2.2 MATHEMATICAL ANALYTICAL TOOLS

Laplace transformation is a method that allows the solutions of linear differential equations to be obtained without complexity.

Laplace transform simplifies the mathematical model of a complex continuous time (analog) control system. A similar conversion technique that converts the discrete time control system model (digital control system) domain equations/model into algebraic equations is z-transform.

The Laplace transformation of a control system mathematically offers the following advantages:

(i) *Ease in finding mathematical solutions*
(ii) *Frequency domain based performance analysis*

2.2.1 Laplace transform technique

The Laplace transform of a time domain function $f(t)$ is given by

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)\, e^{-st}\, dt; \quad t \geq 0 \tag{2.8}$$

Here, the notation L is read as 'Laplace transform of' and s is a complex number given as $s = (\sigma + jw)$. The functions $f(t)$ and $F(s)$ constitute a Laplace transform pair. Similar to Fourier transform, the condition for Laplace transform to exist is

$$\int_{-\infty}^{\infty} | f(t)\, e^{-\sigma t} |\, dt < \infty \tag{2.9}$$

With same finite σ, which is a real number, large enough to ensure absolute convergence. As shown in Eq. (2.8), Laplace transformation thus changes the time domain $f(t)$ to frequency domain $F(s)$. The time domain function $f(t)$ can be obtained back from the frequency domain function $F(s)$ (Laplace transformed function) by the process of inverse Laplace transformation denoted by L^{-1}, as given below.

$$L^{-1}\,[Lf(t)] = L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)\, e^{+st}\, ds \tag{2.10}$$

Here, σ is a real number such that the contour path of integration is in the region of convergence (ROC) of $F(s)$, normally requiring $\sigma > Re(s_p)$ for every singularity s_p of $F(s)$.

The various commonly used control system input signals, like ramp, parabolic functions, etc., cannot be handled by Fourier transform as the integral, shown in Eq. (2.9), is not converging for the above input signals and are not Fourier transformable. Whereas because of convergence factor $e^{-\sigma t}$, the above input functions are Laplace transformable.

Some important Laplace transforms used in control system analysis are given below:

1. $L[f_1(t) + f_2(t) + f_3(t) + \cdots] = L[f_1(t)] + L[f_2(t)] + L[f_3(t)] + \cdots$
$= F_1(s) + F_2(s) + F_3(s) + \cdots$

2. $L[Kf(t)] = K[f(t)] = KF(s)$; Laplace operator obeys linearity.

3. $L[f_1(t) \times f_2(t)] \neq F_1(s) \times F_2(s)$; The multiplication of two or more Laplace functions is not the simple multiplication of two time-domain functions and then their Laplace transforms. It is instead equal to the convolution of two time-domain functions.
4. For differential functions:

(i) $L\,\dfrac{df(t)}{dt} = [sF(s) - f(0^+)]$

(ii) $L\,\dfrac{d^2 f(t)}{dt^2} = [s^2F(s) - s\,f(0^+) - f'(0^+)]$

(iii) $L\,\dfrac{d^3 f(t)}{dt^3} = [s^3F(s) - s^2f(0^+) - s\,f'(0^+) - f''(0^+)]$

and so on

$f'(0^+)$ and $f''(0^+)$ are the functional values $\dfrac{df(t)}{dt}$ and $\dfrac{d^2f(t)}{dt^2}$, respectively, at $t = 0^+$.

5. For integral functions

(i) $L \int f(t)\,dt = \left[\dfrac{F(s)}{s} + \dfrac{f^{-1}(0^+)}{s}\right]$

(ii) $L \iint f(t)\,dt = \left[\dfrac{F(s)}{s^2} + \dfrac{f^{-1}(0^+)}{s^2} + \dfrac{f^{-2}(0^+)}{s}\right]$

(iii) $L \iiint f(t)\,dt = \left[\dfrac{F(s)}{s^3} + \dfrac{f^{-1}(0^+)}{s^3} + \dfrac{f^{-2}(0^+)}{s^2} + \dfrac{f^{-3}(0^+)}{s}\right]$

and so on where $f^{-i}(t) = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{i \text{ times}} f(t)\,dt$

6. Initial value theorem:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,L[f(t)] = \lim_{s \to \infty} sF(s)$$

7. Final value or steady-state value theorem:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,L[f(t)] = \lim_{s \to 0} sF(s)$$

The final value theorem is valid only if $\lim_{x \to \infty} f(t)$ exists.

