11

Assignment Problem

"O.R. produces an integrated solution for the good of entire organisation and not just to effect local improvement"

11:1. INTRODUCTION

The assignment problem is a special case of the transportation problem in which the objective is to **assign** a number of resources to the **equal** number of activities at a minimum cost (or maximum profit).

Assignment problem is completely degenerate form of a transportation problem. The units available at each origin (resource) and units demanded at each destination (activity) are all equal to one. That means exactly one occupied cell in each row and each column of the transportation table, *i.e.*, only n occupied (basic) cells in place of the required $n + n - 1$ $(= 2n - 1)$.

11:2. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a problem of assignment of n resources (workers) to n activities (jobs) so as to minimize the overall cost or time in such a way that each resource can associate with one and only one job. The cost (or effectiveness) matrix (c_{ij}) is given as under :

		Activity				*Available*
		A_1	A_2	$\cdots$	A_n	
Resource	R_1	c_{11}	c_{12}	$\cdots$	c_{1n}	1
	R_2	c_{21}	c_{22}	$\cdots$	c_{2n}	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	R_n	c_{n1}	c_{n2}	$\cdots$	c_{nn}	1
Required		1	1	$\cdots$	1	

This cost matrix is same as that of a transportation problem except that availability at each of the resources and the requirement at each of the destinations is unity (due to the fact that assignments are made on a one-to-one basis).

Let x_{ij} denote the assignment of ith resource to jth activity, such that

$$x_{ij} = \begin{cases} 1, & \text{if resource } i \text{ is assigned to activity } j \\ 0, & \text{otherwise} \end{cases}$$

Then, the mathematical formulation of the assignment problem is

$$\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad \text{subject to the constraints :}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{and} \quad \sum_{j=1}^{n} x_{ij} = 1 ; \quad x_{ij} = 0 \text{ or } 1$$

$$\text{for all } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n.$$

Note. In the above statement c_{ij} is the cost associated with assigning ith resource to jth activity.

Illustration. *Given below is an assignment problem, write it as a transportation problem :*

	A_1	A_2	A_3
R_1	1	2	3
R_2	4	5	1
R_3	2	1	4

Answer. Let x_{ij} denote the assignment of R_i $(i = 1, 2, 3)$ to A_j $(j = 1, 2, 3)$, such that

$$x_{ij} = \begin{cases} 1, & \text{if } R_i \text{ is assigned to } A_j \\ 0, & \text{otherwise} \end{cases}$$

Then the transportation problem is :

$$\text{Minimize} \quad z = 1x_{11} + 2x_{12} + 3x_{13} + 4x_{21} + 5x_{22} + x_{23} + 2x_{31} + x_{32} + 4x_{33}$$

subject to the constraints :

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} &= 1 \\ x_{21} + x_{22} + x_{23} &= 1 \\ x_{31} + x_{32} + x_{33} &= 1 \end{aligned}\right\}, \quad \left.\begin{aligned} x_{11} + x_{21} + x_{31} &= 1 \\ x_{12} + x_{22} + x_{32} &= 1 \\ x_{13} + x_{23} + x_{33} &= 1 \end{aligned}\right\},$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3.$$

Theorem 11-1 **(Reduction Theorem).** *In an assignment problem, if we add or subtract a constant to every element of any row (or column) of the cost matrix* $[c_{ij}]$, *then an assignment that minimizes the total cost on one matrix also minimizes the total cost on the other matrix. In other words, if* $x_{ij} = x_{ij}^*$ *minimizes*

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad \textit{with} \quad \sum_{i=1}^{n} x_{ij} = 1, \ \sum_{j=1}^{n} x_{ij} = 1 ; \ x_{ij} = 0 \textit{ or } 1$$

then x_{ij}^* *also minimizes* $z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij}$, *where* $c_{ij}^* = c_{ij} - u_i - v_j$ *for all* $i, j = 1, 2, \ldots, n$ *and* u_i, v_j *are some real numbers.*

Proof. We write

$$z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{n} x_{ij} \sum_{j=1}^{n} v_j$$

$$= z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j ;$$

since $$\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1.$$

This shows that the minimization of the new objective function z^* yields the same solution as the minimization of original objective function z, because Σu_i and Σv_j are independent of x_{ij}.

Theorem 11-2 *If $c_{ij} \geq 0$, such that minimum $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = 0$, then the feasible solution x_{ij} provides an optimum assignment.*

The proof is left as an exercise to the reader.

The above two theorems form the basis of *Assignment Algorithm.* By selecting suitable constants to be added to or subtracted from the elements of the cost matrix we can ensure that each $c_{ij}^* \geq 0$ and can produce at least one $c_{ij}^* = 0$ in each row and each column and try to make assignments from among these 0 positions. The assignment schedule will be optimal if there is exactly one assignment (*i.e.*, exactly one assigned 0) in each row and each column.

Remarks. It may be noted that assignment problem is a variation of transportation problem with two characteristics (*i*) the cost matrix is a square matrix, and (*ii*) the optimum solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

11:3. SOLUTION METHODS OF ASSIGNMENT PROBLEM

An assignment problem can be solved using the following four methods :

1. Complete Enumeration Method. In this method, a list of all possible assignments among the given resources and activities is prepared. Then an assignment involving the minimum cost, time or distance (or maximum profit) is selected. It represents the optimum solution. In case there are more than one assignment patterns involving the same least cost, then they all represent the optimum solutions — the problem has multiple optima.

In general, if there are n jobs and n workers, there are $n\,!$ possible assignments. Thus, the listing and evaluation of all the possible assignments is a simple matter when n is small. When n is large, this method is not very practical. For example, if there are 8 jobs and 8 workers, we have to evaluate a total of 8 ! or 40,320 assignments. The method, therefore, is not suitable for real world situations.

2. Transportation Method. Since an assignment problem is a special case of the transportation problem, it can be solved by transportation methods discussed in the previous chapter. However, every basic feasible solution of a general assignment problem having a square payoff matrix of order n should have $m + n - 1 = n + n - 1 = 2n - 1$ assignments or basic cells. But due to the special structure of this problem, any basic solution cannot have more than n assignments. Thus, the assignment problem is inherently degenerate. In order to remove degeneracy, $(n - 1)$ dummy allocations will be required to proceed with the transportation method. However, because of the large number of dummy allocations in the solution, the transportation method becomes computationally inefficient for solving an assignment problem.

3. Simplex Method. An assignment problem can be formulated as a transportation problem which, in turn, is itself a special case of an LPP. Accordingly, an assignment problem can be formulated as an LPP with integer valued variables and may be solved using a modified simplex method or otherwise. Here, the decision variables take only one of the two values : 1 or 0.

In general let

$$x_{ij} = \begin{cases} 1 & \text{if } i\text{th person is assigned } j\text{th job} \\ 0 & \text{if } i\text{th person is not assigned } j\text{th job} \end{cases}$$

The mathematical formulation of the assignment problem as a 0–1 integer linear programming problem would be :

$$\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad \text{subject to the constraints :}$$

$$x_{i1} + x_{i2} + \ldots + x_{in} = 1 ; \qquad i = 1, 2, \ldots, n$$

$$x_{1j} + x_{2j} + \ldots + x_{nj} = 1 ; \qquad j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j.$$

As can be seen in the general mathematical formulation of the assignment problem, there are $n \times n$ decision variables and $n + n$ or $2n$ equalities/equations. In particular, for a problem involving 5 workers/jobs, there will be 25 decision variables and 10 equalities. That means a simplex table having 25 columns and 10 rows. It is difficult to solve manually and hence this approach to the solution is not considered.

4. Hungarian Assignment Method. An efficient method for solving an assignment problem is the *Hungarian Assignment Method* (also known as reduced matrix method), which is based on the concept of opportunity cost. Opportunity costs show the relative penalties associated with assignment of resource to an activity as opposed to making the best or least cost assignment. If we can reduce the cost matrix to the extent of having at least one zero in each row and each column, then it will be possible to make optimal assignments (opportunity costs are all zero).

The method of solving an assignment problem (minimization case) consists of the following steps :

Step 1. Determine the cost table from the given problem.

(*i*) If the number of sources is equal to the number of destinations, go to *step 3.*

(*ii*) If the number of sources is not equal to the number of destinations, go to *step 2.*

Step 2. Add a dummy source or dummy destination, so that the cost table becomes a square matrix. The cost entries of dummy source/destinations are always zero.

Step 3. Locate the smallest element in each row of the given cost matrix and then subtract the same from each element of that row.

Step 4. In the reduced matrix obtained in *step 3*, locate the smallest element of each column and then subtract the same from each element of that column. Each column and row now have at least one zero.

Step 5. In the modified matrix obtained in *step 4*, search for an optimal assignment as follows :

(*a*) Examine the rows successively until a row with a single zero is found. Enrectangle this zero (□) and cross off (×) all other zeros in its column. Continue in this manner until all the rows have been taken care of.

(*b*) Repeat the procedure for each column of the reduced matrix.

(*c*) If a row and/or column has two or more zeros and one cannot be chosen by inspection, then assign arbitrary any one of these zeros and cross off all other zeros of that row/column.

(*d*) Repeat (*a*) through (*c*) above successively until the chain of assigning (□) or cross (×) ends.

Step 6. If the number of assignments (□) is equal to n (the order of the cost matrix), an optimum solution is reached.

If the number of assignments is less than n (the order of the matrix), go to the next step.

Step 7. Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced matrix. This can be conveniently done by using a simple procedure :

(*a*) Mark (√) rows that do not have any assigned zero.

(*b*) Mark (√) columns that have zeros in the marked rows.

(*c*) Mark (√) rows that have assigned zeros in the marked columns.

(*d*) Repeat (*b*) and (*c*) above until the chain of marking is completed.

(*e*) Draw lines through all the *unmarked rows* and *marked columns*. This gives us the desired minimum number of lines.

Step 8. Develop the new revised cost matrix as follows :

(*a*) Find the smallest element of the reduced matrix not covered by any of the lines.

(*b*) Subtract this element from all the *uncovered* elements and add the same to all the elements lying at the *intersection* of any two lines.

