5

Duality in Linear Programming

"Man is simply to recognise that there are different approaches for solving problems"

5:1. INTRODUCTION

Associated with every linear programming problem (maximization or minimization) there always exist another Linear Programming problem which is based upon the same data and having the same solution. The original problem is called the **primal problem** while the associated one is called its **dual problem**. It is important to note that either of the two linear programming problems can be treated as primal and the other as its dual. The two problems, thus, constitute a primal-dual pair.

The concept of duality is based on the fact that any linear programming problem must be first put in its standard form before solving the problem by simplex method. Since, all the primal-dual computations are obtained directly from the simplex table, it is logical that we define the dual that may be constituent with the standard form of the primal.

5:2. GENERAL PRIMAL-DUAL PAIR

Based on the standard form of primal, there are two important primal-dual pairs :

Definition 1. **(Standard primal problem)**

Maximize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ *subject to the constraints :*

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i; \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0; \qquad j = 1, 2, \ldots, n$$

Dual Problem

Minimize $z^* = b_1w_1 + b_2w_2 + \ldots + b_mw_m$ *subject to the constraints :*

$$a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m \geq c_j; \qquad j = 1, 2, \ldots, n$$

$$w_i \ (i = 1, 2, \ldots, m) \text{ unrestricted.}$$

Note that x_j's are the primal variables, w_i's the dual variables and the other constants have their usual meanings.

Definition 2. **(Standard primal problem)**

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ *subject to the constraints :*

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i; \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0; \qquad j = 1, 2, \ldots, n$$

Dual Problem

Maximize $z^* = b_1w_1 + b_2w_2 + \ldots + b_mw_m$ *subject to the constraints :*

$$a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m \leq c_j; \qquad j = 1, 2, \ldots, n$$

$$w_i \ (i = 1, 2, \ldots, m) \text{ unrestricted.}$$

Remark. From the above definitions one may easily observe the following:

(*a*) For every primal constraint, there is a dual variable.

(*b*) For every primal variable, there is a dual constraint.

(*c*) The coefficients of the dual variables in the constraints are the same as the coefficients of the primal variables except that they are transposed, *i.e.*, columns in primal coefficient matrix become the rows in the dual coefficient matrix.

(*d*) The number of dual constraints is exactly equal to the number of primal variables, whereas the number of dual variables is exactly equal to the number of primal constraints.

(*e*) The objective coefficients of the primal variables become the right-hand side constants of dual constraints, whereas the right-hand side constants of the dual constraints become the objective coefficients of the primal problem.

The information regarding the primal-dual objective, type of constraints and the signs of the dual variables may be summarized in the following table :

Standard primal objective	*Dual*		
	Objective	*Constraints*	*Variables*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

5:3. FORMULATING A DUAL PROBLEM

Various steps involved in the formulation of a primal-dual pair are :

Step 1. Put the given linear programming problem into its standard form. Consider it as the primal problem.

Step 2. Identify the variables to be used in the dual problem. The number of these variables equals the number of constraint equations in the primal.

Step 3. Write down the objective function of the dual, using the right-hand side constants of the primal constraints.

If the primal problem is of maximization type, the dual will be a minimization problem and vice-versa.

Step 4. Making use of dual variable identified in *Step 2*, write the constraints for the dual problem.

(*a*) If the primal is a maximization prob' m, the dual constraints must be all of '$\geq$' type. If the primal is a minimization problem, the dual constraints must be all of '$\leq$' type.

(*b*) The column coefficients of the primal constraints become the row coefficients of the dual constraints.

(*c*) The coefficients of the primal objective function becomes the right-hand side constants of the dual constraints.

(*d*) The dual variables are defined to be unrestricted in sign.

Step 5. Using *steps 3* and *4*, write down the dual of the given L.P.P.

Note : The dual constraint corresponding to an artificial variable in the standard form of the primal is always *redundant*, hence it is never necessary to consider the dual constraint associated with an artificial variable.

Remarks. 1. If the given linear programming problem is in its canonical form, the primal-dual pair is said to be *symmetric*.

2. If the given linear programming problem is in its standard form, the primal-dual pair is said to be *unsymmetric*.

5:4. PRIMAL-DUAL PAIR IN MATRIX FORM

Standard Primal Problem

Definition 1. **(Standard primal problem).** *Find* $\mathbf{x}^T \in \mathbf{R}^n$ *so as to maximize* $z = \mathbf{cx}$, $\mathbf{c} \in \mathbf{R}^n$ *subject to the constraints :*

$$\mathbf{Ax} = \mathbf{b} \quad \textit{and} \quad \mathbf{x} \geq \mathbf{0},\ \mathbf{b} \in \mathbf{R}^m$$

where $\mathbf{A}$ *is an* $m \times n$ *real matrix.*

Dual Problem. *Find* $\mathbf{w}^T \in \mathbf{R}^m$ *so as to minimize* $z^* = \mathbf{b}^T\mathbf{w}$, $\mathbf{b} \in \mathbf{R}^m$ *subject to the constraints :*

$$\mathbf{A}^T\mathbf{w} \geq \mathbf{c}^T,\ \mathbf{c} \in \mathbf{R}^n$$

where $\mathbf{A}^T$ *is the transpose of an* $m \times n$ *real matrix* $\mathbf{A}$ *and* $\mathbf{w}$ *is unrestricted in sign.*

Definition 2. **(Standard primal problem).** Find $\mathbf{x}^T \in \mathbf{R}^n$ so as to *minimize* $z = \mathbf{cx}$, $\mathbf{c} \in \mathbf{R}^n$ *subject to the constraints :*

$$\mathbf{Ax} = \mathbf{b} \quad \textit{and} \quad \mathbf{x} \geq \mathbf{0},\ \mathbf{b}^T \in \mathbf{R}^m$$

where $\mathbf{A}$ *is an* $m \times n$ *real matrix.*

Dual Problem. *Find* $\mathbf{w}^T \in \mathbf{R}^n$, *so as to maximize* $z^* = \mathbf{b}^T\mathbf{w}$, $\mathbf{b} \in \mathbf{R}^m$ *subject to the constraints :*

$$\mathbf{A}^T\mathbf{w} \leq \mathbf{c}^T,\ \mathbf{c} \in \mathbf{R}^n$$

where $\mathbf{A}^T$ *is the transpose of an* $m \times n$ *real matrix* $\mathbf{A}$ *and* $\mathbf{w}$ *is unrestricted in sign.*

SAMPLE PROBLEMS

501. *Formulate the dual of the following linear programming problem :*

Maximize $z = 5x_1 + 3x_2$ *subject to the constraints :*

$$3x_1 + 5x_2 \leq 15,\ 5x_1 + 2x_2 \leq 10,\ x_1 \geq 0 \ \textit{and}\ x_2 \geq 0.$$

Solution. ***Standard Primal.*** Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$, the standard linear programming problem is :

$$\text{Maximize } z = 5x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2 \text{ subject to the constraints :}$$

$$3x_1 + 5x_2 + s_1 + 0 \cdot s_2 = 15,\ 5x_1 + 2x_2 + 0 \cdot s_1 + s_2 = 10,\ x_1, x_2, s_1, s_2 \geq 0$$

Dual. Let w_1 and w_2 be the dual variables corresponding to the primal constraints. Then, the dual problem will be :

$$\text{Minimize } z^* = 15w_1 + 10w_2 \text{ subject to the constraints :}$$

$$3w_1 + 5w_2 \geq 5,\ 5w_1 + 2w_2 \geq 3$$

$$\left.\begin{array}{l} w_1 + 0 \cdot w_2 \geq 0 \\ 0 \cdot w_1 + w_2 \geq 0 \end{array}\right\} \Rightarrow w_1 \geq 0 \text{ and } w_2 \geq 0$$

w_1 and w_2 unrestricted (redundant).

The dual variables "w_1 and w_2 unrestricted" are dominated by $w_1 \geq 0$ and $w_2 \geq 0$. Eliminating redundancy, the restricted variables are $w_1 \geq 0$ and $w_2 \geq 0$.

502. *Write the dual of the L.P.P. :*

Minimize $z = 4x_1 + 6x_2 + 18x_3$ *subject to the constraints :*

$$x_1 + 3x_2 \geq 3,\ x_2 + 2x_3 \geq 5 \ \textit{and}\ x_1, x_2, x_3 \geq 0. \qquad \textit{[Delhi B.Sc. (Stat.) 2006]}$$

Solution. ***Standard Primal.*** Introducing surplus variables $s_1 \geq 0$ and $s_2 \geq 0$, the standard form of the L.P.P. is:

$$\text{Minimize } z = 4x_1 + 6x_2 + 18x_3 + 0 \cdot s_1 + 0 \cdot s_2 \text{ subject to the constraints :}$$

$$x_1 + 3x_2 - s_1 = 3,\ 0 \cdot x_1 + x_2 + 2x_3 - s_2 = 5,\ x_1, x_2, x_3, s_1, s_2 \geq 0.$$

Dual. If w_1 and w_2 be the dual variables corresponding to each primal constraint, the dual problem will be

$$\text{Maximize } z^* = 3w_1 + 5w_2 \text{ subject to the constraints :}$$

$$w_1 + 0 \cdot w_2 \le 4,\ 3w_1 + w_2 \le 6,\ 0 \cdot w_1 + 2w_2 \le 18,$$

$$-w_1 + 0 \cdot w_2 \le 0, \text{ and } 0 \cdot w_1 - w_2 \le 0,$$

$$w_1 \text{ and } w_2 \text{ unrestricted (redundant).}$$

Eliminating redundancy, the dual problem is :

$$\text{Maximize } z^* = 3w_1 + 5w_2 \text{ subject to the constraints :}$$

$$w_1 \le 4,\ 3w_1 + w_2 \le 6,\ 2w_2 \le 18;\ w_1 \ge 0 \text{ and } w_2 \ge 0.$$

503. *Write the dual of the following linear programming problem :*

$$\textit{Minimize } z = 3x_1 - 2x_2 + 4x_3 \textit{ subject to the constraints :}$$

$$3x_1 + 5x_2 + 4x_3 \ge 7,\ 6x_1 + x_2 + 3x_3 \ge 4$$

$$7x_1 - 2x_2 - x_3 \le 10,\ x_1 - 2x_2 + 5x_3 \ge 3$$

$$4x_1 + 7x_2 - 2x_3 \ge 2,$$

$$x_1 \ge 0,\ x_2 \ge 0, \textit{ and } x_3 \ge 0.$$

[Visvesvaraya M.B.A. (Dec.) 2011]

