Injective Coloring

Advanced Graph Algorithms - CS608

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Abstract. In this paper, we attempt at answering a few questions related to the Injective Coloring problem on graphs. We provide a fixed parameter tractable algorithm to solve the injective coloring problem with parameters such as Vertex Cover, Distance to Clique, Neighborhood Diversity and more. Then, we give proofs of the only 2-critical and 3-critical graphs and provide intuitions for n-critical and $n-1$-critical graphs. Finally, we provide the injective chromatic number of some graph operations on cycles and paths like cartesian product, lexicographic product and tensor product.

Keywords: Graph Coloring · Injective Coloring · Fixed Parameter Tractable · Vertex Cover · Distance to Clique · Neighborhood Diversity · Twin Cover · Distance to Co-Cluster · Cartesian Product · Lexicographic Product · Tensor Product

1 Introduction

Given a graph G and a positive integer k, the injective coloring problem is to decide if there exists an injective k-coloring of G. A graph G has an injective k-coloring if there exists a vertex coloring of the graph G that uses k colors in such a way that no two vertices having a common neighbor have the same color. Thus, an injective coloring need not be a proper coloring.
The injective chromatic number of a graph, $\chi_i(G)$ is defined as the smallest number k for which there exists an injective k-coloring of G. Another way to look at the injective chromatic number is to consider the common neighbor graph $G^{(2)}$ of G defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{[u, v]$:there is a path of length 2 in G joining u and $v\}$. The common neighbor graph is useful because of the result that $\chi_i(G) = \chi(G^{(2)})$.

The rest of the paper is organised as follows. In Section 2, we give fixed parameter algorithms for some selected parameters. In Section 3, we classify the graphs that are k-critical for $k = 2, 3$. We also give a solid intuition for the classification of n-critical graphs. Finally, in Section 4, we give the chromatic number of some graph operations.

2 Fixed Parameter Tractable Algorithms

In the following sub-sections, we prove for each parameter, that there is a fixed parameter tractable algorithm to solve the injective coloring problem.

2.1 Vertex Cover as a parameter

Given a graph G, an integer k and a vertex set $X \subseteq G$ such that X is a vertex cover, we have to decide if G admits an injective k-coloring.

Since X is a vertex cover of G, the rest of the graph $I = G - X$ is an independent set. In order to find the injective chromatic number of G, it is sufficient to find the proper chromatic number of $G^{(2)}$ (as discussed before). We first partition the independent set I based on the type of the vertices. Since the only possible neighbors of the vertices in I belong to X, the number of different partitions is at most $2^{|X|} = 2^d$. We are mainly interested in how the $G^{(2)}$ graph would look for the vertices in I. Since all vertices of the same type have at least one common neighbour, we can say that the vertices belonging to the same partition in G will form a clique in $G^{(2)}$. Thus, each of the 2^d partitions now transform into a clique.

Thus, I becomes a union of cliques. If two partitions exist such that the common neighbors of one partition and the common neighbors of the other partition share a common vertex, then these two partitions also become completely connected with edges. If we now consider the vertices of X and partition each vertex of X into its own separate partition, then the $G^{(2)}$ graph that we have is now bounded in terms of its neighborhood diversity, i.e. the neighborhood diversity of $G^{(2)}$ is at most $2^d + d$. So we need the proper chromatic number of a graph with bounded neighborhood diversity. The proper coloring problem is known to be fixed parameter tractable with neighborhood diversity as a parameter.
Therefore, we say that the injective coloring problem is fixed parameter tractable with vertex cover as a parameter.

2.2 Distance to Clique as a parameter

Given a graph G, an integer k and a vertex set $X \subseteq G$ such that $G - X$ is a clique, we have to decide if G admits an injective k-coloring.

We know that $C = G - X$ is a clique. In order to find the injective chromatic number of G, we use the same idea again and find the $G^{(2)}$ graph of G. In this case too, we are mainly interested in the vertices of C in $G^{(2)}$. Since C is a clique in G, there is a path of length 2 to every other vertex of the clique. So, C remains a clique in $G^{(2)}$ as well. Hence, regardless of what happens

to the vertices of X in $G^{(2)}$, we can say that $G - X$ is a clique in $G^{(2)}$. So we need the proper chromatic number of a graph with distance to clique as a parameter. The proper coloring problem is known to be fixed parameter tractable with distance to clique as a parameter.
Therefore, we say that the injective coloring problem is fixed parameter tractable with distance to clique as a parameter.

