ROOT LOCUS METHOD

7.21 ROOT LOCUS

The stability of a closed-loop control system is determined from the location of the roots* of the characteristic equation $1 + G(s)\, H(s) = 0$. For a system to be stable the roots of its characteristic equation should be located in the L.H.S. of s-plane.

A closed-loop control system having forward path gain K is shown by the block diagram in Fig. 7.21.1.

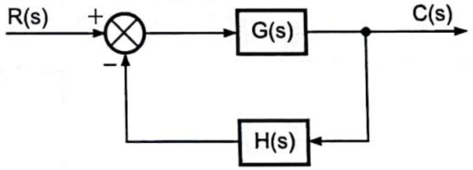


Fig. 7.21.1. Block diagram of closed-loop control system.

The characteristic equation of the system is given by

$$1 + G(s)\, H(s) = 0 \qquad \text{...(7.90)}$$

Any root of Eq. (7.90) satisfies following two conditions :

$$|G(s)\, H(s)| = 1 \qquad \text{...(7.91)}$$

and
$$\angle G(s)\, H(s) = \pm (2k + 1)\, 180°$$

where $k = 0, 1, 2, \ldots$ ...(7.92)

The root locus method of analysis is a process of determining the points in s-plane satisfying Eqs. (7.91) and (7.92). Usually the forward path gain factor K is considered as an independent variable and the roots of $1 + G(s)\, H(s) = 0$ as dependent variables, the roots are plotted in s-plane with K as variable parameter. From the location of the roots in s-plane the nature of time response and system stability can be ascertained.

The root locus for the three examples given below is plotted by determining the roots of the characteristic equation, the variable parameter being K. The information obtained from the root locus plotted as above is used to frame the procedure for plotting the root locus.

*The poles of closed-loop transfer function are also the roots of the characteristic equation.

7.22 SALIENT FEATURES OF ROOT LOCUS PLOT

The salient features as observed from the root locus plots for the examples given in section 7.21.1 are as follows :

1. The root locus starts ($K = 0$) from the open-loop poles and terminates ($K = \infty$) on either finite open-loop zeros or infinity.
2. The number of separate branches of the root locus equals either the number of open-loop poles or number of open-loop zeros whichever is greater.
3. A section of root locus lies on the real axis if the total number of open-loop poles and zeros to the right of the section is odd.
4. The root locus is symmetrical with respect to the real axis.
5. At some point on the root locus between two open-loop poles the root locus breaks away.
6. For higher values of K the root locus can be approximated by asymptotic lines and these asymptotic lines intersect at a point on the real axis.
7. The asymptotic lines make an angle with real axis at an intersection point (centre of gravity).
8. If the root locus intersects the imaginary axis then the points of intersection are conjugate.
9. From the open-loop complex pole the root locus departs making an angle with the horizontal line.

