Introduction to Management Science

Operations Research

KANTI SWARUP
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SULTAN CHAND & SONS

Operations Research

उपदेशलवं शास्त्रं कुरुते धीमतो यतः।
तत्तु प्राप्यैव विस्तारं स्वयमेवोपगच्छति।।

जले तेलं खले गुह्यं पात्रे दानं मनागपि।
प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तिनः।।

"A little instruction and guidance in Science is sufficient for the intelligent student, for this alone will help him develop his knowledge of his own accord. Science instilled into the intelligent mind has sufficient vitality in it to grow and expand by its own force even as a drop of oil on a sheet of water, a piece of secret confined to a villain, or a little act of charity of the deserving person".

— *Bhaskara*

"OPERATIONS RESEARCH, a branch of Applied Mathematics, has witnessed a remarkable phenomenal growth during the last two decades. Its usefulness in the context of our present-day socio-economic problems has come to be widely recognised. The inevitable result has been a scarcity of up-to-date text material for university students sufficient to permit a comprehensive study of its basic methodology and techniques.

This book intends to provide an introduction to the subject — its theory, techniques and their applications. It encompasses the needs of the degree students of Mathematics, Statistics and Engineering. The text will be found equally useful for the post-graduate students of Economics, Commerce and Management. It may also serve as a good reference for professionals".

— from PREFACE to the First Edition

Some Salient Features

- The entire text restructured, rewritten and updated to cater to the revised syllabii of all Indian Universities and institutions.
- Addition of three new chapters on *Investment Analysis*, *Network Routing Problems*, and *Inventory Control — II*.
- Addition of several new topics at appropriate places. These include Complementary Slackness Theorem, Karmarkar Algorithm, Triangular Basis, Maintenance Crew Scheduling, *M/G/*1 and other queueing models, Recruitment and Promotion Problem, Equipment Renewal Problem, Systems of Inventory Control, Negative Float and Negative Slack, Minimal Spanning Tree Problem, Shortest Route Problem, Maximal Flow Problem, Insights into Big Networks, Investment under Uncertainty, Decision Trees Analysis, etc.
- Step-by-step algorithmic format for various computational procedures.
- Sample problems and unsolved problems drawn from the latest examination papers.

An Introduction to Management Science

Operations Research

[*A textbook for the students of Science & Technology, Commerce, Accountancy, and Management of all Indian Universities, Professional Bodies and Civil Services*]

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Preface

To the Ninteenth Revised Edition

It gives us great pleasure and satisfaction to present the Ninteenth Edition of the book before the esteemed readers. The purpose of this edition, as with previous editions, is to provide undergraduate and graduate students with a sound conceptual understanding of the role that management science plays in the decision-making process. The text describes many applications and techniques which are used successfully in Management Science.

The book is divided into thirty four chapters and four appendices. The text begins with the Chapter numbered *0* which provides *Prerequisite* of Linear Algebra. Chapter *1* on *Operations Research — 'An Overview'* highlights the nature, scope and limitations of O.R. The chapter also provides a broad classification of different O.R. models. Chapter *2* concentrates on the mathematical formulation of a *Linear Programming Problem.* In Chapter *3*, *Graphical Method* is explained to give the reader an intuitive idea of what L.P. is supposed to achieve. Chapter *4* develops the *Simplex Method* procedure for solving linear programming problems. The concept of *Duality* is introduced in Chapter *5,* in which rigorous proofs of the Fundamental Theorem of Duality and Complimentary Slackness Theorem are provided. Chapter *6* deals with *Post-optimal Analysis* of a linear programming problem. *Integer Programming* is discussed in Chapter *7,* where Gomory's all-integer method and the Branch-and-Bound Techniques are discussed in detail.

The important concept of *Goal Programming* has been discussed in Chapter *8*. To acquaint the reader with the new developments, some *Advanced Linear Programming Techniques* are presented in Chapter *9*. These include Fractional Programming, Karmarkar Algorithm, Revised Simplex Method, Bounded Variable Technique and Parametric Programming. The special class of linear programming problems—*Transportation Problem* and *Assignment Problem*—are discussed in Chapters *10* and *11,* respectively, while Chapter *12* is specially meant for the *Sequencing Problems* with application to Maintenance Crew Scheduling. Chapter *13* presents an introduction to *Dynamic Programming* with solution procedures and some applications.

To make the book self-contained, a brief review of the concepts of *Probability* and *Markov Analysis* are given in Chapters *14* and *15,* respectively. Decision-making rules both under uncertainty and under risk have been discussed in Chapter *16* on *Decision Analysis*. An application of linear programming to *Games and Strategies* is discussed in Chapter *17,* while *Replacement Problem and System Reliability* are discussed in Chapter *18*. A discussion on Recruitment and Promotion Problem, and Equipment Renewal Problem has also been discussed in this chapter. Chapters *19* and *20* present a variety of deterministic and probabilistic *Inventory Control* problems both with and without price-breaks. Systems of Inventory Control, One-period Inventory Problem and Application of Inventory Control to Dynamic Programming are added features of Chapter *20*. Chapter *21* presents an introduction to *Queueing Theory*—Both Poisson and non-Poisson systems are discussed. Chapter *22* on *Simulation* provides the application of Monte-Carlo Simulation in a variety of situations. Some typical applications in the area of Inventory, Queueing, Sequencing and

Finance are presented. *Investment Analysis and Break-even Analysis* has been discussed in Chapter *23*. The concept is of immense use to the decision-makers who have to tackle the investment decisions.

A new topic on *Network Routing Problems* is discussed in Chapter *24*. It includes Network Flow Problems, Minimal Spanning Tree Problems, Shortest Path Problems, Maximal Flow Problems; and also provides insights into Big Networks. Chapter *25* offers a brief introduction to the techniques of *Network Scheduling by PERT/CPM*. Chapter *26* on *Resource Analysis in Network Scheduling* includes a discussion on Precedence Planning, Updating and Resource Allocation.

The concept of *Non-linear Programming* is introduced in Chapter *27*. A summary of classical optimization techniques for constrained non-linear programming is given and the Kuhn-Tucker conditions are derived elegantly. Chapter *28* provides a number of *Non-linear Programming Methods*, including those of Quadratic and Separable Convex Programming. Chapter *29* discusses *Geometric Programming*, while a brief account of Shannon's *Information Theory* is presented in Chapter *30*. Chapter 31 gives a brief account of the forecasting methods, which are categorised as being qualitative and quantitative. *Concept of Learning Curve* and its relevance in management applications is discussed in Chapter *32*. A novel feature of the text is the inclusion of *Some Cases in O.R.* taken from the actual decision-making situations and are presented in Chapter *33*. The chapter offers challenging decision-making problems to the would be O.R. consultants.

At the end of each section, graded *Sample Problems* and *Unsolved Problems* have been given. These help the student to grasp the theory better, and prepare him/her for the next section. These exercises are thus an integral part of the theory. Well graded *Review Questions* are given at the end of each chapter. Also, at the end of each chapter a set of *Multiple Choice Questions* are given with answers. These include questions on the entire chapter and are carefully selected to represent variety. Sample problems, unsolved problems and review questions have been mostly drawn from the latest examination papers.

Old stereotype problems have been replaced by new problems taken from latest examination question papers of various universities. In particular we would like to mention the inclusion of the following:

- PG Diploma in Global Business Operations Research of Delhi University.
- MIB (Master of International Business) of Delhi University
- MBA of Delhi University and various other Universities/Management Institutes
- M.Com. of Delhi University and various other Universities
- BBA of Delhi University and GGSIP University
- MMS of Pune University.

To enhance further reading, carefully selected *References* are provided in Appendix *A*. *Answers* to unsolved problems are provided in Appendix *B*. A set of questions from recent examination question papers of various universities alongwith their answers are given in the form of Appendix *C*. Necessary *Statistical Tables* are also included in Appendix *D* to make the book self-contained.

It is hoped that the present version of the book is error free and shall be found more useful by the users. Suggestions for improvement will be gratefully received and highly appreciated.

New Delhi

P K GUPTA

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Prerequisite

"I cherish both faces of Mathematics, the pure as a beautiful retreat from reality and the applied as an ardent hope for life"

0:1. MATRICES

***Definition* (Matrix).** *m, n being positive integers, a rectangular array of mn numbers, real or complex, arranged in m rows and n columns, say*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a ***matrix of order*** $m \times n$ *or an* ***m by n matrix.***

The numbers a_{ij} *are called the* ***elements*** *(or* ***entries****) of the matrix.*

The matrix **A** *is called a* ***real matrix*** *if the elements* a_{ij} *are all real numbers.*

Notation. The $m \times n$ matrix **A**, whose (i, j)th element is a_{ij}, will be denoted more compactly by $(a_{ij})_{m \times n}$ and where no ambiguity is involved as to the order of the matrix, simply by the symbol (a_{ij}).

Some Special Types of Matrices

(*i*) A matrix of order $n \times n$ is called a ***square matrix*** of order n.

(*ii*) An $1 \times n$ matrix is called a ***row vector.***

(*iii*) An $m \times 1$ matrix is called a ***column vector.***

(*iv*) A matrix whose every element is zero is called a ***null matrix*** and is denoted by **0**.

(*v*) A square matrix, in which all principle diagonal elements are unity while all other elements are zeros, is called an ***identity matrix.*** It is generally denoted by **I**. For example, a 3×3 identity matrix is given by

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(*vi*) The ***transpose*** of a matrix $\mathbf{A} = (a_{ij})$ denoted by $\mathbf{A}^T$, is such that the (i, j)th element of $\mathbf{A}^T$ is the same as the (j, i)th element of **A**. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 6 & 5 \end{bmatrix}, \text{ then } \mathbf{A}^T = \begin{bmatrix} 2 & 1 & 6 \\ 4 & 3 & 5 \end{bmatrix}.$$

(*vii*) Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are said to be ***equal***, if and only if they are of the same order and $a_{ij} = b_{ij}$ for all i and j.

0:2. OPERATIONS ON MATRICES

We shall now show how we may operate on a matrix or a pair of matrices to obtain a new matrix.

Definition 1 **(Addition of two matrices).** *The* ***sum*** *of two matrices* $\mathbf{A} = (a_{ij})$ *and* $\mathbf{B} = (b_{ij})$ *of the same order, is a matrix* (c_{ij}) *of the same order, such that* $c_{ij} = a_{ij} + b_{ij}$ *for each* (i, j). *This sum is denoted by the symbol* $\mathbf{A} + \mathbf{B}$, *so that*

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij}).$$

Definition 2 **(Multiplication by a scalar).** *The* ***scalar multiple*** *of a matrix* $\mathbf{A} = (a_{ij})$ *by a scalar (real number)* λ, *is a matrix of the same order to be denoted by* $\lambda\mathbf{A}$, *such that*

$$\lambda\mathbf{A} = (\lambda a_{ij}).$$

Definition 3 **(Negative of a matrix).** *The* ***negative*** *of a matrix* $\mathbf{A} = (a_{ij})$ *is the scalar multiple of* $\mathbf{A}$ *by* -1, *that is,*

$$-\mathbf{A} = (-a_{ij}).$$

Definition 4 **(Subtraction).** *The* ***subtraction*** *of a matrix* $\mathbf{B}$ *from a matrix of the same order,* $\mathbf{A}$, *is the matrix* $\mathbf{A} + (-1)\,\mathbf{B}$, *and is denoted by* $\mathbf{A} - \mathbf{B}$. *Thus*

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\,\mathbf{B}.$$

Properties of Matrix Addition

Some general properties of matrix addition are :

(*i*) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(*ii*) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

(*iii*) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

(*iv*) $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{0}$

(*v*) $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$

0:3. MATRIX MULTIPLICATION

Definition **(Product of two matrices).** *Let* $\mathbf{A} = (a_{ij})_{m \times n}$ *and* $\mathbf{B} = (b_{jk})_{n \times p}$ *be two matrices, such that the number of columns in* $\mathbf{A}$ *is the same as the number of rows in* $\mathbf{B}$. *The product of* $\mathbf{A}$ *by* $\mathbf{B}$ *to be denoted by* $\mathbf{AB}$ *is the matrix* $(c_{ik})_{m \times p}$, *where*

$$c_{ik} = \sum_{k=1}^{n} a_{ij} b_{jk}\,; \qquad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, p.$$

Properties of Matrix Multiplication

For three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the following properties of matrix multiplication hold, provided the products involved are defined, *i.e.*, meaningful :

(*i*) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\,\mathbf{C}$

(*ii*) $(\mathbf{A} + \mathbf{B})\,\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(*iii*) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

(*iv*) $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$, where $\mathbf{I}$ is the identity matrix

Remarks 1. When $\mathbf{AB} = \mathbf{0}$, it does not follow that $\mathbf{A}$ or $\mathbf{B}$ must be $\mathbf{0}$. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ then } \mathbf{AB} = \mathbf{0}.$$

2. Even if $\mathbf{AB}$ and $\mathbf{BA}$ both are defined, in general $\mathbf{AB} \neq \mathbf{BA}$. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ then } \mathbf{AB} \neq \mathbf{BA}.$$

PROBLEMS

001. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 2 & 1 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 2 & -1 \\ 3 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 4 & 5 \end{pmatrix}$;

Determine :

(*i*) $\mathbf{A} + \mathbf{B}$, $\mathbf{A} + \mathbf{C}$, $\mathbf{B} + \mathbf{C}$, $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(*ii*) $\mathbf{AB}$, $\mathbf{BC}$, $\mathbf{CA}$, $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$.

(*iii*) $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $(\mathbf{A} + \mathbf{B})\mathbf{C}$

(*iv*) $7\mathbf{A} + 3\mathbf{B} - 4\mathbf{C}$.

002. Give a counter example to show that the matrix multiplication is not commutative.

003. If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, then obtain $\mathbf{AB}$. Give your comments on the result that you obtain.

004. Let $\mathbf{AB} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 & 4 \\ 3 & 0 \end{pmatrix}$. Show that $\mathbf{AB} = \mathbf{AC}$. What conclusion can you draw from this?

0:4. PARTITIONED MATRICES

A matrix **A** can always be put in a more compact form, as a matrix whose elements are submatrices of **A** — this is achieved by drawing imaginary, horizontal and vertical lines to divide the elements of **A** into a number of blocks.

Thus

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & : & a_{13} \\ a_{21} & a_{22} & : & a_{23} \\ \cdots & \cdots & : & \cdots \\ a_{31} & a_{32} & : & a_{33} \end{bmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

where

$$\mathbf{A}_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{A}_{12} = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \quad \mathbf{A}_{21} = (a_{31} \quad a_{32}), \quad \mathbf{A}_{22} = (a_{33}).$$

Matrix **A** so represented is called a ***partitioned matrix.***

Addition of Partitioned Matrices

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two $m \times n$ matrices and let us suppose they have been partitioned in the same way (*identically partitioned), so that*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

where $\mathbf{A}_{ij}$, $\mathbf{B}_{ij}$ are like matrices.

Then, it is easy to verify that

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} \mathbf{A}_{11} + \mathbf{B}_{11} & \mathbf{A}_{12} + \mathbf{B}_{12} \\ \mathbf{A}_{21} + \mathbf{B}_{21} & \mathbf{A}_{22} + \mathbf{B}_{22} \end{pmatrix}$$

Product of Partitioned Matrices

Let **A**, **B** be matrices of order $m \times n$ and $n \times p$ respectively, so that **AB** is defined. Suppose, **A**, **B** have been partitioned in the same way,

$$\therefore \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

where $\mathbf{A}_{ij}$, $\mathbf{B}_{ij}$ are like matrices. Then, it may be easily verified that

$$\mathbf{AB} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{pmatrix}$$

0:5. DETERMINANT OF A SQUARE MATRIX

From his elementary course in Algebra, the reader is familiar with determinants of second and third order. For example :

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

We proceed to define det **A** for any square matrix by induction on n, the order of **A**.

Definition* (Determinant).** *Let* **M** *be the set of all square matrices. Let* $\mathbf{A} \in \mathbf{M}$, *then* ***determinant *of* **A**, *to be denoted by* ***det*** **A**, *is the number found as follows :*

(*i*) *If* $\mathbf{A} = (a)$ *is a* 1×1 *matrix, then* **det A** $= a$,

(*ii*) *For* $n > 1$, *if* $\mathbf{A} = (a_{ij})$ *is an* $n \times n$ *matrix and,* $\mathbf{M}_{ij}$ *denotes the* $(n-1)$ *rowed sub-matrix of* **A**, *obtained by striking off the ith row and jth column of* **A**, *then*

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathbf{M}_{ij} \qquad (i \text{ fixed})$$

We shall also denote det **A** by the more convenient symbol $|\mathbf{A}|$.

For each pair, (i, j)th number

$$\mathbf{A}_{ij} = (-1)^{i+j} \det \mathbf{M}_{ij}$$

is called the *co-factor* of a_{ij} in $|\mathbf{A}|$. Thus, we may also write as

$$|\mathbf{A}| = \det \mathbf{A} = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{ij} \qquad (i \text{ fixed})$$

Properties of the Determinant Function

The following properties follow immediately from the definition :

(*i*) The value of the determinant is not changed if its rows and columns are interchanged.

(*ii*) The value of the determinant is zero, if every element of a row or column is zero.

(*iii*) The value of the determinant is zero, if any two columns (or rows) of the determinant are identical.

(*iv*) If **B** is obtained from a square matrix **A** by an interchange of two columns (or rows), then $|\mathbf{B}| = -|\mathbf{A}|$.

(*v*) If every element of a column (or row) of a determinant is multiplied by a scalar λ, the value of the determinant is multiplied by λ.

(*vi*) The value of the determinant remains the same, if scalar λ times a row (or column) is added to another row (or column).

(*vii*) For any two square matrices **A**, **B** of the same order

$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|.$$

0:6. INVERSE OF A SQUARE MATRIX

***Definition* (Inverse).** *Let* **A** *be a square matrix. If there exists a matrix* **B**, *such that*

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA};$$

then **B** *is called an* ***inverse*** *of* **A** *and is denoted by* $\mathbf{A}^{-1}$.

It can be easily shown that the *inverse of a square matrix, if it exists, is unique.*

From the relation $\mathbf{AA}^{-1} = \mathbf{I}$, it follows that

$$1 = |\mathbf{I}| = |\mathbf{AA}^{-1}| = |\mathbf{A}| \cdot |\mathbf{A}^{-1}|$$

which shows that if **A** has an inverse, then necessarily $|\mathbf{A}| \neq 0$.

A square matrix **A** is called ***singular*** if $|\mathbf{A}| = 0$; and if $|\mathbf{A}| \neq 0$, it is called ***non-singular***. Thus, we shall talk of the inverse of a non-singular matrix only.

For non-singular matrices, the following two results can be easily proved :

1. When **A** and **B** are non-singular square matrices of the same order, then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

2. When **A** is non-singular, then

$$\mathbf{AB} = \mathbf{AC} \text{ implies } \mathbf{B} = \mathbf{C}.$$

Elementary Transformations of a Matrix

An elementary row (column) transformation of a matrix is one of the following types :

Type 1. Interchange of any two rows (columns).

Type 2. Multiplying the elements of a row (column) by a non-zero scalar.

Type 3. Adding to any row (column), a scalar multiple of another row (column).

Computation of the Inverse of a Square Matrix by Elementary Transformations

Let **A** be a non-singular matrix so that **A** has an inverse. We write

$$\mathbf{IA} = \mathbf{A}.$$

By successive elementary row transformations on the pre-factor of the product **IA** and corresponding transformations on the matrix **A** on the right, we reduce the above matrix equation to the form

$$\mathbf{BA} = \mathbf{I}.$$

Then clearly,

$$\mathbf{B} = \mathbf{A}^{-1}$$

Similarly, if we write $\mathbf{AI} = \mathbf{A}$ then using successive elementary column transformations on the post-factor of **AI**, and corresponding transformations on the matrix **A** on the right, we reduce the above matrix equation to the form $\mathbf{AC} = \mathbf{I}$. This implies that $\mathbf{C} = \mathbf{A}^{-1}$.

PROBLEM

005. Compute the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

by (*i*) elementary row transformations, and (*ii*) elementary column transformations.

0:7. RANK OF A MATRIX

Definition* (Rank).** *A positive integer r is said to be the* ***rank *of a matrix* **A**, *denoted by* $\rho(\mathbf{A})$, *if*

(*i*) **A** *possesses at least one r-rowed minor which does not vanish ; and*

(*ii*) **A** *does not possess any non-zero* $(r + 1)$*-rowed minor.*

Example. Let $\mathbf{A} = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$. Clearly, **A** possesses a 2-rowed non-zero minor $\begin{vmatrix} 1 & -3 \\ 9 & 1 \end{vmatrix}$. Thus, $\rho(\mathbf{A}) \geq 2$. Also, since **A** does not possess any 3-rowed minor (they therefore all vanish), we have $\rho(\mathbf{A}) \leq 2$. This gives $\rho(\mathbf{A}) = 2$.

0:8. SOLUTION OF LINEAR EQUATIONS

Let there be a system of m linear equations in n unknowns, $(m < n)$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

The system can be put in a more convenient form, by setting

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ a_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then, the system can be represented by the single matrix equation

$$\mathbf{AX} = \mathbf{b}$$

where **A** is an $m \times n$ matrix, **X** an $n \times 1$ matrix and b an $m \times 1$ matrix.

This system of linear equations is called ***homogeneous*** if $\mathbf{b} = \mathbf{0}$ and ***non-homogeneous*** otherwise.

A set of values x_j $(j = 1, 2, \ldots, n)$ which satisfy the given set of equations is called a ***solution*** of the system of equations. The system of simultaneous linear equations is said to be *consistent*, if it has one or more solutions, and *inconsistent* otherwise.

The coefficient matrix obtained for the system of non-homogeneous linear equations in the form $(\mathbf{A} : \mathbf{b})$ is called an ***augmented matrix.***

If $m = n$, we have a system of n equations in n unknowns. Here, if $\rho(\mathbf{A}) = n$, then $|\mathbf{A}| \neq 0$, and we have a unique solution

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}.$$

In particular, if $\mathbf{b} = \mathbf{0}$, and $|\mathbf{A}| \neq 0$, then the only solution possible is the trivial one, that is

$$x_1 = x_2 = \ldots = x_n = 0.$$

We now state, without proof, two other results which are often used.

(*i*) *If the coefficient matrix of a homogeneous system of linear equations in n unknowns has rank $r < n$, then $n - r$ unknowns can be given arbitrary values and the values of the remaining unknowns can then be determined uniquely.*

(*ii*) *Let* **A** *be an $m \times n$ matrix and* $\mathbf{AX} = \mathbf{b}$ *be a system of m equations in n unknowns. Then, the system is consistent if the coefficient matrix* **A** *and the augmented matrix* $(\mathbf{A} : \mathbf{b})$ *have the same rank.*

PROBLEMS

006. Solve each of the following sets of equations :

(*a*) $x + y - z = 0$
$2x + y + z = 7$
$3x - y + 2z = 7$

(*b*) $3x - y + 2z = 4$
$x + 4y - z = 4$
$x - 9y + 4z = 3.$

007. Determine the value of constant k that will allow the following set of equations to be consistent :

$x - y + z = 1 \quad 2x + 2y - z = 2 \quad 3x - y + 2z = 3 \quad 4x + y + z = k.$

008. Examine if the following equations are consistent and find the solution, if one exists :

$5x + 3y + 7z = 4 \quad 7x + 2y + 10z = 5 \quad 3x + 26y + 2z = 9.$

009. Determine the values of k such that the system of linear equations

$$x - 2y = 1 \qquad x - y + kz = -2 \qquad ky + 4z = 6$$

has a unique solution.

0:9. VECTORS

Row or column matrices are often referred to as *vectors*. A row matrix is called a *row vector* and a column matrix is called a *column vector*. The vectors will generally be symbolised by lower case bold face letters such as **a**, **b**, **x**, **y**, etc.

***Definition 1* (Vector).** *An **n-vector** **a** is an ordered n-tuple of real numbers*

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)$$

The members a_i $(i = 1, 2, \ldots, n)$ are called the components of **a**.

***Definition 2* (Equality of two n-vectors).** *The two n-vectors* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *are said to be **equal** if $a_i = b_i$ for all $i = 1, 2, \ldots, n$.*

Note that if $\mathbf{a} = \mathbf{b}$ then $\mathbf{b} = \mathbf{a}$.

***Definition 3* (Sum of two n-vectors).** *The **sum** of two n-vectors* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, *written as* $\mathbf{a} + \mathbf{b}$, *is defined to be the n-vector*

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

The subtraction of **b** *from* **a**, *denoted by* $\mathbf{a} - \mathbf{b}$ *is given by*

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

In general, given three n-vectors, **a**, **b** and **c**, we have

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}\,;\ (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

***Definition 4* (Multiplication of a vector by a scalar).** *The **product** of an n-vector* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *by a **scalar** λ, written as $\lambda\mathbf{a}$, is defined to be the n-vector*

$$\lambda\mathbf{a} = \lambda(a_1, a_2, \ldots, a_n)$$
$$= (\lambda a_1, \lambda a_2, \ldots, \lambda a_n).$$

In general, given two n-vectors **a** and **b** and the scalars λ and μ, we have

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}\,;\ \lambda(\mu\mathbf{a}) = (\lambda\mu)\,\mathbf{a}.$$

***Definition 5* (Row and Column vectors).** *An n-vector*

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)$$

*is called a **row vector**. Corresponding to each row vector* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, *there corresponds another n-vector written as*

$$\mathbf{a}^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \text{or} \quad \mathbf{a} = [a_1, a_2, \ldots, a_n]$$

*which we shall call a **column vector**.*

Remark. While row vectors and column vectors are both ***n*-vectors**, the two types are distinct. A row (column) vector can only be added to a row (column) vector.

***Definition 6* (Inner product).** *The **Inner Product** of two n-vectors* **a** *and* **b** *is the scalar*

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = \sum_{i=1}^{n} a_i b_i$$

and is denoted by $\mathbf{a}\,.\,\mathbf{b}^T$ *or* $\mathbf{b}\,.\,\mathbf{a}^T$.

***Definition 7* (Length of a vector).** *The **length** or **magnitude** of a vector* **a**, *written as* $|\mathbf{a}|$ *is defined by*

$$|\mathbf{a}| = [\mathbf{a} \,.\, \mathbf{a}^T]^{\frac{1}{2}} = [\sum_{i=1}^{n} a_i^2]^{\frac{1}{2}}.$$

Note that for any scalar λ, $|\lambda \mathbf{a}| = |\lambda| \,.\, |\mathbf{a}|$.

***Definition 8* (Null vector).** *The **null vector** is a vector whose all components are zero.*

It is usually denoted by **0**.

***Definition 9* (Unit vector).** *A **unit vector** $\mathbf{e}_i$ is a vector with its ith component unity and all other components zero.*

For n-component vectors, there are n unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

***Definition 10* (Euclidean space).** *An **n-dimensional euclidean space** is defined to be the coilection of all n-vectors.*

The n-dimensional euclidean space will be denoted by $\mathbf{R}^n$. The vectors that will occur in our study, shall always belong to some euclidean space.

Note. The term 'point' and 'vector' in space $\mathbf{R}^n$, will be used without distinguishing between them.

0:10. VECTOR INEQUALITIES

Definition 1* ($\geq$ Order).** *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$, *then* **a** *is said to be **greater than or equal to **b**, *if and only if,* $a_i \geq b_i$ *for* $i = 1, 2, \ldots, n$. *This is denoted by writing* $\mathbf{a} \geq \mathbf{b}$.

Definition 2* ($\leq$ Order).** *Vector* $\mathbf{a} = (a_1, \ldots, a_n)$ *is said to be **less than or equal to $\mathbf{b} = (b_1, \ldots, b_n)$ *if and only if* $\mathbf{b} \geq \mathbf{a}$. *This is denoted by writing* $\mathbf{a} \leq \mathbf{b}$.

Remarks. The two vectors **a** and **b** are ***order comparable***, if and only if, they have an equal number of components.

Example. Let $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (0, 1, -1)$. Clearly, $\mathbf{a} \geq \mathbf{b}$, or equivalently $\mathbf{b} \leq \mathbf{a}$.

The *vector inequality* $\mathbf{x} \geq \mathbf{0}$ simply indicates that **x** is a vector whose each component is non-negative. Such vector inequalities shall frequently occur in our study.

0:11. LINEAR COMBINATION OF VECTORS

Given two vectors **a** and **b**, we can always perform operations of addition and scalar multiplication to obtain vectors of the form $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $5\mathbf{a} + 3\mathbf{b}$, $2\mathbf{a} - 7\mathbf{b}$, etc. Such vectors are called **linear combinations** of **a** and **b**. In general, for any scalars λ and μ, the vector $\lambda\mathbf{a} + \mu\mathbf{b}$ is a linear combination of **a** and **b**.

Thus, if $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is a set of k vectors from $\mathbf{R}^n$ and $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a set of k scalars, then the vector

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_k \mathbf{a}_k$$

is known as a linear combination of the given set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

***Definition 1* (Linear dependence and independence of vectors).** *A set of vectors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ *of* $\mathbf{R}^n$, *is said to be **linearly dependent**, if there exist scalars* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *not all zero, such that* $\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_k \mathbf{a}_k = \mathbf{0}$.

When it is not possible to have the above equality without all λ_i's being zero, the given set of vectors $\lambda_1, \lambda_2, \ldots, \lambda_k$ is said to be **linearly independent.**

Example 1. The vector $\mathbf{a}_1 = (1, 2)$ and $\mathbf{a}_2 = (2, 4)$ are linearly dependent since there exist $\lambda_1 = 2$ and $\lambda_2 = -1$ for which $\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 = 0$.

Example 2. The vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ are linearly independent since $\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 = \mathbf{0}$

$$\Rightarrow \quad \lambda_1 (1, 0, 0) + \lambda_2 (0, 1, 0) + \lambda_3 (0, 0, 1) = (0, 0, 0)$$

$$\Rightarrow \quad (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$$

$$\Rightarrow \quad \lambda_1 = 0, \ \lambda_2 = 0 \text{ and } \lambda_3 = 0.$$

Note. If a set of vectors is linearly independent, then, any subset of it is also linearly independent. If a set of vectors is linearly dependent, then a superset of it is also linearly dependent.

Definition 2* (Basis set).** *A linearly independent set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ of $\mathbf{R}^n$, having the property that any vector of $\mathbf{R}^n$ can be expressed as linear combination of these vector, is called a* ***basis set *for $\mathbf{R}^n$.*

Theorem 0-1. *The set of n unit vector $\mathbf{e}_i$ $(i = 1, 2, \ldots, n)$ forms a basis for $\mathbf{R}^n$.*

Proof. We shall show that the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbf{R}^n$ are linearly independent and that any vector of $\mathbf{R}^n$ can be expressed as a linear combination of these unit vectors.

Consider,

$$\sum_{i=1}^{n} \lambda_i \mathbf{e}_i = \mathbf{0}, \quad \text{where } \lambda_i \ (i = 1, 2, \ldots, n) \text{ are scalars}$$

or

$$\lambda_1 (1, 0, 0, \ldots, 0) + \lambda_2 (0, 1, 0, \ldots, 0) + \ldots + \lambda_n (0, 0, \ldots, 1) = 0.$$

or

$$(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = (0, 0, 0, \ldots, 0).$$

This implies that $\lambda_i = 0$ for all $i = 1, 2, \ldots, n$.

Thus, the n unit vectors $\mathbf{e}_i$; $i = 1, 2, \ldots, n$ are linearly independent.

Let $\mathbf{x}$ be any other vector of $\mathbf{R}^n$ different from the unit vectors $\mathbf{e}_i$ $(i = 1, 2, \ldots, n)$. This can be written as

$$\mathbf{x} = (x_1, x_2, \ldots, x_k, \ldots, x_n) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \ldots + x_n \mathbf{e}_n = \Sigma x_i \mathbf{e}_i.$$

This happens to be a linear combination of the n unit vectors $\mathbf{e}_i$ $(1 = 1, 2, \ldots, n)$. Since $\mathbf{x}$ is any vector of $\mathbf{R}^n$ different from the n unit vectors, therefore, every vector of $\mathbf{R}^n$ can be expressed as a linear combination of these unit vectors.

Hence $\{ \mathbf{e}_i ; i = 1, 2, \ldots, n \}$ forms a basis for $\mathbf{R}^n$.

Definition 3* (Standard basis).** *The set of vectors $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ is called the* ***standard basis *for $\mathbf{R}^n$.*

Remark. We may state without proof the important fact, that in any basis of $\mathbf{R}^n$, there are exactly n elements.

Theorem 0-2. Let $\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \}$ be any basis set for $\mathbf{R}^n$ and $\mathbf{b} \neq \mathbf{0}$ any other vector in $\mathbf{R}^n$. If $\mathbf{b}$ is expressible as

$$\mathbf{b} = \sum_{i=1}^{n} \delta_i \mathbf{a}_i \quad \text{where } \delta_i \neq 0 \text{ for some } i,$$

then the vector $\mathbf{a}_k$ (for which $\delta_k \neq 0$) can be replaced by the vector $\mathbf{b}$. The new set of n vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n, \mathbf{b}$ is also a basis set for $\mathbf{R}^n$.

Proof. Let $\delta_k \neq 0$. Consider the new set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n, \mathbf{b}$.

To prove that this set is a basis for $\mathbf{R}^n$, we must show

(*i*) $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \mathbf{a}_n, \mathbf{b}$ are linearly independent, and

(*ii*) any vector of $\mathbf{R}^n$ can be expressed as a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n, \mathbf{b}$.

Let us suppose that the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \ldots, \mathbf{a}_n, \mathbf{b}\}$ is not a linearly independent set. Then the equation

$$\sum_{i=1, i \neq k}^{n} \lambda_i \mathbf{a}_i + \lambda \mathbf{b} = \mathbf{0}$$

will hold even with some non-zero scalars λ_i $(i = 1, 2, \ldots, k-1, k+1, \ldots, n)$ and λ.

Now $\lambda = 0$ implies

$$\sum_{i=1, i \neq k}^{n} \lambda_i \mathbf{a}_i = \mathbf{0} \quad \text{with some } \lambda_i \neq 0.$$

This contradicts the fact that the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n\}$ is linearly independent (being a subset of a linearly independent set of vectors).

Thus, $\lambda \neq 0$, and we have

$$\sum_{i=1, i \neq k}^{n} \lambda_i \mathbf{a}_i + \lambda \sum_{i=1}^{n} \delta_i \mathbf{a}_i = \mathbf{0}, \quad \text{since } \mathbf{b} = \sum_{i=1}^{n} \delta_i \mathbf{a}_i$$

or

$$\sum_{i=1, i \neq k}^{n} (\lambda_i + \lambda \delta_i) \mathbf{a}_i + \lambda \delta_k \mathbf{a}_k = \mathbf{0}.$$

Since $\lambda \neq 0$, $\delta_k \neq 0$, therefore the set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is linearly dependent. This contradicts the hypothesis.

Thus, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n, \mathbf{b}\}$ is a linearly independent set.

Now, let $\mathbf{x}$ be any vector in $\mathbf{R}^n$, other than the given basis vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Then, we can express $\mathbf{x}$ as

$$\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_i, \quad \text{where } \lambda_i \ (i = 1, 2, \ldots, n) \text{ are scalars}$$

Now, since

$$\mathbf{b} = \sum_{i=1}^{n} \delta_i \mathbf{a}_i = \sum_{i=1, i \neq k}^{n} \delta_i \mathbf{a}_i + \delta_k \mathbf{a}_k, \qquad \delta_k \neq 0$$

we have

$$\mathbf{a}_k = \frac{1}{\delta_k}\left[\mathbf{b} - \sum_{i=1, i \neq k}^{n} \delta_i \mathbf{a}_i\right] = \frac{\mathbf{b}}{\delta_k} - \sum_{i=1, i \neq k}^{n} \frac{\delta_i}{\delta_k} \mathbf{a}_i$$

This gives

$$\mathbf{x} = \sum_{i=1, i \neq k}^{n} \lambda_i \mathbf{a}_i + \lambda_k \left[\frac{\mathbf{b}}{\delta_k} - \sum_{i=1, i \neq k}^{n} \frac{\delta_i}{\delta_k} \mathbf{a}_i\right]$$

$$= \sum_{i=1, i \neq k}^{n} \left(\lambda_1 - \frac{\lambda_k}{\delta_k} \delta_i\right) \mathbf{a}_i + \frac{\lambda_k}{\delta_k} \mathbf{b}.$$

Thus, $\mathbf{x}$ can be expressed as a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \ldots, \mathbf{a}_n, \mathbf{b}$.

Hence, the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n, \mathbf{b}\}$ forms a basis for $\mathbf{R}^n$.

PROBLEMS

010. If the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ constitute a linearly independent set, and the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k+1}$ constitute a linearly dependent set, then show that the vector $\mathbf{a}_{k+1}$, is expressible as a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

011. Given the basis vectors

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0) \quad \text{and} \quad \mathbf{e}_3 = (0, 0, 1) \text{ for } \mathbf{R}^3.$$

Find a vector that can be removed from the basis and be replaced by a vector $\mathbf{b} = (4, 3, 6)$ such that the new set is also a basis set for $\mathbf{R}^3$.

012. Do the following vectors form a basis for $\mathbf{R}^3$?

(a) $\mathbf{a}_1 = [3, 0, 2]$, $\mathbf{a}_2 = [7, 0, 9]$, $\mathbf{a}_3 = [4, 1, 2]$

(b) $\mathbf{a}_1 = [1, 1, 0]$, $\mathbf{a}_2 = [3, 0, 1]$, $\mathbf{a}_3 = [5, 2, 1]$

(c) $\mathbf{a}_1 = [1, 5, 7]$, $\mathbf{a}_2 = [4, 0, 6]$, $\mathbf{a}_3 = [1, 0, 0]$.

0:12. HYPERPLANES AND HYPERSPHERES

***Definition 1* (Line segment in $\mathbf{R}^n$).** *The line segment joining two points* $\mathbf{a}_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$, *and* $\mathbf{x}_2 = (x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)})$, *is the collection of all points* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, *where*

$$\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \qquad \text{for all values of } \lambda \text{ such that } 0 \le \lambda \le 1.$$

***Definition 2* (Hyperplane).** *In* $\mathbf{R}^n$, *the set of points* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *satisfying*

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = z \qquad (\text{not all } c_i = 0)$$

defines a ***hyperplane*** *for given values of* c_i*'s and* z.

Remarks 1. If $z = 0$, then the hyperplane is said to pass through the origin and its equation is then written as

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = 0 \quad \text{or} \quad \mathbf{cx} = 0.$$

This implies that $\mathbf{c}$ is orthogonal to every vector $\mathbf{x}$ in the hyperplane.

2. If $z \neq 0$ and $P(\mathbf{x}_1)$, $Q(\mathbf{x}_2)$ are any two points on the hyperplane, then

$$\mathbf{cx}_1 = z \quad \text{and} \quad \mathbf{cx}_2 = z.$$

This yields $\mathbf{c}(\mathbf{x}_1 - \mathbf{x}_2) = 0$, which implies that $\mathbf{c}$ is normal to the vector $\overrightarrow{\mathbf{QP}}$, *i.e.*, $\mathbf{x}_2 - \mathbf{x}_1$, and is thus in the direction of the normal to the hyperplane.

In general, $\mathbf{c}$ is called the *normal* to the hyperplane and $\pm \dfrac{\mathbf{c}}{|\mathbf{c}|}$ are called *unit normals.*

Definition 3* (Parallel hyperplanes).** *Two hyperplanes are said to be* ***parallel, *if they have the same unit normals. The hyperplanes* $\mathbf{c}_1\mathbf{x} = z_1$ *and* $\mathbf{c}_2\mathbf{x} = z_2$ *are parallel, if* $\mathbf{c}_1 = \lambda \mathbf{c}_2$ *for some* $\lambda \neq 0$.

***Definition 4* (Closed and Open half space).** *In* $\mathbf{R}^n$, *every hyperplane* $\mathbf{cx} = z$ *determines two open half spaces and two closed half spaces. The sets defined by*

$$\mathbf{H}_+^o = \{\mathbf{x} : \mathbf{cx} > z\} \text{ and } \mathbf{H}_-^o = \{\mathbf{x} : \mathbf{cx} < z\} \text{ are called } \textbf{open half spaces}$$

and

$$\mathbf{H}_+ = \{\mathbf{x} : \mathbf{cx} \ge z\} \text{ and } \mathbf{H}_- = \{\mathbf{x} : \mathbf{cx} \le z\} \text{ are called } \textbf{closed half spaces.}$$

Example. Let $\mathbf{c} = \{2, 3\}$ and $z = 7$. Then

$\mathbf{H}_+ = \{2x_1 + 3x_2 \ge 7\}$ gives a positive closed half space, and

$\mathbf{H}_- = \{2x_1 + 3x_2 \le 7\}$ gives a negative closed half space.

Definition 5* (Polytope).** *The intersection of a finite number of closed half spaces is called a* ***polytope. *The hyperplanes producing the half spaces are called the generating hyperplanes of the polytope.*

Definition 6* (Hypersphere).** *A* ***hypersphere *in* $\mathbf{R}^n$ *with centre at* $\mathbf{a}$ *and radius* $\in > 0$ *is defined to be the set of points*

$$\mathbf{S} = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| = \in\}.$$

Thus, the equation for hypersphere in $\mathbf{R}^n$ is

$$\sum_{i=1}^{n} (x_i - a_i)^2 = \in^2.$$

The inside of a hypersphere with centre at $\mathbf{a}$ and radius $\in$ is the set of points

$$S_i = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| < \in\}.$$

0:13. CONVEX SETS AND THEIR PROPERTIES

Definition 1* (Convex set).** *A subset $S \subset \mathbf{R}^n$, is said to be* ***convex*, if for any two points $\mathbf{x}_1$, $\mathbf{x}_2$ in S the line segment joining the points $\mathbf{x}_1$ and $\mathbf{x}_2$ is also contained in S.*

In other words, a subset $S \subset \mathbf{R}^n$ is convex, if and only if

$$\mathbf{x}_1, \mathbf{x}_2 \in S \Rightarrow \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in S; \quad 0 < \lambda \leq 1.$$

Some convex and non-convex sets in $\mathbf{R}^2$ are given below :

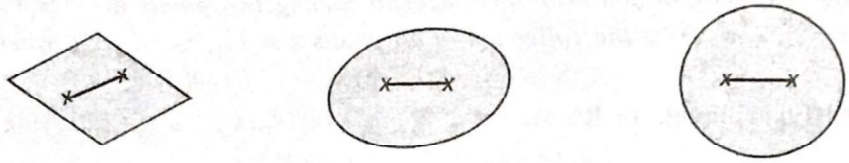

Fig. 0.1. Convex sets

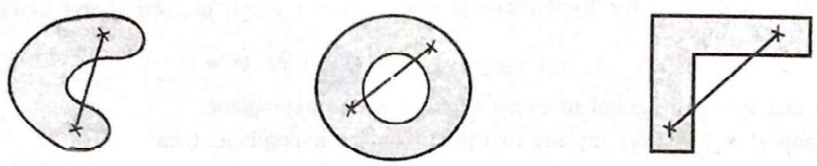

Fig. 0.2. Non-convex sets

Example 1. *Show that the set*

$$S = \{ (\mathbf{x}_1, \mathbf{x}_2) : 3x_1^2 + 2x_2^2 \leq 6 \}$$

is convex.

Solution. Let $\mathbf{x}, \mathbf{y} \in S$ where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

The line segment joining $\mathbf{x}$ and $\mathbf{y}$ is the set

$$\{ \mathbf{u} : \mathbf{u} = \lambda \mathbf{x} + (1 - \lambda)\mathbf{y}, \ 0 \leq \lambda \leq 1 \}$$

For some λ, $0 \leq \lambda \leq 1$, let $\mathbf{u} = (u_1, u_2)$ be a point of this set, so that

$$u_1 = \lambda x_1 + (1 - \lambda) y_1, \quad \text{and} \quad u_2 = \lambda x_2 + (1 - \lambda) y_2.$$

Now,

$$\begin{aligned} 3u_1^2 + 2u_2^2 &= 3[\lambda x_1 + (1 - \lambda) y_1]^2 + 2[\lambda x_2 + (1 - \lambda) y_2]^2 \\ &= \lambda^2 (3x_1^2 + 2x_2^2) + (1 - \lambda)^2 [3y_1^2 + 2y_2^2] + 2\lambda (1 - \lambda)(3x_1 y_1 + 2x_2 y_2) \\ &\leq 6\lambda^2 + 6(1 - \lambda)^2 + 12\lambda(1 - \lambda), \end{aligned}$$

since $(3x_1y_1 + 2x_2y_2) \leq \sqrt{(x_1\sqrt{3})^2 + (x_2\sqrt{2})^2}\ \sqrt{(y_1\sqrt{3})^2 + (y_2\sqrt{2})^2}$.

Thus, $3u_1^2 + 2u_2^2 \leq 6$; and hence $\mathbf{u} = (u_1, u_2)$ is also a point of S.

Hence, S is a convex set.

The following results easily follow from the definition of a convex set :

(*i*) A hyperplane in $\mathbf{R}^n$ is a convex set.

(*ii*) The sets $\mathbf{H}_+$, $\mathbf{H}_-$, $\mathbf{H}_+^o$ and $\mathbf{H}_-^o$ are convex sets.

(*iii*) A closed ball in $\mathbf{R}^n$, namely, $\{ \mathbf{x} : |\mathbf{x} - \mathbf{x}_o| \leq r \}$, where $r > 0$ and $\mathbf{x}_o$, $\mathbf{x} \in \mathbf{R}^n$, is a convex set.

Definition 2* (Extreme point or Vertex of a convex set).** *An* ***extreme point (vertex) *of a convex set is a point of the set which does not lie on any segment joining two other points of the set.*

Thus, a point $\mathbf{x}$ of a convex set S is an extreme point of the set, if there does not exist any pair of points $\mathbf{x}_1$, $\mathbf{x}_2$, in S, such that

$$\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2, \quad 0 < \lambda < 1$$

Note. We should note the strict inequality imposed on λ. An extreme point is a boundary point of the set; however, not all boundary points of a convex set are necessarily extreme points.

***Definition 3* (Convex combination of vectors).** *Given a set of vectors* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, *a linear combination*

$$\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_k \mathbf{x}_k$$

is called a ***convex combination*** *of the given vectors, if*

$$\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

The reader may recall that we defined a line segment joining two points $\mathbf{x}_1$ and $\mathbf{x}_2$, as the set of points

$$\{\mathbf{x} \mid \mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2, \ 0 \leq \lambda \leq 1\}.$$

Thus, a line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ is the set of all convex combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$. Also, clearly, a line segment is a set. You may wonder if we ask : given three vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, will the set of all linear combinations of these vectors be a convex set?

Theorem 0-3. *The set of all convex combinations of a finite number of points of* $\mathbf{S} \subset \mathbf{R}^n$ *is a convex set.*

Proof. Let

$$\mathbf{S} = \{\mathbf{x} : \mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i, \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1\}, \quad \mathbf{x}_i \in S$$

We have to show that $\mathbf{S}$ is a convex set.

Let $\mathbf{x}'$ and $\mathbf{x}''$ be in $\mathbf{S}$, so that

$$\mathbf{x}' = \sum_{i=1}^{m} \lambda_i' \mathbf{x}_i, \text{ where } \lambda_i' \geq 0, \ \sum_{i=1}^{m} \lambda_i' = 1$$

and

$$\mathbf{x}'' = \sum_{i=1}^{m} \lambda_i'', \text{ where } \lambda_i'' \geq 0, \ \sum_{i=1}^{m} \lambda_i'' = 1.$$

Consider now the vector

$$\begin{aligned} \mathbf{x} &= \lambda \mathbf{x}' + (1 - \lambda)\mathbf{x}'' \quad 0 \leq \lambda \leq 1 \\ &= \lambda \sum_{i=1}^{m} \lambda_i' \mathbf{x}_i + (1 - \lambda) \sum_{i=1}^{m} \lambda_i'' \mathbf{x}_i \\ &= \sum_{i=1}^{m} [\lambda \lambda_i' + (1 - \lambda)\lambda_i''] \mathbf{x}_i = \sum_{i=1}^{m} \mu_i \mathbf{x}_i \end{aligned}$$

where $\mu_i = \lambda \lambda_i' + (1 - \lambda)\lambda_i''$, $\quad i = 1, 2, \ldots, m$.

Since $0 \leq \lambda \leq 1$, $\lambda_i' \geq 0$, $\lambda_i'' \geq 0$, therefore $\mu_i \geq 0$ for each i.

Also

$$\begin{aligned} \sum_{i=1}^{m} \mu_i &= \sum_{i=1}^{m} [\lambda \lambda_i' + (1 - \lambda)\lambda_i''] = \lambda \sum_{i=1}^{m} \lambda_i' + (1 - \lambda) \sum_{i=1}^{m} \lambda_i'' \\ &= \lambda + (1 - \lambda) = 1. \end{aligned}$$

Hence $\mathbf{x}$ is a convex combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, or $\mathbf{x} \in S$.

Thus for each pair of points $\mathbf{x}'$, $\mathbf{x}''$ in S, the line segment joining them is contained in the set.

Hence S is a convex set.

Definition 4* (Convex polyhedron).** *The set of all convex combinations of a finite number of linearly independent vectors is called a* ***convex polyhedron*.*

The convex polyhedron generated by the finite set of linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, is the set

$$\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i, \ \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1 \}.$$

Definition 5* (Convex cone).** *A non-empty subset $C \subset \mathbf{R}^n$, is called a* ***cone *if for each* $\mathbf{x}$ *in C, and $\lambda \geq 0$, the vector $\lambda\mathbf{x}$ is also in C.*

A cone is called a ***convex cone*** *if it is a convex set.*

Example. If $\mathbf{A}$ be an $m \times n$ matrix, then the set of n-vectors $\mathbf{x}$ satisfying $\mathbf{Ax} > \mathbf{0}$ is a convex cone in $\mathbf{R}^n$. It is a cone, because if $\mathbf{Ax} > \mathbf{0}$, then $\mathbf{A}\lambda\mathbf{x} > \mathbf{0}$ for non-negative λ. It is convex because if $\mathbf{Ax}^{(1)} \geq \mathbf{0}$ and $\mathbf{Ax}^{(2)} \geq \mathbf{0}$, then $\mathbf{A}\,[\lambda\mathbf{x}^{(1)} + (1 - \lambda)\,\mathbf{x}^{(2)}] \geq \mathbf{0}$.

Definition 6* (Convex hull of a set).** *Give a subset $\mathbf{Y} \subset \mathbf{R}^n$, the smallest convex set containing* $\mathbf{Y}$ *is called the* ***convex hull *of* $\mathbf{Y}$*, and is denoted by* $<\mathbf{Y}>$.

