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Games and Strategies

"When individuals and groups do not exercise self-restraint, the constitution should have to tell them when to stop"

17:1. INTRODUCTION

Many practical problems require decision-making in a *competitive situation* where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent. For example, candidates for an election, advertising and marketing campaigns by competing business firms, countries involved in military battles, etc. have their conflicting interests. In a competitive situation the courses of action (alternatives) for each competitor may be either finite or infinite. A competitive situation will be called a 'Game', if it has the following properties :

(*i*) There are a finite number of competitors (participants) called *players*.

(*ii*) Each player has a finite number of strategies (alternatives) available to him.

(*iii*) A play of the game takes place when each player employs his strategy.

(*iv*) Every game results in an outcome, *e.g.*, loss or gain or a draw, usually called *payoff*, to some player.

17:2. TWO-PERSON ZERO-SUM GAMES

When there are two competitors playing a game, it is called a 'two-person game'. In case the number of competitors exceeds two, say n, then the game is termed as a 'n-person game'.

Games having the 'zero-sum' character that the algebraic sum of gains and losses of all the players is zero are called *zero-sum games*. The play does not add a single paisa to the total wealth of all the players; it merely results in a new distribution of initial money among them. Zero-sum games with *two* players are called *two-person zero-sum* games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non-zero sum character or simply a *non-zero sum game*.

17:3. SOME BASIC TERMS

1. *Player.* The competitors in the game are known as players. A player may be individual or group of individuals, or an organisation.

2. *Strategy.* A strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt. A strategy may be of two types :

(*a*) *Pure strategy.* If the players select the same strategy each time, then it is referred to as pure-strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

(*b*) *Mixed strategy.* When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize expected gains or to minimize expected losses.

3. *Optimum strategy.* A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

4. *Value of the game.* It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair, if it is non-zero.

5. *Payoff matrix.* When the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called the *payoff matrix.* Since the game is zero-sum, therefore gain of one player is equal to the loss of other and vice-versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff only for one of the players.

Let player A have m strategies $A_1, A_2, \ldots, A_m$ and player B have n strategies $B_1, B_2, \ldots, B_n$. Here, it is assumed that each player has his choices from amongst the pure strategies. Also it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to player A is :

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

The payoff matrix to player B is $(-a_{ij})$.

Example. Consider a two-person coin tossing game. Each player tosses an unbiased coin simultaneously. Player B pays Rs. 7 to A, if $\{H, H\}$ occurs and Rs. 4, if $\{T, T\}$ occurs; otherwise player A pays Rs. 3 to B. This two-person game is a zero-sum game, since the winnings of one player are the losses for the other. Each player has his choices from amongst two *pure* strategies H and T. If we agree conventionally to express the outcome of the game in terms of the payoffs to one player only, say A, then the above information yields the following payoff matrix in terms of the payoffs to the player A. Clearly, the entries in B's payoff matrix will be just the negative of the corresponding entries in A's payoff matrix so that the sum of payoff matrices for player A and player B is ultimately a null matrix. We generally display the payoff matrix of that player who is indicated on the left side of the matrix. For example, A's payoff matrix may be displayed as below :

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} H & T \end{array} \\ \text{Player } A \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 7 & -3 \\ -3 & 4 \end{pmatrix} \end{array}$$

17:4. THE MAXIMIN-MINIMAX PRINCIPLE

We shall now explain the so-called *Maximin-Minimax Principle* for the selection of the optimal strategies by the two players.

For player A, minimum value in each row represents the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the *maximin principle*, and the corresponding gain is called the *maximin value of the game*.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player B is called the *minimax principle*, and the corresponding loss is the *minimax value of the game*.

If the maximin value equals the minimax value, then the game is said to have a *saddle (equilibrium) point* and the corresponding strategies are called *optimum strategies*. The amount of payoff at an equilibrium point is known as the *value of the game*.

To illustrate the maximin-minimax principle, let us consider a two-person zero-sum game with the following 3×2 payoff matrix for player A :

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 9 & 2 \\ 8 & \mathbf{6}^{*\dagger} \\ 6 & 4 \end{pmatrix} \end{array}$$

Let the pure strategies of the two players be designated by

$$S_A = \{A_1, A_2, A_3\} \quad \text{and} \quad S_B = \{B_1, B_2\}.$$

Suppose that player A starts the game knowing fully well that whatever strategy he adopts, B will select that particular counter strategy which will minimize the payoff to A. Thus, if A selects the strategy A_1, then B will reply by selecting B_2, as this corresponds to the *minimum* payoff to A in the first row corresponding to A_1. Similarly, if A chooses the strategy A_2, he may gain 8 or 6 depending upon the strategy chosen by B. However, A can guarantee a gain of at least min. $\{8, 6\} = 6$ regardless of the strategy chosen by B. In other words, whatever strategy A may adopt he can guarantee only the minimum of the corresponding row payoffs. Naturally, A would like to maximise his minimum assured gain. In this example the selection of strategy A_2 gives the maximum of the minimum gains to A. We shall call this gain as the *maximin value* of the game and the corresponding strategy as the *maximin strategy*. The maximin value is indicated in bold type with a star.

On the other hand, player B wishes to minimize his losses. If he plays strategy B_1, his loss is at the most max. $\{9, 8, 6\} = 9$ regardless of what strategy A has selected. He can lose no more than max. $\{2, 6, 4\} = 6$ if he plays B_2. This minimum of the maximum losses will be called the *minimax value* of the game and the corresponding strategy the *minimax strategy*. The minimax value is indicated in bold type marked with [†]. We observe that in the present example the maximum of row minima is equal to the minimum of the column maxima. In symbols,

$$\max_i \{r_i\} = 6 = \min_j \{c_j\}$$

or

$$\max_i [\min_j \{a_{ij}\}] = 6 = \min_j [\max_i \{a_{ij}\}],$$

where $i = 1, 2, 3$ and $j = 1, 2$.

Theorem 17-1. *Let (a_{ij}) be the $m \times n$ payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then $\overline{v} \geq \underline{v}$. That is,*

$$\min_{1 \leq j \leq n} [\max_{1 \leq i \leq m} \{a_{ij}\}] \geq \max_{1 \leq i \leq m} [\min_{1 \leq j \leq n} \{a_{ij}\}].$$

Proof. We have

$$\max_{1 \le i \le m} \{a_{ij}\} \ge a_{ij} \qquad \text{for all } j = 1, 2, \ldots, n$$

and

$$\min_{1 \le j \le n} \{a_{ij}\} \le a_{ij} \qquad \text{for all } i = 1, 2, \ldots, m$$

Let the above maximum be attained at $i = i'$ and the minimum be attained at $j = j'$, *i.e.*,

$$\max_{1 \le i \le m} \{a_{ij}\} = a_{i'j} \quad \text{and} \quad \min_{1 \le j \le n} \{a_{ij}\} = a_{ij'}$$

Then, we must have

$$a_{i'j} \ge a_{ij} \ge a_{ij'} \qquad \text{for all } i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, n$$

From this, we get

$$\min_{1 \le j \le n} \{a_{i'j}\} \ge a_{ij} \ge \max_{1 \le i \le m} \{a_{ij'}\} \qquad \text{for all } i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, m$$

$$\therefore \quad \min_{1 \le j \le n} \max_{1 \le i \le m} \{a_{ij}\}] \ge \max_{1 \le i \le m} [\min_{1 \le j \le n} \{a_{ij}\}]$$

or

$$\bar{v} \ge \underline{v}.$$

Remarks 1. A game is said to be fair, if $\underline{v} = 0 = \bar{v}$.

2. A game is said to be strictly determinable, if $\underline{v} = v = \bar{v}$.

Rule for determining a Saddle Point

We may now summarize the procedure of locating the saddle point of a payoff matrix as follows :

Step 1. Select the minimum element of each row of the payoff matrix and mark them [*].

Step 2. Select the greatest element of each column of the payoff matrix and mark them [†].

Step 3. If there appears an element in the payoff matrix marked [*] and [†] both, the position of that element is a saddle point of the payoff matrix.

SAMPLE PROBLEMS

1701. *Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in the case of strictly determinable games :*

$$(a) \quad \begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \end{array} \qquad (b) \quad \begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix} \end{array}$$

[Madurai M.Com. 1997]

Solution. (*a*) The payoff matrix for player A is

Player A	*Player B*		*Row minima*
	B_1	B_2	
A_1	$5^\dagger$	0^*	0
A_2	0^*	$2^\dagger$	0
Column maxima	5	2	

The payoffs marked with [*] represent the minimum payoff in each row and those marked with [†] represent the maximum payoff in each column of the payoff matrix. The largest component of row minima represents $\underline{v}$ (maximin value) and the smallest component of column maxima represents $\bar{v}$ (minimax value).

Thus obviously, we have

$$\underline{v} = 0 \quad \text{and} \quad \bar{v} = 2.$$

Since $\underline{v} \ne \bar{v}$, the game is not strictly determinable.

