Convolution Theorem

10 July 2023
13:02

<u>Inverse by convolution theorem</u>:

Definition: If $f_1(t)$ and $f_2(t)$ are two functions then the integrals $\int_0^t f_1(u)f_2(t-u)du$ is called the convolution (= twisting, coiling, winding together) of $f_1(t)$ and $f_2(t)$ and is denoted by $f_1(t)^* f_2(t)$.
Thus $f_1(t)^* f_2(t) = \int_0^t f_1(t)f_2(t-u)du$

Theorem: Let $L|f_1(t)| = \emptyset_1(s)$ and $L|f_2(t)| = \emptyset_2(s)$, then $L^{-1}|\emptyset_1(s).\emptyset_2(s)| = \int_0^t f_1(u).f_2(t-u)du$
Where $f_1(t) = L^{-1}|\emptyset_1(s)|$ and $f_2(t) = L^{-1}|\emptyset_2(s)|$

Procedure of Applying Convolution Theorem:

To find $L^{-1}(\emptyset_1(s).\emptyset_2(s))$

1. Find $L^{-1}|\emptyset_1(s)| = f_1(u)$, say putting u in place to t.
2. Find $L^{-1}|\emptyset_2(s)| = f_2(t-u)$, say putting $(t-u)$ in place to t.
3. Find $L^{-1}|\emptyset_1(s).\emptyset_2(s)| = \int_0^t f_1(u)f_2(t-u)du$

Find the inverse Laplace Transform of the following functions using convolution theorem

(1) $\frac{s^2}{(s^2+a^2)^2}$

<u>Solution</u> :- let $\phi_1(s) = \frac{s}{s^2+a^2}$, $\phi_2(s) = \frac{s}{s^2+a^2}$

$\therefore f_1(t) = L^{-1}[\phi_1(s)] = \cos at$, $f_2(t) = L^{-1}[\phi_2(s)] = \cos at$

$\therefore L^{-1}[\phi(s)] = L^{-1}[\phi_1(s)\cdot\phi_2(s)]$

$= \int_0^t f_1(u) f_2(t-u) du$

$= \int_0^t \cos au \cos a(t-u) du$

$= \frac{1}{2}\int_0^t \cos at + \cos a(2u-t) du$

$= \frac{1}{2}\left[u\cos at + \frac{1}{2a}\sin a(2u-t)\right]_0^t$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{2a}\left(\sin at - (-\sin at)\right)\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}\left[\sin at + at\cos at\right]$$

② $\phi(s) = \dfrac{1}{(s-2)(s+2)^2}$

Solution: $\phi(s) = \dfrac{1}{(s-2)(s+2)^2} = \dfrac{1}{(s+2)^2}\cdot\dfrac{1}{s-2}$

$$= \phi_1(s)\cdot\phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$$

By convolution Theorem

$$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s)\,\phi_2(s)]$$

$$= \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t u e^{-2u}\, e^{2(t-u)}\, du$$

$$= \int_0^t u\, e^{2t-4u}\, du$$

$$= \left[u\left(\frac{e^{2t-4u}}{-4}\right) - (1)\left(\frac{e^{2t-4u}}{16}\right)\right]_0^t$$

$$= \left[t\left(\frac{e^{-2t}}{-4}\right) - \left(\frac{e^{-2t}}{16}\right)\right] - \left[0 - \frac{e^{2t}}{16}\right]$$

$$= \left(\frac{e^{2t} - e^{-2t}}{16}\right) - \frac{te^{-2t}}{4} = \frac{1}{16}\left[e^{2t} - 4te^{-2t} - e^{-2t}\right]$$

---

③ $\phi(s) = \frac{s}{(s^2+1)^2}$

Solution :- $\phi(s) = \frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$

$$= \phi_1(s)\ \phi_2(s)$$

$$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

By convolution theorem,

$$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s) \cdot \phi_2(s)]$$

$$= \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \cos u \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \sin t + \sin(t-2u)\ du$$

$$= \frac{1}{2}\left[u \sin t - \frac{\cos(t-2u)}{-2}\right]_0^t$$

$$= \frac{1}{2}\left[(t\sin t + \cos t) - (0 + \cos t)\right]$$

$$= \frac{t \sin t}{2}$$

④ $\phi(s) = \dfrac{1}{(s-a)(s-b)}$

Solution :- $\phi(s) = \dfrac{1}{(s-a)(s-b)} = \dfrac{1}{s-a} \cdot \dfrac{1}{s-b}$

$= \phi_1(s) \cdot \phi_2(s)$

$f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}$

$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\dfrac{1}{s-b}\right] = e^{bt}$

By convolution theorem,

$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s)\,\phi_2(s)]$

$= \int_0^t f_1(u)\, f_2(t-u)\, du = \int_0^t e^{au} \cdot e^{b(t-u)}\, du$

$= \int_0^t e^{bt+(a-b)u}\, du = \left[\dfrac{e^{bt+(a-b)u}}{(a-b)}\right]_0^t$

$= \dfrac{1}{a-b}\left[e^{at} - e^{bt}\right]$

---

⑤ $L^{-1}\left[\dfrac{s+3}{(s^2+6s+10)^2}\right]$

Solution :- $\phi(s) = \dfrac{s+3}{(s^2+6s+10)^2} = \dfrac{s+3}{[(s+3)^2+1]^2}$

$L^{-1}[\phi(s)] = L^{-1}\left[\dfrac{s+3}{[(s+3)^2+1]^2}\right] = e^{-3t} L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right]$

