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Problem 5

Problem 5:

a) $S = \{1, (x+1), (x+2)^2, \ldots, (x+i)^i \ldots\}$.

$f : [0,1] \to \mathbb{R}$

→ Let $(x+i_a)^{i_a}$ be a vector in S that is a linear combination of all other vectors:

$$\therefore (x+i_a)^{i_a} = \sum_{i=1}^{n} \alpha_i (x+i)^i$$

if $i_a < n$,

$$(x+i_n)^{i_n} = \alpha_n^{-1}(x+i_a)^{i_a} + \sum_{1}^{n-1} -\alpha_n^{-1}\alpha_i (x+i)^i$$

thus LHS will be the vector with the highest degree.

without loss of generality; we can say:

$i_a > n$.

$\therefore$ no vector exists less than n (size of S) which will be a linear combination of all others.

$\therefore S$ is linearly independent.

b) $C^0[0,1]$ is a set of continuous functions $f: [0,1] \to R$

$\therefore C^0[0,1] = |x-0.5|$
$= |\sin x|$
$= |\cos x|$
$= e^x.$

suppose $e^x = \sum_{j=1}^{n} \alpha_j (x+j)^j$ for $\alpha_j \neq 0$

taking derivative w.r.t. x.
$$\frac{d}{dx} e^x = e^x$$
and $\frac{d}{dx} \sum_{j=1}^{n} \alpha_j (x+j)^j = \sum_{j=1}^{n} j\alpha_j (x+j)^j$

$\therefore e^x \neq \sum j\alpha_j (x+j)^j$

$\therefore$ e^x cannot be represented as a linear combination of vectors in S.
Same can be proved for other functions satisfying $C^0[0,1]$

Problem 6

Problem 6: Linear maps

a) $f: R^{n\times n} \Rightarrow R^{n\times n}$, $f(X) = AX - XA$, where A is given

→ for f to be a linear map, it must satisfy two properties
i) Homogeneity and ii) Additivity

i) $f(\alpha X) = \alpha f(X)$, $\alpha \in F$, $X \in R^{n\times n}$

$$\therefore f(\alpha X) = A(\alpha X) - (\alpha X)A = \alpha AX - \alpha XA$$
$$= \alpha(AX) - \alpha(XA) = \alpha(AX - XA)$$
$$= \alpha f(X)$$

$\therefore \boxed{f(\alpha X) = \alpha f(X)}$ → Satisfied.

ii) $f(X_1 + X_2) = f(X_1) + f(X_2)$, $X_1, X_2 \in R^{n\times n}$.

$$\therefore f(X_1 + X_2) = A(X_1 + X_2) - (X_1 + X_2)A$$
$$= AX_1 + AX_2 - X_1A - X_2A$$
$$= (AX_1 - X_1A) + (AX_2 - X_2A)$$
$$= f(X_1) + f(X_2)$$

$\therefore \boxed{f(X_1 + X_2) = f(X_1) + f(X_2)}$ → Satisfied.

$\therefore f(X)$ is a linear map.

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Matrix representation.

$$\begin{bmatrix} \beta_{11} \\ \beta_{12} \\ \vdots \\ \beta_{nn} \end{bmatrix} = [C] \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \\ \vdots \\ \alpha_{nn} \end{bmatrix}$$

$$b = \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \right\} \text{ is basis of } \mathbb{R}^{n \times n}$$

$$\begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} - [\alpha] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$$\therefore \beta_{ij} = \sum_{k=1}^{n} a_{ik}\alpha_{kj} - \sum_{k=1}^{n} \alpha_{ik} a_{kj}$$

$$\therefore \boxed{C = \begin{bmatrix} 0 & -a_{21} & -a_{31} & -a_{n1} \cdots & a_{12} & a_{13} \cdots & a_{1n} & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ \vdots & & & & & & & & \vdots \\ 0 & a_{n1} & a_{n2} \cdots & \cdots & -a_{1n} & -a_{2n} & -a_{3n} & \cdots & 0 \end{bmatrix}}$$

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b) $f: \mathbb{R}^n \Rightarrow \mathbb{R}^{n\times m}$ $f(x) = xy^T$ where $y \in \mathbb{R}^m$ is given.

→ for f to be a linear map, it should satisfy:

i) $f(\alpha x) = \alpha f(x)$

$\therefore f(\alpha x) = (\alpha x)y^T = \alpha x y^T = \alpha(xy^T)$
$= \alpha f(x)$

$\therefore \boxed{f(\alpha x) = \alpha f(x)}$ → satisfied.

ii) $f(x_1 + x_2) = f(x_1) + f(x_2)$

$\therefore f(x_1 + x_2) = (x_1 + x_2)y^T = x_1 y^T + x_2 y^T$

$\therefore \boxed{f(x_1 + x_2) = f(x_1) + f(x_2)}$ → satisfied.

$\therefore$ both properties hold true, $f(x)$ is a linear map.