(*b*) Here, the payoff matrix for player A is

Player A	*Player B*		*Row minima*
	B_1	B_2	
A_1	$0*^{\dagger}$	2	0
A_2	$-1*$	$4^{\dagger}$	-1
Column maxima	0	4	

Since, the payoffs marked with [*] represent the minimum payoff in each row and those marked with [†] the maximum payoff in each column of the payoff matrix, we have

$$\underline{v} \text{ (maximin value)} = 0 \quad \text{and} \quad \overline{v} \text{ (minimax value)} = 0.$$

As $\underline{v} = \overline{v} = 0$, the game is strictly determinable and fair. Optimum strategies for players A and B are given by

$$S_0 = (A_1, B_1).$$

1702. *Solve the game whose payoff matrix is given by*

$$\begin{array}{cc} & \textit{Player A} \\ & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \textit{Player B} \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

[Bharathidasan B.Com. 1999]

Solution. Consider the set of pure strategies

$$\alpha = \{A_1, A_2, A_3\} \text{ for player } A \quad \text{and} \quad \beta = \{B_1, B_2, B_3\} \text{ for player } B.$$

Assume that player B starts the game knowing fully well that whatever strategy he adopts, A will counter with a strategy that will minimize the payoff to B. Thus, if B selects B_1, then A will reply by selecting A_1 or A_3 as this corresponds to the minimum payoff to B in the first row corresponding to B_1. Similarly, if B chooses the strategy B_2, he may loose 4 or 3 or may neither loose nor gain depending upon the strategy chosen by A. However, B is assured of a gain of at least min. $\{0, -4, -3\}$; *i.e.*, -4 regardless of the strategy chosen by A. In other words, whatever strategy B may adopt, he can be assured of only the minimum of the corresponding row payoffs These corresponding to $B_i \in \beta$ are indicated by forming a column vector $r = \{1, -4, -1\}$ of the *row minima*. Naturally, B would like to maximize his minimum gain, which is just the largest component of r. Thus, maximum value of the game is maximum of $\{1, -4, -1\}$, *i.e.*, 1 which corresponds to B_1, the *maximin strategy*.

On the other hand, player A wishes to minimize his losses. If he plays strategy A_1, his loss is at the most maximum of $\{1, 0, 1\}$, *i.e.*, 1 regardless of what strategy B has adopted. He loses no more than max. $\{3, -4, 5\}$, if he plays A_2 and no more then max. $\{1\ -3, -1\}$ if he plays A_3. These maximum losses, corresponding to each $A_i \in \alpha$ are indicated by forming a row vector $c = (1, 5, 1)$ of the column maxima. The smallest component of c represents the minimum possible loss to A whatever strategy B may adopt. Thus, the minimax value of the game is min. $(1, 5, 1)$, *i.e.*, which corresponds to A_1 and A_3, the *minimax strategies*.

The maximin value is generally marked by $\{*\}$ and the minimax value by $\{\dagger\}$ as shown below :

$$\begin{array}{cc} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \quad \mathbf{r} \\ \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ c: \end{array} & \begin{bmatrix} 1*^{\dagger} & 3 & 1*^{\dagger} \\ 0 & -4* & -3 \\ 1^{\dagger} & 5^{\dagger} & -1* \end{bmatrix} \begin{array}{c} 1* \\ -4 \\ -1 \end{array} \\ & \begin{array}{ccc} 1^{\dagger} & 5 & 1^{\dagger} \end{array} \end{array}$$

We observe from the above that there exist two saddle points (having * and † both) at positions (1, 1) and (1, 3). Thus, the solution to the game is given by

(*i*) the optimum strategy for player B is B_1,

(*ii*) the optimum strategies for player A are A_1 and A_3,

(*iii*) the value of game is 1 for B and –1 for A.

Note : Since $v \neq 0$, the game is not fair, although it is strictly determinable.

1703. *Determine the range of value of p and q that will make the payoff element a_{22}, a saddle point for the game whose payoff matrix (a_{ij}) is given below :*

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix} \end{array}$$

Solution. Let us first of all ignore the values of p and q and determine the maximin and minimax values of the payoff matrix. For this, we have

$$\begin{array}{cccc} & B_1 & B_2 & B_3 & \text{Row minima} \\ A_1 & 2 & 4 & 5 & 2 \\ A_2 & 10 & 7 & q & 7 \\ A_3 & 4 & p & 8 & 4 \\ \text{Column maxima} & 10 & 7 & 8 & \end{array}$$

Obviously, the maximin value ($\underline{v}$) is 7 and the minimax value ($\overline{v}$) is also 7. Thus, there exists a saddle point at position (2, 2).

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$.

Hence, the required range of values of p and q is

$$7 \leq q, \quad p \leq 7.$$

PROBLEMS

1704. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games :

(*a*)

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 10 & 6 \\ 8 & 2 \end{bmatrix} \end{array}$$

[Madurai M.Com. 1993]

1705. Consider the game G with the following payoff matrix :

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

(*a*) Show that G is strictly determinable whatever μ may be.

(*b*) Determine the value of G.

[Amravathi B.E. (Rul.) 1994]

1706. For the game with payoff matrix :

$$\begin{array}{cc} & \text{Player } A \\ \text{Player } B & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

determine the best strategies for players A and B and also the values of the game for them. Is this game (*i*) fair? (*ii*) strictly determinable?

1707. For what value of λ, the game with following payoff matrix is strictly determinable?

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

1708. Solve the following 2-person zero-sum game :

(*a*)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix} \end{array}$$

[Madurai M.Com. 2003]

(*b*)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} -2 & 12 & -4 \\ 1 & 4 & 8 \\ -5 & 2 & 3 \end{bmatrix} \end{array}$$

[Delhi M.B.A. 2011]

1709. Solve the game whose pay-off matrix is given by

(*a*)

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} & \begin{bmatrix} 20 & 15 & 12 & 35 \\ 25 & 14 & 5 & 10 \\ 40 & 2 & 8 & 5 \\ -5 & 4 & 0 & 0 \end{bmatrix} \end{array}$$

[Delhi PG Dip. in Glob. Bus. Oper. 2011]

(*b*)

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 8 & 4 & 1 \\ 10 & 6 & 2 \\ 12 & 8 & 3 \\ 4 & 4 & 4 \end{bmatrix} \end{array}$$

[Gujarat M.B.A. 2009]

1710. Solve the game whose payoff matrix is given below :

$$\begin{bmatrix} 1 & 3 & 2 & 7 & 4 \\ 3 & 4 & 1 & 5 & 6 \\ 6 & 5 & 7 & 6 & 5 \\ 2 & 0 & 6 & 3 & 1 \end{bmatrix}$$

[Madras M.B.A. 2010]

1711. Solve the following game :

$$\begin{bmatrix} 7 & 8 & 4 & 6 & 8 \\ -8 & 6 & 1 & 9 & 6 \end{bmatrix}$$

[Madras M.C.A. (Oct.) 2002]

1712. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following payoff matrix describes the increase in market share for Firm *A* and the decrease in market share for Firm *B*. Determine the optimum strategies for each firm.

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{ccc} \text{No promotion} & \text{Moderate promotion} & \text{Much promotion} \end{array} \\ \text{Firm } A \begin{array}{l} \text{No promotion} \\ \text{Moderate promotion} \\ \text{Much promotion} \end{array} & \begin{bmatrix} 5 & 0 & -10 \\ 10 & 6 & 2 \\ 20 & 15 & 10 \end{bmatrix} \end{array}$$

(*i*) Which firm would be the winner, in terms of market share?

(*ii*) Would the solution strategies necessarily maximize profits for either of the firms? *[Delhi M.B.A. (April) 1999]*

1713. Two competitive manufacturers are producing a new toy under licence from a patent holder. In order to meet the demand, they have the option of running the plant for 8, 16 or 24 hours a day. As the length of production increases, so does the cost. One of the manufacturers say *A*, has set up the matrix given below, in which he estimates the percentage of the market that he could capture and maintain the different production schedules :

Manufacturer A	*Manufacturer B*		
	C_1 : 8 *hrs.*	C_2 : 16 *hrs.*	C_3 : 24 *hrs.*
S_1 : 8 hrs.	60%	56%	34%
S_2 : 16 hrs.	63%	60%	55%
S_3 : 24 hrs.	83%	72%	60%

(*i*) At which level should each produce?

(*ii*) What percentage of the market will manufacturer *B* have? *[Delhi PG Dip. in Glob. Bus. Oper. 2011]*

17:5. GAMES WITHOUT SADDLE POINTS—MIXED STRATEGIES

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position—the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called *mixed* strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents which least player A can expect to win and the least which player B can lose. The expected payoff to a player in a game with arbitrary payoff matrix (a_{ij}) of order $m \times n$ is defined as :

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = \mathbf{p}^T \mathbf{A}\mathbf{q}$$

where $\mathbf{p}$ and $\mathbf{q}$ denote the mixed strategies for players A and B respectively.

Maximin-Minimax Criterion. Consider an $m \times n$ game (a_{ij}) without any saddle point, *i.e.*, strategies are mixed. Let $p_1, p_2, \ldots, p_m$ be the probabilities with which player A will play his moves $A_1, A_2, \ldots, A_m$ respectively; and let $q_1, q_2, \ldots, q_n$ be the probabilities with which player B will play his moves $B_1, B_2, \ldots, B_n$ respectively. Obviously, $p_i \geq 0$ $(i = 1, 2, \ldots, m)$, $q_j \geq 0$ $(j = 1, 2, \ldots, n)$, and $p_1 + p_2 + \ldots + p_m = 1$; $q_1 + q_2 + \ldots + q_n = 1$.