Solution. ***Standard Primal.*** Introducing surplus variables $s_1 \ge 0$, $s_2 \ge 0$, $s_4 \ge 0$, $s_5 \ge 0$ and a slack variable $s_3 \ge 0$, the standard form of L.P.P. is :

$$\text{Minimize } z = 3x_1 - 2x_2 + 4x_3 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_4 + 0 \cdot s_5 \text{ subject to the constraints :}$$

$$3x_1 + 5x_2 + 4x_3 - s_1 = 7$$

$$6x_1 + x_2 + 3x_3 - s_2 = 4$$

$$7x_1 - 2x_2 - x_3 + s_3 = 10$$

$$x_1 - 2x_2 + 5x_3 - s_4 = 3$$

$$4x_1 + 7x_2 - 2x_3 - s_5 = 2$$

$$x_1, x_2, x_3, s_1, s_2, s_3, s_4, s_5 \ge 0.$$

Dual. If $w_j\,(j = 1, 2, 3, 4, 5)$ are the dual variables corresponding to the five primal constraints in the given order, the dual of the standard primal L.P.P. is :

$$\text{Maximize } z^* = 7w_1 + 4w_2 + 10w_3 + 3w_4 + 2w_5 \text{ subject to the constraints :}$$

$$3w_1 + 6w_2 + 7w_3 + w_4 + 4w_5 \le 3$$

$$5w_1 + w_2 - 2w_3 - 2w_4 + 7w_5 \le -2$$

$$4w_1 + 3w_2 - w_3 + 5w_4 - 2w_5 \le 4$$

$$-w_1 \le 0,\ -w_2 \le 0,\ w_3 \le 0,\ -w_4 \le 0,\ -w_5 \le 0$$

$$w_j\,(j = 1, 2, 3, 4, 5) \text{ are unrestricted in sign.}$$

Eliminating redundancy, the dual variables are : $w_1 \ge 0$, $w_2 \ge 0$, $w_3 \le 0$, $w_4 \ge 0$ and $w_5 \ge 0$.

Note. In the primal, if we multiply the third constraint throughout by -1, the dual variable w_3 will be ≥ 0 instead of ≤ 0.

504. *Obtain the dual of the following L.P.P. :*

$$\textit{Maximize } z = 2x_1 + 3x_2 + x_3 \textit{ subject to the constraints :}$$

$$4x_1 + 3x_2 + x_3 = 6,\ x_1 + 2x_2 + 5x_3 = 4;\ x_1, x_2, x_3 \ge 0$$

[Madras B.E. 1999]

Solution. Clearly, the given linear programming problem is in standard form. Considering it as standard primal, its dual is :

$$\text{Minimize } z^* = 6w_1 + 4w_2 \text{ subject to the constraints :}$$

$$4w_1 + w_2 \ge 2,\ 3w_1 + 2w_2 \ge 3,\ w_1 + 5w_2 \ge 1;\ w_1 \text{ and } w_2 \text{ are unrestricted,}$$

where w_1 and w_2 are dual variables.

505. *Obtain the dual problem of the following primal problem :*

Minimize $z = x_1 - 3x_2 - 2x_3$ *subject to the constraints :*

$$3x_1 - x_2 + 2x_3 \le 7,\ 2x_1 - 4x_2 \ge 12,\ -4x_1 + 3x_2 + 8x_3 = 10$$

$x_1, x_2 \ge 0$ *and* x_3 *is unrestricted.* ***[Delhi B.Sc. (Stat.) 1995]***

Solution. ***Standard Primal.*** Introducing a slack variable $s_1 \ge 0$ and a surplus variable $s_2 \ge 0$, the primal problem is restated as

Minimize $z = \mathbf{cx}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$,

where $\mathbf{x} = [x_1, x_2, x_3', x_3'', s_1, s_2]$, $\mathbf{c} = [1, -3, -2, 2, 0, 0]$, $\mathbf{b} = [7, 12, 10]$ and

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 & -2 & 1 & 0 \\ 2 & -4 & 0 & 0 & 0 & -1 \\ -4 & 3 & 8 & -8 & 0 & 0 \end{pmatrix}, \text{ when } x_3 = x_3' - x_3''.$$

Dual. If $\mathbf{w} = (w_1, w_2, w_3)$ are the dual variables, the dual problem of the primal is

Maximize $z^* = 7w_1 + 12w_2 + 10w_3$ subject to the constraints :

$$3w_1 + 2w_2 - 4w_3 \le 1$$

$$-w_1 - 4w_2 + 3w_3 \le -3 \Rightarrow w_1 + 4w_2 - 3w_3 \ge 3$$

$$\left.\begin{aligned} 2w_1 + 8w_3 &\le -2 \\ -2w_1 - 8w_3 &\le 2 \end{aligned}\right\} \Rightarrow -2w_1 - 8w_3 = 2$$

$$\left.\begin{aligned} w_1 &\le 0 \\ -w_2 &\le 0 \end{aligned}\right\} \Rightarrow w_1 \le 0 \text{ and } w_2 \ge 0$$

w_1, w_2 and w_3 unrestricted.

Eliminating redundancy, dual variables are $w_1 \le 0$, $w_2 \ge 0$ and w_3 unrestricted. This is re-written as follows :

Maximize $z^* = 7w_1 + 12w_2 + 10w_3$ subject to the constraints :

$$3w_1 + 2w_2 - 4w_3 \le 1,\ w_1 + 4w_2 - 3w_3 \ge 3,\ -2w_1 - 8w_3 = 2;$$

$$w_1 \le 0 \text{ and } w_2 \ge 0,\ w_3 \text{ unrestricted.}$$

Note. In the primal, if we multiply the first inequation by -1; the dual variable w_1 will be ≥ 0 instead of ≤ 0.

PROBLEMS

506. Formulate dual of the following L.P.P. :

(*a*) Maximize $z = 4x_1 + 2x_2$ subject to the constraints :

$$x_1 + x_2 \ge 3,\ x_1 - x_2 \ge 2;\ x_1 \ge 0 \text{ and } x_2 \ge 0.$$ ***[Annamalai M.B.A. (Nov.) 2009]***

(*b*) Maximize $z = 2000x + 3000y$ subject to the constraints :

$$6x + 9y \le 100,\ 2x + y \le 20;\ x \ge 0 \text{ and } y \ge 0.$$ ***[Osmania M.B.A. (Sept.) 2001]***

507. Formulate the dual of the following L.P.P. :

Maximize $z = 10x_1 + 8x_2$ subject to the constraints :

$$x_1 + 2x_2 \ge 5,\ 2x_1 - x_2 \ge 12,\ x_1 + 3x_2 \ge 4,$$

$x_1 \ge 0$ and x_2 is unrestricted. ***[Osmania M.B.A. 1999]***

508. Obtain the dual problem of the following L.P.P. :

Maximize $f(x) = 2x_1 + 5x_2 + 6x_3$ subject to the constraints :

$$5x_1 + 6x_2 - x_3 \le 3,\ -2x_1 + x_2 + 4x_3 \le 4,\ x_1 - 5x_2 + 3x_3 \le 1,$$

$$-3x_1 - 3x_2 + 7x_3 \le 6,\ x_1, x_2, x_3 \ge 0.$$

509. Write the dual of the following primal :

Minimize $z = x_1 + 2x_2$ subject to the constraints :

$$2x_1 + 4x_2 \le 160,\ x_1 - x_2 = 30,\ x_1 \ge 10;\ x_1 \ge 0 \text{ and } x_2 \ge 0.$$ *[Delhi MIB 2011]*

510. Obtain the dual of the following linear programming problem :

Maximize $z = 2x_1 + 3x_2 + x_3$ subject to the constraints :

$$4x_1 + 3x_2 + x_3 = 6,\ x_1 + 2x_2 + 5x_3 = 4,\ x_1, x_2, x_3 \ge 0.$$ *[Madras B.E. 1999]*

511. Write the dual problem for the following L.P.P. :

Maximize $z = 3x_1 + 5x_2 + 7x_3$ subject to the constraints :

$$x_1 + x_2 + 3x_3 \le 10,\ 4x_1 - x_2 + 2x_3 \ge 15$$

$$x_1,\ x_2 \ge 0 \text{ and } x_3 \text{ is unrestricted.}$$ *[Delhi PG Dip. in Glob. Bus. Oper. 2010]*

512. Obtain the dual of the following linear programming problem :

Minimize $z = x_1 + x_2 + x_3$ subject to the constraints :

$$x_1 - 3x_2 + 4x_3 = 5,\ x_1 - 2x_2 \le 3,\ 2x_2 - x_3 \ge 4$$

$$x_1, x_2 \ge 0 \text{ and } x_3 \text{ unrestricted.}$$ *[Garhwal M.Sc. (Math.) 2002]*

513. Give the dual of the following L.P.P. :

Minimize $z = 2x_1 + 3x_2 + 4x_3$ subject to the constraints :

$$2x_1 + 3x_2 + 5x_3 \ge 2,\ 3x_1 + x_2 + 7x_3 = 3,\ x_1 + 4x_2 + 6x_3 \le 5$$

$$x_1, x_2 \ge 0 \text{ and } x_3 \text{ is unrestricted.}$$

514. Maximize $z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$ subject to the constraints :

$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2,\ 2x_1 + x_2 + 3x_3 + 2x_4 + 9x_5 = 6$$

$$x_1, x_2, x_3, x_4 \ge 0,\ x_5 \text{ unrestricted.}$$

5:5. DUALITY THEOREMS

Theorem 5-1. *The dual of the dual is the primal.*

Proof. Let the primal L.P.P. be to determine $\mathbf{x}^T \in \mathbf{R}^n$ so as to

Maximize $f(\mathbf{x}) = \mathbf{cx},\ \mathbf{c} \in \mathbf{R}^n$ subject to the constraints :

$$\mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0},\ \mathbf{b}^T \in \mathbf{R}^m,$$

where $\mathbf{A}$ is an $m \times n$ real matrix.

The dual of this primal is the L.P.P. of determining $\mathbf{w}^T \in \mathbf{R}^m$ so as to

Minimize $f(\mathbf{w}) = \mathbf{b}^T\mathbf{w},\ \mathbf{b}^T \in \mathbf{R}^m$ subject to the constraints :

$$\mathbf{A}^T\mathbf{w} \ge \mathbf{c}^T,\ \mathbf{w} \text{ unrestricted},\ \mathbf{c} \in \mathbf{R}^n.$$

Now, introduce surplus variables $\mathbf{s} \ge \mathbf{0}$ in the constraints of the dual and write $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$, where $\mathbf{w}_1 \ge \mathbf{0}$ and $\mathbf{w}_2 \ge \mathbf{0}$.

