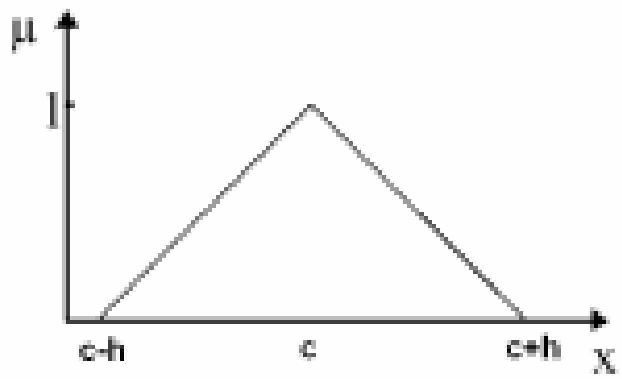
Fuzzification and De-fuzzification

Membership Function of a fuzzy set

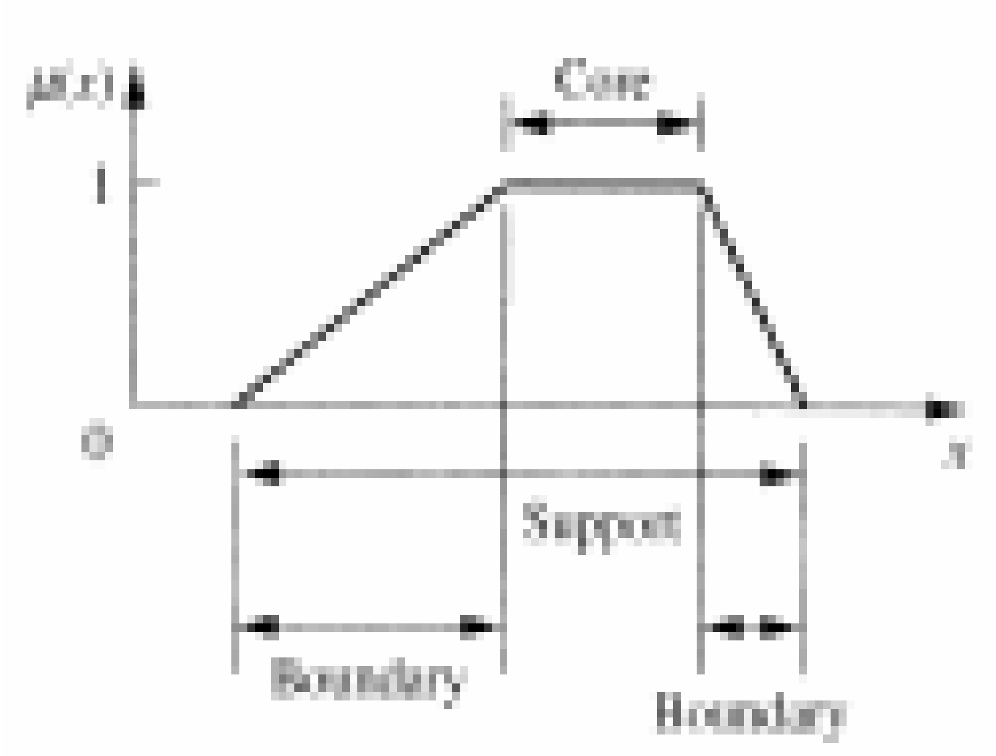
- All information contained in a fuzzy set is described by its membership function.
- For simplicity we take continuous fuzzy sets**.**

Continuous example:

$$\mu_A(x) = \begin{cases} 1 + \dfrac{x-c}{h}, & x \in [c-h, c] \\ 1 - \dfrac{x-c}{h}, & x \in [c, c+h] \\ 0, & \text{otherwise} \end{cases}$$



