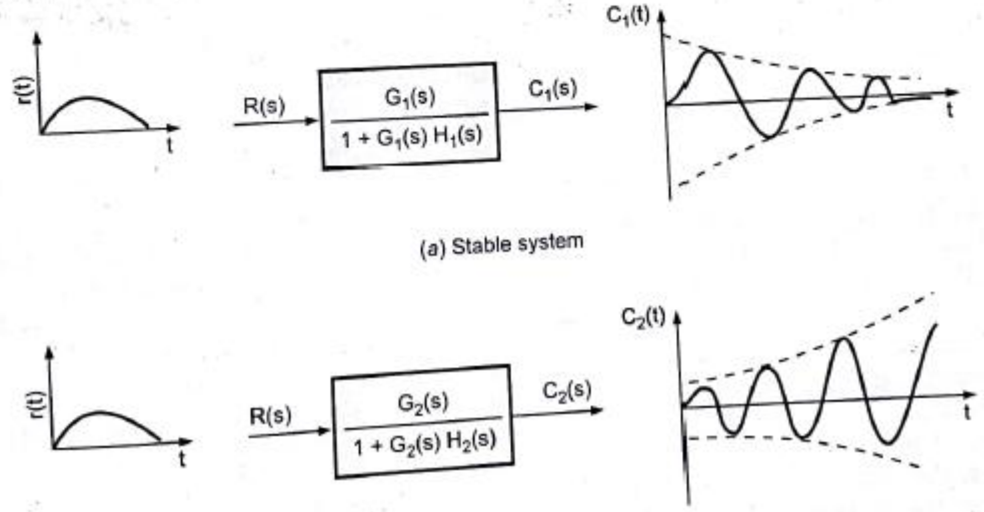
ROUTH – HURWITZ CRITERION

Introduction. In the chapter on time response of control system, the time solution of system equations were obtained. The time solutions thus obtained give the variation of the system output with time as independent variable. The time solution can be classified into two categories, the one wherein the output tending to a finite value and the other wherein the output approaching towards infinite value with the advancement of time. The former case is termed as stable system while the latter is termed as unstable system.

An illustration to explain stability and instability is shown in Fig. 7.0.1.



(a) Stable system

(b) Unstable system

Fig. 7.0.1

A bounded input is applied to systems shown in Fig. 7.0.1. For the time response shown in Fig. 7.0.1 (*a*) the output response is also bounded and the system is said to be stable whereas for the system shown in Fig. 7.0.1 (*b*) the output response is unbounded and the system is said to unstable.

5.1. CONCEPT OF STABILITY

The concept of stability is very important to analyse and design the system. A system is said to be stable if its response cannot be made to increase indefinitely by the application of a bounded input excitation. If the output approaches towards infinite value for sufficiently large time, the system is said to be unstable.

A linear time invariant (LTI) system is stable if

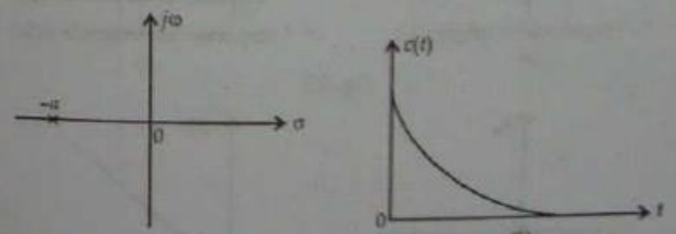
1. the system is excited by a bounded input, the output is bounded. (BIBO stability criteria)
2. in the absence of the input, the output tends towards zero (the equilibrium state of the system). This is known as asymptotic stable.

Consider the transfer function

$$\frac{C(s)}{R(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0} \quad \ldots(5.1)$$

5.2. EFFECT OF LOCATION OF POLES ON STABILITY

(*a*) **poles on negative real axis**



(*a*) simple pole on negative real axis
(*b*) response

Fig. 5.1.

Consider a simple pole at $s = -a$ as shown in fig. 5.1*a*., the corresponding impulse response is given by

$$g(t) = \pounds^{-1}G(s) = \pounds^{-1}\frac{K}{s+a} = K.e^{-at}$$

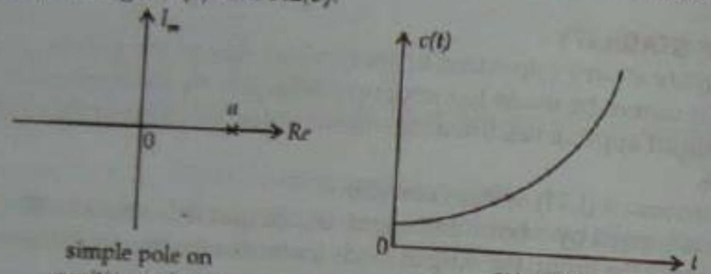
As the time '*t*' increases, the response approaches zero and the system is stable. The response shown in fig 5.1(*b*).