The standard form of dual then is to

Minimize $g(\mathbf{w}) = \mathbf{b}^T(\mathbf{w}_1 - \mathbf{w}_2),\ \mathbf{b}^T \in \mathbf{R}^m$ subject to the constraints :

$$\mathbf{A}^T(\mathbf{w}_1 - \mathbf{w}_2) - \mathbf{I}_n\mathbf{s} = \mathbf{c}^T,\ \mathbf{c} \in \mathbf{R}^n$$

$$\mathbf{w}_1, \mathbf{w}_2 \text{ and } \mathbf{s} \ge \mathbf{0}.$$

Considering this linear programming problem as our standard primal, the associated dual problem will be to

Maximize $h(\mathbf{y}) = \mathbf{cy},\ \mathbf{c} \in \mathbf{R}^n$ subject to the constraints :

$$(\mathbf{A}^T)^T\mathbf{y} \le (\mathbf{b}^T)^T,\ -(\mathbf{A}^T)^T\mathbf{y} \le -(\mathbf{b}^T)^T,$$

$$-\mathbf{y} \le \mathbf{0}\ (\Rightarrow \mathbf{y} \ge \mathbf{0}) \text{ and } \mathbf{y} \text{ unrestricted.}$$

Eliminating redundancy, the dual problem may be re-written as :

$$\text{Maximize } h(\mathbf{y}) = \mathbf{cy}, \ \mathbf{c} \in \mathbf{R}^n \text{ subject to the constraints :}$$

$$\left.\begin{array}{l}\mathbf{Ay} \le \mathbf{b} \\ \mathbf{Ay} \ge \mathbf{b}\end{array}\right\} \Rightarrow \mathbf{Ay} = \mathbf{b}, \ \mathbf{b}^T \in \mathbf{R}^m$$

$$\mathbf{y} \ge \mathbf{0}.$$

This problem, which is the dual of the dual problem, is just the primal problem we had started with.

This completes the proof.

Theorem 5-2 **(Weak Duality Theorem).** *Let* $\mathbf{x}_o$ *be a feasible solution to the primal problem*

$$\textit{Maximize } f(\mathbf{x}) = \mathbf{cx} \textit{ subject to : } \mathbf{Ax} \le \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}$$

where $\mathbf{x}^T$ *and* $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{b}^T \in \mathbf{R}^m$ *and* $\mathbf{A}$ *is an* $m \times n$ *real matrix. If* $\mathbf{w}_o$ *be a feasible solution to the dual of the primal, namely*

$$\textit{Minimize } g(\mathbf{w}) = \mathbf{b}^T\mathbf{w} \textit{ subject to : } \mathbf{A}^T\mathbf{w} \ge \mathbf{c}^T, \ \mathbf{w} \ge \mathbf{0}$$

where $\mathbf{w}^T \in \mathbf{R}^m$, *then* $\mathbf{cx}_o \le \mathbf{b}^T\mathbf{w}_o$.

Proof. Since $\mathbf{x}_o$ and $\mathbf{w}_o$ are the feasible solutions to the primal and its dual respectively, we must have

$$\mathbf{Ax}_o \le \mathbf{b}, \ \mathbf{x}_o \ge \mathbf{0} \text{ and } \mathbf{A}^T\mathbf{w}_o \ge \mathbf{c}^T, \ \mathbf{w}_o \ge \mathbf{0}.$$

Thus $$\mathbf{c} \le \mathbf{w}_o^T\mathbf{A} \text{ or } \mathbf{cx}_o \le \mathbf{w}_o^T\mathbf{Ax}_o \le \mathbf{w}_o^T\mathbf{b}$$

or $$\mathbf{cx}_o \le \mathbf{b}^T\mathbf{w}_o \qquad [\text{since } \mathbf{w}_o^T\mathbf{b} = \mathbf{b}^T\mathbf{w}_o].$$

Theorem 5-3. *Let* $\mathbf{x}_o$ *be a feasible solution to the primal problem*

$$\textit{Maximize } f(\mathbf{x}) = \mathbf{cx} \textit{ subject to : } \mathbf{Ax} \le \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}$$

and $\mathbf{w}_o$ *be a feasible solution to its dual :*

$$\textit{Minimize } g(\mathbf{w}) = \mathbf{b}^T\mathbf{w} \textit{ subject to : } \mathbf{A}^T\mathbf{w} \ge \mathbf{c}^T, \ \mathbf{w} \ge \mathbf{0}$$

where $\mathbf{x}^T$ *and* $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{w}^T$ *and* $\mathbf{b}^T \in \mathbf{R}^m$ *and* $\mathbf{A}$ *is an* $m \times n$ *real matrix.*

If $\mathbf{cx}_o = \mathbf{b}^T\mathbf{w}_o$, *then both* $\mathbf{x}_o$ *and* $\mathbf{w}_o$ *are optimum solutions to the primal and dual respectively.*

Proof. Let $\mathbf{x}_o^*$ be any other feasible solution to the primal problem then theorem 5-2 gives $\mathbf{cx}_o^* \le \mathbf{b}^T\mathbf{w}_o$.

Thus $$\mathbf{cx}_o^* \le \mathbf{cx}_o \qquad (\text{since } \mathbf{cx}_o = \mathbf{b}^T\mathbf{w}_o)$$

and hence $\mathbf{x}_o$ is an optimum solution to the primal problem, because primal is a maximization problem.

Similarly, if $\mathbf{w}_o^*$ is any other feasible solution to the dual problem, then $\mathbf{b}^T\mathbf{w}_o \le \mathbf{b}^T\mathbf{w}_o^*$ and thus $\mathbf{w}_o$ is an optimum solution to the dual problem, because dual is a minimization problem.

Theorem 5-4 **(Basic Duality Theorem).** *Let a primal problem be*

$$\textit{Maximize } f(\mathbf{x}) = \mathbf{cx} \textit{ subject to : } \mathbf{Ax} \le \mathbf{b}, \ \mathbf{x} \ge \mathbf{0} \quad \mathbf{x}^T, \mathbf{c} \in \mathbf{R}^n$$

and the associated dual be

$$\textit{Minimize } g(\mathbf{w}) = \mathbf{b}^T\mathbf{w} \textit{ subject to : } \mathbf{A}^T\mathbf{w} \ge \mathbf{c}^T, \ \mathbf{w} \ge \mathbf{0} \quad \mathbf{w}^T, \mathbf{b}^T \in \mathbf{R}^m.$$

If $\mathbf{x}_o$ $(\mathbf{w}_o)$ *is an optimum solution to the primal (dual), then there exists a feasible solution* $\mathbf{w}_o$ $(\mathbf{x}_o)$ *to the dual (primal) such that*

$$\mathbf{cx}_o = \mathbf{b}^T\mathbf{w}_o.$$

Proof. The standard primal can be written as

$$\text{Maximize } z = \mathbf{cx} \text{ subject to : } \mathbf{Ax} + \mathbf{Ix}_s = \mathbf{b}$$

where $\mathbf{x}_s^T \in \mathbf{R}^m$ is the slack vector and $\mathbf{I}$ is the associated identity matrix.

Let $\mathbf{x}_o = [\mathbf{x}_B, \mathbf{0}]$ be an optimum solution to the primal, where $\mathbf{x}_B{}^T \in \mathbf{R}^m$ is the optimal basic feasible solution given by $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, $\mathbf{B}$ being the optimal basis of $\mathbf{A}$. Then, the optimum primal objective function is

$$z = \mathbf{c}\mathbf{x}_o = \mathbf{c}_B \mathbf{x}_B$$

where $\mathbf{c}_B$ is the cost vector associated with $\mathbf{x}_B$.

Now, the net evaluations in the optimal simplex table are given by

$$z_j - c_j = \mathbf{c}_B \mathbf{y}_j - c_j = \begin{cases} \mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_j - c_j & \text{for all } \mathbf{a}_j \in \mathbf{A} \\ \mathbf{c}_B \mathbf{B}^{-1}\mathbf{e}_j - 0 & \text{for all } \mathbf{e}_j \in \mathbf{I} \end{cases}$$

Since $\mathbf{x}_B$ is optimal, we must have $z_j - c_j \geq 0$ for all j. This gives

$$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j \geq c_j \quad \text{and} \quad \mathbf{c}_B \mathbf{B}^{-1} \mathbf{e}_j \geq \mathbf{0} \qquad \text{for all } j$$

or

$$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{A} \geq \mathbf{c} \quad \text{and} \quad \mathbf{c}_B \mathbf{B}^{-1} \geq \mathbf{0} \qquad \text{(in matrix form)}$$

or

$$\mathbf{A}^T \mathbf{B}^{-1} \mathbf{c}_B{}^T \geq \mathbf{c}^T \quad \text{and} \quad \mathbf{B}^{-1} \mathbf{c}_B{}^T \geq \mathbf{0}.$$

Now, if we let $\mathbf{B}^{-1} \mathbf{c}_B{}^T = \mathbf{w}_o$, the above become

$$\mathbf{A}^T \mathbf{w}_o \geq \mathbf{c}^T \text{ and } \mathbf{w}_o \geq \mathbf{0}, \ \mathbf{w}_o{}^T \in \mathbf{R}^m.$$

This means that $\mathbf{w}_o$ is a feasible solution to the dual problem. Moreover, the corresponding dual objective function value is,

$$\mathbf{b}^T \mathbf{w}_o = \mathbf{w}_o{}^T \mathbf{b} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} = \mathbf{c}_B \mathbf{x}_B = \mathbf{c}\mathbf{x}_o.$$

Thus, given an optimal solution $\mathbf{x}_o$ to the primal, there exists a feasible solution $\mathbf{w}_o$ to the dual such the $\mathbf{c}\mathbf{x}_o = \mathbf{b}^T\mathbf{w}_o$.

Similarly, starting with $\mathbf{w}_o$, the existence of $\mathbf{x}_o$ can be proved.

Corollary. *If $\mathbf{x}_o$ is an optimal solution to the primal, an optimal solution to the dual is given by $\mathbf{w}_o = \mathbf{B}^{-1} \mathbf{c}_B$, where $\mathbf{B}$ is the primal optimal basis.*

Note. Observe that $\mathbf{B}^{-1} \mathbf{c}_B{}^T$ represents optimum $z_j - c_j$ under primal slack columns.

Theorem 5-5 **(Fundamental Theorem of Duality).** *If the primal or the dual has a finite optimum solution, then the other problem also possesses a finite optimum solution and the optimum values of the objective functions of the two problems are equal.*

Proof. Consider the primal-dual pair :

Primal. Maximize $f(\mathbf{x}) = \mathbf{c}\mathbf{x}$ subject to : $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Dual. Minimize $g(\mathbf{w}) = \mathbf{b}^T \mathbf{w}$ subject to : $\mathbf{A}^T\mathbf{w} \geq \mathbf{c}^T$ and $\mathbf{w} \geq \mathbf{0}$.

(*Necessary condition*). Let a feasible solution $\mathbf{x}_o$ ($\mathbf{w}_o$) be an optimum solution to the primal (dual) problem. It then follows from Theorem *5-4* that there exists a feasible solution $\mathbf{w}_o$ ($\mathbf{x}_o$) to the dual (primal) problem, such that $\mathbf{c}\mathbf{x}_o = \mathbf{b}^T \mathbf{w}_o$.