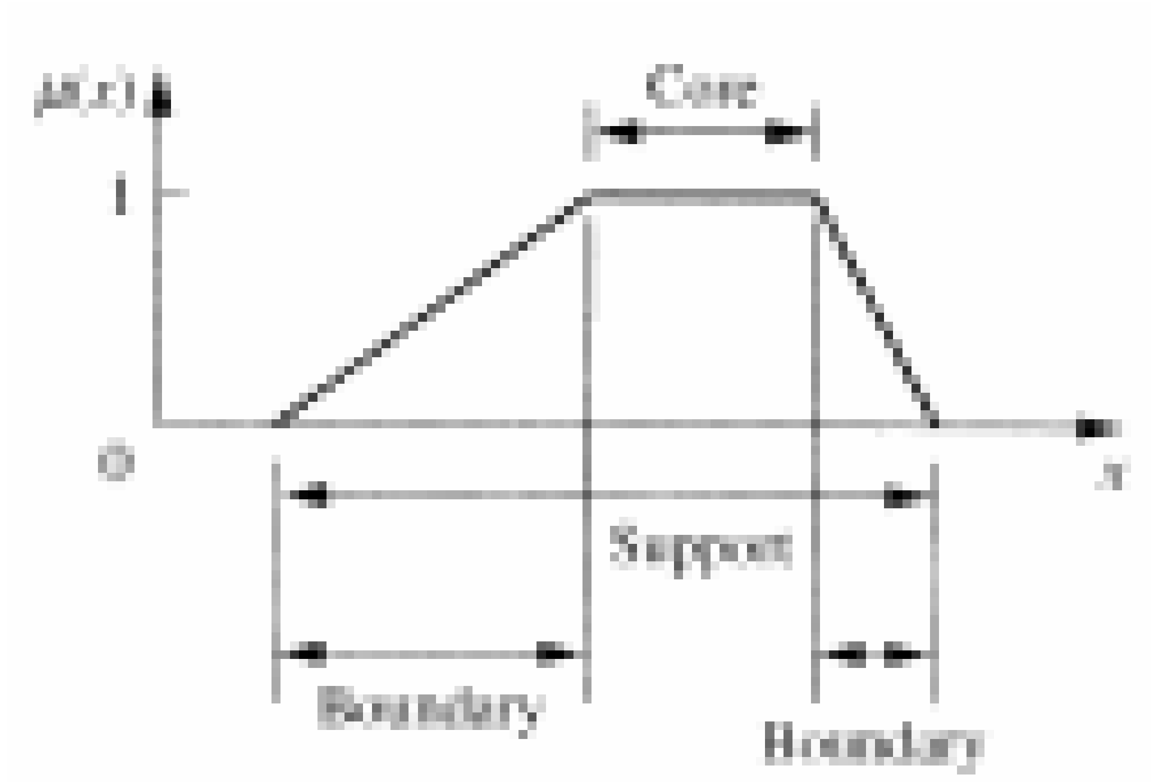
Properties of Membership Function



Core: comprises those elements x of the universe such that $\mu_A(x) = 1$.