Step 9. Go to *Step 6* and repeat the procedure until an optimum solution is attained.

SAMPLE PROBLEMS

1101. *A departmental head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take to perform each task, is given in the matrix below :*

Tasks	*Men*			
	E	*F*	*G*	*H*
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man-hours?

[Andhra B.E. (Mech. & Ind.) 1996]

Solution.

Step 1. Here, the number of tasks and the number of subordinates each equal 4, therefore the problem is balanced and we move on to *step 3*.

Step 3. Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix :

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Step 4. Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

Step 5. Starting with row *1*, we enrectangle (□) (i.e., make assignment) a single zero, if any, and cross (×) all other zeros in the column so marked. Thus, we get

7	11	5	$\boxed{0}$
$\boxed{0}$	11	~~0~~	13
23	$\boxed{0}$	2	~~0~~
9	12	13	~~0~~

In the above matrix, we arbitrarily enrectangled a zero in column *1*, because row *2* had two zeros. It may be noted that column *3* and row *4* do not have any assignment. So, we move on to the next step.

Step 7. (*i*) Since row *4* does not have any assignment, we mark this row (√).

(*ii*) Now there is a zero in the fourth column of the marked row. So, we mark fourth column (√).

(*iii*) Further there is an assignment in the first row of the marked column. So we mark first row (√).

(*iv*) Draw straight lines through all unmarked rows and marked columns. Thus, we have

7	11	5	$\boxed{0}$	√
$\boxed{0}$	11	~~8~~	13	
23	$\boxed{0}$	2	~~8~~	
9	12	13	~~8~~	√
			√	

Step 8. In *step 7*, we observe that the minimum number of lines so drawn is *3*, which is less than the order of the cost matrix, indicating that the current assignment is not optimum.

To increase the minimum number of lines, we generate new zeros in the modified matrix.

The smallest element not covered by the lines is 5. Subtracting this element from all the uncovered elements and adding the same to all the elements lying at the intersection of the lines, we obtain the following new reduced cost matrix :

2	6	0	0
0	11	0	18
23	0	2	5
4	7	8	0

Step 9. Repeating *step 5* on the reduced matrix, we get

2	6	$\boxed{0}$	~~8~~
$\boxed{0}$	11	~~8~~	18
23	$\boxed{0}$	2	5
4	7	8	$\boxed{0}$

Now, since each row and each column has one and only one assignment, an optimal solution is reached. The optimum assignment is :

$$A \to G,\ B \to E,\ C \to F \text{ and } D \to H.$$

The minimum total time for this assignment scheduled is 17 + 13 + 19 + 10 or 59 man-hours.

1102. *A company wishes to assign 3 jobs to 3 machines in such a way that each job is assigned to some machine and no machine works on more than one job. The cost of assigning job i to machine j is given by the matrix below (ijth entry) :*

$$\textit{Cost matrix :} \begin{bmatrix} 8 & 7 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 7 \end{bmatrix}$$

Draw the associated network. Formulate the network LPP and find the minimum cost of making the assignment. ***[GGSIP Univ. B.B.A. 2011; Madras B.Com. 2007]***

Solution. (*a*) Network formulation of the given problem is given as under :

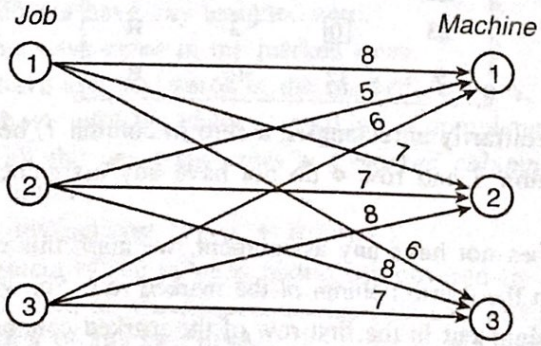

Fig. 11.1

(*b*) Linear programming formulation of the given problem is :

Minimize the total cost involved, *i.e.*,

$$\text{Minimize } z = (8x_{11} + 7x_{12} + 6x_{13}) + (5x_{21} + 7x_{22} + 8x_{23}) + (6x_{31} + 8x_{32} + 7x_{33})$$

subject to the constraints :

$$x_{i1} + x_{i2} + x_{i3} = 1; \qquad i = 1, 2, 3$$

$$x_{1j} + x_{2j} + x_{3j} = 1; \qquad j = 1, 2, 3$$

$$x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j.$$

(*c*) Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix, make assignments in rows and columns having single zeros and cross off all other zeros in rows and columns in the which the assignment has been made. See table *11.1*. Now, draw the minimum number of lines to cover all the zeros. For this, we proceed as follows:

(*i*) Mark (√) third row since it has no assignment.

(*ii*) Mark (√) first column, since third row has a zero in this column.

(*iii*) Mark (√) second row, since marked column has an assignment in the second row.

(*iv*) Since no other row or column can be marked, draw straight lines through the unmarked rows and marked column as shown in table *11.1* :

2	0	$\boxed{0}$	
$\boxed{0}$	1	3	√
~~0~~	1	1	√
√			

Table 11.1

3	$\boxed{0}$	~~0~~
$\boxed{0}$	~~0~~	2
~~0~~	~~0~~	$\boxed{0}$

Table 11.2

Modify the reduced cost matrix (table *11.1*) by selecting the smallest element among all the uncovered elements. Subtract this element from all the uncovered elements including itself and add it to the intersection element (1, 1) which lies at the intersection of two lines. The modified cost matrix so obtained is shown in table *11.2*.

In table *11.2*, we observe that there is no row and column which has single zero. So, we make an assignment arbitrarily at (1, 2) and cross off all zeros of first row and second column. Now, we get a single zero in the second row and therefore an assignment is made at (2, 1). Cross off all zeros in the first column. Finally, we make an assignment at (3, 3) being the single zero in the third row.

Clearly, the number of assignments in table *11.2* is equal to the order of the matrix. Hence, an optimum assignment has been attained, *viz.*,

Job *1* → Machine *2*, Job *2* → Machine *1*, Job *3* → Machine *3*.

Total minimum cost will be (7 + 5 + 7), *i.e.*, 19.

1103. *A pharmaceutical company is producing a single product and is selling it through five agencies located in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to despatch the product to needy cities in such a way that the travelling distance is minimised. The distance between the surplus and deficit cities (in km) is given in the following table :*

		Deficit cities				
		a	*b*	*c*	*d*	*e*
Surplus cities	*A*	85	75	65	125	75
	B	90	78	66	132	78
	C	75	66	57	114	69
	D	80	72	60	120	72
	E	76	64	56	112	68

Determine the optimum assignment schedule. **[Delhi M.Com. 2009]**

Solution. Subtracting the smallest element of each row from every element of that and then subtracting the smallest element of each coloumn from every element of that coloumn, we get the reduced distance table :

	a	*b*	*c*	*d*	*e*
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Table 11.3

In the reduced distance table, we make assignments in rows and coloumns having single zeros and cross off all other zeros is those rows and columns, where assignments have been made. Now draw the minimum number of lines to cover all the zeros. This is done in the following steps :

(*i*) Mark (√) row '*D*' since it has no assignment.

(*ii*) Mark (√) coloumn '*C*' since row '*D*' has zero in this column.

(*iii*) mark (√) row *B* since column '*C*' has an assignment in row '*B*'.

(*iv*) Since no other rows or columns can be marked, draw straight lines through the unmarked rows '*A*', '*C*', and '*E*', and marked coloumn '*C*' as shown in Table *11.4*..

	a	*b*	*c*	*d*	*e*	
A	2	2	~~0~~	4	$\boxed{0}$	
B	6	4	$\boxed{0}$	10	2	√
C	$\boxed{0}$	1	~~0~~	1	2	
D	2	4	~~0~~	4	2	√
E	2	$\boxed{0}$	~~0~~	~~0~~	2	
			√			

Table 11.4

	a	*b*	*c*	*d*	*e*
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

Table 11.5

Modify the reduced distance table (Table *11.4*) by subtracting the smallest element not covered by lines from all the uncovered elements and add the same at the intersection elements of the lines. The modified distance table so obtain is shown in Table *11.5*.

Repeat the above procedure to find the new assignment in table *11.6*.

	a	*b*	*c*	*d*	*e*	
A	2	2	2	4	$\boxed{0}$	√
B	4	2	$\boxed{0}$	8	~~0~~	√
C	$\boxed{0}$	1	2	1	2	√
D	~~0~~	2	~~0~~	2	~~0~~	√
E	2	$\boxed{0}$	2	~~0~~	2	
	√		√		√	

Table 11.6

	a	*b*	*c*	*d*	*e*
A	2	1	2	3	$\boxed{0}$
B	4	1	$\boxed{0}$	7	~~0~~
C	~~0~~	$\boxed{0}$	2	~~0~~	2
D	$\boxed{0}$	1	~~0~~	1	~~0~~
E	3	~~0~~	3	$\boxed{0}$	3

Table 11.7

Clearly the assignment shown in table *11.6* is also not optimum, since only four assignments are made. To get the next solution, we draw the minimum number of horizontal and vertical lines to cover all the zeros in table *11.6*. Subtracting the smallest unconvered element (*viz.*, 1) from all uncovered elements and adding the same to the intersection element of two lines gives us table *11.7*.

The new assignment schedule is shown in table *11.7*. Since, both the rows '*C*' and '*E*' have two zeros, the arbitrary selection of a cell in any of these two rows will give us an alternative solution having the same total distance.

Since the number of assignments is equal to the order of the given matrix, an optimum solution is attained, *viz.*,

$$A \to e,\ B \to c,\ C \to b,\ D \to a,\ E \to d;\ \text{or}\ A \to e,\ B \to c,\ C \to d,\ D \to a,\ E \to b.$$

Total minimum distance in both the cases will be 399 kilometres.