It now follows from Theorem *5-3* that $\mathbf{w}_o$ must be optimal.

This proves the necessity of the condition.

(*Sufficiency*). It follows from Theorem *5-3*.

Theorem 5-6 **(Existence Theorem).** *If either the primal or the dual problem has an unbounded objective function value, then the other problem has no feasible solution.*

Proof. Let the given primal problem have unbounded solution. Then for any value of the objective function, say $+\infty$, there exists a feasible solution say $\mathbf{x}$ yielding this solution, *i.e.*, $\mathbf{c}\mathbf{x} \to \infty$. If possible, let the dual problem have a feasible solution. Then from Theorem *5-2*, for each feasible solution $\mathbf{w}_o$ of the dual, there exists a feasible solution $\mathbf{x}_o$ to the primal such that $\mathbf{c}\mathbf{x}_o \leq \mathbf{b}^T \mathbf{w}_o$. That is, $\mathbf{b}^T \mathbf{w}_o \to \infty$. Now as $\mathbf{b}$ is constant and $\mathbf{w}_o$ has to satisfy the constraint

$\mathbf{A}^T \mathbf{w}_o \geq \mathbf{c}^T$, therefore the dual objective function $\mathbf{b}^T \mathbf{w}_o$ must be finite. This contradicts the result $\mathbf{b}^T \mathbf{w}_o \to \infty$. Hence the dual problem has no feasible solution.

By a similar argument it can be shown that when the dual has an unbounded solution, the primal has no feasible solution.

Standard results on duality can be summarized as follows :

Dual problem	*Primal problem*	
	Feasible solution	*No feasible solution*
Feasible Solution	Optimum	Dual Unbounded
No Feasible Solution	Primal Unbounded	Unbounded or Infeasible

In the following section we shall see how the above existence theorem helps us in understanding the relationship between the optimum values of the primal and dual variables.

5:6. COMPLEMENTARY SLACKNESS THEOREM

Theorem 5-7 **(Complementary Slackness).** *Let* $\mathbf{x}_o$ *and* $\mathbf{w}_o$ *be the feasible solutions to the primal* $\{ \max.\ \mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \}$ *and its dual* $\{ \min.\ \mathbf{b}^T\mathbf{w} \mid \mathbf{A}^T\mathbf{w} \geq \mathbf{c}^T,\ \mathbf{w} \geq \mathbf{0} \}$ *respectively. Then, a necessary and sufficient condition for* $\mathbf{x}_o$ *and* $\mathbf{w}_o$ *to be optimal to their respective problems is that*

$$\mathbf{w}_o^T(\mathbf{b} - \mathbf{A}\mathbf{x}_o) = 0 \quad \text{and} \quad \mathbf{x}_o^T(\mathbf{A}^T\mathbf{w}_o - \mathbf{c}^T) = 0.$$

Proof. *Necessity.* Let $\alpha = \mathbf{w}_o^T(\mathbf{b} - \mathbf{A}\mathbf{x}_o)$ and $\beta = \mathbf{x}_o^T(\mathbf{A}^T\mathbf{w}_o - \mathbf{c}^T)$. Since $\mathbf{x}_o$, $\mathbf{w}_o$ are feasible solutions to the primal and dual respectively, we have

$$\alpha \geq 0,\ \beta \geq 0 \quad \text{and} \quad \alpha + \beta = \mathbf{w}_o^T\mathbf{b} - \mathbf{x}_o^T\mathbf{c}^T.$$

Now, if $\mathbf{x}_o$, $\mathbf{w}_o$ are optimal, then $\mathbf{c}\mathbf{x}_o = \mathbf{b}^T\mathbf{w}_o$ so that $\alpha + \beta = 0$. But since $\alpha \geq 0$ and $\beta \geq 0$, this gives $\alpha = 0$ and $\beta = 0$.

Thus the conditions are necessary.

Sufficiency. Let the given conditions hold for the feasible solutions $\mathbf{x}_o$ and $\mathbf{w}_o$. That is, $\alpha = 0$ and $\beta = 0$.

Then $$\mathbf{0} = \alpha + \beta = \mathbf{w}_o^T\mathbf{b} - \mathbf{x}_o^T\mathbf{c}^T$$

$\Rightarrow$ $$\mathbf{c}\mathbf{x}_o = \mathbf{b}^T\mathbf{w}_o$$

$\Rightarrow$ $\mathbf{x}_o$ and $\mathbf{w}_o$ are optimal.

Thus the conditions are sufficient.

Corollary 1. *If* $\mathbf{x}^o$ *and* $\mathbf{w}^o$ *be feasible solutions to the primal and dual problems respectively, then they will be optimal if and only if*

$$w_i^o\,(b_i - \sum_{j=1}^{n} a_{ij}x_i^o) = 0, \qquad i = 1, 2, \ldots, m$$

and $$x_j^o\,(\sum_{i=1}^{m} a_{ji}w_i^o - c_j) = 0, \qquad j = 1, 2, \ldots, n.$$

Proof. From the above theorem, $\mathbf{x}^o$ and $\mathbf{w}^o$ will be optimal, if and only if

$$\mathbf{w}_o^T(\mathbf{b} - \mathbf{A}\mathbf{x}_o) = 0 \quad \text{and} \quad \mathbf{x}_o^T(\mathbf{c}^T - \mathbf{A}^T\mathbf{w}_o) = 0.$$

Consider the first set of conditions. Since each term in the summation $\mathbf{w}_o^T(\mathbf{b} - \mathbf{A}\mathbf{x}_o)$ is non-negative, it follows that

$$w_i^o\,(b_i - \sum_{i=1}^{n} a_{ij}x_j^o) = 0, \qquad i = 1, 2, \ldots, m.$$

Similarly, the second set of conditions is equivalent to

$$x_j^o \left(\sum_{i=1}^{m} a_{ji} w_i^o - c_j \right) = 0, \qquad j = 1, 2, \ldots, n.$$

Corollary 2. *For optimal feasible solutions of the primal and dual systems, whenever the ith variable is strictly positive in either system, the ith relation of its dual is an equality.*

Proof. It follows from corollary *1*, that

$$w_i^o > 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j^o = b_i \qquad (i\text{th primal relation})$$

and

$$x_j^o > 0 \Rightarrow \sum_{i=1}^{m} a_{ji} w_i^o = c_j \qquad (j\text{th dual relation})$$

Corollary 3. *For optimal feasible solutions of the primal and dual systems, whenever ith relation of either system is satisfied as a strict inequality, then the ith variable of its dual vanishes.*

Proof. If follows from corollary *1* that

$$\sum_{j=1}^{n} a_{ij} x_j^o < b_i \Rightarrow w_i^o = 0$$

and

$$\sum_{i=1}^{m} a_{ji} w_i^o > c_j \Rightarrow x_j^o = 0.$$

Remarks. The conditions of corollary *1* can also be written as

$$w_i^o \cdot x_{n+i} = 0 \qquad i = 1, 2, \ldots, m$$

and

$$x_j^o \cdot w_{m+j} = 0 \qquad j = 1, 2, \ldots, n$$

where x_{n+1} is the ith slack variable in the primal problem and w_{m+j} is the jth surplus variable in the dual.

Thus, the theorem relates the variables of one problem to the slack or surplus variables of the other. The above relations are called 'complementary slackness' because they imply that whenever a constraint in one of the problems holds with strict inequality (so that there is *slack* in the constraint), the *complementary* dual variable vanishes.

5:7. DUALITY AND SIMPLEX METHOD

Since any L.P.P. can be solved by using simplex method, the method is applicable to both the primal as well as to its dual. The fundamental theorem of duality suggests that an optimum solution to the *associated* dual can be obtained from that of its primal and *vice versa*.

If primal is a maximization problem, then following are the set of rules that govern the derivation of the optimum solution :

Rule 1. Corresponding net evaluations of the starting *primal* variables
= Difference between the left and right sides of the dual constraints associated with the starting primal variables.

Rule 2. Negative of the corresponding net evaluations of the starting *dual* variables
= Difference between the left and right sides of the primal constraints associated with dual starting variables.

Rule 3. If the primal (dual) problem is unbounded, then the dual (primal) problem does not have any *feasible solution.*

Note. In *rule 2*, dual problem is to be solved by changing the objective from minimization to the maximization.

SAMPLE PROBLEMS

515. *Use duality to solve the following L.P.P. :*

Maximize $z = 2x_1 + x_2$ *subject to the constraints :*

$$x_1 + 2x_2 \le 10,\ x_1 + x_2 \le 6,\ x_1 - x_2 \le 2,\ x_1 - 2x_2 \le 1;\ x_1, x_2 \ge 0.$$

[Delhi B.Sc. (Stat.) 2006]

Solution. The dual problem of the given primal is

Minimize $z^* = 10w_1 + 6w_2 + 2w_3 + w_4$ subject to the constraints :

$$w_1 + w_2 + w_3 + w_4 \ge 2,\ 2w_1 + w_2 - w_3 - 2w_4 \ge 1,\ w_1, w_2, w_3, w_4 \ge 0.$$

Introducing surplus variables $s_1 \ge 0$, $s_2 \ge 0$ and artificial variables $A_1 \ge 0$, $A_2 \ge 0$, an initial basic feasible solution is $A_1 = 2$ and $A_2 = 1$. (The primal constraints associated with s_1, s_2, A_1 and A_2 are: $-x_1 \le 0$, $-x_2 \le 0$, $x_1 \le M$ and $x_2 \le M$).

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_8$.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$-M$	y_7	2	1	1	1	1	-1	0	1	0
$-M$	y_8	1	2	1	-1	-2	0	-1	0	1
	z^*	$-3M$	$-3M+10$	$-2M+6$	2	$M+1$	M	M	0	0

First Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_7$.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$-M$	y_7	3/2	0	1/2	3/2	2	-1	1/2	1	$-1/2$
-10	y_1	1/2	1	1/2	$-1/2$	-1	0	$-1/2$	0	1/2
	z^*	$-5-3M/2$	0	$1-M/2$	$7-3M/2$	$11-2M$	M	$5-M/2$	0	$-5+3M/2$

Second Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_1$.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-2	y_3	1	0	1/3	1	4/3	$-2/3$	1/3	2/3	$-1/3$
-10	y_1	1	1	2/3	0	$-1/3$	$-1/3$	$-1/3$	1/3	13
	z^*	-12	0	$-4/3$	0	5/3	14/3	8/3	$M-14/3$	$M-8/3$

Final Iteration. Optimum Solution.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
-2	y_3	1/2	$-1/2$	0	1	3/2	$-1/2$	1/2	1/2	$-1/2$
-6	y_2	3/2	3/2	1	0	$-1/2$	$-1/2$	$-1/2$	1/2	1/2
	z^*	-10	2	0	0	1	4	2	$M-4$	$M-2$

Thus, an optimum feasible solution to the dual problem is

$$w_1 = 0,\ w_2 = 3/2 \text{ and } w_3 = 1/2;\ \text{min. } z^* = -(-10) = 10.$$

Also, the primal constraints associated with the dual variables A_1 and A_2 are $x_1 \le M$ and $x_2 \le M$.