$$\operatorname{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$$

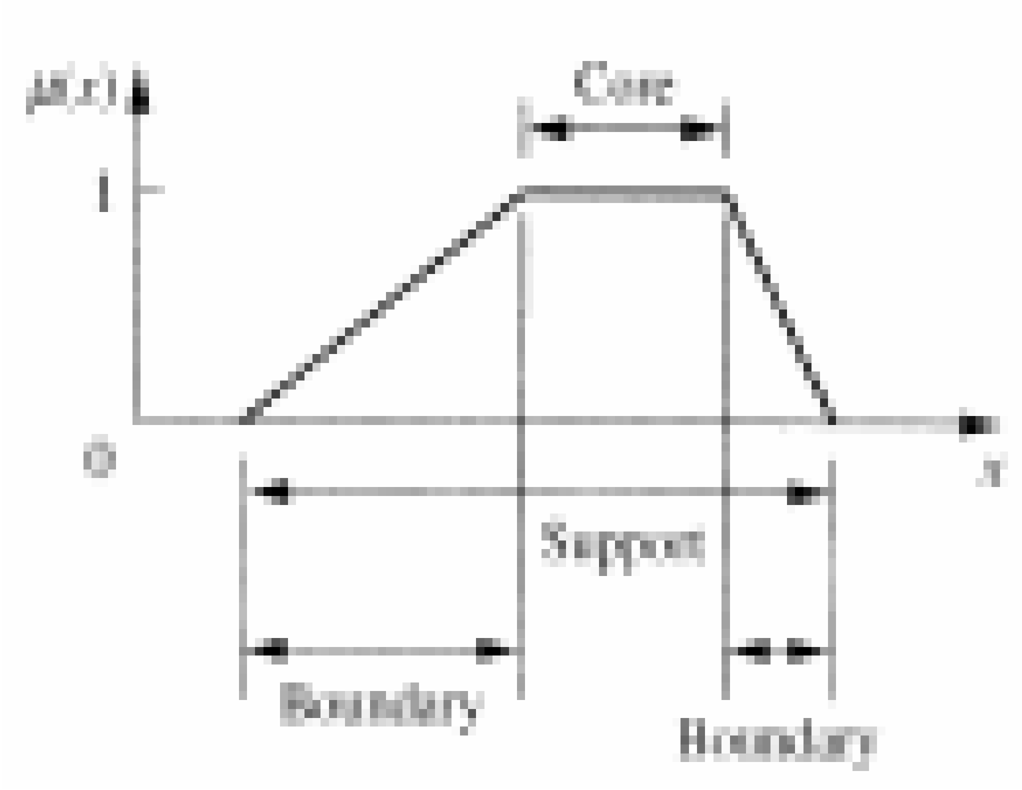
Properties of Membership Function



Support : support of a fuzzy set A is a crisp set that contains all elements of A with non-zero membership grade:

$$\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$$

Properties of Membership Function



Boundary : boundaries comprise those elements x of the universe such that $0 < \mu_A(x) < 1$

$$\text{bnd}(A) = \{x \in X \mid 0 < \mu_A(x) < 1\}$$

Properties of Membership Function

Height of a fuzzy set A, hgt (A) or h (A) is the largest membership grade obtained by any element in that set.

$$\mathrm{hgt}(A) = \sup_{x \in X} \mu_A(x)$$

Set A is called normal if hgt(A)=1
and subnormal if hgt(A)<1

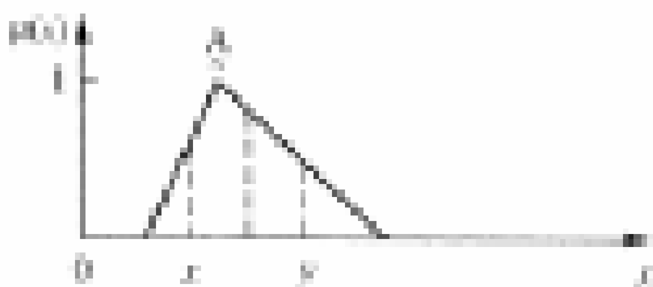
Properties of Membership Function

A convex set is described by membership function whose membership values are:

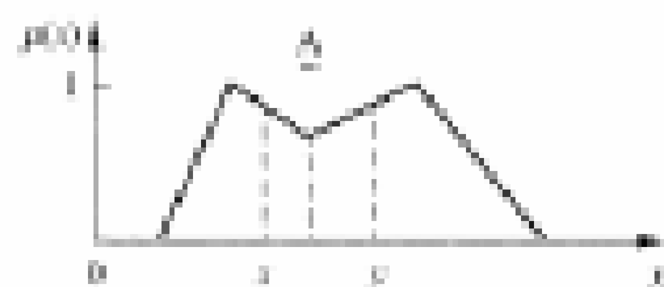
- Either strictly increasing or decreasing.
- Or specifically strictly monotonically increasing then strictly monotonically decreasing.
- For any values of x, y, z in fuzzy set $\underline{A}$, the relation x<z<y implies that,

$$\mu_{\underline{A}}(z) \geq \min[\mu_{\underline{A}}(x), \mu_{\underline{A}}(y)]$$

- Then the set $\underline{A}$ is convex set.
- Also if $\underline{A}$ and $\underline{B}$ are convex then $\underline{A} \cap \underline{B}$ is also convex.