It can easily be seen that the convex hull of a set is the intersection of all convex sets containing $\mathbf{Y}$.

Example. Let $\mathbf{A} = \{ \mathbf{x}_1, \mathbf{x}_2 \}$. Then the line segment joining $\mathbf{x}_1$, $\mathbf{x}_2$ is a convex set. Also if $\mathbf{S}$ is any convex set containing $\mathbf{A}$, then it must contain the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$\therefore \quad <\mathbf{A}> = \{\mathbf{x} : \mathbf{x} = \lambda\mathbf{x}_1 + (1 - \lambda)\,\mathbf{x}_2, \ 0 < \lambda < 1\}.$$

Theorem 0-4. *If* $\mathbf{A}$ *is any finite subset of vectors in $\mathbf{R}^n$, then the convex hull of* $\mathbf{A}$ *is the set of all convex combinations of vectors in* $\mathbf{A}$.

Proof. Let $\mathbf{A}$ be a finite subset of vectors in $\mathbf{R}^n$ and $<\mathbf{A}>$ be its convex hull. Let $\mathbf{S}$ be the set of all convex combinations of vectors in $\mathbf{A}$.

Then, we know that $\mathbf{S}$ is a convex set containing $\mathbf{A}$; and hence $<\mathbf{A}> \subset \mathbf{S}$.

Again $<\mathbf{A}>$ contains $\mathbf{A}$. We shall show that this implies that $<\mathbf{A}> \supset \mathbf{S}$. We use finite induction on k, the number of vectors in $\mathbf{A}$, to prove this result.

For $k = 2$, let $\mathbf{x}_1, \mathbf{x}_2$ be two vectors in $\mathbf{A}$. Then $<\mathbf{A}>$ contains $\mathbf{x}_1, \mathbf{x}_2$; and is a convex set.

$$\therefore \quad \mathbf{x} = \lambda\mathbf{x}_1 + (1 - \lambda)\,\mathbf{x}_2 \qquad 0 \leq \lambda \leq 1$$

is also in $<\mathbf{A}>$.

$\therefore$ All possible convex combinations of $\mathbf{x}_1, \mathbf{x}_2$ are in $<\mathbf{A}>$.

Assume that, for any positive integer k, the result is true for a set of at most $k - 1$ vectors.

Consider now the set $\mathbf{A} = \{ \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \}$. Then $<\mathbf{A}>$ is a convex set containing $\mathbf{A}$, and in particular $\{ \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1} \}$. Let

$$X_1 = \{ \mathbf{x} \mid \mathbf{x} = \sum_{i=1}^{k-1} \mu_i \mathbf{x}_i ; \ \mu_i \geq 0, \ \sum_{i=1}^{k-1} \mu_i = 1 \}$$

Thus, by induction hypothesis $<\mathbf{A}> \supset X_1$.

Now $<\mathbf{A}>$ contains X_1, as well as $\{ \mathbf{x}_k \}$.

$\therefore$ $<\mathbf{A}>$ contains all segments joining $\mathbf{x}_k$ to a point of X_1.

or

$$\mathbf{x} = \delta\mathbf{x}_k + (1 - \delta) \sum_{i=1}^{k-1} \mu_i \mathbf{x}_i, \quad \mu_i \geq 0, \ \sum_{i=1}^{k-1} \mu_i = 1,$$

is a point in $<\mathbf{A}>$.

Now

$$\mathbf{x} = \delta\mathbf{x}_k + (1 - \delta) \sum_{i=1}^{k-1} \mu_i \mathbf{x}_i \quad \Rightarrow \quad \mathbf{x} = \sum_{i=1}^{k-1} \beta_i \mathbf{x}_i,$$

where $\beta_i = (1 - \delta)\,\mu_i$, for $i = 1, 2, \ldots, k - 1$ and $\beta_k = \delta$.

Since, $\mu_i \geq 0$ for each i, and $0 \leq \delta \leq 1$, $\therefore$ $\beta_i \geq 0$ for $j = 1, 2, \ldots, k$.

Also $$\sum_{j=1}^{n} \beta_j = \delta + (1 - \delta) \sum_{i=1}^{k-1} \mu_i = 1.$$

$\therefore$ $\mathbf{x} = \sum_{i=1}^{n} \beta_i \mathbf{x}_i$ is a convex combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ and is in $<\mathbf{A}>$, *i.e.*, $<\mathbf{A}> \supset \mathbf{S}$.

We conclude that $<\mathbf{A}> = \mathbf{S}$.

In view of the above theorem, the convex polyhedron may equivalently be defined as the convex hull of a set point **S**, consisting of a finite number of points in $\mathbf{R}^n$.

***Definition 7* (Simplex).** *A **simplex** in n-dimension is a convex polyhedron having exactly $(n + 1)$ vertices.*

A simplex in zero dimension is a point, in one dimension it is a line segment, in two dimensions it is a triangle, in three dimensions it is a tetrahydron, and so on.

0:14. SUPPORTING AND SEPARATING HYPERPLANES

***Definition* (Supporting hyperplane).** *Let $\mathbf{S} \subset \mathbf{R}^n$ be any closed convex set and $\mathbf{w} \in \mathbf{S}$ be a boundary point. Then a hyperplane $\mathbf{cx} = z$ is called a **supporting hyperplane** of **S** at **w**, if*

(i) $\mathbf{cw} = z$ *and* *(ii)* $\mathbf{S} \subset \mathbf{H}_+$ *or* $\mathbf{S} \subset \mathbf{H}_-$

where $\mathbf{H}_+ = \{\mathbf{x} : \mathbf{cx} \geq z\}$ *and* $\mathbf{H}_- = \{\mathbf{x} : \mathbf{cx} \leq z\}$.

Remarks 1. The supporting hyperplane need not be unique.

2. *S* may intersect the supporting hyperplane in more than one boundary points.

Theorem 0-5. *Let **S** be a closed convex set which is bounded from below. Then **S** has extreme points in every supporting hyperplane.*

Proof. Let **w** be a boundary point of a closed convex set **S**.

Let $\mathbf{cx} = z$ be a supporting hyperplane at $\mathbf{w} \in \mathbf{S}$. Let $\mathbf{B} = \mathbf{S} \cap \{\mathbf{x} \mid \mathbf{cx} = z\}$.

Then **B** is a closed convex set and $\mathbf{B} \neq \varphi$, for $\mathbf{w} \in \mathbf{B}$.

We claim that every extreme point of **B** is also an extreme point of **S**.

Let us assume to the contrary that an extreme point **b** of **B**, is not an extreme point of **S**. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{S}$, such that

$$\mathbf{b} = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \qquad 0 < \lambda < 1$$

$$\therefore \qquad \mathbf{cb} = \lambda \mathbf{cx}_1 + (1 - \lambda) \mathbf{cx}_2. \qquad \ldots(i)$$

Since $\mathbf{cx} = z$ is a supporting hyperplane at **w** and $\mathbf{x}_1, \mathbf{x}_2 \in X$,

$$\mathbf{cx}_1 \leq z \quad \text{and} \quad \mathbf{cx}_2 \leq z$$

or $$\mathbf{cx}_1 \geq z \quad \text{and} \quad \mathbf{cx}_2 \geq z. \qquad \ldots(ii)$$

From *(i)* and *(ii)*

$$\mathbf{cb} \leq \lambda z + (1 - \lambda) z = z \quad \text{or} \quad \mathbf{cb} \geq \lambda z + (1 - \lambda) z = z$$

$\therefore$ **b** is not a point of **B**.

This is a contradiction.

$\therefore$ Every extreme point of **B** is also an extreme point of **S**.

***Theorem 0-6* (Separating hyperplane).** *Let $\mathbf{S} \subset \mathbf{R}^n$ be a closed convex set. Then for any point **y** not is **S**, there is a hyperplane containing **y** such that **S** is contained in one of the open half spaces determined by the hyperplane.*

Proof. We are given that $\mathbf{y} \notin \mathbf{S}$.

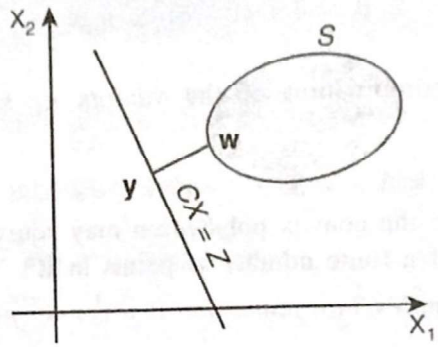

Fig. 0.3. Separating hyperplane

Since, **S** is a closed set, there exist $\mathbf{w} \subset \mathbf{S}$, such that

$$|\mathbf{w} - \mathbf{y}| = \min_{\mathbf{x} \in S} |\mathbf{x} - \mathbf{y}| \qquad ...(i)$$

Let **u** be any point of **S**. Then, since **S** is a convex set

$$[\lambda \mathbf{u} + (1 - \lambda)\mathbf{w}] \in \mathbf{S} \quad \text{for } 0 \le \lambda \le 1 \qquad ...(ii)$$

From (*i*) and (*ii*)

$$|\lambda \mathbf{u} + (1 - \lambda)\mathbf{w} - \mathbf{y}| \ge |\mathbf{w} - \mathbf{y}|$$

$$\Rightarrow \quad |(\mathbf{w} - \mathbf{y}) + \lambda(\mathbf{u} - \mathbf{w})|^2 \ge |\mathbf{w} - \mathbf{y}|^2$$

$$\Rightarrow \quad \lambda^2 |\mathbf{u} - \mathbf{w}|^2 + |\mathbf{w} - \mathbf{y}|^2 + 2\lambda(\mathbf{w} - \mathbf{y})(\mathbf{u} - \mathbf{w}) \ge |\mathbf{w} - \mathbf{y}|^2$$

$$\Rightarrow \quad \lambda^2 |\mathbf{u} - \mathbf{w}|^2 + 2\lambda(\mathbf{w} - \mathbf{y})(\mathbf{u} - \mathbf{w}) \ge 0.$$

Letting $\lambda \to 0$, and $\mathbf{c} = (\mathbf{w} - \mathbf{y})$ such that $|\mathbf{c}| \ne 0$; we get

$$(\mathbf{w} - \mathbf{y})(\mathbf{u} - \mathbf{w}) \ge 0 \quad \text{or} \quad \mathbf{c}(\mathbf{u} - \mathbf{w}) \ge 0$$

or

$$\mathbf{c}(\mathbf{u} - \mathbf{y}) \ge \mathbf{c}(\mathbf{w} - \mathbf{y}) = |\mathbf{c}|^2$$

$$\therefore \quad \mathbf{cu} > \mathbf{cy}.$$

Setting $\mathbf{cy} = z$, we find that $\mathbf{cu} > z$.

Thus **y** lies on the hyperplane $\mathbf{cx} = z$ and for all $\mathbf{u} \in \mathbf{S}$, $\mathbf{cu} \ge z$.

This completes the proof.

PROBLEMS

013. Given the hyperplane $3x_1 + 2x_2 + 4x_3 + 6x_4 = 7$. Find in which half space is the point $\mathbf{x} = \{6, 1, 7, 2\}$.

014. Prove that the set $\mathbf{x} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 4\}$ is a convex set.

015. Examine whether the following set is convex or not :

(*i*) $S = \{(x_1, x_2) : x_1 x_2 \ge 1,\ x_1 \ge 0,\ x_2 \ge 0\}$

(*ii*) $S = \{(x_1, x_2) : 5x_1 + 2x_2 \ge 10,\ 2x_1 + 5x_2 \ge 10\}$

(*iii*) $S = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1,\ x_1 + x_2 \ge 1\}$

016. Show that $S = \{(x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4\} \subset \mathbf{R}^3$ is a convex set.

017. Show that in $\mathbf{R}^3$, the closed ball $x_1^2 + x_2^2 + x_3^2 \le 1$ is a convex set.

018. Let $S = \{(x_1, x_2) : x_1 - x_2 \le 4,\ x_1 + x_2 \ge -3,\ x_2 \le 8\}$.

Find all the extreme points of S and represent $\mathbf{x} = (2, 1)$ as convex combination of the extreme points.

019. Determine the convex hull of the following sets :

(*i*) $A = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$

(*ii*) $A = \{(x_1, x_2)\}$.

020. Graph the convex hull of the points

$$(0, 0),\ (0, 1),\ (1, 2),\ (1, 1),\ (4, 0).$$

Which of these points is an interior point of the convex hull? Express it as a convex combination of the extreme points.

0:15. CONVEX FUNCTIONS

Definition 1* (Convex function).** *Let* **S** *be a non-empty convex subset of* $\mathbf{R}^n$. *A function* $f(\mathbf{x})$ *on* **S** *is said to be* ***convex *if for any two vectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *in* **S**

$$f[\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2] \le \lambda f(\mathbf{x}_1) + (1 - \lambda) f(\mathbf{x}_2), \quad 0 \le \lambda \le 1$$

Definition 2* (Strictly convex function).** *Let* **S** *be a non-empty convex subset of* $\mathbf{R}^n$. *A function* $f(\mathbf{x})$ *on* **S** *is said to be* ***strictly convex *if for any two different vectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *in* **S**

$$f[\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2] < \lambda f(\mathbf{x}_1) + (1 - \lambda) f(\mathbf{x}_2), \quad 0 < \lambda < 1$$

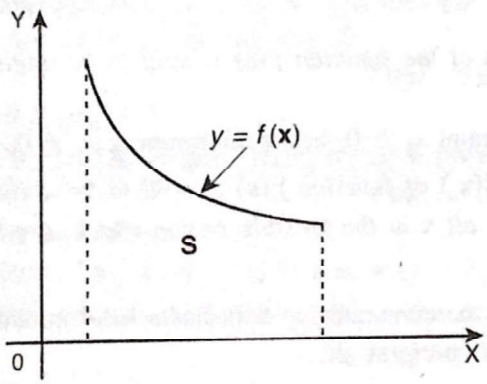

Fig. 0.4. Strictly convex function

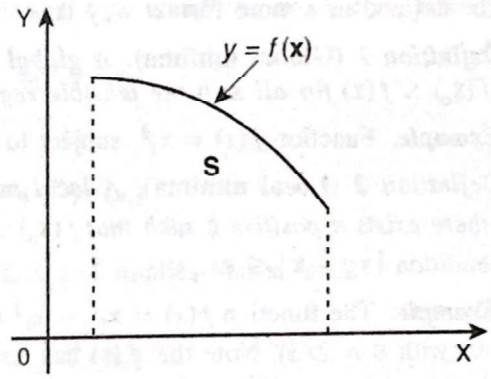

Fig. 0.5. Strictly concave function

It follows from the above two definitions that every strictly convex function is also convex. The graph of a strictly convex function has been illustrated in Fig. 0.4.

***Definition 3* [Concave (strictly concave) function].** *A function* $f(\mathbf{x})$ *on a non-empty subset S of* $\mathbf{R}^n$ *is said to be concave (strictly concave) if* $-f(\mathbf{x})$ *is convex (strictly convex).*

Clearly, every strictly concave function is also concave. The graph of a strictly concave function has been illustrated in Fig. 0.5.

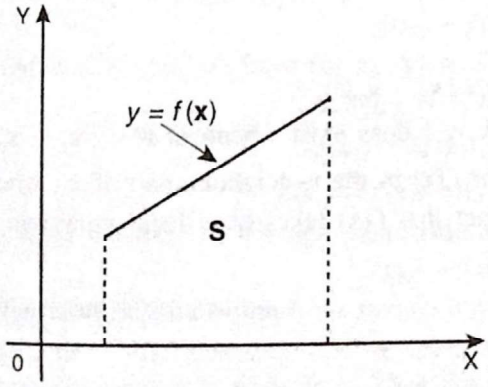

Fig. 0.6. Both convex and concave functions

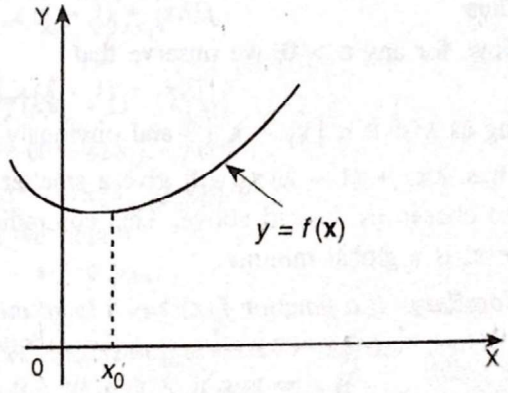

Fig. 0.7

It is possible for a function to be both convex and concave. For example, $f(\mathbf{x}) = \mathbf{x}$ is such a function (Fig. 0.6). The function in Fig. 0.7 is strictly convex for $\mathbf{x} \ge \mathbf{x}_0$ but not strictly convex for $\mathbf{x} < \mathbf{x}_0$.

The following results are the immediate consequences of the above definitions :

(i) *A linear function* $z = \mathbf{c}\,\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^n$ *is a convex (concave) function but not strictly convex (concave).*

(ii) *The sum of convex (concave) functions is convex (concave) and if at least one of the functions is strictly convex (concave) then so is their sum.*

Note. In what follows we shall deal with convex functions only. However, all the results remain valid if we deal with concave functions instead.

0:16. LOCAL AND GLOBAL EXTREMA

In the problems of constrained optimization, one is interested in determining a vector $\mathbf{x}$ that minimises the function $f(\mathbf{x})$ [or maximises $-f(\mathbf{x})$] subject to the 'constraints' $g_i(\mathbf{x}) \le 0$ $(i = 1, 2, \ldots, m)$. The set of the vectors $\mathbf{x}$ satisfying these constraints is usually called the 'feasible region'. These terms shall be defined in a more formal way later.

Definition 1* (Global minima).** *A* ***global minimum *of the function* $f(\mathbf{x})$ *is said to be attained at* $\mathbf{x}_o$ *if* $f(\mathbf{x}_o) \le f(\mathbf{x})$ *for all* $\mathbf{x}$ *in the feasible region.*

Example. Function $f(x) = x_1^2$, subject to the constraint $x_1 \ge 0$, has a minimum at $x_1 = 0$.

Definition 2* (Local minima).** *A* ***local minimum $f(\mathbf{x}_o)$ *of function* $f(\mathbf{x})$ *is said to be attained at* $\mathbf{x}_o$ *if there exists a positive* ε *such that* $f(\mathbf{x}_o) \le f(\mathbf{x})$ *for all* $\mathbf{x}$ *in the feasible region which also satisfy the condition* $|\mathbf{x}_o - \mathbf{x}| \le \varepsilon$.

Example. The function $f(x) = x_1^2 - x_1^3$ subject to the constraint $x_1 \ge 0$, has a local minimum at $x_1 = 0$ (with $\varepsilon = 2/3$). Note that $f(x)$ has no global minimum at all.

Note. The word extremum (plural extrema) is used to indicate either maximum or minimum.

Theorem 0-7. *Let* $f(\mathbf{x})$ *be a convex function of a convex set S. If* $f(\mathbf{x})$ *has a local minimum on S, then this local minimum is also a global minimum on S.*

Proof. Let a local minimum $f(\mathbf{x}_o)$ be attained at $\mathbf{x}_o$. Then, there exists at least one $\mathbf{x}_1$ in S $(\mathbf{x}_1 \ne \mathbf{x}_o)$ such that $f(\mathbf{x}_1) < f(\mathbf{x}_o)$. Since, $f(\mathbf{x})$ is a convex function on S, we have

$$f[\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_o] \le \lambda f(\mathbf{x}_1) + (1 - \lambda) f(\mathbf{x}_o)$$

Also

$$\lambda f(\mathbf{x}_1) + (1 - \lambda) f(\mathbf{x}_o) < \lambda f(\mathbf{x}_o) + (1 - \lambda) f(\mathbf{x}_o) = f(\mathbf{x}_o)$$

Thus

$$f[\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_o] \le f(\mathbf{x}_o).$$

Now, for any $\varepsilon > 0$, we observe that

$$|[\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_o] - \mathbf{x}_o| = \lambda |\mathbf{x}_1 - \mathbf{x}_o| < \varepsilon,$$

so long as $\lambda < \varepsilon < |\mathbf{x}_1 - \mathbf{x}_o|^{-1}$ and obviously such a $\lambda < 1$ does exist whenever $\varepsilon < |\mathbf{x}_1 - \mathbf{x}_o|$.

Thus, $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_o$ will give a smaller value for $f(\mathbf{x})$ in the ε–neigbourhood of $\mathbf{x}_o$, whenever λ is so chosen as desired above. This contradicts the fact that $f(\mathbf{x})$ takes on a local minimum at $\mathbf{x}_o$. Hence $\mathbf{x}_o$ is a global minima.

Corollary. *If a function* $f(\mathbf{x})$ *has a local minimum on a convex set S on which it is strictly convex, then this local minimum is also a global minimum on that set. This global minimum is attained at a single point.*

Proof. Exercise for the reader.

Theorem 0-8. *Let* $f(\mathbf{x})$ *be a convex function on a convex set S. Then the set of points in S at which* $f(\mathbf{x})$ *takes on its global minimum, is a convex set.*

Proof. The result is obvious if the global-minimum is attained at just a single point. Let us assume that the global minimum is attained at two different points $\mathbf{x}_1$ and $\mathbf{x}_2$ of S. Then $f(\mathbf{x}_1) = f(\mathbf{x}_2)$.

Now, since $f(\mathbf{x})$ is convex,

$$f[\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1] \le \lambda f(\mathbf{x}_2) + (1 - \lambda) f(\mathbf{x}_1) = f(\mathbf{x}_2) \qquad 0 \le \lambda \le 1$$

$$\Rightarrow \qquad f[\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1] \le f(\mathbf{x}_2) = f(\mathbf{x}_1).$$

$$\therefore \qquad f[\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1] \le f(\mathbf{x}_1).$$

Thus, every point $\mathbf{x} = \lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1$ corresponds to a global minima. The set of all such $\mathbf{x}$ is, obviously, a convex set.

Corollary. *If the global minimum is attainable at two different points of S, then it is attainable at an infinite number of points of S.*

Proof. Exercise for the reader.

Theorem 0-9. *Let $f(\mathbf{x})$ be differentiable in its domain. If $f(\mathbf{x})$ is defined on an open convex set S, then $f(\mathbf{x})$ is convex if and only if*

$$f(\mathbf{x}_2) - f(\mathbf{x}_1) \ge (\mathbf{x}_2 - \mathbf{x}_1)^T \nabla f(\mathbf{x}_1)$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in S$.

Proof. (*Sufficiency*). Here, we shall prove that if

$$f(\mathbf{x}_2) - f(\mathbf{x}_1) \ge (\mathbf{x}_2 - \mathbf{x}_1)^T \nabla f(\mathbf{x}_1)$$

then $f(\mathbf{x})$ is convex.

Since $\mathbf{x}_1, \mathbf{x}_2 \in S$, $\mathbf{x}_o = \lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1$ for $0 \le \lambda \le 1$ implies that $\mathbf{x}_o \in S$.

Using the above condition for $\mathbf{x}_1$ and $\mathbf{x}_o$, we have

$$f(\mathbf{x}_1) - f(\mathbf{x}_o) \ge (\mathbf{x}_1 - \mathbf{x}_o)^T \nabla f(\mathbf{x}_o). \qquad \text{...}(i)$$

Similarly, for $\mathbf{x}_2$ and $\mathbf{x}_o$,

$$f(\mathbf{x}_2) - f(\mathbf{x}_o) \ge (\mathbf{x}_2 - \mathbf{x}_o)^T \nabla f(\mathbf{x}_o). \qquad \text{...}(ii)$$

Multiplying (*ii*) by λ and (*i*) by $(1 - \lambda)$ and then adding, we get

$$\lambda f(\mathbf{x}_2) + (1 - \lambda) f(\mathbf{x}_1) \ge f(\mathbf{x}_o) \ge f(\mathbf{x}) + [\lambda \mathbf{x}_2^T + (1 - \lambda)\mathbf{x}_1^T] \nabla f(\mathbf{x}_o) - \mathbf{x}_o^T \nabla f(\mathbf{x}_o)$$

Using the definition of $\mathbf{x}_o$, this yields

$$\lambda f(\mathbf{x}_2) + (1 - \lambda) f(\mathbf{x}_1) \ge f[\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1]$$

which implies that $f(\mathbf{x})$ is convex.

(*Necessity*). We now prove that if $f(\mathbf{x})$ is convex, we must have

$$f(\mathbf{x}_2) - f(\mathbf{x}_1) \ge (\mathbf{x}_2 - \mathbf{x}_1)^T \nabla f(\mathbf{x}_1)$$

For $0 \le \lambda \le 1$, we have for $\mathbf{x}_1, \mathbf{x}_2 \in S$ that

$$\lambda f(\mathbf{x}_2) + (1 - \lambda) f(\mathbf{x}_1) \ge f[\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1]$$

i.e.,

$$\lambda f(\mathbf{x}_2) - \lambda f(\mathbf{x}_1) \ge f[\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1] - f(\mathbf{x}_1)$$

or

$$f(\mathbf{x}_2) - f(\mathbf{x}_1) \ge [f\{\lambda \mathbf{x}_2 + (1 - \lambda)\mathbf{x}_1\} - f(\mathbf{x}_1)]/\lambda.$$

Taking the limit of the right hand side as $\lambda \to 0$, we obtain

$$f(\mathbf{x}_2) - f(\mathbf{x}_1) \ge (\mathbf{x}_2 - \mathbf{x}_1)^T \nabla f(\mathbf{x}_1).$$

This completes the proof.

Theorem 0-10. *Let $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$ be a convex continuously differentiable function defined on an open convex set S. Let $\mathbf{x}_o$ be in S. Then $f(\mathbf{x}_o) \le f(\mathbf{x})$ for all $\mathbf{x}$ in S, if and only if*

$$(\mathbf{x} - \mathbf{x}_o)^T \nabla f(\mathbf{x}_o) \ge 0.$$

Proof. (*Necessary condition*). Let $f(\mathbf{x})$ be a convex function on a convex set S, and let $\mathbf{x}_o \in S$ such that $f(\mathbf{x}_o) \le f(\mathbf{x})$ for all $\mathbf{x} \in S$. Since S is open, the point $\mathbf{x}_o$ is an interior point of S.

$$\therefore \quad f(\mathbf{x}_o) \le f[\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}_o]$$

for all $\mathbf{x}$ in S and $0 \le \lambda \le 1$.

For $\lambda > 0$,

$$\frac{f[\mathbf{x}_o + \lambda(\mathbf{x} - \mathbf{x}_o)] - f(\mathbf{x}_o)}{\lambda} \ge 0.$$

Taking limit as λ approaches zero, we get

$$(\mathbf{x} - \mathbf{x}_o)^T \nabla f(\mathbf{x}_o) \ge 0.$$

(*Sufficient condition*). Given that $(\mathbf{x} - \mathbf{x}_o)^T \nabla f(\mathbf{x}_o) \ge 0$ for all $\mathbf{x}$ in S, by convexity of $f(\mathbf{x})$, we have

$$f(\mathbf{x}) - f(\mathbf{x}_o) \ge (\mathbf{x} - \mathbf{x}_o)^T \nabla f(\mathbf{x}_o) \ge 0.$$

Hence, $f(\mathbf{x}) \ge f(\mathbf{x}_o)$ for all $\mathbf{x}$ in and $\mathbf{x}_o$ minimizes $f(\mathbf{x})$ over S.

Remark. From a geometric point of view, the above theorem states that $\mathbf{x}_o$ is a minimum point if the angle between the vectors $\nabla f(\mathbf{x}_o)$ and $(\mathbf{x} - \mathbf{x}_o)^T$ for all $\mathbf{x}$ in S is less than or equal to $\pi/2$.

0:17. QUADRATIC FORMS

***Definition 1* (Quadratic form).** *Let $\mathbf{A} = (a_{ij})$ be an n-rowed square matrix. Then given an $n \times 1$ matrix $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, the function*

$$\mathbf{Q}(\mathbf{x}) = \mathbf{x}^T \mathbf{A}\mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

is called a quadratic form over the domain of all $n \times 1$ matrices $\mathbf{x}$.

Example. The quadratic form

$$\mathbf{Q}(\mathbf{x}) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 6x_1x_3 - 5x_2x_3$$

can be written as

$$[x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 1 & -2 & 3 \\ -2 & 2 & -5/2 \\ 3 & -5/2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Note. The matrix $\mathbf{A}$ can always be assumed symmetric, since each element of every pair of coefficients a_{ij} and a_{ji} $(i \ne j)$ can be replaced by $(a_{ij} + a_{ji})/2$ without changing the value of $\mathbf{Q}(\mathbf{x})$.

Properties of a Quadratic Form

(*i*) *Positive definite.* The quadratic form $\mathbf{Q}(\mathbf{x})$ is positive definite if and only if $\mathbf{Q}(\mathbf{x}) > 0$ (positive) for all $\mathbf{x} \ne \mathbf{0}$.

For example, $\mathbf{Q}(\mathbf{x}) = x_1^2 + 5x_2^2 + 7x_3^2$ is positive definite as it is positive for all values of x_1, x_2, x_3 except when $x_1 = x_2 = x_3 = 0$.

(*ii*) *Positive semi-definite.* The quadratic form $\mathbf{Q}(\mathbf{x})$ is positive semi-definite if and only if $\mathbf{Q}(\mathbf{x}) \ge 0$ (non-negative) for all $\mathbf{x}$ and there exist at least one non-zero vector $\mathbf{x} \ne \mathbf{0}$ for which $\mathbf{Q}(\mathbf{x}) = 0$.

For example, $\mathbf{Q}(\mathbf{x}) = (x_1 - x_2)^2 + 2x_3^2$ is positive semi-definite as it is zero for $x_1 = x_2$ and $x_3 = 0$ and positive for all values of x_1, x_2, x_3 except when $x_1 = x_2 = x_3 = 0$.

(*iii*) *Negative definite.* The quadratic form $\mathbf{Q}(\mathbf{x})$ is negative definite, if and only if, $-\mathbf{Q}(\mathbf{x})$ is positive definite. That is, $\mathbf{Q}(\mathbf{x})$ is negative definite when $\mathbf{Q}(\mathbf{x})$ is negative when $\mathbf{x} \ne \mathbf{0}$.

(*iv*) *Negative semi-definite.* The quadratic form $\mathbf{Q}(\mathbf{x})$ is negative semi-definite, if and only if, $-\mathbf{Q}(\mathbf{x})$ is positive semi-definite. That is, $\mathbf{Q}(\mathbf{x})$ is negative semi-definite when $\mathbf{Q}(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ and zero for at least one $\mathbf{x} \neq \mathbf{0}$.

For example, if $\mathbf{Q}(\mathbf{x}) = -x_1^2 + 2x_1x_2 - x_2^2 = -(x_1^2 - 2x_1x_2 + x_2^2) = -(x_1 - x_2)^2$, then $\mathbf{Q}(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ but zero for $x_1 = x_2$.

(*v*) *Indefinite.* The quadratic form $\mathbf{Q}(\mathbf{x})$ is indefinite if $\mathbf{Q}(\mathbf{x})$ is positive for some $\mathbf{x}$ and negative for some other $\mathbf{x}$.

For example, $\mathbf{Q}(\mathbf{x}) = x_1^2 - 3x_2$ can be positive for $x_1 \neq 0$ and $x_2 = 0$ and negative for $x_1 = 0$ and $x_2 \neq 0$.

PROBLEMS

021. Show that the following functions are convex over $\mathbf{R}^1$:

(*i*) $f(\mathbf{x}) = 3\mathbf{x}^2$ (*ii*) $f(\mathbf{x}) = e^x$ and (*iii*) $f(\mathbf{x}) = |\mathbf{x}|$.

022. Show that the function $f(\mathbf{x}) = 2x_1^2 + x_2^2$ is a convex function over all of $\mathbf{R}^2$.

023. Write the following quadratic forms in the form $\mathbf{x}^T\mathbf{A}\mathbf{x}$

(*i*) $x_1^2 + 8x_1x_2 + 16x_2^2 - 3x_3^2$ (*ii*) $2x_1^2 - 6x_1x_2 + 2x_1x_3 + 2x_3^2 + 6x_2x_3 - 5x_3^2$

024. Write down the quadratic forms whose associated matrices are :

(*i*) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & 2 \\ 1 & 2 & -6 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 14 \end{bmatrix}$

025. Determine whether the quadratic form

$$x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$$

is positive definite.

026. Determine whether each of the following quadratic forms is positive definite or negative definite :

(*a*) $2x_1^2 + 6x_2^2 - 6x_1x_2$, and (*b*) $-x_1^2 - x_2^2 - 4x_3^2 + x_1x_2 - 2x_2x_3$.

MULTIPLE CHOICE QUESTIONS

1. The matrix $\mathbf{A} + \mathbf{B}$ is defined, if

(*a*) the matrix $\mathbf{A}$ has 3 rows and matrix $\mathbf{B}$ has 2 rows.
(*b*) the matrix $\mathbf{A}$ has 2 columns and matrix $\mathbf{B}$ has 4 columns.
(*c*) the matrix $\mathbf{A}$ has 3 rows and the matrix $\mathbf{B}$ has 4 columns.
(*d*) both $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order.

2. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. The matrix $\begin{pmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{pmatrix}$ is equal to

(*a*) $3\mathbf{A} - 2\mathbf{B}$, (*b*) $3\mathbf{B} - 2\mathbf{A}$, (*c*) $2\mathbf{A} + 3\mathbf{B}$, (*d*) none of these.

3. The matrix $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a

(*a*) diagonal matrix, (*b*) scalar matrix, (*c*) unit matrix, (*d*) none of these.

4. For $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(*a*) $\mathbf{AB} = \mathbf{BA}$, (*b*) $\mathbf{AB} \neq \mathbf{BA}$, (*c*) $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$, (*d*) none of these.

5. A is any square matrix, then $\mathbf{A} + \mathbf{A}^T$ will be a
   (*a*) diagonal matrix, (*b*) symmetric matrix, (*c*) skew-symmetric matrix, (*d*) identity matrix.
6. If **A** and **B** are invertible square matrices of the same order, then
   (*a*) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, (*b*) $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$,
   (*c*) $(\mathbf{A}^T\mathbf{B}^T)^{-1} = (\mathbf{B}^{-1})^T(\mathbf{A}^{-1})^T$, (*d*) none of these.
7. The inverse of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is
   (*a*) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, (*c*) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, (*d*) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
8. The system of simultaneous linear equations : $2x + 3y + 4 = 0$, $3x + 4y + 6 = 0$, and $4x + 5y + 8 = 0$ is
   (*a*) consistent, (*b*) inconsistent, (*c*) none of these.
9. Which of the following statements is not true?
   (*a*) A row (column) matrix is called a row (column) vector.
   (*b*) The terms a 'point' in $\mathbf{R}^n$ and a 'vector' in $\mathbf{R}^n$ are used interchangeably meaning the same thing.
   (*c*) A row (column) vector cannot be added to any other row (column) vector.
   (*d*) Two vectors are order comparable, if and only if, they have an equal number of components.
10. Select the wrong statement:
   (*a*) If a set of vectors is linearly independent, then, any subset of it is linearly dependent.
   (*b*) The set of vectors $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ is called the standard basis of $\mathbf{R}^n$.
   (*c*) If $c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$, then the hyperplane is said to pass through the origin.
   (*d*) The intersection of a finite number of closed half spaces is called a polytope.
11. The set $S = \{ (x_1, x_2) : x_1x_2 \geq 1 ;\ x_1 \geq 0 \text{ and } x_2 \geq 0 \}$ is
   (*a*) convex, (*b*) not convex, (*c*) concave.
12. Which of the following statements is not correct?
   (*a*) A cone is called a convex cone if it is a convex set.
   (*b*) The smallest convex set containing **Y** is called the convex hull of **Y**.
   (*c*) A simplex in n-dimension is a convex polyhedron having n vertices.
   (*d*) The supporting hyperplane need not be unique.
13. Choose the correct statement:
   (*a*) A supporting hyperplane must be unique.
   (*b*) A simplex in two dimensions is a triangle.
   (*c*) A function cannot be both convex and concave.
   (*d*) Every strictly convex function may be concave.
14. Identify the incorrect statement : A quadratic form $\mathbf{Q(x)}$ is:
   (*a*) positive definite if and only if $\mathbf{Q(x)} > 0$.
   (*b*) negative definite if and only if $\mathbf{Q(x)} < 0$.
   (*c*) indefinite if $\mathbf{Q(x)} > 0$ for some **x** and $\mathbf{Q(x)} < 0$ for some othr **x**.
   (*d*) positive definite as well negative definite irrespective of sign of $\mathbf{Q(x)}$.
15. The quadratic form with the associated matrix $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -1 & 14 \end{pmatrix}$ is :
   (*a*) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$,
   (*b*) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_1x_3 + 8x_2x_3 - 4x_1x_2$,
   (*c*) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_2x_3 + 8x_1x_3 - 4x_1x_2$,
   (*d*) $x_1^2 + 6x_2^2 + 14x_3^2 + 8x_1x_2 + 4x_1x_3 - 4x_2x_3$.

State whether the following statements are 'True' or 'False' :

16. Matrix addition is both commutative and associative.
17. Non-square matrices are not invertible.
18. Inverse of a square matrix exists provided it is singular.
19. Two matrices **A** and **B** are said to anti-commute, if **AB** = –**BA**.
20. The solution of the system of linear non-homogeneous equations :
$$5x + 2y = 1,\ 7x + 2y = -1 \text{ is } x = 3 \text{ and } y = -7.$$
21. The set of n unit vectors forms a basis for the euclidean n space.
22. The set of all convex combination of a finite number of points of $\mathbf{S} \subset \mathbf{R}^n$ is a convex set.
23. Convex set may intersect the supporting hyperplane in more than one boundary points.
24. The set of all convex combination of a finite number of linearly independent vectors is called a convex cone.
25. A sub-set of convex cone is called the convex hull of a set.
26. It is possible for a function to be both convex and concave.
27. The sum of concave (convex) functions is concave (convex).
28. A global minimum of the function $f(\mathbf{x})$ is said to be attained at $\mathbf{x}_o$ if $f(\mathbf{x}) \geq f(\mathbf{x})$ for all **x** in the feasible region.
29. The quadratic form $\mathbf{Q}(\mathbf{x})$ is indefinite if $\mathbf{Q}(\mathbf{x})$ is positive for some **x** and negative for some other **x**.
30. The quadratic form $\mathbf{x}^T\mathbf{c}\,\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^n$ is convex, if and only if, the symmetric matrix **c** is positive semi-definite.

ANSWERS

1. (*d*)	2. (*b*)	3. (*a*)	4. (*b*)	5. (*b*)
6. (*a*), (*b*) and (*c*)	7. (*c*)	8. (*a*)	9. (*c*)	10. (*a*)
11. (*a*)	12. (*c*)	13. (*b*)	14. (*d*)	15. (*a*)
16. T	17. T	18. F	19. T	20. F
21. T	22. T	23. T	24. F	25. F
26. T	27. T	28. F	29. T	30. T.

REVIEW QUESTIONS

1. If **A** and **B** are two matrices such that **AB** and **A** + **B** are both defined, show that **A**, **B** are square matrices of the same order.
2. If **A** is any $m \times n$ matrix such that **AB** and **BA** are both defined, show that **B** is an $n \times m$ matrix.
3. When is a square matrix said to be non-singular?
4. Given a system of equations, what are the conditions for the existence of a solution to this system of equations?
5. Show that a system of m linear equations in n unknowns will be consistent if and only if the coefficient matrix and the augmented matrix have the same rank.
6. Prove that the set of all convex combinations of a finite number of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a convex set.
7. (*a*) Define and explain the terms : (*i*) Convex set ; and (*ii*) Hyperplane.
   (*b*) Show that the intersection of two convex sets is also a convex set.
8. When is a set $\mathbf{K} \subset \mathbf{B}^n$ said to be convex? Show that for a set **K** to be convex it is necessary and sufficient that every convex linear combination of points in **K** belongs to **K**. What is the role of the theory of convex sets in the solution of the linear programming problems?
9. Let **X** be a closed convex set and $\mathbf{y} \notin \mathbf{X}$ a given point. Show that there exist a hyperplane which contains **y** such that all of **x** is contained in one open half space produced by that hyperplane.

10. (*a*) Define a convex function and prove that a positive linear combination of convex functions is again convex.

(*b*) If $F(\mathbf{x})$ is a convex function (differentiable) of $\mathbf{x} \in \mathbf{R}^n$, with continuous gradient vector $g(\mathbf{x})$, then prove that

$$F(\mathbf{x}_2) - F(\mathbf{x}_1) \geq (\mathbf{x}_2 - \mathbf{x}_1)^T g(\mathbf{x})$$

for any two vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$ where T denotes the transposition.

(*c*) Show that the quadratic form $\mathbf{x}^T \mathbf{c}\mathbf{x}$, for $\mathbf{x} \in \mathbf{R}^n$ is convex, if and only if the symmetric matrix $\mathbf{c}$ is positive semi-definite.

1

Operations Research—An Overview

"Ideas illuminate, while interest only befog"

1:1. INTRODUCTION

The activity "Operations Research" (abbreviated as O.R.) has become increasingly important in the face of fast moving technology and increasing complexities in business and industry. This chapter starts with the historical development of Operations Research and provides an overall view of the subject. It covers some general ideas on the subject, thus providing a perspective.

1:2. ORIGIN AND DEVELOPMENT OF O.R.

No science has ever been born on a specific day. Operations research is no exception. Its roots are as old as science and society. Though the roots of O.R. extend to even early 1800's. It was in 1917, when A.K. Erlang, a Danish mathematician, published his work on the problem of congestion of telephone traffic. The difficulty was that during busy periods, telephone operators were unable to handle the calls the moment they were made, resulting in delayed calls. A few years after its appearance, his work was accepted by the British Post Office as the basis for calculating circuit facilities. During the 1930's, H.C. Levinson applied scientific analysis to the problems of merchandising. His work included scientific study of customers' buying habits, response to advertising and relation of environment to the type of article sold.

The term, *Operational Research*, was first coined in 1940 by McClosky and Trefthen in a small town, Bowdsey, of the United Kingdom. This new science came into existence as a result of research on military operations during World War II. During the war there were strategic and tactical problems which were greatly complicated, to expect adequate solutions from individuals or specialists was unrealistic. Therefore, the military management called on scientists from various disciplines and organised them into teams to assist in solving strategic and tactical problems, *i.e.*, to discuss, evolve and suggest ways and means to improve the execution of various military projects. By their joint efforts, experience and deliberations, they suggested certain approaches that showed remarkable results. This new approach to systematic and scientific study of the operations of the system was called the ***Operations Research*** or ***Operational Research*** (abbreviated as O.R.).

Following the end of World War II, the success of military teams attracted the attention of industrial managers of U.K. who were seeking solutions to their complex executive type problems. It was becoming apparent that these were basically the same problems but in a different context. Thus, keeping in view the critical economic situation which required drastic increase in production efficiency, O.R. activities were diverted from military to civil government and industries. In this way O.R. began to creep into business and industry. By 1948 it had taken good hold in U.K. and was in the process of achieving the same in the United States. It was only in the early 1950's that the

industries in U.S.A. realized the importance of this new science in solving their management problems. Since then, industrial O.R. developed rapidly in U.S.A. as compared to U.K.

The dramatic development and refinement of the various techniques of O.R. and the advent of digital computers are the two prime factors that have contributed to the growth and application of O.R. in the post-war period. In the 1950's O.R. achieved recognition as a subject for study in the universities. Since then the subject has gained more and more importance for the students of management, public administration, behavioural sciences, engineering, mathematics, economics and commerce.

In recent years, Operations Research has had an increasingly great impact on the management of organizations. Both the number and the variety of its applications continue to grow rapidly, and no slowdown is in sight. The subject is also being used widely in other types of organizations, including business and industry. Many industries including aircraft and missile, automobile, communication, computer, electronics, mining, paper, petroleum and transportation have made widespread use of Operations Research in determining their tactical and strategical decisions more scientifically. In addition to the industries, the financial institutions, governmental agencies and hospitals are also rapidly increasing the use of Operations Research.

In India, Operations Research came into existence in 1949 with the opening of an O.R. unit at the Regional Research Laboratory at Hyderabad. At the same time, another group was set up in the Defence Science Laboratory which devoted itself to the problems of stores, purchase and planning. In 1953, an O.R. unit was established in the Indian Statistical Institute, Calcutta, for the application of O.R. methods in national planning and survey. O.R. Society of India was formed in 1957. It became a member of the International Federation of O.R. Societies in 1959.

Towards the applications of O.R. in India, Prof. Mahalonobis made the first important application. He formulated the Second Five-Year Plan with the help of O.R. techniques to forecast the trends of demand, availability of resources and for scheduling the complex schemes necessary for developing our country's economy.

1:3. NATURE AND FEATURES OF O.R.

After tracing the process of establishment and growth of Operations Research, we can consider it as a source to other new sciences. Literally, the word 'operation' may be defined as some action that we apply to some problems or hypotheses and the word 'research' is an organised process of seeking out facts about the same. O.R. has been variously described as the *"science of use"*, *"quantitative common sense"*, *"scientific approach to decision-making problems"*, etc. But only a few are commonly used and widely accepted, namely,

(*i*) "O.R. is the art of giving bad answers to problems which otherwise have worse answers."

— *T.L. Saaty*

(*ii*) "O.R. is a scientific method of providing executive departments with a quantitative basis for decisions under their control."

— *P.M. Morse and G.E. Kimball*

(*iii*) "O.R. is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem."

— *Churchman, Ackoff and Arnoff*

(*iv*) "O.R. is applied *decision theory*. It uses scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough-going rationality in dealing with his decision problems."

— *D.W. Miller and M.K. Starr*

(*v*) "O.R. is a *scientific approach* to problems solving for executive management."

— *H.M. Wagner*

(*vi*) "O.R. is a *scientific knowledge through interdisciplinary team* effort for the purpose of determining the best utilization of limited resources." — *H.A. Taha*

Some significant features of O.R. are highlighted below :

(*a*) *Decision-Making.* Primarily, O.R. is addressed to managerial decision-making or problem-solving. A major premise of O.R. is that decision-making, irrespective of the situation involved, can be considered as a general systematic process. It consists of the following steps :

(*i*) Define the problem and establish a criterion to be used.

(*ii*) Select the alternative courses of action for consideration.

(*iii*) Determine the model to be used and the values of the parameters involved.

(*iv*) Solve the model to choose the best (optimal) alternative.

(*b*) *Scientific Approach.* O.R. employs scientific methods for the purpose of solving problems. It is a formalised process of reasoning.

(*c*) *Objective.* O.R. attempts to locate the best or optimal solution to the problem under consideration. For this purpose, it is necessary that a measure of effectiveness is defined which is based on the goals of the organisation. This measure is then used as the basis to compare the alternative courses of action.

(*d*) *Inter-disciplinary Team Approach.* O.R. is inter-disciplinary in nature and requires a team approach to a solution of the problem. Managerial problems have economic, physical, psychological, biological, sociological and engineering aspects. This requires a blend of people with expertise in the areas of mathematics, statistics, engineering, economics, management, computer science, and so on.

(*e*) *Digital Computer.* Use of a digital computer has become an integral part of the O.R. approach to decision-making. The computer may be required due to the complexity of the model, volume of data required and the computations to be made.

1:4. SCIENTIFIC METHOD IN O.R.

The scientific method in Operations Research is its most important feature. It consists of the following three phases :

Judgement Phase. This phase includes : (*i*) identification of the real-life problem, (*ii*) selection of an appropriate goal and the value of various variables related to the goals, (*iii*) appropriate scale of measurement, and (*iv*) formulation of an appropriate model of the problem, abstracting the essential information so that a solution at the decision-maker's goal can be sought.

Research Phase. This phase is the largest and longest among the other two. However, the remaining two are also equally important as they provide the basis for a scientific method. This phase utilizes : (*i*) observations and data collection for a better understanding of what the problem is, (*ii*) formulation of hypothesis and models, (*iii*) observation and experimentation to test the hypothesis on the basis of additional data, (*iv*) analysis of the available information and verification of the hypothesis using pre-established measures of effectiveness, (*v*) predictions of various results from the hypothesis, and (*vi*) generalization of the results and consideration of alternative methods.

Action Phase. This phase consists of making recommendations for decision process by those who first posed the problem for consideration, or by anyone in a position to make a decision influencing the operation in which the problem occurred.

1:5. MODELLING IN OPERATIONS RESEARCH

A *model* in O.R. is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.

Models, in general, cannot represent every aspect of reality because of innumerable and changing characteristics of the real life problems to be represented. Instead, they are limited approximations of the reality.

The objective of a model is to provide a means for analysing the behaviour of the system for the purpose of improving its performance.

There are several models in each area of business, or industrial activity. For instance, an *account model* is a typical budget in which business accounts are referred to with the intention of providing measurements such as rate of expenses, quantity sold, etc.; a mathematical equation may be considered to be a *mathematical model* in which a relationship between constants and variables is represented. A model which has the possibility of measuring observations may be called a *quantitative model*; a product, a device or any tangible thing used for experimentation may represent a *physical model*.

Following are the main characteristics that a good model for Operations Research study should have :

(*i*) A good model should be capable of taking into account new formulations without having any significant change in its frame.

(*ii*) Assumptions made in the model should be as small as possible.

(*iii*) It should be simple and coherent. Number of variables used should be less.

(*iv*) It should be open to parametric type of treatment.

(*v*) It should not take much time in its construction for any problem.

Classification of Models

Although the classification of models is a subjective problem, they may be distinguished as follows :

Models by degree of abstraction. These models are based on the past data/information of the problems under consideration and can be categorised into (*a*) language models, and (*b*) case studies.

A book may be regarded as an example of a language model.

Models by function. These models consist of (*a*) Descriptive models, (*b*) Predictive models, and (*c*) Normative models.

(*a*) *Descriptive models :* A descriptive model simply describes some aspects of a situation based on observation, survey, questionnaire results, or other available data. The result of an opinion poll represents a descriptive model.

(*b*) *Predictive models :* Such models can answer 'if and what' type of questions, *i.e.*, they make predictions regarding certain events. For example, based on survey results, television networks attempt to explain and predict the election outcome before all the votes are actually counted.

(*c*) *Normative models :* Finally, when a predictive model has been repeatedly successful it can be used to prescribe a source of action. Linear programming is a normative or prescriptive model, because it prescribes what the managers ought to do.

Models by structure. These models are represented by (*a*) Iconic models, (*b*) Analogue models, and (*c*) Symbolic models.