(*b*) Pole on positive real axis

Consider a system having simple pole on positive real axis at $s = a$, the corresponding impulse response is given by

$$g(t) = \pounds^{-1}\frac{K}{s-a} = K.e^{at}$$

The response increases exponentially with time, hence the system is unstable. The simple pole and response are shown in fig 5.2 (*a*) and 5.2(*b*).



simple pole on positive real axis (*a*)
response (*b*)

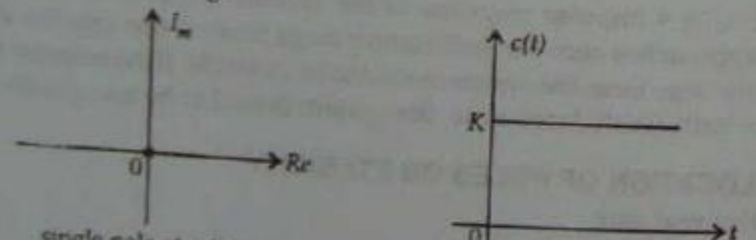
Fig. 5.2.

(*c*) Pole at the origin : Consider a pole at origin

$$\therefore \quad g(t) = \pounds^{-1}\frac{K}{s} = K$$

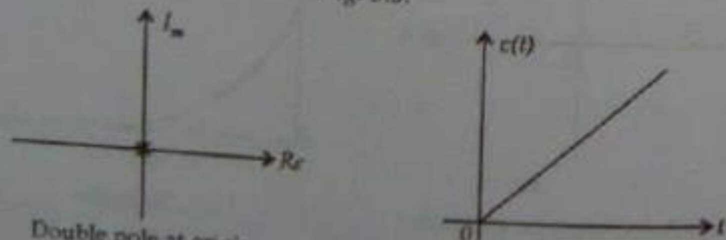
This is constant value, hence the system is marginally stable. If there are two poles at the origin the time response would be

$$g(t) = \pounds^{-1}\frac{K}{s^2} = Kt$$



single pole at origin (*a*)
response due to single pole (*b*)

Fig. 5.3.



Double pole at origin (*a*)
Impulse response (*b*)

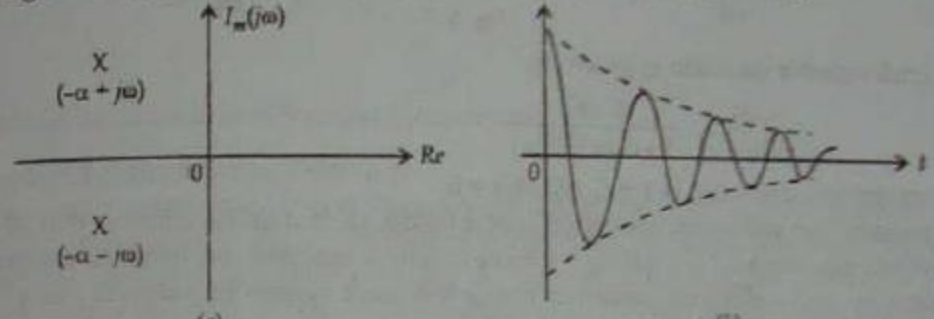
(d) **Complex pole in the left half of s-plane**

Let the transfer function has a complex conjugate poles at $s = -\alpha \pm j\omega$. The time response due to the complex conjugate poles is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{K}{s+\alpha-j\omega} + \frac{K}{s+\alpha+j\omega}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2K(s+\alpha)}{(s+\alpha)^2+\omega^2}\right] = 2K e^{-\alpha t}\cos\omega t \quad ...(5.7)$$

When t increases $g(t)$ approaches zero and the system is stable. The complex poles and corresponding time response is shown in fig. [5.5(*a*)] and [5.5(*b*)] respectively.



(*a*) (*b*)

Fig. 5.5.