Convex, normal fuzzy set



Non convex, normal fuzzy set

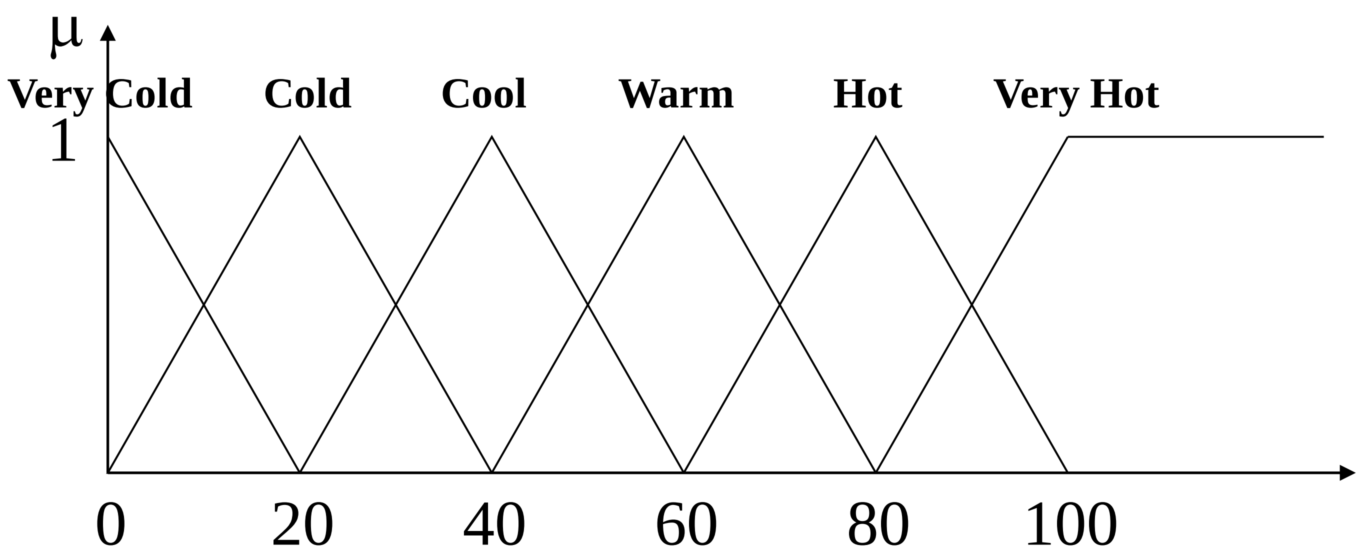
- Membership functions are typically defined on one dimension but can certainly be defined on n-dimension.
- If it is defined on two dimension, then curves become surfaces.
- And for three or more dimension the curve become hypersurfaces.
- Interval valued fuzzy set.

Fuzzification

- Process of making a crisp quantity fuzzy.
- Following are some of the methods of assigning membership values or functions to fuzzy variables.
  - ➢ Intuition
  - ➢ Inference
  - ➢ Rank Ordering
  - ➢ Angular Fuzzy Sets
  - ➢ Neural Networks
  - ➢ Genetic Algorithms
  - ➢ Inductive Reasoning
  - ➢ Soft Partitioning
  - ➢ Meta Rules
  - ➢ Fuzzy Statistics.

Fuzzification

- Intuitive Approach
  - ✓ From human experience.
  - ✓ From knowledge base.
  - ✓ From a detailed study of the system in hand.



Fuzzification

- Inference Approach
  - ✓ We use knowledge to perform deductive reasoning.
  - ✓ We deduce or infer a conclusion.
- Family of Triangles Example
  - Let
    - $\underline{I}$ is a set of fuzzy isosceles triangle
    - $\underline{R}$ is a set of fuzzy right triangle
    - $\underline{IR}$ is a set of fuzzy isosceles and right triangle
    - $\underline{E}$ is a set of fuzzy equilateral triangle
    - $\underline{T}$ is a set of other fuzzy triangles
  - The universe of discourse is defined as

$$U = \{(A, B, C) \mid A \geq B \geq C \geq 0; A + B + C = 180^0\}$$

Fuzzification

$$\mu_{\underline{I}}(A,B,C)=1-\frac{1}{60^0}\min(A-B,B-C)$$

$$\mu_{\underline{R}}(A,B,C)=1-\frac{1}{90^0}\,|\,A-90^0\,|$$

$$\mu_{\underline{E}}(A,B,C)=1-\frac{1}{180^0}(A-C)$$

$$\underline{IR}=\underline{I}\cap\underline{R};\quad \underline{T}=\overline{(\underline{I}\cup\underline{R}\cup\underline{E})}=\overline{\underline{I}}\cap\overline{\underline{R}}\cap\overline{\underline{E}}$$

$$If\quad \{X: A=85^0\geq B=50^0\geq C=45^0; A+B+C=180^0\}$$

$$then\quad \mu_{\underline{R}}(x)=0.94,\mu_{\underline{I}}(x)=0.916,\mu_{\underline{IR}}(x)=0.916,\mu_{\underline{E}}(x)=0.7,\mu_{\underline{T}}(x)=0.05$$