Iconic or Physical models are pictorial representation of real systems and have the appearance of the real thing. Examples of such models are : city maps, houses blueprint's, globe, and so on. An iconic model is said to be 'scaled-down' or 'scaled-up' according as the dimensions of the model are smaller or greater than those of the real item. For instance, in biology, the structure of a cell may be illustrated by an enlarged (scaled-up) iconic model for teaching purposes.

Iconic models are easy to observe, build and describe, but are difficult to manipulate and not very useful for the purposes of prediction. Commonly, these models represent a *static* event.

Analogue models are more abstract than the iconic ones for there is no 'look-alike' correspondence between these models and real life items. They are built by utilizing one set of

properties to represent another set of properties. For instance, a network of pipes through which water is running could be used as a parallel for understanding the distribution of electric currents. Graphs and maps in various colours are analogue models, in which different colours correspond to different characteristics. A flow process chart is an analogue model which represents the order of occurrence of various events to make a product.

Mathematical or Symbolic models are most abstract in nature. They employ a set of mathematical symbols to represent the components (and relationships between them) of the real system. These models are most general and precise. However, it is not always possible to depict a real system in mathematical formulation, sometimes it is easier to use mathematical symbols for describing the relationship of the components, and sometimes an analogue model may express the pattern of its relationship in a better way.

Models by nature of the environment. These models can be classified into (*a*) Deterministic models, and (*b*) Probabilistic models.

In *deterministic models*, all the parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear Programming and Break-even models are the examples of deterministic models.

On the other hand, models in which at least one parameter or decision variable is a random variable are called *probabilistic or stochastic models*. These models reflect, to some extent, the complexity of the real world and the uncertainty surrounding it.

Models by the extent of generality. These models can be categorised into (*a*) Specific models, and (*b*) General models.

When a model presents a system at some specific time, it is known as a *specific model*. In these models if the time factor is not considered, then they are termed as *static models*, and *dynamic models* otherwise. An inventory problem of determining economic order quantity for the next period, assuming that the demand in planning period would remain same as that of today, is an example of a static model. Dynamic Programming may be considered as an example of dynamic model.

Simulation and *Heuristic models* fall under the category of *general models*. These models are mainly used to explore alternative strategies (courses of action) which have been overlooked previously. These models do not yield any optimum solution to the problem, but give a solution to a problem depending on assumptions based on the past experience.

1:6. ADVANTAGES AND LIMITATIONS OF MODELS

Models in O.R. are used as an aid for analysing complex problems. The main *advantages* of a model are :

(*i*) Through a model, the problem under consideration becomes controllable.

(*ii*) It provides some logical and systematic approach to the problem.

(*iii*) It indicates the limitations and scope of an activity.

(*iv*) It helps in incorporating useful tools that eliminate duplication of methods applied to solve any specific problems.

(*v*) It helps in finding avenues for new research and improvements in a system.

(*vi*) It provides economic descriptions and explanations of the operations of the system they represent.

However, besides the above advantages a model has the following *limitations :*

(*a*) Models are only an attempt in understanding operations and should never be considered as absolute in any sense.

(*b*) Validity of any model with regard to corresponding operation can only be verified by carrying out the experiment and observing relevant data characteristics.

(*c*) Construction of models require experts from various disciplines.

1:7. GENERAL SOLUTION METHODS FOR O.R. MODELS

There are, in general, three methods for deriving the solution to an O.R. model: the *analytical methods, the numerical methods* and the *Monte-Carlo methods*. The first one is deductive by nature, while the second one is inductive.

Analytical or Deductive Methods. In these methods classical optimization techniques such as Calculus, Finite Differences, etc., are used for solving an O.R. model. The kind of mathematics required depends upon the nature of the model. For example, in the inventory model, in order to calculate the economic order quantity (EOQ), the analytical method requires the first derivative of the mathematical expression

$$TC = (D/Q)\,C_S + (Q/2)\,C_1,$$

equated to zero, as the first step towards identifying the optimum value of *EOQ, viz.*, $\sqrt{2DC_1/C_S}$. *This is because of the concept of maxima and minima for optimality. Here, TC = total annual cost, D = annual demand, Q = size of an order,* C_S *= set up cost per production run, and* C_1 *= holding cost for a unit in inventory.*

Numerical Methods. Numerical methods are concerned with the iterative or trial and error procedures, through the use of numerical computation at each step. These numerical methods are used when some analytical methods fail to derive the solution.

The numerical method starts with a trial (initial) solution and continued with a set of rules for improving it towards optimality. The trial solution is then replaced by the improved one and the process is repeated until either no further improvement is possible or the cost of further computations cannot be justified. Thus the numerical methods are hit and trial methods that end at a certain step after which no further improvement can be made.

Monte Carlo Methods. These involve the use of probability and sampling concepts. The various steps associated with a Monte Carlo method are as follows :

(*a*) For appropriate model of the system, make sample observations and determine the probability distribution for the variables of interest.

(*b*) Convert the probability distribution to cumulative distribution.

(*c*) Select the sequence of random numbers with the help of random tables.

(*d*) Determine the sequence of values of variables of interest with the sequence of random numbers obtained in the above step.

(*e*) Fit an appropriate standard mathematical function to the values obtained in step (*d*).

The Monte Carlo method is essentially a *simulation technique* in which statistical distribution functions are created by generating a series of random numbers.

1:8. METHODOLOGY OF OPERATIONS RESEARCH

The systematic *methodology* developed for an O.R. study deals with problems involving conflicting multiple objectives, policies and alternatives. O.R. in the final analysis is a scientific methodology which is applied to the study of the operations of large complex organisations or activities with a view to assessing the overall implications of various alternative courses of action, thus providing an improved basis for managerial decisions.

The O.R. approach to problem solving consists of the following seven steps :

1. Formulate the problem.
2. Construct a mathematical model.
3. Acquire the input data.
4. Derive the solution from the model.

5. Validate the model.
6. Establish control over the solution.
7. Implement the final results.

1. *Formulating the Problem.* The first step in O.R. is to develop a clear and concise statement of the problem, *i.e.*, to identify the problem correctly. Once the problem is rightly identified we can say that fifty per cent of the problem is solved. After identification, formulation of the problem is to be carried out. For this it is necessary to study comprehensively the components of the problems, *viz.*, (*a*) the decision-maker, (*b*) the objective, (*c*) the alternative courses of action, and (*d*) the environment. After identifying the components of the problem, the relationship that exists among the components of the problem should be analysed. Also, there must be complete agreement on the above points between persons initiating the O.R. study and the persons performing the study.

2. *Constructing a Mathematical Model.* The next step is to build a suitable mathematical model. Model construction consists of hypothesizing relationships between variables subject to, and not subject to control by decision-maker.

A mathematical model should include mainly the following three basic sets of elements :

(*a*) Decision variables and Parameters,

(*b*) Constraints or Restrictions, and

(*c*) Objective function.

A generalized mathematical model might take the form : $E = f(x_i, y_i)$ where f represents a system of mathematical relationships between the measures of effectiveness of the objective sought (E) and the variables, both controllable (x_i) and uncontrollable (y_i).

3. *Obtaining the Input Data.* Once the mathematical model of the problem has been formulated, the next step is to obtain the data to be used in the model as input. Since the quality of data determines the quality of output, the importance of obtaining accurate and complete data cannot be over-emphasised.

Obtaining correct and relevant data may, however, be a difficult exercise when relatively large problems are involved. A number of sources, such as: company reports and documents, interviews with the company personnel and so on may by used for collecting data.

4. *Deriving the solution from the Model.* Having collected the input data, the next step is to determine the values of decision variables that optimize the given objective function. This deals with the mathematical calculations for obtaining the solution to the model.

It may be noted that the solutions can be categorised as being (*i*) feasible or infeasible, (*ii*) optimum or non-optimum, and (*iii*) unique or multiple.

(*i*) A solution that satisfies all the constraints of the problem including the condition of non-negativity is known as a *feasible solution*, while the infeasible solution is one that does not satisfy the condition of non-negativity of variables.

(*ii*) An *optimum solution* to the problem is one of the feasible solutions which also optimises the objective function. The feasible solutions other than the optimum solution are called non-optimum solutions.

(*iii*) If there exists only one optimum solution to be problem, it is said to be an *unique optimum solution. On the other hand, if two or more optimum solutions to the problem exist, then multiple optimum solutions are said to exist.*

While algorithms exist for most of the standardised problems, there are also some numerical techniques, which yield solutions that are not necessarily optimum. **Heuristics** and **Simulation** illustrate those methods. *Heuristics* are step by step logical rules that yield values of the variables which satisfy all the constraints, but not necessarily provide optimum solution. However, these values provide an *acceptable value* for the objective function. Similarly, the technique of *Simulation* is

applied where a given system is sought to be replicated and experimented with. Solutions using simulation need not be optimal, because the technique is only descriptive in nature.

In addition to the solution of the model, it is also sometimes essential to perform *sensitivity analysis*, *i.e.*, to determine the behaviour of the system, changes in the system's parameters and specifications. This is done because the input data (parameters) may not be accurate or stable, and the structural assumptions of the model may not be valid.

5. *Validation of the Model.* Validating a model requires of determining as if the model can reliably predict the actual system's performance. It also involves testing the structural assumptions of the model to ascertain their validity. A common approach for testing the validity of a model is to compare its performance with past data available for the actual system.

If during validation, the solution cannot be implemented, one needs to find if some of the original constraints were incorrect and need to be modified. In such a case, one must return to the problem formulation step and carefully make the appropriate modifications to represent more realistic situation.

A model must be applicable for a reasonable time period and should be updated from time to time, taking into consideration the past, present and future aspects of the problem.

6. *Establishing control over the solution.* After testing the model and its solution, the next step of the study is to establish control over the solution, by proper feedback of the information on variables which deviated significantly. As soon as one or more of the controlled variables change significantly, the solution goes out of control. In such a situation the model may accordingly be modified.

7. *Implementation of the final results.* Finally, the tested results of the model are implemented to work. This would basically involve a careful explanation of the solution to be adopted and its relationship with the operating realities. This stage of O.R. investigation is executed primarily through the co-operation of both the O.R. experts and those who are responsible for managing and operating the system.

1:9. OPERATIONS RESEARCH AND DECISION-MAKING

Every industrial enterprise or business house works on a properly established management information system. Such a system usually assumes a three-tier position at the level of (*i*) strategic planning, (*ii*) managerial control, and (*iii*) operational control. These levels are identified with the top management, middle management and supervisory management respectively. Many prefer to call these three-tier activities as policy-making, decision-making and activity oriented. For a competent management, decision-making is a major task.

Decision-making is not the headache of only the management, rather all of us make decisions. We daily decide about many minor and major issues. The essential characteristics of all decisions are:

(1) objective; (2) alternatives at the disposal; and (3) influencing factors.

Once these characteristics are known, one can think of improving the characteristics so as to improve upon the decision itself.

Let us consider a situation where a decision concerns spending summer vacations at a hill resort. The problem may be to decide the mode of conveyance from amongst the three alternatives: train, bus and a taxi.

At the *first* level of decision-making, bus is chosen as the mode of conveyance just by intuition. At the *second* level of decision-making, the three modes are compared and it is decided qualitatively that the bus will be preferred since it is less time consuming compared to the train and cheaper as compared to taxi. At the *third* level of decision-making, the three alternatives are compared and it is suggested that the bus will be chosen, as it will be taking only half the time taken by train and shall be 40% ~~less~~ costlier than the taxi.

Although outcome of all these decisions is the same, one can easily judge the quality of each decision. We may brand the first decision as 'bad' since it is highly emotional, while we may call the second decision as 'good' since it is scientific, though qualitative. The third decision is doubtlessly the best as it is scientific as well as quantitative.

It is this *scientific quantification* used in O.R. that helps management to make better decisions.

Operations Research may be regarded as a tool employed to increase the effectiveness of managerial decisions as an objective supplement to the subjective feelings of the decision-maker.

For instance, in distribution or allocation areas, O.R. may suggest the best locations for agencies, warehouses as well as the most economical kind of transportation; in marketing areas, it may aid in indicating the most profitable type, use and size of advertising campaigns, in regard to available financial limits. O.R. may suggest alternative courses of action when a problem is analysed and solution is attempted. However, the study of complex problems by O.R. techniques becomes useful only when a choice between two or more courses of action is possible.

O.R. may be regarded as a tool that enables the decision-maker to be objective in choosing an alternative from among many that he can conceive of. Following are the salient advantages of an operations research study approach in decision-making:

(*i*) ***Better Decisions.*** O.R. models frequently yield actions that do improve on intuitive decision-making. A situation may be so complex that the human mind can never hope to assimilate all the significant factors without the aid of O.R. guided computer analysis.

(*ii*) ***Better Coordination.*** Sometimes, operations research has been instrumental in bringing order out of chaos. For instance, an O.R.-oriented planning model becomes a vehicle for coordinating marketing decisions within the limitations imposed on manufacturing capabilities.

(*iii*) ***Better Control.*** The management of large organizations recognize that it is extremely costly to provide continuous executive supervision over routine decisions. An O.R. approach, thereby regained new freedom to the executive to devote his attention to more pressing matters. The most frequently adopted application in this category deal with production scheduling and inventory replenishment.

(*iv*) ***Better Systems.*** Often, an O.R. study is initiated to analyse a particular decision-problem, such as whether to open a new warehouse. Afterwards, the approach is further developed into a system to be employed repeatedly. Thus, the cost of undertaking the first application may produce benefits.

1:10. APPLICATIONS OF OPERATIONS RESEARCH

O.R. is mainly concerned with the techniques of applying scientific knowledge, besides the development of science. It provides an understanding which gives the expert/manager new insights and capabilities to determine better solutions in his decision-making problems, with great speed, competence and confidence. In recent years, O.R. has successfully entered many different areas of research in Defence, Government, Service Organisations and Industry. We briefly describe some applications of O.R. in the functional areas of management :

Finance, Budgeting and Investment

(*i*) Cash flow analysis, long range capital requirements, dividend policies, investment portfolios.

(*ii*) Credit policies, credit risks and delinquent account procedures.

(*iii*) Claim and complaint procedure.

Marketing

(*i*) Product selection, timing, competitive actions.

(*ii*) Advertising media with respect to cost and time.

(*iii*) Number of salesmen, frequency of calling of account, etc.
(*iv*) Effectiveness of market research.

Physical Distribution

(*i*) Location and size of warehouses, distribution centres, retail outlets, etc.
(*ii*) Distribution policy.

Purchasing, Procurement and Exploration

(*i*) Rules of buying.
(*ii*) Determining the quantity and timing of purchase.
(*iii*) Bidding policies and vendor analysis.
(*iv*) Equipment replacement policies.

Personnel

(*i*) Forecasting the manpower requirement, recruitment policies and job assignments.
(*ii*) Selection of suitable personnel with due consideration for age and skills, etc.
(*iii*) Determination of optimum number of persons for each service centre.

Production

(*i*) Scheduling and sequencing the production run by proper allocation of machines.
(*ii*) Calculating the optimum product mix.
(*iii*) Selection, location and design of the sites for the production plant.

Research and Development

(*i*) Reliability and evaluation of alternative designs.
(*ii*) Control of developed projects.
(*iii*) Co-ordination of multiple research projects.
(*iv*) Determination of time and cost requirements.

Besides the above mentioned applications of O.R. in the context of modern management, its use has now extended to a wide range of problems, such as the problems of communication and information, socio-economic fields and national planning.

1:11. OPPORTUNITIES AND SHORTCOMINGS OF OPERATIONS RESEARCH

The use of O.R. to improve decision-making has become almost universal today. However, this remarkable achievement of O.R. is not totally free from shortcomings. Certain shortcomings result from lack of awareness on the part of managers about their roles, while others result due to avoidance of the behavioural and organisational issues which are a part of every successful application.

Opportunities of O.R.

1. Using O.R. approach, the decision-maker can determine a solution to his routine or repetitive problem. For obtaining solution of such type of problems, it is necessary to build a model so that future solutions can be obtained using the model thus freeing managers to concentrate on more pressing matters.

2. O.R. requires business managers to be quite explicit about their objectives, their assumptions and the way of visualising the constraints.

3. While using O.R. approach, a manager has to consider very carefully all those variables which influence his decisions and the way these variables in a problem interact with each other. He then selects a decision which is best for the organisation as whole.

4. O.R. approach allows a decision-maker to solve a complex problem involving multiple variables much more quickly than if he had to compute them using traditional methods. Sometimes it may not be possible to solve such complex problems without using O.R. methods.

5. Using O.R. approach, a decision-maker can examine a situation from various angles by simulating the model which he has constructed for the real problem. He can change various conditions under which decisions are being made, and examine the effect of these changes through appropriate experiments on the model, to determine the best or optimal solution for the problem under consideration. All these experiments can be carried out without causing any serious damage to the existing system or incurring excessive cost.

Shortcomings of O.R.

1. There are certain problems which a decision-maker may have to solve only once. Constructing a complex O.R. model for solving such problems is often too expensive when compared with the cost of other less sophisticated approaches available to solve them.

2. Sometimes O.R. specialists become too much enamoured with the model they have built and forget the fact that their model does not represent the "real world problem" in which decisions have to be made.

3. Sometimes the basic data are subject to frequent changes. In such cases, modification of O.R. models is a costly affair.

4. Many O.R. models are so complex that they cannot be solved without the use of computer. Also, the solutions obtained from these models are difficult to explain to managers and hence fail to gain their support and confidence.

5. Magnitude of computation involved, lack of consideration for non-quantifiable factors and psychological issues involved in implementation are some of the other shortcomings of O.R.

MULTIPLE CHOICE QUESTIONS

Choose the correct alternative:

1. Operations Research came into existence
(*a*) in the year 1940, (*b*) in the military context,
(*c*) during World War I, (*d*) during World War II.

2. Operations Research achieved recognition as a subject of academic study in the year
(*a*) 1949, (*b*) 1950,
(*c*) 1953, (*d*) 1957.

3. Operations Research is
(*a*) applied decision theory,
(*b*) a scientific approach to problems solving for executive management,
(*c*) the science of use,
(*d*) all of the above.

4. Operations Research approach is
(*a*) intuitive, (*b*) objective,
(*c*) multi-disciplinary, (*d*) all of the above.

5. A model in Operations Research is
(*a*) an approximation, (*b*) an idealization,
(*c*) an essence of reality, (*d*) all of the above.

6. A physical model is an example of
(*a*) ironic model, (*b*) analogue model,
(*c*) verbal model, (*d*) symbolic model.

7. Decision variables in an O.R. model are
   (*a*) controllable, (*b*) uncontrollable,
   (*c*) parameters, (*d*) constants.
8. The scientific method in Operations Research consists of
   (*a*) Judgement Phase, (*b*) Research Phase,
   (*c*) Action Phase, (*d*) all of the above.
9. Operations Research techniques are not applicable in the following situation :
   (*a*) Resources available are unlimited,
   (*b*) Objective can be defined for maximization or minimization,
   (*c*) Sufficient input data is available for formulating the problem.
   (*d*) Scientific methods, techniques and tools may be applied.
10. An optimization model in O.R.
   (*a*) provides decision within its limited context,
   (*b*) mathematically provides the best decision,
   (*c*) helps in evaluating various alternatives constantly,
   (*d*) all of the above.
11. Which of the following is not correct?
   (*a*) A model should be simple and coherent.
   (*b*) A model should not take much time in its construction for any problem.
   (*c*) A model representing the typical budget of business accounts is called 'account model'.
   (*d*) A model which has the probability of measuring observations is called 'qualitative model'.
12. Which of the following is not correct?
   (*a*) Models are only an attempt in understanding operations and should never be considered as absolute in any sense.
   (*b*) Iconic models are pictorial representation of real system.
   (*c*) Heuristic models are particular cases of iconic models.
   (*d*) Analogue models are most abstract than the iconic ones.

ANSWERS

1. (*d*)	**2.** (*b*)	**3.** (*d*)	**4.** (*c*)	**5.** (*d*)
6. (*a*)	**7.** (*a*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*d*)	**12.** (*c*)			

REVIEW QUESTIONS

1. What is Operations Research?
2. What is Operations Research? Account for the growing importance of O.R. in business decisions. *[Madurai M.Com. 2003]*
3. Give any three definitions of Operations Research and explain.
4. State the features of O.R.
5. Explain the nature of Operations Research and its limitations. *[Madras M.B.A. (Nov.) 2006]*
6. What are the essential characteristics of O.R.? Mention different phases in an O.R. study. Explain the role of computers in this field. *[Madurai M.Com. 2003]*
7. Model building is the essence of Operations Research. Discuss.
8. Discuss various classification schemes of models.
9. What are the advantages and disadvantages of Operations Research model? Why is it necessary to test models and how would you go to about testing a model? *[Madurai M.Com., 2002]*
10. "Operations Research is a bunch of mathematical techniques." Comment. *[Lucknow B.M.S. 2008]*

11. How can Operations Research models be classified? What is the best classification in terms of learning and understanding the fundamentals of OR? *[Delhi MBA (HCA) 2002, Madras MBA (Nov.) 2006]*
12. (*a*) List the phases of OR. *[Visvesvaraya M.B.A. (Dec.) 2011]*
    (*b*) What are the steps involved in OR? *[Panjab Tech. Univ. M.B.A. (Dec.) 2012]*
13. Describe briefly the different phases of Operations Research. *[Madurai M.Com. (Nov.) 2003]*
14. Write short note on methodology of Operations Research. *[Panjab Tech. Univ. MBA (Dec.) 2010]*
15. "Operations Research is an aid for the executive in making his decisions based on scientific method analysis." Discuss the above statement in brief.
16. Explain how and why OR methods have been valuable in aiding executive decisions.
17. Comment on the following statements :
    (*a*) Operations Research advocates a system approach and is concerned with optimization.
    (*b*) Operations Research replaces management by personality.
18. Discuss the importance of mathematical models by giving examples in the solution of OR problems. *[Panjab Tech. Univ. M.B.A. (Dec.) 2012]*
19. Discuss descriptive model, predictive model and normative model briefly in Operations Research. Also, list the managerial applications of OR. *[Visvesvaraya M.B.A. (Dec.) 2011]*
20. (*a*) Explain briefly the applications of OR *[Madurai M.Com. 2002, Madras MBA (Nov.) 2006]*
    (*b*) Write a short note on the importance of OR in project management. *[Madras MBA 1999]*
21. Discuss the significance and scope of Operations Research in decision-making with special reference to Health Care Administration. *[Delhi M.B.A. (HCA) 2002]*
22. Briefly describe the scope of Operations Research. *[Madras MCA 2003, MBA (April) 2006]*
23. Discuss the usefulness of OR in decision-making process and the role of computers in this field. *[Madras MCA 1999]*
24. What are the limitations of OR? Suggest three advantages of OR approach in decision-making. *[Madras MBA 2004]*
25. What is Operations Research? Which of the techniques of OR clearly fall under the category of optimal decision-making? Which of the techniques are not directly concerned with optimal decision-making? What are they concerned with? *[Delhi M.Com. 2000]*
26. Name some of the tools of Operations Research. Discuss with examples, how these tools can be useful in Health Care Administration. *[Delhi MBA (HCA) PT 2008]*

2

Linear Programming Problem —Mathematical Formulation

"Resources are scarce and property of the society and their abuse is a social evil"

2:1. INTRODUCTION

Many business and economic situations are concerned with a problem of planning activity. In each case, there are limited *resources* at your disposal and your problem is to make such a use of these resources so as to yield the maximum production or to minimise the cost of production, or to give the maximum profit, etc. Such problems are referred to as the problems of constrained optimisation. *Linear programming* is a technique for determining an optimum schedule of interdependent activities in view of the available resources. *Programming* is just another word for 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. The word *linear* stands for indicating that all relationships involved in a particular problem are linear.

In the present chapter, some applications of linear programming problems and their mathematical formulations are discussed. The concepts are then extended to the general linear programming problem.

2:2. LINEAR PROGRAMMING PROBLEM

A Linear Programming Problem (LPP) consists of three *components*, namely the (*i*) decision variables (activities), (*ii*) the objective (goal), and (*iii*) the constraints (restrictions).

(*i*) The *decision variables* refer to the activities that are competing one another for sharing the resources available. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. All the decision variables are considered as continuous, controllable and non-negative.

(*ii*) A linear programming problem must have an objective which should be clearly identifiable and measurable in quantitative terms. It could be of profit (sales) maximisation, cost (time) minimisation, and so on. The relationship among the variables representing objective must be linear.

(*iii*) There are always certain limitations (or constraints) on the use of resources, such as labour, space, raw material, money, etc. that limit the degree to which an objective can be achieved.

Such constraints must be expressed as linear inequalities or equalities in terms of decision variables.

Basic assumptions. The following four basic assumptions are necessary for all linear programming problems:

(*a*) *Certainty.* In all LPP's, it is assumed that all the parameters; such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit

decision variable must be known and fixed. In other words, this assumption means that all the coefficients in the objective function as well as in the constraints are completely known with certainty and do not change during the period of study.

(*b*) *Divisibility (or continuity).* This implies that solution values of the decision variables and resources can take on any non-negative values, including fractional values of the decision variables. For instance, it is possible to produce 4.35 quintals of wheat or 17.35 thousand kilometers of cloth or 6.52 thousand kilolitres of milk, so these variables are divisible. But it is not possible to produce 2.6 refrigerators. Such variables are not divisible and hence are to be assigned integer values. When it is necessary to have integer variables, the integer programming problem is considered to attain the desired values.

(*c*) *Proportionality.* This requires the contribution of each decision variable in both the objective function and the constraints to be directly proportional to the value of the variable. For example, if production of one unit of a particular product uses 3 hours of a particular resource, then the production of 6 units of that product uses 3×6, *i.e.*, 18 hours of that resource.

(*d*) *Additivity.* The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum of the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision variable respectively. For example, the total profit earned by the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed by A and B must be equal to the sum of resources used for A and B individually.

2:3. MATHEMATICAL FORMULATION OF THE PROBLEM

The procedure for mathematical formulation of a linear programming problem consists of the following major steps :

Step 1. Study the given situation to find the *key decisions* to be made.

Step 2. Identify the *variables* involved and designate them by symbols x_j ($j = 1, 2, \ldots$).

Step 3. State the *feasible alternatives* which generally are : $x_j \geq 0$, for all j.

Step 4. Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables.

Step 5. Identify the objective function and express it as a linear function of the decision variables.

2:4. ILLUSTRATIONS ON MATHEMATICAL FORMULATION OF LPPs

Here are some problems from real life, which have been put in the mathematical format.

SAMPLE PROBLEMS

201. (Product Allocation Problem). *A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of clothes namely suitings, shirtings and woollens yielding a profit of Rs. 2, Rs. 4 and Rs. 3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.*

Formulate the linear programming problem to find the product mix to maximize the profit.

Mathematical Formulation

The data of the problem is summarized below :

	Departments			*Profit*
	Weaving (in minutes)	*Processing (in minutes)*	*Packing (in minutes)*	*(Rs. per metre)*
Suitings	3	2	1	2
Shirtings	4	1	3	4
Woollens	3	3	3	3
Availability (minutes)	60 × 60	40 × 60	80 × 60	

Step 1. The key decision is to determine the weekly rate of production for the three types of clothes.

Step 2. Let us designate the weekly production of suitings, shirtings and woollens by x_1 metres, x_2 meters and x_3 metres respectively.

Step 3. Since it is not possible to produce negative quantities, feasible alternatives are sets of values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step 4. The constraints are the limited availability of three operational departments. One metre of suiting requires 3 minutes of weaving. The quantity being x_1 metres, the requirement for suiting alone will be $3x_1$ units. Similarly, x_2 metres of shirting and x_3 metres of woollen will require $4x_2$ and $3x_3$ minutes respectively. Thus, the total requirement of weaving will be $3x_1 + 4x_2 + 3x_3$, which should not exceed the available 3600 minutes. So, the labour constraint becomes $3x_1 + 4x_2 + 3x_3 \leq 3600$.

Similarly, the constraints for the processing department and packing departments are $2x_1 + x_2 + 3x_3 \leq 2400$ and $x_1 + 3x_2 + 3x_3 \leq 4800$ respectively.

Step 5. The objective is to maximize the total profit from sales. Assuming that whatever is produced is sold in the market, the total profit is given by the linear relation $z = 2x_1 + 4x_2 + 3x_3$.

The linear programming problem can thus be put in the following mathematical format :

Find x_1, x_2 and x_3 so as to maximize

$$z = 2x_1 + 4x_2 + 3x_3$$

subject to the constraints :

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$
$$2x_1 + x_2 + 3x_3 \leq 2400$$
$$x_1 + 3x_2 + 3x_3 \leq 4800$$
$$x_1 \geq 0, \quad x_2 \geq 0 \quad \text{and} \quad x_3 \geq 0.$$

202. (Product Mix Problem). *Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available :*

Machine	*8-ounce bottles*	*16-ounce bottles*
A	*100/minute*	*40/minute*
B	*60/minute*	*75/minute*

Each machine can be run 8-hours per day, 5 days per week. Profit on a 8-ounce bottle is 25 paise and on a 16-ounce bottle is 35 paise. Weekly production of the drink cannot exceed 3,00,000 ounces and the market can absorb 25,000 8-ounce bottles and 7,000 16-ounce bottles per week. The planner wishes to maximize his profit subject, of course, to all the production and marketing restrictions. Formulate this as a linear programming problem. ***[Meerut M.Sc. (Math.) 1998]***

Mathematical Formulation

The data of the problem is summarized as follows :

Resource/ constraint	*Production*		*Availability*
	8-ounce bottle	*16-ounce bottle*	
Machine *A* time	100/minute	40/minute	$8 \times 5 \times 60 = 2400$ minutes
Machine *B* time	60/minute	75/minute	$8 \times 5 \times 60 = 2400$ minutes
Production	1	1	3,00,000 ounces/week
Marketing	1	—	25,000 units/week
	—	1	7,000 units/week
Profit/unit (Rs.)	0.25	0.35	

Step 1. The key decision to be made is to determine the number of bottles (*8-ounce* and *16-ounce*) to be produced per week. Let x and y be the number of *8-ounce* and *16-ounce* bottles respectively, produced per week.

Step 2. Feasible alternatives are the sets of values $x \geq 0$, $y \geq 0$.

Step 3. Constraints are on the availability of machine time and production.

(*i*) *Machine-time constraints.* An *8-ounce* bottle takes 1/100 minutes on machine *A* and 1/60 minutes on machine *B*, while a *16-ounce* bottle takes 1/40 minutes on machine *A* and 1/75 minutes on machine *B*. Since both the machines can run 8 hours per day for 5 days per week, the time available on both the machines is 2,400 minutes per week individually. Thus, the two machine time constraints are :

$$\frac{x}{100} + \frac{y}{40} \leq 2{,}400 \quad (\textit{Machine A})$$

$$\frac{x}{60} + \frac{y}{75} \leq 2{,}400 \quad (\textit{Machine B})$$

(*ii*) *Production constraints.* It is given that the weekly production of the drink should not exceed 3,00,000 ounces and the market can absorb only up to 25,000 (*8-ounce* bottles) and 7,000 (*16-ounce* bottles) per week. Therefore, the two production constraints are :

$$8x + 16y \leq 3{,}00{,}000 \quad (\textit{Production})$$

$$x \leq 25{,}000 \text{ and } y \leq 7{,}000 \quad (\textit{Market})$$

Step 4. The objective is to maximize the total profit, *viz.*, $0.25x + 0.35y$.

The linear programming problem, therefore, can be put in the following mathematical format :

$$\boxed{\begin{array}{l} \text{Maximize} \quad z = 0.25x + 0.35y \\ \text{subject to the constraints :} \\ \qquad 4x + 10y \leq 9{,}60{,}000 \\ \qquad 15x + 12y \leq 21{,}60{,}000 \\ \qquad 8x + 16y \leq 3{,}00{,}000 \\ \qquad x \leq 25{,}000 \text{ and } y \leq 7{,}000 \\ \qquad x \geq 0,\ y \geq 0. \end{array}}$$

203. (Production Problem). *An electronic company is engaged in the production of two components C_1 and C_2 used in T.V. sets. Each unit of C_1 costs the company Rs. 25 in wages and Rs. 25 in material, while each unit of C_2 costs the company Rs. 125 in wages and Rs. 75 in material. The company sells both products on one-period credit terms, but the company's labour and material expenses must be paid in cash. The selling price of C_1 is Rs. 150 per unit and of C_2 is Rs. 350 per unit. Because of the strong monopoly of the company for these components, it is assumed that the company can sell at the prevailing prices as many units as it produces. The company's production*

capacity is, however, limited by two considerations. First, at the beginning of period 1, the company has an initial balance of Rs. 20,000 (cash plus bank credit plus collections from past credit sales). Second, the company has available in each period 4,000 hours of machine time and 2,800 hours of assembly time. The production of each C_1 requires 6 hours of machine time and 4 hours of assembly time, whereas the production of each C_2 requires 4 hours of machine time and 6 hours of assembly time. Formulate this problem as an Linear Programming model so as to maximize the total profit to the company.

Mathematical Formulation

The data of the problem is summarised as below :

Resource/constraint	*Components*		*Total availability*
	C_1	C_2	
Machine time (hours)	6	4	4,000 hours
Assembly time (hours)	4	6	2,800 hours
Budget (Rs.)	50	200	Rs. 20,000
Selling price (Rs.)	150	350	
Cost (= Wages + Material) price in Rs.	(25 + 25)	(125 + 75)	

Step 1. The key decision is to determine the number of units of C_1 and C_2 to be produced.

Step 2. Decision variables : Let x = number of units of C_1 and y = number of units of C_2.

Step 3. Feasible alternatives : $x \geq 0$ and $y \geq 0$.

Step 4. Constraints are on the availability of time and budget as under :

$$6x + 4y \leq 4{,}000 \qquad \textit{(Machine time)}$$

$$4x + 6y \leq 2{,}800 \qquad \textit{(Assembly time)}$$

$$50x + 200y \leq 20{,}000 \qquad \textit{(Budget)}$$

Step 5. The objective is to maximize the total profit from the sale of two type of components. Assuming that whatever is produced is sold in the market, the total profit is given by the relation

$$z = (150 - 50)x + (350 - 200)y \text{ or } z = 100x + 150y$$

The LPP in mathematical format, therefore, is :

$$\boxed{\begin{array}{c} \text{Maximize } z = 100x + 150y \\ \text{subject to the constraints :} \\ 6x + 4y \leq 4{,}000,\ 4x + 6y \leq 2{,}800\,; \\ 50x + 200y \leq 20{,}000\,; \text{ and } x \geq 0,\ y \geq 0. \end{array}}$$

204. (Product Allocation Problem). *An Electronics Company produces three types of parts for automatic washing machine. It purchases casting of the parts from a local foundry and then finishes the part of drilling, shaping and polishing machines.*

The selling prices of part A, B and C respectively are Rs. 8, Rs. 10 and Rs. 14. All parts made can be sold. Castings for parts A, B and C respectively cost Rs. 5, Rs. 6 and Rs. 10.

The shop possesses only one of each type of machine. Costs per hour to run each of the three machines are Rs. 20 for drilling, Rs. 30 for shaping and Rs. 30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the following table :

Machine	*Capacity per hour*		
	Part A	*Part B*	*Part C*
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an Linear Programming model so as to maximize total profit to the company. **[Delhi M.B.A. (Nov.), 2003]**

Mathematical Formulation

Step 1. The key decision is to produce the three type of parts for the automatic washing machine.

Step 2. Decision variables : Let x_1, x_2 and x_3 = number of type *A*, *B* and *C* parts to be produced per hour, respectively.

Step 3. Feasible alternatives : $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step 4. The constraints are on the time. All the three parts are to be processed on each of the three machines. On the drilling-machine, one type *A* part consumes 1/25th of the available hour, a type *B* part consumes 1/40th, and type *C* part consumes 1/25th of an hour. Thus the drilling-machine constraint is

$$\frac{x_1}{25} + \frac{x_2}{40} + \frac{x_3}{25} \leq 1, \text{ i.e., } 0.04x_1 + 0.025x_2 + 0.04x_3 \leq 1$$

Similarly, the other two constraints are :

$$\frac{1}{25}x_1 + \frac{1}{20}x_2 + \frac{1}{20}x_3 \leq 1, \text{ i.e., } 0.04x_1 + 0.05x_2 + 0.05x_3 \leq 1 \qquad \textit{(Shaping-machine)}$$

and

$$\frac{1}{40}x_1 + \frac{1}{30}x_2 + \frac{1}{40}x_3 \leq 1, \text{ i.e., } 0.025x_1 + 0.033x_2 + 0.025x_3 \leq 1 \qquad \textit{(Polishing-machine)}$$

Step 5. Profit must allow not only for the cost of the casting but for the cost of drilling, shaping, and polishing. Since, 25 type *A* parts per hour can be run on the drilling machine at a cost of Rs. 20, then Rs. $20 \times \frac{1}{25}$ = Re. 0.80 is the drilling cost per type *A* part. Similar reasoning for shaping and polishing gives

$$\text{Profit per type } A \text{ part} = (8 - 5) - \left(\frac{20}{25} + \frac{30}{25} + \frac{30}{40}\right) = 3 - 2.75 = 0.25$$

$$\text{Profit for type } B \text{ part} = (10 - 6) - \left(\frac{20}{40} + \frac{30}{20} + \frac{30}{30}\right) = 4 - 3.00 = 1.00$$

$$\text{Profit for type } C \text{ part} = (14 - 10) - \left(\frac{20}{25} + \frac{30}{20} + \frac{30}{40}\right) = 4 - 3.05 = 0.95$$

The objective is to maximize the total profit from sales, *viz.*, $0.25x_1 + x_2 + 0.95x_3$.

The linear programming problem, therefore, can be put in the following mathematical format :

$$\text{Maximize } z = 0.25x_1 + x_2 + 0.95x_3$$

subject to the constraints :

$$0.04x_1 + 0.025x_2 + 0.04x_3 \leq 1,$$
$$0.04x_1 + 0.05x_2 + 0.05x_3 \leq 1,$$
$$0.025x_1 + 0.033x_2 + 0.025x_3 \leq 1,$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

205. (Blending Problem). *The manager of an oil refinery must decide on the optimum mix of two possible blending processes of which the input and output production runs are as follows :*

Process	*Input*		*Output*	
	Crude A	*Crude B*	*Gasoline X*	*Gasoline Y*
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crudes A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 4 and Rs. 5 respectively. Formulate the problem for maximising the profit. **[Madras B.Com. 2005]**

Mathematical Formulation

Step 1. The key decision is to determine the number of units of gasoline produced from process *1* and process 2.

Step 2. Decision variables: Let x_1, x_2 = number of units of gasoline produced from process *1* and 2 respectively.

Step 3. Feasible alternatives: $x_1 \geq 0$ and $x_2 \geq 0$.

Step 4. The constraints are on the availability of crude oil and demand of crude oil, *viz.*,

$$6x_1 + 5x_2 \leq 250 \quad \textit{(Availability of crude A)}$$

$$4x_1 + 6x_2 \leq 200 \quad \textit{(Availability of crude B)}$$

$$6x_1 + 5x_2 \geq 150 \quad \textit{(Demand of gasoline X)}$$

and

$$9x_1 + 5x_2 \geq 130 \quad \textit{(Demand of gasoline Y)}$$

Step 5. The objective is to maximize the total profit from the production of gasoline, *viz.*, $4x_1 + 5x_2$.

The required linear programming problem, therefore, is

$$\boxed{\begin{array}{l} \text{Maximize } z = 4x_1 + 5x_2 \\ \text{subject to the constraints :} \\ \quad 6x_1 + 5x_2 \leq 250,\ 4x_1 + 6x_2 \leq 200, \\ \quad 6x_1 + 5x_2 \geq 150,\ 9x_1 + 5x_2 \geq 130, \\ \quad x_1 \geq 0 \text{ and } x_2 \geq 0. \end{array}}$$

206. (Production Problem). *A complete unit of a certain product consists of four units of component A and three units of component B. The two components (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively, are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components per production run and the recoulting units of each component are given below :*

Department	*Input per run (units)*		*Output per run (units)*	
	Raw material I	*Raw material II*	*Component A*	*Component B*
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulate this problem as a linear programming model so as to determine the number of production runs for each department which will maximize the total number of complete units of the final product. **[Himachal B.Tech. (Mech.) June 2007]**

Mathematical Formulation

Decision variables : Let x_1 = number of production runs for department *1*,

x_2 = number of production runs for department 2, and

x_3 = number of production runs for department *3*.

Objective function :

Since each unit of the final product requires 4 units of component A and 3 units of component B, therefore, maximum number of units of the final product cannot exceed the smaller value of

$$\left\{\frac{\text{Total number of units } A \text{ produced}}{4}, \frac{\text{Total number of units } B \text{ produced}}{3}\right\}$$

i.e., Minimum of $\left\{\dfrac{6x_1 + 5x_2 + 7x_3}{4}, \dfrac{4x_1 + 8x_2 + 3x_3}{3}\right\}$.

Constraints : (*i*) If y is the number of component units of final product, then obviously, we have

$$\frac{6x_1 + 5x_2 + 7x_3}{4} \geq y \quad \text{and} \quad \frac{4x_1 + 8x_2 + 3x_3}{3} \geq y$$

(*ii*) Constraints on raw material are :

$$7x_1 + 4x_2 + 2x_3 \leq 100 \qquad \textit{(Raw material I)}$$

and

$$5x_1 + 8x_2 + 7x_3 \leq 200 \qquad \textit{(Raw material II)}$$

The LPP, therefore, is expressed as follows:

$$\text{Maximize } z = \text{Minimum of } \left\{\frac{6x_1 + 5x_2 + 7x_3}{4}, \frac{4x_1 + 8x_2 + 3x_3}{3}\right\}$$

subject to the constraints :

$$\left.\begin{aligned} 6x_1 + 5x_2 + 7x_3 - 4y \geq 0 \\ 4x_1 + 8x_2 + 3x_3 - 3y \geq 0 \end{aligned}\right\} \qquad \textit{(Number of components of final product)}$$

$$7x_1 + 4x_2 + 2x_3 \leq 100 \qquad \textit{(Raw material I)}$$

$$5x_1 + 8x_2 + 7x_3 \leq 200 \qquad \textit{(Raw material II)}$$

$$x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \geq 0. \qquad \textit{(Non-negative restrictions)}$$

207. (Advertisement Problem). *The owner of Metro Sports wishes to determine how many advertisements to place in the selected three monthly magazines A, B and C. His objective is to advertise in such a way that total exposure to principal buyers of expensive sports good is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used :*

	Magazine		
	A	*B*	*C*
Readers	*1 lakh*	*0.6 lakh*	*0.4 lakh*
Principal Buyers	*20%*	*15%*	*8%*
Cost per Advertisement (Rs.)	*8000*	*6000*	*5000*

The budgeted amount is at most Rs. 1 lakh for the advertisements. The owner has already decided that magazine A should have no more than 15 advertisements and that B and C each have at least 80 advertisements. Formulate on LP model for the problem.

Mathematical Formulation

Decision variables : Let x_1 = number of insertions in magazine A,
x_2 = number of insertions in magazine B, and
x_3 = number of insertions in magazine C.

Objective function :

Maximize (total exposure)

$$z = (20\% \text{ of } 1,00,000)\, x_1 + (15\% \text{ of } 60,000)\, x_2 + (8\% \text{ of } 40,000)\, x_3$$
$$= 20,000\, x_1 + 9,000\, x_2 + 3,200\, x_3$$

Constraints : $8{,}000\,x_1 + 6{,}000\,x_2 + 5{,}000\,x_3 \le 1{,}00{,}000$ *(Budgeting)*

$x_1 \le 15,\ x_2 \ge 8,\ x_3 \ge 8$ *(Advertisement)*

Also, $x_1 \ge 0,\ x_2 \ge 0$ and $x_3 \ge 0.$ *(Non-negative restrictions)*

208. (Agriculturist Problem). *An agriculturist has a farm with 125 acres. He produces Radish, Muttar and Potato. Whatever he raises is fully sold in the market. He gets Rs. 5 for Radish per kg., Rs. 4 for Muttar per kg. and Rs. 5 for Potato per kg. The average yield is 1,500 kg. of Radish per acre, 1,800 kg. of Muttar per acre and 1,200 kg. of Potato per acre. To produce each 100 kg. of Radish and Muttar and to produce each 80 kg. of Potato, a sum of Rs. 12.50 has to be used for manure. Labour required for each acre to raise the crop is 6 man-days for Radish and Potato each and 5 man-days for Muttar. A total of 500 man-days of labour at a rate of Rs. 40 per man-day are available.*

Formulate this as a Linear Programming model to maximize the Agriculturist's total profit.

[Delhi M.B.A. (PT) 2005; Visvesvaraya M.B.A. (June) 2011]

Mathematical Formulation

Decision variables : Let x_1 = acreage for Raddish in kg.,

x_2 = acreage for Muttar in kg., and

x_3 = acreage for Potato in kg.

Objective function : We are given the following information :

	Radish	*Muttar*	*Potato*
Output (in kg.)	1,500	1,800	1,200
Cost of manure (in Rs. per kg.)	$\frac{12.50}{100} = 0.125$	$\frac{12.50}{100} = 0.125$	$\frac{12.50}{80} = 0.156$
Labour cost (in Rs.)	$6 \times 40 = 240$	$5 \times 40 = 200$	$6 \times 40 = 240$
Selling price (per kg.)	Rs. 5	Rs. 4	Rs. 5
Total cost	$0.125 \times 1{,}500 + 240 = 427.50$	$0.125 \times 1{,}800 + 200 = 425.00$	$0.156 \times 1{,}200 + 240 = 427.20$
Total Revenue	$5 \times 1{,}500 = 7{,}500$	$4 \times 1{,}800 = 7{,}200$	$5 \times 1{,}200 = 6{,}000$

Since, total profit = revenue − cost; the *objective function* is to maximize

$$z = (7{,}500 - 427.5)\,x_1 + (7{,}200 - 425)\,x_2 + (6{,}000 - 427.2)\,x_3$$

$$= 7{,}072.5x_1 + 6{,}775x_2 + 5{,}572.8x_3.$$

Constraints : Constraints on the availability of land and man-days are :

$x_1 + x_2 + x_3 \le 125$ *(Land)*

$6x_1 + 5x_2 + 6x_3 \le 500$ *(Man-days)*

Also, $x_1 \ge 0,\ x_2 \ge 0$ and $x_3 \ge 0$ *(Non-negative restrictions)*

209. (Crop and Livestock Problem). *A cooperative farm owns 100 acres of land and has Rs. 25,000 in funds available for investment. The farm members can produce a total of 3,500 man-hours worth of labour during the months September–May and 4,000 man-hours during June–August. If any of these man-hours are not needed, some members of the farm will use them to work on a neighbouring farm for Rs. 2/hour during September–May and Rs. 3/hour during June–August. Cash income can be obtained from the three main crops and two types of livestock : dairy cows and laying hens. No investment funds are needed for the crops. However, each cow will require an investment outlay of Rs. 3,200 and each hen will require Rs. 15.*

Moreover, each cow will require 1.5 acres of land, 100 man-hours of work during September–May and another 50 man-hours during the summer. Each cow will produce a net annual cash income of Rs. 3,500 for the farm. The corresponding figures for each hen are : no acreage,

0.6 man-hours during September–May, 0.4 man-hours during June–August, and an annual net cash income of Rs. 200. The chicken house can accommodate a maximum of 4,000 hens and the size of the cattle-shed limits the numbers to a maximum of 32 cows.

Estimated man-hours and income per acre planted in each of the three crops are :

	Paddy	*Bajra*	*Jowar*
Man-hours:			
September–May	40	20	25
June–August	50	35	40
Net annual cash income (Rs.)	1,200	800	850

The cooperative farm wishes to determine how much acreage should be planted of each of the crops and how many cows and hens should be kept to maximise its net cash income.

Formulate this problem as LPP to maximize net annual cash income.

Mathematical Formulation

The data of the problem is summarised below :

Constraints	*Cows*	*Hens*	*Crop*			*Extra hours*		*Total availability*
			Paddy	*Bajra*	*Jowar*	*Sept.–May*	*June–Aug.*	
Man-hours:								
Sept.–May	100	0.6	40	20	25	1	—	3,500
June–Aug.	50	0.4	50	35	40	—	1	4,000
Land	1.5	—	1	1	1	—	—	100
Cow	1	—	—	—	—	—	—	32
Hens	—	1	—	—	—	—	—	4,000
Net annual cash income (Rs.)	3,500	200	1,200	800	850	2	3	

Decision variables : Let x_1 and x_2 = number of cows and hens respectively;

x_3, x_4 and x_5 = acreage planting of paddy, bajra and jowar respectively.

x_6 and x_7 = extra man-hours utilized during Sept.–May and May–June respectively.

Objective function : Maximization of net annual cash, *viz.*,

$$z = 3{,}500x_1 + 200x_2 + 1{,}200x_3 + 800x_4 + 850x_5 + 2x_6 + 3x_7$$

Constraints :

(*i*)
$$\left.\begin{aligned} 100x_1 + 0.6x_2 + 40x_3 + 20x_4 + 25x_5 + x_6 &= 3{,}500 \\ 50x_1 + 0.4x_2 + 50x_3 + 35x_4 + 40x_5 + x_7 &= 4{,}000 \end{aligned}\right\} \quad \textit{(Man-hours)}$$

(*ii*)
$$1.5x_1 + x_3 + x_4 + x_5 \le 100 \quad \textit{(Land availability)}$$

(*iii*)
$$x_1 \le 32 \text{ and } x_2 \le 4{,}000 \quad \textit{(Livestock constraints)}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0,\ x_6 \ge 0 \text{ and } x_7 \ge 0 \quad \textit{(Non-negative restrictions)}$$

210. (Trim Loss Problem). *Rolls of paper having a fixed length and width of 180 cm are being manufactured by a paper mill. These rolls have to be cut to satisfy the following demand :*

Width :	*80 cm*	*45 cm*	*27 cm*
No. of rolls :	200	120	130

Obtain the linear programming formulation of the problem to determine the cutting pattern, so that the demand is satisfied and wastage of paper is a minimum.

Mathematical Formulation

Here, the key decision is to determine how the paper rolls be cut to the required width so that trim loss (wastage) is a minimum.

Various alternatives for the number of rolls are given below :

Feasible patterns of cutting	*No. of rolls cut*	*Wastage per roll*	Rolls obtained from each roll of width		
			80 cm	*45 cm*	*27 cm*
80 + 80	x_1	20	2	—	—
80 + 45 + 45	x_2	10	1	2	—
80 + 45 + 27 + 27	x_3	1	1	1	2
80 + 27 + 27 + 27	x_4	19	1	—	3
45 + 45 + 45 + 45	x_5	0	—	4	—
45 + 45 + 45 + 27	x_6	18	—	3	1
45 + 45 + 27 + 27 + 27	x_7	9	—	2	3
45 + 27 + 27 + 27 + 27 + 27	x_8	0	—	1	5
27 + 27 + 27 + 27 + 27 + 27	x_9	18	—	—	6

Decision variables : Let x_j $(j = 1, 2, \ldots, 9)$ represent the number of times each cutting alternative is to be used.