1104. *A department head has four tasks to be performed and three subordinates, the subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each man, so as to minimize the total man-hours?*

Task	*Men*		
	1	*2*	*3*
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Solution. Here we have three subordinates who have to perform four tasks. So, the given problem is unbalanced and therefore we add a dummy subordinate (column) with all its entries as zero. The resulting balanced problem is :

Subordinate		*1*	*2*	*3*	*Dummy*
Task	*I*	9	26	15	0
	II	13	27	6	0
	III	35	20	15	0
	IV	18	30	20	0

Now, reduce the balanced time-matrix by subtracting the smallest element of each column from all the elements of that column. In the reduced matrix, make assignments in rows and columns having single zeros and cross off all other zeros of the rows and columns, where assignment have been made. We get the following assignment solution :

	1	*2*	*3*	*Dummy*
I	$\boxed{0}$	6	9	~~0~~
II	4	7	$\boxed{0}$	~~0~~
III	26	$\boxed{0}$	9	~~0~~
IV	9	10	14	$\boxed{0}$

Table 11.8

The optimum assignment is

$I \to 1$, $II \to 3$ and $III \to 2$; while task *IV* should be assigned to a dummy man, *i.e.*, it remains to be done. The minimum time is 35 hours.

PROBLEMS

1105. Four professors are each capable of teaching any one of four different courses. Class preparation time in hours for different topics varies from professor to professor and is given in the table below. Each professor is assigned only one course. Determine an assignment schedule so as to minimize the total course preparation time for all courses :

Professor	Linear Programmes	Queueing Theory	Dynamic Programme	Regression Analysis
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	4	15	13	9

[Delhi B.Sc. (Stat.) 2004; J.N.T. M.B.A. 1995]

1106. An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of man-hours that would be required for each job-man combination. This is given in the matrix form in the following table :

Man	Job			
	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Find the optimum assignment that will result in minimum man-hours needed. *[Lucknow M.B.A. 2008]*

1107. Solve the following assignment problems :

(*a*)
$$\begin{array}{c} \\ I \\ II \\ III \\ IV \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 4 & 6 & 3 \\ 9 & 7 & 10 & 9 \\ 4 & 5 & 11 & 7 \\ 8 & 7 & 8 & 5 \end{bmatrix} \end{array}$$

(*b*)
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 10 & 25 & 15 & 20 \\ 15 & 30 & 5 & 15 \\ 35 & 20 & 12 & 24 \\ 17 & 25 & 24 & 20 \end{pmatrix} \end{array}$$

(*c*)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 10 & 12 & 19 & 11 \\ 5 & 10 & 7 & 8 \\ 12 & 14 & 13 & 11 \\ 8 & 15 & 11 & 9 \end{bmatrix} \end{array}$$

[Delhi M.B.A. (Oct.) 2010]

(*d*)
$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 5 & 8 & 3 & 2 \\ 10 & 7 & 5 & 8 \\ 4 & 10 & 12 & 10 \\ 8 & 6 & 9 & 4 \end{bmatrix} \end{array}$$

[Sambalpur M.B.A. 1994]

1108. Four operators O_1, O_2, O_3 and O_4 are available to a manager who has to get four jobs J_1, J_2, J_3 and J_4 done by assigning one job to each operator. Given the times needed by different operators for different jobs in the matrix below :

$$\begin{array}{c} \\ O_1 \\ O_2 \\ O_3 \\ O_4 \end{array}\begin{array}{c} \begin{array}{cccc} J_1 & J_2 & J_3 & J_4 \end{array} \\ \begin{bmatrix} 12 & 10 & 10 & 8 \\ 14 & 12 & 15 & 11 \\ 6 & 10 & 16 & 4 \\ 8 & 10 & 9 & 7 \end{bmatrix} \end{array}$$

(*i*) How should the manager assign the jobs so that the total time needed for all jobs is minimum?

(*ii*) If job J_2 is not to be assigned to operator O_2, what should be the assignment and how much additional total time will be required?

[Kurukshetra M.B.A. 2008]

1109. Four operators *A–D* are available to a manager who has to get four jobs I–IV done by assigning one job to each operator. Given the times needed by different operators for different jobs in the matrix below :

Jobs	Operator			
	A	B	C	D
I	15	13	14	17
II	11	12	15	13
III	18	12	10	11
IV	15	17	14	16

(*i*) How should the manager assign the jobs so that the total time needed for all the jobs is minimum?

(*ii*) If job *II* is not to be assigned to operator *A* what should be the assignment and how much additional total time will be required? *[C.A. Final (May) 1994]*

1110. (*i*) Suggest optimal assignment of the workers to jobs if the completion time (in hours) of different jobs by different workers is as given below :

$$\text{Jobs}\quad \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{Workers} \\ \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 8 & 7 & 9 & 10 \\ 7 & 9 & 9 & 8 \\ 10 & 8 & 7 & 11 \\ 10 & 6 & 8 & 7 \end{bmatrix} \end{array}$$

(*ii*) What are the similarities between transportation and assignment techniques? *[Delhi M.Com. 2000]*

1111. Consider the problem of assigning five operators to five machines. The assignment costs are given below :

Operators	Machines				
	A	*B*	*C*	*D*	*E*
I	10	3	10	7	7
II	5	9	7	11	9
III	13	18	2	9	10
IV	15	3	2	7	4
V	16	6	2	12	12

Assign the operators to different machines so that total cost is minimised. *[Madurai M.Com. (Nov.) 2003]*

1112. Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows :

$$\text{Person}\quad \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \text{Job} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{bmatrix} \end{array}$$

Determine the optimum assignment schedule. *[Pondicherry B.Sc. (Math.) 2006]*

1113. Five men are available to do five different jobs. From past records, the time (in hours) that each man takes to do each job is known and given in the following table :

$$\text{Men}\quad \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \text{Jobs} \\ \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \boxed{\begin{array}{ccccc} 2 & 9 & 2 & 7 & 1 \\ 6 & 8 & 7 & 6 & 1 \\ 4 & 6 & 5 & 3 & 1 \\ 4 & 2 & 7 & 3 & 1 \\ 5 & 3 & 9 & 5 & 1 \end{array}} \end{array}$$

Find the assignment of men to jobs that will minimize the total time taken. *[Delhi M.B.A. (PT.) 2009]*

1114. The *XYZ* company has 5 jobs *I, II, III, IV, V* to be done and 5 men *A, B, C, D, E* to do these jobs. The number of hours each man would take to accomplish each job is given by the following table :

$$\begin{array}{c} I \\ II \\ III \\ IV \\ V \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 16 & 13 & 17 & 19 & 20 \\ 14 & 12 & 13 & 16 & 17 \\ 14 & 11 & 12 & 17 & 18 \\ 5 & 5 & 8 & 8 & 11 \\ 5 & 3 & 8 & 8 & 10 \end{bmatrix} \end{array}$$

Work out the optimal assignment and the total minimum time taken. *[Lucknow M.B.A. 2009]*

1115. There are five jobs to be assigned to five machines. Costs of completion of the jobs on the respective machine are as given in the table below :

		Machine				
		M_1	M_2	M_3	M_4	M_5
Job	J_1	65	40	90	80	90
	J_2	60	35	100	85	85
	J_3	60	38	105	90	95
	J_4	70	45	120	90	100
	J_5	65	40	105	87	90

Employ the Hungarian method of optimal assignment of jobs to the machines so as to minimize the total cost of all the jobs. Is the solution obtained unique ? If not, work out an alternative solution and calculate the total cost implied by it. *[Delhi M.Com. 2008]*

1116. Solve the following assignment problem using Hungarian method. The matrix entries are processing times in hours :

		Operator				
		O_1	O_2	O_3	O_4	O_5
Job	J_1	20	22	35	22	18
	J_2	4	26	24	24	7
	J_3	23	14	17	19	19
	J_4	17	15	16	18	15
	J_5	16	19	21	19	25

[Madras M.B.A. (April) 2006]

1117. A car hire company has one car at each of the five depots *a, b, c, d* and *e*. A customer in each of the five towns *A, B, C, D* and *E* requires a car. The distance (in miles) between the depots (origins) and the towns (destinations) where the customers are, is given in the following distance-matrix :

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How should the cars be assigned to the customers so as to minimize the distance travelled?

[Delhi M.B.A. (PT) 2008; IAS 1994]

1118. A national truck-rental service has a surplus of one truck in each of the cities *1, 2, 3, 4, 5* and *6*; and a deficit of one truck in each of the cities *7, 8, 9, 10, 11* and *12*. The distances (in kilometres) between the cities with a surplus and the cities with a deficit are displayed below :

		7	8	9	10	11	12
From	1	31	62	29	42	15	41
	2	12	19	39	55	71	40
	3	17	29	50	41	22	22
	4	35	40	38	42	27	33
	5	19	30	29	16	20	23
	6	72	30	30	50	41	20

How should the trucks be dispersed so as to minimize the total distance travelled?

[Osmania M.B.A. 2000; Delhi M.B.A. (PT) 2002; Meerut M.Sc. (Math.) 1995]

1119. A private firm employs typists on hourly page rate basis for their daily work. Five typists are working in that firm and their charges and speeds are different. On the basis of some earlier understanding, only one job is given to one typist and the typist is paid for full hours even when he or she works for a fraction of an hour. Find the least cost allocation for the following data :

Typist	Rate per hour (Rs.)	Number of pages typed per hr.
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of pages
P	199
Q	175
R	145
S	298
T	178

[Delhi M.Com. 2009]

[*Hint :* The cost table will be same as in sample problem *1103*.]

1120. A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table :

Job	Machine			
	A	B	C	D
1	18	24	28	32
2	8	13	17	19
3	10	15	19	22

Determine the optimum assignment. *[Gauhati M.C.A. 1992]*

1121. To stimulate interest and provide an atmosphere for intellectual discussion, a finance faculty in a management school decides to hold special seminars on four contemporary topics. Such seminars should be held once per week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows :

	Leasing	Portfolio management	Private mutual fund	Swaps & options
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule of the seminars. Also find out the total number of students who will be missing at least one seminar. *[C.A. Final (May) 1999]*

1122. A company has one surplus truck in each of the cities *A*, *B*, *C*, *D* and *E* and one deficit truck in each of the cities *1*, *2*, *3*, *4*, *5* and *6*.

