Classical & Fuzzy Relations

Cartesian Product

- An ordered sequence of r elements in the form $(a_1, a_2, a_3, \ldots, a_r)$ is called an ordered tuple.
- If r=2, it forms an ordered pair.
- For crisp sets $A_1, A_2, A_3, \ldots, A_r$
  - $(a_1, a_2, a_3, \ldots, a_r)$, where $a_1 \in A_1$, $a_2 \in A_2$ etc. is called the cartesian product $A_1 \times A_2 \times A_3 \ldots \times A_r$.
- Cartesian product is not the same as arithmetic product.
- When all A_r are identical, $A_1 \times A_2 \times A_3 \ldots \times A_r$ can be denoted as A^r.

Classical Relations

- A subset of the cartesian product of *r* sets is called a *r-ary relation* over $A_1, A_2, A_3, \ldots, A_r$.
- If r=2, it is a binary relation from A_1 into A_2 (A_1 x A_2).
- For a cartesian product of two universe X and Y,

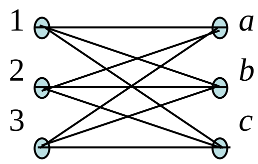
the *strength* of the relationship between the pairs of elements is measured by a characteristic function χ. If χ=1, it implies complete relationship and χ=0 implies no relationship, i.e.,

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

$$\chi_{X \times Y} = \begin{cases} 1 & (x, y) \in X \times Y \\ 0 & (x, y) \notin X \times Y \end{cases}$$

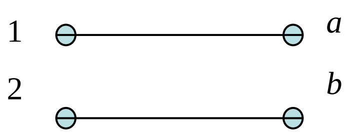
Classical Relations

- A binary relationship can be represented by a two-dimensional matrix.



$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

This is an example of unconstrained or universal relation



$$R = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

This is an example of constrained or identity relation

Cardinality of Crisp Relations

- If n elements of set X are related to n elements of set Y, then the cardinality of the relation R between these two sets is the product of the cardinalities of the two sets, i.e., $n_X * n_Y$.

Composition of Crisp Relations

- Given a relation R between X & Y and
  Given a relation S between Y & Z, find
  a relation T between X & Z.

Composition of Crisp Relations

- Two forms of composition operations are there.
  - max-min composition, *T* = *R* O *S*

$$\chi_T(x,z) = \vee_{y \in Y}(\chi_R(x,y) \wedge \chi_S(y,z))$$

  - max-product (max dot) composition, *T* = *R* O *S*

$$\chi_T(x,z) = \vee_{y \in Y}(\chi_R(x,y) \bullet \chi_S(y,z))$$

Example

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$$\chi_T(x_1, z_1) = \max[\min(1,0), \min(0,0), \min(1,0), \min(0,0)] = 0$$

$$\chi_T(x_1, z_2) = \max[\min(1,1), \min(0,0), \min(1,1), \min(0,0)] = 1$$

$$\therefore T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy Relations

- It is a mapping $\underline{R}$ from the cartesian product $X \times Y$ to the interval [0, 1], where the strength of $\underline{R}$ is expressed by the membership function of the relation, $\mu_{\underline{R}}(x, y)$.

Let $\underline{R}$ be a fuzzy relation between two sets X = {Calcutta, Bhuvaneshwar} and Y={Bombay, Pune, Calcutta} representing "very far". Then R can be represented by

$\underline{R}$ = 0.8/(Calcutta, Bombay) + 1.0/(Calcutta, Pune) + 0.0/(Calcutta, Calcutta) + 0.7/(Bhuvaneshwar, Bombay) + 0.9/ (Bhuvaneshwar, Pune) + 0.4/ (Bhuvaneshwar, Calcutta)

	Calcutta	Bhuvaneshwar
Bombay	0.8	0.7
Pune	1.0	0.9
Calcutta	0.0	0.4

2-D Matrix

Operations on Fuzzy Relations

If $\underline{R}$ & $\underline{S}$ be fuzzy relations for $X \times Y$, the following operations can be applied:

- Union $$\mu_{\underline{R}\cup\underline{S}}(x,y)=\max(\mu_{\underline{R}}(x,y),\mu_{\underline{S}}(x,y))$$

- Intersection $$\mu_{\underline{R}\cap\underline{S}}(x,y)=\min(\mu_{\underline{R}}(x,y),\mu_{\underline{S}}(x,y))$$

- Complement $$\mu_{\underline{R}}(x,y)=1-\mu_{\underline{R}}(x,y)$$

- Containment $$\underline{R}\subset\underline{S}=\mu_{\underline{R}}(x,y)\le\mu_{\underline{S}}(x,y)$$

Fuzzy Cartesian Product

- For two fuzzy sets $\underline{A}$ on universe X and $\underline{B}$ on universe Y, the cartesian product (or relation) $\underline{R}$ between $\underline{A}$ and $\underline{B}$ is given by

$$\underline{A} \times \underline{B} = \underline{R} \subset X \times Y$$

$$\mu_{\underline{R}}(x, y) = \mu_{\underline{A} \times \underline{B}}(x, y) = \min(\mu_{\underline{A}}(x), \mu_{\underline{B}}(y))$$

Example

$$\underline{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}$$

If $\underline{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}$

Then $$\underline{A} \times \underline{B} = \underline{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Composition of Fuzzy Relations

