}\times\bar{b}\right).\left[\left(\bar{b}\times\bar{c}\right)\times(\bar{c}\times\bar{a})\right]$

$=\left(\bar{a}\times\bar{b}\right)\cdot\left\{\left[\bar{b}\;\bar{c}\;\bar{a}\right]\bar{c}-\left[\bar{b}\;\bar{c}\;\bar{c}\right]\bar{a}\right\}$

$=\left(\bar{a}\times\bar{b}\right)\cdot\left[\bar{b}\;\bar{c}\;\bar{a}\right]\bar{c}\qquad\because\left[\bar{b}\;\bar{c}\;\bar{c}\right]=0$

$=\left(\left(\bar{a}\times\bar{b}\right)\cdot\bar{c}\right)\left[\bar{b}\;\bar{c}\;\bar{a}\right]$

$=\left[\bar{a}\;\bar{b}\;\bar{c}\right]\left[\bar{a}\;\bar{b}\;\bar{c}\right]$

$=\left[\left(\bar{a}\times\bar{b}\right)\cdot\bar{c}\right]^2=$ RHS

27. Prove that $\left[\bar{b}\times\bar{c}\;\;\bar{a}\times\bar{c}\;\;\bar{a}\times\bar{b}\right]=\left[\bar{a}\,\bar{b}\,\bar{c}\right]^2$

Solution: Since $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$, we have

LHS $=\left(\bar{b}\times\bar{c}\right)\cdot\left[(\bar{c}\times\bar{a})\times\left(\bar{a}\times\bar{b}\right)\right]$

$=\left(\bar{b}\times\bar{c}\right)\cdot\left\{\left[\bar{c}\;\bar{a}\;\bar{b}\right]\bar{a}-\left[\bar{c}\;\bar{a}\;\bar{a}\right]\bar{b}\right\}$

$=\left(\bar{b}\times\bar{c}\right)\cdot\left\{\left[\bar{a}\,\bar{b}\,\bar{c}\right]\bar{a}-0\right\}\because\left[\bar{c}\;\bar{a}\;\bar{a}\right]=0$

$=\left[\bar{a}\,\bar{b}\,\bar{c}\right]\left(\left(\bar{b}\times\bar{c}\right)\cdot\bar{a}\right)\quad=\left[\bar{a}\,\bar{b}\,\bar{c}\right]\left[\bar{b}\,\bar{c}\,\bar{a}\right]$

$=\left[\bar{a}\,\bar{b}\,\bar{c}\right]\left[\bar{a}\,\bar{b}\,\bar{c}\right]\qquad=\left[\bar{a}\,\bar{b}\,\bar{c}\right]^2$

28. Prove that $\left(\bar{a}\times\bar{b}\right).\left(\bar{c}\times\bar{d}\right)+\left(\bar{b}\times\bar{c}\right).\left(\bar{a}\times\bar{d}\right)+(\bar{c}\times\bar{a}).\left(\bar{b}\times\bar{d}\right)=0.$

Solution: By the above Lagrange's identity

LHS $=\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}+\begin{vmatrix}\bar{b}\cdot\bar{a} & \bar{c}\cdot\bar{a}\\ \bar{b}\cdot\bar{d} & \bar{c}\cdot\bar{d}\end{vmatrix}+\begin{vmatrix}\bar{c}\cdot\bar{b} & \bar{a}\cdot\bar{b}\\ \bar{c}\cdot\bar{d} & \bar{a}\cdot\bar{d}\end{vmatrix}=(\bar{a}\cdot\bar{c})\left(\bar{b}\cdot\bar{d}\right)-\left(\bar{a}\cdot\bar{d}\right)\left(\bar{b}\cdot\bar{c}\right)+\left(\bar{b}\cdot\bar{a}\right)\left(\bar{c}\cdot\bar{d}\right)-\left(\bar{b}\cdot\bar{d}\right)(\bar{c}\cdot$
$\bar{a})+\left(\bar{c}\cdot\bar{b}\right)\left(\bar{a}\cdot\bar{d}\right)-\left(\bar{c}\cdot\bar{d}\right)\left(\bar{a}\cdot\bar{b}\right)=0$