	Truck →	1	2	3	4	5	6
	A	12	10	15	22	18	08
	B	10	18	25	15	16	12
City	C	11	10	03	08	05	09
	D	06	14	10	13	13	12
	E	08	12	11	07	13	10

The distance between the cities in kilometers is shown in the table. Find the assignment of trucks from cities in surplus to cities in deficit so that the total distance covered by the vehicles is minimum.

[Madras M.B.A. (Nov.) 2008]

1123. The marketing director of a multi-unit company is faced with a problem of assigning 5 senior managers to six zones. From the past experience he knows that the efficiency percentage judged by sales, operating cost etc. depends on manager-zone combination. The efficiency of different managers is given below :

Manager	Zone					
	I	II	III	IV	V	VI
A	73	91	87	82	78	80
B	81	85	69	76	74	85
C	75	72	83	84	78	91
D	93	96	86	91	83	82
E	90	91	79	89	69	76

Find out which zone will be managed by a junior manager due to non-availability of senior manager.

[IGNOU M.B.A. 2008]

11:4. SPECIAL CASES IN ASSIGNMENT PROBLEMS

Maximization Case in Assignment Problem. In some cases, the pay off elements of the assignment problem may represent revenues or profits instead of costs so that the objective will be to maximize the total revenue or profit. The Hungarian method explained earlier can also be used for maximization case. The problem of maximization can be converted into a minimization one by selecting the largest element of the profit matrix and then subtracting from it all the elements of the matrix. This gives us an *opportunity loss matrix*. We then proceed as usual and obtain the optimum assignment schedule. Maximum revenue/profit is then obtained by restoring the original values of those cells in which the assignments have been made.

Prohibited Assignments. Sometimes due to certain reasons, a particular resource (say a man or a machine) cannot be assigned to perform a particular activity (say a territory or a job). In such cases, the cost of performing that particular activity by a particular resource is considered to be very large (written as M or ∞) so as to prohibit the entry of this pair of resource-activity into the final solution.

SAMPLE PROBLEMS

1124. *A manufacturing company has four zones A, B, C, D and four sales engineers P, Q, R, S respectively for assignment. Since the zones are not equally rich in sales potential, it is estimated that a particular engineer operating in a particular zone will bring the following sales :*

Zone A : 4,20,000, Zone B : 3,36,000, Zone C : 2,94,000, Zone D : 4,62,000

The engineers are having different sales ability. Working under the same conditions their yearly sales are proportional to 14, 9, 11 and 8 respectively. The criteria of maximum expected total sales is to be met by assigning the best engineer to the richest zone, the next best to the second richest zone and so on.

Find the optimum assignment and the maximum sales. **[C.A. Final (May) 1998]**

Solution. *Step 1.* ***Construct the Effectiveness Matrix.*** To avoid the fractional values of annual sales of each sales engineer in each zone, for convenience consider their yearly sales as 42 (*i.e.*, the sum of sales proportions), taking Rs. 1000 as one unit. Now divide the individual sales in each zone by 42 to obtain the required annual sales by each sales engineer. For example, if sales engineer P is assigned to zone A, the annual sales is $14 \times 4,20,000/42$, *i.e.*, 1,40,000. Thus, the proportional sales in the four zones are given in the following effective table (matrix) :

		Zones				Sales proportion
		A	B	C	D	
Sales engineer	P	140	112	98	154	14
	Q	90	72	63	99	9
	R	110	88	77	121	11
	S	80	64	56	88	8

Step 2. ***Converting Maximization problem into Minimization problem :*** The given maximization assignment problem (table) can be converted into a minimization assignment problem by subtracting from the highest element (*i.e.*, 154), all the elements of the given table. The resulting opportunity loss matrix is :

	A	B	C	D
P	14	42	56	0
Q	64	82	91	55
R	44	66	77	33
S	74	90	98	66

Step 3. ***Optimum Assignment :*** Now, in the opportunity loss matrix subtract the minimum element of each row from all the elements of that row. Then subtract the minimum element of each

column from all the elements of that column. In the reduced matrix make assignments in rows and columns that have single zeros as usual. Thus, we have :

Initial Iteration. Draw the minimum number of lines to cover all the zeros of the reduced matrix. See table *11.9*.

First Iteration. Modify the reduced opportunity loss matrix by subtracting element '1' from all the elements not covered by the lines and adding the same at the intersection of two lines. Thus, we get table *11.10*.

	A	B	C	D	
P	6	18	24	$\boxed{0}$	√
Q	1	3	4	~~0~~	√
R	3	9	12	~~0~~	√
S	$\boxed{0}$	~~0~~	~~0~~	~~0~~	
				√	

Table 11.9

	A	B	C	D	
P	5	17	23	$\boxed{0}$	√
Q	$\boxed{0}$	2	3	~~0~~	
R	2	8	11	~~0~~	√
S	~~0~~	$\boxed{0}$	~~0~~	1	
				√	

Table 11.10

Second Iteration. Modify further the reduced table *11.10* by subtracting element '2' from all the elements not covered by the lines and adding the same at the intersection of two lines. Thus, we get table *11.11*.

Final Iteration. Modify further the reduced table *11.11* by subtracting element '2' from all the elements not covered by the lines and adding the same at the intersection of two lines. Thus, we get table *11.12*.

	A	B	C	D	
P	3	15	21	$\boxed{0}$	√
Q	$\boxed{0}$	2	3	2	√
R	~~0~~	6	9	~~0~~	√
S	~~0~~	$\boxed{0}$	~~0~~	3	
	√			√	

Table 11.11

	A	B	C	D
P	3	13	19	$\boxed{0}$
Q	~~0~~	$\boxed{0}$	1	2
R	$\boxed{0}$	4	7	~~0~~
S	2	~~0~~	$\boxed{0}$	5

Table 11.12

The optimum solution is obtained and the assignment is :

$$P \to D,\ Q \to B,\ R \to A \text{ and } S \to C$$

The solution shows that the best salesman P is assigned to the richest zone D and the worse salesman S to the poorest zone C. The second best salesman to the next richest zone A, and so on.

Maximum sales = Rs. 154 + 72 + 110 + 56 = Rs. 392 thousands.

Problem 1125. *A student has to select one and only one elective in each semester and the same elective should not be selected in different semesters. Due to various reasons, the expected grades in each subject, if selected in different semesters, vary and they are given below:*

Semester	*Analysis*	*Statistics*	*Graph Theory*	*Algebra*
I	*F*	*E*	*D*	*C*
II	*E*	*E*	*C*	*C*
III	*C*	*D*	*C*	*A*
IV	*B*	*A*	*H*	*H*

The grade points are: $H = 10$, $A = 9$, $B = 8$, $C = 7$, $D = 6$, $E = 5$ and $F = 4$. How will the student select the electives in order to maximize the total expected points and what will be his maximum expected total points? ***[Delhi B.Sc. (Stat.) 2006]***

Solution. Since the given problem is to maximize the total expected points, we convert the problem into that of minimization by subtracting all the elements of the grade points matrix from the highest grade point, *viz.*, $H (= 10)$. The reduced matrix is as shown in table *11.13*.

Now subtract the minimum element of each row (column) from all the elements of the respective row (column) in table *11.13* and then make assignments in the rows and columns that have single zeros. Thus, we have optimum table *11.14* :

6	5	4	3
5	5	3	3
3	4	3	1
2	1	0	0

Table 11.13

1	1	1	$\boxed{0}$
~~0~~	1	$\boxed{0}$	~~0~~
$\boxed{0}$	2	2	~~0~~
~~0~~	$\boxed{0}$	~~0~~	~~0~~

Table 11.14

From table *11.14*, we observe that the optimum assignment is to select in

$I \rightarrow$ Algebra, $II \rightarrow$ Graph Theory, $III \rightarrow$ Analysis and $IV \rightarrow$ Statistics.

Maximum expected total points will be $C + C + C + A$, *i.e.*, $3 \times 7 + 9$ or 30.

1126. *The following is the cost matrix of assigning 4 clerks to 4 key punching jobs. Find the optimal assignment if clerk 1 cannot be assigned to job 1 :*

Clerk	*Job*			
	1	*2*	*3*	*4*
1	—	5	2	0
2	4	7	5	6
3	5	8	4	3
4	3	6	6	2

What is the minimum total cost? ***[Dibrugarh M.Sc. (Stat.) 1994; C.A. Final (Nov.) 2002]***

Solution. Reduce the cost matrix by subtracting smallest element of each row (column) from the corresponding row (column) elements. In the reduced matrix make assignments in rows and columns that have single zeros. Thus, we have :

Initial Iteration. Draw the minimum number of lines to cover all the zeros of the reduced matrix. See table *11.15*.

Final Iteration. Modify the reduced cost matrix by subtracting element '1' from all the elements not covered by the lines and adding the same at the intersection of two lines. See table *11.16*.

M	2	1	$\boxed{0}$	√
~~0~~	$\boxed{0}$	~~0~~	2	
2	2	$\boxed{0}$	~~0~~	
1	1	3	~~0~~	√
			√	

Table 11.15

M	1	~~0~~	$\boxed{0}$
$\boxed{0}$	~~0~~	~~0~~	3
2	2	$\boxed{0}$	1
~~0~~	$\boxed{0}$	2	~~0~~

Table 11.16

Since the number of assignments is equal to the order of the matrix, an optimum solution is reached. Also, since there are at least two zeros for assignment in row 2 and row 4 as well as in column 1 and column 2, there exists an alternative assignment schedule. The optimum solution is :

Assign Clerk *1* to Job *4*, Clerk *2* to Job *1*, Clerk *3* to job *3* and Clerk *4* to Job *2*;

or Assign Clerk *1* to Job *4*, Clerk *2* to Job *2*, Clerk *3* to Job *3* and Clerk *4* to Job *1*.

Total minimum cost will be 14.