Proof of Laplace transform of differential functions

The time domain function in this case is $\dfrac{df(t)}{dt}$. The Laplace transform of a time-domain function $f(t)$ is given by

Laplace Transform of Some Commonly Used Control System Input Signals

(i) Impulse function

It is defined as

$$f(t) = \lim_{t_0 \to 0} \frac{K}{t_0} \quad \text{for } 0 < t < t_0$$

(where, k is a constant)

$$= 0 \quad \text{for } t < 0,\ t_0 < t$$

In the above case, its Laplace transform is given by

$$L[f(t)] = \lim_{t_0 \to 0} \frac{K}{t_0 s}(1 - e^{-st_0})$$

$$= \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}\left[K(1-e^{-st_0})\right]}{\frac{d}{dt_0}(t_0 s)}$$

$$= \frac{Ks}{s} = K$$

The Laplace transform of an impulse function is equal to the area under the impulse.

(ii) Step function:

It is defined as

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= K \text{ (some constant)} && \text{for } t > 0 \end{aligned}$$

Taking the Laplace transform of the above function

$$L[f(t)] = \int_0^\infty Ke^{-st}\, dt = \frac{K}{s}$$

(iii) Ramp function:

It is defined as

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= Kt && \text{for } t \geq 0; \quad \text{where } K \text{ is a constant} \end{aligned}$$

Taking the Laplace transform of the above function

$$\begin{aligned} L[f(t)] &= K\int_0^\infty t\, e^{-st}\, dt \\ &= Kt\frac{e^{-st}}{s}\Bigg|_0^\infty - \int_0^\infty K\frac{e^{-st}}{-s}\, dt \\ &= \frac{K}{s}\int_0^\infty e^{-st}\, dt \\ &= \frac{K}{s^2} \end{aligned}$$

(iv) Parabolic function

It is defined as

$$\begin{aligned} f(t) &= 0 \text{ for } t < 0 \\ &= \frac{Kt^2}{2} \text{ for } t \geq 0 \quad \text{(unit parabolic function)} \end{aligned}$$

Here $$\int_0^\infty f(t)\, e^{-st} dt = \int_0^\infty \frac{Kt^2}{2} e^{-st} dt = \frac{K}{2}\int_0^\infty t^2\, e^{-st} dt \quad (2.24)$$

Taking $t^2/2$ as the first term and e^{-st} as the second term and integrating it by parts, we get

$$\int_0^{\infty} f(t)e^{-st}\,dt = \frac{K}{2}\left[\left[\frac{-t^2e^{-st}}{s}\right]_0^{\infty} + \frac{2}{s}\int_0^{\infty} t\,e^{-st}dt\right] \tag{2.25}$$

Taking t as the first term and e^{-st} as the second term in the integral on RHS and integrating by parts we get

$$\int_0^{\infty} f(t)e^{-st}\,dt = \frac{K}{2}\left[0 + \frac{2}{s}\left(\frac{-t\,e^{-st}}{s}\Bigg|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}\,dt\right)\right] \tag{2.26}$$

or

$$\int_0^{\infty} f(t)e^{-st}\,dt = \frac{K}{2}\times\frac{2}{s}\left[-\frac{1}{s^2}\,e^{-st}\Big|_0^{\infty}\right]$$

or

$$\int_0^{\infty} f(t)e^{-st}\,dt = \frac{K}{s^3} \tag{2.27}$$

Alternatively, $$L\left[\int_0^{\infty} f(t)\,e^{-st}\,dt\right] = \frac{K}{2}\int_0^{\infty} t^2\,e^{-st}\,dt$$