Thus, using duality rules, the optimum solution to the primal is obtained below :

Starting dual variables	A_1	A_2
Corresponding $\{-(z_j - c_j)\}$	$-(M-4)$	$-(M-2)$
Left minus right sides of the primal constraint associated with starting dual variables	$x_1 - M$	$x_2 - M$

Making use of *Rule 2*, we get

$$x_1 - M = -M + 4 \text{ and } x_2 - M = -M + 2$$

$$\therefore \quad x_1 = 4 \text{ and } x_2 = 2$$

Hence $\quad$ Maximum z = Minimum $z^* = 10$.

516. *Consider the L.P.P. : Maximize* $z = 2x_1 + 4x_2 + 4x_3 - 3x_4$ *subject to the constraints :*

$$x_1 + x_2 + x_3 = 4,\ x_1 + 4x_2 + x_4 = 8,\ x_1, x_2, x_3, x_4 \geq 0.$$

By using x_3 and x_4 as the starting variables, the optimum table is given by

Basis	*Solution*	x_1	x_2	x_3	x_4
x_3	2	3/4	0	1	−1/4
x_2	2	1/4	1	0	1/4
z	16	2	0	0	3

Write the dual problem and find its solution from the optimum primal table.

[Delhi B.Sc. (Stat.) 2002]

Solution. Since, the primal problem is in standard form with non-negative variables, the dual problem is :

$$\text{Minimize } z^* = 4w_1 + 8w_3 \text{ subject to the constraints :}$$

$$w_1 + w_2 \geq 2,\ w_1 + 4w_2 \geq 4,\ w_1 \geq 4 \text{ and } w_2 \geq -3$$

$$w_1 \text{ and } w_2 \text{ unrestricted (redundant).}$$

Now, x_3 and x_4 are the starting variables in the primal, and their respective net evaluations are 0 and 3 in the optimum primal table. The dual constraints associated with x_3 and x_4 are $w_1 \geq 4$ and $w_2 \geq -3$. Therefore, we write this information as below :

Starting primal variables	x_3	x_4
Corresponding net evaluations	0	3
Left minus right sides of the dual constraint associated with primal starting variables	$w_1 - 4$	$w_2 - (-3)$

Making use of *Rule 1*, we get

$$w_1 - 4 = 0 \text{ and } w_2 - (-3) = 3, \text{ i.e., } w_1 = 4 \text{ and } w_2 = 0$$

The optimum value of the dual objective is $\quad$ Min. $z^* = 16 =$ max. z.

517. *Consider the following LP problem :*

$$\textit{Maximize } z = 3x_1 + 2x_2 + 5x_3 \textit{ subject to :}$$

$$x_1 + 2x_2 + x_3 \leq a_1,\ 3x_1 + 2x_3 \leq a_2,\ x_1 + 4x_2 \leq a_3$$

where a_1, a_2 and a_3 are constants. For specific values of a_1, a_2 and a_3 the optimal solution is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	*Solution*
z	4	0	0	c_1	c_2	0	1,350
x_2	b_1	1	0	1/2	−1/4	0	100
x_3	b_2	0	1	0	1/2	0	c_3
x_6	b_3	0	0	−2	1	1	20

where b_i's and c_i's are constants. Determine :

(*i*) *the values of a_1, a_2 and a_3 that yield the given optimal solution.*

(*ii*) *the values of b_1, b_2, b_3 and c_1, c_2, c_3 in the optimal tableau.*

(*iii*) *the optimal dual solution.*

[Delhi B.Sc. (Stat.) 1996]

Solution. The optimal table indicates that slack variables x_4, x_5 and x_6 are introduced in the three primal constraints. They happen to be the starting primal basic variables also. Thus the optimal basis inverse is given by $\mathbf{B}^{-1} = [\mathbf{y}_4 \ \ \mathbf{y}_5 \ \ \mathbf{y}_6]$ from the optimal table.

(*i*) We have $\mathbf{B}^{-1}\,\mathbf{b} = \mathbf{x}_B$

$$\Rightarrow \quad \begin{pmatrix} \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 100 \\ c_3 \\ 20 \end{bmatrix}$$

$$\Rightarrow \quad \tfrac{1}{2}a_1 - \tfrac{1}{4}a_2 = 100,\ \tfrac{1}{2}a_2 = c_3,\ -2a_1 + a_2 + a_3 = 20.$$

Also $\quad z = \mathbf{c}_B\mathbf{x}_B \Rightarrow 1{,}350 = 200 + 5c_3 \Rightarrow c_3 = 230,$ where $\mathbf{c}_B = [2 \quad 5 \quad 0]$.

Thus, we get $a_1 = 430,\ a_2 = 460$ and $a_3 = 480$.

(*ii*) The z-row gives :

$$4 = \mathbf{c}_B\mathbf{y}_1 - c_1 = 2b_1 + 5b_2 - 3 \Rightarrow 2b_1 + 5b_2 = 7$$
$$c_1 = \mathbf{c}_B\mathbf{y}_4 - c_4 = 1 - 0 = 1$$
$$c_2 = \mathbf{c}_B\mathbf{y}_5 - c_5 = -\tfrac{1}{2} + \tfrac{5}{2} - 0 = 2$$

To obtain the values of b_1, b_2 and b_3, we perform iteration on the starting primal table :

Initial Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_5$.

			3	2	5	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	$a_1 = 430$	1	2	1	1	0	0
0	$\mathbf{y}_5$	$a_2 = 460$	3	0	2	0	1	0
0	$\mathbf{y}_6$	$a_3 = 480$	1	4	0	0	0	1
		0	–3	–2	–5	0	0	0

First Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	200	–1/2	2	0	1	–1/2	0
5	$\mathbf{y}_3$	230	3/2	0	1	0	1/2	0
0	$\mathbf{y}_6$	480	1	4	0	0	0	1
		1150	9/2	–2	0	0	5/2	0

Second Iteration. Optimum Solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
2	$\mathbf{y}_2$	100	–1/4	1	0	1/2	–1/4	0
5	$\mathbf{y}_3$	230	3/2	0	1	0	1/2	0
0	$\mathbf{y}_6$	80	–4	0	0	–2	1/2	1
		1350	4	0	0	1	2	0

Comparing it with the given optimal table, we get

$$b_1 = -1/4,\ b_2 = 3/2 \text{ and } b_3 = -4.$$

(Note that the values of c_1, c_2 are also readily available.)

(*iii*) The dual problem is

Minimize $z^* = a_1w_1 + a_2w_1 + a_3w_3$ subject to the constraints :

$$w_1 + 3w_2 + w_3 \geq 3,\ 2w_1 + w_2 + 4w_3 \geq 2,\ w_1 + 2w_2 + 0w_3 \geq 5$$
$$w_1 \geq 0,\ w_2 \geq 0 \text{ and } w_3 \geq 0.$$

The dual constraints associated with the starting primal variables x_4, x_5 and x_6 are $w_1 \geq 0$, $w_2 \geq 0$ and $w_3 \geq 0$.

Thus, we have the following information :

Starting primal variables	x_4	x_5	x_6
Left minus right sides of the associated dual constraint	$w_1 - 0$	$w_2 - 0$	$w_3 - 0$
Net evaluation primal optimal table	c_1	c_2	0

Thus using *Rule 1* we get

$$c_1 = w_1 - 0 \Rightarrow w_1^* = c_1 = 1$$
$$c_2 = w_2 - 0 \Rightarrow w_2^* = c_2 = 2$$
$$0 = w_3 - 0 \Rightarrow w_3^* = 0$$

The optimal dual objective is Min. $z^* = 1{,}350 = a_1 w_1^* + a_2 w_2^*$.

PROBLEMS

Write down the dual of the following L.P.P. and solve :

518. Maximize $z = 8x_1 + 4x_2$ subject to the constraints :

$$4x_1 + 2x_2 \le 30,\ 2x_1 + 4x_2 \le 24,\ x_1, x_2 \ge 0.$$ *[Osmania M.B.A. 2000]*

519. Minimize $z = 15x_1 + 10x_2$ subject to the constraints :

$$3x_1 + 5x_2 \ge 5,\ 5x_1 + 2x_2 \ge 3,\ x_1, x_2 \ge 0.$$

520. Maximize $z = 5x_1 + 2x_2$ subject to the constraints :

$$6x_1 + x_2 \ge 6,\ 4x_1 + 3x_2 \ge 12,\ x_1 + 2x_2 \ge 4,\ x_1, x_2 \ge 0.$$ *[Sambalpur MBA 1994]*

521. Minimize $z = 2x_1 + 9x_2 + x_3$ subject to the constraints :

$$x_1 + 4x_2 + 2x_3 \ge 5,\ 3x_1 + x_2 + 2x_3 \ge 4 \text{ and } x_1, x_2, x_3 \ge 0.$$

522. Consider the following L.P.P. :

Maximize $z = x_1 + 5x_2 + 3x_3$ subject to the constraints :

$$x_1 + 2x_2 + x_3 = 3,\ 2x_1 - x_2 = 4;\ x_1, x_2, x_3 \ge 0.$$

(*a*) Write the associated dual problem.

(*b*) Given the information that the optimal basic variables are x_1 and x_3, determine the associated optimal dual solution.

523. Given the L.P.P. :

Maximize $z = 2x_1 + 4x_2 + 4x_3 - 3x_4$ subject to the constraints :

$$x_1 + x_2 + x_3 = 4,\ x_1 + 4x_2 + x_4 = 8;\ x_1, x_2, x_3, x_4 \ge 0.$$

Use the dual problem to verify that the basic solution (x_1, x_2) is not optimal.

524. Find the optimal value of the objective function for the following problem by only inspecting its dual :

Minimize $z = 10x_1 + 4x_2 + 5x_3 + x_4$ subject to the constraints :

$$5x_1 - 7x_2 + 3x_3 + 0.5x_4 \ge 150;\ x_j \ge 0 \quad (j = 1, 2, 3, 4)$$ *[Karnataka B.E. (Prod.) 1994]*

525. Use duality to solve the following L.P.P. :

Maximize $z = x_1 - x_2 + 3x_3 + 2x_4$ subject to the constraints :

$$x_1 + x_2 \ge -1,\ x_1 - 3x_2 - x_3 \le 7,\ x_1 + x_3 - 3x_4 = -2;\ x_j \ge 0 \quad (j = 1, 2, 3, 4).$$

[Delhi B.Sc. (Math.) 1995]

526. Use duality in solving the L.P.P. :

Minimize $z = 2x_1 - x_2 + x_3 + 5x_4 - 3x_5$ subject to the constraints :

$$4x_1 + 2x_2 + x_3 + x_4 = 3,\ 2x_1 + 2x_2 + x_3 + x_5 = 2;\ x_j \ge 0 \quad (j = 1, 2, 3, 4, 5).$$

527. Consider the problem :

Maximize $z = 2x_2 - 5x_3$ subject to the constraints :

$$x_1 + x_3 \ge 2,\ 2x_1 + x_2 + 6x_3 \le 6,\ x_1 - x_2 + 3x_3 = 0;\ x_1, x_2, x_3 \ge 0.$$

(*i*) Write its dual.