1127. *In the modification of a plant layout of a factory, four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost of placing of machine i at place j (in hundred rupees) is shown below :*

		Location				
		A	B	C	D	E
Machine	M_1	9	11	15	10	11
	M_2	12	9	—	10	9
	M_3	—	11	14	11	7
	M_4	14	8	12	7	8

Find the optimal assignment schedule. **[Delhi M.B.A. (Nov.) 2009, MIB 2011]**

Solution. Since, the number of machines ≠ number of vacant places, the cost matrix is unbalanced. So, we add a dummy machine (row) with all its elements as zero. Also, assign a very high cost M, to places (3, 1) and (2, 3). The cost matrix so obtained is given in table *11.17*. Reduce this cost matrix by subtracting the smallest element of each row (column) from all the elements of that row (column). In the reduced matrix make assignments in rows and columns that have single zeros and cross off all other zeros of the row/column in which an assignment is made. Thus, we get table *11.18* :

	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	M	10	9
M_3	M	11	14	11	7
M_4	14	8	12	7	8
Dummy	0	0	0	0	0

Table 11.17

	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	M	1	~~0~~
M_3	M	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
Dummy	~~0~~	~~0~~	$\boxed{0}$	~~0~~	~~0~~

Table 11.18

Since the number of assignments in table *11.18* is equal to the order of the matrix, an optimum solution is reached. The optimum solution is :

Assign M_1 at A, M_2 at B, M_3 at E and M_4 at D ; at the minimum total cost of Rs. 3,200.

PROBLEMS

1128. *MCS* Inc is a software company that has three projects of Y2K with the departments of health, education, and housing of Maharashtra Government. Based on the background and experiences of the project leaders, they differ in terms of their performance at various projects. The performance score matrix is given below :

Project leaders	Projects		
	Health	Education	Housing
P_1	20	26	42
P_2	24	32	50
P_3	32	34	44

Help the management by determining the optimal assignment that maximises the total performance score. **[Mumbai D.M.S. 1998]**

1129. Suggest the optimal assignment schedule for the following assignment problem :

Sales (Rs. in lakh)

Salesmen	Markets			
	I	II	III	IV
A	80	70	75	72
B	75	75	80	85
C	78	78	82	78

What will be the total maximum sale? **[Delhi M.Com. 2001]**

1130. Solve the following assignment problem to find the maximum total expected sale

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Area} & I & II & III & IV \end{array} \\ \textit{Salesman} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 42 & 35 & 28 & 21 \\ 30 & 25 & 20 & 15 \\ 30 & 25 & 20 & 15 \\ 24 & 20 & 16 & 12 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2002]

1131. *XYZ* Company has 4 sales managers to be assigned to 5 different sales territories. The weekly wages estimated for each sales manager in different territories in thousands of rupees are shown in the following table :

Sales managers	*Sales territories*				
	I	*II*	*III*	*IV*	*V*
A	75	80	85	70	90
B	91	71	82	75	85
C	78	90	85	80	80
D	65	75	88	85	90

Suggest optimal assignment for the sales manager. Which sales territory will remain unassigned.

[Delhi PG Dip. in Glob. Bus. Oper. 2011]

1132. Suggest optimal assignment for four salesmen to sales territories where the estimated sales (in lakh rupees) to be made by each of the salesmen in different sales territories are given below :

Salesmen	*Sales territories*				
	I	*II*	*III*	*IV*	*V*
A	16	15	17	10	8
B	16	16	20	15	12
C	12	8	10	13	15
D	18	16	17	12	10

What will be the total maximum sale.

[Delhi M.B.A. (Nov.) 2003]

1133. A firm produces four products. There are tour operators who are capable of producing any of these four products. The processing time varies from operator to operator. The firm records 8 hours a day and allows 30 minutes for lunch. The processing time in minutes and the profit for each of the product are given below :

Operator	*Product*			
	A	*B*	*C*	*D*
1	15	9	10	6
2	10	6	9	6
3	25	15	15	9
4	15	9	10	10
Profit (Rs. per unit)	8	6	5	4

Find the optimum assignment of product to operators.

[C.A. (Nov.) 1997]

[***Hint :*** Obtain the number of items that can be produced by dividing 450 with the processing times. Then multiply these with the corresponding profit.]

1134. Parul Corporation has four plants each of which can manufacture any one of four products. Production costs differ from one plant to another as do sales revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximise profit :

Plant	*Sales revenue (Rs. '000s)* *Product*				*Production cost (Rs. '000s)* *Product*			
	1	*2*	*3*	*4*	*1*	*2*	*3*	*4*
A	50	68	49	62	49	60	45	61
B	60	70	51	74	55	63	45	69
C	55	67	53	60	52	62	49	58
D	58	65	54	69	55	64	48	66

[***Hint :*** Profit = Sales revenue – Production cost]

1135. A marketing manager has 5 salesmen and 5 sales districts. Considering the capabilities of the salesman and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :

Salesman	*Sales district*				
	A	*B*	*C*	*D*	*E*
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

What is the maximum sales that may be expected if an optimum assignment is made?

[UP Tech. Univ. M.B.A. 2006; Delhi M.Com. 2005]

1136. Five lathes are to be allotted to five operators (one for each). The following table gives weekly output figures (in pieces) :

Operator	*Weekly output in Lathe*				
	L_1	L_2	L_3	L_4	L_5
P	20	22	27	32	36
Q	19	23	29	34	40
R	23	28	35	39	34
S	21	24	31	37	42
T	24	28	31	36	41

Profit per piece is Rs. 25. Find the maximum profit per week.

1137. A company marketing consumer items has divided its area of operation into four zones (*A*, *B*, *C* and *D*). There are four sales teams available. Based on previous experience the company has estimated the annual sales (in Rs. crores) if different teams are assigned to different zones as given in the table below. Find the optimal allocation of the sales teams to different regions so that the total annual sales is maximised.

Sales teams	*Zones*			
	A	*B*	*C*	*D*
1	5	11	8	9
2	5	7	9	7
3	7	8	9	9
4	6	8	11	12

[Annamalai M.B.A. 2002]

1138. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows :

Job	*Machine*			
	A	*B*	*C*	*D*
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solve the problem to maximize the total profits. ***[Annamalai M.B.A. 2010; Madurai M.Com. 2003]***

1139. The Captain of a cricket team has to allot five middle batting positions to five batsmen. The average runs scored by each batsman at these positions are as follows :

Batsman	*Batting positions*				
	I	*II*	*III*	*IV*	*V*
P	40	40	35	25	50
Q	42	30	16	25	27
R	50	48	40	60	50
S	20	19	20	18	25
T	58	60	59	55	53

(*i*) Find the assignment of batsmen to positions, which would give the maximum number of runs.

[Delhi M.B.A. (PT.) 2015]

(*ii*) If another batsman 'U' with the following average runs in batting positions :

Batting position :	*I*	*II*	*III*	*IV*	*V*
Average runs :	45	52	38	50	49

is added to the team, should he be included to play in the team? If so, who will be replaced by him?

[Delhi M.B.A. 2011, PG Dip. in Glob. Bus. Oper. 2011]

1140. Sales persons *A, B, C, D, E* and *F* are to be assigned to areas *U, V, W, X, Y* and *Z*. The sales figures for different possible assignments are tabulated. Find the optimal assignment of the sales persons to different areas :

	U	*V*	*W*	*X*	*Y*	*Z*
A	5	3	2	7	8	6
B	8	6	1	2	4	5
C	7	2	3	5	6	2
D	3	7	2	8	10	1
E	10	6	3	2	4	5
F	12	8	11	6	3	2

[Madras M.B.A. 2000]

1141. Determine an optimum assignment schedule for the following assignment problem. The cost matrix is given below :

Job	*Machine*					
	1	*2*	*3*	*4*	*5*	*6*
A	11	17	8	16	20	15
B	9	7	12	6	15	13
C	13	16	15	12	16	8
D	21	24	17	28	26	15
E	14	10	12	11	15	6

If job *C* cannot be assigned to machine 6, will the optimum solution change? *[Madras B.Sc. (Math.) 1995]*

1142. The secretary of a school is taking bids on the City's four school bus routes. Four companies have made the bids as detailed in the following table :

Bids

	Route 1	*Route 2*	*Route 3*	*Route 4*
Company 1	Rs. 4,000	Rs. 5,000	—	—
Company 2	—	Rs. 4,000	—	Rs. 4,000
Company 3	Rs. 3,000	—	Rs. 2000	—
Company 4	—	—	Rs. 4,000	Rs. 5,000

Suppose each bidder can be assigned only one route. Use the assignment model to minimize the school's cost of running the four bus routes.

[Delhi PG Dip. in Glob. Bus. Oper. 2010]

1143. A company has four zones open and four marketing managers available for assignment. The zones are no equal in sales potentials. It is estimated that a typical marketing manager operating in each zone would bring in the following annual sales :

Zones :	*East*	*West*	*North*	*South*
Annual Sales (in Rs.) :	2,40,000	1,92,000	1,44,000	1,20,000

The four marketing managers are also different in ability. It is estimated that working under the same conditions, their yearly sales would be proportional as under :

Manager :	*M*	*N*	*O*	*P*
Proportion :	8	7	5	4

Required: If the criterion is maximum expected total sales, find the optimum assignment and the maximum sales.

[C.A. Final (Nov.) 2007]

1144. Imagine yourself to be the Executive Director of a 5-Star Hotel which has four banquet halls that can be used for all functions including weddings. The halls are all about the same size and the facilities in each hall differed. During a heavy marriage season, 4 parties approached you to reserve a hall for the marriage to be celebrated on the same day. These marriage parties were told that the first choice among these 4 halls would cost Rs. 10,000 for the day. They were also required to indicate the second, third and fourth preferences and the price that they would be willing to pay. Marriage parties *A* and *B* indicated that they won't be interested in Halls 3 and 4. Other particulars are given in the following table :

REVENUE/HALL

Marriage Party	*Hall*			
	1	*2*	*3*	*4*
A	10,000	9,000	×	×
B	8,000	10,000	8,000	5,000
C	7,000	10,000	6,000	8,000
D	10,000	8,000	×	×

where × indicates that the party does not want that hall. Decide on an allocation that will maximise the revenue to your hotel. ***[Delhi PG Dip. in Glob. Bus. Oper. 2011]***

1145. An airline company has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of 10. The higher the number, the greater is the preference. Some of these flights are not suitable to some pilots due to domestic reasons. These have been marked with *X*.