$$= \frac{K}{2}\left[\frac{e^{-st}}{s}t^2\Bigg|_0^{\infty}\right] + \frac{1}{s}\int_0^{\infty} 2t\,e^{-st}\,dt$$

$$= \frac{K}{s}\left[\frac{e^{-st}}{s}t\Bigg|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}\,dt\right]$$

$$= \frac{K}{s^2}\left[\frac{e^{-st}}{-s}\Bigg|_0^{\infty}\right] = \frac{K}{s^3}$$

(v) Exponential function

It is defined as

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= Ke^{-at} && \text{for } t \geq 0 \end{aligned}$$

Where, K and a are constants.

The Laplace transform of the above function is given as

$$L[f(t)] = \int_0^{\infty} K\,e^{-at}\,e^{-st}\,dt = K\int_0^{\infty} e^{-(a+s)t}\,dt$$

$$= \frac{K}{(s+a)}$$

Table 2.1

$f(t)$	$F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{-at}	$\dfrac{1}{(s+a)}$
e^{at}	$\dfrac{1}{(s-a)}$
$e^{-at} f(t)$	$F(s+a)$
te^{-at}	$\dfrac{1}{(s+a)^2}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\dfrac{\omega}{(s^2+\omega^2)}$
$\cos \omega t$	$\dfrac{s}{(s^2+\omega^2)}$
$e^{-at} \sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\dfrac{(s+a)}{(s+a)^2+\omega^2}$
$\sinh at$	$\dfrac{a}{(s^2-a^2)}$
$\cosh at$	$\dfrac{s}{(s^2-a^2)}$

Example 2.7 Solve the following differential equations:

(i) $4\dfrac{dx}{dt} + 3x = 10;\quad x(0^+) = 1$

(ii) $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 10x = 0;\quad x(0^+) = 0$ and $x'(0^+) = 1$

Solution

(i) Taking Laplace transform of both sides, we get

$$4[sX(s) - x(0^+)] + 3X(s) = \frac{10}{s} \tag{2.34}$$

Putting the value of $x(0^+)$ and rearranging Eq. (2.34), we get

$$X(s) = \frac{\left(\dfrac{10}{s} + 4\right)}{(4s+3)} \tag{2.35}$$

or

$$X(s) = \frac{10}{s(4s+3)} + \frac{4}{(4s+3)} \tag{2.36}$$

or

$$X(s) = \frac{10}{4s(s+3/4)} + \frac{1}{(s+3/4)} \tag{2.37}$$

Taking inverse Laplace transform of Eq. (2.37), we get

$$x(t) = L^{-1}[X(s)] = \underbrace{L^{-1}\left[\frac{10}{4s(s+3/4)}\right]}_{\text{Part I}} + \underbrace{L^{-1}\left[\frac{1}{(s+3/4)}\right]}_{\text{Part II}} \tag{2.38}$$

Taking Part I first,

$$L^{-1}\left[\frac{10}{4s(s+3/4)}\right] = L^{-1}\left[\frac{10}{4}\left(\frac{A}{s} + \frac{B}{(s+3/4)}\right)\right]$$

$$= L^{-1}\left[\frac{10}{4}\left(\frac{4}{3s} - \frac{4}{3} \times \frac{1}{(s+3/4)}\right)\right]$$

$$= \frac{10}{4} \times \frac{4}{3} - \frac{10}{4} \cdot \frac{4}{3} \cdot e^{-(3/4)t}$$

$$= (10/3)[1 - e^{-(3/4)t}]$$

Taking Part II from Eq. (2.38) for inverse Laplace transform, we have

$$L^{-1}\left[\frac{1}{(s+3/4)}\right] = e^{-(3/4)t}$$

Putting both the inverse Laplace transforms together, we have

$$x(t) = \left(\frac{10}{3} - \frac{7}{3}e^{-(3/4)t}\right)$$

(ii) Taking the Laplace transform of the given time-domain (differential) equation, we get