29. Prove that $\left[\left(\bar{a}\times\bar{b}\right)\times(\bar{a}\times\bar{c})\right].\,\bar{d}=\left[\bar{a}\,\bar{b}\,\bar{c}\right]\left(\bar{a}\,.\,\bar{d}\right)$

Solution: LHS $=\left[\left(\bar{a}\times\bar{b}\right)\times(\bar{a}\times\bar{c})\right].\,\bar{d}$

$=\left(\left[\bar{a}\,\bar{b}\,\bar{c}\right]\bar{a}-\left[\bar{a}\,\bar{b}\,\bar{a}\right]\bar{c}\right)\cdot\bar{d}$

$= \left[[\overline{a}\,\overline{b}\,\overline{c}]\overline{a} - 0\right] \cdot \overline{d} \qquad \because [\overline{a}\,\overline{b}\,\overline{a}] = 0$

$= [\overline{a}\,\overline{b}\,\overline{c}]\,(\overline{a}\,.\,\overline{d})$

30. Prove that $(\overline{a} \times \overline{b}) \times (\overline{a} \times \overline{c}) = [(\overline{a} \times \overline{b})\,.\,\overline{c}\,]\overline{a}$

Solution: LHS $= (\overline{a} \times \overline{b}) \times (\overline{a} \times \overline{c})$

$= [\overline{a}\,\overline{b}\,\overline{c}]\overline{a} - [\overline{a}\,\overline{b}\,\overline{a}]\overline{c}$

$= [\overline{a}\,\overline{b}\,\overline{c}]\overline{a} - 0 \qquad \because [\overline{a}\,\overline{b}\,\overline{a}] = 0$

$= [(\overline{a} \times \overline{b}) \cdot \overline{c}]\overline{a}$

31. Prove that $(\overline{b} \times \overline{c}) \times (\overline{a} \times \overline{d}) + (\overline{c} \times \overline{a}) \times (\overline{b} \times \overline{d}) + (\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = -2[\overline{a}\,\overline{b}\,\overline{c}]\,\overline{d}$

Solution: LHS $= (\overline{b} \times \overline{c}) \times (\overline{a} \times \overline{d}) + (\overline{c} \times \overline{a}) \times (\overline{b} \times \overline{d}) + (\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d})$

$= ([\overline{b}\,\overline{c}\,\overline{d}]\overline{a} - [\overline{b}\,\overline{c}\,\overline{a}]\overline{d}) + ([\overline{c}\,\overline{b}\,\overline{d}]\overline{a} - [\overline{a}\,\overline{b}\,\overline{d}]\overline{c}) + ([\overline{a}\,\overline{b}\,\overline{d}]\overline{c} - [\overline{a}\,\overline{b}\,\overline{c}]\overline{d})$

$= ([\overline{b}\,\overline{c}\,\overline{d}]\overline{a} - [\overline{b}\,\overline{c}\,\overline{a}]\overline{d}) + (-[\overline{b}\,\overline{c}\,\overline{d}]\overline{a} - [\overline{a}\,\overline{b}\,\overline{d}]\overline{c}) + ([\overline{a}\,\overline{b}\,\overline{d}]\overline{c} - [\overline{a}\,\overline{b}\,\overline{c}]\overline{d})$

$\therefore$ LHS $= -[\overline{b}\,\overline{c}\,\overline{a}]\overline{d} - [\overline{a}\,\overline{b}\,\overline{c}]\overline{d}$

$= -[\overline{a}\,\overline{b}\,\overline{c}]\overline{d} - [\overline{a}\,\overline{b}\,\overline{c}]\overline{d}$

$= -2[\overline{a}\,\overline{b}\,\overline{c}]\overline{d}$