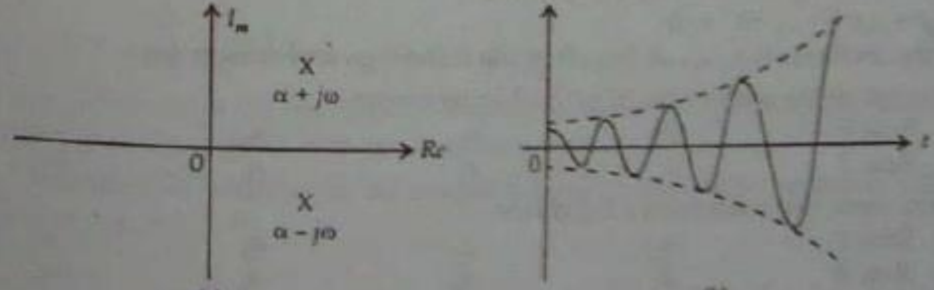
(e) **Complex poles in the right half of s-plane**

Suppose the system has complex conjugate poles at $s = \alpha \pm j\omega$. The time response is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{A}{s-\alpha-j\omega} + \frac{A}{s-\alpha+j\omega}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2A(s-\alpha)}{(s-\alpha)^2+\omega^2}\right] = 2A e^{\alpha t}\cos\omega t \quad ...(5.8)$$

Hence, the response increases exponentially sinusoid with time and therefore the response is unstable. The poles and time response shown in fig [5.6(*a*)] and [5.6(*b*)] respectively.



(*a*) (*b*)

Fig. 5.6.

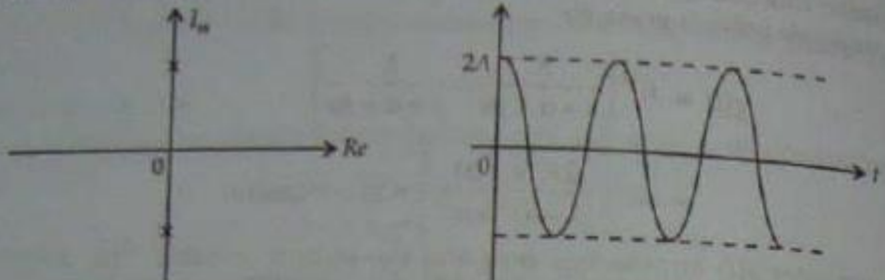
(f) **Poles on $j\omega$-axis**

If the system having the complex poles on $j\omega$-axis the corresponding time response would be

$$g(t) = \mathcal{L}^{-1}\left[\frac{A}{s+j\omega} + \frac{A}{s-j\omega}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2As}{s^2+\omega^2}\right] = 2A\cos\omega t \quad ...(5.9)$$

The response is marginally stable. The equation (5.9) shows the sustained oscillations of constant amplitude. This situation will also be considered unstable.



Poles on imaginary axis (*a*) time response (*b*)

Fig. 5.7.

The overall transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad ...(5.10)$$

The characteristic equation is $1 + G(s)H(s) = 0$ ...(5.11)

The necessary and sufficient condition that a feedback system be stable is that all the zeros of the characteristic equation $1 + G(s)H(s) = 0$ have negative real part. or, in terms of poles we can say that the necessary and sufficient condition that a feedback system be stable is that all the poles of overall transfer function have negative real part.

5.3. NECESSARY BUT NOT SUFFICIENT CONDITIONS FOR STABILITY

Consider a system with characteristicequation

$$a_0 s^m + a_1 s^{m-1} + \dots + a_m = 0 \quad \text{...(5.12)}$$

(a) All the coefficients of the equation should have same sign,

(b) There should be no missing term.

If above two conditions are not satisfied the system will be unstable. But if all the coefficients have same sign and there is no missing term we have no guarantee that the system will be stable. For stability we use ROUTH-HURWITZ CRITERION.

5.4.1. Statement of Routh-Hurwitz Criterion

Routh-Hurwitz Criterion states that the system is stable if and only if all the elements in the first column have the same algebaric sign. If all elements are not of the same sign then the number of sign changes of the elements in first column equals the number of roots of the characteristic equation in the right half of the s-plane (or equals to the number of roots with positive real parts)

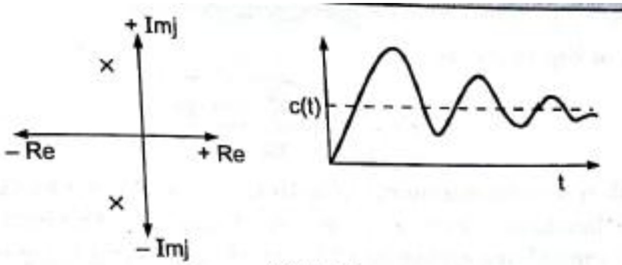
in polynomial form.