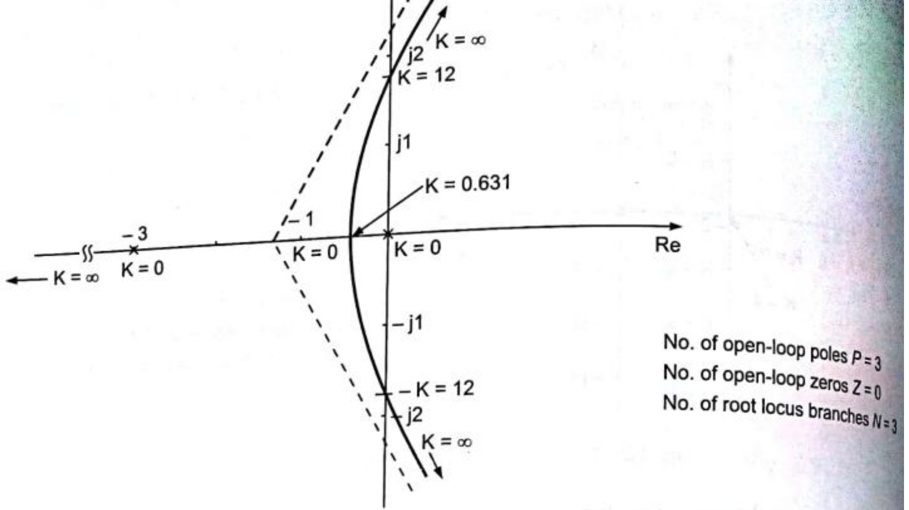
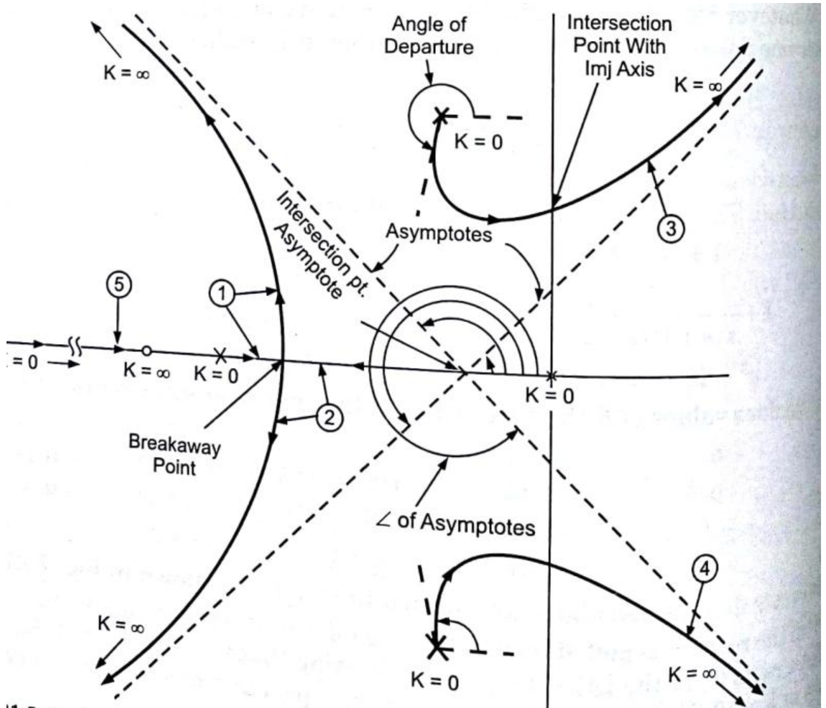


Fig. 7.21.4. Root locus for $G(s)\,H(s) = K/s(s + 1)(s + 3)$.



1.5. Typical root locus plot for a system having Number of open-loop poles $P = 5$ and Number of open-loop zero $Z = 1$. No. of root locus branches $N = 5$.

7.23 THE PROCEDURE FOR PLOTTING ROOT LOCUS

The stepwise procedure for plotting the root locus for a given open-loop transfer function based on salient features mentioned in section 7.22 is given below :

1. **Starting points.** The root locus starts ($K = 0$) from the open-loop poles.

2. **Ending points.** The root locus terminates ($K = \infty$) either on open-loop zero or infinity.

3. **Number of branches.** The number of branches of the root locus are

$$\begin{aligned} N &= P, \quad \text{if } P > Z \\ &= Z, \quad \text{if } Z > P \end{aligned}$$

Usually, $P > Z$, therefore, $N = P$.

4. **Existence on real axis.** The existence of the root locus on a section of real axis i confirmed if the sum of the open-loop poles and zeros to the right of the section is odd.

5. **Breakaway points.** On the root locus between two open-loop poles the roots mov towards each other as the gain factor K is increased till they are coincident. At the coinciden point the value of K is maximum as far as the portion of the root locus between the two open loop poles is concerned. Any further increase in the value of K, breaks the root locus in tw parts. The breakaway point can be determined by rewriting the characteristic equation an therefrom solving for the value of s from the equation given below :

$$\frac{dK}{ds} = 0$$

6. **The angle of asymptotes.** For higher values of K the root locus branches ar approximated by asymptotic lines making an angle with the real axis given by

$$\frac{(2k+1)\times 180°}{P-Z} \quad \text{where } k = 0, 1, 2, \ldots \text{ upto } (P-Z)-1$$

7. **Intersection of asymptotes on real axis.** The asymptotes intersect at a point on the real axis given by

$$x = \frac{\Sigma \text{ Poles} - \Sigma \text{ Zeros}}{P-Z}$$

8. **Intersection points with imaginary axis.** The value of K and the point at which the root locus branch crosses the imaginary axis is determined by applying Routh criterion to the characteristic equation. The roots at the intersection point are imaginary.

9. **The angle of departure from complex pole.** The angle of departure of the root locus from a complex pole is given by

$$\phi_d = 180° - (\phi_p - \phi_z)$$

where ϕ_p is the sum of all the angles subtended by remaining poles.

and ϕ_z is the sum of all the angles subtended by zeros.