2.3 Twin Cover as a parameter

Given a graph G, an integer k and a vertex set $X \subseteq G$ such that X is a twin cover, we have to decide if G admits an injective k-coloring.

Since X is a twin cover, $G - X$ only has edges between two vertices of the same type. Thus, we can partition X into some number of cliques, $X = C_1 \cup C_2 \cup \cdots \cup C_r$. In order to find the injective chromatic number of G, yet again we use the idea of computing the $G^{(2)}$ graph. Since each C_i is a clique in G, it will remain a clique in $G^{(2)}$ as well. Also two cliques C_i and C_j may become completely connected in $G^{(2)}$ if the type of both cliques share a common vertex. We can use the same idea that we used for vertex cover in this case also. Considering each vertex in X to be in a separate partition and considering each clique C_i to be a partition, we have a graph with neighborhood diversity bounded by $2^d + d$. So we need the proper chromatic number of a graph with bounded neighborhood diversity. The proper coloring problem is known to be fixed parameter tractable with neighborhood diversity as a parameter.
Therefore, we say that the injective coloring problem is fixed paramater tractable with twin cover as a parameter.

2.4 Neighborhood Diversity as a parameter

Given a graph G and an integer k such that the neighborhood diversity of the graph G is at most d, we have to decide if G admits an injective k-coloring.

The idea of computing the $G^{(2)}$ graph of G works in this case as well. Since the neighborhood diversity of the graph is $\leq d$, we can divide the vertices of the graph into d partitions such that all vertices in a partition have the same type, i.e. they have the same neighbors. So any two vertices from the same partition have the exact same neighbors. Now the same can be extended for the neighbors of these common neighbors, i.e. any two vertices in the same partition not only have common neighbors, but the vertices that are at distance 2 are also common. Hence, the $G^{(2)}$ graph of this graph will also have bounded neighborhood diversity of at most d. So we need the proper chromatic number of a graph with bounded neighborhood diversity. The proper coloring problem is known to be fixed parameter tractable with

neighborhood diversity as a parameter.
Therefore, we say that the injective coloring problem is fixed parameter tractable with neighborhood diversity as a parameter.

2.5 Distance to Co-Cluster as a parameter

Given a graph G, an integer k and a vertex set $X \subseteq V(G)$ such that $G - X$ is a co-cluster, we have to decide if G admits an injective k-coloring.

We again compute the $G^{(2)}$ graph of G. A co-cluster is the complement of a cluster, so it is the complement of a disjoint union of cliques. Thus, a co-cluster is a set of multiple independent sets that are completely connected. Each independent set becomes a clique in $G^{(2)}$, becuase there is a path of length 2 between two vertices of the independent set by going through any vertex in a different independent set. Also the individual vertex sets remain completely connected as there is again a path of length 2 between any two vertices of different independent sets by going through any vertex of a third independent set. Thus in $G^{(2)}$, we have a set of multiple cliques that are each completely connected. Thus, this whole structure becomes a clique, i.e. $G-X$ is a larger clique in $G^{(2)}$. So we need the proper chromatic number of a graph with distance to clique as a parameter. The proper coloring problem is known to be fixed parameter tractable with distance to clique as a parameter. Therefore, we say that the injective coloring problem is fixed parameter tractable with distance to co-cluster as a parameter.

3 *k*-critical graphs

3.1 2-critical graphs

P_3

Claim. *A graph G is 2-critical if and only if it is* $\mathbf{P_3}$.

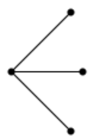
We prove this claim by first showing the reverse direction, i.e. P_3 is 2-critical. Let the vertices of P_3 be $\{v_1, v_2, v_3\}$ and the edges of P_3 be $\{(v_1, v_2), (v_2, v_3)\}$. Since v_1 and v_3 have a common neighbor (v_2), they must be coloured differently and v_2 can be colored with either of the 2 colors. Thus, P_3 is itself injectively 2-colorable. Now we note that every subset of P_3 (either by deleting vertices or edges) is either P_2 or an independent set. The injective chromatic number of both these graphs are less than 2 ($\chi_i(P_2) = 1$).

Hence P_3 is 2-critical.