Objective function : The objective is to minimize the wastage produced, *i.e.*,

$$\text{Minimize} \quad z = 20x_1 + 10x_2 + x_3 + 19x_4 + 18x_6 + 9x_7 + 18x_9$$

Constraints : Constraints on the availability of rolls and the requirement of desired width of rolls are :

$$2x_1 + x_2 + x_3 + x_4 = 200 \qquad \textit{(80 cm rolls)}$$

$$2x_2 + x_3 + 4x_5 + 3x_6 + 2x_7 + x_8 = 120 \qquad \textit{(45 cm rolls)}$$

$$2x_3 + 3x_4 + x_6 + 3x_7 + 5x_8 + 6x_9 = 130 \qquad \textit{(27 cm rolls)}$$

$$x_j \geq 0 ; \quad j = 1, 2, 3, \ldots, 9. \qquad \textit{(Non-negativity)}$$

211. (Marketing Problem). *The PQR Stone Company sells stone secured from any of the three adjacent quarries. The stone sold by the company must conform to the following specifications :*

Material X equal to 30 per cent ; Material Y equal to or less than 40 per cent ; Material Z between 30 per cent and 40 per cent.

Stone from quarry A costs Rs. 100 per tonne and has the following properties :

Material X : 20 per cent, material Y : 60 per cent, and material Z : 20 per cent.

Stone from quarry B costs Rs. 120 per tonne and has the following properties :

Material X : 40 per cent, material Y : 30 per cent, and material Z : 30 per cent.

Stone from quarry C costs Rs. 150 per tonne and has the following properties :

Material X : 10 per cent, material Y : 40 per cent, and material Z : 50 per cent.

Formulate the above as an LPP to minimise cost per tonne.

Mathematical Formulation

The data of the problem is summarised below :

Material	*Quarry*			*Specification*
	A	*B*	*C*	
X	20%	40%	10%	30%
Y	60%	30%	40%	less than or equal to 40%
Z	20%	30%	50%	between 30% and 40%
Cost per tonne (Rs.)	100	120	150	

Decision variables : Let x_1, x_2, x_3 represent the proportion of stone in tonne to be produced from quarries *A*, *B* and *C* respectively.

Objective function : The objective is to minimize the total cost, *i.e.*,

$$\text{Minimize } z = 100x_1 + 120x_2 + 150x_3$$

Constraints : The constraints are on the three types of material of the stone. These are :

$$20x_1 + 40x_2 + 10x_3 = 30 \qquad \text{(Material } X\text{)}$$

$$60x_1 + 30x_2 + 40x_3 \le 40 \qquad \text{(Material } Y\text{)}$$

$$\left.\begin{aligned} 20x_1 + 30x_2 + 50x_3 &\le 40 \\ 20x_1 + 30x_2 + 50x_3 &\ge 30 \end{aligned}\right\} \qquad \text{(Material } Z\text{)}$$

Also,

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \qquad \text{(Non-negative restrictions)}$$

212. (Blending Problem). *Three grades of coal A, B and C contain ash and phosphorus as impurities. In a particular industrial process a fuel obtained by blending the above grades containing not more than 25% ash and 0.03% phosphorus is required. The maximum demand of the fuel is 100 tons. Percentage impurities and costs of the various grades of coal are shown below. Assuming that there is an unlimited supply of each grade of coal and there is no loss in blending, formulate the blending problem to minimise the cost.*

Coal grade	*% ash*	*% phosphorus*	*Cost per ton (in Rs.)*
A	*30*	*0.02*	240
B	*20*	*0.04*	300
C	*35*	*0.03*	280

[Pune M.B.A. 2009]

Mathematical Formulation

Decision variables : Let x_1 = tons of grade *A* coal,

x_2 = tons of grade *B* coal, and x_3 = tons of grade *C* coal.

Objective function : Minimize $z = 240x_1 + 300x_2 + 280x_3$

Constraints : $0.3x_1 + 0.2x_2 + 0.35x_3 \le 0.25(x_1 + x_2 + x_3)$

or

$$x_1 - x_2 + 2x_3 \le 0 \qquad \text{(ash)}$$

$$\frac{0.02}{100}x_1 + \frac{0.04}{100}x_2 + \frac{0.03}{100}x_3 \le \frac{0.03}{100}(x_1 + x_2 + x_3)$$

or

$$-x_1 + x_2 \le 0 \qquad \text{(phosphorus)}$$

$$x_1 + x_2 + x_3 \le 100 \qquad \text{(demand of fuel)}$$

$$x_1 \ge 0,\ x_2 \ge 0 \text{ and } x_3 \ge 0. \qquad \text{(Non-negative restrictions)}$$

213. (Capital Budgeting Problem). *An engineering company is planning to diversify its operations during the year 2006-07. The company has allocated capital expenditure budget equal to Rs. 5.15 crore in the year 2006 and Rs. 6.50 crore in the year 2007. The company has five investment projects under consideration. The estimated net returns at present value and expected cash expenditures of each project in the two years are as follows :*

Project	*Estimated net returns (in '000 Rs.)*	*Cash expenditure (in '000 Rs.)*	
		Year 2006	*Year 2007*
A	*240*	*120*	*320*
B	*390*	*550*	*594*
C	*80*	*118*	*202*
D	*150*	*250*	*340*
E	*182*	*324*	*474*

Assume that the returns from a particular project would be in direct proportion to the investment in it, so that, for example, if in a project, say A, 20% (of 120 in 2006 and of 320 in 2007) is invested, then the resulting net returns in it would be 20% (of 240). This assumption also implies that

individuality of the project should be ignored. Formulate this capital budgeting problem as a Linear Programming model to maximize the net returns.

Mathematical Formulation

Decision variables : Let x_1, x_2, x_3, x_4 and x_5 represent the proportion of investment in project *A, B, C, D* and *E* respectively.

Objective function : The objective is to maximize the net returns. Thus, the objective function is :

$$\text{Maximize} \quad z = 240x_1 + 390x_2 + 80x_3 + 150x_4 + 182x_5$$

Constraints : Capital expenditure budget constraints are :

$$120x_1 + 550x_2 + 118x_3 + 250x_4 + 324x_5 \le 515 \qquad \textit{(for 2006)}$$

$$320x_1 + 594x_2 + 202x_3 + 340x_4 + 474x_5 \le 650 \qquad \textit{(for 2007)}$$

Also, $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0$ and $x_5 \ge 0$. *(Non-negative restrictions)*

In addition to the above, there is a requirement of 0–1 (*i.e.*, of integrality). So,

$$x_j = 1 \text{ or } 0 \quad (j = 1, 2, 3, 4, 5).$$

214. (Media Problem). *The Marketing Department of Everest Company has collected information on the problem of advertising for its products. This relates to the advertising media available, the number of families expected to be reached with each alternative, cost per advertisement, the maximum availability of each medium and the expected exposure of each one (Measured as the relative value of one advertisement in each of the media).*

The information is given as under :

Advertising media	*No. of families to cover*	*Cost/ad. (Rs.)*	*Maximum availability (No. of times)*	*Expected exposure (Units)*
TV (30 sec.)	*3,000*	*8,000*	*8*	*80*
Radio (15 sec.)	*7,000*	*3,000*	*30*	*20*
Sunday Edition (1/4 page)	*5,000*	*4,000*	*4*	*50*
Magazine (1 page)	*2,000*	*3,000*	*2*	*60*

Other information and requirements :

(*a*) *The advertising budget is Rs. 70,000.* (*b*) *At least 40,000 families should be covered.*

(*c*) *At least 2 insertions be given in Sunday Edition of Daily but not more than 4 advertisements should be given on the TV.*

Formulate this as a linear programming problem. The company's objective is to maximize the expected exposure.

Mathematical Formulation

The key decision to be made is to determine the number of advertisements to be brought in Television, Radio, Sunday Edition and Magazine.

Decision variables. Let x_1, x_2, x_3 and x_4 represent the number of advertisements in TV, Radio, Sunday Edition and Magazine respectively.

Using the given information, the appropriate linear programming problem is :

$$\text{Maximize (total expected exposure)} \quad z = 80x_1 + 20x_2 + 50x_3 + 60x_3$$

subject to the constraints :

$$3{,}000x_1 + 7{,}000x_2 + 5{,}000x_3 + 2{,}000x_4 \ge 40{,}000 \qquad \textit{(Families)}$$

$$8{,}000x_1 + 3{,}000x_2 + 4{,}000x_3 + 3{,}000x_4 \ge 70{,}000 \qquad \textit{(Budget)}$$

$$x_1 \le 8,\ x_2 \le 30,\ x_3 \le 4,\ x_4 \le 2 \qquad \textit{(Availability)}$$

$$x_3 \ge 2 \text{ and } x_1 \le 4 \qquad \textit{(No. of advt.)}$$

Also, $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$ and $x_4 \ge 0$. *(Non-negative restrictions)*

215. (Investment Problem). *A person is interested in investing Rs. 5,00,000 in a mix of investments. The investment choices and expected rates of return on each one of them are :*

Investment	*Projected rate of return*
Mutual Fund A	*0.12*
Mutual Fund B	*0.09*
Money Market Fund	*0.08*
Government Bonds	*0.085*
Share Y	*0.16*
Share X	*0.18*

The investor wants at least 35 per cent of his investment in government bonds. Because of the higher perceived risk of the two shares, he has specified that the combined investment in these not to exceed Rs. 80,000. The investor has also specified that at least 20 per cent of the investment should be in the money market fund and that the amount of money invested in shares should not exceed the amount invested in mutual funds. His final investment condition is that the amount invested in mutual fund A should be no more than the amount invested in mutual fund B. The problem is to decide the amount of money to invest in each alternative so as to obtain the highest annual total return. Formulate the above as linear programming problem.

Mathematical Formulation

The *key decision* is to determine the amount to be invested in various options available to the investor.

Decision variables : Let the decision variables x_1, x_2, x_3, x_4, x_5 and x_6 represent the amount invested in mutual fund *A*, mutual fund *B*, money-market fund, government bonds, shares *Y* and shares *X* respectively.

Then, the appropriate L.P.P. is :

$$\text{Maximize } z = 0.12x_1 + 0.09x_2 + 0.08x_3 + 0.085x_4 + 0.16x_5 + 0.18x_6$$

subject to the constraints :

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 5{,}00{,}000 \qquad \textit{(Availability of money)}$$

$$x_4 \ge 1{,}75{,}000 \qquad \textit{(Government bond)}$$

$$x_5 + x_6 \le 80{,}000 \qquad \textit{(Shares)}$$

$$x_3 \ge 1{,}00{,}000 \qquad \textit{(Money market fund)}$$

$$x_5 + x_6 \le x_1 + x_2 \qquad \textit{(Shares vers. Mutual fund)}$$

$$x_1 \le x_2 \qquad \textit{(Mutual fund)}$$

Also, $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0$ and $x_6 \ge 0$. *(Non-negative restrictions)*

216. (Personnel-Manpower Problem). *The super bazar in a city daily needs between 32 and 40 workers depending on the time of day. The rush hours are between noon and 2 p.m. The table below indicates the number of workers needed at various hours when the bazar is open.*

Time period	*Number of workers needed*
9 a.m.—11 a.m.	*32*
11 a.m.— 1 p.m.	*40*
1 p.m.— 3 p.m.	*35*
3 p.m.— 5 p.m.	*33*

The super bazar now employs 34 full-time workers, but needs a few part-time workers also. A part-time worker must put in exactly 4 hours per day, but can start any time between 9 a.m. and 1 p.m. Full-time workers work from 9 a.m. to 5 p.m. but are allowed an hour for lunch (half of the full-timers eat at 12 noon, the other half at 1 p.m.).

The management of the super bazar limits part-time hours to a maximum of 50 per cent of the day's total requirement.

Part-timers earn Rs. 48 per day on the average, while full-timers earn Rs. 140 per day in salary and benefits on the average. The management wants to set a schedule that would minimize total manpower costs. Formulate this problem as an Linear Programming model to minimize total daily manpower cost.

Mathematical Formulation

The *key decision* is to determine the number of full-time and part-time workers needed in Super Bazar.

Decision variables : Let y represents the number of full-time workers and x_j represents the part-time workers starting at 9 a.m., 11 a.m. and 1 p.m. respectively. (j = 1, 2, 3).

Using the given information, the appropriate LPP is :

$$\text{Minimize (total daily manpower cost)} \quad z = 140y + 48(x_1 + x_2 + x_3)$$

subject to the constraints :

$$y + x_1 \geq 32 \qquad \textit{(9 a.m.—11 a.m. need)}$$

$$\tfrac{1}{2}y + x_1 + x_2 \geq 40 \qquad \textit{(11 a.m.—1 p.m. need)}$$

$$\tfrac{1}{2}y + x_2 + x_3 \geq 35 \qquad \textit{(1 p.m.—3 p.m. need)}$$

$$y + x_3 \geq 33 \qquad \textit{(3 p.m.—5 p.m. need)}$$

$$y \leq 34 \qquad \textit{(Full timers available)}$$

$$4(x_1 + x_2 + x_3) \leq 0.50(32 + 40 + 35 + 33)$$

Part timers' hour cannot exceed 50% of total hours required each day which is the sum of the workers needed each hour)

Also, $y \geq 0$ and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$ *(Non-negative restrictions)*

217. (Manpower Schedule Problem). *A city hospital has the following minimal daily requirements for nurses :*

Period	*Clock time (24 hour day)*	*Minimal number of nurses required*
1	*6 a.m.—10 a.m.*	*2*
2	*10 a.m.— 2 p.m.*	*7*
3	*2 p.m.— 6 p.m.*	*15*
4	*6 p.m.—10 p.m.*	*8*
5	*10 p.m.— 2 a.m.*	*20*
6	*2 a.m.— 6 a.m.*	*6*

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimal number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this problem as a linear programming model by setting up appropriate constraints and objective function.

[Delhi M.B.A. (October) 2002, B.Com. (Hons.) 2006]

Mathematical Formulation

The *key decision* to be made is to determine the number of nurses to be employed in the city hospital.

Decision variables : Let x_j (j = 1, 2, 3, 4, 5, 6) be the number of nurses reporting to work on shift j.

Also, since negative values of x_j are meaningless, we must have $x_j \geq 0$ for all j.

Objective function : If the number of nurses reporting to work on shifts 1, 2, 3, ..., 6 is known, then the manpower schedule can be decided. Thus, the objective is to minimize the total number of nurses (manpower for all the six periods together). As such, the objective is to minimize the linear function : $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$.

Constraints : In any 8-hour period, the number of nurses reporting at the beginning of the period and the number of nurses continuing from the earlier period should be at least equal to the minimum requirement.

Thus, we have the following constraints :

$$x_1 + x_2 \geq 7, \quad x_2 + x_3 \geq 15, \quad x_3 + x_4 \geq 8,$$
$$x_4 + x_5 \geq 20, \quad x_5 + x_6 \geq 6, \quad x_6 + x_1 \geq 2.$$

$\therefore$ The given *linear programming problem* is :

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to the constraints :

$$x_1 + x_2 \geq 7, \quad x_2 + x_3 \geq 15, \quad x_3 + x_4 \geq 8,$$
$$x_4 + x_5 \geq 20, \quad x_5 + x_6 \geq 6, \quad x_6 + x_1 \geq 2;$$

and $$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0, \ x_5 \geq 0, \ x_6 \geq 0.$$

218. (Transportation Problem). *ABC manufacturing company wishes to develop a monthly production schedule for the next three months. Depending upon the sales commitments, the company can either keep the production constant, allowing fluctuation in inventory or inventories can be maintained at a constant level, with fluctuating production. Fluctuating production makes overtime work necessary, the cost of which is estimated to be double the normal production cost of Rs. 12 per unit. Fluctuating inventories result in an inventory carrying cost of Rs. 2 per unit/month. If the company fails to fulfil its sales commitment, it incurs a shortage cost of Rs. 4 per unit/month. The production capacities for the next three months are shown below :*

Month	*Production capacity*		*Sales*
	Regular	*Overtime*	
1	*50*	*30*	*60*
2	*50*	*0*	*120*
3	*60*	*30*	*40*

Formulate this problem as an LP model so as to minimize the total cost. **[Delhi M.B.A. 2000]**

Mathematical Formulation

We are given the following information :

Normal production cost : Rs. 12 per unit, Overtime cost : Rs. 24 per unit,

Inventory carrying cost : Rs. 2 per unit per month, Shortage cost : Rs. 4 per unit per month.

Let there be five sources of supply—three regular and two overtime (because second month overtime production is zero) production capacities. The demand for the three months will be the sales during these months.

It is assumed that all supplies against order have to be fulfilled and can be made in the subsequent month, if not possible during the current month of order with additional cost equivalent to shortage cost, *i.e.*, in month 2. The cumulative production of first two months in regular and overtime is of 130 units, while the orders are for 180 units. The balance can be supplied during month 3 at an additional production cost of Rs. 4.

The total cost (normal cost + inventory carrying cost + shortage cost, if any) is given below :

Month	*Cost (Rs.)*			*Overtime cost (Rs.)*		*Sales*
	Month 1	*Month 2*	*Month 3*	*Month 1*	*Month 2*	
Month 1	Rs. 12	Rs. 16	Rs. 20	Rs. 24	Rs. 32	60
Month 2	Rs. 14	Rs. 12	Rs. 16	Rs. 26	Rs. 28	120
Month 3	Rs. 16	Rs. 14	Rs. 12	Rs. 28	Rs. 24	40
Supply (units)	50	50	60	30	30	220

Decision variables : Let x_{ij} represents the number of units to be shipped from source of supply i (i = 1, 2, 3) to destination j (j = 1, 2, 3, 4, 5).

Objective function : The objective is to minimize the total cost involved, *i.e.*, to minimize the expression

$$z = \{(12x_{11} + 16x_{12} + 20x_{13} + 24x_{14} + 32x_{15}) + (14x_{21} + 12x_{22} + 16x_{23} + 26x_{24} + 28x_{25}) + (16x_{31} + 14x_{32} + 12x_{33} + 28x_{34} + 24x_{35})\}.$$

Constraints : The constraints on demand and supply are :

(*i*)
$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 60,$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 120, \text{ and}$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 40.$$

(*ii*)
$$x_{11} + x_{21} + x_{31} = 50, \quad x_{12} + x_{22} + x_{32} = 50,$$
$$x_{13} + x_{23} + x_{33} = 60, \quad x_{14} + x_{24} + x_{34} = 30,$$
$$x_{15} + x_{25} + x_{35} = 30,$$

(*iii*) $x_{ij} \geq 0$ for all i and j (*Non-negative restrictions*)

219. (Cargo Loading Problem). *A ship has three cargo holds, forward, aft and centre; the capacity limits are :*

Forward	*2,000 tonnes*	*100,000 m³*
Centre	*3,000 tonnes*	*135,000 m³*
Aft	*1,500 tonnes*	*30,000 m³*

The following cargoes are offered; the ship owners may accept all or any part of each commodity :

Commodity	*Amount (tonnes)*	*Volume per tonne (m³)*	*Profit per tonne (Rs.)*
A	*6,000*	*60*	*60*
B	*4,000*	*50*	*80*
C	*2,000*	*25*	*50*

In order to preserve the trim of the ship, the weight in each hold must be proportional to the capacity in tonnes. The objective is to maximize the profit. Formulate the linear programming model for this problem.

Mathematical Formulation

Let x_{1A}, x_{2A}, x_{3A} be the weights (in kg.) of the commodity A to be accommodated in *forward, centre* and *aft* portions respectively. Similarly, let x_{1B}, x_{2B}, x_{3B} and x_{1C}, x_{2C}, x_{3C} be the corresponding weights (in kg.) of B and C. Then the *objective* is to maximize profit :

$$z = 60(x_{1A} + x_{2A} + x_{3A}) + 80(x_{1B} + x_{2B} + x_{3B}) + 50(x_{1C} + x_{2C} + x_{3C})$$

Various constraints are :

$$\left.\begin{aligned} x_{1A} + x_{2A} + x_{3A} &\leq 6000 \\ x_{1B} + x_{2B} + x_{3B} &\leq 4000 \\ x_{1C} + x_{2C} + x_{3C} &\leq 2000 \end{aligned}\right\} \quad \textit{(Commodity cargo)}$$

$$\left.\begin{aligned} x_{1A} + x_{1B} + x_{1C} &\leq 2000 \\ x_{2A} + x_{2B} + x_{2C} &\leq 3000 \\ x_{3A} + x_{3B} + x_{3C} &\leq 1500 \end{aligned}\right\} \quad \textit{(Weight capacity)}$$

$$\left.\begin{aligned} 60x_{1A} + 50x_{1B} + 25x_{1C} &\leq 1,00,000 \\ 60x_{2A} + 50x_{2B} + 25x_{2C} &\leq 1,35,000 \\ 60x_{3A} + 50x_{3B} + 25x_{3C} &\leq 30,000 \end{aligned}\right\} \quad \textit{(Volume capacity)}$$

$$x_{iA} \geq 0, \; x_{iB} \geq 0 \text{ and } x_{iC} \geq 0 \text{ for } i = 1, 2, 3. \quad \textit{(Non-negative restrictions)}$$

220. *WELLTYPE manufacturing company produces three types of typewriter : Manual typewriters, Electronic typewriters, and Deluxe Electronic typewriters. All the three models are required to be machined first and then assembled. The time required for the various models are as follows :*

Types	*Machine time (in hours)*	*Assembly time (in hours)*
Manual Typewriter	*15*	*4*
Electronic Typewriter	*12*	*3*
Deluxe Electronic Typewriter	*14*	*5*

The total available machine time and assembly time are 3,000 hours and 1,200 hours respectively. The data regarding the selling price and variable costs for the three types are :

	Manual	*Electronic*	*Deluxe*
Selling Price (Rs.)	*4,100*	*7,500*	*14,600*
Labour, material and other variable costs (Rs.)	2,500	*4,500*	*9,000*

The company sells all the three types on credit basis, but will collect the amounts on the first of next month. The labour, material and other variable expenses will have to be paid in cash. This company has taken a loan of Rs. 40,000 from a cooperative bank and this company will have to repay it to the bank on 1-4-2006. The TNC Bank from whom this company has borrowed Rs. 60,000 has expressed its approval to renew the loan.

The Balance Sheet of this Company as on 31-03-2006 is as follows :

Liabilities	Rs.	*Assets*	Rs.
Equity Share Capital	*1,50,000*	*Land*	*90,000*
Capital Reserve	*15,000*	*Building*	*70,000*
General Reserve	*1,10,000*	*Plant & Machinery*	*1,00,000*
Profit & Loss A/c	*25,000*	*Furniture & Fixtures*	*15,000*
Long-term Loan	*1,00,000*	*Vehicles*	*30,000*
Loan from TNC Bank	*60,000*	*Inventory*	*5,000*
Loan from Co-operative Bank	*40,000*	*Receivables*	*50,000*
		Cash	*1,40,000*
	Total 5,00,000		*Total 5,00,000*

The company will have to pay a sum of Rs. 10,000 towards the salary for top management executives and other fixed overheads for the month. Interest on long-term loans is to be paid every month at 24% per annum. Interest on loan from TNC and Co-operative Banks may be taken to be Rs. 1,200 for the month. Also, this company has promised to deliver 2 Manual typewriters and 8 Deluxe Electronic typewriters to one of its valued customers next month. Also, make sure that the level of operations in this company is subject to the availability of cash next month. This company will also be able to sell all their types of typewriters in the market. The Senior Manager of this company desires to know as to how many units of each typewriter must be manufactured in the factory next month so as to maximise the profits of the company. Formulate this as a linear programming problem. The formulated problem need not be solved.

Mathematical Formulation

Let x_1, x_2 and x_3 denote the number of manual, electronic and deluxe electronic typewriters to be produced by the company next month. The profit contributions from the above three products are : Rs. 1,600, Rs. 3,000 and Rs. 5,600 respectively.

The objective, therefore, is to maximize $z = 1{,}600x_1 + 3{,}000x_2 + 5{,}600x_3$ subject to the constraints :

$$15x_1 + 12x_2 + 14x_3 \le 3{,}000 \qquad \textit{(Machine time restriction)}$$

$$4x_1 + 3x_2 + 5x_3 \le 1{,}200 \qquad \textit{(Assembly time restriction)}$$

From the balance-sheet, we observe that the cash availability

= Cash Balance + Receivables – Co-operative bank loan paid off – Top management salary *plus* fixed overheads – Interest on long-term loan – Interest on short-term loan.

= 1,40,000 + 50,000 – 40,000 – 10,000 – 2,000 – 1,200

= 1,36,800 (Interest on long-term loan is 24% p.a. paid every month, *viz.*, Rs. 2,000 p.m.)

Thus, $$2{,}500x_1 + 4{,}500x_2 + 9{,}000x_3 \le 1{,}36{,}800 \qquad \textit{(Cash Requirement Restriction)}$$

Also, $$x_1 \ge 2,\ x_2 \ge 0 \text{ and } x_3 \ge 8.$$

All x_j (j = 1, 2, 3) are also integral valued.

PROBLEMS

221. A firm manufactures headache pills in two sizes *A* and *B*. Size *A* contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size *B* contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP. ***[GGSIP Univ. B.B.A. 2011]***

222. Old hens can be bought for Rs. 2.00 each and young ones cost Rs. 5.00 each. The old hens lay 3 eggs per week and the young ones, 5 eggs per week, each being worth 30 paise. A hen costs Re. 1.00 per week to feed. If I have only Rs. 80.00 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6.00 per week, assuming that I cannot house more than 20 hens? Write a mathematical model of the problem. ***[Pune M.B.A. 1999; Delhi B.F.I.A. 2008]***

223. An animal feed company must produce 200 lbs of a mixture containing the ingredients X_1 and X_2. X_1 costs Rs. 3 per lb. and X_2 costs Rs. 8 per lb. Not more than 80 lbs. of X_1 can be used and minimum quantity to be used for X_2 is 60 lbs. Find how much of each ingredient should be used if the company wants to minimise the cost. Formulate.

224. A factory engaged in the manufacturing of pistons, rings and valves for which the profits per unit are Rs. 10, 6 and 4 respectively wants to decide the most profitable mix. It takes one hour of preparatory work, ten hours of machining and two hours of packing and allied formalities for a piston. Corresponding time requirements for rings and valves are 1, 4 and 2 and 1, 5 and 6 hours respectively. The total number of hours available for preparatory work, machining and packing and allied formalities are 100, 600 and 300 respectively. Determine the most profitable mix, assuming that what all produced can be sold. Formulate the LPP.

225. A ship is to carry 3 types of liquid cargo—*X*, *Y* and *Z*. There are 3,000 litres of *X* available, 2,000 litres of *Y* available and 1,500 litres of *Z* available. Each litre of *X*, *Y* and *Z* sold fetches a profit of Rs. 20, Rs. 35 and Rs. 40 respectively. The ship has 3 cargo holds—*A*, *B* and *C*, of capacities 2,000, 2,500 and 3,000 litres respectively. From stability considerations, it is required that each hold be filled in the same proportion. Formulate the problem of loading the ship as a linear programming problem. State clearly all decision variables and constraints.

226. A firm produces three products *A*, *B* and *C*. It uses two types of raw materials *I* and *II* of which 5,000 and 7,500 units respectively are available. The raw material requirements per unit of the products are given below :

Raw material	*Requirement per unit of product*		
	A	*B*	*C*
I	3	4	5
II	5	3	5

The labour time for each unit of product *A* is twice that of product *B* and three times that of product *C*. The entire labour force of the firm can produce the equivalent of 3,000 units. The minimum demand of the three

products is 600, 650 and 500 units respectively. Also the ratios of the number of units produced must be equal to 2 : 3 : 4. Assuming the profits per unit of *A*, *B* and *C* as Rs. 50, 50 and 80 respectively.

Formulate the problem, as a linear programming model in order to determine the number of units of each product which will maximize the profit. *[Madras MCA 1999]*

227. A manufacturing company has four machine centres—machining, grinding, assembling and painting, to produce four products *A*, *B*, *C* and *D*. The available number of hours per month in each of these centres are 150, 40, 750 and 500 hours respectively. The number of hours required by each product in each of the centre are given below :

		A	*B*	*C*	*D*
Machining	...	1.5	1	2	1.5
Grinding	...	3	3	4	3
Assembling	...	6	5	6	7
Painting	...	3	4	4	3

The profit contribution for products *A*, *B*, *C* and *D* is Rs. 10, Rs. 15, Rs. 12 and Rs. 20 respectively. Assuming there is enough demand for these products, how much of each product should the company manufacture to maximise the total profit?

228. A firm buys castings of *P* and *Q* type of parts and sells them as finished product after machining, boring and polishing. The purchasing cost for castings are Rs. 3 and Rs. 4 each for parts *P* and *Q* and selling costs are Rs. 8 and Rs. 10 respectively. The per hour capacity of machines used for machining, boring and polishing for two products is given below :

Capacity (per hour)	*Parts*	
	P	*Q*
Machining	30	50
Boring	30	45
Polishing	45	30

The running costs for machining, boring and polishing are Rs. 30, Rs. 22.5 and Rs. 22.5 per hour respectively.

Formulate the linear programming problem to find out the product mix to maximize the profit. *[Delhi M.B.A. (PT) 2002; Panjab B.Com. 2007]*

229. A Mutual Fund Company has Rs. 20 lakhs available for investment in Government Bonds, blue chip stocks, speculative stocks and short-term bank deposits. The annual expected return and risk factor are given below :

Type of investment	*Annual expected return (%)*	*Risk factor (0-100)*
Government Bonds	14	12
Blue Chip Stocks	19	24
Speculative Stocks	23	48
Short-term Deposits	12	6

Mutual fund is required to keep at least Rs. 2 lakhs in short-term deposits and not to exceed an average risk factor of 42. Speculative stocks must be at most 20 per cent of the total amount invested. How should mutual fund invest the funds so as to maximize its total expected annual return? Formulate this as a Linear Programming Problem. Do not solve it. *[Himachal B.Tech. (Mech.) 2006]*

230. An advertising company wishes to plan an advertising campaign in three different media—television, radio and magazines. The purpose of the advertising is to reach as many potential customers as possible. Results of a market study are given below (in Rs.) :

	Television		*Radio*	*Magazine*
	Prime day	*Prime time*		
Cost of an advertising unit	40,000	75,000	30,000	15,000
Number of potential customers reached per unit	4,00,000	9,00,000	5,00,000	2,00,000
Number of women customers reached per unit	3,00,000	4,00,000	2,00,000	1,00,000

The company does not want to spend more than Rs. 8 lakhs on advertising. It is further required that

(*i*) at least 2 million exposures take place among women;

(*ii*) advertising on television be limited to Rs. 5,00,000;

(*iii*) at least 3 advertising units be bought in prime day and two units during prime time; and

(*iv*) the number of advertising units on radio and magazine should each be between 5 and 10.

231. Formulate the following linear programming problems. Do not solve.

A publishing house publishes three weekly magazines—*Daily Life, Agriculture Today* and *Surf's Up*. Publication of one issue of each of the magazines requires the following amounts of production time and paper :

Magazine	*Production (hour)*	*Paper (kg)*
Daily Life	0.01	0.2
Agriculture Today	0.03	0.5
Surf's Up	0.02	0.3

Each week the publisher has available 120 hours of production time and 3,000 kg. of paper. Total circulation for all three magazines must exceed, 5,000 issues per week, if the company is to keep its advertisers. The selling price per issue is Rs. 22.50 for *Daily Life*, Rs. 40.00 for *Agriculture Today*, and Rs. 15.00 for *Surf's Up*. Based on past sales the publisher knows that the maximum weekly demand for *Daily Life* is 3,000 issues; for *Agriculture Today*, 2000 issues; and for *Surf's Up*, 6,000 issues. The production manager wants to know the number of issues of each magazine to produce weekly in order to maximize total sales revenue. ***[Delhi M.B.A. (Nov.) 2008]***

232. A media specialist plans to allocate advertising expenditure in three media whose unit costs of a message are Rs. 7,500, Rs. 6,250 and Rs. 5,000 respectively. The total advertising budget available for the year is Rs. 2,50,000. The first medium is monthly magazine and it is desired to advertise not more than once in one issue. At least five advertisements should appear in the second medium and the number of advertisements in the third medium should strictly lie between 6 and 10. The effective audience for unit advertisement in the three media is given below:

Medium	:	I	II	III
Expected affective audience	:	100,000	80,000	50,000

Formulate a linear programming problem to find the optimum allocation of advertisements in three media that would maximize the total effective audience. ***[Delhi PG Dip. in Glob. Bus. Oper. 2011]***

233. Evening shift resident doctors in a government hospital work five consecutive days and have two consecutive days off. Their five days of work can start on any day of the week and the schedule rotates indefinitely. The hospital requires the following minimum number of doctors working :

Sun.	*Mon.*	*Tues.*	*Wed.*	*Thurs.*	*Fri.*	*Sat.*
35	55	60	50	60	50	45

No more than 40 doctors can start their five working days on the same day. Formulate the general linear programming model to minimize the number of doctors employed by the hospital.

234. *A* Company manufactures three grades of paints—Venus, Diana and Aurora. The plant operates on a three-shift basis and the following data are available from the production records :

Requirement of resource	*Grade*			*Availability (capacity per month)*
	Venus	*Diana*	*Aurora*	
Special additive (kgs. per litre)	0.30	0.15	0.75	600 tonnes
Milling (kilolitres per machine shift)	2.0	3.0	5.0	100 machine shifts
Packing (kilolitres per shift)	12.0	12.0	12.0	80 shifts

There are no limitations on other resources. The particulars of sale forecasts and estimated contribution to overheads and profits are given below :

	Venus	*Diana*	*Aurora*
Maximum possible sales per month (in kilolitres)	100	400	600
Contribution (Rs. per kilolitre)	4,000	3,500	2,000

Due to commitments already made, a minimum of 200 kilolitres per month of *Aurora* has to be necessarily supplied during the next year.

Just as the company was able to finalise the monthly production programme for the next 12 months, an offer was received from a nearby competitor for hiring 40 machine shifts per month of milling capacity for grinding *Diana* paint, that can be spared for at least a year. However, due to additional handling and the profit margin of the competitor involved, by using this facility, the contribution from *Diana* will get reduced by Re. 1 per litre. Formulate (do not solve) the linear programming model for determining the monthly production programme to maximize contribution. *[Delhi M.B.A. (Nov.) 2003]*

235. Two alloys *A* and *B* are made from four different metals, *I, II, III* and *IV* according to the following specifications :

A : at most 80% of *I*
at most 30% of *II*
at least 50% of *III*

B : between 40% & 60% of *II*
at least 30% of *III*
at most 70% of *IV*

The four metals are extracted from three different ores whose constituents percentage of these metals, maximum available quantity and cost per tonne are as follows :

Ore	*Maximum quantity (tonnes)*	*Constituent %*					*Price (Rs./tonne)*
		I	*II*	*III*	*IV*	*Others*	
1	1,000	20	10	30	30	10	30
2	2,000	10	20	30	30	10	40
3	3,000	5	5	70	20	0	50

Assuming the selling prices of alloys *A* and *B* are Rs. 200 and Rs. 300 per tonne respectively, formulate the above as a linear programming problem selecting appropriate objective function and constraints. *[Poona M.B.A. 1998]*

236. A manufacturer of biscuits is considering four types of gift packs containing three types of biscuits—Orange cream (OC), Chocolate cream (CC), and Wafers (W). Market research study conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand :

Assortments	*Contents*	*Selling price per kg. in Rs.*
A	Not less than 40% of OC Not more than 20% of CC Any quantity of W	20
B	Not less than 20% of OC Not more than 40% of CC Any quantity of W.	25
C	Not less than 50% of OC Not more than 10% of CC Any quantity of W	22
D	No restrictions	12

For the biscuits, the manufacturing capacity and costs are given below :

Type of biscuit	*Plant capacity (kg/day)*	*Manufacturing cost (Rs./kg)*
OC	200	8
CC	200	9
W	150	7

Formulate the LP model to find the production schedule which maximise the profit assuming that there are no market restrictions. *[Delhi M.B.A. April 1999]*

237. A 24-hour B.P.O. Company has the following minimal requirements for IT personnel and software engineers:

Time of day	Minimum number of IT personnel and software engineers required
Midnight — 4 A.M.	7
4 A.M. — 8 A.M.	20
8 A.M. — Noon	14
Noon — 4 P.M.	20
4 P.M. — 8 P.M.	10
8 P.M. — Midnight	5

Shift 1 follows immediately after shift 6. A B.P.O. employee works eight consecutive hours, starting at the beginning of one of the six periods. The company wants to determine how many IT experts should work each shift in order to minimize the total number of persons employed while still satisfying the staffing requirements.

Formulate the problem as a linear programming problem. *[Delhi PG Dip. in Glob. Bus. Oper. 2010]*

238. A company manufacturing laminated sheets is considering to adopt an optimum advertising strategy to advertise its product in South India for which it has an advertising budget of Rs. 5 lakh for the coming year. The company has decided to advertise in three magazines, say, M_1, M_2 and M_3. M_1 and M_3 are monthly and M_2 is a fortnightly. The company is targeting the audience having age 20—40 years, monthly income above Rs. 10,000, and education above SSC. These three characteristics have been assigned weights as 30 per cent, 50 per cent and 20 per cent respectively. The following table gives, for each of the three magazines, the percentage of readers having the above characteristics:

Characteristics	*Percentage of readers of magazine*		
	M_1	M_2	M_3
Age: 20—40 years	70	60	90
Monthly income: > Rs. 10,000	50	40	75
Education: above SSC	80	70	80

The advertisement may be in colour, or black and white. The efficiency index for colour advertisement may be taken to be 0.3 and that for black and white 0.2. The cost per insertion of colour, black and white advertisements and the readership for the three magazines are given below:

Magazine	*Cost (in Rs.) per insertion*		*Readership (in lakh)*
	Colour	*Black and white*	
M_1	18,000	12,000	4
M_2	16,000	10,000	3
M_3	19,000	15,000	2

To create the desired impact on the audience, it has been felt that during the coming year, at least five insertions are necessary in each of M_1 and M_3, and at least four in M_2. No issue will have more than one insertion.

Formulate the above as a linear programming problem to find the optimum advertising strategy for the coming year in order to maximize the expected effective exposure. (Do not solve the LPP.) *[Delhi M.I.B. (Apr.) 2001]*

239. Dr. Shilpa Soni, the head administrator at *XYZ* hospital must determine a schedule for nurses to make sure that there are enough nurses on duty throughout the day. During the day, the demand for nurses varies. Dr. Shilpa has broken the day into 12 two-hour periods. The slowest time of the day encompasses the three periods from 12:00 mid-night to 6:00 A.M., which beginning at midnight, require a minimum of 30, 20 and 40 nurses, respectively.

The demand for nurses steadily increases during the next four daytime periods. Beginning with the 6:00 A.M. to 8:00 A.M. period, a minimum of 50, 60, 80 and 80 nurses are required for these four periods, respectively. After 2:00 P.M. the demand for nurses decreases during the afternoon and evening hours. For the five two-hours periods beginning at 2:00 P.M. and ending at midnight 70, 70, 60, 50 and 50 nurses are required, respectively. A nurse reports for duty at the beginning of one of the two hour periods and works eight consecutive hours (which is required in the nurses contract). Dr. Shilpa Soni wants to determine a nursing schedule that will meet the hospital's minimum requirements throughout the day while using the minimum number of nurses.

[Delhi M.B.A. (Nov.) 2008]

240. In a chemical industry, two products A and B are made involving two operations. The production of B also results in a by-product C. The product A can be sold at a profit of Rs. 3 per unit and B at a profit of Rs. 8 per unit. The by-product C has a profit of Rs. 2 per unit. Forecasts show that up to 5 units of C can be sold. The company gets 3 units of C for each unit of B produced. The manufacturing times are 3 hours per unit for A on each of the operation one and two and 4 hours and 5 hours per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18 hours and 21 hours of operation one and two respectively. The company desires to know as to much A and B should be produced keeping C in mind to make the highest profit. Formulate LP model for this problem.

[C.A. Final (May) 2000]

241. A computer company produces three types of models, which are first required to be machined and then assembled. The time (in hours) for these operations for each model is given below:

Model	*Machine time*	*Assembly time*
P III	20	5
P II	15	4
Celeron	12	3

The total available machine-time and assembly-time are 1,000 hours and 1,500 hours, respectively. The selling price and other variable costs for three models are:

	P III	*P II*	*Celeron*
Selling price (Rs.)	3,000	5,000	15,000
Labour, material, and other variable costs (Rs.)	2,000	4,000	8,000

The company has taken a loan of Rs. 50,000 from a nationalised bank, which is required to be re-paid on 1.4.2001. In addition, the company has borrowed Rs. 1,00,000 from *XYZ* Cooperative Bank. However, this bank has given its consent to renew the loan.

The balance sheet of the company as on 31.3.2001 is as follows:

Liabilities	*(Rs.)*	*Assets*	*(Rs.)*
Equity share capital	1,00,000	Land	80,000
Capital reserve	20,000	Buildings	50,000
Profit and loss account	30,000	Plant and machinery	1,00,000
Long-term loan	2,00,000	Furniture, etc.	20,000
Loan from *XYZ* cooperative bank	1,00,000	Vehicles	40,000
Loan from nationalised bank	50,000	Cash	2,10,000
Total	5,00,000		5,00,000

The company is required to pay a sum of Rs. 15,000 towards the salary. Interest on long-term loan is to be paid every month at the rate of 18 per cent per annum. Interest on loan from *XYZ* cooperative and nationalised banks may be taken as Rs. 1,500 per month. The company has promised to deliver three *P* III, two *P* II, and five Celeron type of computers to M/s *ABC* Ltd. next month. The level of operation in the company is subject to the availability of cash next month.

The company manager is willing to know that how many units of each model must be manufactured next month, so as to maximise the profit.

Formulate a linear programming problem for the above. ***[C.A. Final (May) 2001]***

242. A firm places an order for a particular product at the beginning of each month and the product is received at the end of the month. The firm sells the same product during the month from the stocks and it can sell any quantity.

The prices at which the firm buys and sells vary every month. The following table shows the projected buying and selling prices for the next four months :

Month	*Selling price (Rs.) (During the month)*	*Purchase price (Rs.) (Beginning of the month)*
April	—	75
May	90	75
June	60	60
July	75	—

The firm has no stock on hand as on April 1, and does not wish to have any stock at the end of July. The firm has a warehouse of limited size, which can hold a maximum of 150 units of the product.

Formulate this problem as an LP model to determine the number of units to buy and sell each month so as to maximize the profits from its operations.

[Delhi M.B.A. (Nov.) 2009]

MULTIPLE CHOICE QUESTIONS

1. Linear programming problem (LPP) must have an
(*a*) Objective (goal) that we aim to maximize or minimize.
(*b*) Constraints (restrictions) that we need to specify.
(*c*) Decision variables (activities) that we need to determine.
(*d*) all of the above.

2. Which of the following is not correct about LPP ?
(*a*) All constraints must be linear relationships.
(*b*) Objective function must be linear.
(*c*) All the constraints and decision variables must be of either '≤' or '≥' type.
(*d*) All decision variables must be non-negative.

3. Which of the following is not associated with an LPP ?
(*a*) Proportionality (*b*) Uncertainty
(*c*) Additivity (*d*) Divisibility.

4. Which of the following is correct ?
(*a*) Linear programming takes into consideration the effect of time and uncertainty.
(*b*) An LPP can have only two decision variables.
(*c*) Decision variables in an LPP may be more or less than the number of constraints.
(*d*) Linear programming deals with problems involving only a single objective.

5. A constraint in an LPP restricts
(*a*) value of objective function (*b*) value of a decision variable
(*c*) use of available resource (*d*) uncertainty of optimum value.

6. A constraint in an LPP is expressed as
(*a*) an equation with = sign (*b*) inequality with ≥ sign
(*c*) inequality with ≤ sign (*d*) any of the above.

7. Non-negativity condition is an important component of LPP, because
(*a*) variables are interrelated in terms of limited resources
(*b*) value of variables make sense and correspond to real world problems
(*c*) value of variables should remain under the control of decision-maker
(*d*) none of the above.

8. Minimization of objective function in LPP means
(*a*) least value chosen among the allowable decisions
(*b*) greatest value chosen among the allowable decisions
(*c*) both (*a*) and (*b*)
(*d*) none of the above.

9. Which of the following is not correct ?
For the application of linear programming, the following requirements must hold :
(*a*) There should be an objective which should be clearly identifiable and measurable in quantitative terms.
(*b*) The activities to be included should be distinctly identifiable and measurable in quantitative terms.
(*c*) The relationships representing the objective as well as the resource limitation considerations must be linear in nature.
(*d*) There should be a series of infeasible alternative courses of action available to the decision-maker.

10. Which of the following is correct ?
(*a*) Variables can be unrestricted in the context of an LPP.
(*b*) For an LPP having n-decision variables, there must be an equal number of constraints.
(*c*) Objective function specifies the dependent relationship between the decision variables and the objective fu
(*d*) Linear programming is probabilistic in nature.

ANSWERS

1. (*d*)	**2.** (*c*)	**3.** (*b*)	**4.** (*c*)	**5.** (*c*)
6. (*d*)	**7.** (*b*)	**8.** (*a*)	**9.** (*d*)	**10.** (*c*)

REVIEW QUESTIONS

1. Explain the linear programming problem giving two examples. *[Madurai M.Com. 2002]*
2. What are the components of linear programming model ? Explain them in brief. *[Madras M.B.A. 2004]*
3. (*a*) What is linear programming? Explain by taking an example, what are its limitations? Discuss. *[Delhi M.B.A. (PT.) 2008]*

   (*b*) What is linear programming? Discuss its importance using one application to a business problem. *[Delhi M.B.A. (PT.) 2015]*
4. Two of the major limitations of linear programming are: assumptions of 'additivity' and 'single objective'. Elaborate by giving appropriate examples. *[Delhi M.B.A. 2008]*
5. Give the general form of a basic model of linear programming. *[Madurai M.Com. 2002]*
6. State any two applications of linear programming. *[Bharthidasan M.Com. 2007]*
7. Explain how the linear programming technique can be helpful in decision-making in the areas of marketing and finance. *[Delhi M.Com. 1998]*
8. Describe the major applications of linear programming in business and industry, pointing out limitations, if any. *[Delhi M.B.A. 2002]*
9. Explain the advantages of linear programming and its limitations. *[Pune M.B.A. 1999]*
10. Explain the major characteristics of a linear programming problem. *[Delhi M.Com. 2000]*
11. What are the major assumptions and limitations of linear programming ? *[Madurai M.Com. 2003; Madras M.B.A. 2004]*
12. Give the mathematical and economic structure of linear programming problems. What requirements should be met in order to apply linear programming ?
13. Describe the role of linear programming in Health Care Administration, bringing out limitations, if any. *[Delhi M.B.A. (HCA) PT 2003]*
14. Discuss the assumptions in linear programming. *[Madras M.B.A. (Nov.) 2006]*
15. "Linear Programming has no real application." Do you agree? Discuss. *[Delhi M.B.A. 2009]*
16. "Linear Programming is useful Management Science Technique, but it has some limitations." Discuss it and give appropriate examples. *[Delhi M.B.A. (Nov.) 2009]*

3

Linear Programming Problem —Graphical Solution and Extension

"Graph is the best tool to visualise any concept"

3:1. INTRODUCTION

Linear programming problems involving two decision variables can easily be solved by *graphical method*. The method also provides an insight into the concepts of *Simplex Method*—a powerful technique to solve the linear programming problems involving three or more decision variables.

3:2. GRAPHICAL SOLUTION METHOD

The major steps in the solution of a linear programming problem by graphical method are summarised as follows :

Step 1. Identify the problem — the decision variables, the objective and the restrictions.

Step 2. Set up the mathematical formulation of the problem.

Step 3. Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step 4. The feasible region obtained in *step 3* may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step 5. Find out the value of the objective function at each corner (solution) point determined in *step 4*.

Step 6. Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the *optimum feasible solution*.

Remarks. 1. The above method is known as *Search Approach Method*.

2. Another method known as *Iso-Profit* or *Iso-cost approach*, involves the following steps :

(*a*) First *four steps* are same as in the Search Approach. In the *fifth step* we choose a convenient profit (or cost) and draw iso-profit (iso-cost) line so that it falls within the feasible region.

(*b*) Move this iso-profit (or iso-cost) line parallel to itself farther (closer) from (to) the origin.

(*c*) Identify the optimum solution as the coordinates of that point on the feasible region touched by the highest possible iso-profit line (or lower-possible iso-cost line).

(*d*) Compute the *optimum feasible solution*.

SAMPLE PROBLEMS

301. *A company makes two kinds of leather belts. Belt A is a high quality belt, and belt B is of lower quality. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires*

twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (Both A and B combined). Belt A requires a fancy buckle and only 400 buckles per day are available. There are only 700 buckles a day available for belt B. Determine the optimal product mix. **[Madras M.B.A. 2006; Delhi M.Com. 2005; M.B.A. (Nov.) 2009]**

Solution.

Step 1. The appropriate mathematical formulation of the given linear programming problem is :

Maximize $z = 4x_1 + 3x_2$ subject to the constraints :

$$2x_1 + x_2 \le 1{,}000 \qquad \textit{(Time constraint)}$$

$$x_1 + x_2 \le 800 \qquad \textit{(Availability of Leather)}$$

$$x_1 \le 400 \text{ and } x_2 \le 700 \qquad \textit{(Availability of Buckles)}$$

$$x_1 \ge 0 \text{ and } x_2 \ge 0,$$

where x_1 = number of belts of type A, and x_2 = number of belts of type B.

Step 2. Next we construct the graph by considering the cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2) ; any point satisfying the conditions $x_1 \ge 0$ and $x_2 \ge 0$ lies in the first quadrant.

Now, the inequalities are graphed taking them as equations, *e.g.*, the first constraint $2x_1 + x_2 \le 1000$ will be graphed as $2x_1 + x_2 = 1000$. The equation is re-written as $\frac{x_1}{500} + \frac{x_2}{1000} = 1$.

This equation indicates that when it is plotted on the graph, it cuts an x_1-intercept of 500 and x_2-intercept of 1000. These two points are then connected by a straight line which is shown in Fig. 3.1(*a*) as line *AB*. Any point (representing a combination of x_1 and x_2) that falls on this line or in the area below it, is acceptable in so far as this constraint is concerned. The region *OAB* formed by two axes and the line representing the equation $2x_1 + x_2 = 1000$ is the region containing acceptable values of x_1 and x_2 in respect of this constraint.