The angle of departure is tangent to the root locus at the complex pole.

10. **The angle of arrival at complex zeros.** The angle of arrival of the root locus a a complex zero is given by

$$\phi_a = 180° - (\phi_z - \phi_p)$$

where ϕ_z is the sum of all the angles subtended by remaining zeros.

and ϕ_p is the sum of all the angles subtended by poles.

The angle of arrival is tangent to the root locus at the complex zero.

Example 7.24.1. *A unity feedback control system has an open-loop transfer function*

$$G(s) = \frac{K}{s\,(s+4)}$$

Draw the root locus and determine the value of K, if the damping ratio ζ is to be 0.707.

Solution. 1. The root loci start ($K = 0$) from $s = 0$ and $s = -4$.

2. As there is no open-loop zero root loci terminate ($K = \infty$) at infinity.

3. As the number of poles is 2 the number of root locus branches $N = 2$.

4. The root locus on the real axis exists between $s = 0$ and $s = -4$.

5. Breakaway points.

The characteristic equation is

$$s\,(s + 4) + K = 0 \quad \text{or} \quad K = -(s^2 + 4s)$$

$$\therefore \quad \frac{dK}{ds} = -(2s + 4)$$

Put $$\frac{dK}{ds} = 0 \quad \therefore \quad (2s + 4) = 0$$

$\therefore$ $s = -2$ is a breakaway point.

6. The angle of asymptotes is given by

$$\frac{(2k+1)\,180°}{P-Z} \quad \text{where } k = 0 \text{ and } 1$$

The two asymptotes are at an angle of

(*i*) $$\frac{(2\times 0+1)\,180°}{2-0} = 90°$$

(*ii*) $$\frac{(2\times 1+1)\,180°}{2-0} = 270°$$

7. The asymptotes intersect on real axis at a point given by

$$x = \frac{\Sigma\,\text{Poles} - \Sigma\,\text{Zeros}}{P-Z} = \frac{(0-4)-0}{2} = -2$$

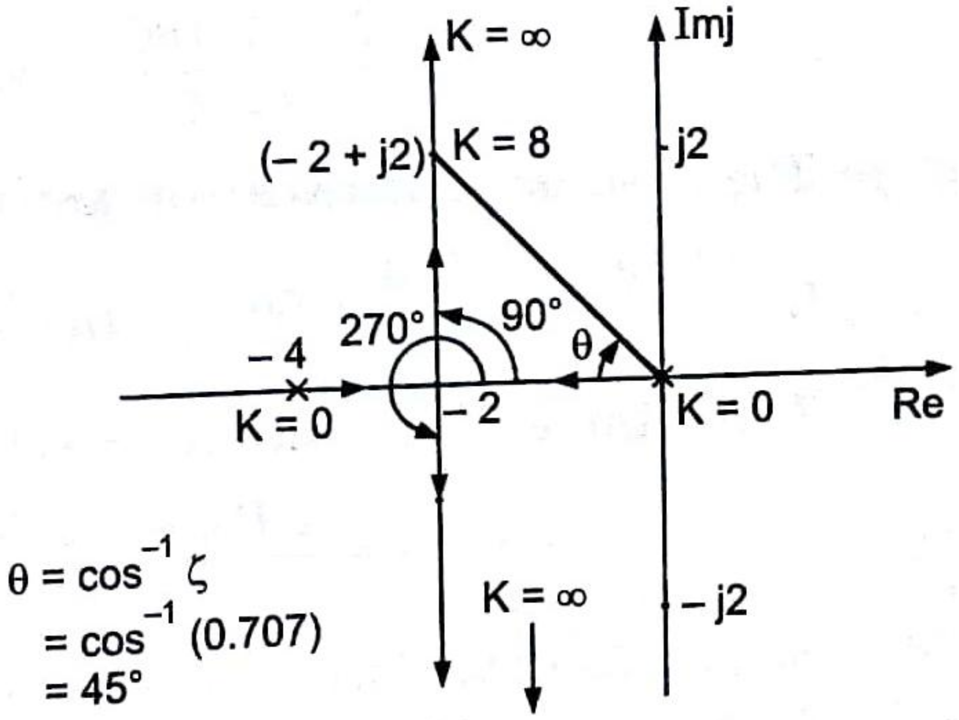


Fig. 7.24.1. Root locus for $G(s)\,H(s) = K/s(s+4)$.