Now we prove the forward direction, i.e. P_3 is the only 2-critical graph. The only other connected graph with $n = 3$ is K_3. It is trivial to notice that $\chi_i(K_3) = 3$. Now for any vertex with $n \geq 4$, we know that there exists at least one vertex v such that $deg(v) \geq 2$. (If not, then $2m = \sum deg(v) \leq n$. However, for a connected graph we need $m \geq n - 1$) As soon as we have a vertex with degree 2 or more, P_3 becomes a subgraph of the graph. Thus, a subgraph of the original graph has an injective 2-coloring, so the original graph cannot be a 2-critical.

Thus, the only 2-critical graph is P_3.

3.2 3-critical



$K_{1,3}$

Claim. *A graph G is 3-critical if and only if it is* $\mathbf{C_{2k+1}}, \mathbf{C_{4k+2}}$ *or* $\mathbf{K_{1,3}}$

We prove this claim by first showing the reverse directions, i.e. C_{2k+1}, C_{4k+2} and $K_{1,3}$ are all 3-critical. Note that $\chi_i(P_n) = 2 \forall n \geq 3$.

- It is easy to see that $\chi_i(C_{2k+1}) = 3$, i.e. every odd cycle needs 3 colors for an injective coloring. Also, every subgraph of an odd cycle is a path. Thus, every subgraph has injective chromatic number less than 3.
- It is also easy to see that $\chi_i(C_{4k+2}) = 3$. Also, every subgraph of an even cycle is again a path or a disjoint union of two paths. Both can be colored injectively with 2 colors. Thus, every subgraph has injective chromatic number less than 3.
- $K_{1,3}$ has a vertex of degree 3. Therefore, each of its 3 neighbors will have to receive a different color and this vertex can be colored with any of the 3 colors. Also, every subgraph of $K_{1,3}$ is either P_3 or an independent set. Thus, every subgraph has injective chromatic number less than 3.

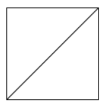
Now we prove the forward direction of the proof, i.e. C_{2k+1}, C_{4k+2} and $K_{1,3}$ are the only 3-critical graphs. For any graph, if there exists a vertex of degree 3 or more, then $K_{1,3}$ is a subgraph of the original graph. So the chromatic number of a subgraph is 3, therefore the original graph cannot be 3-critical.

Therefore, we are only left with considering graphs with $\Delta(G) \leq 2$. Any graph with $\Delta(G) \leq 2$ is either a path, cycle or P_2. The injective chromatic number of P_2 is 1, so it cannot be 3-critical. The injective chromatic number of any path is 2, so it cannot be 3-critical. Finally, the injective chromatic number of an even cycle C_{4k+2} is 2, so it cannot be 3-critical.

Thus, the only graphs that can be 3-critical are C_{2k+1}, C_{4k+2} and $K_{1,3}$.

3.3 n-critical

For a graph G to be n-critical, we first need $\chi_i(G) = n$. In the paper 'On the injective chromatic number of graphs', the authors have proved a lemma that states that $\chi_i(G) = n$ if and only if **either G is a complete graph** or **G has a diameter 2 and every edge of G is contained in a triangle**. It is easy to notice that both K_4 and the graph shown below have injective chromatic number equal to 4. Since the graph shown below is a subgraph of K_4, K_4 cannot be n-critical. However, we can check that the graph in the image is indeed 4-critical by computing its $G^{(2)}$ graph. (It will be a clique)



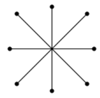
Therefore, not all graphs that satisfy the above conditions are n-critical. So our intuition is that we must find a graph on n vertices such that it uses as few triangles as possible to completely cover the vertices. We were not able to provide a general classification of such graphs nor are we sure that these graphs are the only n-critical graphs, but the following images depict how we can construct an n-critical graph on any number of vertices. We would also

like to report that our construction uses $2n - 3$ edges, while the complete graph on n vertices uses ${}^nC_2$ edges. Thus, our graph removes ${}^nC_2 - (2n - 3)$ edges from the complete graph.

n	Edges removed
3	0
4	1
5	3
6	6
7	10

3.4 $n-1$-critical

For $n-1$-critical graphs we were not able to generalize any class of graphs. However, we found that the star graph shown in the figure below is $n-1$-critical. This is because of we compute the $G^{(2)}$ graph of the star graph, we get a $n-1$ complete graph with the vertex at the centre disconnected from the rest of the graph. Any removal of a vertex or edge would reduce the injective chromatic number of the star graph to $n-2$, hence the star graph is $n-1$-critical.