		Flights				
		1	2	*3*	*4*	*5*
	A	8	2	×	5	4
	B	10	9	2	8	4
Pilots	*C*	5	4	9	6	×
	D	3	6	2	8	7
	E	5	6	10	4	3

What should be the allocation of the pilots to the flights in order to meet as many preferences as possible? ***[Delhi MIB 2011]***

1146. A company has four sales representatives who are to be assigned to four different sales territories. The monthly sales increase estimated for each sales representative for different sales territories (in lakh rupees), are shown in the following table :

		Sales territories			
		I	*II*	*III*	*IV*
	A	200	150	170	220
	B	160	120	150	140
Sales representative	*C*	190	195	190	200
	D	180	175	160	190

(*i*) Suggest optimum assignment and the total maximum sales increase per month.

(*ii*) If for certain reasons, sales representative *B* cannot be assigned to sales territory *III*, will the optimum assignment schedule be different? If so, find that schedule and the effect on total sales. ***[Delhi M.Com. 1997]***

11.5. A TYPICAL ASSIGNMENT PROBLEM

SAMPLE PROBLEM

1147. *Kapil Airlines that operates seven days a week has a time-table as shown below. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing, the crew will be based at the city that results in the smaller layover :*

Delhi–Jaipur			Jaipur–Delhi		
Flight No.	*Depart*	*Arrive*	*Flight No.*	*Depart*	*Arrive*
1	7.00 A.M.	8.00 A.M.	*101*	8.00 A.M.	9.15 A.M.
2	8.00 A.M.	9.00 A.M.	*102*	8.30 A.M.	9.45 A.M.
3	1.30 P.M.	2.30 P.M.	*103*	12.00 Noon	1.15 P.M.
4	6.30 P.M.	7.30 P.M.	*104*	5.30 P.M.	6.45 P.M.

For each pair also mention the town where the crew should be based. ***[Meerut M.Sc. (Math.) 2009]***

Solution. It is assumed that a plane flying from Delhi for Jaipur must come back to Delhi at the immediate next opportunity. It is further assumed that each plane will make only one forward and one return trip and thus there must be four planes for four forward and return flights. The problem is to determine the optimum pairing.

As the objective is to minimize the total layover time, let us at first instance determine the layover time both at Delhi and at Jaipur for all possible pairings. For instance, for flight 1 and 101 pair, the plane leaves Delhi at 7 a.m. reaching Jaipur at 8 a.m. If it has to avail of 101 return flight, it can start at 8 a.m. next day, thereby making the layover of 24 hours at Jaipur. Similarly, for the same pairing, the plane reaches Delhi at 9.15 a.m. and leaves for Jaipur at 7 a.m. next morning making the layover of 21.75 hours at Delhi. Likewise, we determine layover time at Jaipur and Delhi for all possible pairings. These are shown in the following tables :

Layover Time in Hours

Crew based in Delhi

	101	*102*	*103*	*104*
1	24	24.5	28	9.5
2	23	23.5	27	8.5
3	17.5	18	21.5	27
4	12.5	13	16.5	22

Table 11.19

Crew based in Jaipur

	101	*102*	*103*	*104*
1	21.75	21.25	17.75	12.25
2	22.75	22.25	18.75	13.25
3	28.25	27.25	24.25	18.75
4	9.25	8.75	5.25	23.75

Table 11.20

Next construct the table for minimum layover time between flights with the help of above tables. The layover time marked '*' denotes that the crew is based in Jaipur. Thus, we get the following table :

Minimum layover time

	101	*102*	*103*	*104*
1	21.75*	21.25*	17.75*	9.5
2	22.75*	22.25*	18.75*	8.5
3	17.5	18	21.5	18.75*
4	9.25*	8.75*	5.25*	22

Table 11.21

Now, subtract the minimum element of each row from all the elements of that row. Then subtract the minimum element of each column from all the elements of that column. In the reduced matrix make assignments in rows and columns that have single zeros. Thus, we have :

Initial Iteration. Draw the minimum number of lines to cover all the zeros of the reduced matrix. See table *11.22.*

First Iteration. Modify the reduced matrix by subtracting the element '8.25' from all the elements not covered by the lines and adding the same at the intersection of two lines. See table *11.23 :*

12.25	11.25	8.25	$\boxed{0}$	√
14.25	13.25	10.25	~~0~~	√
~~0~~	$\boxed{0}$	3.50	0.75	
4.0	3.0	$\boxed{0}$	6.75	
				√

Table 11.22

4	3	$\boxed{0}$	~~0~~	√
6	5	2	$\boxed{0}$	√
$\boxed{0}$	~~0~~	3.5	9	
4	3	~~0~~	25	√
		√	√	

Table 11.23

Final Iteration. Modify further the reduced table *11.23* by subtracting element '3' from all the elements not covered by the lines and adding the same at the intersection of two lines. Thus, we get table *11.24* and table *11.25* :

	101	*102*	*103*	*104*
1	1	$\boxed{0}$	~~0~~	~~0~~
2	3	2	2	$\boxed{0}$
3	$\boxed{0}$	~~0~~	6.5	12
4	1	~~0~~	$\boxed{0}$	25

Table 11.24

or

	101	*102*	*103*	*104*
1	1	~~0~~	$\boxed{0}$	~~0~~
2	3	2	2	$\boxed{0}$
3	$\boxed{0}$	~~0~~	6.5	12
4	1	$\boxed{0}$	~~0~~	25

Table 11.25

Thus the optimum solution is obtained with the following assignment schedule :

Flights	:	$1 \to 102$ or $1 \to 103$	$2 \to 104$	$3 \to 101$	$4 \to 103$ or $4 \to 102$
Base Crew	:	Jaipur	Delhi	Delhi	Jaipur

Total layover time = 52.5 hours.

PROBLEMS

1148. *XYZ* Airline operating 7 days a week has given the following time-table. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimizes layover time away from home. For any given pairing the crew will be based at the city results in the smaller layover :

Chennai—Mumbai			*Mumbai—Chennai*		
Flight Number	*Depart*	*Arrive*	*Flight Number*	*Depart*	*Arrive*
A_1	6 AM	8 AM	B_1	8 AM	10 AM
A_2	8 AM	10 AM	B_2	9 AM	11 AM
A_3	2 PM	4 PM	B_3	2 PM	4 PM
A_4	8 PM	10 PM	B_4	7 PM	9 PM

[Delhi M.Com. 2010]

1149. A trip from Chennai to Bangalore takes six hours by bus. A typical time table of the bus service in both directions is given below :

Departure from Chennai	*Route number*	*Arrival at Bangalore*	*Arrival at Chennai*	*Route number*	*Departure from Bangalore*
06.00	$\xrightarrow{a}$	12.00	11.30	$\xleftarrow{1}$	05.30
07.30	$\xrightarrow{b}$	13.30	15.00	$\xleftarrow{2}$	09.00
11.30	$\xrightarrow{c}$	17.30	21.00	$\xleftarrow{3}$	15.00
19.00	$\xrightarrow{d}$	01.00	00.30	$\xleftarrow{4}$	18.30
00.30	$\xrightarrow{e}$	06.30	06.00	$\xleftarrow{5}$	00.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service time. There are five crews. There is a

constraint that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip. The company has residential facilities for the crew at Chennai as well as at Bangalore. Find which crew be assigned which line of service or which service line be connected with which other line, so as to reduce the waiting time to be the minimum.

[***Hint :*** When crew is based at Chennai (*i.e.*, they start from and come back to Chennai with minimum halt at Bangalore), waiting time for different services is obtained in table *11.26*. Similarly, when crew is based at Bangalore, waiting time for different services is obtained in table *11.27*.

17.5	21	3	6.5	12
16	19.5	1.5	5	10.5
12	15.5	21.5	1	6.5
4.5	8	14	17.5	23
23	2.5	8.5	12	17.5

Table 11.26

18.5	15	9	5.5	0
20	16.5	10.5	7	1.5
0	20.5	14.5	11	5.5
7.5	4	22	18.5	13
13	9.5	3.5	0	18.5

Table 11.27

The minimum waiting time for different route connections by choosing minimum value out of the two waiting times is given in table *11.28*, where the value of waiting time is more than 4 hours (minimum desired rest).

17.5	15	9	5.5	12
16	16.5	10.5	5	10.5
12	15.5	14.5	11	5.5
4.5	8	14	17.5	13
13	9.5	8.5	12	17.5

Table 11.28

Use now assignment method for optimum solution.]

11:6. DUAL OF THE ASSIGNMENT PROBLEM

While solving an assignment problem, we determine the assignment matrix $\mathbf{X} = (x_{ij})$, such that

$$\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1 \quad (\text{for } i, j = 1, 2, \ldots, n)$$

with $\quad$ minimum $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$, where $c_{ij} \geq 0$ for i and $j = 1, 2, \ldots, n$.

The two constraint equations are the conditions that jointly specify :

$$x_{ij} = \begin{cases} 1 & \text{if resource } i \text{ is assigned to activity } j. \\ 0 & \text{if resource } i \text{ is not assigned to activity } j. \end{cases}$$

Also, each row and each column of the matrix $\mathbf{X}$ has a single unity element and all other elements are equal to zero.

For the dual of the given assignment problem, we proceed as in the case of a transportation problem and determine a set of u_i's and v_j's such that the following conditions are satisfied :

$$c_{ij} = u_i + v_j \text{ for } x_{ij} > 0 \text{ and } c_{ij} \geq u_i + v_j \text{ for } x_{ij} = 0$$

or

$$\sum_i \sum_j u_i x_{ij} + \sum_i \sum_j v_j x_{ij} \leq \sum_i \sum_j c_{ij} x_{ij}.$$

or

$$\sum_i u_i (\sum_j x_{ij}) + \sum_j v_j (\sum_i x_{ij}) \leq z$$

or

$$\sum_i u_i + \sum_j v_j \leq z, \quad \text{since } \sum_i x_{ij} = \sum_j x_{ij} = 1.$$

If we denote the L.H.S. of this expression by z^*, then the problem associated with minimizing $\sum_i \sum_j c_{ij} x_{ij}$ is identical to the following dual problem :

Maximize $z^* = \sum_i u_i + \sum_j v_j$ subject to the constraints :

$$u_i + v_j \le c_{ij} \qquad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$$

u_i and v_j unrestricted in sign.