Example 7.24.2. *Sketch the root locus for the open-loop transfer function of a unity feedback control system given below and determine*

(i) The value of K for $\zeta = 0.5$ (ii) the value of K for marginal stability (iii) the value of K at $s = -4$ and obtain the closed-loop transfer function for $K = 1.66$.

$$G(s) = \frac{K}{s\,(s+1)\,(s+3)}.$$

Solution. 1. The root loci start ($K = 0$) at $s = 0$, $s = -1$, $s = -3$.

2. As there is no open-loop zero the root loci terminate at infinity.

3. $P = 3$, $Z = 0$ $(P > Z)$, therefore, number of branches of the root locus $N = 3$.

4. The root locus on the real axis exists between $s = 0$ and $s = -1$ and beyond $s = -3$.

5. Between open-loop poles $s = 0$ and $s = -1$, there exists a breakaway point as calculated below :

The characteristic equation is

$$s\,(s+1)\,(s+3) + K = 0$$

or $$K = -(s^3 + 4s^2 + 3s)$$

$\therefore$ $$\frac{dK}{ds} = -(3s^2 + 8s + 3)$$

Put $$\frac{dK}{ds} = 0$$

$\therefore$ $$3s^2 + 8s + 3 = 0$$

or $$s = -1.33 \pm 0.88$$

$\therefore$ $$s_1 = -0.45 \text{ and } s_1 = -2.21$$

As the breakaway point has to lie between $s = 0$ and $s = -1$, the breakaway point is $s = -0.45$.

6. The angle of asymptotes is given by

$$\frac{(2k+1)\,180°}{P-Z} \quad \text{where } k = 0, 1 \text{ and } 2$$

The three asymptotes are at any angle of

$$(i)\ \frac{(2\times0+1)\times180°}{(3-0)} = 60° \qquad (ii)\ \frac{(2\times1+1)\times180°}{(3-0)} = 180° \qquad (iii)\ \frac{(2\times2+1)\times180°}{(3-0)} = 300°$$

7. The three asymptotes intersect on the real axis at

$$x = \frac{\Sigma\,\text{Poles} - \Sigma\,\text{Zeros}}{P-Z} = \frac{(0-1-3)-(0)}{3-0} = -1.33.$$

As per the data calculated above the root locus plot is sketched in Fig. 7.24.2.

8. The value of K for marginal stability is determined by applying Routh criterion to the characteristic equation.

$$s\,(s+1)\,(s+3) + K = 0 \quad \text{or} \quad s^3 + 4s^2 + 3s + K = 0$$

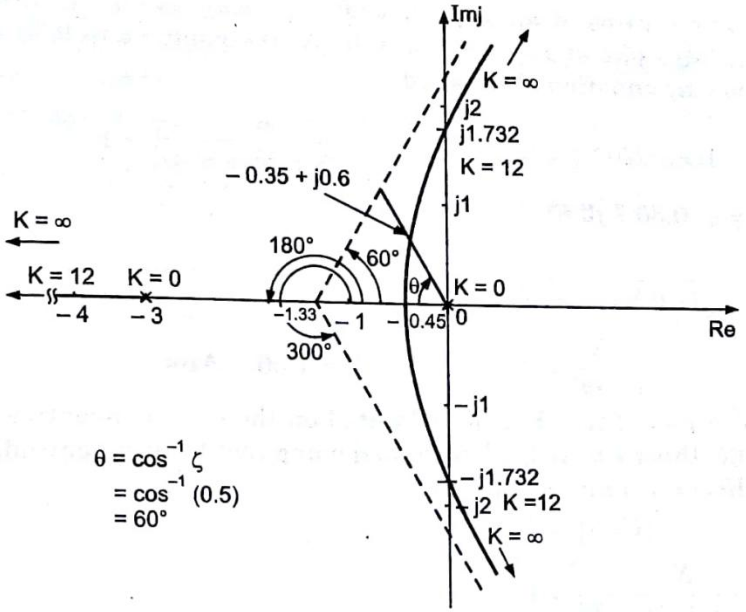


Fig. 7.24.2. Root locus for $G(s) = K/s(s+1)(s+3)$.