The expected payoff function for player A, therefore, will be given by

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

Making use of maximin-minimax criterion, we have

For Player A.

$$\underline{v} = \max_{\mathbf{p}}. \min_{\mathbf{q}}. E(\mathbf{p}, \mathbf{q}) = \max_{\mathbf{p}}. \left[\min_{j}. \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right]$$

$$= \max_{\mathbf{p}}. \left[\min_{j}. \left\{ \sum_{i=1}^{m} p_i a_{i1}, \sum_{i=1}^{m} p_i a_{i2}, \ldots, \sum_{i=1}^{m} p_i a_{in} \right\} \right]$$

Here, $\min. \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\}$ denotes the expected gain to player A, when player B uses his jth pure strategy.

For player B.

$$\overline{v} = \min_{\mathbf{q}}. \left[\max_{i}. \left\{ \sum_{j=1}^{n} q_j a_{1j}, \sum_{j=1}^{n} q_j a_{2j}, \ldots, \sum_{j=1}^{n} q_j a_{mj} \right\} \right].$$

Here $\max. \left\{ \sum_{j=1}^{n} q_j a_{ij} \right\}$ denotes the expected loss to player B when player A uses his ith strategy.

The relationship $\underline{v} \leq \overline{v}$ holds good in general and when p_i and q_j correspond to the optimal strategies the relation holds in 'equality' sense and the expected value for both the players becomes equal to the optimum expected value of the game.

Definition. *A pair of strategies* $(\mathbf{p}, \mathbf{q})$ *for which* $\underline{v} = \overline{v} = v$ *is called a* ***saddle point*** *of* $E(\mathbf{p}, \mathbf{q})$.

Theorem 17-2. *For any* 2×2 *two-person zero-sum game without any saddle point having the payoff matrix for player A*

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array},$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad \textit{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix},$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

where $p_1 + p_2 = 1$ *and* $q_1 + q_2 = 1$. *The value* v *of the game to* A *is given by*

$$v = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Proof. Let a mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, where $p_1 + p_2 = 1$. Thus, if player B moves B_1 the net expected gain of A will be

$$E_1(p) = a_{11}\, p_1 + a_{21}\, p_2$$

and if B moves B_2, the net expected gain of A will be

$$E_2(p) = a_{12}\, p_1 + a_{22}\, p_2.$$

Similarly, if B plays his mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$, where $q_1 + q_2 = 1$, then B's net expected loss will be

$$E_1(q) = a_{11}\, q_1 + a_{12}\, q_2$$

if A plays A_1, and

$$E_2(q) = a_{21}\, q_1 + a_{22}\, q_2$$

if A plays A_2.

The expected gain of player A, when B mixes his moves with probabilities q_1 and q_2 is, therefore, given by

$$E(\mathbf{p}, \mathbf{q}) = q_1\,[a_{11}\, p_1 + a_{21}\, p_2] + q_2\,[a_{12}\, p_1 + a_{22}\, p_2].$$

Player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

Now,

$$\begin{aligned} E(\mathbf{p}, \mathbf{q}) &= q_1\,[a_{11}\, p_1 + a_{21}\,(1 - p_1)] + (1 - q_1)\,[a_{12}\, p_1 + a_{22}\,(1 - p_1)] \\ &= [a_{11} + a_{22} - (a_{12} + a_{21})]\, p_1 q_1 + (a_{12} - a_{22})\, p_1 + (a_{21} - a_{22})\, q_1 + a_{22} \\ &= \lambda\left(p_1 - \frac{a_{22} - a_{21}}{\lambda}\right)\left(q_1 - \frac{a_{22} - a_{12}}{\lambda}\right) + \frac{a_{11}\, a_{22} - a_{12}\, a_{21}}{\lambda}, \end{aligned}$$

where $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$.

We see that if A chooses $p_1 = \dfrac{a_{22} - a_{21}}{\lambda}$, he ensures an expected gain of at least $(a_{11}\, a_{22} - a_{12}\, a_{21})/\lambda$. Similarly, if B chooses $q_1 = \dfrac{a_{22} - a_{12}}{\lambda}$, then B will limit his expected loss to at most $(a_{11}\, a_{22} - a_{12}\, a_{21})/\lambda$. These choices of p_1 and q_1 will thus be optimal to the two players.

Thus, we get

$$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \text{and} \quad p_2 = 1 - p_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \text{and} \quad q_2 = 1 - q_1 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

and

$$v = \frac{a_{11}\, a_{22} - a_{12}\, a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Hence, we have

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}; \quad \text{and} \quad v = \frac{a_{11}\,a_{22} - a_{12}\,a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Note : The above formulae for p_1, p_2, q_1, q_2 and v are valid only for 2×2 games without saddle points.

SAMPLE PROBLEMS

1714. *For the game with the following payoff matrix, determine the optimum strategies and the value of the game :*

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

[ICSI (June) 1996; Madurai M.Com. (Nov.) 2002]

Solution. Clearly, the given matrix is without a saddle point. So, the mixed strategies of P_1 and P_2 are :

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix}, \quad S_{P_2} = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix}; \quad p_1 + p_2 = 1 \quad \text{and} \quad q_1 + q_2 = 1$$

If $E(p, q)$ denotes the expected payoff function, then

$$E(p, q) = 5p_1q_1 + 3(1-p_1)\,q_1 + p_1(1-q_1) + 4(1-p_1)(1-q_1)$$
$$= 5p_1q_1 - 3p_1 - q_1 + 4 = 5(p_1 - 1/5)(q_1 - 3/5) + 17/5.$$

If P_1 chooses $p_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $q_1 = 3/5$, P_2 can keep $E(p_1, q_1)$ down to 17/5. So P_1 might as well settle for 17/5 and P_2 reconcile to 17/5. Hence, the optimum strategies for P_1 and P_2 are

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ 1/5 & 4/5 \end{bmatrix}, \quad S_{P_2} = \begin{bmatrix} 1 & 2 \\ 3/5 & 2/5 \end{bmatrix}$$

and the value of the game is $v = 17/5$.

1715. *Consider a "modified" form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Re. 1.00 if the coins turn both tails. The non-matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?*

[Delhi M.B.A. 1999, 2007]

Solution. The payoff matrix for the matching player is given by

$$\begin{array}{ccc} & & \textit{Non-matching Player} \\ & & \begin{array}{cc} H & T \end{array} \\ \text{Matching Player} & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Clearly, the payoff matrix does not possess any saddle point. The players will use mixed strategies. The optimum mixed strategy for matching player is determined by

$$p_1 = \frac{1 - (-3)}{8 + 1 - (-3 - 3)} = \frac{4}{15}, \quad p_2 = \frac{11}{15}$$

and for the non-matching player, by

$$q_1 = \frac{1 - (-3)}{8 + 1 - (-3 - 3)} = \frac{4}{15}, \quad q_2 = \frac{11}{15}.$$

The expected value of the game (corresponding to the above strategies) is given by

$$v = \frac{8 - 3(-3)(-3)}{8 + 1 - 1(-3 - 3)} = -\frac{1}{15}.$$

Thus, the optimum mixed strategies for matching player and non-matching player are given by

$$S_{match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix} \quad \text{and} \quad S_{non\text{-}match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix}.$$

Clearly, we would like to be the non-matching player.

PROBLEMS

1716. Solve the following game and determine the value of the game :

(*a*)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array},$$

[Madras B.E. (Mech.) 1999]

(*b*)
$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

[Allahabad M.B.A. 1998]

1717. In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A.

[Amravathi B.E. (Rul.) 1994]

1718. Two players A and B match coins. If the coins match, then A wins two units of value. If the coins do not match, then B wins 2 units of value. Determine the optimum strategies for the players and the value of the game. *[Madras M.B.A. (Nov.) 2006; Delhi M.Com. 2008]*

1719. A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly then the loser has to pay him as many rupees as the sum of the number held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players. *[Jodhpur M.Sc. (Math.) 1994]*

17:6. GRAPHIC SOLUTION OF $2 \times n$ AND $m \times 2$ GAMES

The procedure described in the last section will generally be applicable for any game with 2×2 payoff matrix unless it possesses a saddle point. Moreover, the procedure can be extended to any square payoff matrix of any order. But it will not work for the game whose payoff matrix happens to be a *rectangular* one, say $m \times n$. In such cases a very simple graphical method is available if either m or n is two. The graphic short-cut enables us to reduce the original $2 \times n$ or $m \times 2$ game to a much simpler 2×2 game. Consider the following $2 \times n$ game :

$$\begin{array}{ccc} & & \textit{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \textit{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \end{array}.$$

It is assumed that the game does not have a saddle point. Let the optimum mixed strategy for A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$. The average (expected) payoff for A when he plays S_A against B's pure moves $B_1, B_2, \ldots, B_n$ is given by

B's pure move	*A's expected payoff* $E(p)$
B_1	$E_1(p_1) = a_{11}p_1 + a_{21}p_2 = a_{11}p_1 + a_{21}(1-p_1)$
B_2	$E_2(p_1) = a_{12}p_1 + a_{22}p_2 = a_{12}p_1 + a_{22}(1-p_1)$
$\vdots$	$\vdots$
B_n	$E_n(p_1) = a_{1n}p_1 + a_{2n}p_2 = a_{1n}p_1 + a_{2n}(1-p_1).$

According to the maximin criterion for mixed strategy games, player A should select the values of p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the expected payoff lines :

$$E_j(p_1) = (a_{1j} - a_{2j})\,p_1 + a_{2j} \qquad (j = 1, 2, \ldots, n).$$

The highest point on the *lower envelope* of these lines will give maximum of the minimum (*i.e.*, maximin) expected payoffs to player A as also the maximum value of p_i.