- Two forms of composition operations are there.
  - Max-min composition, $\underline{T} = \underline{R} \circ \underline{S}$

$$\mu_{\underline{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\underline{R}}(x,y) \wedge \mu_{\underline{S}}(y,z))$$

  - max-product (max dot) composition, $\underline{T} = \underline{R} \circ \underline{S}$

$$\mu_{\underline{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\underline{R}}(x,y) \bullet \mu_{\underline{S}}(y,z))$$

But $\underline{\boldsymbol{R}} \circ \underline{\boldsymbol{S}} \neq \underline{\boldsymbol{S}} \circ \underline{\boldsymbol{R}}$

Example (max-min composition)

$$Let \quad X = \{x_1, x_2\}; \quad Y = \{y_1, y_2\}; \quad Z = \{z_1, z_2, z_3\}$$

$$\underline{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \qquad \underline{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$\therefore \mu_{\underline{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

$$Thus, \quad \underline{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Example (max-dot composition)

$$Let \quad X = \{x_1, x_2\}; \quad Y = \{y_1, y_2\}; \quad Z = \{z_1, z_2, z_3\}$$

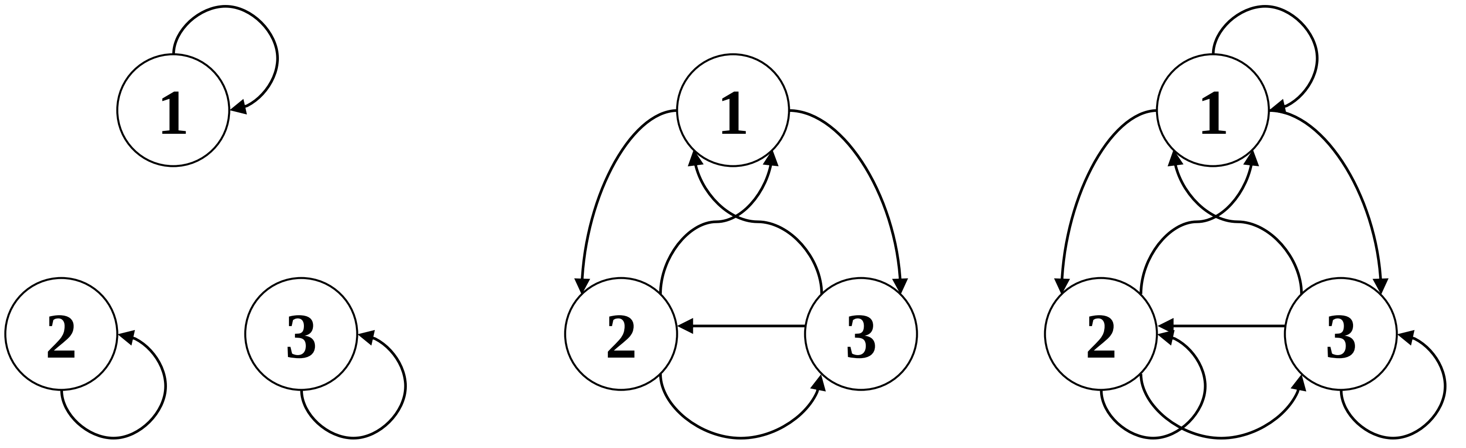
$$\underline{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \qquad \underline{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$\therefore \mu_{\underline{T}}(x_2, z_2) = \max[(0.8 \bullet 0.6), (0.4 \bullet 0.7)] = 0.48$$

$$Thus, \quad \underline{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}$$

Properties of Relations

- Any relation can have the properties of reflexivity, symmetry and transitivity.



Reflexivity **Symmetry** **Transitivity**

Crisp Equivalence

- Mathematically, a crisp relation R from X to X is an equivalence relation if it possesses all the three properties.

  - Reflexivity
    - $(x_i, x_i) \in R$ or $\chi_R(x_i, x_i) = 1$
  - Symmetry
    - $(x_i, x_j) \in R$ implies $(x_j, x_i) \in R$ or $\chi_R(x_i, x_j) = \chi_R(x_j, x_i)$
  - Transitivity
    - $(x_i, x_j) \in R$ and $(x_j, x_k) \in R$ implies $(x_i, x_k) \in R$

    or $\chi_R(x_i, x_j)$ and $\chi_R(x_j, x_k) = 1$ implies $\chi_R(x_i, x_k) = 1$

Fuzzy Equivalence or Similarity Relation

- Reflexivity
  - $\mu_{\underline{R}}(x_i, x_i) = 1$
- Symmetry
  - $\mu_{\underline{R}}(x_i, x_j) = \mu_{\underline{R}}(x_j, x_i)$
- Transitivity
  - $\mu_{\underline{R}}(x_i, x_j) = \lambda_1$ and $\mu_{\underline{R}}(x_j, x_k) = \lambda_2$ implies $\mu_{\underline{R}}(x_i, x_k) = \lambda$ where $\lambda \geq \min(\lambda_1, \lambda_2)$

Fuzzy Tolerance/ Proximity

- Exhibits only reflexivity and symmetry.

- Any fuzzy tolerance relation R_1 can be reformed into fuzzy equivalence relation by at most (n-1) compositions just like crisp equivalence relation is formed from crisp tolerance relation.

  $R_1 = R_1 \text{ o } R_1 \text{ o } R_1 \ldots\ldots \text{ O } R_1 = R$