The Routh array is tabulated below

s^3	1	3
s^2	4	K
s^1	$\dfrac{(12-K)}{4}$	
s^0	K	

The value of K at which the root locus plot crosses the imaginary axis is determined by equating the first term in s^1 row to zero, therefore,

$$\frac{(12-K)}{4} = 0 \qquad \therefore \quad K = 12$$

(*i*) For $K > 12$ the roots lie in the R.H.S. of s-plane, hence $K = 12$ is the value for marginal stability. **Ans.**

At $K = 12$ the root locus plot intersects the imaginary axis and the value of s at the intersection point is determined by solving auxiliary equation formed from the s^2 terms in Routh array, therefore,

$$4s^2 + K = 0 \quad \text{or} \quad 4s^2 + 12 = 0 \qquad \therefore \quad s = \pm j1.73$$

At $K = 12$ the system exhibits sustained oscillations, the frequency of these oscillations is given by the intersection point of the root locus plot with the imaginary axis, therefore, the frequency of sustained oscillations is $\omega = 1.73$ rad/sec.

Example 7.24.3. *The open-loop transfer function of a control system is given by*

$$G(s)\,H(s) = \frac{K}{s\,(s+6)\,(s^2+4s+13)}$$

Sketch the root locus and determine : (a) The breakaway points, (b) The angle of departure from complex poles and (c) The stability condition.

Solution. The four open-loop poles $P = 4$ are

$$s = 0,\ s = -6,\ s = (-2 + j3) \text{ and } s = (-2 - j3)$$

The number of open-loop zeros $Z = 0$.

The number of root locus branches $N = P = 4$. On the real axis the root locus lies between two poles $s = 0$ and $s = -6$.

The angle of asymptotes is given by

$$\frac{(2k+1)\times 180°}{P - Z} \qquad \text{where } k = 0, 1, 2, \text{ and } 3$$

The four angles of asymptotes are

(i) $\dfrac{(2\times 0+1)\times 180°}{4-0} = 45°$ (ii) $\dfrac{(2\times 1+1)\times 180°}{4-0} = 135°$

(iii) $\dfrac{(2\times 2+1)\times 180°}{4-0} = 225°$ (iv) $\dfrac{(2\times 3+1)\times 180°}{4-0} = 315°$

The asymptotes intersect on the real axis at

$$x = \frac{\Sigma \text{ Poles} - \Sigma \text{ Zeros}}{P - Z} = \frac{(0 - 6 - 2 + j3 - 2 - j3) - 0}{4 - 0} = -\frac{10}{4} = -2.5$$

The characteristic equation is

$$s\,(s+6)\,(s^2+4s+13) + K = 0$$

or $$K = -(s^4 + 10s^3 + 37s^2 + 78s)$$

$$\therefore \quad \frac{dK}{ds} = -(4s^3 + 30s^2 + 74s + 78)$$

Put $$\frac{dK}{ds} = 0 \quad \therefore \quad 4s^3 + 30s^2 + 74s + 78 = 0$$

The breakaway point on the real axis is given by the real root of the above equation, the real root is determined approximately as follows :

Let $$f(s) = 4s^3 + 30s^2 + 74s + 78$$

It is known that the breakaway point on the real axis lies somewhere between 0 and – 6.

The intersection of the root locus plot with the imaginary axis is determined by applying Routh criterion to the characteristic equation,

i.e. $$s^4 + 10s^3 + 37s^2 + 78s + K = 0$$

The Routh array is arranged below :

s^4	1	37	K
s^3	10	78	
s^2	29.2	K	
s^1	$(78 - 0.34\,K)$		
s^0	K		

equation is satisfied ...

The angle of departure from the pole $(-2+j3)$ is calculated below (Refer ...

$$\phi_{(-2+j3)} = 180° - (\phi_{p1} + \phi_{p2} + \phi_{p3})$$

$$\phi_{p1} = 180° - \tan^{-1}\left(\frac{3}{2}\right) = 123.6°$$

$$\phi_{p2} = \tan^{-1}\left(\frac{3}{4}\right) = 36.86°$$

$$\phi_{p3} = 90°$$

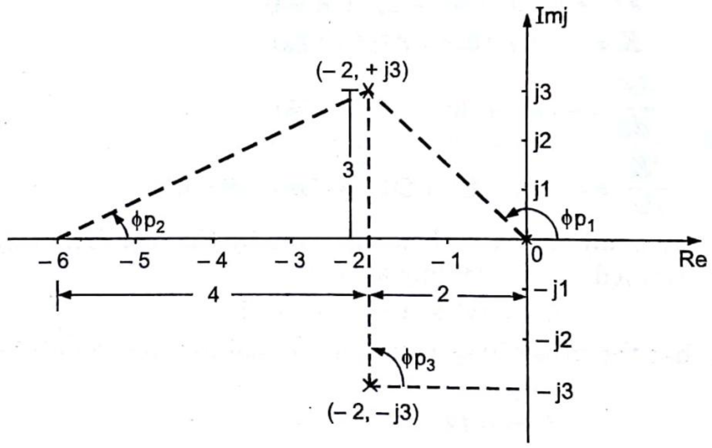


Fig. 7.24.4. Pole-zero configuration for $G(s)\,H(s) = K/s(s+6)(s^2+4s+13)$.