$$[s^2X(s) - sx(0^+) - x'(0^+)] + [10sX(s) - 10x(0^+)] + 10X(s) = 0$$

Substituting the given values of initial conditions and then rearranging, we have

$$X(s) = \frac{1}{(s^2 + 10s + 10)} \tag{2.39}$$

Modifying the expression shown in Eq. (2.39), we get

$$X(s) = \frac{1}{s^2 + 10s + (5)^2 - 15}$$

$$= \frac{1}{\sqrt{15}}\left(\frac{\sqrt{15}}{(s+5)^2 - (\sqrt{15})^2}\right) \tag{2.40}$$

Taking inverse Laplace transform of Eq. (2.40), we get

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{1}{\sqrt{15}}\left(\frac{\sqrt{15}}{(s+5)^2 - (\sqrt{15})^2}\right)\right] = \frac{1}{\sqrt{15}}\, e^{-5t} \sin h\, \sqrt{15}\, t$$

Example 2.8 ***Solve the differential equation given below:***

$$2\frac{dx}{dt} + 8x = 10e^t;\ \text{given } x(0^+) = 2$$

Example 2.10 *A series circuit consisting of resistance R and a capacitor C is connected to a DC supply voltage of V volts. Find the initial value of the current flowing in the circuit.*

Solution

Applying KVL to the given circuit, we get

$$Ri(t) + \frac{1}{C}\int i(t)\,dt = V \tag{2.56}$$

Taking Laplace transform of Eq. (2.56), we get

$$RI(s) + \left(\frac{I(s)}{Cs} + \frac{i^{-1}(0^+)}{Cs}\right) = \frac{V}{s} \tag{2.57}$$

Assuming $i^{-1}(0^+) = 0$, we find

$$RI(s) + \frac{1}{Cs}\,I(s) = \frac{V}{s} \tag{2.58}$$

or

$$I(s) = \frac{V.Cs}{s(RCs+1)} \tag{2.59}$$

Applying the initial value theorem to Eq. (2.59), for initial value of the current, we get

$$i(0^+) = \lim_{t\to 0} i(t) = \lim_{s\to\infty} s\,I(s) = \lim_{s\to\infty} s \times \left(\frac{V.Cs}{s(CRs+1)}\right)$$

$$= \lim_{s\to\infty}\left(\frac{V.C}{(CR+1/s)}\right)$$

$$= \frac{VC}{CR} = \frac{V}{R} \text{ amperes}$$

Example 2.11 *The Laplace-transformed expression for series current passing through a series circuit consisting of resistance RΩ, an inductance L henry, and a DC supply voltage of V volts is given by*

$$I(s) = \frac{1}{(R+Ls)} \cdot \frac{V}{s}$$

The component values are $V = 100$ volts, $R = 1$ MΩ, and $L = 0.1$ mH. Calculate the steady-state value of the series current flowing in the circuit.

Solution

It is given that

$$I(s) = \frac{1}{(R+Ls)} \cdot \frac{V}{s} \tag{2.60}$$

Applying the final value theorem to Eq. (2.60), we have

$$i_{ss} = \lim_{t\to\infty} i(t) = \lim_{s\to 0} sI(s)$$

$$= \lim_{s\to 0} s \cdot \frac{1}{(R+Ls)} \cdot \frac{V}{s}$$

$$= \lim_{s\to 0} \cdot \frac{V}{(R+Ls)} = \frac{V}{R}$$

Putting the values of V and R as given, we get

$$i_{ss} = \frac{100}{1\times 10^6} = 100\ \mu$$

or

$$L[f(t)] = -\log\left(1+\frac{1}{s}\right) = -\log\left(\frac{s+1}{s}\right) = \log\left(\frac{s}{s+1}\right)$$

Q. 2.3 **Find the inverse Laplace transform of the following frequency domain function:**

$$F(s) = \frac{s^2+3s+4}{s^3+9s^2+27s+27}$$

Solution

Here, the denominator can be expressed as $(s+3)^3$.