4 Chromatic Number of some Graph Operations

Theorem 2.1 of Section 2 of Injective Coloring on Graph Operations paper states that "If G and H are connected graphs that are both distinct from K2, then $X_i(G\square H) \leq X_i(G) \times X_i(H)$." Where $\square$ represents the cartesian product operation. Theorem 2.3 of the same paper proves that injective chromatic number of a grid graph is 4. The second theorem can be obtained from the first theorem simply substituting the injective chromatic number of paths to be 2. We also find that the cartesian product of 2 paths (P_n) forms a grid (see Fig 2.). With the same theorem, we can give the following bounds:

Table 1: Last column contains max possible X_i of cartesian product graph

G	$\mathbf{X_i(G)}$	**H**	$\mathbf{X_i(H)}$	$\mathbf{X_i(G\square H)}$
P_m	2	P_n	2	4
P_m	2	C_n^{*1}	2	4
P_m	2	C_n^{*2}	3	6
C_m^{*1}	2	C_n^{*1}	2	4
C_m^{*1}	2	C_n^{*2}	3	6
C_m^{*2}	3	C_n^{2}	3	9

^{*1}: Cycles of type $m\%4 \equiv 0, n\%4 \equiv 0$
^{*2}: Cycles of type $m\%4\epsilon\{1,2,3\}, n\%4\epsilon\{1,2,3\}$

We find the following properties for lexicographic product of 2 graphs as follows:

- The product of a path, P_n (or cycle C_n) with any arbitrary graph H returns n partitions, each a copy of H. Each partition is completely

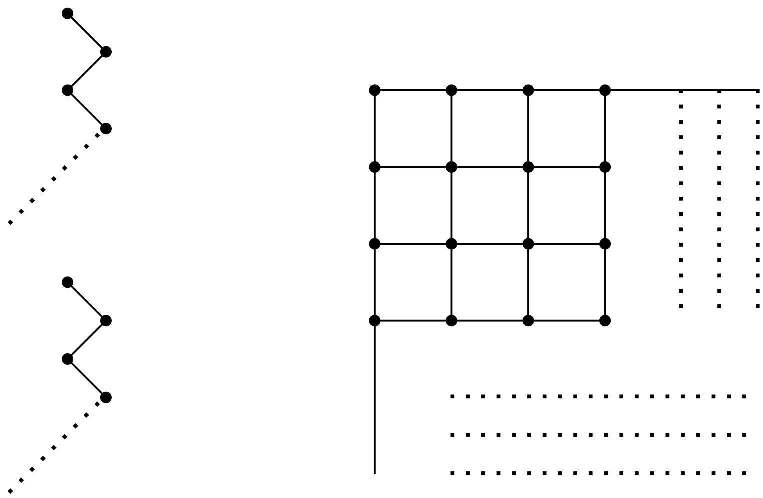


Figure 2: Cartesian Product of 2 paths

connected to the next partition, analogous to vertices in the path (cycle) graph.

- Each partition can be colored in no less than $\|H\|$ colors
- Any partition of $\|H\|$ vertices can't be injectively colored in the same color as any vertex in a partition adjacent to it or at a distance 2 from it.
  - Adjacent partition vertices can't be colored in the same color because vertices in any 2 adjacent partitions will be completely connected across partitions in the lexicographic product graph
  - Partitions at distance 2 can't be colored with the same color because for each vertex in any partition (say A) there exists an edge to every vertex in the neighboring partition (say B), and in turn every vertex in B as adjacent some vertex in the partition C at distance 2 from A and neighboring B.

Hence, the injective coloring of lexicographic product of a path and an arbitrary graph is simple and has a chromatic number of $3\|H\|$. However we get the following cases for the injective coloring of the lexicographic product of a cycle and an arbitrary graph:

- $X_i(G \circ H) = 3 \times \|V(H)\|$, if $G = C_{3k}$
- $X_i(G \circ H) = 4 \times \|V(H)\|$, if $G = C_{3k+1}$
- $X_i(G \circ H) = 5 \times \|V(H)\|$, if $G = C_{3k+2}$

Finally, coming to the tensor product:

- The tensor product of 2 paths results in 2 disconnected components which happen to be grids. From the earlier theorem, we know that grids can be injectively colored with atmost 4 colors.