Illustration. *Write the dual of the following assignment problem and obtain the optimal values of the dual variables :*

$$\begin{array}{cc} & \textit{Programmers} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \textit{Programmes} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 120 & 100 & 80 \\ 80 & 90 & 110 \\ 110 & 140 & 120 \end{bmatrix} \end{array}$$

Answer. The given problem can be written as a transportation problem :

Programmes	*Programmers*			*Supply*
	A	*B*	*C*	
1	120	100	80	1
2	80	90	110	1
3	110	140	120	1
Demand	1	1	1	3

The mathematical formulation of this T.P. is as follows :

Minimize $z = (120x_{11} + 100x_{12} + 80x_{13}) + (80x_{21} + 90x_{22} + 110x_{23}) + (110x_{31} + 140x_{32} + 120x_{33})$

subject to the constraints :

$$120x_{11} + 100x_{12} + 80x_{13} = 1,\ 80x_{21} + 90x_{22} + 110x_{23} = 1,\ 110x_{31} + 140x_{32} + 120x_{33} = 1$$

$$120x_{11} + 80x_{21} + 110x_{31} = 1,\ 100x_{12} + 90x_{22} + 140x_{32} = 1,\ 80x_{13} + 110x_{23} + 120x_{33} = 1$$

$$x_{ij} = 1 \text{ or } 0, \text{ for all } i = 1, 2, 3 \text{ and } j = 1, 2, 3.$$

Let $u_i\ (i = 1, 2, 3)$ and $v_j\ (j = 1, 2, 3)$ are the dual variables corresponding to the constraints of the above T.P. Then the dual problem is :

Maximize $z^* = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3)$ subject to the constraints :

$$u_1 + v_1 \le 120,\ u_1 + v_2 \le 100,\ u_1 + v_3 \le 80$$

$$u_2 + v_1 \le 80,\ u_2 + v_2 \le 90,\ u_2 + v_3 \le 110$$

$$u_3 + v_1 \le 110,\ u_3 + v_2 \le 140,\ u_3 + v_3 \le 120.$$

$$u_i \ge 0\ (i = 1, 2, 3) \text{ and } v_j \ge 0\ (j = 1, 2, 3)$$

For the optimum values of the dual variables, an optimum solution to the given problem is to be obtained by transportation method.

Therefore, using VAM for initial basic feasible solution and MODI method for optimality, the optimum solution to the given T.P. is obtained, as displayed in the following transportation table :

				u_i
	120	100	1 / 80	0
∈ / 80	1 / 90	110		10
1 / 110	140	∈ / 120		40
v_j 70	80	80		

Table 11.29

We observe that the optimum values of the dual variables are : $u_1 = 0$, $u_2 = 10$, $u_3 = 40$, $v_1 = 70$, $v_2 = 80$ and $v_3 = 80$. Sum of these values is 280.

Also, $$\text{Minimum } z = 1 \times 80 + \varepsilon \times 80 + 1 \times 90 + 1 \times 110 + \varepsilon \times 120$$
$$= 280 + 200\,\varepsilon = 280 \text{ as } \varepsilon \to 0.$$

This shows that minimum z = maximum $z^* = 280$.

Further, it may be observed that optimum values of dual variables satisfy all the constraints and are not restricted in sign.

11:7. THE TRAVELLING SALESMAN PROBLEM

Suppose a salesman has to visit n cities. He wishes to start from a particular city, visit each city once, and then return to his starting point. The objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized. Clearly, starting from a given city, the salesman will have a total of $(n - 1)!$ different sequences (possible round trips). Further, since the salesman has to visit all the n cities, the optimal solution remains independent of selection of the starting point.

The problem can be represented as a network where the nodes and arcs represent the cities and the distances between them, respectively. Let in a five-city problem, a round trip of the salesman be given by the following arcs :

$$(3, 1), (1, 2), (2, 4), (4, 5), (5, 3)$$

These arcs, taken in order, are called the first, second, third, fourth and fifth *directed arcs* for the trip. In general the kth directed arc represents the kth leg of the trip, *i.e.*, on leg k the salesman travels from city i to city j $(i, j = 1, 2, \ldots, n : i \neq j)$.

To formulate the problem whose solution will yield the minimum travelling time, let the variables x_{ijk} be defined as

$$x_{ijk} = \begin{cases} 1, & \text{if } k\text{th directed arc is from city } i \text{ to city } j, \\ 0, & \text{otherwise,} \end{cases}$$

where i, j and k are integers that vary between 1 and n.

Following are the constraints of the problem :

(*a*) Only one directed arc may be assigned to a specific k, thus,

$$\sum_{j} \sum_{k} x_{ijk} = 1, \qquad k = 1, 2, \ldots, n.$$
$$i \neq j$$

(*b*) Only one other city may be reached from a specific city i, thus

$$\sum_{j} \sum_{k} x_{ijk} = 1, \qquad i = 1, 2, \ldots, n.$$

(*c*) Only one other city can initiate a direct arc to a specified city j, thus

$$\sum_{i} \sum_{k} x_{ijk} = 1, \qquad j = 1, 2, \ldots, n.$$

(*d*) Given the kth directed arc ends at some specific city j, the $(k + 1)$th directed arc must start at the same city j; thus

$$\sum_{i \neq j} x_{ijk} = \sum_{r \neq j} x_{jr(k+1)}, \qquad \text{for all } j \text{ and } k.$$

These constraints ensure that the round trip will consist of connected directed arcs. The objective function is to minimize

$$z = \sum_{i} \sum_{j} \sum_{k} d_{ij} x_{ijk}, \qquad i \neq j$$

where d_{ij} is the distance from city i to city j.

Formulation of Travelling Salesman Problem as an A.P.

The travelling salesman problem is very similar to the assignment problem except that in the former, there is an additional restriction. A similar problem arises when n items say A_i, $i = 1, 2, \ldots, n$ are to be processed on a machine in a scheduled time. The problem then becomes of choosing the sequence of these items. Let the setup cost of the machine when item A_i is followed by A_j be c_{ij}. Also let $x_{ij} = 1$ if item A_i is followed by A_j directly and 0 otherwise. It is to be noticed that $c_{ij} = \infty$ when $i = j$, *i.e.*, the item A_i is not processed again after A_i. It is important to note that only one $x_{ij} = 1$ for each value of i and for each value of j.

In view of the above, the assignment problem can be solved and one may hope that the solution satisfies the additional restriction also.

If the solution to the assignment problem does not satisfy the additional restriction, then after solving the problem by assignment technique, we use the method of enumeration.

The procedure is best illustrated with the help of the following sample problems.

SAMPLE PROBLEMS

1150. *A machine operator processes five types of items on his machine each week, and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table :*

From item	*To item*				
	A	*B*	*C*	*D*	*E*
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

[Madras B.E. 1999; Meerut M.Sc. (Math.) 2000; M.S. Tirunvelli B.E. (Electronics) 1996]

Solution. Reduce the cost matrix and make assignments in rows and columns having single zeros.

Initial Iteration. Draw the minimum number of lines to cover all the zeros. See table *11.30*.

First Iteration. Modify table *11.30* by subtracting the lowest element from all the elements not covered by lines and adding the same at the intersection of two lines. See table *11.31*. We observe that this table gives us an optimum assignment schedule.

∞	1	3	$\boxed{0}$	1	√
1	∞	2	~~0~~	1	√
2	1	∞	2	$\boxed{0}$	
~~0~~	$\boxed{0}$	3	∞	4	
$\boxed{0}$	~~0~~	0	3	∞	
			√		

Table 11.30

∞	~~0~~	2	$\boxed{0}$	~~0~~
$\boxed{0}$	∞	1	~~0~~	~~0~~
2	1	∞	3	$\boxed{0}$
~~0~~	$\boxed{0}$	3	∞	4
~~0~~	~~0~~	$\boxed{0}$	4	∞

Table 11.31

The optimum assignment is $A \to D$, $B \to A$, $C \to E$, $D \to B$ and $E \to C$ with a minimum cost of 20.

This assignment schedule does not provide us the solution of travelling salesman problem as it gives $A \to D$, $D \to B$, $B \to A$, while B is not allowed to follow A unless C and E are processed.

Second Iteration. We try to find the next best solution which satisfies this extra restriction. The next minimum (non-zero) element in the matrix is 1. So we try to bring 1 into the solution. But the

element '1' occurs at two places. We shall consider all the cases separately until the acceptable solution is reached.

We start with making an assignment at (2, 3) instead of zero assignment at (2, 1). The resulting feasible solution, then will be

$$A \to D,\ D \to B,\ B \to C,\ C \to E,\ E \to A.$$

When an assignment is made at (3, 2) instead of zero assignment at (3, 5), the resulting feasible solution will be

$$A \to E,\ E \to C,\ C \to B,\ B \to D,\ D \to A$$

The total set-up cost in both the programmes comes out to be 21.

1151. *A salesman is planning to tour cities B, C, D and E starting from his home city A. The inter-city distances are shown in the following table :*

City :	A	B	C	D	E
A	∞	103	188	136	38
B	103	∞	262	176	52
C	188	262	∞	85	275
D	136	176	85	∞	162
E	38	52	275	162	∞

How should he plan his tour so that (i) he visits each of the cities only once, and (ii) travels the minimum distance. **[Delhi M.B.A. (Nov.) 2003]**

Solution. Since the salesman has to visit all the cities before returning back to his home city, the assignments from the city to itself is prohibited, *i.e.*, we consider the huge penalty of assigning A to A, B to B, C to C, D to D and E to E.

Now, reduce the cost (distance) matrix and make assignments in rows and columns having single zeros.

Initial Iteration. Draw the minimum number of lines to cover all the zeros of the reduced matrix. See table *11.32*.