The two lines* passing through the *maximin* point identify the two *critical* moves of B which, combined with two of A, yield the 2×2 matrix that can be used to determine the optimum strategies of the two players, for the original game, using the results of the previous section.

The $(m \times 2)$ games are also treated in the same way where the upper envelope of the straight lines corresponding to B's expected payoffs will give the maximum expected payoff to player B and the lowest point on this then gives the minimum expected payoff (minimax value) and the optimum value of q_1.

SAMPLE PROBLEMS

1720. *Solve the following 2×2 game graphically :*

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Math.) 1996]

Solution. Clearly, the problem does not possess a saddle point. Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_2 = 1 - p_1$, against B. Then A's expected payoffs against B's pure moves are given by

B's pure move	*A's expected payoff* $E(p_1)$
B_1	$E_1(p_1) = p_1 + 1$
B_2	$E_2(p_1) = p_1$
B_3	$E_3(p_1) = -3p_1 + 3$
B_4	$E_4(p_1) = -4p_1 + 2$

These expected payoff equations are then plotted as functions of p_1 as shown in Fig. 17.1 which shows the payoffs of each column represented as points on two vertical axis *1* and *2*, unit distance apart. Thus line B_1 joins the first payoff element 2 in the first column represented by +2 on axis *2*, and the second payoff element 1 in the first column represented by +1 on axis *1*. Similarly, lines B_2, B_3 and B_4 join the corresponding representation of payoff elements in the second, third and fourth columns. Since the player A wishes to maximize his minimum expected payoff we consider the highest point of intersection H on the *lower envelope* of the A's expected payoff equations. This point H represents the *maximin* expected value of the game for A. The lines B_2 and B_4, passing through H, define the two relevant moves B_2 and B_4 that alone B needs to play. The solution to the original 2×4 game, therefore, boils down that of the simpler game with the 2×2 payoff matrix :

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \end{array}$$

*If there are more than two lines passing through the maximin point, there are ties for the optimum mixed strategies for player B. Thus any two such lines with *opposite sign* slopes will define an *alternative optimum* for B.

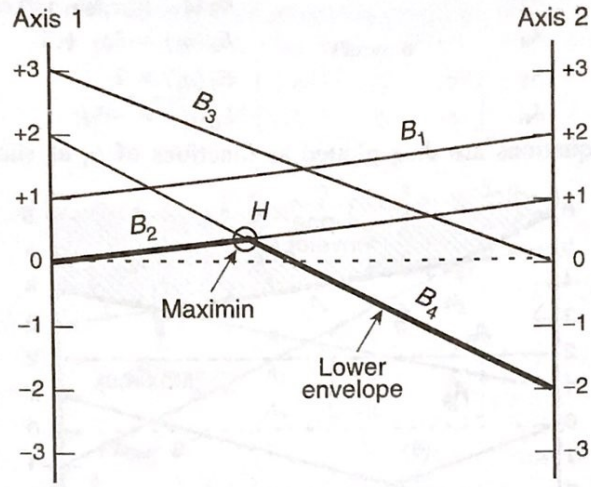

Fig. 17.1. The maximin value

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_2 & B_4 \\ q_2 & q_4 \end{bmatrix}$

be the optimum strategies for A and B, then we have

$$p_1 = \frac{2-0}{1+2-(-2)} = 2/5, \quad p_2 = 1 - p_1 = 3/5,$$

$$q_2 = \frac{2-(-2)}{1+2-(-2)} = 4/5, \quad q_4 = 1 - q_1 = 1/5.$$

Hence, the solution to the game is

(*i*) the optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$,

(*ii*) the optimum strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$

and (*iii*) the expected value of the game is $v = \dfrac{2 \times 1 - 0 \times (-2)}{1 + 2 - (0 - 2)} = \dfrac{2}{5}$.

1721. *Obtain the optimal strategies for both-persons and the value of the game for zero-sum two-person game whose payoff matrix is as follows :*

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

[Guru Nanak Dev Univ. B.Com. 2006]

Solution. Clearly, the given problem does not possess any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then B's expected payoffs against A's pure moves are given by

A's pure move	*B's expected payoff* $E(q_1)$
A_1	$E_1(q_1) = 4q_1 - 3$
A_2	$E_2(q_1) = -2q_1 + 5$

A_3	$E_3(q_1) = -7q_1 + 6$
A_4	$E_4(q_1) = 3q_1 + 1$
A_5	$E_5(q_1) = 2$
A_6	$E_6(q_1) = -5q_1$

The expected payoff equations are then plotted as functions of q_1 as shown in Fig. 17.2 :

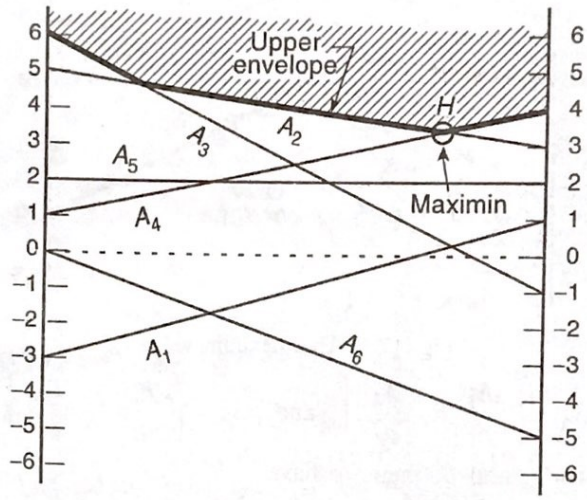

Fig. 17.2. The minimax value

Since, the player B wishes to minimize his maximum expected payoff, we consider the lowest point of intersection H on the *upper envelope* of B's expected payoff equations. This point H represents the minimax expected value of the game for player B. The lines A_2 and A_4 passing through H, define the two relevant moves A_2 and A_4 that alone the player A needs to play. The solution to the original 6×2 game, therefore, reduces to that of the simpler game with 2×2 payoff matrix :

$$\begin{array}{cc} & \textit{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

If we now, let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix}, \quad p_1 + p_2 = 1; \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \quad q_1 + q_2 = 1$$

then using the usual method of solution for 2×2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}$$

and the value of the game as $v = 17/5$.

PROBLEMS

1722. Solve the following problem graphically :

$$\begin{array}{cc} & \textit{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

[Jodhpur M.Sc. (Math. 1993]

1723. Use graphical method in solving the following game :

$$\begin{array}{cc} & \textit{Player } A \\ \text{Player } B & \begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

[Panjab Tech. Univ. M.B.A. (Dec.) 2010]

Solve the following games graphically :

1724.

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix} \end{array}$$

[Jammu M.B.A. 2008]

1725.

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} \end{array}$$

1726.

$$\begin{array}{ccc} & & \textit{B's strategy} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{A's strategy} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{bmatrix} \end{array}$$

[Gujarat M.B.A. 1998]

1727. (*a*)

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix} \end{array}$$

[Madras M.E. (Struct.) 2000]

1728.

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} -4 & 3 \\ -7 & 1 \\ -2 & -4 \\ -5 & -2 \\ -1 & -6 \end{bmatrix} \end{array}$$

[Karnataka B.E. (Ind.) 1994]

1729. Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs. 400 per colour set and Rs. 300 per black & white set. Firm B can, on the other hand, make either 300 colour sets, or 150 colour and 150 black & white sets, or 300 black & white sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set.

Write the payoff matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game. *[Bombay M.M.S. 1997; Delhi M.B.A. (Nov.) 2009]*

1730. A soft drink company calculated the market share of two products against its major competitor having three products and found out the impact of additional advertisement in any one of its products against the other :

$$\begin{array}{ccc} & & \textit{Competitor B} \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Company } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 & 7 & 15 \\ 20 & 12 & 10 \end{bmatrix} \end{array}$$

What is the best strategy for the company as well as the competitor? What is the payoff obtained by the company and the competitor in the long run? Use graphical method to obtain the solution.

[Meerut M.Sc. (Math.) 1999; Delhi M.B.A. (April) 1998]

17:7. DOMINANCE PROPERTY

Sometimes, it is observed that one of the pure strategies of either player is always *inferior* to at least one of the remaining ones. The superior strategies are said to *dominate* the inferior ones. Clearly, a player would have no incentive to use inferior strategies which are dominated by the superior ones. In

such cases of dominance, we can reduce the size of the payoff matrix by deleting those strategies which are dominated by the others. Thus if each element in one row, say kth of the payoff matrix (a_{ij}) is less than or equal to the corresponding elements in some other row, say rth, then player A will never choose kth strategy. In other words, probability $p_k = P$ (choosing the kth strategy) is zero, if $a_{kj} \leq a_{rj}$ for all $j = 1, \ldots, n$.

The value of the game and the non-zero choice of probabilities remain unchanged even after the deletion of kth row from the payoff matrix. In such a case the kth strategy is said to be dominated by the rth one.

General rules for dominance are :

(*a*) If all the elements of a row, say kth, are less than or equal to the corresponding elements of any other row, say rth, then kth row is *dominated* by rth row.

(*b*) If all the elements of a column, say kth are greater than or equal to the corresponding elements of any other column, say rth, then kth column is dominated by the rth column.

(*c*) Dominated rows or columns may be deleted to reduce the size of payoff matrix, as the optimal strategies will remain unaffected.

