

# MATH 231 : Numerical ODEs

## Random Examples

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### Question 1: Eigenvalues of special tridiagonal matrices

This question is about finding eigenvalues of tridiagonal linear systems arising from applications, specifically finding the eigenvalues of an  $n \times n$  matrix of the form,

$$A = \begin{pmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{pmatrix}$$

where  $a, b, c$  are real numbers with  $bc > 0$  (i.e.  $b$  and  $c$  have the same signs).

- (a) Show that the eigenvalue problem of  $A$  is equivalent to the equations

$$cv_{j-1} + (a - \lambda)v_j + bv_{j+1} = 0, \quad j = 1, \dots, n$$

$$v_0 = 0 = v_{n+1}$$

where  $\mathbf{v} = (v_1, \dots, v_n)^T$  is an eigenvector of  $A$  associated with the eigenvalue  $\lambda$ .

- (b) The recurrence relation (1) is a second order linear difference equation and can be solved similar to second order linear differential equations. By guessing  $v_j = r^j$  for some constant  $r$ , show that  $r$  satisfies

$$r_{\pm} = \frac{\lambda - a \pm \sqrt{(\lambda - a)^2 - 4bc}}{2b}, \quad \text{with} \quad r_+ r_- = \frac{c}{b}$$

- (c) Show by contradiction that  $r_{\pm}$  must be distinct.

Hint: if  $r_{\pm} = r$  are repeated, then  $v_j = Ar^j + Br^j$  for some constants  $A, B$ .

- (d) Since  $r_{\pm}$  are distinct, the general solution for (1) is  $v_j = Ar_+^j + Br_-^j$  for constants  $A, B$ . Use this to conclude from (2) and (3) that,

$$\left(\frac{br_+^2}{c}\right)^{n+1} = 1$$

- (e) From part (c), (3) and (4), show that  $r_{\pm}$  must be complex valued and conclude that (4) has the solutions for  $k = 1, \dots, n$ ,

$$r_{\pm, k} = \sqrt{\frac{c}{b}} \exp\left(\frac{\pm ik\pi}{n+1}\right), \quad \text{where } i = \sqrt{-1}$$

- (f) Using part (e), conclude that the eigenvalues of  $A$  is given by

$$\lambda_k = a + 2 \operatorname{sgn}(\sqrt{bc}) \cos\left(\frac{\pi k}{n+1}\right), \quad k = 1, \dots, n$$

- (g) Find the eigenvalues of the  $n \times n$  finite difference matrix  $A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$ , where

$$h = \frac{1}{n+1}.$$

Conclude that  $A_h$  is symmetric positive definite and find its condition number  $\kappa(A_h)$  with respect to  $\|\cdot\|_2$ . Show that  $\kappa(A_h) = \mathcal{O}(h^{-2})$  as number of grid points  $n$  increases. What does this mean for solving  $A_h \mathbf{x} = \mathbf{b}$  when  $n$  is large?

## Solution

- (a) Let  $(\lambda, \vec{v})$  be an eigenpair of  $A$

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ (A - \lambda I)\vec{v} &= \vec{0} \\ \begin{pmatrix} (a - \lambda)v_1 + bv_2 \\ cv_1 + (a - \lambda)v_2 + bv_3 \\ \vdots \\ cv_{n-2} + (a - \lambda)v_{n-1} + b_n \\ c_{n-1} + (a - \lambda)v_n \end{pmatrix} &= \vec{0}. \end{aligned}$$

We can write the above relation as the following,

$$cv_{j-1} + (a - \lambda)v_j + bv_{j+1} = 0. \quad (1)$$

Where  $0 \leq j \leq n + 1$  and  $v_0 = 0 = v_{n+1}$   $\oplus$

- (b) Using the hint we guess the following form of the solution  $v_j = r^j$ . Substituting in 1,

$$\begin{aligned} cr^{j-1} + (a - \lambda)r^j + br^{j+1} &= 0 \\ c + (a - \lambda)r + br^2 &= 0 \\ &. \end{aligned}$$

Using the quadratic formula, we get

$$r_{\pm} = \frac{\lambda - a \pm \sqrt{(a - \lambda)^2 - 4bc}}{2b}..$$

As  $r_{\pm}$  are the roots to a quadratic, hence  $r_+r_- = \frac{c}{b}$

- (c) If 1 has a repeated root, say  $r_{\pm} = r$ , then solution to the recursion would look like,

$$v_j = Ar^j + Bjr^j.$$

Checking the boundary conditions,  $v_0 = 0 = v_{n+1}$

$$v_0 = Ar^0 + B(0)r^0 = A = 0. \quad (2)$$

$$v_{n+1} = (0)r^{n+1} + B(n+1)r^{n+1} = B(n+1)r^{n+1} = 0 \implies B = 0. \quad (3)$$

Combining 2 & 3 gives,

$$v_j = 0.$$

Which is the trivial eigenvector. Hence, we cannot have a repeated root if we want a non-zero eigenvector.

