3 (a)

<u>To Prove</u> : $\nabla_x (x^T a) = \nabla_x (a^T x) = a$

for any two column vectors $x, a \in \mathbb{R}^n$

<u>Proof</u> : $x = [x_1 \dots x_n]^T \Rightarrow x^T = [x_1 \dots x_n]$

$a = [a_1 \dots a_n]^T \Rightarrow a^T = [a_1 \dots a_n]$

$$f(x) = (x^T . a) = [x_1 \dots x_n][a_1 \dots a_n]^T \Rightarrow f(x) = [x_1 a_1 \dots x_n a_n]$$

$$\cancel{\Rightarrow f(x) = x_1 a_1 \dots x_n a_n}$$

$$\therefore \nabla_x f(x) = \left[\frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_n} \right]^T$$

$$= [a_1 \dots a_n]^T$$

$$\Rightarrow \nabla_x (x^T a) = a \quad \text{--- } (1)$$

Say, $g(x) = (a^T x) = [a_1 \dots a_n][x_1 \dots x_n]^T$

$$= [a_1 x_1 \dots a_n x_n] = x^T a = f(x)$$

$$\therefore g(x) = f(x) \Rightarrow \nabla_x g(x) = \nabla_x f(x) \quad \text{--- } (2)$$

from (1) & (2), $\nabla_x (x^T a) = \nabla_x (a^T x) = a$ ∎

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

3 (b)

<u>To Prove</u> : $\nabla_x (x^T A x) = (A + A^T) x$

for col. vector $x \in \mathbb{R}^n$ & $n \times n$ mtx A.

<u>Proof</u>: $x = [x_1 \; \cdots\cdots \; x_n]^T \Rightarrow x^T = [x_1 \; \cdots \; x_n]$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

For $x^T A x$, we multiply left to right.

$$x^T A = [x_1 \; \cdots \; x_n]_{1\times n} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}_{n\times n}$$

$$= \Big[\underbrace{(x_1 a_{11} + \cdots\cdots + x_n a_{n1}) \;\; \cdots\cdots \;\; (x_1 a_{1n} + \cdots + x_n a_{nn})} \Big]_{1\times n}$$

then, $x^T A x = [\quad \downarrow \quad]_{1\times n} \cdot [x_1 \; \cdots\cdots \; x_n]^T_{n\times 1}$

$$\Rightarrow f(x) = x^T A x = \Big[x_1 (x_1 a_{11} + \cdots + x_n a_{n1}) + \cdots + x_n (x_1 a_{1n} + \cdots + x_n a_{nn}) \Big]_{1\times 1}$$

$$\therefore f(x) = \Big[\left(x_1^2 a_{11} + x_2^2 a_{22} + \cdots + x_n^2 a_{nn} \right)$$
$$+ x_1 x_2 a_{12} + x_1 x_2 a_{21}$$
$$+ x_1 x_3 a_{13} + x_1 x_3 a_{31}$$
$$+ \cdots$$
$$+ x_1 x_n a_{1n} + x_1 x_n a_{n1}$$
$$+ \vdots \cdots \Big]_{1 \times 1}$$

$$= \Big[\left(x_1^2 a_{11} + x_2^2 a_{22} + \cdots + x_n^2 a_{nn} \right) + x_1 \left(x_2 (a_{12} + a_{21}) + \cdots + x_n (a_{n1} + a_{1n}) \right)$$
$$+ \cdots + x_n \left(x_1 (a_{n1} + a_{1n}) + \cdots + x_{n-1} (a_{n,n-1} + a_{n-1,n}) \right) \Big]_{1 \times 1}$$

$$D_x f(x) = \begin{bmatrix} \left(2x_1 a_{11} + x_2(a_{21} + a_{12}) + \cdots\cdots + x_n(a_{n1} + a_{1n})\right) \\ \vdots \\ \left(2x_n a_{nn} + x_1(a_{n1} + a_{1n}) + \cdots + x_{n-1}(a_{n-1,1} + a_{1,n-1})\right) \end{bmatrix}_{n\times 1}$$

$\to$ 2

Solving for $(A + A^T) x = \bar{f}(x)$.

$$A + A^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}_{n\times n} + \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \\ a_{1n} & & a_{nn} \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} 2a_{11} & \cdots & a_{1n} + a_{n1} \\ \vdots & \ddots & \\ a_{n1} + a_{1n} & & 2a_{nn} \end{bmatrix}_{n\times n}$$

$$(A + A^T) x = \underbrace{}_{\left[\,\cdot\,\right]_{n\times n} \cdot \left[x_1 \cdots x_n\right]^T_{n\times 1}}$$

$$= \begin{bmatrix} (2a_{11} x_1 + (a_{12} + a_{21}) x_2 + \cdots + (a_{1n} + a_{n1}) x_n) \\ \vdots \\ ((a_{n1} + a_{1n}) x_1 + \cdots + 2a_{nn} x_n) \end{bmatrix}_{n\times 1}$$

$\rightarrow$ 1

from ① & ②, we have,

$$\nabla_x f(x) = \nabla_x (x^T A x) = (A + A^T) x. \quad \blacksquare$$

3(c) Say, $A \rightarrow$ symmetric.

then by property of symmetric matrices,

$$A + A^T = 2A.$$

thus, $\nabla_x f(x) = \nabla x(x^T A x) = (A + A^T)\, x = 2Ax.$

3(d)

$$f(x) = (Ax + b)^T (Ax+b)$$
$$= (x^T A^T + b^T)(Ax+b)$$
$$= \underbrace{x^T A^T A x} + x^T \underbrace{A^T b} + \underbrace{b^T A} x + b^T b$$
$$= x^T \cdot (A^T A) \cdot x + x^T \cdot (A^T b) + (b^T A) \cdot x + (b^T b)$$
$$\nabla_x f(x) = (A^T A + (A^T A)^T) x + A^T b + b^T A + 0$$
$$= (A^T A + A A^T) x + A^T b + b^T A$$
$$= 2(A^T A) x + 2(A^T b) \quad \text{(due to symmetry \& transpose properties)}$$
$$= 2A^T (Ax + b).$$

Linear Regression

(i) Training Cost = 39.243

(ii) Testing Cost = 206.796