The *Modified Dominance Property*. The dominance property is not always based on the superiority of pure strategies only. A given strategy can also be said to be dominated if it is inferior to an average of two or more other pure strategies. More generally, if some convex linear combination of some rows dominates the ith row, then ith row will be deleted. Similar arguments follow for columns.

SAMPLE PROBLEMS

1731. *Two firms are competing for business under the condition so that one firm's gain is another firm's loss. Firm A's payoff matrix is given below :*

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{ccc} \textit{No ad} & \textit{Medium ad} & \textit{Heavy ad} \end{array} \\ \begin{array}{c} \\ \textit{Firm } A \end{array} & \begin{array}{l} \textit{No advertising} \\ \textit{Medium advertising} \\ \textit{Heavy advertising} \end{array} \begin{bmatrix} 10 & 5 & -2 \\ 13 & 12 & 15 \\ 16 & 14 & 10 \end{bmatrix} \end{array}$$

Suggest optimum strategies for the two firms and the net outcome thereof. **[Delhi MIB 2011]**

Solution. Clearly, the first column is dominated by the second column as all the elements of the first column are greater than elements of second column. Thus eliminating first column, we get

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{ccc} \textit{Medium ad} & \textit{Heavy ad} \\ B_2 & B_3 \end{array} \\ \begin{array}{c} \\ \textit{Firm } A \end{array} & \begin{array}{ll} \text{No advertising} & A_1 \\ \text{Medium advertising} & A_2 \\ \text{Heavy advertising} & A_3 \end{array} \begin{bmatrix} 5 & -2 \\ 12 & 15 \\ 14 & 10 \end{bmatrix} \end{array}$$

Again, first row is dominated by second and third row as all the elements of first row are less than the respective elements of second, and third row. Hence eliminating first row, we obtain the following 2×2 payoff matrix.

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{ccc} \textit{Medium ad} & \textit{Heavy ad} \\ B_2 & B_3 \end{array} \\ \begin{array}{c} \\ \textit{Firm } A \end{array} & \begin{array}{ll} \text{Medium advertising} & A_2 \\ \text{Heavy advertising} & A_3 \end{array} \begin{bmatrix} 12 & 15 \\ 14 & 10 \end{bmatrix} \end{array}$$

Since the reduced payoff matrix do not have any saddle point, the strategies are mixed. So, let

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}, \quad p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$$

Using the usual method for the solution of 2×2 payoff matrices, the optimum strategies for the two players and the value of the game can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 4/7 & 3/7 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 5/7 & 2/7 \end{bmatrix} \quad \text{and} \quad v = 90/7.$$

Hence, firm A should adopt strategy A_2 and A_3 with 57% of time and 43% of time respectively (or with 57% and 43% probability on any one play of the game respectively). Similarly, firm B should adopt strategy B_2 and B_3 with 71% of time and 29% of time respectively (or with 71% and 29% probability on any one play of the game respectively).

1732. *Solve the following game :*

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cccc} I & II & III & IV \end{array} \\ \text{Player } A \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix} \end{array}$$

[Madurai M.Com. 2003; Panjab M.C.A. 2007]

Solution. From the above payoff matrix, we observe that first row is dominated by third row and first column is dominated by third column. The reduced payoff matrix is

$$\begin{array}{c} \\ II \\ III \\ IV \end{array} \begin{array}{c} \begin{array}{ccc} II & III & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} \end{array}$$

Now, none of the pure strategies of player B is inferior to any of his other strategies. However, a *convex linear combination* (average) of strategies *III* and *IV* dominates strategy *II* of player B, yielding the reduced payoff matrix

$$\begin{array}{c} \\ II \\ III \\ IV \end{array} \begin{array}{c} \begin{array}{cc} III & IV \end{array} \\ \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ 0 & 8 \end{bmatrix} \end{array}$$

Again, we observe that none of the pure strategies of player A is inferior to any of his other strategies. However, a convex linear combination of strategies *III* and *IV* dominates strategy *II* of player A, yielding the reduced payoff matrix

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} III & IV \end{array} \\ \text{Player } A \begin{array}{c} III \\ IV \end{array} & \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \end{array}$$

Now, letting

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & p_1 & p_2 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_1 & q_2 \end{bmatrix},$$

where $p_1 + p_2 = 1$, $q_1 + q_2 = 1$, and then using the method of solving 2×2 games, we can easily obtain the optimum strategies as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 2/3 & 1/3 \end{bmatrix}$$

and the value of game as $v = 8/3$.

1732. (*a*) *Solve the game whose payoff matrix is given below :*

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \\ -2 & 1 & -2 & 0 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2006]

Solution. Clearly A_3 is dominated by A_2. Thus, the deletion of third row yields the reduced payoff matrix

$$\begin{array}{c} \\ A \end{array} \begin{array}{c} B \\ \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

Now, let A play the strategies A_1 and A_2 with respective probabilities p_1 and $1-p_1$. Then, the expected payoff for A, when B chooses to play B_1, shall be $4p_1 - (1-p_1)$, *i.e.*, $5p_1 - 1$. Similarly, payoffs in respect of other strategy plays can be determined. These are presented graphically in Fig. 17.3. Point H represents the maximin expected value of game to player A :

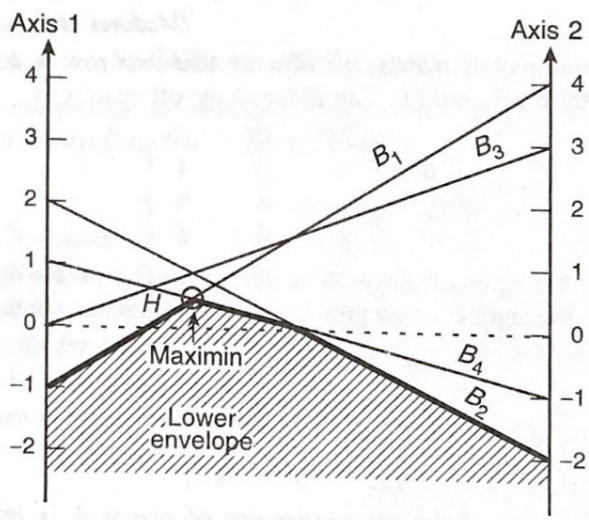

Fig. 17.3

Alternatively, the solution to the original game reduces to that of the simpler game with 2×2 payoff matrix as

$$\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

If, we let

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}, \quad p_1 + p_2 = 1 \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}, \quad q_1 + q_2 = 1 ;$$

then the optimum strategies can easily be determined as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/7 & 5/7 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 2/7 & 0 & 0 & 5/7 \end{bmatrix}; \quad \text{and} \quad v = 3/7.$$

1733. *Two firms A and B have for years been selling a competitive product which forms a part of both firms' total sales. The marketing executive of firm A raised the question. "What should be the firm's strategies in terms of advertising for the product in question?" The market research team of firm A developed the following data for varying degrees of advertising :*

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm A with no advertising : 40% of the market with medium advertising by firm B and 28% of the market with large advertising by firm B.

(iii) Firm A using medium advertising : 70% of market with no advertising by firm B and 45% of the market with large advertising by firm B.

(iv) Firm A using large advertising : 75% of the market with no advertising by firm B and 47.5% of the market with medium advertising by firm B.

(a) Based upon the foregoing information, answer the marketing executive's questions.

(b) What advertising policy should firm A pursue when consideration is given to the above factors : selling price Rs. 4.00 per unit; variable cost of product Rs. 2.50 per unit; annual volume of 30,000 units for firm A; cost of annual medium advertising Rs. 5,000 and cost of annual large advertising Rs. 15,000? What contribution, before other fixed costs, is available to the firm?

[Delhi M.B.A. 2000]

Solution. (*a*) Using the given information, the pay-off matrix of the game between firms *A* and *B* is as follows :

Firm A	*Firm B*		
	No advertising	*Medium advertising*	*Large advertising*
No Advertising	50	40	28
Medium Advertising	70	50	45
Large Advertising	75	47.5	50

Clearly, the first row is dominated by the second row and first column is dominated by the second column. Eliminating first row and first column, the reduced pay-off matrix, therefore, is

$$\begin{array}{c} \\ \textit{Medium Adv.} \\ \textit{Large Adv.} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Medium Adv.} & \textit{Large Adv.} \end{array} \\ \begin{bmatrix} 50 & 45 \\ 47.5 & 50 \end{bmatrix} \end{array}$$

Since, the reduced pay-off matrix do not have any saddle point, the strategies are mixed. So, let

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

and $$p_1 + p_2 + p_3 = 1, \quad q_1 + q_2 + q_3 = 1.$$

Using the usual method for solving a 2×2 pay-off matrices, the optimum strategies for the two players and the value of the game are easily obtained :

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 1/3 & 2/3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 2/3 & 1/3 \end{bmatrix} \text{ and } v = 145/3.$$

Hence, the optimum strategy for firm *A* is to apply strategy A_2 (medium strategy) with probability 0.33 and strategy A_3 (large strategy) with probability 0.67 on any one play of the game. With this policy, the firm may expect to gain 145/3 (= 48.3) per cent of the market share.