So,

$$F(s) = \frac{s^2+3s+4}{(s+3)^3}$$

Using partial fraction expansion, we get

$$F(s) = \frac{k_1}{(s+3)} + \frac{k_2}{(s+3)^2} + \frac{k_3}{(s+3)^3}$$

or $$\frac{s^2+3s+4}{(s+3)^3} = \frac{k_1}{(s+3)} + \frac{k_2}{(s+3)^2} + \frac{k_3}{(s+3)^3}$$

or $$s^3+3s+4 = k_1(s+3)^2 + k_2(s+3) + k_3 = k_1 s^2 + (6k_1 + k_2)s + (9k_1 + 3k_2 + k_3)$$

Equating coefficients of s^2, we get

$$k_1 = 1$$

Similarly, equating coefficients of s, we get

$$k_2 = 3 - 6 = -3$$

and lastly equating constant terms, we get

$$k_3 = 4 - 9 + 9 = 4$$

Thus $$F(s) = \frac{1}{s+3} - \frac{3}{(s+3)^2} + \frac{4}{(s+3)^3}$$

Now, taking its inverse Laplace transform on both sides, we have

$$L^{-1}F(s) = L^{-1}\left[\frac{1}{(s+3)} - \frac{3}{(s+3)^2} + \frac{4}{(s+3)^3}\right]$$

or $$L^{-1}F(s) = L^{-1}\frac{1}{s+3} - L^{-1}\frac{3}{(s+3)^2} + L^{-1}\frac{4}{(s+3)^3}$$

or $$f(t) = e^{-3t} - 3te^{-3t} + \frac{4}{3}t^2 e^{-3t}$$

Finally, $$f(t) = e^{-3t}\left[1 - t(3 - \frac{4}{3}t)\right]$$

Q. 2.4 Find Laplace transform and then find time solution of the following differential equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 6e^t$$

Here, $y(0^+) = 0$ and $y'(0^+) = 4$.

Solution

Taking Laplace transform on both sides, we get

$$L\left[\frac{d^2y}{dt^2}\right] + L\left[\frac{3dy}{dt}\right] + L[2y] = L[6e^t]$$

or $$[s^2Y(s) - sy(0^+) - y'(0^+)] + 3[sY(s) - y(0^+)] + 2[Y(s)] = \frac{6}{s-1}$$

or $$[s^2Y(s) - 0 - 4] + 3[sY(s) - 0] + 2Y(s) = \frac{6}{s-1}$$

[Substituting $y(0^+) = 0$ and $y'(0^+) = 4$]

or
$$(s^2 + 3s + 2)Y(s) - 4 = \frac{6}{s-1}$$

or
$$Y(s) = \frac{\frac{6}{s-1} + 4}{(s^2 + 3s + 2)} = \frac{6 + 4s - 4}{(s-1)(s+2)(s+1)}$$

$$= \frac{4s+2}{(s-1)(s+2)(s+1)}$$

Further, obtaining partial fractions expansion,

$$Y(s) = \frac{4s+2}{(s-1)(s+1)(s+2)} = \frac{k_1}{s-1} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

Also $\quad 4s + 2 = k_1(s + 1)(s + 2) + k_2(s - 1)(s + 2) + k_3(s^2 - 1)$

or $\quad 4s + 2 = k_1(s^2 + 3s + 2) + k_2(s^2 + s - 2) + k_3(s^2 - 1)$

Now, equating coefficients of s^2, we get

$$0 = k_1 + k_2 + k_3 \tag{2.93}$$

Similarly, equating coefficients of s, we get

$$4 = 3k_1 + k_2 \tag{2.94}$$

and lastly equating constants, we get $2 = 2k_1 - 2k_2 - k_3$ (2.95)