Similarly, the constraint $x_1 + x_2 \le 800$ can be plotted. The line *CD* in Fig. 3.1(*b*) represents the equation $x_1 + x_2 = 800$. The region *OCD*, formed by the two axes and this line represents the area in which any point would satisfy this constraint of leather availability. Further, the constraints $x_1 \le 400$ and $x_2 \le 700$ are also plotted on the graph which represents the area between the two axes and the lines $x_1 = 400$ and $x_2 = 700$ as shown in Fig. 3.1(*b*).

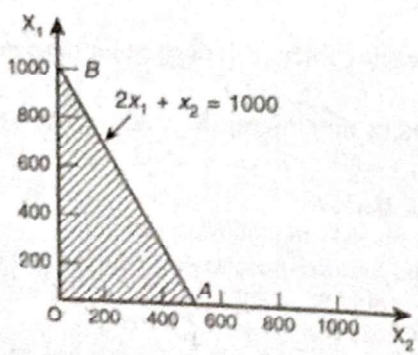

Fig. 3.1(*a*)

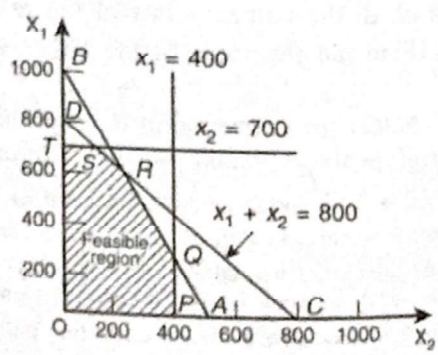

Fig. 3.1(*b*)

Now all the constraints have been graphed. The area bounded by all these constraints, called feasible *region* or *solution space*, is as shown in Fig. 3.1(*b*) by the shaded area *OPQRST*.

Step 3. The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of the extreme points are :

$O = (0, 0)$, $P = (400, 0)$, $Q = (400, 200)$, $R = (200, 600)$, $S = (100, 700)$, and $T = (0, 700)$.

Step 4. We now compute the z-values corresponding to the extreme points :

Extreme point	(x_1, x_2)	$z = 4x_1 + 3x_2$
O	(0, 0)	0
P	(400, 0)	1600
Q	(400, 200)	2200
R	(200, 600)	**2600** ← maximum
S	(100, 700)	2500
T	(0, 700)	2100

Step 5. The optimum solution is that extreme point for which the objective function has the largest value. Thus, the optimum solution occurs at the point R, *i.e.*, $x_1 = 200$ and $x_2 = 600$ with the objective function value of Rs. 2600.

Hence, to maximize profit, the company should produce 200 belts of type A and 600 belts of type B per day.

Alternative Method (Iso-profit approach)

The feasible region (solution space) obtained in *step 2* is as shown in Fig. 3.1(*c*) by the shaded area *OPQRST*.

Let the profit to the company (arbitrary) is Rs. 1200. The objective function then becomes :

$$4x_1 + 3x_2 = 1200.$$

We draw this equation as a straight line in the feasible region shown in Fig. 3.1(*c*). This line is known as *iso-profit line.*

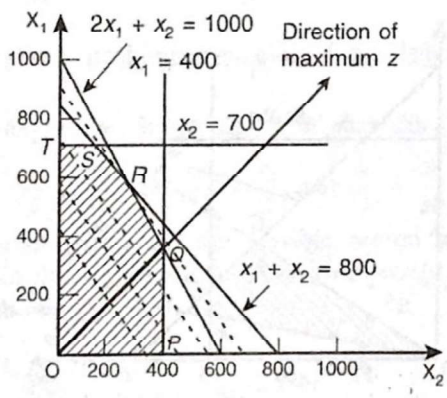

Fig. 3.1(*c*)

It may be noted that the iso-profit function (objective value function) is a straight line on which every point has the same total profit.

Now, we move the iso-profit line parallel to itself farther from the origin. We observe that one of the iso-profit line touches only point R before leaving the feasible region. This iso-profit line is termed as highest possible iso-profit line and point R gives the extreme point of the solution space.

Hence, the optimum feasible solution is :

$$x_1 = 200 \text{ and } x_2 = 600 \text{ with Maximum } z = \text{Rs. } 2600.$$

302. *Let us assume that you have inherited Rs. 1,00,000 from your father-in-law that can be invested in a combination of only two stock portfolios, with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return, you will*

not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?

Formulate this as a Linear Programming Problem and solve it by Graphic Method.

[C.A. Final (May) 1999]

Solution.

Step 1. The appropriate mathematical formulation of the linear programming problem is :

$$\text{Maximize } z = 0.10x_1 + 0.20x_2 \text{ subject to the constraints :}$$

$$x_1 + x_2 \le 1{,}00{,}000, \quad x_1 \le 75{,}000, \quad x_2 \le 75{,}000$$

$$0.10x_1 + 0.20x_2 \ge 0.12(x_1 + x_2) \quad \text{or} \quad -0.02x_1 + 0.08x_2 \ge 0$$

$$4x_1 + 9x_2 \le 6(x_1 + x_2) \quad \text{or} \quad -2x_1 + 3x_2 \le 0$$

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0$$

where x_1 = amount invested in portfolio *1*, and x_2 = amount invested in portfolio 2.

Step 2. The first constraint $x_1 + x_2 \le 1{,}00{,}000$ can be graphed by plotting the straight line $\frac{x_1}{1{,}00{,}000} + \frac{x_2}{1{,}00{,}000} = 1$. This cuts a x_1-intercept and x_2-intercept of 1,00,000 each. The area below this line represents the feasible area in respect of this constraint. Similarly, the other constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 75{,}000$, $x_2 = 75{,}000$, $-2x_1 + 3x_2 = 0$, and $-0.02x_1 + 0.08x_2 = 0$. Here, the area below the first three lines and beyond the fourth line gives the feasible region in respect of these four constraints.

Thus, the feasible region in respect of the given problem is as shown in Fig. 3.2.

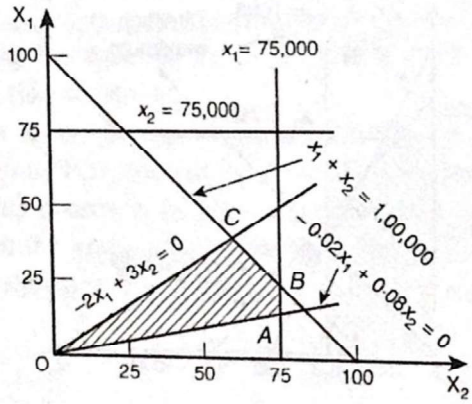

Fig. 3.2

Step 3. The coordinates of the extreme points are : $O = (0, 0)$, $A = (75{,}000, 18{,}750)$, $B = (75{,}000, 25{,}000)$ and $C = (60{,}000, 40{,}000)$.

Step 4. The z-values corresponding to the extreme points are :

Extreme point	(x_1, x_2)	$z = 0.10x_1 + 0.20x_2$
O	(0, 0)	0
A	(75,000, 18,750)	11,250
B	(75,000, 25,000)	12,500
C	(60,000, 40,000)	**14,000** ⟵ maximum

Hence, the optimum solution is :

$$x_1 = 60{,}000, \quad x_2 = 40{,}000 \quad \text{and} \quad \text{maximum return} = \text{Rs. } 14{,}000.$$

303. *A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary*

to buy two additional products, say, A and B. One unit of product A contains 36 units of X, 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs Rs. 20 per unit and product B Rs. 40 per unit.

Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphic method. **[C.A. Final (May) 2002]**

Solution. *Step 1.* The data of the given problem can be summarised as follows :

Nutrient constituents	*Nutrient content in product*		*Minimum amount of nutrient*
	A	*B*	
X	36	06	108
Y	03	12	36
Z	20	10	100
Cost of Product	Rs. 20	Rs. 40	

Making use of above information, the appropriate mathematical formulation of the linear programming problem is :

$$\text{Minimize } z = 20x_1 + 40x_2 \text{ subject to the constraints :}$$

$$36x_1 + 6x_2 \geq 108,\ 3x_1 + 12x_2 \geq 36,\ 20x_1 + 10x_2 \geq 100, \text{ and } x_1, x_2 \geq 0.$$

where x_1 = number of units of product A, and x_2 = number of units of product B.

Step 2. Consider now a set of cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2), any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The constraints of the given problem are plotted as described earlier by treating them as equations :

$$36x_1 + 6x_2 = 108,\ 3x_1 + 12x_2 = 36 \text{ and } 20x_1 + 10x_2 = 100$$

or

$$\frac{x_1}{3} + \frac{x_2}{18} = 1,\ \frac{x_1}{12} + \frac{x_2}{3} \text{ and } \frac{x_1}{5} + \frac{x_2}{10} = 1.$$

The area beyond these lines represents the feasible region in respect of these constraints; any point on the straight lines or in the region above these lines would satisfy the constraints. The feasible region of the problem is as shown in Fig. 3.3.

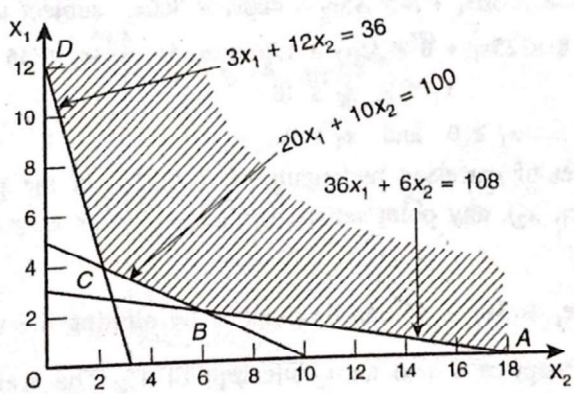

Fig. 3.3

Step 3. The coordinates of the extreme points of the feasible region are :

$$A = (0, 18),\ B = (2, 6),\ C = (4, 2) \text{ and } D = (12, 0).$$

Step 4. The value of objective function at each of the extreme points can be evaluated as follows :

Extreme point	(x_1, x_2)	$z = 20x_1 + 40x_2$
A	(0, 18)	720
B	(2, 6)	280
C	(4, 2)	**160** ←——minimum
D	(12, 0)	240

Hence, the optimum solution is to purchase 4 units of product *A* and 2 units of product *B* in order to maintain a minimum cost of Rs. 160.

304. *A company has two grades of inspectors 1 and 2 who are to be assigned to a quality inspection work. It is required that at least 1,800 pieces are inspected per 8-hour day. Grade 1 inspectors can check pieces at the rate of 25 per hour with an accuracy of 98%. Grade 2 inspectors can check at the rate of 15 pieces per hour with an accuracy of 95%. The wage rate for grade 1 inspector is Rs. 40 per hour while that of grade 2 is Rs. 30 per hour. Each time an error is caused by the inspector the cost to the company is Rs. 20. The company has eight grade 1 and ten grade 2 inspectors. The company wants to determine the optimal assignment of inspectors to minimise total inspection cost. Formulate it as LPP and solve using graphical method.* ***[Annamalai M.B.A. 2002]***

Solution.

Step 1. The data of the given problem can be summarised as follows :

	Grade 1 inspector	*Grade 2 inspector*
Number of inspectors	8	10
Rate of checking per hour	25 pieces	15 pieces
Inaccuracy in checking	1 – 0.98 = 0.02	1 – 0.95 = 0.05
Cost of inaccuracy in checking	Rs. 20	Rs. 20
Wage rate per hour	Rs. 40	Rs. 30

Let x_1 and x_2 designate the number of Grade 1 and Grade 2 inspectors, respectively.

Hourly costs of each Grade *1* and Grade *2* inspectors are given by :

Grade *1* inspector : Rs. $(40 + 20 \times 0.02 \times 25)$ = Rs. 50.

Grade *2* inspector : Rs. $(30 + 20 \times 0.05 \times 15)$ = Rs. 45.

Using the above information, the appropriate linear programming problem is :

Minimize $z = 8 \times 50x_1 + 8 \times 45x_2 = 400x_1 + 360x_2$ subject to the constraints :

$$8 \times 25x_1 + 8 \times 15x_2 \geq 1{,}800 \text{ or } 5x_1 + 3x_2 \geq 45 \qquad \textit{(Number of pieces)}$$

$$x_1 \leq 8, \; x_2 \leq 10 \qquad \textit{(Number of inspectors)}$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \qquad \textit{(Non-negative restriction)}$$

Step 2. Consider a set of cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2), any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The first constraint $5x_1 + 3x_2 \geq 45$ can be graphed by plotting the straight line $\frac{x_1}{9} + \frac{x_2}{15} = 1$.

This gives an x_1-intercept of 9 and an x_2-intercept of 15. The area away from the origin, *i.e.*, above this line represents the feasible area of this constraint. Similarly, the other two constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 8$ and $x_2 = 10$. Here, the area below these lines (towards the origin) gives the feasible area. The common area *ABC* which satisfies all the three constraints is the solution space.

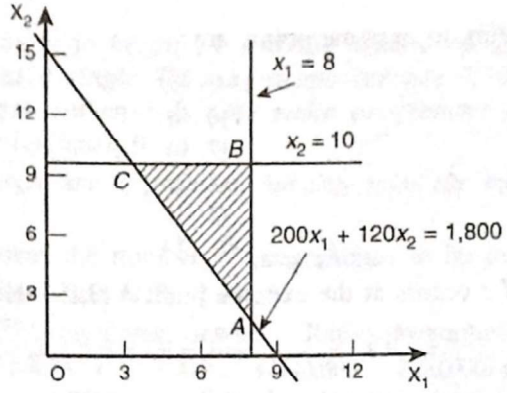

Fig. 3.4

Step 3. The coordinates of the extreme points are : $A = (8, 5/3)$, $B = (8, 10)$, and $C = (3, 10)$.

Step 4. The z-value corresponding to the extreme points are :

Extreme point	(x_1, x_2)	$z = 400x_1 + 360x_2$
A	$(8, 5/3)$	**3,800** ⟵ minimum
B	$(8, 10)$	6,800
C	$(3, 10)$	4,800

Hence, the optimum solution is :

$$x_1 = 8 \text{ and } x_2 = 1.7 \text{ or } 2 \text{ (approximately), minimum } z = 3{,}800$$

Thus, 8 grade *1* inspectors and 2 grade *2* inspectors should be assigned to have Rs. 3,800 as the total minimum inspection cost.

305. *Use the graphical method to solve the following LPP :*

$$\textit{Minimize } z = -x_1 + 2x_2; \textit{ subject to the constraints :}$$

$$-x_1 + 3x_2 \le 10, \quad x_1 + x_2 \le 6,$$

$$x_1 - x_2 \le 2, \quad \textit{and} \quad x_1 \ge 0, \; x_2 \ge 0.$$

Solution. The constraints are re-written in the intercept form. Thus, we write

$$\frac{x_1}{-10} + \frac{x_2}{10/3} \le 1, \quad \frac{x_1}{6} + \frac{x_2}{6} \le 1, \quad \text{and} \quad \frac{x_1}{2} + \frac{x_2}{-2} \le 1$$

First, we treat these inequalities as equations and graph them as straight lines. Now, considering the inequalities, shade the feasible region for each constraint. The common region *OABCD* gives the solution space of the LPP. It may be noted that we have considered the feasible region only in the first quadrant of the cartesian axis, as $x_1 \ge 0$ and $x_2 \ge 0$.

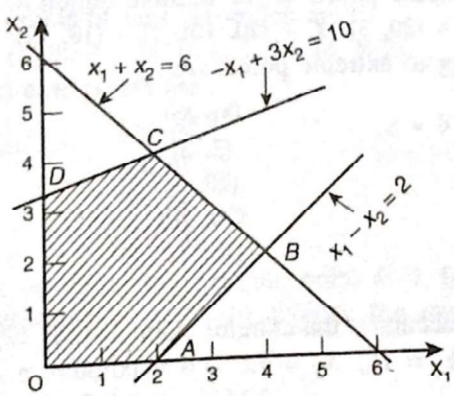

Fig. 3.5

The coordinates of the extreme points are :

$$O = (0, 0), \; A = (2, 0), \; B = (4, 2), \; C = (2, 4) \text{ and } D = \left(0, \frac{10}{3}\right).$$

The z-value corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = -x_1 + 2x_2$
O	(0, 0)	0
A	(2, 0)	**−2** ⟵ minimum
B	(4, 2)	0
C	(2, 4)	6
D	$(0, \frac{10}{3})$	$\frac{20}{3}$

The minimum value of z occurs at the extreme point A (2, 0). Hence, the optimum solution of the LPP is :

$$x_1 = 2,\ x_2 = 0 \text{ and minimum } z = -2.$$

306. *Use the graphical method to solve the following LPP :*

$$\textit{Maximize } z = 2x_1 + 3x_2;\ \textit{subject to the constraints :}$$

$$x_1 + x_2 \le 30,\ x_1 - x_2 \ge 0,\ x_2 \ge 3,$$

$$0 \le x_1 \le 20 \text{ and } 0 \le x_2 \le 12.$$

Solution. To graph the given inequalities, we first treat them as equations $x_1 + x_2 = 30$, $x_1 - x_2 = 0$, $x_2 = 3$, $x_1 = 20$ and $x_2 = 12$ and plot each of these equations as straight lines. We use the inequality condition of each constraint to plot the corresponding feasible region. The common region $ABCDE$ satisfying all the inequalities (feasible region) is shown in shaded Fig. 3.6.

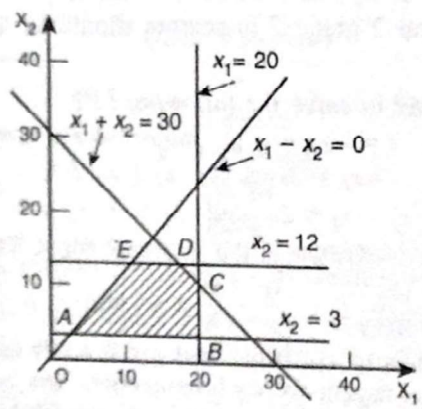

Fig. 3.6

The coordinates of the extreme points of the feasible region are :

$$A = (3, 3),\ B = (20, 3),\ C = (20, 10),\ D = (18, 12), \text{ and } E = (12, 12).$$

The z-values corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = 2x_1 + 3x_2$
A	(3, 3)	15
B	(20, 3)	49
C	(20, 10)	70
D	(18, 12)	**72** ⟵ maximum
E	(12, 12)	60

The maximum value of z occurs at the extreme point D (18, 12). Hence, the optimum solution is :

$$x_1 = 18,\ x_2 = 12 \text{ and maximum } z = 72.$$

307. *The advertising agency wishes to reach two types of audiences, customers with annual incomes greater than Rs. 40,000 (target audience A) and customers with annual incomes of less than Rs. 40,000 (target audience B). The total advertising budget is Rs. 2,00,000. One programme of TV advertising costs Rs. 50,000; one programme of radio advertising costs Rs. 20,000. For contract*

reasons, at least 3 programmes ought to be on TV and the number of radio programmes must be limited to 5. Surveys indicate that a single TV programme reaches 7,50,000 customers in target audience A and 1,50,000 in target audience B. One radio programme reaches 40,000 in target audience A and 2,60,000 in target audience B.

Formulate this as a linear programming problem and determine the media mix to maximize the total reach. **[Delhi M.B.A. (Oct.) 2010]**

Solution. Let x_1 and x_2 represent the number of programmes to be released on TV and radio, respectively. The mathematical formulation of the LPP is :

$$\text{Maximize } z = \text{TV programme viewers} + \text{Radio programme viewers}$$
$$= (7{,}50{,}000 + 1{,}50{,}000)x_1 + (40{,}000 + 2{,}60{,}000)x_2$$
$$= 9{,}00{,}000x_1 + 3{,}00{,}000x_2$$

subject to the constraints :

$$50{,}000x_1 + 20{,}000x_2 \le 2{,}00{,}000 \qquad \textit{(Budget constraint)}$$
$$x_1 \ge 3 \text{ and } x_2 \le 5 \qquad \textit{(Programme constraints)}$$
$$x_1 \ge 0 \text{ and } x_2 \ge 0 \qquad \textit{(Non-negative restriction)}$$

For solving this L.P.P. graphically, we first consider the inequalities as equations and draw the straight lines of these equations. Now, consider the inequalities and shade the feasible region satisfying all the inequalities, *viz.*, the region *ABC* consisting of the extreme points *A, B* and *C*.

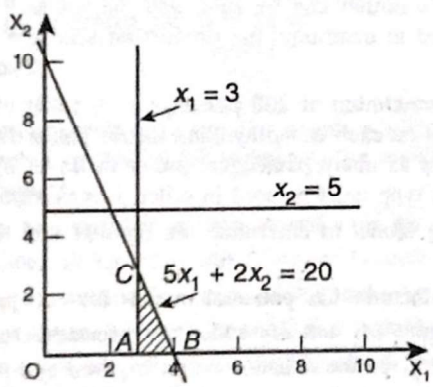

Fig. 3.7

The coordinates of the extreme points of feasible region are :

$$A = (3, 0),\ B = (4, 0) \text{ and } C = (3, 5/2).$$

The z-value corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = 9{,}00{,}000x_1 + 3{,}00{,}000x_2$
A	(3, 0)	27,00,000
B	(4, 0)	**36,00,000** ⟵ maximum
C	(3, 5/2)	34,50,000

Since, the maximum value of z occurs at the extreme point B (4, 0) ; the agency must release 4 programmes on TV and no programme on radio to achieve the maximum reach of 36,00,000 audiences.

PROBLEMS

308. A firm manufactures two products *A* and *B* on which the profits earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product *A* requires one minute of processing time on M_1 and two minutes on M_2 while *B* requires one minute on M_1 and one minute on M_2. Machine M_1 is

available for not more than 7 hours and 30 minutes, while machine M_2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactured to get maximum profit.

[Madurai M.Com. (Nov.) 2003]

309. A farmer has a supply of chemical fertilizer of Type I which contains 10% nitrogen and 6% phosphoric acid and Type II fertilizer which contains 5% nitrogen and 10% phosphoric acid. After testing the soil conditions of a field, it is found that at least 14 kg. of nitrogen and 14 kg. of phosphoric acid is required for a good crop. The fertilizer Type I costs Rs. 2 per kg. and the Type II costs Rs. 3 per kg. How many kilograms of each fertilizer should be used to meet the requirement and the cost be minimum? (Use graphic method).

[Jiwaji M.B.A. March 2010]

310. The *ABC* Company has been a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and marketing of *AM* and *AM-FM* radios. It has built a new plant that can operate 48 hours per week. Production of an *AM* radio in the new plant will require 2 hours and production of an *AM-FM* radio will require 3 hours. Each *AM* radio will contribute Rs. 40 to profits while an *AM-FM* radio will contribute Rs. 80 to profits. The marketing departments, after extensive research, has determined that a maximum of 15 *AM* radios and 10 *AM-FM* radios can be sold each week.

(*i*) Formulate a linear programming model to determine the optimum production mix of *AM-FM* radios that will maximize profits.

(*ii*) Solve the above problem using the graphic model. *[Delhi M.B.A. (Nov.) 1998]*

311. A firm makes two products X and Y and has a total production capacity of 9 tonnes per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of X and at least 3 tonnes of Y per day to another company. Each tonne of X requires 20 machine hours production time and each tonne of Y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All the firm's output can be sold, and the product made is Rs. 80 per tonne of X and Rs. 120 per tonne of Y. It is required to determine the production schedule for maximum profit and to calculate this profit. *[Delhi M.B.A. 2009; C.A. Final (Nov.) 2010]*

312. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each economy class ticket. The airline reserves at least 20 tickets for first class seats. However, at least 4 times as many passengers prefer to travel by economy class as to the first class. Determine how many tickets of each type must be sold in order to maximize the total profit for the airline.

Formulate a linear programming model to determine the optimal mix that will maximize profit. Solve it by graphic method. *[M.D. Univ. M.B.A. (Nov.) 2010]*

313. A consumer durable manufacturer has potential market for two products P_1 and P_2. In the production shop he finds that two of his machines M_1 and M_2 which are necessary for the products P_1 and P_2 are having limited capacity. The following table gives the machine hours required and machine hours available :

MACHINE-HOURS REQUIRED PER UNIT

Product	M_1	M_2
P_1	3	2
P_2	4	1
Available machine hours	36	16

The costing section has arrived at the contribution of P_1 as Rs. 30 per piece and of P_2 as Rs. 50 per piece. The sales division has indicated that they can sell any amount of P_2 but only 10 units of P_1. Find out the best manufacturing strategy.

314. A small scale manufacturer has production facilities for producing two different products. Each of the products requires three different operations: grinding, assembling and testing. Product A requires 15, 20 and 10 minutes to grind, assembly and test respectively, whereas product B requires 7.5, 40 and 45 minutes for grinding, assembling and testing. The production run calls for at least 7.5 hours of grinding time, at least 20 hours of assembling time, and at least 15 hours of testing time. If product A costs Rs. 60 and product B costs Rs. 90 to manufacture, determine the number of units of each product the firm should produce in order to minimize the cost of operations. *[IGNOU M.B.A. 2008]*

315. An oil refinery can blend three grades of crude oil to produce quality R and quality S petrol. Two possible blending processes are available. For each production run the older process uses 5 units of crude A, 7 units

of crude B and 2 units of crude C to produce 9 units of R and 7 units of S. The newer process uses 3 units of crude A, 9 units of crude B and 4 units of crude C to produce 5 units of R and 9 units of S petrol. Because of prior contract commitments, the refinery must produce at least 500 units of R and at least 300 units of S for the next month. It has available 1500 units of crude A, 1900 units of crude B and 1000 units of crude C. For each unit of R the refinery receives Rs. 60, while for each unit of S it receives Rs. 90. Write down the linear programming formulation of the problem so as to maximize the revenue and solve it by graphical method.

316. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the inputs and outputs per production run are as follows :

Process	*Input (units)*		*Output (units)*	
	Grade A	*Grade B*	*Gasoline X*	*Gasoline Y*
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crudes A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run for process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Solve the LP problem by the graphical method. *[Rajasthan M.B.A. 2008]*

317. A scrap metal dealer has received an order from a customer for at least 2,000 kilograms of scrap metal. The customer requires that at least 1,000 kilograms of the shipment of metal must be high quality copper that can be melted down and used to produce copper tubings. Furthermore, the customer will not accept delivery of the order if it contains more than 175 kilograms of metal that he deems unfit for commercial use, *i.e.*, metal that contains an excessive amount of impurities and cannot be melted down and refined profitably.

The dealer can purchase scrap metal from two different supplies in unlimited quantities with the following percentages (by weight) of high quality copper and unfit scrap.

	Supplier A	*Supplier B*
Copper	25%	75%
Unfit scrap	5%	10%

The costs per kilogram of metal purchase from supplier A and supplier B are Re. 1 and Rs. 4 respectively. The problem is to determine (using graphic method of LP technique) the optimum quantities of metal for the dealer to purchase from each of the two suppliers. *[Allahabad M.B.A. 2009]*

318. A manufacturer produces two different models, X and Y of the same product. The raw materials r_1 and r_2 are required for production. At least 18 kg. of r_1 and 12 kg. of r_2 must be used daily. Also at most 34 hours of labour are to be utilized. 2 kg. of r_1 are needed for each model X and 1 kg. of r_1 for each model Y. For each model of X and Y, 1 kg. of r_2 is required. It takes 3 hours to manufacture a model X and 2 hours to manufacture a model Y. The profit is Rs. 50 for each model X and Rs. 30 for each model Y. How many units of each model should be produced to maximize the profit. Use graphic model. *[Gujarat M.B.A. 2008]*

319. A company has two grades of inspectors 1 and 2, who are to be assigned for a quality control inspection. It is required at least 2000 pieces be inspected per 8-hour day. Grade 1 inspector can check pieces at the rate of 40 per hour, with an accuracy of 97%. Grade 2 inspector checks at the rate of 30 pieces per hour with an accuracy of 95%.

The wage rate of a Grade 1 inspector is Rs. 5 per hour, while that of a Grade 2 inspector is Rs. 4 per hour. An error made by an inspector costs Rs. 3 to the company. There are only nine Grade 1 inspectors and eleven Grade 2 inspectors available in the company. The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection. Formulate this problem as a linear programming model and solve it by graphical method. *[Delhi M.B.A. 2004; Panjab B.Com. 2008]*

320. A manufacturer employs three inputs: man-hours, machine-hours and cloth material to manufacture two types of dresses. Type A dress fetches him a profit of Rs. 160 per piece, while type B, that of Rs. 180 per piece. The manufacturer has enough man-hours to manufacture 50 pieces of type A or 20 pieces of type B dresses per day, while the machine-hours he possesses suffice only for 36 pieces of type A or for 24 pieces for type B dresses. Cloth material available per day is limited but sufficient enough for 30 pieces of either type of dresses. Formulate the linear programming model and solve it graphically. *[Delhi M.Com. 2005]*

$\left[\textbf{Hint: } \text{Constraints are } \frac{x}{50} + \frac{y}{20} \le 1,\ \frac{x}{36} + \frac{y}{24} \le 1,\ \frac{x}{30} + \frac{y}{30} \le 1.\right]$

321. The standard weight of a special purpose brick is 5 kg. and it contains two basic ingredients B_1 and B_2. B_1 costs Rs. 5 per kg. and B_2 costs Rs. 8 per kg. Strength considerations dictate that the brick contains not more

than 4 kg. of B_1 and minimum of 2 kg. of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.

[Bangalore M.B.A. 1997]

322. Find the maximum value of $z = 50x_1 + 60x_2$ subject to the constraints :

$$2x_1 + 3x_2 \le 1500,\ 3x_1 + 2x_2 \le 1500,\ 0 \le x_1 \le 400,\ 0 \le x_2 \le 400.$$

323. Maximize $z = 1.75x_1 + 1.50x_2$ subject to the constraints :

$$8x_1 + 5x_2 \le 320,\ 4x_1 + 5x_2 \le 200,\ 0 \le x_1 \ge 15,\ 0 \le x_2 \ge 10.$$

324. Find the maximum value of $z = 7x_1 + 3x_2$ subject to the constraints :

$$x_1 + 2x_2 \ge 3,\ x_1 + x_2 \le 4,\ 0 \le x_2 \le 5/2,\ 0 \le x_2 \le 3/2.$$

325. Show graphically that the maximum or minimum values of the objective functions for the following problem are same :

Maximize (or Minimize) $z = 5x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 6,\ 2x_1 + 3x_2 \ge 3,\ 0 \le x_1 \ge 3,\ 0 \le x_2 \ge 3.$$

[Purvanchal M.C.A. 1996]

326. Find the maximum value of $z = 5x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 6,\ 2x_1 + 3x_2 \ge 6,\ 0 \le x_1 \le 4,\ 0 \le x_2 \le 3.$$

[Madurai M.B.A. (DPL) 2009]

327. Solve graphically the following L.P.P. :

Maximize $z = 3x_1 + 2x_2$ subject to the constraints :

$$-2x_1 + x_2 = 1,\ x_1 \le 2,\ x_1 + x_2 \le 3,\ x_1, x_2 \ge 0.$$

328. Find the minimum or/and maximum value of $z = 3x_1 + 5x_2$ subject to the constraints :

(*a*) $-3x_1 + 4x_2 \le 12,\ 2x_1 - x_2 \ge -2$

$2x_1 + 3x_2 \ge 12,\ 4 \ge x_2 \ge 0,\ 2 \ge x_2 \ge 0.$

(*b*) $x_1 \le 4,\ 2x_2 \le 6,\ 3x_1 + 2x_2 \le 18$

$x_1 + x_2 \le 9,\ x_1, x_2 \ge 0.$

3:3. SOME EXCEPTIONAL CASES

In the preceding sections, we discussed some linear programming problems which may be called 'well-behaved' problems. In each case, a solution was obtained, in some cases it took less effort while in some others it took less effort while in some others it took a little more. But a solution was finally obtained. It should not be taken as a rule. There may be an L.P.P. for which no solution exists or for which the only solution obtained is an unbounded one. Though such problems seldom occur in real situations, it will be an omission, if at this stage, the reader is not exposed to such exceptional cases.

This section considers the following three special cases that arise in the application of the graphical method :

(*i*) Alternative optima, (*ii*) Unbounded solution, (*iii*) Infeasible (or non-existing) solution.

1. **Alternative optima.** When the objective function is parallel to a binding constraint (*i.e.*, a constraint that is staisfied as an equation by the optimal solution), the objective function will assume the same optimum value at more than one solution point. For this reason they are called alternative optima. The problem given below shows that there is an infinity number of such solutions.

SAMPLE PROBLEMS

329. *Use graphical method to solve the L.P.P. :*

Maximize $z = 2x_1 + 4x_2$ *subject to the constraints :*

$$x_1 + 2x_2 \le 5,\ x_1 + x_2 \le 4;\ \text{and}\ x_1, x_2 \ge 0.$$

Solution. The problem is depicted graphically in Fig. 3.8. The extreme points of the feasible region are *O, A, B* and *C*.

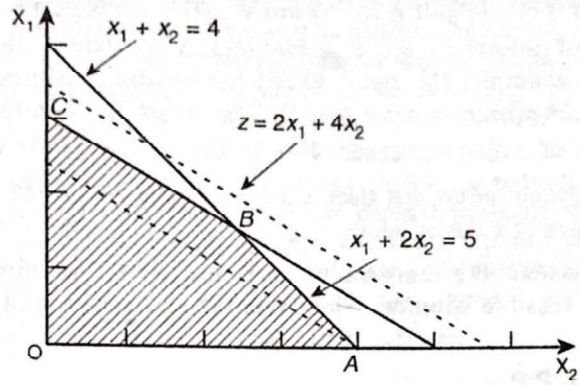

Fig. 3.8

We observe that our objective function (iso-profit line) is parallel to the line BC (or the first constraint), which forms the boundary of the feasible region. Thus as we move the iso-profit line away from the origin, it coincides with the portion BC of the constraint line which forms the boundary of the feasible region. This implies that any point including extreme points B and C on the same line between B and C is an optimal solution. Therefore, in fact an infinite number of values of x_1, x_2 give the same value of objective function.

Now, the value of objective function at each of the extreme points is evaluated as follows :

Extreme point	(x_1, x_2)	$z = 2x_1 + 4x_2$
O	(0, 0)	0
A	(4, 0)	8
B	(3, 1)	**10** ⟵ maximum
C	(0, 2.5)	**10** ⟵ maximum

Since, any point on the line segment BC gives the maximum value (z = 10) of the objective function, there exists an alternative optima.

2. **Unbounded solution.** When the values of the decision variables may be increased indefinitely without violating any of the constraints, the solution space (feasible region) is unbounded. The value of objective function, in such cases, may increase (for maximization) or decrease (for minimization) indefinitely. Thus, both the solution space and the objective function value are unbounded.

330. *Use graphical method to solve the following L.P.P. :*

Maximize $z = 6x_1 + x_2$ *subject to the constraints :*

$$2x_1 + x_2 \geq 3, \quad x_2 - x_1 \geq 0 \quad \text{and} \quad x_1, x_2 \geq 0.$$

Solution. The problem is depicted graphically in Fig. 3.9. The two extreme points of the feasible region are A and B.

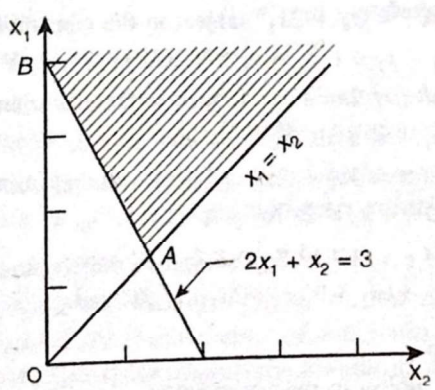

Fig. 3.9

We observe, that the feasible region (solution space) is unbounded. The value of the objective function at the extreme points A (1, 1) and B (0, 3) are 7 and 3 respectively.

But there exist number of points in the feasible region for which the value of the objective function is more than 7. For example, the point (3, 6) lies in the feasible region and the objective function value at this point is 24 which is more than 7. Thus both the variables x_1 and x_2 can be made arbitrarily large and the value of z also increased. Hence, the problem has an unbounded solution.

Remark. An unbounded solution means that there exist an infinite number of solutions to the given problem and the optimal value of z lies at infinity.

3. **Infeasible solution.** When the constraints are not satisfied simultaneously, the linear programming problem has no feasible solution. This situation can never occur if all the constraints are of the '$\leq$' type.

331. *Solve the following L.P.P. :*

$$\text{Maximixe } z = x_1 + x_2 \text{ subject to the constraints :}$$
$$x_1 + x_2 \leq 1, \; -3x_1 + x_2 \geq 3$$
$$x_1 \geq 0, \; x_2 \geq 0.$$

[Guru Nanak Dev Univ. B.Com. 2006]

Solution. The problem is depicted graphically in Fig. 3.10. As shown in the figure, there is no point (x_1, x_2) which can lie in both the regions (satisfy both the constraints), there exists no solution to the given problem. Hence, there is infeasible solution.

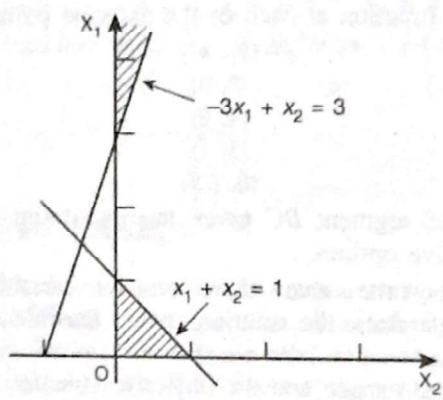

Fig. 3.10

PROBLEMS

332. Find the maximum value of $z = 10x_1 + 6x_2$ subject to the constraints :

$$5x_1 + 3x_2 \leq 30, \; x_1 + 2x_2 \leq 18 \text{ and } x_1, x_2 \geq 0.$$

333. Find the maximum value of $z = x_1 + 2x_2$ subject to the constraints :

$$x_1 - x_2 \leq 1, \; x_1 + x_2 \geq 3 \text{ and } x_1 \geq 0, \; x_2 \geq 0.$$

334. Find the maximum value of $z = 2x_1 + x_2$ subject to the constraints :

$$x_1 - x_2 \leq 10, \; x_1 \leq 20 \text{ and } x_1 \geq 0, \; x_2 \geq 0.$$

335. Find the maximum value of $z = 3x_1 + 2x_2$ subject to the constraints :

$$2x_1 + x_2 \leq 2, \; 3x_1 + 4x_2 \geq 12; \; x_1, x_2 \geq 0.$$

336. Find the maximum value of $z = 3x_1 + 2x_2$ subject to the constraints :

$$-2x_1 + 3x_2 \leq 9, \; x_1 - 5x_2 \geq -20 \text{ and } x_1, x_2 \geq 0.$$

[Panjab Tech. Univ. M.B.A. (Dec.) 2009]

337. Consider the following LPP :

Maximize $z = 3x_1 + 7x_2$ subject to the constraints :

$$4x_1 + 5x_2 \leq 20, \; 2x_1 + x_2 \leq 6, \; 2x_1 \geq 7, \; 2x_2 \leq 7; \; x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Demonstrate whether it is possible to get a solution to the above LPP. ***[Delhi M.B.A. (IICA) 2015]***

338. A company buying scrap metal has two types of scrap available to them. The first type of scrap metal has 20% of metal A, 10% of impurity and 20% of metal B by weight. The second type of scrap has 30% of metal A, 10% of impurity and 15% of metal B by weight. The company requires at least 120 kg. of metal A, at most 40 kg. of impurity and at least 90 kg. of metal B. The prices for the two scraps are Rs. 200 and Rs. 300 per kg. respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost. *[Nagpur M.B.A. 1997]*

339. A manufacturing firm has a long history of production troubles. It produces two products A and B, which are equally profitable. Recently, the company has entered into contract to supply 40 units of A and 20 units of B per week to another company. The technology of the chemical process implies that production of A must always be at least as large as of B. There are two raw material constraints to be satisfied:

$$5A + 8B \leq 400 \text{ and } 55A + 50B \leq 2{,}750.$$

Solve the problem graphically and comment on the solution obtained.

3:4. GENERAL LINEAR PROGRAMMING PROBLEM

We shall now consider the L.P.P. in the general context, that is, when the number of variables is more than two.

Definition 1 **(General Linear Programming Problem).** *Let z be a linear function on R^n defined by*

(a) $$z = c_1x_2 + c_2x_2 + \ldots + c_nx_n$$

where c_j's are constants. Let (a_{ij}) be an $m \times n$ real matrix and $\{b_1, b_2, \ldots, b_m\}$ be a set of constants such that

(b) $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq or \leq or = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq or \leq or = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq or \leq or = b_m \end{cases}$$

and finally let

(c) $$x_j \geq 0, \quad j = 1, 2, \ldots, n.$$

The problem of determining an n-tuple $(x_1, x_2, \ldots, x_n)$ which makes z a minimum (or maximum) and satisfies (b) and (c) is called the ***general linear programming problem.***

Objective function. *The linear function*

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

which is to be minimized (or maximized) is called the ***objective function*** *of the General L.P.P.*

Constraints. *The inequations (b) are called the* ***constraints*** *of the General L.P.P.*

Non-negative restrictions. *The set of inequations (c) is usually known as the set of* ***non-negative restrictions*** *of the General L.P.P.*

Definition 2 **(Solution).** *An n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers which satisfies the constraints of a General L.P.P. is called a* ***solution*** *to the General L.P.P.*

Definition 3 **(Feasible solution).** *Any solution to a General L.P.P. which also satisfies the non-negative restrictions of the problem, is called a* ***feasible solution*** *to the General L.P.P.*

Definition 4 **(Optimum solution).** *Any feasible solution which optimizes (minimizes or maximizes) the bjective function of a General L.P.P. is called an* ***optimum solution*** *to the General L.P.P.*

Note. The term *optimal solution* is also used for optimum solution.

Example of a general L.P.P. **(Diet problem).** *Given the nutrient contents of a number of different foodstuffs and the daily minimum requirement of each nutrient for a diet, determine the balanced diet which staisfied the minimum daily requirements and at the same time has the minimum cost.*

Mathematical Formulation

Let there be n different types of foodstuffs available and m different types of nutrients required. Let a_{ij} denote the number of units of nutrient i in one unit of foodstuff j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Let x_j be the number of units of food j in the desired diet. Then, the total number of units of nutrient i in the desired diet is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$$

Let b_i be the number of units of the minimum daily requirement of nutrient i. Then, we must have

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i \qquad i = 1, 2, \ldots, m.$$

Also, each x_j must be either positive or zero. Thus, we also have

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n.$$

Finally, consider the cost. Let c_j be the cost per unit of food j. Thus, the total cost of the diet is given by

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Thus, the problem of selecting the best diet reduces to the following mathematical form :

Find an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers, such that

(*a*) $$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i \qquad i = 1, 2, \ldots, n$$

(*b*) $$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

and for which the expression (objective function)

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

may be a minimum (least).

This is a General L.P.P. It is general in the sense that the data b_i, a_{ij} and c_j are parameters, which for different sets of values will give rise to different problems.

Slack and Surplus Variables

Definition 1 **(Slack variables).** *Let the constraints of a General L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i = 1, 2, \ldots, k$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \qquad i = 1, 2, \ldots, k$$

are called ***slack variables.***

Definition 2 **(Surplus variables).** *Let the constraints of a General L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad i = k+1, k+2, \ldots, l$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \qquad i = k+1, k+2, \ldots, l$$

are called ***surplus (or negative slack) variables.***

3:5. CANONICAL AND STANDARD FORMS OF L.P.P.

After the formulation of linear programming problem (L.P.P.), the next step is to obtain its solution. But for the solution of any linear programming problem, the problem must be available in a particular form. Two forms are dealt with here, the *canonical form* and the *standard form*.

The Canonical Form

The general formulation of linear programming problem discussed in the previous section can always be put in the following form :

$$\text{Maximize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \text{ subject to the constraints :}$$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i; \qquad i = 1, 2, \ldots, m$$

$$x_1, x_2, \ldots, x_n \ge 0$$

by making use of some elementary transformations. This form of L.P.P. *is called the* **canonical form** of L.P.P. The characteristics of this form are :

(*i*) *The objective function is of the maximization type.*

The minimization of a function, $f(x)$, is equivalent to the maximization of the negative expression of this function, $-f(x)$, *i.e.*,

$$\text{Minimize } f(x) = -\text{Maximize}\{-f(x)\}$$

For example, the linear objective function

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

is equivalent to

$$-\text{Maximize } h = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$$

with $z = -h$.

(*ii*) *All the constraints are of the "≤" type, except for the non-negative restrictions.*

An inequality of "≥" type can be changed to an inequality of the "≤" type by multiplying both sides of the inequality by –1.

For example, the linear constraint

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ge b_i$$

is equivalent to

$$-a_{i1}x_1 - a_{i1}x_2 - \ldots - a_{in}x_n \le -b_i$$

An equation may be replaced by two weak inequalities in opposite directions. For example,

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$$

is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i$$

and

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ge b_i.$$

(*iii*) *All the variables are non-negative.*

A variable which is *unrestricted* in sign (*i.e.*, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x_j is unrestricted in sign, it can be replaced by $(x_j' - x_j'')$, where x_j' and x_j'' are both non-negative, *i.e.*,

$$x_j = x_j' - x_j'', \qquad \text{where } x_j' \ge 0 \text{ and } x_j'' \ge 0.$$

The Standard Form

The general linear programming problem in the form

$$\text{Maximize or Minimize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \text{ subject to the constraints :}$$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i, \qquad i = 1, 2, \ldots, n$$

$$x_1, x_2, \ldots, x_n \ge 0$$

is known as in **standard form**. The characteristics of this form are :

(*i*) All the constraints are expressed in the form of equations, except for the non-negative restrictions.

(*ii*) The right hand side of each constraint equation is non-negative.

The inequality constraint can be changed into equation by introducing a non-negative variable on the left hand side of such constraint. It is to be added (slack variable) if the constraint is of "$\leq$" type and subtracted (surplus variable) if the constraint is of "$\geq$" type.

In matrix notation the standard form of L.P.P. can be expressed as :

$$\text{Maximize or Minimize } z = \mathbf{cx} \qquad \textit{(objective function)}$$

subject to the constraints :

$$\mathbf{Ax} = \mathbf{b}, \qquad \textit{(constraints)}$$

$$\mathbf{x} \geq \mathbf{0} \qquad \textit{(non-negative restrictions)}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, $\mathbf{b}^T = (b_1, b_2, \ldots, b_m)$ and $\mathbf{A} = (a_{ij})$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Remarks 1. The coefficients of slack and/or surplus variables in the objective function are always assumed to be zero, so that the conversion of the constraints to a system of simultaneous linear equations does not change the objective function under consideration.

2. The linear programming form :

$$\text{Maximize } z = \mathbf{cx} \text{ subject to the constraints : } \mathbf{Ax} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$$

is known as the **canonical form** of the L.P.P.

Theorem 2-1. *The set of feasible solutions to an L.P.P. is a convex set.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to maximize the linear function $z = \mathbf{c}^T\mathbf{x}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Let $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ be two feasible solutions to this problem, so that

$$\mathbf{Ax}^{(1)} = \mathbf{b};\ \mathbf{Ax}^{(2)} = \mathbf{b};\ \mathbf{x}^{(1)} \geq \mathbf{0} \text{ and } \mathbf{x}^{(2)} \geq \mathbf{0}.$$

Now, consider convex combination of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, namely,

$$\mathbf{x} = \lambda\mathbf{x}^{(1)} + (1 - \lambda)\mathbf{x}^{(2)},\quad 0 \leq \lambda \leq 1.$$

Clearly

$$\begin{aligned}\mathbf{Ax} &= \mathbf{A}[\lambda\mathbf{x}^{(1)} + (1 - \lambda)\mathbf{x}^{(2)}] = \lambda\mathbf{Ax}^{(1)} + (1 - \lambda)\mathbf{Ax}^{(2)} \\ &= \lambda\mathbf{b} + (1 - \lambda)\mathbf{b} = \mathbf{b}\end{aligned}$$

Again, since $\mathbf{x}^{(1)} \geq \mathbf{0}$, $\mathbf{x}^{(2)} \geq \mathbf{0}$, and $\lambda, 1 - \lambda \geq 0$, $\therefore\ \mathbf{x} \geq \mathbf{0}$.

Hence, $\mathbf{x}$ is also feasible solution to the problem. Thus, the set

$$S = \{\mathbf{x} \mid \mathbf{x} \text{ is a feasible solution to an L.P.P.}\}$$

is a convex set.

Remark. In general a convex set S is either empty or unbounded or closed.

The *empty set* occurs when the constraints are not satisfied simultaneously. In this case the system yields no solution. An unbounded set implies that the region of feasible solution is not constrained in at least one direction.

A *closed set* implies that the region of feasible solution is a convex polyhedron, since it is defined by the intersections of a finite number of linear constraints.

SAMPLE PROBLEMS

340. *Rewrite in standard form the following linear programming problem :*

$$\textit{Minimize } z = 2x_1 + x_2 + 4x_3 \textit{ subject to the constraints :}$$

$$-2x_1 + 4x_2 \leq 4,\ x_1 + 2x_2 + x_3 \geq 5,\ 2x_1 + 3x_3 \leq 2,$$

$$x_1, x_2 \geq 0 \textit{ and } x_3 \textit{ unrestricted in sign.}$$

Solution. As the constraints are in the form of inequalities, we introduce slack variables $x_4 \geq 0$ and $x_5 \geq 0$ in the first and third inequalities and a surplus variable $x_6 \geq 0$ in the second inequality of the constraints.

Further since x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$, where $x_3' \geq 0$ and $x_3'' \geq 0$. Thus the constraints of the problem become :

$$-2x_1 + 4x_2 + x_4 = 4$$
$$x_1 + 2x_2 + (x_3' - x_3'') - x_6 = 5$$
$$2x_1 + 3(x_3' - x_3'') + x_5 = 2$$

The non-negative restrictions are :

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \text{ and } x_6 \geq 0.$$

The objective function of the problem is converted into that of maximization by multiplying it by (–1). Also we assign a cost zero to each slack/surplus variable. Thus, the revised objective function is :

$$\text{Maximize } z^* = -2x_2 - x_2 - 4(x_3' - x_3'') + 0.x_4 + 0.x_5 + 0.x_6$$

where $z^* = -z$.