(*ii*) Solve the primal and then find the solution of the dual.

[Delhi B.Sc. (Stat.) 1996]

528. Solve the following L.P.P. :

$$\text{Maximize } z = 2x_1 - x_2 + x_3 \text{ subject to the constraints :}$$

$$x_1 + x_2 - 3x_3 \le 8,\ 4x_1 - x_2 + x_3 \ge 2,\ 2x_1 + 3x_2 - x_3 \ge 4,$$

$$x_1, x_2, x_3 \ge 0.$$

Find the solution of the corresponding dual problem from the final simplex tableau. *[Delhi B.Sc. (Stat.) 1998]*

529. Consider the problem :

$$\text{Maximize } z = 4x_1 + 5x_2 - 3x_3 \text{ subject to the constraints :}$$

$$x_1 + x_2 + x_3 = 10,\ x_1 - x_2 \ge 1,\ 2x_1 + 3x_2 + x_3 \le 40,$$

$$x_1, x_2, x_3 \ge 0.$$

Write the dual of this problem. Solve the dual and from there find the solution of the primal problem. *[Delhi B.Sc. (Stat.) 2000]*

530. In a product-mix problem x_1, x_2, x_3 and x_4 indicate the units of products A, B, C and D respectively and we have to

$$\text{Maximize } z = 4x_1 + 6x_2 + 3x_3 + x_4 \text{ subject to the constraints :}$$

$$1.5x_1 + 2x_2 + 4x_3 + 3x_4 \le 550 \quad \text{(Machine } I\text{, hours)}$$

$$4x_1 + x_2 + 2x_3 + x_4 \le 700 \quad \text{(Machine } II\text{, hours)}$$

$$2x_1 + 3x_2 + x_3 + 2x_4 \le 200 \quad \text{(Machine } III\text{, hours)}$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

(*a*) Formulate the dual of the above problem.

(*b*) Solve the dual. From the solution of the dual, read the solution of the primal problem.

(*c*) If machine *I* remains out of order for 24 hours, what will be the loss to the manufacturer?

(*d*) If the profit contribution from product *A* increases from Rs. 4 to Rs. 5 per unit, what will be the effect on optimal product mix? *[Delhi M.Com. 2007]*

531. Consider the following LPP :

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3 \text{ subject to the constraints :}$$

$$x_1 + 5x_2 + 2x_3 \le b_1,\ x_1 - 5x_2 - 6x_3 \le b_2,\ x_1, x_2, x_3 \ge 0$$

where b_1, b_2 are constants. For specific values of b_1, b_2, the following optimal table is obtained:

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
5	y_1	30	1	b	2	1	0
0	y_5	10	0	c	−8	−1	1
z – row		150	0	a	7	d	e

The constants a, b, c, d and e can be obtained from the data of the given LPP and the optimal feasible condition of the table. Determine :

(*a*) the values of R.H.S. constants b_1, b_2.

(*b*) the optimal dual solution (from the given table).

(*c*) the values of a, b and c associated with the variable x_2.

532. Given below are the objective function, the constraints and the simplex tableau for a linear programming product mix problem :

$$\text{Maximize } z = 2x_1 + 5x_2 + 8x_3 + 0s_1 + 0s_2 + 0s_3 \text{ subject to the constraints :}$$

$$6x_1 + 8x_2 + 4x_3 \le 96 \quad \text{(hours/week, Department } I\text{)}$$

$$2x_1 + x_2 + 2x_3 \le 40 \quad \text{(hours/week, Department } II\text{)}$$

$$5x_1 + 3x_2 + 2x_3 \le 60 \quad \text{(hours/week, Department } III\text{)}$$

$$x_1, x_2, x_3 \ge 0.$$

c_j	Product mix	x_1	x_2	x_3	s_1	s_2	s_3	Quantity
5	x_2	1/3	1	0	1/6	−1/3	0	8/3
8	x_3	5/6	0	1	−1/12	2/3	0	56/3
0	s_3	7/3	0	0	−1/3	−1/3	1	44/3
	c_j	2	5	8	0	0	0	
	z_j	25/3	5	8	1/6	11/3	0	488/3
	$c_j - z_j$	−19/3	0	0	−1/6	−11/3	0	

Answer the following questions, giving reasons in brief :

(*i*) Is the above solution optimal? Explain.

(*ii*) Write the optimal product mix and the profit contribution shown by the optimal solution.

(*iii*) If the company wishes to expand the production capacity, which of the three departments should be given priority.

(*iv*) What are the shadow prices of three constraints?

(*v*) Write the dual of the given problem.

(*vi*) Obtain the optimal values of the dual variables from the final simplex table of the primal problem.

[Delhi PG Dip. in Glob. Bus. Oper. 2010]

5:8. ECONOMIC INTERPRETATION OF DUALITY

The primal-dual relationship discussed so far is not only related mathematically, they are also related in economic sense.

Illustration

Let the following table give the amounts of two vitamins v_1 and v_2 per unit, present in two different foods f_1 and f_2 respectively :

Vitamin	Food f_1	Food f_2	Daily requirement
v_1	6	8	100
v_2	7	12	120
Cost per unit	12	20	

The last column of the table represents the number of units of the minimum daily requirement for the two vitamins, whereas the last row represents the cost per unit for the two foods. The problem is to determine the minimum quantities of the two foods f_1 and f_2 so that the minimum daily requirement of the two vitamins is met and that, at the same time, the cost of purchasing these quantities of f_1 and f_2 is minimum. To formulate the problem mathematically let x_j be the number of units of food f_j ($j = 1, 2$) to be purchased, then the above problem is to determine two real numbers x_1 and x_2 so as to

Minimize $z = 12x_1 + 20x_2$ subject to the constraints :

$$6x_1 + 8x_2 \geq 100, \quad 7x_1 + 12x_2 \geq 120 ; \quad x_1, x_2 \geq 0.$$

Now, let us consider a different problem, which is associated with the above problem. Suppose there is a wholesale dealer who sells the two vitamins v_1 and v_2 along with some other commodities. The local shopkeepers purchase the vitamins from him and form two foods f_1 and f_2 (the details are same as in the given table). The dealer knows very well that the foods f_1 and f_2 have their market values only because of their vitamin contents. The problem of the dealer is to fix the maximum per unit selling prices for the two vitamins v_1 and v_2, in such a way that the resulting prices of foods f_1 and f_2 do not exceed their existing market prices.

The selling prices are generally known as **dual prices** or **shadow prices** or **imputed prices**, indicating the worth of the resources.

To formulate the problem mathematically, let the dealer decide to fix up the two prices at w_1 and w_2 per unit, respectively. Then, the dealer's problem can be stated mathematically as to determine two real numbers w_1 and w_2 so as to

$$\text{Maximize } z^* = 100w_1 + 120w_2 \text{ subject to the constraints :}$$
$$6w_1 + 7w_2 \le 12,\ 8w_1 + 12w_2 \le 20;\ w_1, w_2 \ge 0.$$

The importance of the duality in linear programming is in terms of the information which the dual L.P.P. provides about the value of the resources. The managerial significance of the study of duality may be given as under :

(*a*) Dual variables provide the decision-maker a basis for deciding how much to pay for additional units of resources.

(*b*) Dual variables equal the marginal values of resources. The total marginal value of the resources equals the optimum value of the objective function.

(*c*) The maximum amount that should be paid for one additional unit of a resource is termed as *shadow price* or *imputed price*.

Note. The value of *i*th dual variable is the rate at which the primal objective function value will increase as the *i*th resource increases, assuming that all other information is unchanged.

SAMPLE PROBLEM

533. *One unit of product A contributes Rs. 7 and requires 3 units of raw material and 2 hours of labour. One unit of product B contributes Rs. 5 and requires 2 units of raw material and 3 hours of labour. Availability of raw material at present is of 240 units and there are 200 hours of labour.*

(*i*) *Formulate it as a linear programming problem.*

(*ii*) *Write its dual and also its economic interpretation.*

(*iii*) *Solve the dual by simplex method and find the optimal product mix, the shadow prices of the raw material and labour.*

(*iv*) *Use shadow prices for checking the optimal solution of the given problem.*

Solution. (*i*) Let x_1 denote the number of units of product *A* and x_2 denote the number of units of product *B*. Then, the appropriate mathematical formulation of L.P.P. is :

$$\text{Maximize } z = 7x_1 + 5x_2 \text{ subject to the constraints :} \qquad \textit{(Profit function)}$$
$$3x_1 + 2x_2 \le 240 \qquad \textit{(Raw material constraint)}$$
$$2x_1 + 3x_2 \le 200 \qquad \textit{(Labour hours constraint)}$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

(*ii*) Now, suppose that the producer is approached by an individual who would like to rent the facilities for one week. The owner has its assets in the form of 240 units of raw material and 200 labour hours. If we let w_1 represent the rental rate per unit of raw material and w_2 the rental rate per labour hour, the owner would receive a total rental equal to $240w_1 + 200w_2$. We shall compute the minimum value of the rental so that the owner will know as to what minimum offer shall be economically acceptable to him. The constraints can be set up by keeping in mind that the alternative of renting must be at least as favourable as the other ones. The rental rates of the resources should be at least as attractive as producing products *A* and *B*. We know that production of one unit of product *A* requires 3 units of raw material and 2 labour hours. Thus, the total rental for these amounts of resources should be greater than, or equal to, the profit obtainable from one unit of product *A*, *i.e.*, Rs. 7. Thus, we should have

$$3w_1 + 2w_2 \ge 7.$$

In a similar way, the resources consumed in producing one unit of product *B* are 2 units of raw material and 3 labour hours. The total rental of these resources should equal at least Rs. 5, the unit profit of product *B*. Thus, we should have

$$2w_1 + 3w_2 \ge 5.$$

Also, the rentals can never be negative. This puts the condition that w_1 and w_2 both should be non-negative.