(*b*) When the annual value of firm *A* is of 30,000 units, its market share is :

$$\begin{array}{cc} & \\ \textit{Firm A} & \begin{array}{c} \textit{No Adv.} \\ \textit{Medium Adv.} \\ \textit{Large Adv.} \end{array} \end{array} \begin{array}{c} \textit{Firm B} \\ \begin{array}{ccc} \textit{No Adv.} & \textit{Medium Adv.} & \textit{Large Adv.} \end{array} \\ \begin{bmatrix} 0.50 \times 30{,}000 = 15{,}000 & 0.40 \times 30{,}000 = 12{,}000 & 0.28 \times 30{,}000 = 8{,}400 \\ 0.70 \times 30{,}000 = 21{,}000 & 0.50 \times 30{,}000 = 15{,}000 & 0.45 \times 30{,}000 = 13{,}500 \\ 0.75 \times 30{,}000 = 22{,}500 & 0.475 \times 30{,}000 = 14{,}250 & 0.50 \times 30{,}000 = 15{,}000 \end{bmatrix} \end{array}$$

Given the expenditure on medium and large advertisements as Rs. 5,000 and Rs. 15,000 respectively, net profit to firm A can be calculated as follows :

$$\text{Net profit} = (\text{Sales price} - \text{Cost price}) \times \text{Sales volume} - \text{Advertising expenditure}$$
$$= (4 - 2.50) \times \text{Sales volume} - \text{Advertising expenditure}$$

The net profit to firm A is shown below :

$$\begin{array}{cc} & \text{Firm } B \\ & \begin{array}{ccc} \textit{No Advvertising} & \textit{Medium Advertising} & \textit{Large Advertising} \end{array} \\ \begin{array}{c} \textit{No Adv.} \\ \text{Firm } A \quad \textit{Medium Adv.} \\ \textit{Large Adv.} \end{array} & \begin{bmatrix} 1.5 \times 15{,}000 - 0 = 22{,}500 & 1.5 \times 12{,}000 - 0 = 18{,}000 & 1.5 \times 8{,}400 - 0 = 12{,}600 \\ 1.5 \times 21{,}000 - 5{,}000 = 26{,}500 & 1.5 \times 15{,}000 - 5{,}000 = 17{,}500 & 1.5 \times 13{,}500 - 5{,}000 = 15{,}250 \\ 1.5 \times 22{,}500 - 15{,}000 = 18{,}750 & 1.5 \times 14{,}250 - 15{,}000 = 6{,}375 & 1.5 \times 15{,}000 - 15{,}000 = 7{,}500 \end{bmatrix} \end{array}$$

From the above pay-off matrix, we observe the following :

(*i*) If firm A chooses the strategy of 'No advertising', then minimum profit is of Rs. 12,600, because firm B can adopt its strategy 'Large advertising'.

(*ii*) If firm A chooses the strategy of 'Medium advertising', then minimum profit is of Rs. 15,250 because firm B can again adopt its strategy 'Large advertising'.

(*iii*) If firm A chooses the strategy of 'Large advertising', then minimum profit is of Rs. 6,375 because firm B can adopt its strategy 'Medium advertising'.

Based on these observations, the firm A must adopt the policy of 'Medium advertising' to gain maximum profit of Rs. 15,250 among these three alternatives and must spend Rs. 5,000 for advertising.

PROBLEMS

1734. A and B play game in which each has three coins, a 5 p., 10 p. and a 20 p. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins B's coin. If the sum is even, B wins A's coin. Find the best strategy for each player and the value of the game.

[Annamalai M.B.A. (Nov.) 2009; Rohilkhand M.Sc. (Maths.) 2009]

1735. Use the principal of dominance to solve the following 5×5 game problem :

$$\begin{array}{cc} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 8 & 12 & 14 & 10 & 11 \\ 9 & 11 & 15 & 10 & 13 \\ 7 & 8 & 6 & 11 & 12 \\ 10 & 9 & 7 & 9 & 9 \\ 12 & 13 & 10 & 12 & 10 \end{bmatrix} \end{array}$$

[Madurai M.B.A. 2010]

1736. Find optimal strategies for two companies competing with each other on the basis of the following pay-off matrix showing gain or loss of customers :

Company A	*Company B*		
	Quantity discounts	*CSR initiatives*	*Events sponsorship*
Quantity Discounts	20	60	30
CSR Initiative	–10	30	25
Event Sponsorship	40	50	–30

(*i*) State the required assumptions for finding optimal strategies for two companies,

(*ii*) Find out value of the game. Use dominance rule.

[Delhi M.Com. 2010]

1737. In an election campaign, the strategies adopted by the ruling and opposition party alongwith pay-offs (ruling party's % share in votes polled) are given below :

Ruling party's strategies	*Opposition party's strategies*		
	Campaign one day in each city	*Campaign two days in large towns*	*Spend two days in large rural sectors*
Campaign one day in each city	55	40	35
Campaign two days in large towns	70	70	55
Spend two days in large rural sectors	75	55	65

Assume a zero-sum game. Find optimum strategies for both parties and expected pay-off to ruling party.

[Kurukshetra M.B.A. 2007]

1738. In a small town, there are two discount stores *ABC* and *XYZ*. They are the only stores that handle sundry goods. The total number of customers is equally divided between the two, because the price and quality of goods sold are equal. Both stores have good reputation in the community, and they render equally good customer services. Assume that a gain of customers by *ABC* is a loss to *XYZ*, and vice-versa.

Both stores plan to run annual pre-Diwali sales during the first week of October. Sales are advertised through the local newspaper, radio and television media. With the aid of an advertising firm *ABC* store constructed the game matrix given below (Figures in the matrix represent a gain or loss of customer.) Find the optimum strategies for both stores and the value of the game :

Store ABC	*Store XYZ*		
	Newspaper	*Radio*	*Television*
Newspaper	30	40	–80
Radio	0	15	–20
Television	90	20	50

[Delhi M.B.A. (Nov.) 2010; Visvesvaraya M.B.A. (Dec.) 2011]

1739. Solve the following '2-person, zero-sum game' :

Player A	*Player B*		
	B_1	B_2	B_3
A_1	15	10	4
A_2	10	–3	10
A_3	12	23	0

[Delhi M.B.A. PT. (Nov.) 2003; Delhi M.Com. 1997]

1740. Determine the optimum strategies and the value of the game from the following 2×5 pay- off matrix game for X :

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 6 & 3 & -1 & 0 & -3 \\ 3 & 2 & -4 & 2 & -1 \end{bmatrix} \end{array}$$

[Madurai M.Com. 2002]

1741. The following matrix represents the payoff to P_1 in a rectangular game between two persons P_1 and P_2 :

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

By the notion of dominance, reduce the game to a 2×4 game and solve it graphically.

[Delhi B.Sc. (Stat.) 2002]

1742. The following pay-off matrix describes the increase in sales of Firm A from each of its strategies A_1, A_2, A_3 and A_4 against each of the strategies B_1, B_2, B_3 and B_4 of the rival Firm B.

$$\text{Firm } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \text{Firm } B \\ \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 43 & 73 & 33 & 13 \\ 38 & 28 & 23 & 8 \\ 48 & 58 & 8 & 18 \\ 63 & 68 & 18 & 23 \end{bmatrix} \end{array}$$

Assuming a zero-sum game, show that no optimal strategy-mix exists in pure strategies irrespective of whether we use the principle of dominance or that of maximin and minimax. The game, however, has a solution in mixed strategies but not without the aid of the principle of dominance. Solve the game and find its value.

[Delhi M.Com. 2007]

1743. Solve the following 2-person zero-sum game :

(*a*)

$$\text{Player } A \quad \begin{array}{c} \text{Player } B \\ \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 0 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2003]

(*b*)

$$\text{Player } A \quad \begin{array}{c} \text{Player } B \\ \begin{bmatrix} 4 & -2 & 3 & -1 \\ 1 & 2 & 0 & 1 \\ -2 & 1 & -2 & 0 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2006]

1744. (*a*) Use the notion of dominance to simplify the rectangular game with the following pay-off :

$$\text{Player } A \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{bmatrix} 18 & 4 & 6 & 4 \\ 6 & 2 & 13 & 7 \\ 11 & 5 & 17 & 3 \\ 7 & 6 & 12 & 2 \end{bmatrix} \end{array}$$

Find its graphical solution.

[Madras M.B.A. 2000; Delhi MIB 2011]

(*b*) Following is the pay-off matrix for player A :

$$\text{Player } A \quad \begin{array}{c} \\ I \\ II \\ III \\ IV \end{array} \begin{array}{c} \\ \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 2 & 4 & 3 & 8 & 4 \\ 5 & 6 & 3 & 7 & 8 \\ 6 & 7 & 9 & 8 & 7 \\ 4 & 2 & 8 & 4 & 3 \end{bmatrix} \end{array}$$

Using dominance property, obtain the optimal strategies for both the players and determine the value of the game.

[Delhi PG Dip. in Glob. Bus. Oper. 2011]

1745. A steel company is negotiating with its union for revision of wages to its employees. The management, with the help of a mediator, has prepared a payoff matrix shown below. Plus sign represents wage increase, while negative sign stands for wage decrease. Union has also constructed a table which is comparable to that developed by management. The management does not have the specific knowledge of game theory to select the best strategy or (strategies) for the firm. You have to assist the management on the problem. What game value and strategies are available to the opposing group?

Steel Co. strategies	*Additional costs to Steel Co. (Rs.)* *Union strategies*			
	U_1	U_2	U_3	U_4
C_1	0.25	0.27	0.35	–0.02
C_2	0.20	0.16	0.08	0.08
C_3	0.14	0.12	+0.15	0.13
C_4	0.30	0.14	0.19	0.00

[Delhi M.B.A. (PT.) Dec. 2015]

1746. Two breakfast food manufacturers, *ABC* and *XYZ* are competing for an increased market share. The pay-off matrix, shown in the following table, describes the increase in market share of *ABC* and decrease in market share of *XYZ*.