While applying Routh-Hurwitz criterion it is imperative that no powers of s in the characteristic equation be absent. Any absence of such powers indicate the presence of at least one positive real part root and confirms system instability by inspection.

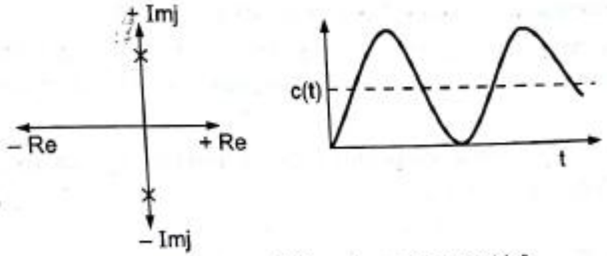
However, if the characteristic equation contains either only odd or even powers of s, this indicates that the roots have no real parts and possess only imaginary parts and therefore the system has sustained oscillations in its output response.

Following difficulties are faced while applying Routh-Hurwitz criterion :

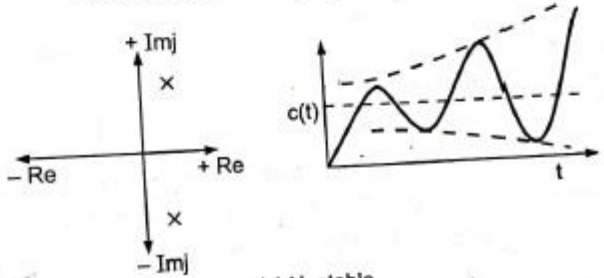
1. The element in one of the first columns of Routh array is zero.
2. All the elements in one of the rows of Routh array are zero.



(a) Stable



(b) Sustained oscillations (marginally stable)



(c) Unstable

Fig. 7.1.1. Time response and location of roots of characteristic equation.

Example 7.5.1. *A closed-loop control system has the characteristic equation given by,* $s^3 + 4.5s^2 + 3.5s + 1.5 = 0$. *Investigate the stability using Routh-Hurwitz criterion.*

therefore, the system is stable.

Example 7.5.3. *Determine the stability of a system whose overall transfer function is given below*

$$\frac{C(s)}{R(s)} = \frac{2s + 5}{s^5 + 1.5s^4 + 2s^3 + 4s^2 + 5s + 10}$$

If the system is found unstable, how many roots it has with positive real part ?

Example 7.5.4. *Determine the stability of a closed-loop control system whose characteristic equation is*

$$s^5 + s^4 + 2s^3 + 2s^2 + 11s + 10 = 0.$$

SPECIAL CASES

Case 1 : If a first column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then multiply the original equation by a factor $(s + a)$ where 'a' is any positive real number. The simplest value of 'a' is 1 (take $a = 1$). Consider the following ecample.

Example 5.7. Investigate the stability

$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

Solution :

s^5	1	2	3
s^4	1	2	5
s^3	0		
s^2			
s^1			
s^0			

Now, multiply the characteristic equation by $(s + 1)$

$(s + 1)(s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5) = 0$

or, $s^6 + 2s^5 + 3s^4 + 4s^3 + 5s^2 + 8s + 5 = 0$

s^6	1	3	5	5
s^5	2	4	8	
s^4	1	1	5	
s^3	2	–2		
s^2	2	5		
s^1	–7			
s^0	1			

From the above table :

No. of sign change in the first column = 2

No. of roots in the right half of s-plane = 2

Hence, system is unstable.

The presence of a zero in the first column of the Routh's tabulation leads to following conclusions :

(*i*) equal real roots with opposite signs. As one of the roots is $+ve$, the system is unstable as indicated by the sign change in the first column.

(*ii*) pair of conjugate root on imaginary axis. This gives marginal stability provided there is no sign change in the first column.

In the both cases as per procedure 0 is replaced by ε.

The conclusions as above are verified below :

(*i*) $F(s) = (s-1)(s+1)^2(s+2)$: two equal roots with opposite sign $s = -1, s = +1$

or $$F(s) = s^4 + 3s^3 + s^2 - 3s + 2$$

The Routh's tabulation is formed below :

s^4	1	1	2
s^3	3	-3	
s^2	2	-2	
s^1	0 (ε)		+ as $\varepsilon \to 0$
s^0	-2		

There is one sign change in the first column ; the system being unstable.