The dual problem, therefore, is

$$\text{Minimize } z^* = 240w_1 + 200w_2 \text{ subject to the constraints :}$$

$$3w_1 + 2w_2 \geq 7,\ 2w_1 + 3w_2 \geq 5, \text{ and } w_1 \geq 0,\ w_2 \geq 0.$$

(*iii*) Using simplex method, an optimum simplex table of the dual L.P.P. can easily be obtained as under :

Final Iteration. Optimum Solution.

c_B	y_B	w_B	y_1	y_2	y_3	y_4	y_5	y_6
–240	y_1	11/5	1	0	–3/5	2/5	3/5	–2/5
–200	y_2	1/5	0	1	2/5	–3/5	–2/5	3/5
	z^*	–568	0	0	64	24	$M-64$	$M-24$

The optimum solution of the dual is :

$$w_1 = \frac{11}{5} \text{ and } w_2 = \frac{1}{5}$$

For the optimum solution of primal, we write

Starting dual variables	A_1	A_2
Corresponding $\{-(z_j - c_j)\}$	$-(M-64)$	$-(M-24)$
For starting primal variables	$x_1 - M$	$x_2 - M$

$\therefore \quad x_1 - M = -(M-64)$ and $x_2 - M = -(M-24)$ *i.e.*, $x_1 = 64$ and $x_2 = 24$.

$$\text{Maximum } z = -\text{Minimum } z^* = 568.$$

Hence, the optimum solution to the given L.P.P. indicates that producing of 64 units of product A and 24 units of product B per week yield the maximum profit of Rs. 568. The optimum solution to the dual indicates that the rental rates w_1 and w_2 should be 11/5 (= 2.20) and 1/5 (= 0.20) respectively so as to have the minimum rental of Rs. 568.

(*iv*) The individual rental rates w_1 and w_2 are called the *shadow prices* or *imputed prices* for raw material and labour, indicating the worth of resources. The economic nature of the shadow prices of raw material and labour in more detail is discussed below :

Of the two products A and B, we know that each unit of product A contributes Rs. 7 to the profit. The imputed price of raw material and labour being, Rs. 2.20 per unit and Re. 0.20 per hour. The total imputed cost of the resources used in making a unit of product would be :

3 units @ Rs. 2.20 per unit = Rs. 6.60
2 units @ Re. 0.20 per unit = Re. 0.40
Total Rs. 7.00

Thus, the total imputed cost of producing one unit of product A equals the unit profit obtainable from it.

Similarly, from each unit of product B, the total imputed cost of resources employed would be :

2 units @ Rs. 2.20 per unit = Rs. 4.40
3 units @ Re. 0.20 per unit = Re. 0.60
Total Rs. 5.00

This, obviously equals the profit per unit of product B.

534. *A manufacturing company makes three products, each of which requires three operations as part of the manufacturing process. The company can sell all of the products it can manufacture but its production capacity is limited by the capacity of its operations centre. Additional data concerning the company as shown in the table given below:*

Product	Manufacturing requirements (hours/unit)			Cost (Rs.)	Selling price (Rs.)
	Centre I	Centre II	Centre III		
A	1	3	2	11	15
B	3	4	1	12	20
C	2	2	2	10	16
Hours available	160	120	80		

What should the product mix be? Write the dual of the given problem. Give its economic interpretation and use it for checking the optimal solution of the given problem.

[Delhi M.Com. 2009]

Solution. If x_1 units of product A, x_2 units of product B, and x_3 units of product C are produced, the LPP may be stated as follows:

$$\text{Maximize } z = 4x_1 + 8x_2 + 6x_3 \text{ subject to the constraints :}$$

$$x_1 + 3x_2 + 2x_3 \le 160,\ 3x_1 + 4x_2 + 2x_3 \le 120,\ 2x_1 + x_2 + 2x_3 \le 80;\ x_1 \ge 0,\ x_2 \ge 0 \text{ and } x_3 \ge 0.$$

Using simplex method, the optimum solution is obtained in the following simplex table:

Final Iteration. Optimum solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	160/3	–5/3	0	0	1	–2/3	–1/3
8	y_2	40/3	1/3	1	0	0	1/3	–1/3
6	y_3	100/3	5/6	0	1	0	–1/6	2/3
	z	920/3	11/3	0	0	0	5/3	4/3

Thus, optimum solution to the problem is to produce Product A = nil, Product B = 40/3 units and Product C = 100/3 units. Total profit = Rs. 920/3.

Dual: If w_1, w_2 and w_3 are dual variables, the dual may be stated as :

$$\text{Minimize } z^* = 160w_1 + 120w_2 + 80w_3 \text{ subject to the constraints :}$$

$$w_1 + 3w_2 + 2w_3 \ge 4,\ 3w_1 + 4w_2 + w_3 \ge 8,\ 2w_1 + 2w_2 + 2w_3 \ge 6 \text{ and } w_1, w_2, w_3 \ge 0.$$

For the optimum solution of the dual, we write

Starting primal variables	Slack variables		
	s_1	s_2	s_3
Corresponding $(z_j - c_j)$	0	5/3	4/3
For starting dual variables	w_1	w_2	w_3

$$\therefore\ w_1 = 0,\ w_2 = \frac{5}{3} \text{ and } w_3 = \frac{4}{3} \text{ and minimum } z^* = 160 \times 0 + 120 \times \frac{5}{3} + 80 \times \frac{4}{3} = \frac{920}{3}.$$

Economic Interpretation

The dual variable represent the worth of the resources of the organisation. The variables w_1, w_2 and w_3 represent the marginal profitability of the capacity of centres 1, 2 and 3 respectively. If the company were to rent out its capacity, the minimum rental for each centre would be equal to its marginal profitability. Since the centre 1 has unused capacity, its marginal profitability is nil, while for centres 2 and 3, these are, respectively, Rs. 5/3 and Rs. 4/3, per hour.

With these shadow prices or imputed values, we can make the following observations :

For Product A. Here the decision reached is not to produce it. This is because it yields a profit of Rs. 4, while the value of resource it consumes is :

1 hour @ Rs. 0	= Rs. 1 × 0	= Rs. 0
3 hours @ Rs. 5/3	= Rs. 3 × 5/3	= Rs. 5
2 hours @ Rs. 4/3	= Rs. 2 × 4/3 = Rs. 8/3	= Rs. 2.67
	Total :	Rs. 7.67

Since, the total value of resource required by a unit of product A is Rs. 7.67 while the profit associated is only Rs. 4, obviously it should not be produced. In case a unit of product A is produced, the manufacturer will incurr a loss of (Rs. 7.67 – Rs. 4), *i.e.*, of Rs. 3.67.

For Product B and C, the value of resources required for their production is :

Product B	*Product C*
3 hours @ Rs. 0 = Rs. 0	2 hours @ Rs. 0 = Rs. 0
4 hours @ Rs. 5/3= Rs. 20/3	2 hours @ Rs. 5/3 = Rs. 10/3
1 hours @ Rs. 4/3= Rs. 4/3	2 hours @ Rs. 4/3 = Rs. 8/3
Total : Rs. 24/3 = Rs. 8.00	Total : Rs. 18/3 = Rs. 6.00
Profit per unit = Rs. 8.00	Profit per unit = Rs. 6.00

Clearly, for each of the product B and C, the opportunity cost is equal to zero.

PROBLEMS

535. A diet-conscious housewife wishes to ensure certain minimum intake of vitamins A, B and C for the family. The minimum daily needs of the vitamins A, B and C for the family are 60, 40 and 32 units, respectively. For the supply of these vitamins the housewife relies on two fresh foods F_1 and F_2. One unit of F_1 contains 14, 10, and 4 units of vitamins A, B and C respectively. One unit of F_2 contains 4, 8 and 16 units of the three vitamins respectively. F_1 costs Rs. 3 per unit and F_2 Rs. 2 per unit. The problem is to find how many units of each food the housewife should buy every day to keep her food bill as low as possible.

(*a*) Formulate it as a linear programming problem to minimise the total cost;

(*b*) Write the dual of the problem as formulated in (*a*) and present the economic interpretation of the dual.

(*c*) Solve the dual by using simplex method and read therefrom the answer to the primal.

536. An electronics firm is undecided as to the most profitable mix for its products. The products now manufactured are transistors, resistors, and electron tubes, with a profit (per 100 units) of Rs. 100, Rs. 60 and Rs. 40 respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering service, 10 hours of direct labour, and 2 hours of administration service. To produce 100 resistors requires 1 hour, 4 hours and 2 hours of engineering, direct labour and administration time, respectively. To produce one shipment of electron tubes (100 units) requires 1 hour of engineering, 5 hours of direct labour and 6 hours of administration. There are 100 hours of engineering services available, 600 hours of direct labour and 300 hours of administration. What is the most profitable product mix? Write the *dual* of the given problem, use it to check the optimal solution and give its economic interpretation. ***[Delhi M.B.A. (Nov.) 2005]***

5:9. DUAL SIMPLEX METHOD

Dual simplex method is applicable to those linear programming problems that start with *infeasible* but otherwise *optimum* solution. The method may be summarized as follows :

Step 1. Write the given linear programming problem in its standard form and obtain a starting basic solution.

Step 2. (*a*) If the current basic solution is feasible, use simplex method to obtain an optimum solution.

(*b*) If the current basic solution is infeasible, *i.e.*, values of basic variables are ≤ 0, go to the next step.

Step 3. Check whether the solution is optimum.

(*a*) If the solution is not optimum, add an artificial constraint in such a way that the condition of optimality is satisfied.

(*b*) If the solution is *optimum*, go to next *step*.

Step 4. Select the basic variable having the most negative value. This basic variable becomes the leaving variable and the row corresponding to it becomes the *key row*.

Step 5. Obtain the ratios of the net evaluations to the corresponding coefficients in the key row. Ignore the ratios associated with positive and zero denominators. The entering vector is the one with the smallest absolute value of the ratios. Column corresponding to the entering vector becomes the key column.

Step 6. Reduce the leading element into unity and all other entries of the key column to zero by elementary row operations.

Step 7. Go to *Step 2* and repeat the procedure until an optimum basic feasible solution is attained.

SAMPLE PROBLEMS

537. *Use dual simplex method to solve the following L.P.P. :*

Minimize $z = 3x_1 + x_2$ *subject to the constraints :*

$$x_1 + x_2 \geq 1,\ 2x_1 + 3x_2 \geq 2,\ x_1, x_2 \geq 0.$$ ***[IAS 1990]***

Solution. Converting the given constraints into $\leq$ type, introduce slack variables $s_1 \geq 0$ and $s_2 \geq 0$. An initial basic (infeasible) solution of the modified linear programming problem will be $s_1 = -1$ and $s_2 = -2$.

The iterative dual simplex tables are :

Initial Iteration. Drop $\mathbf{y}_4$ and enter $\mathbf{y}_2$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
0	$\mathbf{y}_3$	−1	−1	−1	1	0
0	$\mathbf{y}_4$	−2	−2	−3	0	1
	$(z_j - c_j)$	0	3	1	0	0

Since all $z_j - c_j \geq 0$ and $x_{B1}\ (= s_1) < 0$, $x_{B2}\ (= s_2) < 0$; an optimum but infeasible solution has been attained. In order to obtain a feasible optimum solution, we select a basis vector to leave the basis and non-basis vector to enter the basis as follows :

Since, $\min\limits_i \{x_{Bi}, x_{Bi} < 0\} = \min\{-1, -2\} = -2\ (= x_{B2})$; basis vector $\mathbf{y}_4$ will leave the basis.