Matrix representation:

$$\begin{bmatrix}\beta_1\\ \vdots \\ \beta_n\end{bmatrix} = C\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}$$

Basis for $\mathbb{R}^n$:

$$b_1 = \left\{\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\ \vdots\\1\\0\end{pmatrix}\dots\right\}$$

verification of basis:

$$\alpha_1\begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}\dots\dots = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix} \quad \forall\, \alpha_i \in \mathbb{R}.$$

Basis for $\mathbb{R}^{m\times n}$:

$$b_2 \left\{\begin{bmatrix}1&0&0&\cdots&0\\0&&&&\vdots\\0&&\ddots&&\\ \vdots&&&&0\\0&\cdots&&&0\end{bmatrix}, \begin{bmatrix}0&1&0&\cdots&0\\ \vdots&&\ddots&&\vdots\\0&\cdots&&&0\end{bmatrix}\cdots \begin{bmatrix}0&0&\cdots&0\\ &\ddots&&\vdots\\ 0&\cdots&&1\end{bmatrix}\right\}$$

b_1 & b_2 both are linearly independent.

we know:

$$\beta_1 = C_{11}\alpha_1 + C_{12}\alpha_2 + \dots + C_{1n}\alpha_n$$

but $\beta_1 = y_1\alpha_1$

$\therefore C_{11} = y_1,\ C_{12} = C_{13} = \dots C_{1n} = 0.$

$$\therefore \begin{bmatrix}\beta_1\\ \vdots\\ \beta_{nm}\end{bmatrix} = \begin{bmatrix}y_1&\cdots&0&\cdots&0&\cdots&0\\ \vdots&&0&&&&\\ y_m&&0&&&&\\ &&y_1&&&&\\ \vdots&&\vdots&\cdots&&&y_1\\ &&&&&&y_m\end{bmatrix}\begin{bmatrix}\alpha_1\\ \vdots\\ \vdots\\ \alpha_n\end{bmatrix}$$

Problem 6

c) $f: \mathbb{F}_2^2 \to \mathbb{F}_2^2$

$$f[(x_1, x_2)] = \begin{bmatrix} x_1^2 + x_2^3 \\ x_1 + x_2 \end{bmatrix}$$

For a finite field $\mathbb{F}_2$, $x \cdot x = x$.

$\Rightarrow x^2 = x$

$\therefore x^3 = x^2 \cdot x = x$.

$\therefore x^2 = x^3 = x$.

$$\therefore f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2^3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$$

For some $u, v \in \mathbb{F}_2^2$ & $\alpha \in \mathbb{F}_2$

we need to show:

i) $f(u+v) = f(u) + f(v)$

$u = (x_1, x_2)$, $v = (x_3, x_4)$

$$\therefore f(u+v) = f(x_1, x_2) + f(x_3, x_4)$$

$$= \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 + x_2) + (x_3 + x_4) \\ (x_1 + x_2) + (x_3 + x_4) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} x_3 + x_4 \\ x_3 + x_4 \end{bmatrix} = f(u) + f(v)$$

$\therefore$ satisfied.

ii) $f(\alpha v) = \alpha f(v)$

$$f(\alpha(x_3, x_4)) = f(\alpha x_3, \alpha x_4)$$

$$= \begin{bmatrix} \alpha x_3 + \alpha x_4 \\ \alpha x_3 + \alpha x_4 \end{bmatrix} = \begin{bmatrix} \alpha(x_3 + x_4) \\ \alpha(x_3 + x_4) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_3 + x_4 \\ x_3 + x_4 \end{bmatrix} = \alpha f(v) \to \text{satisfied.}$$

$\therefore f: \mathbb{F}_2^2 \to \mathbb{F}_2^2$ is a linear map.

Matrix representation:

$$\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = [C]\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}$$

Basis for $F_2^2 = \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ → spans F_2^2.

$$\alpha_1\begin{bmatrix}0\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \quad \forall \alpha_1, \alpha_2 \in F_2.$$

Let some $u, v \in F_2^2$.

$$u = \alpha_1\begin{bmatrix}0\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\0\end{bmatrix}$$

$$v = \beta_1\begin{bmatrix}0\\1\end{bmatrix} + \beta_2\begin{bmatrix}1\\0\end{bmatrix}$$

$v = f(u)$.

$$\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = f(\alpha_1, \alpha_2)$$

$$\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = \begin{bmatrix}\alpha_1 + \alpha_2\\ \alpha_1 + \alpha_2\end{bmatrix} \Rightarrow \beta_1 = \alpha_1 + \alpha_2 = \beta_2 \quad -①$$

$$\therefore \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}$$

for ① to satisfy: $C = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ → matrix representation

d) $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = \begin{cases} x_1 & \text{, if } x_1 + x_n = 0 \\ x_n & \text{, otherwise} \end{cases}$

Let $u, v \in \mathbb{R}^n$

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ s.t. $u_1 + u_n = 0$ | $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ s.t. $v_1 + v_n \neq 0$

$\therefore f(u+v) = f\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = f\left(\begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}\right)$

we have, $u_1 + v_1 + u_n + v_n = (u_1 + u_n) + (v_1 + v_n)$
$= 0 + v_1 + v_n \neq 0$

$f\left(\begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}\right) = u_n + v_n$

also, $f(u) + f(v) = u_1 + v_n$

$\therefore \boxed{f(u+v) \neq f(u) + f(v)}$ → Property not satisfied

$\therefore f$ is not a linear map.