ABC	*XYZ*			
	Give coupons	*Decrease price*	*Maintain present strategy*	*Increase advertising*
Give coupons	2	–2	4	1
Decrease price	6	1	12	3
Maintain present strategy	–3	2	0	6
Increase advertising	2	–3	7	7

Determine the optimum strategies for both the manufacturers and value of the game.

[Delhi M.B.A. (Oct.) 2010; PG Dip. in Glob. Bus. Oper. 2010]

17:8. ARITHMETIC METHOD FOR $n \times n$ GAMES

When a game matrix cannot be reduced in size by graphical method or by using the principal of dominance, then for $n \times n$ games an easy method for finding the optimum strategies of the two players consists of the following steps :

Step 1. Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ payoff matrix. Obtain a new matrix **C**, whose first column is obtained from **A** by subtracting its 2nd column from 1st; second column is obtained by subtracting **A**'s 3rd column from 2nd and so on till the last column of **A** has been taken care of. Thus **C** is an $n \times (n - 1)$ matrix.

Step 2. Obtain a new matrix **R**, from **A**, by subtracting its successive rows from the preceding ones, in exactly the same manner as was done for columns in *step 1*. Thus **R** is an $(n - 1) \times n$ matrix.

Step 3. Determine the magnitude of *oddments* corresponding to each row and each column of **A**. The oddment corresponding to ith row of **A** is defined as the determinant $|\mathbf{C}_i|$, where $\mathbf{C}_i$ is obtained from **C** by deleting its ith row.

Similarly, oddment (jth column of **A**) = $|\mathbf{R}_j|$, where $\mathbf{R}_j$ is obtained from **R** by deleting its jth column.

Step 4. Write the magnitude of oddments (after ignoring negative signs, if any, against their respective rows and columns).

Step 5. Check whether the sum of row oddments is equal to the sum of column oddments. If so, the oddments expressed as fractions of the grand total yield the optimum strategies. If not, the method fails.

Step 6. Calculate the expected value of game corresponding to the optimum mixed strategy determined above for the Row Player (against any move of the column player).

SAMPLE PROBLEMS

1747. *Consider the following 2×2 game :*

$$\begin{pmatrix} 4 & 7 \\ 6 & 5 \end{pmatrix}$$

(*a*) *Does it have a saddle point?*

(*b*) *Is it correct to state that the value of game G, will satisfy $5 < G < 6$?*

(*c*) *Determine the frequency of optimum strategies by matrix oddment method and find the value of game.*

Solution. (*a*) From the given payoff matrix, we have maximin value = 5 and minimax value = 6. Since maximin value ≠ minimax value, there does not exist a saddle point.

(*b*) Since maximin value ≤ G ≤ minimax value, we must have $5 \le G \le 6$.

(*c*) Subtracting the second column from first, we get

$$\mathbf{C} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

This gives $\quad |\mathbf{C}_1| = 1$ and $|\mathbf{C}_2| = -3.$

Now, subtracting the second row from first, we get

$$\mathbf{R} = (-2 \quad 2).$$

Thus $\quad |\mathbf{R}_1| = 2$ and $|\mathbf{R}_2| = -2.$

∴ The augmented payoff matrix is :

				Row oddment
	4	7		1
	6	5		3
Column oddment	2	2		4

Since, the sums of row oddments and column oddments each total 4, the optimum strategies of the two players are :

$$\text{Row player}: \left(\tfrac{1}{4}, \tfrac{3}{4}\right) \text{ and Column player}: \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

The value of the game is :

$$G = \tfrac{1}{4} \times 4 + \tfrac{3}{4} \times 6 = 11/2 \quad \text{or} \quad 5.5.$$

1748. *Use matrix oddment method to solve the following 3 × 3 game :*

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Solution. We compute the matrices **C** and **R** from the given payoff matrix; and then obtain the row and column oddments. Thus, we have

$$\mathbf{C} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

The oddments are :

$$\mathbf{C}_1 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \quad \mathbf{C}_2 = \begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} = -3, \quad \mathbf{C}_3 = \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} = 3,$$

and $$\mathbf{R}_1 = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3, \quad \mathbf{R}_2 = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3, \quad \mathbf{R}_3 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3.$$

∴ The augmented payoff matrix is

				Row oddments
	0	1	2	3
	2	0	1	3
	1	2	0	3
Column oddments	3	3	3	9

Since the sum of row and column oddments are both equal to 9, the optimum strategies are :

$$\text{Row player}: \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) \text{ and Column player}: \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right).$$

The value of the game $= \tfrac{1}{3} \times 0 + \tfrac{1}{3} \times 2 + \tfrac{1}{3} \times 1 = 1.$

PROBLEMS

1749. Use matrix oddment method to solve the following games :

(*a*)

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

(*b*)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix}$$

1750. Solve the following 3×3 games :

(*a*)

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 7 & 1 & 7 \\ 9 & -1 & 1 \\ 5 & 7 & 6 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{pmatrix} 2 & -1 & -2 \\ 1 & 0 & 1 \\ -2 & -1 & 2 \end{pmatrix} \end{array}$$

[Meerut M.Sc. (Math.) 2000]

1751. Solve the following 3×3 game :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix} \end{array}$$

1752. Solve the following game :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} \end{array}$$

17:9. GENERAL SOLUTION OF $m \times n$ RECTANGULAR GAMES

In a rectangular game with an $m \times n$ payoff matrix, if there does not exist any saddle point and also it is not possible to reduce the size of the game to $m \times 2$ or $2 \times n$ payoff matrix by using the concept of dominance, the following methods are generally used to solve the game :

(*a*) Linear Programming Method, and

(*b*) Iterative Method for approximate solution.

Linear Programming Method

To illustrate the connection between a game problem and a linear programming problem, let us consider an $m \times n$ payoff matrix (a_{ij}) for player A. Let

$$S_m = \begin{bmatrix} A_1, \ldots, A_m \\ p_1, \ldots, p_m \end{bmatrix} \text{ and } S_n = \begin{bmatrix} B_1, \ldots, B_n \\ q_1, \ldots, q_n \end{bmatrix} \text{ where } \sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1,$$

be the mixed strategies for the two players respectively.

Then, the expected gains g_j $(j = 1, \ldots, n)$ of player A against B's pure strategies will be

$$g_1 = a_{11} p_1 + a_{21} p_2 + \ldots + a_{m1} p_m$$
$$g_2 = a_{12} p_1 + a_{22} p_2 + \ldots + a_{m2} p_m$$
$$\vdots$$
$$g_n = a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m$$

and the expected losses l_i $(i = 1, \ldots, m)$ of player B against A's pure strategies will be

$$l_1 = a_{11} q_1 + a_{12} q_2 + \ldots + a_{1n} q_n$$
$$l_2 = a_{21} q_1 + a_{22} q_2 + \ldots + a_{2n} q_n$$
$$\vdots$$
$$l_m = a_{m1} q_1 + a_{m2} q_2 + \ldots + a_{mn} q_n$$

The objective of player A is to select p_i $(i = 1, 2, \ldots, m)$ such that he can maximize his minimum expected gains; and the player B desires to select q_j $(j = 1, 2, \ldots, n)$ that will minimize his expected losses.

Thus, if we let $u = \min_j \sum_{i=1}^{m} a_{ij} p_i$ $(j = 1, 2, \ldots, n)$ and $v = \max_i \sum_{j=1}^{n} a_{ij} q_j$ $(i = 1, 2, \ldots, m)$; the problem of two players could be written as :

Player A

$$\text{Maximize } u = \text{Minimize } \frac{1}{u} = \sum_{i=1}^{m} \frac{p_i}{u} \quad \text{subject to the constraints :}$$

$$\sum_{i=1}^{m} a_{ij} p_i \geq u \quad \text{and} \quad \Sigma p_i = 1, \; p_i \geq 0 \quad (i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n)$$

Player B

$$\text{Minimize } v = \text{Maximize } \frac{1}{v} = \sum_{j=1}^{n} \frac{q_j}{v} \quad \text{subject to the constraints :}$$

$$\sum_{j=1}^{n} a_{ij} q_j \leq v \quad \text{and} \quad \Sigma q_j = 1, \; q_j \geq 0 \quad (j = 1, 2, \ldots, n; \; i = 1, 2, \ldots, m)$$

Assuming that $u > 0$ and $v > 0$, we introduce new variables defined by

$$p_i' = p_i/u \quad \text{and} \quad q_j' = q_j/v \qquad (i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n)$$

Then, the pair of linear programming problems can be re-written as :

Player A

$$\text{Minimize } p_0 = p_1' + p_2' + \ldots + p_m' \quad \text{subject to the constraints :}$$
$$a_{1j} p_1' + a_{2j} p_2' + \ldots + a_{mj} p_m' \geq 1$$
$$p_i' \geq 0 \qquad (i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n)$$

Player B

$$\text{Maximize } q_0 = q_1' + q_2' + \ldots + q_n' \quad \text{subject to the constraints :}$$
$$a_{i1} q_1' + a_{i2} q_2' + \ldots + a_{in} q_n' \leq 1$$
$$q_j' \geq 0 \qquad (i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n)$$

Remarks 1. It is easy to note that the L.P.P.'s of the two players represent a primal-dual pair. Therefore by fundamental theorem of duality one can read off the optimal solution of one player, just from the optimum simplex table of the opponent. That is, we need to solve just one player's L.P.P. by simplex method.