First Iteration. Modify table *11.32* by subtracting the lowest element from all the elements not covered by lines and adding the same at the intersection of two lines. See table *11.33* :

	A	B	C	D	E	
A	—	51	150	98	$\boxed{0}$	√
B	51	—	210	124	~~0~~	√
C	103	163	—	$\boxed{0}$	190	
D	51	77	$\boxed{0}$	—	77	
E	~~0~~	$\boxed{0}$	237	124	—	
					√	

Table 11.32

	A	B	C	D	E
A	—	~~0~~	99	47	$\boxed{0}$
B	$\boxed{0}$	—	159	73	~~0~~
C	103	163	—	$\boxed{0}$	241
D	51	77	$\boxed{0}$	—	128
E	~~0~~	$\boxed{0}$	237	124	—

Table 11.33

The optimum assignment is $A \to E \to B \to A$ and $C \to D \to C$ or $A \to B \to E \to A$ and $C \to D \to C$ with a minimum distance of 363 in both the cases.

This assignment schedule does not provide us the solution of travelling salesman problem as in both the cases the salesman returns to his starting city A without visiting cities C and D.

Second Iteration. Now, we try to find the next best solution that satisfies the extra restriction. For this, we choose the next lowest element (in place of zero) in the second row, *i.e.*, for city B. Thus, we make the assignment at (2, 4) being the lowest element after 0 (not covered by lines). The resulting feasible solution, then will be : $A \to E$, $E \to B$, $B \to D$, $D \to C$, $C \to A$.

The total distance in this case comes out to be 539, *i.e.*, an increase of 176 over the optimum assignment.

PROBLEMS

1152. Given the following matrix of set-up costs, show how to sequence production so as to minimize set-up cost per cycle :

From	To				
	A	B	C	D	E
A	∞	2	5	7	1
B	6	∞	3	8	2
C	8	7	∞	4	7
D	12	4	6	∞	5
E	1	3	2	8	∞

[Karnataka B.E. (Prod.) 1994; Delhi M.B.A. (Nov.) 2004, M.B.A. (PT.) 2008]

1153. Solve the following travelling salesman problem so as to minimize the cost per cycle.

From	To				
	A	B	C	D	E
A	—	3	6	2	3
B	3	—	5	2	3
C	6	5	—	6	4
D	2	2	6	—	6
E	3	3	4	6	—

[Panjab M.B.A. 2007; Pondicherry B.Sc. (Math.) 2002]

1154. Solve the travelling salesman problem given by the following data :

$$c_{12} = 20,\ c_{13} = 4,\ c_{14} = 10,\ c_{23} = 5,\ c_{34} = 6$$

$$c_{25} = 10,\ c_{35} = 6,\ c_{45} = 20,\ \text{where } c_{ij} = c_{ji}.$$

and there is no route between cities i and j if a value for c_{ij} is not shown. *[IAS 1991]*

1155. The *ABC* Ice Cream Company has a distribution depot in Greater Kailash Part I for distributing ice-cream in South Delhi. There are four vendors located in different parts of South Delhi (call them *A, B, C* and *D*) who have to be supplied ice-cream every day. The following matrix displays the distances (in kilometres) between the depot and the four vendors :

From	To				
	Depot	Vendor A	Vendor B	Vendor C	Vendor D
Depot	—	3.5	3	4	2
Vendor *A*	3.5	—	4	2.5	3
Vendor *B*	3	4	—	4.5	3.5
Vendor *C*	4	2.5	4.5	—	4
Vendor *D*	2	3	3.5	4	—

What route should the company van follow so that the total distance travelled is minimized?

[Delhi M.B.A. (Nov.) 2009]

1156. A salesman must travel from city to city to maintain his accounts. This week he has to leave his home base and visit each other city and return home. The table shows the distance (in kilometres) between the various cities. The home city is city *A*. Use the assignment method to determine the tour that will minimize the total distances of visiting all cities and returning home.

From city	To city				
	A	B	C	D	E
A	—	375	600	150	190
B	375	—	300	350	175
C	600	300	—	350	500
D	160	350	350	—	300
E	190	175	500	300	—

[Delhi M.B.A. (PT.) 2009]

1157. Distances between six islands in the Indian Ocean are given in the table of distance below :

From islands	*To islands*					
	A	*B*	*C*	*D*	*E*	*F*
A	—	120	90	100	110	160
B	120	—	190	200	210	270
C	90	190	—	30	40	90
D	100	200	30	—	30	80
E	110	210	40	30	—	70
F	160	270	90	80	70	—

It is proposed to lay submarine cable of smallest length to connect all the islands by telephone connections. Find the minimum spanning tree for the six islands mentioned.

MULTIPLE CHOICE QUESTIONS

1. The assignment problem is a special case of transportation problem in which number of origins
(*a*) equals the number of destinations.
(*b*) is grater than the number of destinations.
(*c*) is less than the number of destinations.
(*d*) is less than or equal to the number of destinations.

2. An assignment problem is considered as a particular case of transportation problem, because
(*a*) all the rim conditions are 1. (*b*) all x_{ij} are either 1 or 0.
(*c*) the number of rows equals columns. (*d*) all of the above.

3. If there are n workers and n jobs, there would be
(*a*) n solutions (*b*) n ! solutions
(*c*) $(n-1)$! solutions (*d*) $(n!)^n$ solutions.

4. In an assignment problem involving four workers and three jobs, total number of assignments possible are
(*a*) 4 (*b*) 3
(*c*) 7 (*d*) 12.

5. The minimum number of lines covering all zeros in a reduced cost matrix of order n can be
(*a*) at the most n (*b*) at the least n
(*c*) $n-1$ (*d*) $n+1$.

6. An assignment problem can be
(*a*) designed and solved as a transportation problem.
(*b*) of maximization type.
(*c*) solved only if number of rows equals the number of columns.
(*d*) all of the above.

7. The method used for solving an assignment problem is called
(*a*) MODI method (*b*) reduced matrix method
(*c*) Hungarian method (*d*) none of the above.

8. An assignment problem
(*a*) is unbalanced when each of the given workers cannot do each of the given jobs.
(*b*) may not be balanced if the number of jobs matches with the number of workers.
(*c*) can be balanced by introducing a dummy row or dummy column.
(*d*) can be balanced by introducing dummy rows as well as dummy columns.

9. In marking assignments, which of the following should be preferred :
(*a*) only that row which have single zero.
(*b*) only that column which have single zero.

(*c*) only a row/column that have single zero.
(*d*) only column having more than one zero.

10. While solving an assignment problem, an activity is assigned to a resource with zero opportunity cost because the objective is to
(*a*) reduce the cost of assignment to zero.
(*b*) minimize total cost of assignment.
(*c*) reduce the cost of that particular assignment to zero.
(*d*) all of the above.

11. An assignment problem can be solved by
(*a*) transportation method (*b*) simplex method
(*c*) both (*a*) and (*b*) (*d*) none of the above.

12. Maximization assignment problem is transformed into a minimization problem by
(*a*) subtracting each element of the profit matrix from the highest element of the matrix.
(*b*) subtracting all the elements in a column from the highest element of that column.
(*c*) subtracting all the elements in a row from the highest element of that row.
(*d*) any of the above.

13. For a salesman, who has to visit n cities, following are the ways of his tour plan :
(*a*) n (*b*) $n!$
(*c*) $(n+1)!$ (*d*) $(n-1)!$

14. Which of the following statement is not correct ?
(*a*) For every prohibited assignment, the given cost element is replaced by M, which is a very large value.
(*b*) Multiple optimum assignment schedules may have different objective function values.
(*c*) The dual variables for an assignment problem are given by the u_i and v_j values of the optimum solution to the transformed transportation problem.
(*d*) The variables of the dual problem of a given assignment problem may be unrestricted in sign.

ANSWERS

1. (*a*)	**2.** (*d*)	**3.** (*b*)	**4.** (*b*)	**5.** (*a*)
6. (*d*)	**7.** (*c*)	**8.** (*c*)	**9.** (*a*)	**10.** (*b*)
11. (*c*)	**12.** (*a*)	**13.** (*d*)	**14.** (*b*)	

REVIEW QUESTIONS

1. What is an assignment problem ? Explain. *[Madurai M.Com. (Nov.) 2003]*

2. What is an assignment problem ? Give two applications in health care administration. *[Delhi M.B.A. (HCA) PT 2000]*

3. Give the mathematical formulation of an assignment problem.

4. "An assignment problem is a special case of a transportation problem". Explain. *[Delhi B.Sc. (Stat.) 2003]*

5. What is an assignment problem, and how do you interpret it as a linear programming model? *[Meerut M.Sc. (Math.) 2001]*

6. Explain the difference between a transportation problem and an assignment problem. *[Panjab Tech. Univ. M.B.A. (Dec.) 2012; Delhi PG Dip. in Glob. Bus. Oper. 2010]*

7. If in an assignment problem, we add a constant to every element of a row of the cost matrix, then prove that an assignment plan which minimizes the total cost for the new matrix, also minimizes the total cost for the original cost matrix. *[Meerut M.Sc. (Math.) 1996]*

8. Give in detail the computational procedure of solving the assignment problem. *[Madras B.Sc. (Math.) 1995]*

9. Explain Hungarian algorithm. *[Annamalai M.B.A. 2009]*

10. Can there be multiple optimal solutions to an assignment problem? How would you identify the existence of multiple solutions, if any?

11. How will you solve an assignment problem where a particular assignment is prohibited in case of minimization? Name the method used for solving a maximization assignment problem. *[Delhi MIB 2011]*
12. Describe a method of drawing minimum number of lines in the context of assignment problem. Name the method.
13. Explain the method of solving a maximization assignment model. *[Madras M.B.A. 2000]*
14. How does travelling salesman problem differ from an assignment model? *[Annamalai M.B.A. (Nov.) 2010]*
15. Write a short note on Travelling Salesman Problem. *[Meerut M.Sc. (Math.) 2001]*
16. How will you solve an assignment problem where a particular assignment is prohibited? Give an example to show that an assignment problem can be formulated as a linear programming problem. *[Delhi M.B.A. (PT.) HCA Nov. 2009]*