From Equations (2.93) and (2.95), we get

$$0 = k_1 + k_2 + k_3$$
$$2 = 2k_1 - 2k_2 - k_3$$
$$\overline{2 = 3k_1 - k_2} \tag{2.96}$$
$$4 = 3k_1 + k_2 \tag{2.97}$$
$$\overline{6 = 6k_1 \Rightarrow k_1 = 1}$$

Thus, from Eq. (2.96), we get

$$k_2 = 4 - 3k_1 = 4 - 3 = 1$$

From Eq. (2.95), we get

$$k_3 = -(k_1 + k_2) = -(1 + 1) = -2$$

Finally,
$$Y(s) = \frac{1}{s-1} + \frac{1}{s+1} - \frac{2}{s+2}$$

Now, taking its inverse Laplace transform on both sides, we get

$$L^{-1}Y(s) = L^{-1}\left[\frac{1}{s-1} + \frac{1}{s+1} - \frac{2}{s+2}\right]$$

$$y(t) = e^{t} + e^{-t} - 2e^{-2t}$$

INTRODUCTION

Transfer function and gain describe the input–output relationship(s) of a given control system. This relationship may be helpful in analysing the behaviour of control systems. The above relationship(s) will include the effect of dependency of one variable on another along with the contribution by system parameters.

3.1 TRANSFER FUNCTION

Transfer function is defined as the ratio of Laplace transform of the output variable to Laplace transform of the input variable with all the initial conditions pertaining to the system being zero. A general system is shown in Fig. 3.1.

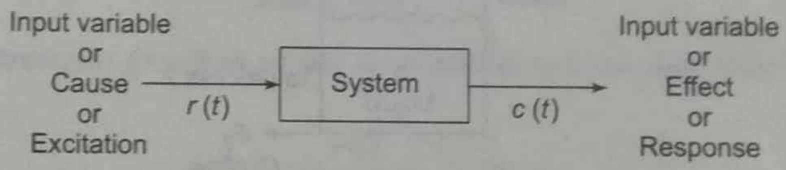

Figure 3.1 A system with input and output variables

For the system, shown in Fig. 3.1, the transfer function will be

$$\text{Transfer function (TF)} = (G(s)) = \left.\frac{C(s)}{R(s)}\right|_{\text{initial condition (s) being zero}} \tag{3.1}$$

$G(s)$ is termed as the transfer function because it represents the way an output signal is related to the input signal.

The transfer function involves Laplace operator s, where

$$s = j\omega \tag{3.2}$$

It makes the transfer function a property of the system in frequency domain. The transfer function is totally a theoretical approach for analysing the system for its behaviour.

The transfer function relates one output variable with one input variable only. In the industrial control systems falling in the categories of MISO, SIMO, and MIMO, the individual transfer functions

Example 3.1 ***Find the transfer function of the electrical circuit shown in Fig. 3.3.***

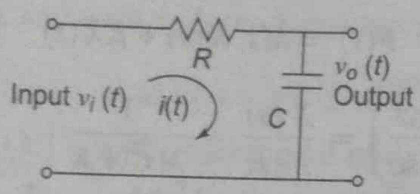

Figure 3.3 Electrical circuit

Solution

Writing KVL equation for the above circuit, we get

$$v_i(t) = Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt \tag{3.5}$$

and
$$v_0(t) = \frac{1}{c}\int_0^t i(t)\,dt \tag{3.6}$$

Taking Laplace transform of Eqs (3.5) and (3.6) and keeping all initial conditions as zero, we obtain

$$V_i(s) = RI(s) + \frac{1}{Cs}I(s) \tag{3.7}$$

and
$$V_0(s) = \frac{1}{Cs}I(s) \tag{3.8}$$

Taking the ratio of $V_0(s)$ to $V_I(s)$ from Eqs (3.8) and (3.7), we get the desired transfer function as

$$\frac{V_o(s)}{V_I(s)} = \frac{1}{1+RCs}$$