$$\phi_{(-2+j3)} = 180° - (123.6° + 36.86° + 90°) = -70.46°. \quad \textbf{Ans.}$$

$$\phi_{(-2-j3)} = +70.46°. \quad \textbf{Ans.}$$

The intersection of the root locus plot with the imaginary axis is given by the value of K obtained by solving following equation :

$$78 - 0.34\,K = 0$$

$$\therefore \qquad K = 229.4$$

The intersection point is determined by solving auxiliary equation for s^2 term

$$\therefore \qquad 29.2\,s^2 + K = 0$$

or $$29.2\,s^2 + 229.4 = 0$$

or $$s = \pm j2.98$$

The root locus plot is shown in Fig. 7.24.5.

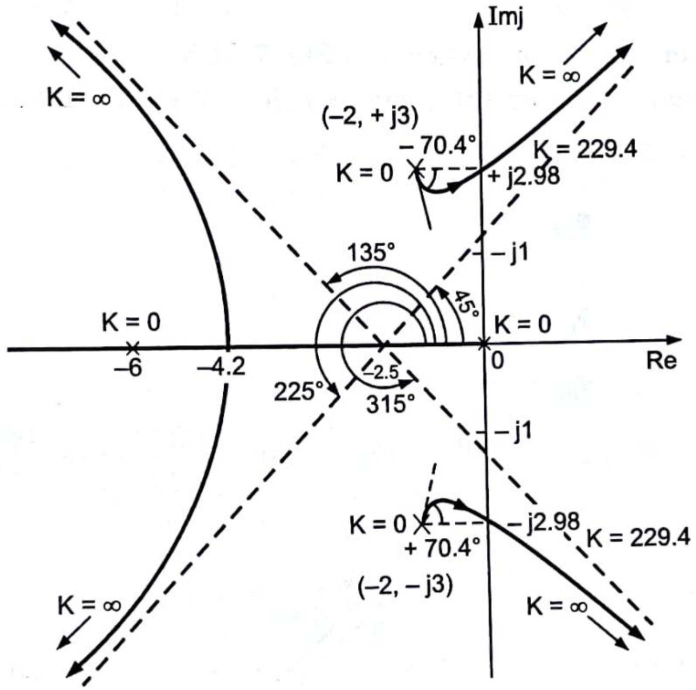


Fig. 7.24.5. Root locus plot for $G(s)\,H(s) = K/s(s+6)(s^2+4s+13)$.

From the root locus plot (Fig. 7.24.5) it is noted that for $K > 229.4$ the two roots have positive real part, therefore, the system is stable if,

$$K < 229.4. \quad \textbf{Ans.}$$

Example 7.24.4. *Sketch the root locus plot for the system having open-loop transfer function is given by*

$$G(s)\,H(s) = \frac{K}{s\,(s+4)\,(s^2+4s+13)}.$$

Solution. The number of open-loop poles $P = 4$, $s = 0$, $s = -4$, $s = (-2 + j3)$ and $s = (-2 - j3)$

The number of open-loop zeros $Z = 0$

Therefore, the number of branches of root locus $N = P = 4$

The four angles of asymptotes are given by $\dfrac{(2k+1)\times 180°}{P - Z}$, $k = 0, 1, 2,$ and 3

(*i*) $\dfrac{(2\times 0+1)+180°}{4-0} = 45°$ (*ii*) $\dfrac{(2\times 1+1)\times 180°}{4-0} = 135°$

(*iii*) $\dfrac{(2\times 2+1)\times 180°}{4-0} = 225°$ (*iv*) $\dfrac{(2\times 3+1)\times 180°}{4-0} = 315°$

The asymptotes intersect on the real axis at

$$x = \frac{\Sigma \text{ Poles} - \Sigma \text{ Zeros}}{P - Z} = \frac{(0-4-2+j3-2-j3)-(0)}{4-0} = \frac{-8}{4} = -2$$

The pole-zero configuration is shown in Fig. 7.24.6.