Thus, the given LPP in standard form is :

Maximize $z^* = -2x_1 - x_2 - 4x_3' + 4x_3'' + 0.x_4 + 0.x_5 + 0.x_6$ subject to the constraints :

$$-2x_1 + 4x_2 + x_4 = 4,\ x_1 + 2x_2 + x_3' - x_3'' - x_6 = 5$$
$$2x_1 + 3x_3' - 3x_3'' + x_5 = 2,\ x_j \geq 0;\ j = 1, 2, 3', 3'', 4, 5, 6.$$

341. *Reduce the following LPP to its standard form :*

Maximize $z^* = 3x_1 + 4x_2 + 6x_3$ *subject to the constraints :*

$$2x_1 + x_2 + 2x_3 \geq 6,\ 3x_1 + 2x_2 = 8,$$
$$7x_1 - 3x_2 + 5x_3 \geq 9;\ x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \text{ unrestricted in sign.}$$

Solution. Since, the first and third constraints are inequalities with $\geq$ sign, we introduce surplus variables $x_4 \geq 0$ and $x_5 \geq 0$ in the respective inequalities.

Further since x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$, where $x_3' \geq 0$ and $x_3'' \geq 0$.

Thus, the constraints of the given LPP are :

$$2x_1 + x_2 + 2(x_3' - x_3'') + x_4 = 6,$$
$$3x_1 + 2x_2 = 8,$$
$$7x_1 - 3x_2 + 5(x_3' - x_3'') + x_5 = 9$$

The objective function will be :

$$\text{Maximize } z = 3x_1 + 4x_2 + 6(x_3' - x_3'').$$

$\therefore$ The given LPP in its standard form is :

Maximize $z = 3x_1 + 4x_2 + 6x_3' - 6x_3'' + 0 \cdot x_4 + 0 \cdot x_5$ subject to the constraints :

$$2x_1 + x_2 + 2x_3' - 2x_3'' + x_4 = 6,\ 3x_1 + 2x_2 = 8$$
$$7x_1 - 3x_2 + 5x_3' - 5x_3'' + x_5 = 9$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ x_4 \geq 0 \text{ and } x_5 \geq 0.$$

PROBLEMS

342. Reduce the following L.P.P. to its standard form :

Maximize $z = x_1 - 3x_2$ subject to the constraints :

$$-x_1 + 2x_2 \leq 15,\ x_1 + 3x_2 = 10$$

x_1 and x_2 unrestricted in sign.

343. Write down the following LPP in standard form :

Maximize $z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints :

$$2x_1 - 3x_2 \leq 3,\ x_1 + 2x_2 + 3x_3 \geq 5,\ 3x_1 + 2x_3 \leq 2$$
$$x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \geq 0.$$

344. Transform the following LPP into the form where all the constraints are of equality type :

$$\text{Maximize } z = 2x_1 + x_2 - 6x_3 - 4x_4 \text{ subject to the constraints :}$$

$$3x_1 + x_4 \le 25,\ x_1 + x_2 + x_3 + x_4 = 20$$

$$4x_1 + 6x_3 \ge 5,\ 2 \le x_1 + 3x_3 + 2x_4 \le 30;\ x_j \ge 0,\ (j = 1, 2, 3, 4).$$

345. Consider the Linear Programme

$$\text{Minimize } z = -3x_1 + x_2 + x_3 \text{ subject to the constraints :}$$

$$x_1 - 2x_2 + x_3 \le 11$$

$$-4x_1 + x_2 + 2x_3 \ge 3,\ 2x_1 - x_3 = -1$$

$$x_1, x_2 \ge 0,\ x_3 \ge \text{ or } < 0.$$

Cast it in standard form, adding variables and/or constraints if necessary. How many decision variables and constraints are present in this problem?

346. Transform the following LP problem into the form where all constraints are of equality type :

$$\text{Maximize } z = 2x_1 + x_2 - 5x_3 + 3x_4 \text{ subject to the constraints :}$$

$$3x_1 + 2x_2 \le 15,\ 4x_1 + 5x_2 \ge 20,\ x_1 + x_2 - x_3 + 2x_4 = 10$$

$$2 \le 2x_1 + 4x_2 - x_3 \le 30,\ x_i \ge 0;\ i = 1, 2, 3, 4$$

Identify the variables introduced.

3:6. INSIGHTS INTO THE SIMPLEX METHOD

While solving an LPP graphically, we noticed the following important characteristics :

(*a*) Whenever feasible solutions existed, the region of feasible solutions was *convex*, bounded by *lines* or *planes* (more than two variable case). For each such convex region, there were *corners* (or vertices) on the boundary and *edges* joining these corners.

(*b*) For each value of z, the objective function could be represented by a line or plane and whenever the maximum or minimum value of z was finite, the optimal solution occurred at some corner (or vertex) of the convex region of feasible solutions. If the optimal solution were not unique, there were points (on an edge) that were optimal, but in every event at least one vertex was optimal. In the case of unbounded optimal solution; however, no corner point was optimal.

Interestingly, these observations, derived from simple graphical examples, hold true for the general LPP also, if we think of a geometrical representation in n dimensional space. The region of the feasible solutions is a convex region or a convex set, has corners or the vertices. The linear constraints of the problem are represented by half spaces determined by the hyperplanes. The optimal solution, if it exists, corresponds to some vertex of the convex region.

The problem essentially is that of determining which particular vertex of the convex region corresponds to the optimal solution. No method is so far known that will locate this optimal vertex in a single step and one has to adopt an iterative procedure for locating the same.

The best known and the most widely used procedure for locating the optimal vertex is the "simplex method" that determines the optimal vertex in a finite number of steps. The *simplex method* is a procedure which consists in moving step by step from a given vertex to an optimal vertex. At each step, it is possible to move only to an adjacent vertex. The method consists of moving along an 'edge' of the convex region of feasible solutions from one vertex to an adjacent one. Of all the adjacent vertices the one yielding an *improved* value of the objective function over that of the preceding vertex, is chosen. At each vertex point, the method tells us whether that extreme point is optimal, and if not, the procedure of jumping from one extreme point to another is repeated. Since the number of vertices is finite, simplex method leads to an optimal vertex in a finite number of steps. If at any stage, the procedure leads us to a vertex which has an edge leading to infinity and if the objective function value can be further improved by moving along that edge, the simplex method tells us that there is an unbounded solution.

MULTIPLE CHOICE QUESTIONS

1. Which of the following is not correct ?
(*a*) The graphic approach to an LPP is most suitable when there are only two decision variables.
(*b*) A possible solution on the graph corresponds to every point (x, y).
(*c*) The graphic approach to an LPP is applicable when the number of decision variables are more than the number of constraints.
(*d*) The common region that satisfies all the contraints is called the feasible (convex) region.

2. A feasible solution to an LPP
(*a*) must satisfy all of the problem's constraints simultaneously.
(*b*) must be a corner point of the feasible region.
(*c*) need not satisfy all of the constraints, only some of them.
(*d*) must optimize the value of the objective function.

3. Which of the following is not correct ?
(*a*) The graphic approach to the solution of LPP's cannot handle problems with more than three variables.
(*b*) A feasible solution to an LPP is one that satisfies at least one of the constraints of the problem.
(*c*) An optimum solution to an LPP is a feasible solution which optimizes the objective function.
(*d*) The feasible region is also termed as the solution space.

4. Which of the following is not correct ?
(*a*) Feasible solution of an LPP is independent of the objective function.
(*b*) The feasible region of an LPP must be a convex set.
(*c*) The feasible region for a constraint is restricted if its '$\geq$' or '$\leq$' sign is replaced by a '=' sign.
(*d*) It is not possible to obtain feasible solution of an LPP by graphical method.

5. An iso-profit line represents
(*a*) an infinite number of solutions all of which yield the same profit.
(*b*) an infinite number of optimum solutions.
(*c*) an infinite number of solutions all of which yield the same cost.
(*d*) a boundary of the feasible region.

6. If an iso-profit line yielding the optimum solution coincides with a constraint line, then
(*a*) the solution is unbounded.
(*b*) the solution is infeasible.
(*c*) the constraint which coincides is redundant.
(*d*) none of the above.

7. If two constraints do not intersect in the positive quadrant of the graph, then
(*a*) one of the constraint is redundant.
(*b*) the solution is infeasible.
(*c*) the solution is unbounded.
(*d*) none of these.

8. Which of the following is not correct ?
(*a*) An infeasible solution exists, when there is no feasible solution.
(*b*) Iso-profit lines on a graph of an LPP would always be parallel to each other.
(*c*) An iso-cost line cannot be parallel to the line of any constraint.
(*d*) Every LPP has a unique optimum solution.

9. Using graphic method, the optimum solution of the LPP of maximizing $z = 10x + 15y$ subject to $2x + y \leq 26$, $x + 2y \leq 28$, $y - x \leq 5$ and $x \geq 0$, $y \geq 0$ is obtained as
(*a*) $x = 8$ and $y = 10$.
(*b*) $x = 6$ and $y = 1$.
(*c*) $x = 6$ and $y = 10$.
(*d*) $x = 8$ and $y = 8$.

10. Given an LPP to maximize $z = -5x_2$ subject to $x_1 + x_2 \leq 1$, $0.5x_1 + 5x_2 \geq 0$, and $x_1 \geq 0$, $x_2 \geq 0$. Using graphical method, we have
(*a*) no feasible solution.
(*b*) unbounded solution.
(*c*) unique optimum solution.
(*d*) multiple optimum solutions.

11. Which of the following is not correct ?
   (*a*) An LPP with an unbounded feasible region would obviously have unbounded solution.
   (*b*) For an LPP, the feasible region may change if non-binding constraints are deleted.
   (*c*) A redundant constraint represents an abundant resource.
   (*d*) It is possible for the objective function value of an LPP to be the same at two distinct extreme points.
12. The general linear programming problem is in standard form, if
   (*a*) the constraints are strict equations.
   (*b*) the constraints are inequalities of '≤' type.
   (*c*) the constraints are inequalities of '≥' type.
   (*d*) the decision variables are unrestricted in sign.
13. Which of the following is not correct ?
   (*a*) Graphical method of linear programming is not useful when there are only two decision variables.
   (*b*) Solution of a maximization LPP when permitted to be infinitely large is called unbounded.
   (*c*) Optimum solution to an LPP always lies at least on the two vertices of the feasible region.
   (*d*) It is possible for the objective function value of an LPP to be the same at two distinct extreme points.
14. Which of the following statements is wrong ?
   (*a*) Slack variables are used to convert the inequalities of the type '≤' into equations.
   (*b*) Surplus variables are used to convert the inequalities of the type '≥' into equations.
   (*c*) An LPP with all its constraints are of the type '≥' is said to be in *standard form*.
   (*d*) An LPP with all its constraints are of the type '≤' is said to be in *canonical form*.

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)
6. (*d*)	**7.** (*b*)	**8.** (*c*)	**9.** (*a*)	**10.** (*a*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*).	

REVIEW QUESTIONS

1. Write down the standard form of a LPP. *[Annamalai M.B.A. 2010]*
2. Explain the graphical method of solving a LPP involving two variables. *[Madurai M.Com. (Nov.) 2002]*
3. What is feasible region ? Is it necessary that it should always be a convex set ? *[Madurai M.Com. 2003]*
4. Define iso-profit and iso-cost lines. How do these help in obtaining an optimum solution to an LPP ?
5. Discuss the special cases of infeasible and unbounded solution by giving examples in LPP. *[Panjab Tech. Univ. M.B.A. (Dec.) 2011]*
6. Graphical solution is not possible for LPP with more than two constraints. True or False ? Justify your answer. *[Annamalai M.B.A. (Nov.) 2002]*
7. What is meant by the term feasible solution space? What determines the region? *[Panjab Tech. Univ. M.B.A. (June) 2011]*
8. Illustrate graphically the following special cases of linear programming problems :
   (*i*) Multiple optimal solutions, (*ii*) No feasible solution, (*iii*) Unbounded solution.
9. Define and explain the following terms :
   (*i*) Optimum solution, (*ii*) Feasible solution, (*iii*) Degenerate basic feasible solution. *[Madras M.B.A. 2004]*
10. Define slack and surplus variables in LPP. *[Delhi B.Sc. (Stat.) 2000; Madras B.E. 1999]*
11. State the general linear programming problem in : (*a*) Standard form, (*b*) Canonical form. *[Delhi B.Sc. (Stat,) 2000]*
12. With the help of suitable sketches, define convex, non-convex, and infeasible region in relation to the graphical solution of a linear programming problem. *[Delhi M.Com. 2008]*

4

Linear Programming Problem —Simplex Method

"One step for better is good enough"

4:1. INTRODUCTION

As discussed earlier, the *Simplex Method*, also called the '*Simplex Technique*' or the '*Simplex Algorithm*' is an iterative procedure for solving a linear programming problem in a finite number of steps. The method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less (or more, as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

In the present chapter, we shall introduce and explain the computational procedure of the simplex method. The theory behind the method will be first developed and then the computational techniques explained and illustrated. The following important definitions are necessary for understanding and developing the theory that follows :

***Definition* (Basic solution).** *Given a system of m simultaneous linear equations in n unknowns* $(m < n)$

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x}^T \in \mathbf{R}^n,$$

where **A** *is an* $m \times n$ *matrix of rank m. Let* **B** *be any* $m \times n$ *submatrix, formed by m linearly independent columns of* **A**. *Then, a solution obtained by setting* $n - m$ *variables not associated with the columns of* **B**, *equal to zero, and solving the resulting system, is called a* ***basic solution*** *to the given system of equations.*

The m variables, which may be all different from zero, are called ***basic variables***. *The* $m \times m$ *non-singular submatrix* **B** *is called a* ***basis matrix*** *with the columns of* **B** *as* ***basis vectors.***

Remarks. The name *basic solution*, as used above, merits a word of caution. If B is the basis sub-matrix chosen, then the basic solution to the system is

$$\mathbf{x}_B = \mathbf{B}^{-1}\,\mathbf{b}.$$

But $\mathbf{x}_B{}^T \in \mathbf{R}^m$, and as such cannot be called a solution of the given system. Truly speaking, if $\mathbf{x}_B$ is a basic solution, then a solution to the given system is $[\mathbf{x}_B{}^T, \mathbf{0}]$ where $\mathbf{x}_B{}^T \in \mathbf{R}^m$, and $\mathbf{0} \in \mathbf{R}^{n-m}$. However, we shall follow the current usage and call $\mathbf{x}_B$ a basic solution of the given system, remembering all the time that the actual solution is $[\mathbf{x}_B{}^T, \mathbf{0}]$.

***Definition* (Degenerate solution).** *A basic solution to the system is called* **degenerate** *if one or more of the basic variables vanish.*

SAMPLE PROBLEMS

401. *Obtain all the basic solutions to the following system of linear equation :*

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5.$$

Solution. The given system of equations can be written in the matrix form as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Since, rank of **A** is 2, the maximum number of linearly independent columns of **A** is 2. Thus, we can take any of the following, 2×2 sub-matrices as *basis matrix* **B** :

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

The variables not associated with the columns of **B** are x_3, x_2 and x_1 respectively, in the three different cases.

Let us first take $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_3 = 0$, and solving the system

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Thus, a basic (non-basic) solution to the given system is :

(Basic) $x_1 = 2, \ x_2 = 1$; (Non-basic) $x_3 = 0$

Similarly, the other two basic and non-basic solutions are :

(Basic) $x_1 = 5, \ x_3 = -1$; (Non-basic) $x_2 = 0$

and

(Basic) $x_2 = 5/3, \ x_3 = 2/3$; (Non-basic) $x_1 = 0$

We observe that all the above three basic solutions are non-degenerate.

402. *Show that the following system of linear equations has a degenerate solution :*

$$2x_1 + x_2 - x_3 = 2$$
$$3x_1 + 2x_2 + x_3 = 3.$$

Solution. The given system of equations can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since rank of **A** is 2, the maximum number of linearly independent columns of **A** is 2. Thus, we can take any of the following 2×2 sub-matrices of **A**, as basis matrix **B** :

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}.$$

The variables not associated with the columns of these submatrices are, respectively, x_3, x_1 and x_2.

Considering $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, a basic solution to the given system is obtained by setting $x_3 = 0$ and solving the system

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus, basic solution to the given problem is

(Basic) $x_1 = 1, \ x_2 = 0$; (Non-basic) $x_3 = 0$.

Similarly, the other two solutions are :

(Basic) $x_2 = 5/3,\ x_3 = -1/3$; (Non-basic) $x_1 = 0$.

and (Basic) $x_1 = 1,\ x_3 = 0$; (Non-basic) $x_2 = 0$.

In each of the two basic solutions, at least one of the basic variables is zero. Hence, two of the basic solutions are degenerate solutions.

Definition* (Basic feasible solution).** *A feasible solution to an L.P.P., which is also a basic solution to the problem is called a* ***basic feasible solution *to the L.P.P.*

Illustrations

1. In sample problem *401* observe that [5, 0, −1] is not a feasible solution. Only basic feasible solutions are :

(*i*) $x_2 = 5/3$ and $x_3 = 2/3$; (*ii*) $x_1 = 2$ and $x_2 = 1$.

2. In sample problem *402*, the non-degenerate solution [0, 5/3, − 1/3] is not feasible. Only basic feasible degenerate solutions are :

(*i*) $x_1 = 1$ and $x_2 = 0$; (*ii*) $x_1 = 1$ and $x_3 = 0$.

***Definition* (Associated cost vector).** *Let* $\mathbf{x}_B$ *be a basic feasible solution to the L.P.P. :*

$$\textit{Maximize } z = \mathbf{cx} \textit{ subject to : } \mathbf{Ax} = \mathbf{b} \textit{ and } \mathbf{x} \geq \mathbf{0}.$$

Then, the vector

$$\mathbf{c}_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}),$$

where c_{Bi} *are components of* $\mathbf{c}$ *associated with the basic variables, is called the* ***cost vector associated*** *with the basic feasible solution* $\mathbf{x}_B$.

It is obvious that the value of the objective function for the basic feasible solution $\mathbf{x}_B$, is given by

$$z_0 = \mathbf{c}_B \mathbf{x}_B.$$

Definition* (Improved basic feasible solution).** *Let* $\mathbf{x}_B$ *and* $\hat{\mathbf{x}}_B$ *be two feasible solutions to the standard L.P.P. Then* $\hat{\mathbf{x}}_B$ *is said to be an* ***improved basic feasible solution*, as compared to* $\mathbf{x}_B$, *if*

$$\hat{\mathbf{c}}_B \hat{\mathbf{x}}_B \geq \mathbf{c}_B \mathbf{x}_B$$

where $\hat{\mathbf{c}}_B$ *is constituted of cost components corresponding to* $\hat{\mathbf{x}}_B$.

***Definition* (Optimum basic feasible solution).** *A basic feasible solution* $\mathbf{x}_B$ *to the L.P.P. :*

$$\textit{Maximize } z = \mathbf{cx} \textit{ subject to : } \mathbf{Ax} = \mathbf{b} \textit{ and } \mathbf{x} \geq \mathbf{0}$$

is called an ***optimum basic feasible solution*** *if* $z_0 = \mathbf{c}_B \mathbf{x}_B \geq z^*$ *where* z^* *is the value of objective function for any feasible solution.*

PROBLEMS

403. Find all the basic feasible solutions of the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

404. Find all the basic feasible solutions of the following L.P.P. without using the simplex alogorithm :

$$\text{Maximize } z = 2x_1 + 3x_2 + 4x_3 + 7x_4 \text{ subject to :}$$
$$2x_1 + 3x_2 + x_3 + 4x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$
$$x_i \geq 0,\ i = 1, 2, 3, 4$$

and choose that one which maximises z.

405. If $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 7, x_5 = 12, x_6 = 10$ is an extreme point solution of the set of equations

$$3x_1 - x_2 + 2x_3 + x_4 = 7$$
$$-2x_1 + 4x_2 + x_5 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + x_6 = 10 ;$$

find two more extreme point solutions of the above problem.

4:2. FUNDAMENTAL PROPERTIES OF SOLUTIONS

We are now well set to discuss the theory behind the computational procedure for obtaining an optimum basic feasible solution to a given L.P.P. The fundamental properties of the solutions to the L.P.P., that now follow, will allow us to arrive at the much awaited **simplex algorithm**.

***Theorem 4-1* (Reduction of a Feasible Solution to a Basic Feasible Solution).** *If an L.P.P. has a feasible solution, then it also has a basic feasible solution.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to maximize

$$z = \mathbf{c}\mathbf{x} \qquad \mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$$

subject to the constraints : $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$; where $\mathbf{A}$ is an $m \times n$ real matrix and $\mathbf{b}$, $\mathbf{c}$ are $m \times 1$ and $1 \times n$ real matrices respectively. Let $\rho(\mathbf{A}) = m$.

Since there does exist a feasible solution, we must have

$$\rho(\mathbf{A}, \mathbf{b}) = \rho(\mathbf{A}) \text{ and } m < n.$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a feasible solution so that $x_j \geq 0$ for all j.

To be precise, let us suppose that $\mathbf{x}$ has p positive components and let the remaining $n - p$ components be zero.

Let us so relabel components that the positive components are the first p components and assume that the columns of $\mathbf{A}$ have been relabelled accordingly. Then

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_p x_p = \mathbf{b}.$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are the first p columns of $\mathbf{A}$.

Two cases now do arise :

(*i*) *The vectors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ *form a linearly independent set*. Then $p \leq m$.

If $p = m$, the given solution is a non-degenerate basic feasible solution, with $x_1, x_2, \ldots, x_p$ as the basic variables.

If $p < m$, then the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_p\}$ can be extended to $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_m\}$ to form a basis for the columns of $\mathbf{A}$.

Then, we have

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_m x_m = \mathbf{b},$$

where $x_j = 0$ for $j = p+1, p+2, \ldots, m$.

Thus we have, in this case, a degenerate basic feasible solution with $m - p$ of the basic variables zero.

(*ii*) *The set* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ *is linearly dependent*. Obviously $p > m$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be a set of constants (not all zero) such that

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_p \mathbf{a}_p = \mathbf{0}.$$

Suppose that for any index r, $\alpha_r \neq 0$. Then $\mathbf{a}_r = \sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} \mathbf{a}_j$.

$$\therefore \qquad \sum_{j \neq r}^{p} \mathbf{a}_j x_j + \left(-\sum_{j \neq r}^{p} \frac{\alpha_j}{\alpha_r} \mathbf{a}_j \right) x_r = \mathbf{b}$$

or

$$\sum_{j \neq r}^{p} \left(x_j - x_r \frac{\alpha_j}{\alpha_r} \right) \mathbf{a}_j = \mathbf{b}.$$

Thus, we have a solution with not more than $p - 1$ non-zero components. To ensure that these are positive, we shall choose $\mathbf{a}_r$ in such a way that

$$x_j - x_r \frac{\alpha_j}{\alpha_r} \geq 0 \quad \text{for all } j \neq r.$$

This requires that either $\alpha_j = 0$ or

$$\frac{x_j}{\alpha_j} \geq \frac{x_r}{\alpha_r}, \text{ if } \alpha_j > 0 \text{ and } \frac{x_j}{\alpha_j} \leq \frac{x_r}{\alpha_r}, \text{ if } \alpha_j < 0.$$

Thus, if we select $\mathbf{a}_r$ such that

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j}, \alpha_j > 0 \right\},$$

then for each of the $p-1$ variables $x_j - x_r \dfrac{\alpha_j}{\alpha_r}$ is non-negative, and so we have a feasible solution with not more than $p-1$ non-zero components.

Consider now this new feasible solution with not more than $p-1$ non-zero components. If the corresponding set of $p-1$ columns of $\mathbf{A}$ is linearly independent, case (*i*) applies and we have arrived at a basic feasible solution. If this set is again linearly dependent, we may repeat the process to arrive at a feasible solution with not more than $p-2$ non-zero components. The argument can be repeated. Ultimately, we get a feasible solution with associated set of column vectors of $\mathbf{A}$ linearly independent. The discussion of case (*i*) then applies and we do get a basic feasible solution.

This completes the proof.

Corollary. *There exists only finite number of basic feasible solutions to an L.P.P.*

Proof. Exercise for the reader.

***Theorem 4-2* (Extreme Point Correspondence).** *A basic feasible solution to an L.P.P. must correspond to an extreme point of the set of all feasible solutions and conversely.*

Proof. Let the L.P.P. be :

$$\text{Maximize} \quad z = \mathbf{c}^T\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n \quad \text{subject to the constraints:}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ are real $m \times n$, $m \times 1$ and $n \times 1$ matrices respectively. Let $\rho(\mathbf{A}) = m$.

Let S be the set of all feasible solutions to the L.P.P. Also suppose that $\mathbf{x}$ is a basic feasible solution

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{x}_B$ is an $m \times 1$ vector, such that for a non-singular sub-matrix $\mathbf{B}$ of $\mathbf{A}$, $\mathbf{B}\mathbf{x}_B = \mathbf{b}$.

If possible, let $\mathbf{x}$ be a point of S, such that there exist $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that $\mathbf{x}_1 \neq \mathbf{x}_2$ and

$$\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \qquad 0 < \lambda < 1.$$

Let

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{v}_2 \end{bmatrix}$$

where $\mathbf{u}_1, \mathbf{u}_2$ are $m \times 1$ vectors and $\mathbf{v}_1, \mathbf{v}_2$ are $(n-m) \times 1$ vectors. Then

$$\begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{v}_2 \end{bmatrix}$$

$\therefore$

$$\mathbf{x}_B = \lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2$$

and

$$\mathbf{0} = \lambda\mathbf{v}_1 + (1-\lambda)\mathbf{v}_2 \qquad 0 < \lambda < 1.$$

Since $\mathbf{x}_1, \mathbf{x}_2$ are feasible solutions, therefore $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \geq \mathbf{0}$.

Now

$$0 < \lambda < 1 \quad \text{and} \quad \mathbf{0} = \lambda\mathbf{v}_1 + (1-\lambda)\mathbf{v}_2$$

$\therefore$ We must have

$$\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}.$$

Thus

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{0} \end{bmatrix}$$

Again, since $\mathbf{x}_1$, $\mathbf{x}_2$ satisfy $\mathbf{A}\mathbf{x}_1 = \mathbf{b}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{b}$, therefore

$$\mathbf{B}\mathbf{u}_1 = \mathbf{b} \quad \text{and} \quad \mathbf{B}\mathbf{u}_2 = \mathbf{b}.$$

Also, since $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ and since expression of $\mathbf{b}$ as a linear combination of basis vectors must be unique, therefore, $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{x}_B$. Hence

$$\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2.$$

This is a contradiction, for by assumption $\mathbf{x}_1 \neq \mathbf{x}_2$.

Hence $\mathbf{x}$ is an extreme point of S.

Conversely. We prove that an extreme point $\mathbf{x}^* = [x_1, x_2, \ldots, x_n]$ of the set of all feasible solutions is a basic feasible solution. That is, the column vectors of $\mathbf{A}$ associated with non-zero variables are linearly independent.

Since some components (variables) of $\mathbf{x}^*$ may be zero, without any loss of generality let us suppose that the first p components of $\mathbf{x}^*$ are positive and the remaining $(n - p)$ are zero. Then $\mathbf{A}\mathbf{x}^* = \mathbf{b}$ implies that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \ldots + x_p \mathbf{a}_p = \mathbf{b},$$

where $\mathbf{a}_i \in \mathbf{A}$.

If possible, let us assume that the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ are not linearly independent. Then, there must exist some scalars α_j not all zero, (that is at least one $\alpha_j \neq 0$) such that

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \ldots + \alpha_p \mathbf{a}_p = \mathbf{0} \quad \text{or} \quad \sum_{j=1}^{p} \alpha_j \mathbf{a}_j = \mathbf{0}.$$

Therefore, for some arbitrary $\varepsilon > 0$, we can write

$$\sum_{j=1}^{p} x_j \mathbf{a}_j \pm \varepsilon \sum_{j=1}^{p} \alpha_j \mathbf{a}_j = \mathbf{b}.$$

or

$$\sum_{j=1}^{p} (x_j \pm \varepsilon \alpha_j) \mathbf{a}_j = \mathbf{b}. \qquad \text{(some } \alpha_j \neq 0)$$

Thus, the two different points

$$\mathbf{x}^{(1)} = (x_1 + \varepsilon \alpha_1,\ x_2 + \varepsilon \alpha_2,\ \ldots,\ x_p + \varepsilon \alpha_p,\ 0,\ 0,\ \ldots,\ 0)$$

and

$$\mathbf{x}^{(2)} = (x_1 - \varepsilon \alpha_1,\ x_2 - \varepsilon \alpha_2,\ \ldots,\ x_p - \varepsilon \alpha_p,\ 0,\ 0,\ \ldots,\ 0)$$

satisfy the constraints $\mathbf{A}\mathbf{x}^* = \mathbf{b}$.

Also, since $x_j > 0$, by selecting ε such that

$$0 < \varepsilon < \min \left\{ \frac{x_j}{|\alpha_j|} \right\} \qquad j = 1, 2, \ldots, p$$

the first p components of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ will always be positive. As the remaining $(n - p)$ components in $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are zero, it follows that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are feasible solutions different from $\mathbf{x}^*$. We observe that

$$\tfrac{1}{2}\mathbf{x}^{(1)} + \tfrac{1}{2}\mathbf{x}^{(2)} = (x_1, x_2, \ldots, x_n) = \mathbf{x}^*.$$

This shows that $\mathbf{x}^*$ can be expressed as a convex combination of two distinct feasible solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ by selecting $\lambda = \frac{1}{2}$, which contradicts the assumption that $\mathbf{x}^*$ is an extreme point.

Consequently, for $\mathbf{x}^*$ to be an extreme point, the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ must be linearly independent and hence $\mathbf{x}^*$ is a basic solution.

Now, since there are m equations in the system, there can be at most m linearly independent vectors in $\mathbf{R}^m$. Therefore, if $p = m$ (a non-degenerate extreme point), then by definition, there is a unique basic feasible solution corresponding to the extreme point. On the other hand, if $p < m$ (a

degenerate extreme point), then one can select $(m - p)$ additional vectorts from $\mathbf{A}$ with their corresponding variables equal to zero, such that the resulting set of vectors are linearly independent.

This completes the proof.

Remarks *1. The number of extreme points of convex set of feasible solutions is finite.* It follows from the preceding theorem that there is only one extreme point for a given basic feasible solution and conversely. So there is one-to-one correspondence between the extreme points and the basic feasible solutions. But basic feasible solutions are finite in number, therefore, the extreme points of the convex set of feasible solutions are also finite in number.

2. An extreme point can have at most m positive x_j's where m is the number of constraints.

Theorem 4-3 **(Fundamental Theorem of Linear Programming).** *If the feasible region of an L.P.P. is a convex polyhedron, then there exists an optimal solution to the L.P.P. and at least one basic feasible solution must be optimal.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to

$$\text{Maximize} \quad z = \mathbf{c}^T\mathbf{x} \quad \text{subject to : } \mathbf{Ax} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{c}, \mathbf{x} \in \mathbf{R}^n$.

Then the feasible region S of the L.P.P. is given by

$$S = \{ \mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0} \}.$$

Since S is a convex polyhedron, it is non-empty, closed and bounded.

The objective function $z = \mathbf{c}^T\mathbf{x}$, $\mathbf{x} \in S$ is continuous on S, which is non-empty, closed and bounded, therefore, z attains its maximum on S. This proves the existence of an optimal solution.

Now, since S is a convex polyhedron, it has finite number of extreme points. Let these be $\mathbf{x}_1$, $\mathbf{x}_2, \ldots, \mathbf{x}_k \in S$. Clearly $S = \langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \rangle$. Therefore, any $\mathbf{x} \in S$ can be expressed as a convex combination of the extreme points, say

$$\mathbf{x} = \sum_{j=1}^{k} \alpha_j \mathbf{x}_j ; \quad \alpha_j \geq 0, \quad \Sigma\alpha_j = 1 \quad \text{for all } j = 1, 2, \ldots, k$$

Let $\quad z_0 = \max. \{ \mathbf{c}^T\mathbf{x}_j \ \ j = 1, 2, \ldots, k \}$. Then for any $\mathbf{x} \in S$,

$$z = \mathbf{c}^T\mathbf{x} = \mathbf{c}^T (\Sigma \alpha_j \mathbf{x}_j) = \Sigma \alpha_j (\mathbf{c}^T \mathbf{x}_j) \leq \Sigma \alpha_j z_0 = z_0$$

$$\therefore \qquad z \leq z_0 \quad \text{for any } \mathbf{x} \in S.$$

Thus, the maximum value of z is attained only at one of the extreme points of S. That is, at least one extreme point of S yields an optimal solution.

Now, since each extreme point of S corresponds to a basic feasible solution of the L.P.P. therefore, at least one basic feasible solution is optimal. This completes the proof.

Remark. It is easily seen that if the optimal value of z is attained at more than one extreme points of S, then *every* convex combination of such extreme points also provides an optimal solution to the L.P.P. Thus, the optimal solution to an L.P.P. is either **unique** or **infinite** in number.

SAMPLE PROBLEM

406. *Let $x_1 = 2$, $x_2 = 4$ and $x_3 = 1$ be a feasible solution to the system of equations*

$$2x_1 - x_2 + 2x_3 = 2$$
$$x_1 + 4x_2 = 18.$$

Reduce the given feasible solution to a basic feasible solution.

Solution. The given system of equations can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 4 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$$

Since, rank (**A**) = 2, there are only two linearly independent columns of **A**. Thus, if the column vectors of **A** be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively, then

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \alpha_3 \mathbf{a}_3 = \mathbf{0};$$

where α_i's are scalars not all zero. If we let $\alpha_3 = -1$, then $\mathbf{a}_3 = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2$.

This gives $2 = 2\alpha_1 - \alpha_2$ and $0 = \alpha_1 + 4\alpha_2$. Solving these equations, we get $\alpha_1 = 8/9$ and $\alpha_2 = -2/9$.

Now,
$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j}, \alpha_j > 0 \right\} = \min \left\{ \frac{2}{8/9} \right\} = \frac{9}{4}.$$

This indicates that vector $\mathbf{a}_1$ can be removed. Hence, the new solution with not more than two non-zero variables is given by :

$$\hat{\mathbf{x}}_1 = 2 - \frac{9}{4} \times \frac{8}{9} = 0, \quad \hat{\mathbf{x}}_2 = 4 - \frac{9}{4} \times \left(\frac{-2}{9} \right) = \frac{9}{2}.$$

$$\hat{\mathbf{x}}_3 = 1 - \frac{9}{4} \times (-1) = \frac{13}{4}.$$

Thus, a basic feasible solution to the given L.P.P. is $x_2 = 9/2$ and $x_3 = 13/4$ with $x_1 = 0$ (non-basic).

Note. If we let $\alpha_3 = 1$, another basic feasible solution is obtained as

$$x_1 = 26/9 \text{ and } x_2 = 34/9 \text{ with } x_3 = 0 \text{ (non-basic).}$$

PROBLEMS

407. The column vector [1, 1, 1] is a feasible solution to the system of equations

$$x_1 + x_2 + 2x_3 = 4$$
$$2x_1 - x_2 + x_3 = 2$$

Reduce the given feasible solution to a basic feasible solution.

408. Consider the system of equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 3x_3 - 2x_4 = 4.$$

Here $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ and $x_4 = 0$ is a feasible solution. Reduce this feasible solution to two different basic feasible solutions. *[Meerut M.Sc. (Math.) 1998; Delhi B.Sc. (Stat.) 2000]*

409. If $x_1 = 2$, $x_2 = 3$, $x_3 = 1$ be a feasible solution of the following Linear Programming problem, then find the basic feasible solution :

Maximize $z = x_1 + 2x_2 + 4x_3$ subject to the constraints :

$$2x_1 + x_2 + 4x_3 = 11, \quad 3x_1 + x_2 + 5x_3 = 14,$$
$$x_1, x_2, x_3 \geq 0.$$

410. Show that the vector [1, 0, 1, 6] is not a basic feasible solution to the system

$$x_1 + x_2 + x_3 = 2$$
$$x_1 - x_2 + x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 - x_4 = 0.$$

***Theorem 4-4* (Replacement of a Basis Vector).** *Let an L.P.P. have a basic feasible solution. If we drop one of the basis vectors and introduce a non-basis vector in the basis set, then the new solution obtained is also a basic feasible solution.*

Proof. Let us consider the L.P.P. of maximizing $z = \mathbf{cx}$ subject to the constraints $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$; where $\mathbf{x}^T, \mathbf{c} \in \mathbf{R}^n$ and **A**, **b** are $m \times n$ and $m \times 1$ real matrices respectively.

Let $\rho(\mathbf{A}) = m$.

Suppose that there does exist a basic feasible solution $\mathbf{x}_B$, so that

(1) $$\mathbf{B}\mathbf{x}_B = \mathbf{b}, \quad \mathbf{x}_B \geq \mathbf{0}$$

where $\mathbf{B}$ forms a basis set for the column vectors of $\mathbf{A}$. Clearly, for any column vector $\mathbf{a}_j \in \mathbf{A}$, we have

$$\mathbf{a}_j = y_{1j}\mathbf{b}_1 + y_{2j}\mathbf{b}_2 + \ldots + y_{mj}\mathbf{b}_m = \mathbf{B}\mathbf{y}_j$$

where $\mathbf{b}_i \in \mathbf{B}$ and y_{ij} are suitable scalars.

Now, we know that if the basis vector $\mathbf{b}_r$, for which the coefficient y_{rj} is non-zero, is replaced by $\mathbf{a}_j \in \mathbf{A}$, then the new set of vectors also forms a basis.

Now, for $y_{rj} \neq 0$, we can write

$$\mathbf{b}_r = \frac{\mathbf{a}_j}{y_{rj}} - \sum_{i=1}^{m} \frac{y_{ij}}{y_{rj}} \mathbf{b}_i \qquad i \neq r$$

and therefore (1) gives

$$\mathbf{b} = \sum_{i=1}^{m} x_{Bi}\mathbf{b}_i + x_{Br}\left[\frac{\mathbf{a}_j}{y_{rj}} - \sum_{i=1}^{m} \frac{y_{ij}}{y_{rj}} \mathbf{b}_i\right] \qquad i \neq r$$

$$= \sum_{i=1}^{m} \left[x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right]\mathbf{b}_i + \frac{x_{Br}}{y_{rj}}\mathbf{a}_j. \qquad i \neq r$$

Thus, the new basic solution is $\hat{\mathbf{x}}_B$ having as its components, the variables

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \quad (i = 1, 2, \ldots, m;\ i \neq r) \quad \text{and} \quad \hat{x}_{Br} = \frac{x_{Br}}{y_{rj}}.$$

We shall now show that $\hat{\mathbf{x}}_B$ is feasible also, that is, the new basic variables $\hat{x}_{Bi}$ are also non-negative. Two cases do arise :

Case (i). $x_{Br} = 0$. In this case the new set of basic variables is obviously non-negative, since we have assumed the existence of a basic feasible solution $\mathbf{x}_B$.

Case (ii). $x_{Br} \neq 0$. In this case we must have $y_{rj} > 0$. This requires that for the remaining $y_{ij}\,(i \neq r)$ either $y_{rj} = 0$ for $i \neq r$, or

$$\frac{x_{Bi}}{y_{ij}} \geq \frac{x_{Br}}{y_{rj}}, \quad \text{for } y_{ij} > 0 \text{ and } i \neq r \quad \text{and} \quad \frac{x_{Bi}}{y_{ij}} \leq \frac{x_{Br}}{y_{rj}}, \quad \text{for } y_{ij} < 0 \text{ and } i \neq r.$$

So, if we select the index $r\,(y_{rj} \neq 0)$ in such a way that

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{\text{Min}}\left\{\frac{x_{Bi}}{y_{ij}};\ y_{ij} > 0,\ i \neq r\right\}$$

then, the new set of basic variables are non-negative; and hence the basic solution $\hat{\mathbf{x}}_B$ is feasible.

Remark. If after replacement of basis vector, the new basis matrix is

$$\hat{\mathbf{B}} = (\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \ldots, \hat{\mathbf{b}}_m)$$

where $\hat{\mathbf{b}}_i = \mathbf{b}_i$ for $i \neq r$ and $\hat{\mathbf{b}}_r = \mathbf{a}_j$; then, the new basic feasible solution is

$$\hat{\mathbf{x}}_B = \hat{\mathbf{B}}^{-1}\mathbf{b}$$

where $\hat{x}_{Bi} = x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}},\ i \neq r$ and $\hat{x}_{Br} = \dfrac{x_{Br}}{y_{rj}}$ are the basic variables.

***Definition* (Net evaluation).** *Let $\mathbf{x}_B$ be a basic feasible solution to the L.P.P. :*

$$\textit{Maximize} \quad z = \mathbf{c}\mathbf{x}, \quad \textit{where} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad \textit{and} \quad \mathbf{x} \geq \mathbf{0}.$$

Let $\mathbf{c}_B$ be the cost vector corresponding to $\mathbf{x}_B$. For each column vector $\mathbf{a}_j$ in $\mathbf{A}$, which is not a column vector of $\mathbf{B}$, let

$$\mathbf{a}_j = \sum_{i=1}^{m} y_{ij}\mathbf{b}_i.$$

Then the number

$$z_j = \sum_{i=1}^{m} c_{Bi} y_{ij}$$

is called the ***evaluation*** *corresponding to* $\mathbf{a}_j$ *and the number* $z_j - c_j$ *is called the* ***net evaluation*** *corresponding to* $\mathbf{a}_j$.

Theorem 4-5 **(Improved Basic Feasible Solution).** *Let* $\mathbf{x}_B$ *be a basic feasible solution to the L.P.P. : Maximize* $z = \mathbf{cx}$ *subject to* $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Let $\hat{\mathbf{x}}_B$ *be another basic feasible solution obtained by admitting a non-basis column vector* $\mathbf{a}_j$ *in the basis, for which the net evaluation* $z_j - c_j$ *is negative. Then* $\hat{\mathbf{x}}_B$ *is an improved basic feasible solution to the problem, that is*

$$\hat{\mathbf{c}}_B \hat{\mathbf{x}}_B > \mathbf{c}_B \mathbf{x}_B.$$

Proof. The L.P.P. is to determine $\mathbf{x}$, so as to Maximize $z = \mathbf{cx}$; $\mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, $\mathbf{b}^T \in \mathbf{R}^m$, where $\mathbf{A}$ is an $m \times n$ real matrix.

We are given that $\mathbf{x}_B$ is a basic feasible solution. Let $z_0 = \mathbf{c}_B \mathbf{x}_B$.

Let $\hat{\mathbf{a}}_j$ be the column vector introduced in the basis, such that $z_j - c_j < 0$.

Let $\mathbf{b}_r$ be vector removed from the basis and let $\hat{\mathbf{x}}_B$ be the new basic feasible solution, then

$$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \quad \text{and} \quad \hat{x}_{Br} = \frac{x_{Br}}{y_{rj}}$$

Thus, the new value of the objective function is

$$\begin{aligned}
\hat{z} &= \sum_{i=1}^{m} \hat{c}_{Bi}\hat{x}_{Bi} \\
&= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right) + \hat{c}_{Br}\frac{x_{Br}}{y_{rj}} \\
&= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right) + c_j\frac{x_{Br}}{y_{rj}} && (\because \hat{c}_{Br} = c_j) \\
&= z_0 - (z_j - c_j)\frac{x_{Br}}{y_{rj}} \\
&> z_0 && [\because x_{Br}/y_{rj} > 0].
\end{aligned}$$

Hence, the new basic feasible solution $\hat{\mathbf{x}}_B$ gives an improved value of the objective function.

Remark. If the given basic solution happens to be a non-degenerate one, then $x_{Br}/y_{rj} > 0$ and a definite increase in the value of z_0 occurs. If the given feasible solution happens to be a degenerate one, then an increase in the value of z_0 depends on x_{Br}/y_{rj}. In any case, $z_j - c_j < 0$ implies that $\hat{z}$ can never be less than z_0.

Corollary. If $z_j - c_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, \ldots, m$; then another basic feasible solution is obtained which gives an unchanged value of the objective function.

Proof. It follows from the above theorem that

$$\begin{aligned}
\hat{z} &= z_0 - (z_j - c_j)\frac{x_{Br}}{y_{rj}}, \quad y_{rj} > 0 \\
&= z_0, \quad \text{since } z_j - c_j = 0.
\end{aligned}$$

Theorem 4-6 **(Unbounded Solution).** *Let there exist a basic feasible solution to a given L.P.P. If for at least one j, for which* $y_{ij} \leq 0$ $(i = 1, 2, \ldots, m)$, *and* $z_j - c_j$ *is negative, then there does not exist any optimum solution to this L.P.P.*

Proof. Let the given L.P.P. be to determine $\mathbf{x}$ so as to

$$\text{Maximize } z = \mathbf{cx}; \quad \mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n \quad \text{subject to the constraints :}$$

$$\mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}, \ \mathbf{b}^T \in \mathbf{R}^m$$

where $\mathbf{A}$ is an $m \times n$ real matrix.

Let a basic feasible solution to this problem be $\mathbf{x}_B$, so that

$$\mathbf{Bx}_B = \mathbf{b} \quad \text{and} \quad \mathbf{x}_B \geq \mathbf{0}$$

with the value of the objective function

$$z_0 = \mathbf{c}_B \mathbf{x}_B = \sum_{i=1}^{m} c_{Bi} x_{Bi}.$$

Now, we can write

$$\mathbf{b} = \mathbf{Bx}_B + \xi \mathbf{a}_j - \xi \mathbf{a}_j \qquad \mathbf{a}_j \in \mathbf{A}, \ \xi \text{ is a scalar}$$

$$= \sum_{i=1}^{m} x_{Bi} \mathbf{b}_i + \xi \mathbf{a}_j - \xi \sum_{i=1}^{m} y_{ij} \mathbf{b}_i$$

$$= \sum_{i=1}^{m} (x_{Bi} - \xi y_{ij}) \mathbf{b}_i + \xi \mathbf{a}_j.$$

If $\xi > 0$, then $(x_{Bi} - \xi y_{ij}) \geq 0$ since $y_{ij} \leq 0$. This shows that there exists a feasible solution whose $(m + 1)$ components may be strictly positive. But, in general, it may not be a basic solution.

The value of the objective function for these $(m + 1)$ variables is given by

$$\hat{z} = \sum_{i=1}^{m} c_{Bi} (x_{Bi} - \xi y_{ij}) + \xi c_j = \sum_{i=1}^{m} c_{Bi} x_{Bi} - \xi \left(\sum_{i=1}^{m} c_{Bi} y_{ij} - c_j \right)$$

$$= z_0 - \xi (z_j - c_j).$$

But $\qquad z_j - c_j < 0 \text{ and } \xi > 0$

$\therefore \qquad \hat{z} \to +\infty \text{ as } \xi \to +\infty.$

Hence, there is no limit to the optimum value of z and, hence there exists an unbounded solution to the given LPP.

Remarks 1. If for some $\mathbf{a}_j$ for which $z_j - c_j > 0$ and $y_{ij} \leq 0$, $i = 1, 2, \ldots, m$, then there exists a feasible solution to the LPP, such that the $(m + 1)$ components of the solution may be strictly positive and

$$\hat{z} \to -\infty \text{ as } \xi \to +\infty.$$

2. If $y_{ij} \leq 0$ for at least one j for which $z_j - c_j = 0$, then $\hat{z} = z_0$.

Theorem 4-7 **(Conditions of Optimality).** *A sufficient condition for a basic feasible solution to an LPP to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which the column vector $\mathbf{a}_j \in \mathbf{A}$ is not in the basis* $\mathbf{B}$.

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to

$$\text{Maximize } \mathbf{z} = \mathbf{cx}; \quad \mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n \quad \text{subject to the constraints :}$$

$$\mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{A}$ and $\mathbf{b}$ are $m \times n$ and $m \times 1$ real matrices respectively. Let $\rho(\mathbf{A}) = m$ so that we can choose an $m \times m$ submatrix $\mathbf{B}$ of $\mathbf{A}$ as a basis matrix.

Let us assume that there exists a basic feasible solution $\mathbf{x}_B$ to this L.P.P. Let $\mathbf{c}_B$ be the cost vector corresponding to the basic variables.

Then $\qquad \mathbf{Bx}_B = \mathbf{b}, \ \mathbf{x}_B \geq \mathbf{0} \text{ and } z_0 = \mathbf{c}_B \mathbf{x}_B.$

Now, for all those j for which $\mathbf{a}_j \notin \mathbf{B}$, we are given that $z_j - c_j \geq 0$.

Let $\mathbf{a}_j = \mathbf{b}_j$ for all such j for which $\mathbf{a}_j \in \mathbf{B}$. Then

$$\mathbf{y}_j = \mathbf{B}^{-1} \mathbf{b}_j = \mathbf{e}_j, \text{ the unit vector} \qquad (\text{since } \mathbf{y}_j = \mathbf{B}^{-1} \mathbf{a}_j)$$

and
$$z_j - c_j = \mathbf{c}_B \mathbf{y}_j - c_j = \mathbf{c}_B \mathbf{e}_j - c_j$$
$$= c_{Bj} - c_j = 0 \qquad \text{(since } \mathbf{a}_j \in \mathbf{B},\ c_{Bj} = c_j\text{)}$$

Thus, $z_j - c_j \geq 0$ for all j for which $\mathbf{a}_j \in \mathbf{A}$.

Now, let $\mathbf{x}$ be a feasible solution. Then
$$\sum_{j=1}^{n} (z_j - c_j)\, x_j \geq 0, \qquad \text{(since } x_j \geq 0\text{)}$$

or
$$\sum_{j=1}^{n} z_j x_j \geq \sum_{j=1}^{n} c_j x_j \quad \text{or} \quad \sum_{j=1}^{n} \mathbf{c}_B \mathbf{y}_j x_j \geq \sum_{j=1}^{n} c_j x_j \qquad \text{(since } z_j = \mathbf{c}_B \mathbf{y}_j\text{)}$$

or
$$\sum_{i=1}^{m} c_{Bi} \sum_{j=1}^{n} y_{ij} x_j \geq \sum_{j=1}^{n} c_j x_j, \qquad \text{(since } \mathbf{c}_B \mathbf{y}_j = \sum_{i=1}^{m} c_{Bi} y_{ij}\text{)}$$

for all j for which $\mathbf{a}_j \notin \mathbf{B}$.

Now, since
$$\mathbf{x}_B = \mathbf{B}^{-1}(\mathbf{A}\mathbf{x}) = (\mathbf{B}^{-1}\mathbf{A})\,\mathbf{x} = \mathbf{Y}\mathbf{x}$$

or
$$x_{Bi} = \sum_{j=1}^{n} y_{ij} x_j \qquad i = 1, 2, \ldots, m$$

therefore, the above inequation can be written as
$$\sum_{i=1}^{m} c_{Bi} x_{Bi} \geq \sum_{j=1}^{n} c_j x_j \quad \text{or} \quad \mathbf{c}_B \mathbf{x}_B \geq \mathbf{c}\mathbf{x} \quad \text{or} \quad z_0 \geq z^*$$

where z^* is the value of the objective function for the feasible solution $\mathbf{x}$.