(*ii*) $F(s) = (s+1)(s+2)(s^2+1)$: pair of conjugate root on imaginary axis $s = \pm j1$

or $$F(s) = s^4 + 3s^3 + 3s^2 + 2$$

The Routh's tabulation is formed below :

s^4	1	3	2
s^3	3	3	
s^2	2	2	
s^1	0 (ε)		
s^0	1		+ as $\varepsilon \to 0$

There is no sign change in the first column and noting that the polynomial $F(s)$ has a … of conjugate root on imaginary axis, the system is marginally stable.

Example 7.5.5. *Determine the stability of a system having following characteristic equation :*

$$s^6 + s^5 + 5s^4 + 3s^3 + 2s^2 - 4s - 8 = 0.$$

Solution. The Routh table is formed below :

s^6	1	5	2	−8
s^5	1	3	−4	0
s^4	2	6	−8	0
s^3	0	0	0	0
s^2				

$\boxed{\textit{Auxiliary equation}}$

It is observed in this example that all the elements in the fourth row vanish and the application of Routh criterion fails. This situation occurs when the array has two consecutive rows having the same ratio of corresponding elements.

This difficulty faced is overcome by forming an auxiliary equation using elements of the last but one vanishing row. The derivative of this auxiliary equation is taken w.r.t. s and the coefficients of the differentiated equation are taken as the elements of the following row :

For the example under consideration the auxiliary equation is

$$A(s) = (2s^4 + 6s^2 - 8)$$

and

$$\frac{dA(s)}{ds} = (8s^3 + 12s - 0)$$

The coefficients of the fourth row are thus 8, 12 and 0. The modified Routh array is given below :

s^6	1	5	2	−8
s^5	1	3	−4	0
s^4	2	6	−8	0
$*s^3$	8	12	0	0
s^2	3	−8	0	0
s^1	$\frac{100}{3}$	0	0	0
s^0	−8	0	0	0

**coefficients of differentiated auxiliary eqn.*

Sign change

As there is one sign change in the first column, the system has one root with positive real part indicating that the system is unstable.

All the elements of a row in Routh's tabulation being zero indicate a pair of conjugate root on imaginary axis and on substitution of coefficients of $dA(s)/ds$ in the zero element row on sign change is noted in the first column of Routh's tabulation, then the system is marginally stable.

However, if any two rows of a Routh's tabulation are zero it can be concluded that a pair of conjugate roots on imaginary axis having multiplicity order of two is present. This shows unstable system even though no change of sign in the first column is observed on following usual procedure for completing Routh's tabulation.

The statements mentioned above are verified as per the examples given below :

$F(s) = (s^2 + 1)(s + 1)(s + 2)(s + 3)$: pair of conjugate root on imaginary axis $s = \pm j1$

$$= s^5 + 6s^4 + 6s^3 + 12s^2 + 5s + 6$$

s^5	1	6	5
s^4	6	12	6
s^3	4	4	
s^2	6	6	
s^1	0(12)*	0	
s^0	6		

Aux. equation $A(s) = 6s^2 + 6$

$$\frac{dA(s)}{ds} = 12s$$

* coeff. of $\frac{dA(s)}{ds}$

There is no sign change in the first column, the system is marginally stable.

$F(s) = (s^2 + 2)^2 (s + 1)(s + 2)$: pair of conjugate root on imaginary axis of multiplicity order two *i.e.* $s = \pm j\sqrt{2}$, $s = \pm j\sqrt{2}$

$$F(s) = s^6 + 3s^5 + 6s^4 + 12s^3 + 12s^2 + 12s + 8$$

s^6	1	6	12	8
s^5	3	12	12	
s^4	2	8	8	
s^3	0(8)*	0(16)*	0	
s^2	4	8		
s^1	0(8)**	0		
s^0	8			

(1) $A(s) = 2s^4 + 8s^2 + 8$

$\frac{dA(s)}{ds} = 8s^2 + 16s$ *coeff. of $\frac{dA(s)}{ds}$

(2) $A(s) = 4s^2 + 8$

$\frac{dA(s)}{ds} = 8$ ** coeff. of $\frac{dA(s)}{ds}$

There is no change of sign in the first column of Routh's tabulation but existence of two rows having zero elements make the system unstable.

