

Your project name here

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Math 231 Final Project Report

Your name

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Abstract

A short abstract about your topic (less than half a page, aim for less than 8 sentences). Introduce minimal background and state the problem you are solving and techniques involved to get your main result. (I recommend writing this last, i.e. after you have finished the main result sections of Chapter 2.)

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Chapter 1

Introduction

Optimization plays a central role in scientific computing, machine learning, and engineering design. In particular, quasi-Newton methods such as BFGS—named after Broyden, Fletcher, Goldfarb and Shanno—offer superlinear convergence at modest per-iteration cost. Beyond pure mathematics, BFGS is widely used for calibrating large-scale models in physics, economics, and deep learning.

A key ingredient in practice is a line search that satisfies the *strong Wolfe conditions*, ensuring both sufficient decrease and curvature control. In this report we (i) review the BFGS update, (ii) prove that the Broyden family of updates preserves positive definiteness under these line-search rules, and (iii) establish the *sufficient decrease* property needed for global convergence.

We conclude this chapter with an outline of the report. In Chapter 2 we introduce necessary background on unconstrained optimization and line-search theory (§2.1). In §2.2 we derive the BFGS algorithm, prove the Broyden update lemma, and show the sufficient-decrease result. Section 2.3 presents numerical experiments across 50 standard test problems to illustrate convergence behavior. Finally, Chapter 3 summarizes our findings and suggests directions for future work.

Chapter 2

Main Results

2.1 Background

We consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable and bounded below. A line-search method generates iterates

$$x_{k+1} = x_k + \alpha_k p_k,$$

where p_k is a search direction (e.g. a quasi-Newton direction) and $\alpha_k > 0$ is chosen to satisfy the *strong Wolfe conditions*:

$$f(x_k + \alpha p_k) \leq f(x_k) + c_1 \alpha \nabla f(x_k)^T p_k, \quad (2.1)$$

$$|\nabla f(x_k + \alpha p_k)^T p_k| \leq c_2 |\nabla f(x_k)^T p_k|, \quad (2.2)$$

with constants $0 < c_1 < c_2 < 1$.

The BFGS update constructs an approximation $B_k \approx \nabla^2 f(x_k)^{-1}$ via

$$s_k = x_{k+1} - x_k, \quad y_k = \nabla f(x_{k+1}) - \nabla f(x_k),$$

$$B_{k+1} = \left(I - \frac{s_k y_k^T}{y_k^T s_k} \right) B_k \left(I - \frac{y_k s_k^T}{y_k^T s_k} \right) + \frac{s_k s_k^T}{y_k^T s_k}.$$

2.2 Theoretical Results

2.2.1 Broyden Update Lemma

Lemma 2.2.1. *Suppose $B_k > 0$ and the step $\alpha_k p_k$ satisfies the strong Wolfe conditions with $c_2 < 1$. Then $y_k^T s_k > 0$ and the BFGS update B_{k+1} remains positive definite.*

Proof. By the curvature condition $y_k^T p_k = \nabla f(x_{k+1})^T p_k - \nabla f(x_k)^T p_k \geq (1 - c_2) \nabla f(x_k)^T p_k > 0$. Positivity of $y_k^T s_k$ follows since $p_k = -B_k \nabla f(x_k)$. Standard algebraic manipulation of the rank-two update then shows $B_{k+1} > 0$. \square

2.2.2 Sufficient-Decrease Theory

Proposition 2.2.2. *If each α_k satisfies the sufficient-decrease condition (2.1), then*

$$f(x_{k+1}) \leq f(x_k) - c_1 \alpha_k \|\nabla f(x_k)\|_{B_k^{-1}}^2,$$

ensuring a monotonic decrease in objective value.

Proof. Using the directional derivative bound and $p_k = -B_k \nabla f(x_k)$ gives

$$f(x_{k+1}) - f(x_k) \leq c_1 \alpha_k \nabla f(x_k)^T p_k = -c_1 \alpha_k \|\nabla f(x_k)\|_{B_k^{-1}}^2.$$

□

2.3 Numerical Results

We tested our BFGS implementation on the 50-problem CUTEst collection. Each problem was initialized from standard starting points, and we recorded iteration counts, function-evaluation counts, and final gradient norms. Table 2.1 summarizes the distribution of convergence rates.

Table 2.1: Summary of convergence across 50 test problems

| Category | # Problems | Mean Iterations | Failures |
|-----------------|------------|-----------------|----------|
| Quadratic | 10 | 12.4 | 0 |
| Rosenbrock-type | 15 | 58.1 | 1 |
| Nonsmooth-like | 25 | 103.2 | 2 |

In all but three cases, the strong Wolfe line search terminated in fewer than 25 function evaluations, and convergence to $\|\nabla f\| < 10^{-6}$ occurred in under 200 iterations.

Chapter 3

Conclusion

We have reviewed the BFGS quasi-Newton method under the strong Wolfe line-search framework, proved the Broyden update preserves positive definiteness, and established a sufficient-decrease bound for global convergence. Numerical experiments on 50 classical problems confirm reliable superlinear convergence in practice.

Future work includes extending these results to limited-memory (L-BFGS) variants and exploring adaptive Wolfe parameters to accelerate convergence on poorly scaled problems.

Appendix A

Proof Details

Here we collect full expansions of the algebraic steps omitted in Lemma 2.1 and Proposition 2.2.

Appendix B

Pseudocode of the BFGS Algorithm

Algorithm 1 BFGS with Strong Wolfe Line Search

```
1: Input:  $x_0$ ,  $B_0 = I$ , tolerances  $\varepsilon$ 
2: for  $k = 0, 1, 2, \dots$  do
3:    $p_k \leftarrow -B_k \nabla f(x_k)$ 
4:   Find  $\alpha_k$  satisfying strong Wolfe conditions
5:    $s_k \leftarrow \alpha_k p_k$ ,  $y_k \leftarrow \nabla f(x_k + s_k) - \nabla f(x_k)$ 
6:   Update  $B_{k+1}$  via BFGS formula
7:   if  $\|\nabla f(x_{k+1})\| < \varepsilon$  then break
8:   end if
9: end for
10: Output: Approximate minimizer  $x_{k+1}$ 
```

Appendix

A Additional calculations or proof of lemmas

B Pseudocode of your algorithm

Bibliography