Hence, z_0 is an optimum for that basic feasible solutions for which $z_j - c_j > 0$ for all j such that $\mathbf{a}_j \notin \mathbf{B}$.

Corollary. *A necessary and sufficient condition for a basic feasible solution to an L.P.P. to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which $\mathbf{a}_j \notin \mathbf{B}$.*

Proof. It follows immediately from theorems *4-5 and 4-6* that the condition is necessary. Theorem *4-7* proves the sufficiency of the condition.

Theorem 4-8. *Any convex combination of k different optimum solutions to an L.P.P. is again an optimum solution to the problem.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be k different optimum solutions to the L.P.P. :

Maximize $z = \mathbf{c}\mathbf{x}$; $\mathbf{c}, \mathbf{x}^T \in \mathbf{R}^n$ subject to the constraints :
$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}$$

where $\mathbf{A}$ and $\mathbf{b}$ are $m \times n$ and $m \times 1$ real matrices respectively.

Let z_0 be the optimum value of z.

Obviously, $z_0 = \mathbf{c}\mathbf{x}_1 = \mathbf{c}\mathbf{x}_2 = \ldots = \mathbf{c}\mathbf{x}_k$ and $\mathbf{b} = \mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2 = \ldots = \mathbf{A}\mathbf{x}_k$

where $\mathbf{x}_1 \geq \mathbf{0}, \mathbf{x}_2 \geq \mathbf{0}, \ldots, \mathbf{x}_k \geq \mathbf{0}$.

Now, any combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ can be written as $\mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{x}_i$ where α_i's are positive scalars such that $\sum_{i=1} \alpha_i = 1$.

$\therefore$
$$\mathbf{A}\mathbf{x} = \mathbf{A}\left(\sum_{i=1}^{k} \alpha_i \mathbf{x}_i\right) = \sum_{i=1}^{k} \alpha_i \mathbf{A}\mathbf{x}_i = \sum_{i=1}^{k} \alpha_i \mathbf{b} = \mathbf{b} \sum_{i=1} \alpha_i$$
$$= \mathbf{b} \quad (\text{since } \Sigma\, \alpha_i = 1).$$

Further, since $\mathbf{x}_i \geq \mathbf{0}$ and $\alpha_i \geq 0$ for all $i = 1, 2, \ldots, k$ therefore every component of $\mathbf{x}$ will be non-negative.

Thus $\mathbf{x}$ is a feasible solution to the L.P.P.

Again, $$\mathbf{cx} = \mathbf{c}\left(\sum_{i=1}^{k} \alpha_i \mathbf{x}_i\right) = \sum_{i=1}^{k} \alpha_i \mathbf{c}\mathbf{x}_i = \sum_{i=1}^{k} \alpha_i z_0 = z_0$$

Hence, a convex combination of k different optimum solutions to an L.P.P. is also an optimum solution to the L.P.P.

4:3. THE COMPUTATIONAL PROCEDURE

The two fundamental conditions on which the simplex method is based are :

(*i*) *Condition of feasibility.* It assumes that if the initial (starting) solution is basic feasible, then during computation only basic feasible solutions will be obtained.

(*ii*) *Condition of optimality.* It guarantees that only better solutions will be encountered.

The computational procedure of simplex method requires the construction of the simplex table. The initial simplex table is constructed by writing out the coefficients of the variables in the objective function and constraints of LPP in a systematic tabular form.

Following is a specimen of the *simplex table* which conveniently displays the basic feasible solution $\mathbf{x}_B$, the associated cost vector $\mathbf{c}_B$, the basic variables, column vectors $\mathbf{y}_j$'s corresponding to basic as well as non-basic variables, the corresponding cost coefficients and the net evaluations in a tabular form :

			c_1	c_2	...	c_n
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	...	$\mathbf{y}_n$
$\mathbf{c}_{B1}$	$\mathbf{y}_{B1}$	$\mathbf{x}_{B1}$	$\mathbf{y}_{11}$	$\mathbf{y}_{12}$	...	$\mathbf{y}_{1n}$
$\mathbf{c}_{B2}$	$\mathbf{y}_{B2}$	$\mathbf{x}_{B2}$	$\mathbf{y}_{21}$	$\mathbf{y}_{22}$	...	$\mathbf{y}_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\mathbf{c}_{Bm}$	$\mathbf{y}_{Bm}$	$\mathbf{x}_{Bm}$	$\mathbf{y}_{m1}$	$\mathbf{y}_{m2}$	...	$\mathbf{y}_{mn}$
		z_0	$z_1 - c_1$	$z_2 - c_2$	...	$z_n - c_n$

The optimal solution to a General L.P.P. (when it exists) is obtained in the following major steps :

Step 1. Select an initial (starting) basic feasible solution to initiate the algorithm. (Use theorem *4-1* for the existence of a basic solution.)

Step 2. Check the objective function to see whether there is some non-basic variable that would improve the objective function if brought in the basis. If such a variable exists, go to *step 3*, otherwise stop. (Use Theorems *4-5* and *4-6* for the check.)

Step 3. Determine how large the variable found in *step 2* can be made until one of the basic variables in the current solution becomes zero. Eliminate the later variable and let the next *trial* solution contain the newly found variable instead. (Use Theorem *4-4*.)

Step 4. Check for optimality the current solution. (Use Theorem *4-7*.)

Step 5. Continue the iterations until either an optimum solution is attained or there is an indication that an unbounded solution exists.

Note. To simplify the computations, we usually consider the starting basis matrix $\mathbf{B}$ as an identity matrix $\mathbf{I}$, *i.e.*, $\mathbf{B} = \mathbf{I}$. Therefore $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j = \mathbf{I}^{-1}\mathbf{a}_j = \mathbf{I}\mathbf{a}_j = \mathbf{a}_j$,

$$\mathbf{I}_2\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = \mathbf{y}_1.$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{I}\mathbf{b} = \mathbf{b}, \text{ etc.}$$

The Simplex Algorithm

For the solution of any L.P.P. by **simplex algorithm**, the existence of an *initial* basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows :

Step 1. Check whether the objective function of the given L.P.P. is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximizing it by using the result

$$\text{Minimum } z = -\text{ Maximum } (-z).$$

Step 2. Check whether all b_i $(i = 1, 2, \ldots, m)$ are non-negative. If any one of b_i is negative then multiply the corresponding inequation of the constraints by -1, so as to get all b_i $(i = 1, 2, \ldots, m)$ non-negative.

Step 3. Convert all the inequations of the constraints into equations by introducing slack and/or surplus variables in the constraints. Put the costs of these variables equal to zero.

Step 4. Obtain an initial basic feasible solution to the problem in the form $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and put it in the first column of the simple table.

Step 5. Compute the net evaluations $z_j - c_j$ $(j = 1, 2, \ldots, n)$ by using the relation $z_j - c_j = \mathbf{c}_B \mathbf{y}_j - c_j$, where $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$.

Examine the sign $z_j - c_j$.

(*i*) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution $\mathbf{x}_B$ is an optimum basic feasible solution.

(*ii*) If at least one $(z_j - c_j) < 0$, proceed on to the next step.

Step 6. If there are more than one negative $z_j - c_j$, then choose the most negative of them. Let it be $z_r - c_r$ for some $j = r$.

(*i*) If all $(z_j - c_j) \geq 0$, then the initial basic feasible solution $\mathbf{x}_B$ is an optimum basic feasible solution.

(*ii*) If at least one $(z_j - c_j) < 0$, proceed on to next step.

Step 7. Compute the ratios $\left\{ \dfrac{x_{Bi}}{y_{ir}},\ y_{ir} > 0,\ i = 1, 2, \ldots, m \right\}$ and choose the minimum of them. Let the minimum of these ratios be x_{Bk}/y_{kr}. Then the vector $\mathbf{y}_k$ will level the basis $\mathbf{y}_B$. The common element y_{kr}, which is in the kth row and the rth column is known as the **leading element (or pivotal element)** of the table.

Step 8. Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeroes by making use of the relations :

$$\hat{y}_{ij} = y_{ij} - \frac{y_{kj}}{y_{kr}} y_{ir} \qquad i = 1, 2, \ldots, m+1;\ i \neq k$$

and

$$\hat{y}_{kj} = \frac{y_{kj}}{y_{kr}} \qquad j = 0, 1, 2, \ldots, n$$

Step 9. Go to *Step 5* and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

SAMPLE PROBLEMS

411. *Use simplex method to solve the following L.P.P. :*

Maximize $z = 4x_1 + 10x_2$ subject to the constraints :

$$2x_1 + x_2 \leq 50,\ 2x_1 + 5x_2 \leq 100,\ 2x_1 + 3x_2 \leq 90;\ x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Solution.

Step 1. By introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$ respectively, the set of constraints of the given L.P.P. are converted into the system of equations :

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$$

The modified objective function is to maximize

$$z = 4x_1 + 10x_2 + 0.s_1 + 0.s_2 + 0.s_3.$$

Step 2. An obvious initial basic feasible solution is given by $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

i.e.,

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$$

since $\quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{B}^{-1}$, and $\mathbf{b} = \begin{bmatrix} 50 \\ 100 \\ 90 \end{bmatrix}$.

Step 3. Compute $\mathbf{y}_j$ and $(z_j - c_j)$ as follows :

$$\mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

$$\mathbf{y}_3 = \mathbf{B}^{-1}\mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{y}_4 = \mathbf{B}^{-1}\mathbf{e}_2 = \mathbf{e}_2 \text{ and } \mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{e}_3 = \mathbf{e}_3$$

$$z_1 - c_1 = \mathbf{c}_B\mathbf{y}_1 - c_1 = (0, 0, 0) \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - 4 = -4$$

$$z_2 - c_2 = \mathbf{c}_B\mathbf{y}_2 - c_2 = (0, 0, 0) \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} - 10 = -10$$

$$z_3 - c_3 = \mathbf{c}_B\mathbf{y}_3 - c_3 = (0, 0, 0) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0 = 0$$

$$z_4 - c_4 = \mathbf{c}_B\mathbf{y}_4 - c_4 = (0, 0, 0) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = 0$$

$$z_5 - c_5 = \mathbf{c}_B\mathbf{y}_5 - c_5 = (0, 0, 0) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 0$$

Step 4. The initial basic feasible solution is now represented in the simplex table below :

		c_j	4	10	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	50	2	1	1	0	0
0	$\mathbf{y}_4$	100	2	5	0	1	0
0	$\mathbf{y}_5$	90	2	3	0	0	1
	z_j		0	0	0	0	0
	$z_j - c_j$	$z(= 0)$	−4	−10	0	0	0

From the tableau, it is apparent that there are two $z_j - c_j$ which are negative. We choose the most negative of these, *viz.*, − 10. The corresponding column vector $\mathbf{y}_2$ enters the basis.

Step 5. Since all the entries of $\mathbf{y}_2$ are positive, we compute $\min. \left\{ \frac{x_{Bi}}{y_{ir}}, y_{ir} > 0 \right\}$; *i.e.,* $\min. \left\{ \frac{50}{1}, \frac{100}{5}, \frac{90}{3} \right\} = \frac{100}{5}$. This occurs for the element $y_{22} = (= 5)$. Thus the vector $\mathbf{y}_4$ will leave the basis $\mathbf{y}_B$ and the common element y_{22} becomes the leading element for the first iteration.

Step 6. Convert the leading element y_{22} to unity and all other elements of $\mathbf{y}_2$ to zeroes by making use of the following transformation :

$$\hat{y}_{ij} = y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2}; \qquad i = 1, 2, 3, 4, \; i \neq 2$$

and

$$\hat{y}_{2j} = \frac{y_{2j}}{y_{22}}; \qquad j = 0, 1, 2, 3, 4, 5.$$

$$\therefore \quad \hat{y}_{21} = \frac{y_{21}}{y_{22}} = \frac{2}{5}; \; y_{20} = \frac{y_{20}}{y_{22}} = \frac{100}{5} \text{ or } 20, \text{etc.}$$

$$\hat{y}_{10} = y_{10} - \frac{y_{20}}{y_{22}} y_{12} = 50 - \frac{100}{5} \times 1 = 30$$

$$\hat{y}_{30} = y_{30} - \frac{y_{20}}{y_{22}} y_{32} = 90 - \frac{100}{5} \times 3 = 30$$

$$\hat{y}_{31} = y_{31} - \frac{y_{21}}{y_{22}} y_{32} = 2 - \frac{2}{5} \times 3 = \frac{4}{5}$$

$$\hat{y}_{11} = y_{11} - \frac{y_{21}}{y_{22}} y_{12} = 2 - \frac{2}{5} \times 1 = \frac{8}{5}$$

$$\hat{y}_{14} = y_{14} - \frac{y_{24}}{y_{22}} y_{12} = 0 - \frac{1}{5} \times 1 = -\frac{1}{5}, \text{ and so on.}$$

Step 7. Using the above computations, the iterated simplex table is :

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	30	8/5	0	1	−1/5	0
10	$\mathbf{y}_2$	20	2/5	1	0	1/5	0
0	$\mathbf{y}_5$	30	4/5	0	0	−3/5	1
	z_j		4	10	0	2	0
	$z_j - c_j$	z (= 200)	0	0	0	2	0

The above simplex table yields a new basic feasible solution with increased value of z. Moreover, no further improvement in the value of z is possible, since all $z_j - c_j \geq 0$. Hence, we have attained our desired optimum solution with basic variables x_2, s_1 and s_3. The maximal basic feasible solution to the given *L.P.P.*, therefore, is

$$x_1 = 0, \; x_2 = 20 \text{ with maximum } z = 200.$$

Note. It is evident from the net evaluations of the optimum table that the net evaluation corresponding to non-basic variable x_1 is zero.

This is an indication that an alternative basic solution exists. Thus, we can bring $\mathbf{y}_1$ into the basis in place of $\mathbf{y}_3$ or $\mathbf{y}_5$. the resulting new basic feasible solution will also be an optimum one.

Now, by introducing $\mathbf{y}_1$ into the basis in place of $\mathbf{y}_3$, the new optimum simplex table is as follows :

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
4	$\mathbf{y}_1$	150/8	1	0	5/8	−1/8	0
10	$\mathbf{y}_2$	25/2	0	1	−1/4	1/4	0
0	$\mathbf{y}_5$	15	0	0	−1/2	−1/2	1
	z_j		4	10	0	2	0
	$z_j - c_j$	z (= 200)	0	0	0	2	0

We observe from this table that the basic feasible solution has been changed but the optimum value of z remains the same.

If two basic (optimum) feasible solutions are known, an infinite number of basic (optimum) feasible solutions can be obtained by taking any weighted average of the two solutions. Thus, if

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 20 \\ 30 \\ 0 \\ 30 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 150/8 \\ 25/2 \\ 0 \\ 0 \\ 15 \end{bmatrix} \text{ are two BFSs, then a new BFS is :}$$

$$\mathbf{x}^* = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 = \begin{bmatrix} (1-\lambda)\,150/8 \\ 20\lambda + (1-\lambda)\,25/2 \\ 30\lambda \\ 0 \\ 30\lambda + (1-\lambda)\,15 \end{bmatrix} = \begin{bmatrix} (1-\lambda)\,150/8 \\ (25 - 15\lambda)/2 \\ 30\lambda \\ 0 \\ 15 + 15\lambda \end{bmatrix}, \quad 0 \le \lambda \le 1.$$

It can be verified that the solution $\mathbf{x}^*$ will always give the same value 200 of z, for all $0 \le \lambda \le 1$.

Remark. Interpretation of the final simplex table. The simplex table helps to predict the effect of changes in the resources and the profit margin :

(*a*) *Opportunity cost.* The terms in z_j-row indicate the opportunity cost. If we do not utilize one unit of x_1, the loss of profit is 4 units and non-utilization of one unit of x_2 costs 10 units.

(*b*) *Shadow cost.* The coefficient of non-basic variables, *i.e.*, of s_1 and s_2 are called 'shadow costs'. They represent the decrease in the optimum value of the objective function resulting from a unit increase in a non-basic variable.

412. *Use simplex method to*

$$\textit{Minimize } z = x_2 - 3x_3 + 2x_5 \textit{ subject to the constraints :}$$

$$3x_2 - x_3 + 2x_5 \le 7, \; -2x_2 + 4x_3 \le 12,$$

$$-4x_2 + 3x_3 + 8x_5 \le 10; \; x_2 \ge 0, \; x_3 \ge 0 \text{ and } x_5 \ge 0.$$

[Garhwal M.Sc. (Math.) 2002]

Solution. Introducing slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective inequalities; and converting the objective function into that of maximization; the linear programming problem is :

$$\text{Maximize } z^* = -(x_2 - 3x_3 + 2x_5) + 0.s_1 + 0.s_2 + 0.s_3 \text{ subject to the constraints :}$$

$$3x_2 - x_3 + 2x_5 + s_1 = 7$$

$$-2x_2 + 4x_3 + 0.x_5 + s_2 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + s_3 = 10$$

$$x_2, x_3, x_5, s_1, s_2, s_3 \ge 0.$$

The set of constraints can be written in matrix form as

$$\begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix} \text{ or } \mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{pmatrix}$, $\mathbf{x} = [x_2 \;\; x_3 \;\; x_5 \;\; s_1 \;\; s_2 \;\; s_3]$ and $\mathbf{b} = [7 \;\; 12 \;\; 10]$.

An obvious initial basic feasible solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, where $\mathbf{B} = \mathbf{I}_3$, and

$\mathbf{x}_B$ = basic variables corresponding to coloumns of basis matrix $\mathbf{B}$ (= $\mathbf{I}$).

$$\therefore \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$$

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_5$.

			−1	3	−2	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	7	3	−1	2	1	0	0
0	$\mathbf{y}_5$	12	−2	4	0	0	1	0
0	$\mathbf{y}_6$	10	−4	3	8	0	0	1
	z	0	1	−3	2	0	0	0

At least one $z_j - c_j$, *viz.*, $z_2 - c_2$ is negative and therefore the current basic feasible solution is not optimum. We choose the column corresponding to $z_2 - c_2$, *i.e.*, column vector $\mathbf{y}_2$ enters the basis $\mathbf{y}_B$ (since at least one $y_{i2} > 0$). Further, since minimum $\left\{\frac{x_{Bi}}{y_{i2}}, y_{i2} > 0\right\}$ is $\frac{12}{4}$ (= 3), current basis vector $\mathbf{y}_5$ leaves the basis. This gives y_{22} (= 4) as the leading element. Now using E-Row operations, we convert the leading element into unity and all other elements of the entering column vector $\mathbf{y}_2$ to zero. We get the improved basic feasible solution as shown in the next simplex table.

First Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	10	5/2	0	2	1	1/4	0
3	$\mathbf{y}_2$	3	−1/2	1	0	0	1/4	0
0	$\mathbf{y}_6$	1	−5/2	0	8	0	−3/4	1
	z	9	−1/2	0	2	0	3/4	0

Observe that $z_1 - c_1$ is negative and thus the current basic feasible solution is not optimum. The column vector corresponding to $z_1 - c_1$ enters the next basis $\mathbf{y}_B$ (since $y_{i1} > 0$). Further, since only $y_{11} > 0$ both $y_{12} < 0$ and $y_{13} < 0$); current basis vector $\mathbf{y}_4$ leaves the basis. This gives $y_{11}\left(=\frac{5}{2}\right)$ as the leading element. Using E-Row operations, we convert the leading element into unity and all other entries in its column $\mathbf{y}_1$ to zero. The improved basic feasible solution is obtained as shown in the next simplex table.

Final Iteration. Optimum Solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
−1	$\mathbf{y}_1$	4	1	0	4/5	2/5	1/10	0
3	$\mathbf{y}_2$	5	0	1	2/5	1/5	3/10	10
0	$\mathbf{y}_6$	11	0	0	10	2/5	−1/2	1
	z	11	0	0	12/5	1/5	8/10	0

Since all $z_j - c_j \geq 0$, an optimal basic feasible solution has been attained. Thus, the optimum solution to the given L.P.P. is

$$\text{Minimum } z = -\text{Maximum } z^* = -11 \text{ with } x_2 = 4,\ x_3 = 5 \text{ and } x_5 = 0.$$

413. *Find the maximum value of* $z = 107x_1 + x_2 + 2x_3$ *subject to the constraints :*

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7,\quad 16x_1 + x_2 - 6x_3 \leq 5,\ 3x_1 - x_2 - x_3 \leq 0;$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution. By introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$ respectively, the set of constraints of the given linear programming problem is converted into the system of equations :

$$\begin{pmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

An obvious initial basic feasible solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, where $\mathbf{B} = \mathbf{I}_3$, $\mathbf{x}_B = [x_4\ s_1\ s_2]$, and $\mathbf{b} = [7/3\ \ 5\ \ 0]$.

$$\therefore \quad \begin{bmatrix} x_4 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}.$$

Using now simplex method, the iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_6$.

			107	1	2	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	7/3	14/3	1/3	−2	1	0	0
0	$\mathbf{y}_5$	5	16	1	−6	0	1	0
0	$\mathbf{y}_6$	0	3	−1	−1	0	0	1
	z_j		0	0	0	0	0	0
	$z_j - c_j$	$z(=0)$	−107	−1	−2	0	0	0

It is apparent from the table that there are three $z_j - c_j$ which are negative. We choose the most negative of these, *viz.*, − 107. The corresponding column vector $\mathbf{y}_1$, therefore, enters the basis $\mathbf{y}_B$ (since all $y_{i1} > 0$). Further, since minimum $\left\{ \frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0 \right\}$ is $\frac{0}{3}$ which occurs for the element y_{31} (= 3), vector $\mathbf{y}_6$ leaves the basis $\mathbf{y}_B$ and the element y_{31} becomes the leading element. Reducing the leading element into unity all other entries of $\mathbf{y}_1$ to zero, we get

Final Iteration. Unbounded solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	7/3	0	17/9	−4/9	1	0	−14/9
0	$\mathbf{y}_5$	5	0	19/3	−2/3	0	1	−16/3
107	$\mathbf{y}_1$	0	1	−1/3	−1/3	0	0	1/3
	z_j		107	−107/3	−107/3	0	0	107/3
	$z_j - c_j$	$z(=0)$	0	−110/3	−113/3	0	0	107/3

It is apparent from the table that the net evaluations $z_2 - c_2$ and $z_3 - c_3$ are negative, $z_3 - c_3$ being the most negative one. The column vector corresponding to $z_3 - c_3$ is $\mathbf{y}_3$. Now $\mathbf{y}_3$ will not enter the basis $\mathbf{y}_B$, since y_{i3} (i = 1, 2, 3) are non-positive. Instead, this is an indication that there is an unbounded solution to the given L.P.P.

PROBLEMS

414. Use simplex method to solve the L.P.P. :

(*a*) Maximize $z = 3x_1 + 2x_2$ subject to the constraints :

(*i*) $x_1 + x_2 \le 4,\ x_1 - x_2 \le 2,\ x_1 \ge 0,\ x_2 \ge 0.$ *[IAS 1992]*

(*ii*) $x_1 + x_2 \le 6,\ 2x_1 + x_2 \le 6,\ x_1, x_2 \ge 0.$

(*b*) Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$x_1 + x_2 \le 4,\ -x_1 + x_2 \le 1$ and $x_1 + 2x_2 \le 5,\ x_1 \ge 0,\ x_2 \ge 0.$

415. Use simplex method to maximise $z = 5x_1 + 3x_2$ subject to the constraints :

(*a*) $x_1 + x_2 \le 2,\ 5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12,\ x_1, x_2 \ge 0.$

[Purvanchal M.C.A. 1996]

(*b*) $x_1 \le 4,\ x_2 \le 3\ \ x_1 + 2x_2 \le 18$
$x_1 + x_2 \le 9,\ x_1, x_2 \ge 0.$

[Nagarjuna M.Com. 1995]

416. Use simplex method to maximize $z = x_1 + 2x_2 + 3x_3$ subject to the constraints :

$$x_1 + 2x_2 + 3x_3 \le 10,\ x_1 + x_2 \le 5,\ x_1, x_2, x_3 \ge 0.$$

417. Use simplex method to

Maximize $z = 2x_1 - x_2 + x_3$ subject to the constraints :

$$3x_1 + x_2 + x_3 \le 60,\ x_1 - x_2 + 2x_3 \le 10,$$
$$x_1 + x_2 - x_3 \le 20 \text{ and } x_1, x_2, x_3 \ge 0.$$

418. Use simplex method to

Maximize $z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints :

$$x_1 + 2x_2 + x_3 \le 430,\ 3x_1 + 2x_3 \le 460,\ x_1 + 4x_3 \le 420\ ;\ x_1, x_2, x_3 \ge 0.$$

419. Use simplex method to solve the L.P.P. :

Maximize $z = 2x_1 + 4x_2 + x_3 + x_4$ subject to the constraints :

$$x_1 + 3x_2 + x_4 \le 4,\ 2x_1 + x_2 \le 3,\ x_2 + 4x_3 + x_4 \le 3,\ x_1, x_2, x_3, x_4 \ge 0.$$ *[Madras M.B.A. 2010]*

420. Use simplex method to

Maximize $z = 4x_1 + 3x_2 + 4x_3 + 6x_4$ subject to the constraints :

$$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80,\ 2x_1 + 2x_3 + x_4 \le 60$$
$$3x_1 + 3x_2 + x_3 + x_4 \le 80,\ x_1, x_2, x_3, x_4 \ge 0.$$

421. Using simplex method solve the L.P.P. :

Maximize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$ subject to the constraints :

$$x_1 + x_2 + x_3 + x_4 \le 15,\ 7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100,\ x_1, x_2, x_3, x_4 \ge 0.$$ *[Delhi B.Sc. (Stat.) 1999]*

422. Solve the following problem by the Simplex Method :

Maximize $15x_1 + 6x_2 + 9x_3 + 2x_4$ subject to the constraints :

$$2x_1 + x_2 + 5x_3 + 0.6x_4 \le 10,\ 3x_1 + x_2 + 3x_3 + 0.25x_4 \le 12$$
$$7x_1 + x_4 \le 35,\ x_j \ge 0 \text{ for } j = 1, 2, 3, 4.$$

423. Show that the L.P. P.

Maximize $z = 4x_1 + x_2 + 3x_3 + 5x_4$ subject to the constraints :

$$4x_1 - 6x_2 - 5x_3 - 4x_4 \ge -20,\ 3x_1 - 2x_2 + 4x_3 + x_4 \le 10$$
$$8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20,\ x_1, x_2, x_3, x_4 \ge 0$$

has an unbounded solution.

[Purvanchal M.C.A. 1996]

4:4. USE OF ARTIFICIAL VARIABLES

We have seen that in the computational procedure of the simplex method, it is most convenient to have the slack variables as the starting (initial) basic variables. Thus, if the original constraint is an equation or is of the type (≥) we may no longer have a *ready* starting basic feasible solution.

In order to obtain an initial basic feasible solution, we first put the given L.P.P. into its standard form and then a non-negative variable is added to the left side of each of equation that lacks the much needed starting basic variables. The so-added variable is called an *artificial variable* and plays the same role as a slack variable in providing the initial basic feasible solution. However, since such artificial variables have no physical meaning from the standpoint of the original problem, the method will be valid only if we are able to force these variables to be out or at zero level when the optimum

solution is attained. In other words, to get back to the original problem, artificial variables must be driven to zero in the final solution; otherwise the resulting solution may be infeasible.

Two methods are generally employed for the solution of linear programming problems having artificial variables :

1. Two-Phase Method; and
2. Big-M Method or Method of Penalties.

TWO-PHASE METHOD

In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints (known as auxiliary L.P.P.) to get a basic feasible solution to the original L.P.P. Second phase then optimizes the original objective function starting with the basic feasible solution obtained at the end of *Phase 1*.

The iterative procedure of the algorithm may be summarise as below :

Step 1. Write the given L.P.P. into its standard form and check whether there exists a starting basic feasible solution.

(*a*) If there is a ready starting basic feasible solution, go to *Phase 2*.

(*b*) If there does not exist a ready starting basic feasible solution, go on to the next step.

PHASE 1

Step 2. Add the artificial variable to the left side of the each equation that lacks the needed starting basic variables. Construct an auxiliary objective function aimed at minimising the sum of all artificial variables.

Thus, the new objective is to

$$\text{Minimize} \quad z = A_1 + A_2 + \ldots + A_n$$

i.e.,

$$\text{Maximize} \quad z^* = -A_1 - A_2 - \ldots - A_n$$

where A_i ($i = 1, 2, \ldots, m$) are the non-negative artificial variables.

Step 3. Apply simplex algorithm to the specially constructed L.P.P. The following three cases may arise at the least interaction :

(*a*) max $z^* < 0$ and at least one artificial variable is present in the basis with positive value. In such a case, the original L.P.P. does not possess any feasible solution.

(*b*) max $z^* = 0$ and at least one artificial variable is present in the basis at zero value. In such a case, the original L.P.P. possess the feasible solution. In order to get basic feasible solution we may proceed directly to *Phase 2* or else eliminate the artificial basic variable and then proceed to *Phase 2*.

(*c*) maximum $z^* = 0$ and no artificial variable is present in the basis. In such a case, a basic feasible solution to the original L.P.P. has been found. Go to *Phase 2*.

PHASE 2

Step 4. Consider the optimum basic feasible solution of *Phase 1* as a starting basic feasible solution for the original L.P.P. Assign actual coefficients to the variables in the objective function and a value zero to the artificial variables that appear at zero value in the final simplex table of *Phase 1*.

Apply usual simplex algorithm to the modified simplex table to get the optimum solution of the original problem.

Note : Artificial variables that do not appear in the basic solution may be deleted from the simplex table totally.

BIG-M METHOD (*METHOD OF PENALTIES*)

The Big-M method is an alternative method of solving a linear programming problem involving artificial variables. In this method we assign a very high penalty (say M) to the artificial variables in the objective function.

The iterative procedure of the algorithm is given below :

Step 1. Write the given L.P.P. into its standard form and check whether there exists a starting basic feasible solution.

(*a*) If there is a ready starting basic feasible solution, move on to *Step 3.*

(*b*) If there does not exist a ready starting basic feasible solution, move on to *Step 2.*

Step 2. Add artificial variables to the left side of each equation that has no obvious starting basic variables. Assign a very high penalty (say M) to these variables in the objective function.

Step 3. Apply simplex method to the modified L.P.P. Following cases may arise at the last iteration :

(*a*) At least one artificial variable is present in the basis with zero value. In such a case the current optimum basic feasible solution is degenerate.

(*b*) At least one artificial variable is present in the basis with a positive value. In such a case, the given L.P.P. does not possess an optimum basic feasible solution. The given problem is said to have a pseudo-optimum basic feasible solution.

SAMPLE PROBLEMS

424. *Use two-phase simplex method to maximize* $z = 5x_1 + 3x_2$ *subject to the constraints :*

$$2x_1 + x_2 \le 1, \quad x_1 + 4x_2 \ge 6 \quad \text{and} \quad x_1, x_2 \ge 0.$$

Solution. Introducing a slack variable $s_1 \ge 0$, a surplus variable $s_2 \ge 0$ and an artificial variable $A_1 \ge 0$ in the constraints of the L.P.P., an initial basic feasible solution is : $s_1 = 1$ and $A_1 = 6$ with I_2 as the basis matrix.

Phase I. The objective function of the auxiliary L.P.P. is to maximize $z^* = -A_1$.

Using now simplex algorithm to the auxiliary linear programming problem, the iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_3$.

			0	0	0	0	−1
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	1	2	1	1	0	0
−1	$\mathbf{y}_5$	6	1	4	0	−1	1
	z_j	—	−1	−4	0	1	−1
	$z_j - c_j$	$z\,(=-6)$	−1	−4	0	1	0

Since $z_1 - c_1$ and $z_2 - c_2$ are negative, we choose the most negative of these, *viz.*, −4. The corresponding column vector $\mathbf{y}_2$ enters the basis. Further, since min. $\left\{ \frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0 \right\}$ is 1, which occurs for element y_{12}, $\mathbf{y}_3$ leaves the basis.

Final Iteration. Optimum solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_2$	1	2	1	1	0	0
−1	$\mathbf{y}_5$	2	−7	0	−4	−1	1
		$z\,(=-2)$	7	0	4	1	0

Since all $(z_j - c_j) \geq 0$, an optimum basic feasible solution to the auxiliary L.P.P. is obtained. But max. $z^* < 0$ and an artificial variable is in the basis at a positive level. Therefore, the original L.P.P. does not possess any feasible solution.

425. *Use two-phase simplex method to*

$$\text{Maximize } z = 5x_1 - 4x_2 + 3x_3 \text{ subject to the constraints :}$$

$$2x_1 + x_2 - 6x_3 = 20,\ 6x_1 + 5x_2 + 10x_3 \leq 76$$

$$8x_1 - 3x_2 + 6x_3 \leq 50,\ x_1, x_2, x_3 \geq 0.$$

Solution. Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$, the L.P.P. in its standard form is :

$$\text{Maximize } z = 5x_1 - 4x_2 + 3x_3 \text{ subject to the constraints :}$$

$$2x_1 + x_2 - 6x_3 = 20,\ 6x_1 + 5x_2 + 10x_3 + s_1 = 76$$

$$8x_1 - 3x_2 + 6x_3 + s_2 = 50,\ x_1, x_2, x_3, s_1, s_2 \geq 0.$$

The set of constraints in matrix form is

$$\begin{pmatrix} 2 & 1 & -6 & 0 & 0 \\ 6 & 5 & 10 & 1 & 0 \\ 8 & -3 & 6 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix} = \begin{bmatrix} 20 \\ 76 \\ 50 \end{bmatrix}$$

Since, there does not exist columns of an identity matrix which can be considered for the starting basis matrix. We add requisite identity column(s) so as to complete the identity basis matrix. That is, insert the identity column [1 0 0] as the new column $\mathbf{y}_6$. Clearly, this amounts to adding an *artificial* variable $A_1 \geq 0$ in the 1st constraint.

Now, an initial basic feasible solution is $A_1 = 20$, $s_1 = 76$ and $s_2 = 50$.

Phase 1. The objective function of the auxiliary L.P.P. is $z^* = -A_1$.

The iterative simplex tables for the auxiliary L.P.P. are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_5$.

			0	0	0	0	0	−1
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
−1	$\mathbf{y}_6$	20	2	1	−6	0	0	1
0	$\mathbf{y}_4$	76	6	5	10	1	0	0
0	$\mathbf{y}_5$	50	8	−3	6	0	1	0
		z^* (= −20)	−2	−1	6	0	0	0

Since $z_1 - c_1$ is negative, the column vector $\mathbf{y}_1$ enters the basis. Further, since min. $\left\{ \frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0 \right\} = \frac{50}{8}$; $\mathbf{y}_5$ leaves the basis. The element $\mathbf{y}_{31}$ (= 8) becomes the leading element.

First Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_6$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
−1	$\mathbf{y}_6$	15/2	0	7/4	−15/2	0	−1/4	1
0	$\mathbf{y}_4$	77/2	0	29/4	11/2	1	−3/4	0
0	$\mathbf{y}_1$	25/4	1	−3/8	3/4	0	1/8	0
		z^* (= −15/2)	0	−7/4	15/2	0	1/4	0

Here, $z_2 - c_2$ is the only negative $z_j - c_j$. This indicates that $\mathbf{y}_2$ enters the basis. Also min. $\left\{ \frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0 \right\} = \frac{15/2}{7/4}$ suggests that $\mathbf{y}_6$ must leave the basis, thereby y_{12} $\left(\frac{-7}{4}\right)$ becomes the leading element.

Final Iteration. Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_2	30/7	0	1	−30/7	0	−1/7	4/7
0	y_4	52/7	0	0	256/7	1	2/7	−29/7
0	y_1	55/7	1	0	−6/7	0	1/14	3/14
		z^* (= 0)	0	0	0	0	0	1

Since, all $z_j - c_j \geq 0$ an optimum solution to the auxiliary L.P.P. has been reached. Furthermore, it is apparent from the table that no artificial variable appears in the basis.

Phase 2. Now, we consider the actual costs associated with the original variables. So, the objective function is

$$z = 5x_1 - 4x_2 + 3x_3 + 0 \cdot s_1 + 0 \cdot s_2$$

The iterative simplex table for this phase is :

			5	−4	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
−4	y_2	30/7	0	1	−30/7	0	−1/7
0	y_4	52/7	0	0	256/7	1	2/7
5	y_1	55/7	1	0	−6/7	0	1/14
		z^* (= 155/7)	0	0	69/7	0	13/4

Since, all $z_j - c_j \geq 0$ an optimum basic feasible solution has been reached. Hence, an optimum basic feasible solution to the given L.P.P. is

$$x_1 = 55/7, \quad x_2 = 30/7, \quad x_3 = 0 \text{; maximum } z = 155/7.$$

426. *Use penalty (or Big M) method to*

Maximize $z = 6x_1 + 4x_2$ *subject to the constraints :*

$$2x_1 + 3x_2 \leq 30, \quad 3x_1 + 2x_2 \leq 24, \quad x_1 + x_2 \geq 3,$$

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

Is the solution unique? If not, give two different solutions.

Solution. Introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and surplus variable $s_3 \geq 0$ in the constraints, the standard form of L.P.P. is :

Maximize $z = 6x_1 + 4x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$ subject to the constraints :

$$2x_1 + 3x_2 + s_1 = 30, \quad 3x_1 + 2x_2 + s_2 = 24, \quad x_1 + x_2 - s_3 = 3,$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Clearly, we do not have a initial (starting) basic feasible solution. So, we introduce an artificial variable $A_1 \geq 0$ in the third constraint. Then, an obvious initial basic feasible solution is

$$s_1 = 30, \quad s_2 = 24 \quad \text{and} \quad A_1 = 3.$$

Now, corresponding to the basic variables s_1, s_2 and A_1, the matrix $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$ (where $\mathbf{B}$ = I the identity matrix) and the net evaluations $z_j - c_j$ $(j = 1, 2, 3, 4, 5, 6)$ are computed where $\mathbf{c}_B = (0, 0, -M)$.

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_6$.

			6	4	0	0	0	−M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	30	2	3	1	0	0	0
0	y_4	24	3	2	0	1	0	0
−M	y_6	3	1	1	0	0	−1	1
		z (= −3M)	−M − 6	−M − 4	0	0	M	0

In the above table $z_1 - c_1$ and $z_2 - c_2$ are negative. Among these two $z_1 - c_1$ gives the most negative value (since M is very large), therefore $\mathbf{y}_1$ enters the basis. Further, min. $\left\{ \frac{x_{Bi}}{y_{i1}} \cdot y_{i1} > 0 \right\} = \frac{3}{1}$.

This indicates $\mathbf{y}_6$ leaves the basis and y_{31} becomes the leading element. Since $\mathbf{y}_6$ corresponds to artificial variable A_1, we drop it from the objective function.

First Iteration. Introduce $\mathbf{y}_5$ and drop $\mathbf{y}_4$.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	24	0	1	1	0	2
0	y_4	15	0	-1	0	1	3
6	y_1	3	1	1	0	0	-1
		z (= 18)	0	2	0	0	-6

Since $z_5 - c_5 < 0$, $\mathbf{y}_5$ enters the basis. Further, min. $\left\{ \frac{x_{Bi}}{y_{i5}} \cdot y_{i5} > 0 \right\} = \frac{15}{3}$.

$\therefore$ $\mathbf{y}_4$ leaves the basis and y_{25} becomes the leading element.

Final Iteration. Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	14	0	5/3	1	-2/3	0
0	y_5	5	0	-1/3	0	1/3	1
6	y_1	8	1	2/3	0	1/3	0
		z (= 48)	0	0	0	2	0

Since all $z_j - c_j \geq 0$, an optimum solution has been reached. Thus, an optimum basic feasible solution to the given L.P.P. is

$$x_1 = 8 \text{ and } x_2 = 0 \text{ with max. } z = 48.$$

Alternative Solution. It is evident from the net evaluations of the optimum table that the net evaluation corresponding to non-basic variable x_2 is zero. This is an indication that the current solution is not unique and an alternative solution exists. Thus we can bring $\mathbf{y}_2$ into the basis in place of $\mathbf{y}_3$ or $\mathbf{y}_5$. The resulting new basic feasible solution will also be an optimum one.

Therefore, by introducing $\mathbf{y}_2$ into the basis in place of $\mathbf{y}_3$, the new optimum simplex table is

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
4	y_2	42/5	0	1	3/5	-2/5	0
0	y_5	39/5	0	0	1/5	1/5	1
6	y_1	12/5	1	0	-2/5	3/5	0
		z (= 48)	0	0	0	2	0

We observe from this table that a different basic feasible solution results, but the optimum value of z remains the same. Hence, the two basic (optimum) feasible solutions to the given L.P.P. are :

$$\mathbf{x}_1 = [8, 0] \text{ and } \mathbf{x}_2 = [12/5, 42/5].$$

427. *Maximize* $z = 3x_1 + 2x_2$ *subject to the constraints :*

$$2x_1 + x_2 \leq 2, \quad 3x_1 + 4x_2 \geq 12, \quad x_1, x_2 \geq 0.$$

Solution. Introducing slack variable $s_1 \geq 0$, surplus variable $s_2 \geq 0$ and an artificial variable $A_1 \geq 0$, the reformulated L.P.P. can be written as :

$$\text{Maximize } z = 3x_1 + 2x_2 + 0 \cdot s_1 + 0 \cdot s_2 - MA_1 \text{ subject to the constraints :}$$

$$2x_1 + x_2 + s_1 = 2, \quad 3x_1 + 4x_2 - s_2 + A_1 = 12, \quad x_1, x_2, s_1, s_2 \geq 0 \text{ and } A_1 \geq 0.$$

An obvious starting basic feasible solution is : $s_1 = 2$ and $A_1 = 12$.

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_3$.

			3	2	0	0	$-M$
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	2	2	1	1	0	0
$-M$	$\mathbf{y}_5$	12	3	4	0	-1	1
	z	$-12M-3$	$-3M-2$	$-4M$	0	M	0

Clearly, $z_2 - c_2$ is the most negative and hence $\mathbf{y}_2$ enters the current basis. Further, $\min. \left\{ \frac{x_{Bi}}{y_{i2}} \cdot y_{i2} > 0 \right\} = \frac{2}{1}$. So, $\mathbf{y}_3$ leaves the current basis.

Final Iteration. No solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
2	$\mathbf{y}_2$	2	2	1	0	1	0
$-M$	$\mathbf{y}_5$	4	-5	0	-4	-1	1
	z	$-4M+4$	$5M+1$	0	$4M+2$	M	0

Here, the coefficient of M in each $z_j - c_j$ is non-negative and an artificial vector appears in the basis, not at the zero level. Thus the given L.P.P. does not possess any feasible solution.

428. *Use penalty (or Big M) method to*

Maximize $z = x_1 + 2x_2 + 3x_3 - x_4$ *subject to the constraints :*

$$x_1 + 2x_2 + 3x_3 = 15,\ 2x_1 + x_2 + 5x_3 = 20,\ x_1 + 2x_2 + x_3 + x_4 = 10,\ x_1, x_2, x_3, x_4 \geq 0.$$

[I.A.S. 1995; Garhwal M.Sc. (Math.) 2002]

Solution. From the constraints of the problem, we observe that we do not have requisite identity column to get the starting $\mathbf{B}$ as an identity matrix. So, we introduce artificial variables $A_1 \geq 0$ and $A_2 \geq 0$ in the first and second constraints respectively. An initial basic feasible solution, then, is

$$A_1 = 15,\ A_2 = 20 \text{ and } x_4 = 10$$

Now, corresponding to basic variables A_1, A_2 and x_4, the basis matrix $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$ and the net evaluations $z_j - c_j$ $(j = 1, 2, 3, 4, 5, 6)$ are computed, where $\mathbf{c}_B = (-M\ \ -M\ \ -1)$. The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_5$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$-M$	$\mathbf{y}_6$	15	1	2	3	0	0	1
$-M$	$\mathbf{y}_5$	20	2	1	5	0	1	0
-1	$\mathbf{y}_4$	10	1	2	1	1	0	0
	z	$-35M-10$	$-3M-2$	$-3M-4$	$-8M-4$	0	0	0

Since, the most negative $z_j - c_j$ $(= z_3 - c_3)$ corresponds to $\mathbf{y}_3$, it enters the basis. Further, $\min. \left\{ \frac{x_{Bi}}{y_{i3}} \cdot y_{i3} > 0 \right\} = \frac{20}{5}$, the current basis vector $\mathbf{y}_5$ leaves the basis and y_{23} becomes the leading element.

As $\mathbf{y}_5$ corresponds to an artificial variable A_2, we drop the $\mathbf{y}_5$ column from subsequent simplex tables.

First Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_6$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_6$
$-M$	$\mathbf{y}_6$	3	$-1/5$	7/5	0	0	1
3	$\mathbf{y}_3$	4	2/5	1/5	1	0	0
-1	$\mathbf{y}_4$	6	3/5	9/5	0	1	0
	z	$-3M+6$	$\frac{M}{5} - \frac{2}{5}$	$\frac{-7M}{5} - \frac{16}{5}$	0	0	0

Clearly, $z_2 - c_2$ is the only negative $z_j - c_j$ (since M is very large) and hence $\mathbf{y}_2$ enters the basis. Further, min. $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\}$ corresponds to $\mathbf{y}_6$. So, $\mathbf{y}_6$ leaves the basis and y_{12} becomes the leading element. Again, $\mathbf{y}_6$ corresponds to artificial variable A_1 and therefore we drop the artificial column $\mathbf{y}_6$ in the subsequent tables.

Second Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
2	$\mathbf{y}_2$	15/7	−1/7	1	0	0
3	$\mathbf{y}_3$	25/7	3/7	0	1	0
−1	$\mathbf{y}_4$	15/7	6/7	0	0	1
	z	90/7	−6/7	0	0	0

Clearly, $z_1 - c_1 < 0$ and, therefore, $\mathbf{y}_1$ enters the basis. Further, min. $\left\{\frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0\right\}$ corresponds to $\mathbf{y}_4$. So, $\mathbf{y}_4$ leaves the basis and y_{31} becomes the leading element.

Final Iteration. Optimum Solution.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
2	$\mathbf{y}_2$	15/6	0	1	0	1/6
3	$\mathbf{y}_3$	15/6	0	0	1	−3/6
1	$\mathbf{y}_1$	15/6	1	0	0	7/6
	z	15	0	0	0	1

Since, all $z_j - c_j$ are positive, therefore, an optimum basic feasible solution has been attained. Thus, an optimum solution to the given L.P.P. is

$$\text{Maximum } z = 15;\ x_1 = x_2 = x_3 = 5/2 \text{ and } x_4 = 0.$$

PROBLEMS

429. Use two-phase simplex method to

(*a*) Maximize $z = 10x_1 + 20x_2$ subject to the constraints :

$$2x_1 + x_2 = 1,\ x_1 + 2x_2 = 5,\ x_1 \geq 0,\ x_2 \geq 0.$$ *[Madurai M.Com. 2002]*

(*b*) Minimize $z = 2x_1 + 4x_2$ subject to the constraints :

$$2x_1 + x_2 \geq 14,\ x_1 + 3x_2 \geq 18,\ x_1 + x_2 \geq 12;\ x_1\ x_2 \geq 0.$$ *[Panjab Tech. Univ. M.B.A. (Dec.) 2010]*

430. Use two-phase simplex method to

Maximize $z = 3x_1 - x_2$ subject to the constraints :

$$2x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 2,\ x_2 \leq 4;\ x_1, x_2 \geq 0.$$ *[Meerut M.Sc. (Math.) 1998]*

431. Use two-phase simplex method to

Maximize $z = 5x_1 + 8x_2$ subject to the constraints :

$$3x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4,\ x_1 + x_2 \leq 5,\ x_1, x_2 \geq 0.$$

432. Use two-phase simplex method to

Minimize $z = x_1 + x_2 + x_3$ subject to the constraints :

$$x_1 - 3x_2 + 4x_3 = 5,\ x_1 - 2x_2 \leq 3,\ 2x_2 + x_3 \geq 4$$

$$x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \text{ is unrestricted.}$$

433. Use two-phase simplex method to

Maximize $z = x_1 + 2x_2 + 3x_3$ subject to the constraints :

$$x_1 - x_2 + x_3 \geq 4,\ x_1 + x_2 + 2x_3 \leq 8,\ x_1 - x_3 \geq 2,\ x_1, x_2, x_3 \geq 0.$$

434. Use two-phase simplex method to :

Maximize $z = 12x_1 + 15x_2 + 9x_3$ subject to the constraints :

$$8x_1 + 16x_2 + 12x_3 \le 250,\ 4x_1 + 8x_2 + 10x_3 \ge 80,\ 7x_1 + 9x_2 + 8x_3 = 105;$$
$$x_1, x_2, x_3 \ge 0.$$

435. Use penalty method to

Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$$x_1 + 2x_2 \le 4\ \ x_1 + x_2 = 3;\ x_1 \ge 0 \text{ and } x_2 \ge 0.$$

436. Use penalty method to

Minimize $z = 12x_1 + 20x_2$ subject to the constraints :

$$6x_1 + 8x_2 \ge 100,\ 7x_1 + 12x_2 \ge 120 \text{ and } x_1, x_2 \ge 0.$$

[Bharathidasan B.Sc. (Math.) 1995]

437. Use penalty method to

Minimize $z = 2x_1 + x_2$ subject to the constraints :

$$3x_1 + x_2 = 3,\ 4x_1 + 3x_2 \ge 6,\ x_1 + 2x_2 \le 3, \text{ and } x_1 \ge 0, x_2 \ge 0.$$

[Delhi B.Sc. (Math.) 1996; Madras B.E. 1999]

438. Use Big M method to

Maximize $z = 8x_2$ subject to the constraints :

$$x_1 - x_2 \ge 0,\ 2x_1 + 3x_2 \le 6,\ x_1, x_2 \text{ unrestricted.}$$

[Bharathidasan B.Sc. (Allied) 1995]

439. Use penalty (or Big M) method to

Maximise $z = 2x_1 + x_2 + 3x_3$ subject to the constraints :

$$x_1 + x_2 + 2x_3 \le 5,\ 2x_1 + 3x_2 + 4x_3 = 12;\ x_1, x_2, x_3 \ge 0.$$ *[Delhi B.Sc. (Math.) 1996]*

440. Use Big M method to

Minimize $z = 5x_1 - 6x_2 - 7x_3$ subject to the constraints :

$$x_1 + 5x_2 - 3x_3 \ge 15,\ 5x_1 - 6x_2 + 10x_3 \ge 0,\ x_1 + x_2 + x_3 = 5;\ x_1, x_2, x_3 \ge 0.$$

441. Solve the following L.P.P. by simplex method :

Maximize $z = 3x_1 + 2x_2 + x_3$ subject to the constraints :

$$-3x_1 + 2x_2 + 2x_3 = 8,\ -3x_1 + 4x_2 + x_3 = 7;\ x_1, x_2, x_3 \ge 0.$$

442. Using simplex method, solve the following LPP :

Maximize $z = 3x_1 + 2x_2 + x_3$ subject to the constraints :

$$2x_1 + 5x_2 + x_3 = 12,\ 3x_1 + 4x_2 = 11;\ x_2, x_3 \ge 0 \text{ and } x_1 \text{ unrestricted.}$$

[Guru Nanak Dev Univ. M.C.A. 2006]

4:5. DEGENERACY IN LINEAR PROGRAMMING

The phenomenon of obtaining a degenerate basic feasible solution in a linear programming problem is known as *Degeneracy*.