Further, since

$$\max_j \left\{ \frac{z_j - c_j}{y_{2j}},\ y_{2j} < 0 \right\} = \max\left\{ \frac{3}{-2}, \frac{1}{-3} \right\} = \frac{-1}{3} \left(= \frac{z_2 - c_2}{y_{22}} \right);$$

the non-basis vector $\mathbf{y}_2$ will enter the basis.

First Iteration. Drop $\mathbf{y}_3$ and enter $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
0	$\mathbf{y}_3$	−1/3	−1/3	0	1	−1/3
−1	$\mathbf{y}_2$	2/3	2/3	1	0	−1/3
	$(z_j - c_j)$	−2/3	7/3	0	0	1/3

In the above table,

$$\min_j \{x_{Bi}, x_{Bi} \leq 0\} = -\frac{1}{3}\ (= x_{B1}),\ \text{and}$$

$$\max_j \left\{ \frac{z_j - c_j}{y_{1j}},\ y_{1j} < 0 \right\} = \max\left\{ \frac{7/3}{-1/3}, \frac{1/3}{-1/3} \right\} = -1 \left(= \frac{z_4 - c_4}{y_{14}} \right).$$

Thus $\mathbf{y}_3$ leaves the current basis and non-basis $\mathbf{y}_4$ enters the basis.

Final Iteration. Optimum Solution.

c_B	y_B	w_B	y_1	y_2	y_3	y_4
0	y_4	1	1	0	−3	1
−1	y_2	1	1	1	−1	0
		z^* (= −1)	2	0	1	0

It is now apparent that since all $z_j - c_j \geq 0$ and also all $x_{Bi} \geq 0$, an optimum basic feasible solution has been reached.

Hence, the optimum basic feasible solution to the given L.P.P. is :

$$x_1 = 0 \text{ and } x_2 = 1 \text{ with minimum } z = -\max z^* = -(-1) = 1.$$

PROBLEMS

Use dual simplex method to solve the following linear programming problems :

538. Minimize $z = x_1 + x_2$ subject to the constraints :

$$2x_1 + x_2 \geq 4,\ x_1 + 7x_2 \geq 7;\ x_1, x_2 \geq 0.$$

539. Minimize $z = x_1 + x_2$ subject to the constraints :

$$2x_1 + x_2 \geq 2,\ -x - x_2 \geq 1;\ x_1 \geq 0,\ x_2 \geq 0.$$ *[Annamalai M.B.A (Nov.) 2002]*

540. Maximize $z = -2x_1 - x_2$ subject to the constraints :

$$3x_1 + x_2 \geq 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \geq 3;\ x_1, x_2 \geq 0.$$ *[Madras B.E. (Mech.) 1999]*

541. Minimize $z = 10x_1 + 6x_2 + 2x_3$ subject to the constraints :

$$-x_1 + x_2 + x_3 \geq 1,\ 3x_1 + x_2 - x_3 \geq 2,\ x_1, x_2, x_3 \geq 0.$$

[M.S. Tirunvelli B.Sc. (Math.) 1996]

542. Minimize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$ subject to the constraints :

$$5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12,\ x_2 + 5x_3 - 6x_4 \geq 10,\ 2x_1 + 5x_2 + x_3 + x_4 \geq 8,\ x_1, x_2, x_3, x_4 \geq 0.$$

[Meerut M.Sc. (Math.) 1998]

543. Use dual simplex method to solve the L.P.P. :

Minimize $z = 3x_1 + 2x_2 + x_3 + 4x_4$ subject to the constraints :

$$2x_1 + 4x_2 + 5x_3 + x_4 \geq 10,\ 3x_1 - x_2 + 7x_3 - 2x_4 \geq 2,\ 5x_1 + 2x_2 + x_3 + 6x_4 \geq 15,\ x_1, x_2, x_3, x_4 \geq 0.$$

[Garhwal M.Sc. (Math.) 2002]

544. Solve the following linear programming problem by dual simplex method :

Minimize $z = 2x_1 + 9x_2 + 24x_3 + 8x_4 + 5x_5$ subject to the constraints :

$$x_1 + x_2 + 2x_3 - x_5 - x_6 = 1,\ -2x_1 + x_3 + x_4 + x_5 - x_7 = 2,\ x_j \geq 0;\ j = 1, 2, \ldots, 7$$

545. Maximize $z = 2x_1 + x_2 + x_3$ subject to the constraints :

$$2x_1 + 3x_2 - 5x_3 = 4,\ -x_1 + 9x_2 - x_3 \geq 3,\ 4x_1 + 6x_2 + 3x_3 \leq 8,\ x_1, x_2, x_3 \geq 0.$$

[***Hint :*** Since the initial solution is infeasible and also not optimum, add an artificial constraint $x_1 + x_2 + x_3 \leq M$ (where $M > 0$ is a large number). With the help of this additional constraint, eliminate x_1 from the objective function and constraints of the given problem.]

546. Use dual simplex method to solve the L.P.P.

Minimize $z = x_1 + 2x_2 + 3x_3$ subject to the constraints :

$$x_1 - x_2 + x_3 \geq 4,\ x_1 + x_2 + 2x_3 \leq 8,\ x_2 - x_3 \geq 2,\ x_1, x_2, x_3 \geq 0.$$

547. Using artificial constraint, solve the following L.P.P. by dual simplex method.

Maximize $z = 2x_3$ subject to the constraints :

$$-x_1 + 2x_2 - 2x_3 \geq 8,\ -x_1 + x_2 + x_3 \leq 4,\ 2x_1 - x_2 + 4x_3 \leq 10;\ x_1, x_2, x_3 \geq 0.$$

MULTIPLE CHOICE QUESTIONS

1. Which of the following is not correct ?

(*a*) Associated with every LPP, there is always another LPP which is based on the same data and having the same solution.

(*b*) Given LPP is called the primal problem while the associated LPP is called its dual.

(*c*) It is always necessary to convert the inequality constraints into equality constraints for writing the dual to an LPP.

(*d*) The dual of dual is primal.

2. The dual of the primal maximization LPP having m constraints and n non-negative variables should

(*a*) be a minimization LPP. (*b*) have n constraints and m non-negative variables.

(*c*) both (*a*) and (*b*). (*d*) none of the above.

3. The right hand side constant of a constraint in a primal problem appears in the corresponding dual as

(*a*) a coefficient in the objective function.

(*b*) a right hand side constant of a constraint in a dual problem.

(*c*) an input-output coefficient.

(*d*) none of the above.

4. If dual has an unbounded solution, primal has

(*a*) an unbounded solution (*b*) an infeasible solution

(*c*) a feasible solution (*d*) none of the above.

5. For any primal and its dual,

(*a*) optimum value of the objective function is same.

(*b*) both primal and dual cannot be feasible.

(*c*) primal will have an optimum solution if and only if dual does too.

(*d*) all the above.

6. Dual simplex method is applicable to these LPP's that start with

(*a*) an infeasible solution (*b*) an infeasible but optimum solution

(*c*) a feasible solution (*d*) a feasible and optimum solution.

7. Which of the following is not correct ?

(*a*) The dual represents an alternative formulation of an LPP with decision variables being having implicit values.

(*b*) If the variables in a primal problem are all greater than, or equal to, zero, then the variables in the dual problem would be all less than or equal to zero.

(*c*) Corresponding to every unrestricted primal variable there is an equality dual constraint.

(*d*) The primal and its dual do not have the same number of variables.

8. Which of the following is not correct ?

(*a*) Duality does not play any role in the Post-Optimal Analysis of an LPP.

(*b*) The optimum simplex table provides information about the status and worth of the resources in addition to the optimum values of the decision variables.

(*c*) Using dual simplex method, we can never have an unbounded solution.

(*d*) In dual simplex method, the procedure ends at the iteration when feasibility is reduced.

9. Which of the following is a correct statement?

(*a*) If the primal problem is in its standard form, dual variables will be non-negative.

(*b*) Dual simplex method is applicable to an LPP, if initial basic feasible solution is not optimum.

(*c*) Dual simplex method always leads to degenerate basic feasible solution.

(*d*) If the number of primal variables is very small and the number of constraints is very large, then it is more efficient to solve the dual rather than the primal problem.

10. Which of the following is not correct ?

(*a*) Shadow prices are also known as imputed values of the resources.

(*b*) The shadow price of a non-binding constraint is always equal to zero.

(*c*) All scarce resources have marginal profitability equal to zero.

(*d*) The shadow price indicates the change in the value of the objective function per unit increase in the value of the right hand side.

ANSWERS

1. (*c*)	**2.** (*c*)	**3.** (*a*)	**4.** (*b*)	**5.** (*a*)
6. (*b*)	**7.** (*b*)	**8.** (*a*)	**9.** (*d*)	**10.** (*c*).

REVIEW QUESTIONS

1. Explain duality in linear programming. *[Panjab Tech. Univ. M.B.A. (Dec.) 2010]*

2. Define dual. Give any *two* relationships between a primal and a dual. *[Delhi MIB 2011]*

3. Write the dual of the following linear programming model :

Maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ subject to the constraints :

$$a_{i1} + a_{i2} + \ldots + a_{in} = b_i; \; i = 1, 2, \ldots, n$$

$$x_j \geq 0 \quad (j = 1, 2, \ldots, n)$$

[Delhi M.Com. 1995]

4. Prove that the dual of the dual of a given primal is again primal. *[Delhi B.Sc. (Stat.) 2004]*

5. State the general rules for forming a dual L.P.P. from its primal.

6. State the duality theorem and explain its implications.

7. Write short note on 'complementary slackness'. *[Delhi B.Sc. (Math.) 1995]*

8. State and prove the theorem of dual-primal relationship.

9. State the fundamental theorem of duality and explain the Dual Simplex Method. *[M.S. Tirunvelli B.Sc. (Math.) 1996]*

10. Explain the economic interpretation of dual variables. *[Delhi B.Sc. (Stat.) 1998]*

11. What is shadow price? How does the concept relate to the dual of an LPP? How does it relate to primal?

12. How the concept of duality can be useful in managerial decision-making?

13. Give the applications of duality theory. *[Annamalai M.B.A. (Nov.) 2009]*

14. What is shadow price? How can we read the shadow price from a simplex table? *[Delhi PG Dip. in Glob. Bus. Oper. 2010; Delhi MIB 2011]*

15. What is the physical significance of shadow price of a resource? Explain with the help of an example. *[Delhi M.B.A. (HCA) 2015]*