Different MF Forms

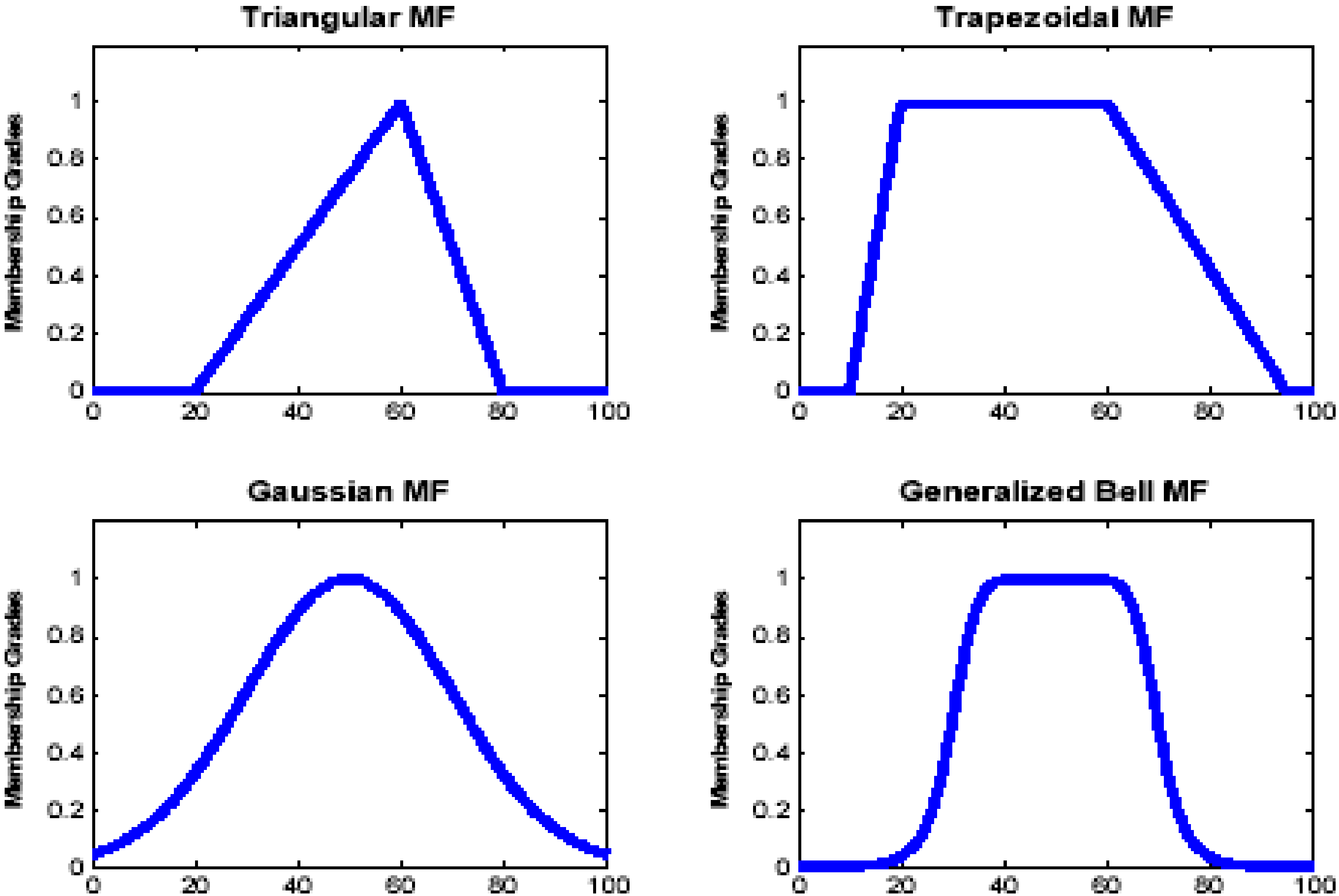
- Triangular MF $trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b} \right), 0 \right)$

- Trapezoidal MF $trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c} \right), 0 \right)$