Degeneracy in an L.P.P. may arise (*i*) at the initial stage, and (*ii*) at any subsequent iteration stage.

In case (*i*) at least one basic variable is zero in the initial basic feasible solution, whereas in case (*ii*) at any iteration of the simplex method more than one vector is eligible to leave the basis, and hence the next simplex iteration produces a degenerate solution in which at least one basic variable is zero. This means that the subsequent iterations may not produce improvements in the value of the objective function. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solution. This concept is known as *Cycling*.

In general, degeneracy does not pose a problem, except only when cycling occurs. Mostly, whenever there is a tie in the replacement ratios, selecting a row arbitrarily is sufficient. However, the

number of iterations required to arrive at the optimum solution can be minimized by adopting the following rules:

(*i*) Divide the coefficients of basic variables (elements of the column vectors of Basis Matrix), in the simplex table where degeneracy is detected, by the corresponding positive elements of the entering coloumn vector from left to right.

(*ii*) The row which contains smallest ratio comparing from left to right column wise becomes the pivot row and the corresponding current basis vector leaves the basis.

SAMPLE PROBLEM

443. *Maximize* $z = 22x_1 + 30x_2 + 25x_3$ *subject to the constraints :*

$$2x_1 + 2x_2 \le 100,\ 2x_1 + x_2 + x_3 \le 100,\ x_1 + 2x_2 + 2x_3 \le 100;\ x_1, x_2, x_3 \ge 0.$$

[Delhi M.Com. 2008]

Solution. By introducing slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective inequalities, the set of constraints can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{bmatrix},\quad \mathbf{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = [x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \quad s_3].$$

An obvious initial (starting) basic feasible solution is $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, where $\mathbf{x}_B = [s_1 \quad s_2 \quad s_3]$, $\mathbf{B} = \mathbf{I}_3$ and $\mathbf{b} = [100 \quad 100 \quad 100]$.

$\therefore$ $\mathbf{x}_B = \mathbf{I}^{-1}\mathbf{b} = \mathbf{Ib}$ gives $[s_1 \quad s_2 \quad s_3] = [100 \quad 100 \quad 100]$

Using now simplex method, the iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_6$.

			22	30	25	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	100	2	2	0	1	0	0
0	$\mathbf{y}_5$	100	2	1	1	0	1	0
0	$\mathbf{y}_6$	100	1	2	2	0	0	1
	z	0	–22	–30	–25	0	0	0

Since, $z_2 - c_2$ is the most negative $z_j - c_j$, $\mathbf{y}_2$ enters the basis. Further, minimum $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\} = 50$ occurs for the elements y_{12} and y_{32}. Thus, there is a tie among the ratios in the first and third rows, *i.e.*, among the basis vectors $\mathbf{y}_4$ and $\mathbf{y}_6$. To obtain the unique current basis vector that will leave the basis, we compute the ratios $\left\{\frac{y_{ij}}{y_{i2}},\ y_{i2} > 0\right\}$ instead of $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\}$ for those column vectors which are in the basis. Here, since $\mathbf{y}_4$, $\mathbf{y}_5$ and $\mathbf{y}_6$ are in the basis and there is a tie among $\mathbf{y}_4$ and $\mathbf{y}_6$ for leaving the basis, we write the coefficients (elements) from the above table :

	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$\mathbf{y}_4$	1	0	0
$\mathbf{y}_6$	0	0	1

Dividing these coefficients by the corresponding element of the entering column, *i.e.*, of $\mathbf{y}_2$, we obtain the following ratios :

	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
$\mathbf{y}_4$	1/2	0/2	0/2
$\mathbf{y}_6$	0/2	0/2	1/2

On comparison of ratios in the first column, $\mathbf{y}_6$-row yields the smallest ratio and hence $\mathbf{y}_6$ leaves the basis.

First Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	0	1	0	−2	1	0	−1
0	$\mathbf{y}_5$	50	3/2	0	0	0	1	−1/2
30	$\mathbf{y}_2$	50	1/2	1	1	0	0	1/2
	z	1,500	−7	0	5	0	0	15

It is apparent from the above table that, $(z_1 - c_1) < 0$ and therefore $\mathbf{y}_1$ enters the basis. Further, since min. $\left\{ \frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0 \right\} = \frac{0}{1} = 0$, the current basis vector $\mathbf{y}_4$ leaves the basis and y_{11} becomes the leading element.

Second Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_5$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
22	$\mathbf{y}_1$	0	1	0	−2	1	0	−1
0	$\mathbf{y}_5$	50	0	0	3	−3/2	1	1
30	$\mathbf{y}_2$	50	0	1	2	−1/2	0	1
	z	1,500	0	0	−9	7	0	8

Clearly, the solution is still not optimum, since $(z_3 - c_3) < 0$. So, $\mathbf{y}_3$ enters the basis. Further, since min. $\left\{ \frac{x_{Bi}}{y_{i3}},\ y_{i3} > 0 \right\} = \frac{50}{3}$, the current basis vector $\mathbf{y}_5$ leaves the basis and y_{23} becomes the leading element.

Final Iteration. Optimum Solution.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
22	$\mathbf{y}_1$	100/3	1	0	0	0	2/3	−1/3
25	$\mathbf{y}_3$	50/3	0	0	1	−1/2	1/3	1/3
30	$\mathbf{y}_2$	50/3	0	1	0	1/2	−2/3	1/3
	z	1,650	0	0	0	5/2	3	11

Since, all $(z_j - c_j) \geq 0$, an optimum basic feasible solution is

$$x_1 = 100/3,\ x_2 = 50/3,\ x_3 = 50/3 \text{ and maximum } z = 1{,}650.$$

PROBLEMS

444. Solve the following L.P.P. :

Maximize $z = 2x_1 + 3x_2 + 10x_3$ subject to the constraints :

$$x_1 + 2x_2 = 0,\ x_2 + x_3 = 1;\ x_1, x_2, x_3 \geq 0.$$

445. Solve the L.P.P. :

Maximize $z = 3x_1 + 2x_2 + 3x_3$ subject to the constraints :

$$2x_1 + x_2 + x_3 \leq 2,\ 3x_1 + 4x_2 + 2x_3 \geq 8;\ x_1, x_2, x_3 \geq 0.$$ *[Madras M.C.A. (Oct.) 2002]*

446. Solve the L.P.P. :

Maximize $z = 5x_1 - 2x_2 + 3x_3$ subject to the constraints :

$$2x_1 + 2x_2 - x_3 \geq 2,\ 3x_1 - 4x_2 \leq 3,\ x_2 + 3x_3 \leq 5;\ x_1, x_2, x_3 \geq 0.$$

447. Solve the following L.P.P. :

Maximize $z = x_1 + 1.5x_2 + 2x_3 + 5x_4$ subject to the constraints :

$$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6,\ 2x_1 + x_2 + x_3 + 5x_4 \leq 4,\ 2x_1 + 6x_2 - 8x_3 + 4x_4 = 0,$$
$$x_1 + 3x_2 - 4x_3 + 3x_4 = 0;\ x_1, x_2, x_3, x_4 \geq 0.$$

4:6. SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Let us consider the following system of m simultaneous linear equations in n unknowns :

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is an $m \times n$ real matrix, $\mathbf{x}^T \in \mathbf{R}^n$ and $\mathbf{b}$ is an $m \times 1$ real matrix.

For the solution of the simultaneous equations we introduce a dummy objective function

$$z = 0 \cdot \mathbf{x} - 1 \cdot \mathbf{x}_a$$

where $\mathbf{x}_a$ are artificial variables, and $\mathbf{x} = \mathbf{x}_r' - \mathbf{x}_r''$, where $\mathbf{x}_r' \geq \mathbf{0}$, $\mathbf{x}_r'' \geq \mathbf{0}$.

The reformulated L.P.P. is then solved by simplex method. The optimal solution of this L.P.P. gives the values of the variables $\mathbf{x}$.

4:7. INVERTING A MATRIX USING SIMPLEX METHOD

Let $\mathbf{A}$ be any $n \times n$ real matrix. Let $\mathbf{x} \in \mathbf{R}^n$ and let $\mathbf{b}$ be any $n \times 1$ real matrix. Consider the system of equations

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}.$$

By introducing a *dummy* objective function

$$z = \mathbf{0} \cdot \mathbf{x} - \mathbf{1} \cdot \mathbf{x}_a$$

$\mathbf{x}_a \geq \mathbf{0}$ being an artificial variable, we seek a solution to the L.P.P. of maximizing z subject to the constraints :

$$\mathbf{Ax} + \mathbf{1x}_a = \mathbf{b}, \quad \mathbf{x}, \mathbf{x}_a \geq \mathbf{0}.$$

If we get the optimal solution to the given L.P.P. in which the basis contains all the variables of vector $\mathbf{x}$, then the inverse of $\mathbf{A}$ is directly read off from the optimum (final) simplex table. It consists of those column vectors in the last iteration of the simplex method which were present in the initial basis.

If the optimum (final) simplex table does not contain all the variables of vector $\mathbf{x}$ in the basis, we continue with the simplex procedure until all the variables of vector $\mathbf{x}$ are in the basis and at the same time the solution remains optimum, may by *feasible* or *infeasible*.

Note. The dummy vector $\mathbf{b}$ can be constructed easily by assigning an arbitrary value to the variables of vector $\mathbf{x}$.

SAMPLE PROBLEMS

448. *Solve the following system of simultaneous linear equations by using the simplex method :*

$$x_1 + x_2 = 1$$
$$2x_1 + x_2 = 3$$

[Meerut M.Sc. (Math.) 2000]

Solution. Since the range for x_1 and x_2 is not given, they are taken to be unrestricted in sign. So, let

$$x_1 = x_1' - x_1'' \text{ and } x_2 = x_2' - x_2''$$

where x_1', x_1'' ; x_2', x_2'' are all non-negative.

Introduce the artificial variables $A_1 \geq 0$ and $A_2 \geq 0$, each having a cost -1 associated with them. Thus, the problem of solving the given simultaneous equations can be considered to be the problem of maximizing

$$z = 0 \cdot x_1 + 0 \cdot x_2 - A_1 - A_2$$

The iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1'$ and drop $\mathbf{y}_3$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1'$	$\mathbf{y}_1''$	$\mathbf{y}_2'$	$\mathbf{y}_2''$	$\mathbf{y}_3$	$\mathbf{y}_4$
-1	$\mathbf{y}_3$	1	1	-1	1	-1	1	0
-1	$\mathbf{y}_4$	3	2	-2	1	-1	0	1
	z	-4	-3	3	-2	2	0	0

Clearly $\mathbf{y}_1'$ enters the basis (since $z_1 - c_1$ is most negative). Also $\mathbf{y}_3$ leaves the current basis as min. $\left\{ \frac{x_{Bi}}{y_{i1}} \cdot y_{i1} > 0 \right\} = 1$.

First Iteration. Introduce $\mathbf{y}_2''$ and drop $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1'$	$\mathbf{y}_1''$	$\mathbf{y}_2'$	$\mathbf{y}_2''$	$\mathbf{y}_4$
0	$\mathbf{y}_1'$	1	1	–1	1	–1	0
–1	$\mathbf{y}_4$	1	0	0	–1	1	1
	z	–1	0	0	1	–1	1

Here $z_4 - c_4 < 0$ and all other $z_j - c_j \geq 0$. So, $\mathbf{y}_2''$ enters the basis. Further, since $y_{24} > 0$ and $y_{14} < 0$; $\mathbf{y}_4$ leaves the current basis.

Final Iteration. Optimum Solution.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1'$	$\mathbf{y}_1''$	$\mathbf{y}_2'$	$\mathbf{y}_2''$
0	$\mathbf{y}_1'$	2	1	–1	1	0
0	$\mathbf{y}_2''$	1	0	0	–1	1
	z	0	0	0	0	0

Now, since all $z_j - c_j$ are non-negative an optimum solution has been attained. Thus, an optimum solution to the given system of equations is

$$x_1' = 2,\ x_2'' = 1;\ x_1'' = 0 = x_2'$$

Hence,

$$x_1 = 2,\ x_2 = -1.$$

449. *Use simplex method to obtain the inverse of the matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}.$$

[Meerut M.Sc. (Math.) 2001]

Solution. Let us consider the system of equations

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}; \quad x_1, x_2 \geq 0$$

where [4 6] is a dummy column vector. We use simplex method to maximize the objective function

$$z = 0 \cdot x_1 + 0 \cdot x_2 - 1 \cdot x_3 - 1 \cdot x_4$$

subject to the constraints on x_1 and x_2, where $x_3 \geq 0$ and $x_4 \geq 0$ are treated as artificial variables.

Clearly, an obvious basic feasible solution to the above L.P.P. is

$$\mathbf{x}_B^T = [4 \quad 6] \quad (x_3 \text{ and } x_4 \text{ basic})$$

Making use of simplex method, now, the iterative simplex tables are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_3$.

			0	0	–1	–1
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
–1	$\mathbf{y}_3$	4	3	2	1	0
–1	$\mathbf{y}_4$	6	4	–1	0	1
	z	–10	–7	–1	0	0

First Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_4$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
0	$\mathbf{y}_7$	4/3	1	2/3	1/3	0
–1	$\mathbf{y}_4$	2/3	0	–11/3	–4/3	1
	z	–2/3	0	11/3	7/3	0

Since all $z_j - c_j$ are non-negative, therefore an optimum solution has been attained. But **A** is still not converted into a unit matrix. In order to do this, we introduce $\mathbf{y}_2$ in the basis and drop $\mathbf{y}_4$ from the basis.

Final Iteration. Inverse achieved.

c_B	y_B	x_B	y_1	y_2	y_3	y_4
0	y_1	16/11	1	0	1/11	2/11
0	y_2	–2/11	0	1	4/11	–3/11
	z	0	0	0	1	1

Again, we see that we arrive at an optimum feasible solution. Now, since the initial basis consisted of the column vectors $\mathbf{y}_3$ and $\mathbf{y}_4$, therefore the inverse of matrix **A** is $\begin{pmatrix} 1/11 & 2/11 \\ 4/11 & -3/11 \end{pmatrix}$

PROBLEMS

450. Using Simplex method, solve the equations :

(*a*) $2x_1 + x_2 = 1$, $x_1 + 2x_2 = 5$ *[Delhi B.Sc. (Math.) 1995]*

(*b*) $8x_1 + 2x_2 = 15$, $5x_1 + 3x_2 = 19$ *[Delhi B.Sc. (Math.) 1996]*

451. Solve the following system of linear equations, using simplex method :

(*a*) $x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$

(*b*) $x_1 - x_3 + 4x_4 = 3$, $2x_1 - x_2 = 3$, $3x_1 - 2x_2 - x_4 = 1$, $x_1, x_2, x_3, x_4 \geq 0$. *[Meerut M.Com. 1997]*

452. Use simplex method to obtain the inverse of the following matrices :

(*a*) $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 9 \end{pmatrix}$

(*b*) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ *[Meerut M.Sc. (Math.) 1996]*

(*c*) $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ *[Meerut M.Sc. (Math.) 1997]*

4:8. APPLICATIONS OF SIMPLEX METHOD

SAMPLE PROBLEMS

453. *A manufacturer of leather belts makes three types of belts A, B and C which are processed on three machines M_1, M_2 and M_3. Belt A requires 2 hours on machine M_1 and 3 hours on machine M_3. Belt B requires 3 hours on machine M_1, 2 hours on machine M_2 and 2 hours on machine M_3 ; and Belt C requires 5 hours on machine M_2 and 4 hours on machine M_3. There are 8 hours of time per day available on machine M_1, 10 hours of time per day available on machine M_2 and 15 hours of time per day available on machine M_3. The profit gained from belt A is Rs. 3.00 per unit, from Belt B is Rs. 5.00 per unit, from belt C is Rs. 4.00 per unit. What should be the daily production of each type of belts so that the profit is maximum.* *[Meerut M.Sc. (Math.) 1998]*

Mathematical Formulation.

Let the manufacturer decide to produce x_1 units of type *A* belt, x_2 units of type *B* belt and x_3 units of type *C* belt. Then the given problem can be put in the following mathematical form :

Maximize $z = 3x_1 + 5x_2 + 4x_3$ subject to the constraints :

$$2x_1 + 3x_2 \le 8 \qquad \textit{(daily time on } M_1\textit{)}$$
$$2x_2 + 5x_3 \le 10 \qquad \textit{(daily time on } M_2\textit{)}$$
$$3x_1 + 2x_2 + 4x_3 \le 15 \qquad \textit{(daily time on } M_3\textit{)}$$

and
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$$

Solution by simplex method. Introducing the slack variables $s_1 \ge 0$, $s_2 \ge 0$, and $s_3 \ge 0$. the set of inequalities representing the constraints of the given LPP is converted into equations :

$$2x_1 + 3x_2 + 0 \cdot x_3 + s_1 = 8,$$
$$0 \cdot x_1 + 2x_2 + 5x_3 + s_2 = 10,$$
and
$$3x_1 + 2x_2 + 4x_3 + s_3 = 15.$$

An obvious initial (starting) basic feasible solution is $s_1 = 8$, $s_2 = 10$ and $s_3 = 15$, *i.e.*,

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \mathbf{b} \text{ or } [s_1\ \ s_2\ \ s_3] = [8\ \ 10\ \ 15].$$

The iterative simplex tables are as follows :

Initial Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_4$.

			3	5	4	0	0	0
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_4$	8	2	3	0	1	0	0
0	$\mathbf{y}_5$	10	0	2	5	0	1	0
0	$\mathbf{y}_6$	15	3	2	4	0	0	1
	z	0	–3	–5	–4	0	0	0

Since all $z_j - c_j$ are not non-negative, and $z_2 - c_2$ is most negative, $\mathbf{y}_2$ enters the basis. Further, since min. $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\} = \frac{8}{3}$, $\mathbf{y}_4$ leaves the current basis.

First Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_5$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
5	$\mathbf{y}_2$	8/3	2/3	1	0	1/3	0	0
0	$\mathbf{y}_5$	14/3	–4/3	0	5	–2/3	1	0
0	$\mathbf{y}_6$	29/3	5/3	0	4	–2/3	0	1
	z	40/3	1/3	0	–4	5/3	0	0

Since $z_3 - c_3$ is only negative, $\mathbf{y}_3$ enters the basis. Further, since min. $\left\{\frac{x_{Bi}}{y_{i3}},\ y_{i3} > 0\right\} = \frac{14/3}{5} = \frac{14}{15}$. $\mathbf{y}_5$ leaves the basis.

Second Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_6$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
5	$\mathbf{y}_2$	8/3	2/3	1	0	1/3	0	0
4	$\mathbf{y}_3$	14/15	–4/15	0	1	–2/15	1/5	0
0	$\mathbf{y}_6$	89/15	41/15	0	0	–2/15	–4/5	1
	z	256/15	–11/15	0	0	17/15	4/5	0

Clearly, $z_1 - c_1 < 0$ and all other $z_j - c_j \ge 0$. Therefore $\mathbf{y}_1$ enters the basis. Further min. $\left\{\frac{x_{Bi}}{y_{i1}},\ y_{i1} > 0\right\} = \frac{89}{41}$, $\mathbf{y}_6$ leaves the basis.

Final Iteration. Optimum solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
5	y_2	50/41	0	1	0	15/41	8/41	–10/41
4	y_3	62/41	0	0	1	–6/41	5/41	4/41
3	y_1	89/41	1	0	0	–2/41	–12/41	15/41
	z	765/41	0	0	0	45/41	24/41	11/41

Now, since all $z_j - c_j$ are non-negative, an optimum basic feasible solution is

$$x_1 = 89/41,\ x_2 = 50/41 \text{ and } x_3 = 62/41;\ \max. z = 765/41.$$

Hence, in order to achieve a maximum income of Rs. 765/41, the firm should produce 89/41 units of belt A, 50/41 units of belt B and 62/41 units of belt C.

454. *Air Force is experimenting with three types of bombs P, Q and R in which three kinds of explosive, viz., A, B and C will be used. Taking the various factors into consideration, it has been decided to use at most 600 kg. of explosive A, at least 480 kg. of explosive B and exactly 540 kg. of explosive C. Bomb P requires 3, 2, 2 kg. of A, B and C respectively. Bomb Q requires 1, 4, 3 kg. of A, B and C respectively. Bomb R requires 6, 2, 3 kg. of A, B and C respectively. Now bomb P will give the equivalent of a 2 tonne explosion, bomb Q will give a 3 tonne explosion and Bomb R will give a 4 tonne explosion. Under what production schedule can be Air Force make the biggest target? Solve by Two-Phase Simple Method.* **[Delhi Dip. in Glob. Bus. Oper. 2011]**

Mathematical Formulation

Let the Air Force experiment with x_1 units of bomb P, x_2 units of bomb Q and x_3 units of bomb R. Then, using the given information, an appropriate mathematical formulation of the given problem is :

Maximize $z = 2x_1 + 3x_2 + 4x_3$ subject to the constraints :

$$3x_1 + x_2 + 6x_3 \le 600 \quad \textit{(Explosive A constraint)}$$
$$2x_1 + 4x_2 + 2x_3 \ge 480 \quad \textit{(Explosive B constraint)}$$
$$2x_1 + 3x_2 + 3x_3 = 540 \quad \textit{(Explosive C constraint)}$$

and $x_1 \ge 0,\ x_2 \ge 0,$ and $x_3 \ge 0.$

Solution by Simplex Method. Introducing slack variable $s_1 \ge 0$, surplus variable $s_2 \ge 0$, artificial variables $A_1 \ge 0$ and $A_2 \ge 0$, standard form of the LPP is

Maximize $z = 2x_1 + 3x_2 + 4x_3 + 0.s_1 + 0.s_2 - MA_1 - MA_2$ subject to the constraints :

$$3x_1 + x_2 + 6x_3 + s_1 = 600$$
$$2x_1 + 4x_2 + 2x_3 - s_2 + A_1 = 480$$
$$2x_1 + 3x_2 + 3x_3 + A_2 = 540$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \ge 0.$$

An obvious initial (starting) basic feasible solution is $s_1 = 600$, $A_1 = 480$ and $A_2 = 540$.

Phase I. The objective function here is maximize $z^* = -A_1 - A_2$

The iterative simplex tables for the auxiliary L.P.P. are :

Initial Iteration. Introduce y_2 and drop y_6.

			0	0	0	0	0	–1	–1
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_4	600	3	1	6	1	0	0	0
–1	y_6	480	2	4	2	0	–1	1	0
–1	y_7	540	2	3	3	0	0	0	1
		–1,020	–4	–7	–5	0	1	0	0

Above table shows that $\mathbf{y}_2$ enters the basis, since $z_2 - c_2$ is most negative and $\mathbf{y}_6$ leaves the basis, since min. $\left\{ \frac{x_{Bi}}{y_{i2}} \cdot y_{i2} > 0 \right\} = \frac{480}{4} = 120$. Leading element is y_{22} (= 4).

First Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_7$
0	$\mathbf{y}_4$	480	5/2	0	11/2	1	1/4	0
0	$\mathbf{y}_2$	120	1/2	1	1/2	0	−1/4	0
−1	$\mathbf{y}_7$	180	1/2	0	3/2	0	3/4	1
	z	−180	−1/2	0	−3/2	0	−3/4	0

$\mathbf{y}_3$ enters the basis, since $z_3 - c_3$ is most negative. Further, since min. $\left\{ \frac{x_{Bi}}{y_{i3}} \cdot y_{i3} > 0 \right\} = \frac{480}{11/2}$ = 87.27 which corresponds to first row, $\mathbf{y}_4$ leaves the basis. Leading element is $y_{13} \left(= \frac{11}{2} \right)$.

Second Iteration. Introduce $\mathbf{y}_5$ and drop $\mathbf{y}_7$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_7$
0	$\mathbf{y}_3$	87.27	5/11	0	1	2/11	1/22	0
0	$\mathbf{y}_2$	76.36	3/11	1	0	−1/11	−3/11	0
−1	$\mathbf{y}_7$	49.0	−2/11	0	0	−3/11	15/22	1
	z	−49	2/11	0	0	3/11	−15/22	0

Since $z_5 - c_5$ is the only negative, $\mathbf{y}_5$ enters the basis. Further, since min. $\left\{ \frac{x_{Bi}}{y_{ir}} \cdot y_{ir} > 0 \right\} = \frac{49}{15/22}$ = 71.86, $\mathbf{y}_7$ leaves the current basis and y_{35} becomes the leading element.

Final Iteration. Optimum Solution.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	84	7/15	0	1	1/5	0
0	$\mathbf{y}_2$	56.76	1/5	1	0	−1/5	0
0	$\mathbf{y}_5$	71.86	−4/15	0	0	−2/5	1
	z	0	0	0	0	0	0

Since all $z_j - c_j \geq 0$, an optimum solution to the auxiliary L.P.P. has been reached. It is apparent from the table that no artificial variable appears in the basis.

Phase 2. Now, we restore the actual coefficients of the original variables in the objective function. So, the objective function is

$$\text{Maximize } z = 2x_1 + 3x_2 + 4x_3 + 0.s_1 + 0.s_2$$

The starting simplex table for this phase is :

			2	3	4	0	0
c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
4	$\mathbf{y}_3$	84	7/15	0	1	1/5	0
3	$\mathbf{y}_2$	56.76	1/5	1	0	−1/5	0
0	$\mathbf{y}_5$	71.86	−4/15	0	0	−2/5	1
		506.28	7/15	0	0	1/5	0

Since all $(z_j - c_j) \geq 0$, an optimum solution has been reached. Hence, an optimum basic feasible solution is : $x_1 = 0$, $x_2 = 57.76$, $x_3 = 84$ and maximum $z = 506.28$.

Therefore, Air Force should use 56.76 (or approximately 57) units of Bomb Q and 84 units of Bomb R only in order to achieve the maximum target of 506 targets.

455. *An advertising agency wishes to reach two types of audiences: Customers with annual income greater than Rs. 60,000 (Category A) and customers with annual income less than Rs. 60,000 (category B). The total advertising budget is Rs. 2,00,000. One programme of TV advertising costs Rs. 50,000; one programme of radio advertising costs Rs. 20,000. For contract reasons, at least 3 programmes ought to be of TV, and the number of radio programmes must be limited to five. Surveys indicate that a single TV programme reaches 4,50,000 customers in category A and 50,000 customers in category B. One radio programme reaches 20,000 customers of category A and 80,000 customers of category B. Determine the media mix to maximize the total reach. Is there any alternate optimal solution to this problem (Use Simplex Method).* **[Delhi MIB 2011]**

Mathematical Formulation

Let the number of insertions in TV and radio be x_1 and x_2, respectively. Then, using the given information, an appropriate mathematical formulation of the given problem is :

$$\text{Maximize } z = (4{,}50{,}000 + 50{,}000)x_1 + (20{,}000 + 80{,}000)x_2 = 5{,}00{,}000x_1 + 1{,}00{,}000x_2$$

subject to the constraints :

$$50{,}000x_1 + 20{,}000x_2 \le 2{,}00{,}000 \qquad \textit{(Constraint on cost)}$$

$$x_1 \ge 3 \text{ and } x_2 \le 5 \qquad \textit{(Constraint on programmes)}$$

and

$$x_1 \ge 0,\ x_2 \ge 0. \qquad \textit{(Non-negative constraint)}$$

Solution by Simplex Method. Introducing slack variables $s_1 \ge 0$, $s_3 \ge 0$, surplus variable $s_2 \ge 0$ and an artificial variable $A_1 \ge 0$, standard form of the LPP is:

$$\text{Maximize } z = 5x_1 + x_2 \qquad \textit{(Considering 1,00,000 as one unit)}$$

subject to the constraints :

$$5x_1 + 2x_2 + s_1 = 20$$

$$x_1 - s_2 + A_1 = 3$$

$$x_2 + s_3 = 5$$

$$x_1, x_2, s_1, s_2, s_3, A_1 \ge 0.$$

The iterative simplex tables for the LPP are :

Initial Iteration. Introduce $\mathbf{y}_1$ and drop $\mathbf{y}_6$.

			5	1	0	0	0	$-M$
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
0	$\mathbf{y}_3$	20	5	2	1	0	0	0
$-M$	$\mathbf{y}_6$	3	1	0	0	-1	0	1
0	$\mathbf{y}_5$	5	0	1	0	0	1	0
	z	$-3M$	$-M-5$	-1	0	M	0	0

Since $(z_1 - c_1)$ is most negative, $\mathbf{y}_1$ enters the basis. Further, since min. $\left\{\dfrac{x_{Bi}}{y_{i1}},\ y_{i1} > 0\right\} = 3$, $\mathbf{y}_6$ leaves the current basis.

First Iteration. Introduce $\mathbf{y}_4$ and drop $\mathbf{y}_3$.

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
0	$\mathbf{y}_3$	5	0	2	1	5	0
5	$\mathbf{y}_1$	3	1	0	0	-1	0
0	$\mathbf{y}_5$	5	0	1	0	0	1
	z	15	0	-1	0	-5	0

Since $(z_4 - c_4)$ is most negative, $\mathbf{y}_4$ enters the basis. Further since y_{13} (= 5) is the only positive element, y_{14} (= 5) becomes the leading element and hence $\mathbf{y}_3$ leaves the basis.

Final Iteration. Optimum solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_4	1	0	2/5	1/5	1	0
5	y_1	4	1	2/5	1/5	0	0
0	y_5	5	0	1	0	0	1
	z	20	0	1	1	0	0

Since all $(z_j - c_j) \geq 0$, an optimum solution is reached. Thus, $x_1 = 4$ and $x_2 = 0$. Hence, the total number of insertions in TV should be 4, while there should not be any insertion in radio. Maximum audiance will be 20,00,000.

PROBLEMS

456. A diet for a sick person must contain at least 4,000 units of vitamins, 50 units of minerals and 1,400 calories. Two foods *A* and *B* are available at a cost of Rs. 4 and Rs. 3 per unit respectively. If one unit of *A* contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of food *B* contains 100 units of vitamins, 2 units of minerals and 40 calories. What combination of foods be used to have least cost?

[Calicut M.Com. 2009]

457. A firm plans to purchase at least 200 quintals of scrap containing high quality metal *X* and low quality metal *Y*. It decides that the scrap to be purchased must contain at least 100 quintals of *X metal and no more than 35 quintals of Y* metal. The firm can purchase the scrap from two suppliers (*A* and *B*) in unlimited quantities. The percentage of *X* and *Y* metals in terms of weight of the scraps supplied by *A* and *B* is given below :

Metals	*Supplier A*	*Supplier B*
X	25%	75%
Y	10%	20%

The price of *A*'s scrap is Rs. 200 per quintal and that of *B*'s Rs. 400 per quintal. The firm wants to determine the quantities that it should buy from the two suppliers so that total cost is minimized. Formulate the problem as LP model and solve by simplex method. ***[Lucknow M.B.A. (April) 2008]***

458. A firm wants to reach two types of customers : Households having Rs. 5 lakh or more of family income and those having less than 5 lakh. The first group purchases twice as much as the second group. One unit of TV advertisement reaches 2,000 families of first group and 8,000 families of second group and costs Rs. 20,000 while advertisements in a magazine costs Rs. 12,000 and reaches 6,000 and 3,000 families respectively. No more than 12 advertisements can be given in magazine and at least 6 TV advertisements have to be given. The advertisement budget is Rs. 1,80,000. Assume that every contracted family purchases the company product. The firm wants to maximize its sales.

Write the above problem as LP model and solve by simplex method. ***[Delhi M.Com. 2006]***

459. Product *A* yields a profit of Rs. 25 per unit and Product *B* yields a profit of Rs. 40 per unit. To manufacture the products—leather, wood and glue are required the amount of which is shown below :

RESOURCES REQUIRED FOR ONE UNIT

Product	*Leather (in kg.)*	*Wood (in sq. metres)*	*Glue (in litres)*
A	0.50	4	0.2
B	0.25	7	0.2

Available resources include 2,200 kg. of leather; 28,000 square metres of wood and 1,400 litres of glue :

(*i*) State the objective function and constraints in mathematical form.

(*ii*) Find the optimum solution.

(*iii*) Which of the resources are fully consumed? How much of each resource remains unused?

(*iv*) What are the shadow prices of resources?

[C.S. Final (June) 1997]

460. Ashok Chemical Company manufactures two chemicals *A* and *B* which are sold to the manufacturers of soaps and detergents. On the basis of the next month's demand, the management has decided that the total production for chemicals *A* and *B* should be at least 350 kilograms. Moreover, a major customer's order for 125 kg. of product *A* must also be supplied. Product *A* requires 2 hours of processing time per kg. and product *B* requires one hour of processing time per kg. For the coming month, 600 hours of processing time are available.

The company wants to meet the above requirements at minimum total production cost. The production costs are Rs. 2 per kg. for product *A* and Rs. 3 per kg. for product *B*.

Ashok Chemicals Company wants to determine its optimal product mix and the total minimum cost relevant there to.

(*i*) Formulate the above as LPP.

(*ii*) Solve the problem with the simplex method.

(*iii*) Does the problem have multiple optimal solution? Why? ***[Delhi M.Com. 1998]***

461. A cabinet manufacturer produces wood cabinets for TV sets, stereosystems and radios, each of which must be assembled, decorated and crated. Each TV cabinet requires 3 hours to assemble, 5 hours to decorate and 1/10 hour to crate and returns a profit of Rs. 10. Each stereo cabinet requires 10 hours to assemble, 8 hours to decorate and 3/5 hour to crate and returns a profit Rs. 25. Each radio cabinet requires 1 hour to assemble, 1 hour to decorate and 1/10 hour to crate and returns a profit of Rs. 3. The manufacturer has the maximum of 30,000, 40,000 and 120 hours available for assembling, decorating and crating respectively.

(*i*) Formulate the above problem as a Linear Programming Problem.

(*ii*) Use simplex method to find how many units of each product should be manufactured to achieve maximum profit.

(*iii*) Does the problem have multiple solutions? ***[Delhi M.B.A. (Nov.) 1998]***

462. A firm produces three products *A*, *B* and *C*, each of which passes through three departments : Fabrication, Finishing and Packaging. Each unit of product *A* requires 3, 4 and 2 hours, a unit of product *B* requires 5, 4 and 4 hours while each unit of product *C* requires 2, 4 and 5 hours respectively in the three departments. Every day, 60 hours are available in the fabrication department, 72 hours in the finishing department and 100 hours in the packaging department.

If the unit contribution of product *A* is Rs. 5, of product *B* is Rs. 10, and of product *C* is Rs. 8; determine the number of units of each of the products, that should be made each day to maximize the total contribution. Also determine, if any, the capacity that would remain unutilized.

463. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat costs Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans costs Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8,000 man-days of work, how many acres should be allocated to each crop to maximize profits ? ***[Jammu M.B.A. 2006]***

464. A company manufactures three products *A*, *B* and *C*. All the three products require to be processed on two machines *D* and *E*. Product *A* requires 1 hour on machine *D* and 3 hours on machine *E*. Product *B* requires 1 hour on machine *D* and 4 hours on machine *E*. Product *C* requires 1 hour on machine *D* and 4 hours on machine *E*. The total available time on machine *D* and *E* are 20 hours and 80 hours respectively. The profit per unit of products *A*, *B* and *C* are Rs. 2, Rs. 4 and Rs. 6 respectively. Find out the optimum product mix.

465. A Pharmaceutical Company has 100 kg. of *A*, 180 kg. of *B* and 120 kg. of *C* available per month. They can use these materials to make three basic pharmaceutical products namely *5-10-5*, *5-5-10* and *20-5-10* where the numbers in each case represent the percentage by weight of *A*, *B* and *C* respectively in each of the products. The costs of these raw materials are given below :

Ingredient	*Cost per kg. (Rs.)*
A	80
B	20
C	50
Inert ingredients	20

Selling prices of these products are Rs. 40.5, Rs. 43 and Rs. 45 per kg. respectively. There is a capacity restriction of the company for the product *5-10-5*, so as they cannot produce more than 30 kg. per month. Determine how much of each product should be produced to maximize profit. ***[Delhi M.B.A. 2008]***

466. A firm assembles and sells two different types of out-board motors, *A* and *B*, using four resources. The production process can be described as follows :

Resources	*Capacity per month*
Motor unit shop resource	400 Type *A* units or 250 Type *B* units or any linear combination of the two
Type *A* gear and drive shop resource	175 Type *A* units
Type *B* gear and drive shop resource	225 Type *B* units
Final assembly resource	200 Type *A* units or 350 Type *B* units or any linear combination of the two

Per unit profit of type A and B are Rs. 90 and Rs. 60 each respectively. Formulate the above as a linear programming problem to maximize profit and solve the same by simplex method. Also, write its dual and determine the shadow price of the resources. *[Delhi B.Com. (Hons.) 2009]*

467. A transistor radio company manufacturers four models A, B, C and D which have profit contributions of Rs. 8, Rs. 15 and Rs. 25 on models A, B and C respectively and a loss of Re. 1 on model D. Each type of radio requires a certain amount of time for the manufacturing of components for assembling and for packing. Specially a dozen units of model A require one hour of manufacturing, two hours for assembling and one hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2 and for a dozen units of C are 3, 5 and 1, while a dozen units of model D require 1 hour of packing only. During the forthcoming week, the company will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Obtain the optimal production schedule for the company.

468. A metal products company produces waste cans, filing cabinets, file boxes for correspondence, and lunch boxes. Its inputs are sheet metal of two different thickness, called A and B, and manual labour. Input-output relationships for the company are shown in the table given below :

	Waste cans	*Filing cabinets*	*Correspondence boxes*	*Lunch boxes*
Sheet metal—A	6	0	2	3
Sheet metal—B	0	10	0	0
Manual labour	4	8	2	3

The sales revenue per unit of waste cans, filing cabinet, correspondence boxes, and lunch boxes are Rs. 20, Rs. 400, Rs. 90, and Rs. 20 respectively. There are 225 units of sheet metal A available in the company's inventory, 300 of sheet metal B, and 190 units of manual labour. What is the company optimal sales revenue ? *[Delhi M.B.A. 2000]*

469. A factory manufactures three products which are processed through three different production stages. The time required to manufacture one unit of each of the three products and the daily capacity of the stages are given in the following table :

Stage	*Time per unit (in minutes)*			*Stage capacity (in minutes)*
	Product 1	*Product 2*	*Product 3*	
1	1	2	1	430
2	3	—	2	460
3	1	4	—	420
Profit per unit	Rs. 3	Rs. 2	Rs. 5	

(*i*) Set the data in simplex table.

(*ii*) Find the table for optimum solution.

(*iii*) State from the table—maximum profit, production pattern and surplus capacity at any stage.

(*iv*) What is the meaning of shadow price? Where is it shown in this table? Explain it in respect of resource of stages having shadow price.

(*v*) How many units of other resources will be required so as to completely utilise the surplus resource? *[Osmania M.B.A. 1997]*

470. A manufacturer produces three products daily X, Y and Z. The three products are each processed through three production operations with time constraints and then stored. The problem has been formulated as :

Maximize $z = 32x_1 + 35x_2 + 45x_3$ subject to the constraints : (*Profit in Rs.*)

$$2x_1 + 3x_2 + 2x_3 \le 120 \quad \text{(Operation 1, hour)}$$

$$4x_1 + 3x_2 + x_3 \le 160 \quad \text{(Operation 2, hour)}$$

$$3x_1 + 2x_2 + 4x_3 \le 100 \quad \text{(Operation 3, hour)}$$

$$x_1 + x_2 + x_3 \le 40 \quad \text{(Storage, } ft^2\text{)}$$

$$x_1, x_2, x_3 \ge 0.$$

The final optimal simplex tableau for this problem is as follows :

C_B	Basic variables (x_B)	Solution values	x_1	x_2	x_3	s_1	s_2	s_3	s_4
0	s_1	10	−1/2	0	0	1	0	1/2	−4
0	s_2	60	2	0	0	0	1	1	−5
45	x_3	10	1/2	0	1	0	0	1/2	−1
35	x_2	30	1/2	1	0	0	0	−1/2	2
	$(z_j - c_j)$	1500	8	0	0	0	0	5	25

On the basis of the above information, answer the following questions :

(*i*) Is the above solution feasible ?

(*ii*) Is the above solution optional ? If yes, what it is ?

(*iii*) Is the above solution unbounded ?

(*iv*) Is the above solution degenerate ?

(*v*) Does it have multiple solutions ?

(*vi*) Why does the company find product X not profitable for the production ?

(*vii*) If at all it is decided to produce product X, then by what minimum amount its price should be increased ?

[Delhi Dip. in Glob. Bus. Oper. 2011]

[**Hint:** (*i*) Yes, because each of the basic variables is non-negative.

(*ii*) Yes; $x_1 = 0$, $x_2 = 30$ and $x_3 = 10$; maximum $z = 1500$.

(*iii*) No, because there exists an optimal solution.

(*iv*) No, since all the basic variables are non-negative.

(*v*) No, because corresponding to non-basic variables none of $(z_j - c_j)$ is zero.

(*vi*) Because its marginal contribution is (–8.00) rupees per unit.

(*vii*) In order to produce product X, price of the product should be increased by Rs. 8.00.]

MULTIPLE CHOICE QUESTIONS

1. Given a system of m simultaneous linear equations in n unknowns ($m < n$), the number of basic variables will be

(*a*) m, (*b*) n,

(*c*) $n - m$, (*d*) $n + m$.

2. Which of the following is not correct ?

(*a*) The general LPP with m constraints in n unknowns ($m < n$) gives $\binom{n}{m}$ basic solutions.

(*b*) A basic solution that satisfies all the constraints and non- negative restrictions is called a basic feasible solution.

(*c*) A basic solution in which at least one basic variable vanishes is zero is called a non-degenerate basic solution.

(*d*) The basic feasible solution which maximizes or minimizes the objective function is called an optimum basic feasible solution.

3. Which of the following is not correct ?

(*a*) Simplex method is an iterative process which involves the substitution of variables for obtaining successively better solutions.

(*b*) Solution by Simplex method requires that an LPP should have at least one non-negative values in the right hand side of the constraints.

(*c*) For solving an LPP by Simplex method, it is essential that all variables involved are non-negative.

(*d*) Simplex method requires that all the constraints should be first converted into equations.

4. For maximization linear programming problem, the simplex method is terminated when all the net-evaluations are

(*a*) negative, (*b*) non-negative,

(*c*) zero, (*d*) non-positive.

5. A necessary and sufficient condition for a basic feasible solution to a minimization LPP to be an optimum is that (for all j) :
   (*a*) $z_j - c_j \geq 0$. (*b*) $z_j - c_j \leq 0$,
   (*c*) $z_j - c_j = 0$, (*d*) $z_j - c_j > 0$ or $z_j - c_j < 0$.
6. At any iteration of the usual Simplex method, if there is at least one basic variable in the basis at zero level and all $(z_j - c_j) \geq 0$, the current solution is
   (*a*) Infeasible, (*b*) Unbounded,
   (*c*) Non-degenerate, (*d*) Degenerate.
7. The role of artificial variables in simplex method is
   (*a*) to aid in finding initial basic feasible solution.
   (*b*) to start phases of simplex method.
   (*c*) to find shadow prices from the final simplex table.
   (*d*) none of the above.
8. For maximization LPP, the objective function coefficient for an artificial variable is
   (*a*) $+M$, (*b*) $-M$,
   (*c*) $+1$, (*d*) zero.
9. In a maximization LPP, if at least one artificial variable is in the basis, but not at zero level and the coefficient of M in each of the net evaluation $(z_j - c_j)$ is non-negative, then we have
   (*a*) a feasible solution, (*b*) no feasible solution
   (*c*) an unbounded solution, (*d*) an optimum solution.
10. In final (optimum) simplex table, if $z_j - c_j = 0$ for at least one non-basic variable, then there will be
   (*a*) an unbounded solution, (*b*) infeasible solution,
   (*c*) alternative solution, (*d*) cycling.
11. If a negative value appears in the solution values (x_B) coloumn of the simplex method, then
   (*a*) the basic solution is optimum, (*b*) the basic solution is infeasible,
   (*c*) the basic solution is unbounded, (*d*) all of the above.
12. If an optimum solution is degenerate, then
   (*a*) the solution is infeasible. (*b*) there are alternative optimum solutions.
   (*c*) the solution is of no use to the decision maker. (*d*) none of the above.
13. Which of the following is not correct ?
   (*a*) Degeneracy in an LPP may arise at the initial stage.
   (*b*) A degenerate solution can never be optimum.
   (*c*) Degeneracy may be a temporary phenomenon.
   (*d*) The Big-M method and the Two- Phase method do not require the same number of iterations for solving an LPP.
14. When we solve a system of simultaneous linear equations by using *Two-Phase Simplex Method*, the values of decision variables may be
   (*a*) positive, (*b*) negative,
   (*c*) zero, (*d*) positive and/or negative.
15. Which of the following is not correct ?
   (*a*) An LPP may have no optimum solution, a unique and optimum solution, multiple optimum solution, or an unbounded solution.
   (*b*) In the optimum solution to an LPP if a basic variable is zero, then multiple optimum solutions are indicated.
   (*c*) Multiple optimum solutions to an LPP would all have the same objective function values and would consume the same amount of the resources.
   (*d*) The inverse of a square matrix can be obtained by using *Simplex method*.
16. In the context of LPP, which of the following is not correct ?
   (*a*) When the constraints are of '$\geq$' type, surplus variables are introduced to convert them into equations.
   (*b*) Artificial variables are no tangible relationship with the decision variables.

(*c*) Surplus variables cannot appear in the basis of the optimum solution to an LPP.

(*d*) An artificial variable can be dropped for further calculations once it becomes non-basic or gets removed at any stage.

ANSWERS

1. (*a*)	**2.** (*c*)	**3.** (*b*)	**4.** (*b*)	**5.** (*b*)
6. (*d*)	**7.** (*a*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)
11. (*b*)	**12.** (*c*)	**13.** (*b*)	**14.** (*d*)	**15.** (*b*)
16. (*c*).				

REVIEW QUESTIONS

1. Define the following terms :

(*i*) Basic variable, (*ii*) Basic solution, (*iii*) Basic feasible solution, and (*iv*) Degenerate basic solution. *[Madras B.E.(Mech.) 1999]*

2. Establish the difference between (*i*) Feasible solution, (*ii*) Basic feasible solution, (*iii*) Degenerate basic feasible solution, and (*iv*) Optimum basic feasible solution. *[Delhi M.Com. 1998]*

3. Differentiate between basic and non-basic variables. What is an artificial variable and why should it be introduced into a solution. *[Panjab Tech. Univ. M.B.A. (Dec.) 2011]*

4. Prove that the set of all feasible solutions of a linear programming problem forms a convex set. *[Delhi B.Sc. (Stat.) 2000]*

5. Show that if there is a feasible solution of the system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ where $\mathbf{A}$ is an $m \times n$ matrix and $m < n$, then there is also a basic feasible solution.

6. Prove that every basic feasible solution of an LPP is an extreme point of the convex set of feasible solutions. *[Delhi B.Sc. (Stat.) 2000]*

7. Show that a linear objective function takes its minimum at an extreme point of the convex set generated by the set of feasible solutions to the linear programming problem. If it assumes its minimum at more than one extreme point, then it takes on the same value for every convex combination of these particular points. *[Delhi B.Sc. (Stat.) 1999]*

8. Explain the phenomenon of 'infeasibility' in an LPP. What are the indications of such a phenomenon? How can it be handled? *[Delhi M.Com. 2000]*

9. What are the two conditions to be satisfied to perform optimality test? *[Annamalai M.B.A. 2003]*

10. Explain the simplex procedure to solve a linear programming problem. *[Meerut M.Sc. (Math.) 2000]*

11. How do you recognise optimality in the simplex method?

12. Write the role of pivot element in simplex table.

13. In solving a linear programming problem by simplex method, explain how you will move from a given basic feasible solution to another basic feasible with an improved value of the objective function. *[Delhi M.Com. 1997]*

14. What is the criterion for testing optimality of solution to a linear programming problem in case of maximization as well as minimization? *[Delhi MIB 2011]*

15. Show that if there is an optimal basic feasible solution to a linear programming problem and, for some $\mathbf{a}_j$ not in the basis, $z_j - c_j = 0$, $y_{ij} > 0$ for all i, then a non-basic alternative optima will exist.

16. While solving a linear programming problem, Maximise $z = \mathbf{cx}$ *s.t.* $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$; what indicates the following ?

(*i*) Unique optimal solution; (*ii*) Presence of redundant constraints; (*iii*) No feasible solution. *[Delhi B.Sc. (Math.) 1995]*

17. How does the simplex algorithm indicate that (*i*) the problem has unbounded optimal solution, (*ii*) the problem has no feasible solution, (*iii*) the problem has alternative optimum solution? *[Delhi PG Dip. in Glob. Bus. Oper. 2011]*

18. How do you identify multiple solution in a linear programming problem? *[Madras B.E. 1999]*

19. How do you conclude that a LPP has an infeasible solution while using simplex method. *[Osmania M.B.A. 2000]*

20. Write the difference in the simplex solution procedure for a maximization problem and a minimization problem of linear programming. *[Delhi M.Com. 1998]*
21. Explain the use of artificial variables in L.P. *[Madras B.E. (Mech.) 1999]*
22. What do you mean by the two-phase method for solving a given L.P.P. ? Why is it used?
23. Explain the penalty method or 'Big-M' method for solving a linear programming problem.
24. How will you identify the existence of the following in the optimum simplex table and what are the effects of these cases on the final answer?

    (*i*) Alternative optimum solution, (*ii*) Unbounded solution, (*iii*) Degeneracy, (*iv*) Infeasible solution.

    [Delhi PG Dip. in Glob. Bus. Oper. 2010]