Now consider $\phi_1(s) = \dfrac{s}{s^2+1}$, $\phi_2(s) = \dfrac{1}{s^2+1}$

$\therefore f_1(t) = L^{-1}[\phi_1(s)] = L^{-1}\left[\dfrac{s}{s^2+1}\right] = \cos t$

$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t$

$$f_2(t) = L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = L^{-1}\left[\phi_1(s)\,\phi_2(s)\right] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t \cos u \,\sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \sin t + \sin(t-2u)\, du$$

$$= \frac{1}{2}\left[u \sin t - \frac{\cos(t-2u)}{-2}\right]_0^t$$

$$= \frac{1}{2}\left[t\sin t + \cos(t) - 0 - \cos t\right]$$

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2} t \sin t$$

$$\therefore L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right] = e^{-3t} L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = e^{-3t}\cdot\frac{1}{2} t\sin t$$

---

6 $$L^{-1}\left[\frac{s^2+5}{(s^2+4s+13)^2}\right]$$

Solution :- $$L^{-1}\left[\frac{s^2+5}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{(s^2+4s+13)-(4s+8)}{(s^2+4s+13)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+4s+13}\right] - 4L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] - 4L^{-1}\left[\frac{s+2}{((s+2)^2+3^2)^2}\right]$$

$$= e^{-2t} L^{-1}\left(\frac{1}{s^2+3^2}\right) - 4e^{-2t} L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right] \quad \text{——— (1)}$$

For solving $L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right]$ convolution theorem can be

For solving $L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right]$, convolution theorem can be used

$$\phi_1(s) = \frac{s}{s^2+3^2} \Rightarrow f_1(t) = L^{-1}[\phi_1(s)] = \cos 3t$$

$$\phi_2(s) = \frac{1}{s^2+3^2} \Rightarrow f_2(t) = L^{-1}[\phi_2(s)] = \frac{1}{3}\sin 3t$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right] = \int_0^t f_1(u) f_2(t-u)\,du = \int_0^t \cos 3u \cdot \frac{1}{3}\sin(3t-3u)\,du$$

$$= \frac{1}{6}\int_0^t \sin 3t + \sin(3t-6u)\,du$$

$$= \frac{1}{6}\left[u\sin 3t + \frac{\cos(3t-6u)}{6}\right]_0^t$$

$$= \frac{1}{6}t\sin 3t$$

Substituting in (1)

$$L^{-1}\left[\frac{s^2+5}{(s^2+6s+13)^2}\right] = e^{-2t}\cdot\frac{1}{3}\sin 3t - 4e^{-2t}\cdot\frac{1}{6}t\sin 3t$$

$$= \frac{1}{3}e^{-2t}\sin 3t - \frac{2}{3}e^{-2t}t\sin 3t$$

$$= \frac{1}{3}e^{-2t}(1-2t)\sin 3t$$

---

(7) $L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right]$

Solution :- $\phi(s) = \frac{1}{s\sqrt{s+4}} = \frac{1}{s}\cdot\frac{1}{\sqrt{s+4}} = \phi_1(s)\cdot\phi_2(s)$

$$f_2(t) = L^{-1}\left[\frac{1}{s}\right] = 1 \qquad f_1(t) = L^{-1}\left[\frac{1}{\sqrt{s+4}}\right] = e^{-4t}L^{-1}\left[\frac{1}{s^{1/2}}\right]$$

$$= e^{-4t}\cdot\frac{t^{-1/2}}{\Gamma(1/2)}$$

$$\therefore L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right] = \int_0^t f_1(u) f_2(t-u)\,du$$

$$= \int_0^t e^{-4u} \frac{u^{-1/2}}{\Gamma 1/2}\, du = \frac{1}{\sqrt{\pi}} \int_0^t e^{-4u} u^{-1/2}\, du$$

put $4u = x^2$, $du = \frac{2x\,dx}{4} = \frac{x\,dx}{2} = \sqrt{u}\,dx$

$\therefore u^{-1/2} du = dx$

$$\therefore L^{-1}\left[\frac{1}{s\sqrt{s+4}}\right] = \frac{1}{\sqrt{\pi}} \int_0^{2\sqrt{t}} e^{-x^2} dx = \frac{1}{2} \operatorname{erf}(2\sqrt{t})$$

$$\left[\because \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx\right]$$

Corollary:- If $\phi(s) = \frac{1}{s} \cdot \phi_1(s)$

then $f_1(t) = L^{-1}[\phi_1(s)]$ & $f_2(t) = L^{-1}\left[\frac{1}{s}\right] = 1$

$$\therefore L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{s}\phi_1(s)\right] = \int_0^t f_1(u)\, du$$

(8) $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Solution:- $L^{-1}\left[\frac{1}{s(s+a)}\right] = L^{-1}\left[\frac{1}{s+a} \cdot \frac{1}{s}\right]$

$\phi_1(s) = \frac{1}{s+a}$ $\qquad f_1(t) = L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$\phi_2(s) = \frac{1}{s}$ $\qquad f_2(t) = L^{-1}\left[\frac{1}{s}\right] = 1.$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t f_1(u) f_2(t-u)\, du = \int_0^t e^{-au} du = \left(\frac{e^{-au}}{-a}\right)_0^t$$

$$= -\frac{1}{a}\left(e^{-at} - 1\right) = \frac{1 - e^{-at}}{a}$$

Ex:- $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$ (H.W) Ans:- $\frac{1-\cos at}{a^2}$

Ex :- $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$ (H.W) Ans :- $\frac{1-\cos at}{a^2}$