- (d)

## Question 2: Classical iterative methods for strictly diagonally dominant matrices

- (a) Show that the diagonal part of any strictly diagonally dominant (S.D.D.) matrix is invertible.
- (b) Recall the Gershgorin's theorem below, which can give useful information about the eigenvalues of a matrix. The eigenvalues of a complex valued matrix  $A$  lies in the union of  $n$  discs  $\bigcup_{i=1}^n D_i$  on the

complex plane, where

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}$$

Using Gershgorin's theorem, conclude S.D.D. matrices are invertible. Hint: Show that  $0 \notin D_i$  for all  $i = 1, \dots, n$ . The next two parts are about showing convergence of Jacobi and Gauss-Seidel iterations for S.D.D. matrices.

- (c) Recall the matrix  $-M^{-1}N$  associated with the Jacobi iteration takes the form  $-D^{-1}(L + U)$ , where  $A = L + D + U$ .
- (d) Let  $A$  be S.D.D. and  $\lambda$  be any eigenvalue of  $-D^{-1}(L + U)$ . Show that  $\det(L + U + \lambda D) = 0$  using part (a). (ii) Now suppose  $|\lambda| \geq 1$ . Deduce from  $A$  being S.D.D. that  $L + U + \lambda D$  must also be S.D.D. (iii) Deduce a contradiction by applying the result from part (b) to  $L + U + \lambda D$ , and conclude that  $|\lambda| < 1$ . (iv) Combine parts (i)-(iii) to conclude that Jacobi iteration converges for S.D.D. matrices.
- (e) Follow a similar argument as part (c) to show that Gauss-Seidel iterations converges for S.D.D. matrices.

### Question 3: Classical iterative methods for symmetric positive definite matrices

This question is about coding and comparing classical iterative methods for the S.P.D. matrix  $A_h$  from Q1(g).

- (a) Write a pseudocode for the classical iterative methods: Richardson, optimal Richardson, Jacobi, Gauss-Seidel, S.O.R., and optimal S.O.R.
- (b) Implement a program to solve  $A_h \mathbf{x} = \mathbf{b}$  with  $\mathbf{b} = (1, \dots, 1)^T \in \mathbb{R}^{20}$  and  $\mathbf{x}_0 = (1, 0, \dots, 0)^T \in \mathbb{R}^{20}$  using Richardson (with  $\omega = \lambda_{\max}^{-1}$ ), optimal Richardson, Jacobi, Gauss-Seidel, S.O.R. (with  $\theta = 1.2$ ) and optimal S.O.R. Generate a plot comparing the log of their residual in  $\ell_2$  norm versus iterations up to 5000. Rank the performance of each method by comparing the iterations needed to reach the residual tolerance of  $10^{-14}$ . Use sparse representation when appropriate. Hint: Use Q1(g) to find parameters for Richardson and vary  $\theta$  to find an approximate optimal parameter for S.O.R.
- (c) Comment on the decreases in performance when  $n = 1000$ . Explain briefly how this relates to  $\kappa(A_h) = O(h^{-2})$ .

### Question 4: Steepest Descent and Conjugate Gradient

- (a) Let  $A$  be a S.P.D. matrix. Show that  $(\mathbf{x}, \mathbf{y})_A := \mathbf{x}^T A \mathbf{y}$  for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  forms an inner product on  $\mathbb{R}^n$ .
- (b) Using part (a), conclude that  $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})_A^{1/2}$  for  $\mathbf{x} \in \mathbb{R}^n$  is a norm on  $\mathbb{R}^n$ .  
Hint: You can assume the Cauchy-Schwarz inequality  $|(\mathbf{x}, \mathbf{y})_A| \leq \|\mathbf{x}\|_A \|\mathbf{y}\|_A$  holds.
- (c) For the method of Steepest Descent, show that  $\nabla f(\mathbf{x}_k)$  and  $\nabla f(\mathbf{x}_{k+1})$  are orthogonal (i.e. zig-zaging behavior), where  $f(\mathbf{y}) = \frac{1}{2} \mathbf{y}^T A \mathbf{y} - \mathbf{y}^T \mathbf{b}$ . Hint: Recall how the step size for Steepest Descent is determined.
- (d) Repeat the experiment from Q3(b) with  $\mathbf{b} = (1, \dots, 1)^T \in \mathbb{R}^{1000}$  and  $\mathbf{x}_0 = (1, 0, \dots, 0)^T \in \mathbb{R}^{1000}$  using the method of Steepest Descent and Conjugate Gradient. Generate a plot comparing the log of their residual in  $\ell_2$  norm versus iterations up to 5000. Rank their performance by comparing the iterations needed to reach the residual tolerance of  $10^{-14}$ , as well as versus the classical iterative methods. Verify your CG method terminates after the desired number of iterations. Use sparse representation when appropriate.