- Gaussian MF $gaussmf(x;\sigma,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$

- Generalized bell MF $gbellmf(x;a,b,c) = \dfrac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$

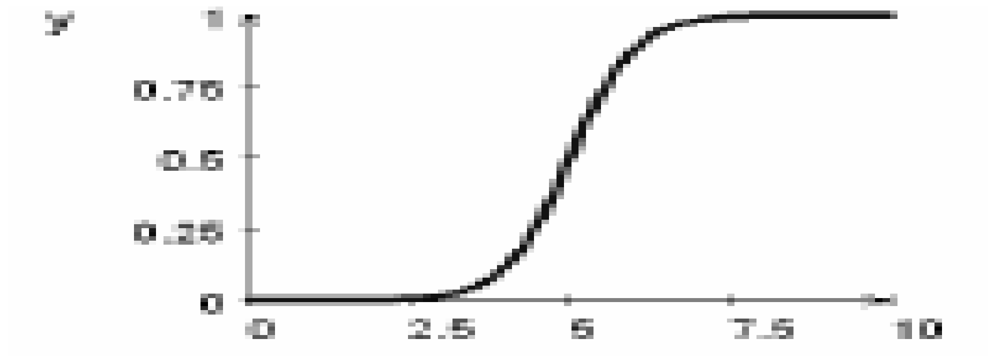
Different MF Forms



- Sigmoidal MF

$$sigmf(x;a,c) = \frac{1}{1+e^{-a(x-c)}}$$

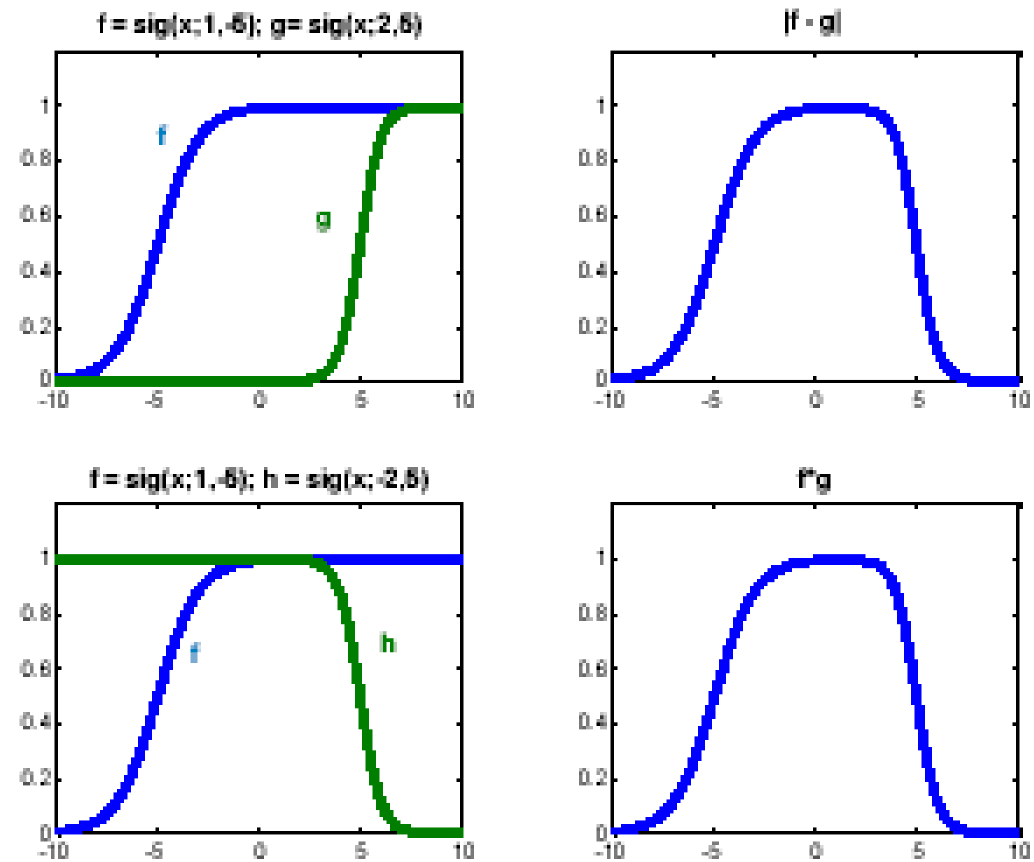
$$\frac{1}{1+e^{-2.5(x-6)}}$$



Extensions of Sigmoidal MF:

- Absolute difference of two sig. MF

- Product of two sig. MF



λ-cuts or α cuts for fuzzy sets

- ✓ $[1 \geq \lambda \geq 0]$
- ✓ A λ-cut set is a fuzzy set containing elements with membership values greater than equal λ.

$$A_\lambda = \{x \mid \mu_{\underline{A}}(x) \geq \lambda\}$$

❖ Any element $x \in A_\lambda$ belongs to $\underline{A}$ with $\mu \geq \lambda$.

$$\underline{A} = \{\frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f}\} \qquad \lambda = 1, 0.9, 0.6, 0.3, 0^+, 0$$

$$A_1 = \{a\}; \quad A_{0.9} = \{a, b\}; \quad A_{0.6} = \{a, b, c\}; \quad A_{0.3} = \{a, b, c, d\}$$

$$A_{0^+} = \{a, b, c, d, e\}; \quad A_0 = \{a, b, c, d, e, f\}$$

λ-cuts for Fuzzy Relations

$$\underline{R} = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix} \Rightarrow \lambda = 1$$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_\lambda = \{(x, y) \mid \mu_R(x, y) \geq \lambda\}$$

If $\underline{R}$ is a two-dimensional array defined on the universe X and Y, then any pair $(x,y) \in R_\lambda$ belongs to $\underline{R}$ with a strength of relation greater than or equal to λ.

λ-cuts for Fuzzy Sets & Relations

- Any λ-cut set of a fuzzy set is referred to as the nearest ordinary set to the fuzzy set.
- Any λ-cut relation of a fuzzy relation is referred to as the nearest ordinary relation to the fuzzy relation.

Cardinality of a Fuzzy Set

- The cardinality (scalar cardinality) of a fuzzy set $\underline{A}$ is the summation of the membership grades of all elements in the set $\underline{A}$.
- It is given by

$$|\underline{A}| = \sum_{x \in U} \mu_{\underline{A}}(x)$$

- The relative cardinality of $\underline{A}$ is

$$|\underline{A}|_{rel} = \frac{|\underline{A}|}{|U|}$$

Defuzzification Methods

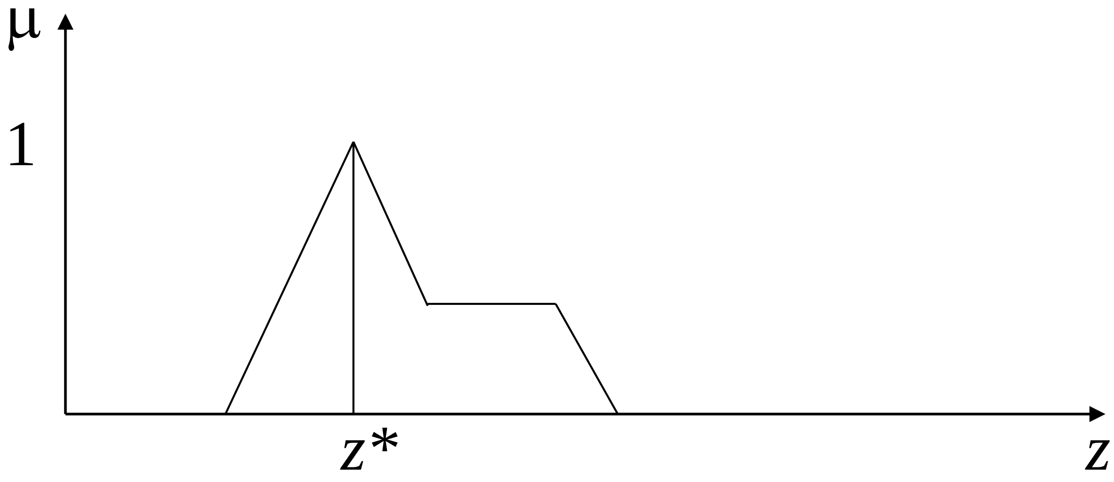
- The process of obtaining back the crisp values from the fuzzified state.
- It involves the conversion of a fuzzy quantity to a precise quantity.
- Several methods for defuzzifying fuzzy output:
  - Max-membership principle
  - Centroid method
  - Weighted average method
  - Mean-max membership
  - Centre of sums
  - Center of largest area
  - First (or last) of maxima

Max-membership principle

✓Also referred to as height method.