The angle of departure from the complex pole $(-2 + j3)$ is given by (Refer Fig. 7.24.6)

$$\phi_{(-2+j3)} = 180° - (\phi_{p1} + \phi_{p2} + \phi_{p3})$$

$$\phi_{p1} = 180° - \tan^{-1}\left(\frac{3}{2}\right)$$

$$\phi_{p2} = \tan^{-1}\left(\frac{3}{2}\right)$$

$$\phi_{p3} = 90°$$

$$\therefore \quad \phi_{(-2+j3)} = 180° - \left[180° - \tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{3}{2}\right) + 90°\right] = -90°$$

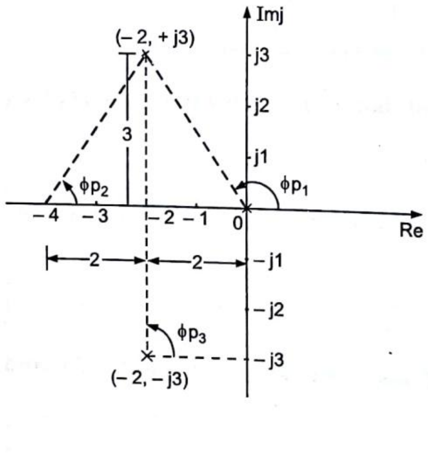


Fig. 7.24.6. Pole-zero configuration for $G(s)\,H(s) = K/s(s+4)(s^2+4s+13)$.

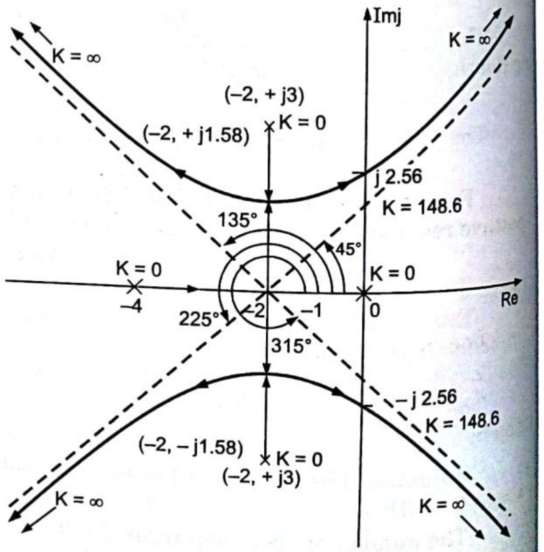


Fig. 7.24.7. Root locus plot for $G(s)\,H(s) = K/s(s+4)(s^2+4s+13)$.

and the angle of departure from the complex pole $(-2 - j3)$ is

$$\phi_{(-2-j3)} = +90°.$$

The root locus plot takes the shape as shown in Fig. 7.24.7. There are three breakaway points. One of the breakaway point is identified at $s = -2$, the other two breakaway points are complex and can be obtained as follows :

The characteristic equation is

$$s(s+4)\,(s^2+4s+13) + K = 0$$

or

$$K = -(s^4 + 8s^3 + 29s^2 + 52s)$$

$$\therefore \quad \frac{dK}{ds} = (-4s^3 + 24s^2 + 58s + 52)$$

Put $\dfrac{dK}{ds} = 0$ $\quad \therefore \quad (4s^3 + 24s^2 + 58s + 52) = 0$

The breakaway points are given by the roots of the above equation, one of the root is identified as $s = -2$ (at $s = -2$ above equation is satisfied), therefore,

$$\frac{(4s^3 + 24s^2 + 58s + 52)}{(s+2)} = (4s^2 + 16s + 26)$$

and thus

$$(s+2)\,(4s^2 + 16s + 26) = 0$$

The two complex breakaway points are given by the roots of the equation $(4s^2 + 16s + 26) = 0$, therefore the complex breakaway points are

$$s = \frac{-16 \pm \sqrt{(16^2 - 4\times 4\times 26)}}{2\times 4} = (-2 \pm j1.58)$$

The intersection of the root locus plot with the imaginary axis is determined by applying Routh criterion to the characteristic equation

$$s^4 + 8s^3 + 29s^2 + 52s + K = 0$$

The Routh array is arranged below

s^4	1	29	K
s^3	8	52	
s^2	22.5	K	