Example 7.5.6. *Using Routh criterion investigate the stability of a unity feedback control system whose open-loop transfer function is given by*

$$G(s) = 2\frac{e^{-sT}}{s(s+2)}$$

Solution. The overall transfer function is

$$\frac{C(s)}{R(s)} = \frac{\dfrac{e^{-sT}}{s(s+2)}}{\left[1+\dfrac{e^{-sT}}{s(s+2)}\cdot 1\right]}$$

$$= \frac{e^{-sT}}{(s^2+2s+e^{-sT})} \qquad \text{...(1)}$$

The characteristic equation is

$$s^2 + 2s + e^{-sT} = 0 \qquad \text{...(2)}$$

Since $$e^{-sT} = \left(1 - sT + \frac{s^2\,T^2}{2!} + \cdots\right) \qquad \text{...(3)}$$

Considering only first two terms of series (3), the characteristic equation can be written as

$$s^2 + 2s + (1 - sT) = 0$$

or $$s^2 + (2 - T)\,s + 1 = 0 \qquad \text{...(4)}$$

The Rough array for equation (4) is formed below :

s^2	1	1
s^1	$(2 - T)$	0
s^0	1	0

For the system to be stable the sign of the first element in the Routh table should be positive, therefore, the condition for the stability of the system is given by

$$(2 - T) > 0$$

$\therefore$ $T < 2$. **Ans.**

... can be determined.

5.5. APPLICATION OF ROUTH'S STABILITY CRITERION TO CONTROL SYSTEM ANALYSIS

Routh stability criterion is also used for the determination of stability of the linear feedback system. Consider the following example.

Example 5.11. The open loop transfer function of unity feedback system is $\dfrac{K}{s(1+0.4s)(1+0.25s)}$. Find the restriction of K so that the closed loop system is absolutely stable.

(R.M.L. University Faizabad 20..)

Solution : Given that

$$G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$$

$$H(s) = 1$$

The characteristic equation $1 + G(s)\,H(s) = 0$

$$1+\frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

or, $\quad s(1+0.4s)(1+0.25s) + K = 0$

or, $\quad s^3 + 6.5s^2 + 10s + 10K = 0$

s^3	1	10
s^2	6.5	10K
s^1	$\dfrac{65-10K}{6.5}$	
s^0	10K	

For absolute stability, there should be no sign change in the first column i.e no root of the characteristic equation should lie in right half of s-plane. This is possible only when

$K > 0$ and $65 - 10K > 0$

Hence, for closed loop stability

$0 < K < 6.5$

5.6. RELATIVE STABILITY ANALYSIS

Routh stability criterion gives the information about the absolute stability. But we are more interested for the relative stability of the system. Relative stability can be examined by shifting the s-plane and then apply Routh stability criterion. The characteristic equation is modified by shifting the origin of s-plane to $s_1 = -\sigma$ by the substitution

$$s = z - s_1$$

Now apply the Routh stability criterion, the number of sign change in the first column is equal to the number of roots to the right of the vertical line $s_1 = -\sigma$ or in other words the roots of the original characteristic equation are more negative than $-\sigma_1$.

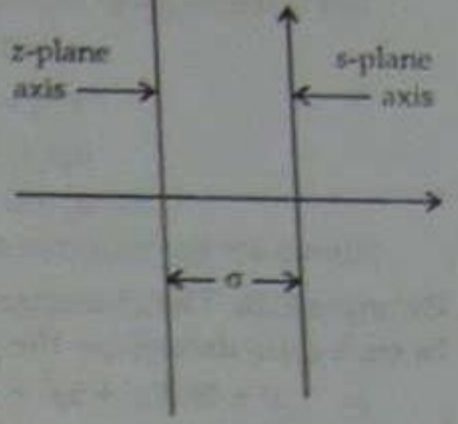


Fig. 5.8.

Example 7.5.7. *(b) Determine the value of K such that the roots of the characteristic equation given below lie to the left of line s = −1*

$$s^3 + 10s^2 + 18s + K = 0.$$

Solution. Put $s = s_1 - 1$

$$\therefore \quad (s_1 - 1)^3 + 10\,(s_1 - 1)^2 + 18(s_1 - 1) + K = 0$$

or

$$s_1^3 + 7s_1^2 + s_1 + (K - 9) = 0$$

The Routh's tabulation is given below :

s_1^3	1	1
s_1^2	7	$(K - 9)$
s_1^1	$\dfrac{7-(K-9)}{7}$	
s_1^0	$K - 9$	

$$K < 16 \text{ and } K > 9,$$

K to have values between 9 and 16. **Ans.**