✓Limited to peak output functions.

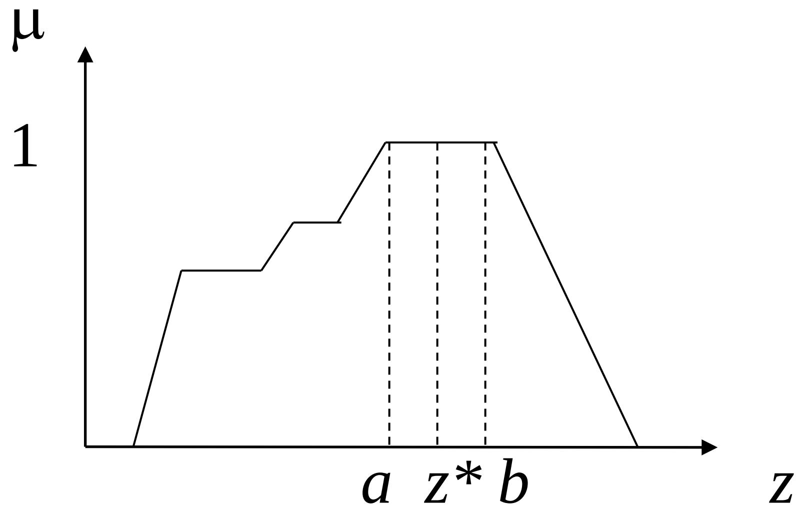
$$\mu_{\underline{C}}(z^*) \geq \mu_{\underline{C}}(z) \quad \forall z \in Z$$



Mean-max membership method

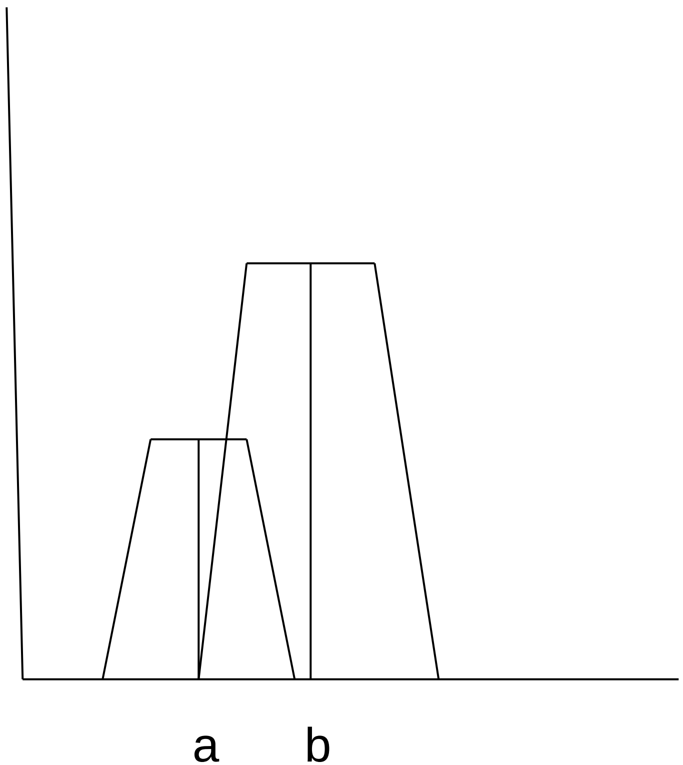
- ✓ Also referred to as the middle-of-maxima method.
- ✓ Similar to max-membership method, but the locations of the maximum membership can be non-unique.
- ✓ The maximum membership can be a plateau rather than a single point.

$$z^* = \frac{a+b}{2}$$



Weighted average method :

Valid for symmetrical output membership functions



$$z^* = \frac{\sum \mu_{\underline{C}}(\bar{z}).z}{\sum \mu_{\underline{C}}(\bar{z})}$$

$$z^* = \frac{a(0.5)+b(0.9)}{0.5+0.9}$$

a and b are values of means of their respective shapes.