2. Linear programming technique requires all variables to be non-negative and, therefore, to obtain a non-negative value v of the game, the data to the problem, *i.e.*, a_{ij} in the payoff table should all be non-negative. If there are some negative elements in the payoff table, a constant to every element in the payoff table must be added so as to make the smallest element zero; the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

SAMPLE PROBLEM

1753. *Solve the following game by linear programming technique :*

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{pmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 2000]

Solution. Since some of the entries in the payoff matrix are negative, we add a suitable constant to such of the entries to ensure them all positive. Thus, adding a constant $c = 4$ to each element, we get the following revised payoff matrix :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{pmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{pmatrix} \end{array}$$

Let the strategies of the two players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix},$$

where $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$.

The linear programming formulation for the 'two players' problems are :

For Player A

$$\text{Maximize } u \equiv \text{Minimize } \frac{1}{u} = x_1 + x_2 + x_3 \quad \text{subject to the constraints :}$$

$$5x_1 + 7x_2 + 10x_3 \geq 1, \; 3x_1 + 9x_2 + 6x_3 \geq 1,$$

$$7x_1 + x_2 + 2x_3 \geq 1; \quad \text{and} \quad x_j \geq 0 \qquad (j = 1, 2, 3)$$

For Player B

$$\text{Minimize } v = \text{Maximize } \frac{1}{v} = y_1 + y_2 + y_3 \quad \text{subject to the constraints :}$$

$$5y_1 + 3y_2 + 7y_3 \leq 1, \; 7y_1 + 9y_2 + y_3 \leq 1,$$

$$10y_1 + 6y_2 + 2y_3 \leq 1; \quad \text{and} \quad y_j \geq 0 \qquad (j = 1, 2, 3)$$

where $x_j = p_j / u$, $(j = 1, 2, 3)$ and $y_j = q_j / v$, $(j = 1, 2, 3)$; u = minimum expected gain to Player A and v = miximum expected loss to player B.

Let us now solve the problem for player B. By introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$; the iterative simplex table are :

Initial Iteration. Introduce $\mathbf{y}_3$ and drop $\mathbf{y}_4$.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	1	5	3	7	1	0	0
0	y_5	1	7	9	1	0	1	0
0	y_6	1	10	6	2	0	0	1
	$\frac{1}{v}$	0	–1	–1	–1	0	0	0

First Iteration. Introduce $\mathbf{y}_2$ and drop $\mathbf{y}_5$.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
1	y_3	1/7	5/7	3/7	1	1/7	0	0
0	y_5	6/7	44/7	60/7	0	–1/7	1	0
0	y_6	5/8	60/8	36/7	0	–2/7	0	1
	$\frac{1}{v}$	1/7	–2/7	–4/7	0	1/7	0	0

Final Iteration. Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
1	y_3	1/10	2/5	0	1	3/20	−1/20	0
1	y_2	1/10	11/5	1	0	−1/60	7/60	0
0	y_6	1/5	24/5	0	0	−1/5	−3/5	1
	$\frac{1}{v}$	1/5	2/15	0	0	2/15	1/15	0

The expected value or game, therefore, is obtained as

$$v^0 = v - 4 = 5 - 4 = 1.$$

Optimum strategies for player B are

$$q_1^0 = 0, \quad q_2^0 = \frac{1}{10} \times 5 = \frac{1}{2} \quad \text{and} \quad q_3^0 = \frac{1}{10} \times 5 = \frac{1}{2}.$$

Making use of duality, the optimum strategies for player A are obtained as

$$p_1^0 = \frac{2}{15} \times 5 = \frac{2}{3}, \quad p_2^0 = \frac{1}{15} \times 5 = \frac{1}{3} \quad \text{and} \quad p_3^0 = 0.$$

Hence the optimum solution to the given problem is

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad v = 1.$$

PROBLEMS

1754. Two companies A and B are competing for the same product. Their different strategies are given in the following payoff matrix :

$$\begin{array}{cc} & \textit{Company } A \\ \text{Company } B & \begin{bmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{bmatrix} \end{array}$$

Use linear programming to determine the best strategies for both the players. *[Meerut M.Sc. (Math.) 2001]*

1755. For the following payoff matrix, find the value of the game and the strategies of players A and B by using linear programming :

$$\begin{array}{ccc} & & \textit{Player } B \\ & & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{Player } A & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 3 & -1 & 4 \\ 6 & 7 & -2 \end{bmatrix} \end{array}$$

[Delhi M.B.A. 2008]

1756. For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method :

$$\begin{array}{ccc} & & \textit{Player } Q \\ & & \begin{array}{ccc} Q_1 & Q_2 & Q_3 \end{array} \\ \text{Player } P & \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} & \begin{bmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{bmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 1995]

1757. For the following two-person zero-sum game, find the strategy of each player and the value of game :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 8 & -3 & 7 \\ 6 & -4 & 5 \\ -2 & 2 & -3 \end{bmatrix} \end{array}$$

1758. Solve the following 3×3 game :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & -2 & 6 \end{bmatrix} \end{array}$$

[Delhi M.B.A. (PT.) 2009]

1759. In a town there are only two discount stores *ABC* and *XYZ*. Both stores run annual pre-Diwali sales during the first week of October. Sales are advertised through local newspapers with the aid of an advertising firm. *ABC* stores constructed following payoff in units of Rs. 1,00,000. Find the optimum strategies for both stores and the value of the game :

$$\begin{array}{ccc} & & \textit{Strategies of XYZ} \\ & & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix} \\ \text{Strategies of } ABC & & \end{array}$$

[Bombay M.M.S. 1995]

1760. Firm *X* is fighting for its life against the determination of firm *Y* to drive it out of the industry. Firm *X* has the choice of increasing price, leaving it unchanged, or lowering it. Firm *Y* has the same three options. Firm *X*'s gross sales in the event of each of the pairs of choices are shown below :

Firm X's pricing strategies	*Firm Y's pricing strategies*		
	Increase price	*Do not change*	*Reduce price*
Increase price	90	80	110
Do not change	110	100	90
Reduce price	120	70	80

Assuming firm *X* as the maximizing one, formulate the problem as a linear programming problem.

[Delhi M.Com. 2009]

1761. Two players *A* and *B* play the following game. *A* has a bag containing three coins, one worth 4 units, one 6 units and the rest 9 units. *A* takes one coin from the bag and before exposure, *B* guesses. If *B* is right he takes the coin and if wrong he pays to *A* the same worth. Find *A*'s and *B*'s best strategies by making use of simplex method.

Iterative Method of Approximate Solution

Iterative method can be applied to solve 3×3 or higher games which cannot be solved by Arithmetic method and are extremely tedious to be solved by method of linear programming. This method gives an approximate solution for the value of the game and the true value can be determined to any degree of accuracy. Optimum strategies can also be determined but not so satisfactorily. The method assumes that each player acts under the assumption that past is the best guide for the future and will play in such a manner so as to maximize the expected gain or to minimize the expected loss.

The method is best explained through a sample problem given below :

SAMPLE PROBLEM

1762. *Solve the following game approximately :*

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{pmatrix} \end{array}$$

[Delhi B.Sc. (Stat.) 1999; Bharathidasan B.Com. 1999]

Solution. Let the player *A* select the second row, being the superior one of his other strategies and place it under the matrix. Player *B* examines this row and chooses a column corresponding to the

smallest number of this row. This is first column. First column is then placed to the right of the matrix. Player A examines this column and chooses a row corresponding to the *largest* number in this column. This is the first row. First row is then added to the row last chosen. Player B then chooses the column corresponding to the smallest number in the new row and adds this column to the column last chosen. In case of a tie the player will select that row or column which is different from his last choice. The procedure is repeated for a finite number of iterations. Here below, ten iterations are presented with the smallest elements in each succeeding row with the largest elements in each succeeding column being encircled.

	Player B											
Player A	1	−1	−1	1	0	1	0	−1	0	−1	0	4/10
	−1	−1	3	−1	−2	−3	−4	−1	−2	1	0	3/10
	−1	2	−1	−1	1	0	2	1	0	−1	−2	3/10
	−1	−1	3									
	0	−2	2									
	−1	0	1									
	0	−1	0									
	−1	1	−1									
	−2	3	−2									
	−1	2	−3									
	−2	1	0									
	−1	0	−1									
	−2	−1	2									
	4/10	2/10	4/10									

The approximate strategies after 10 iterations are found by dividing the number of encircled elements in each row or column by total number of iterations. Thus, we shall have

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 4/10 & 3/10 & 3/10 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 4/10 & 2/10 & 4/10 \end{bmatrix}$$

The upper bound for the value of the game can be determined by dividing the largest element '2', in the last column by the total number of iterations. Likewise, the lower bound can be determined by dividing the smallest element '−2', in the last row by the total number of iterations.

Thus the approximate value of game is

$$-\frac{2}{10} \le v \le \frac{2}{10}, \quad i.e., \quad (-1/5) \le v \le (1/5).$$

PROBLEMS

1763. Find approximate value of the game and strategies by iterative method whose payoff matrix is given below. (Try 10 iterations) :

$$\text{Player } A \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{ccc} 1 & 2 & 1 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \end{array}$$

[Madurai M.B.A. 2009]

1764. Solve the following game approximately :

$$\begin{array}{cc} & \textit{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix} \end{array}$$

1765. Solve the following matrix game approximately :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{pmatrix} 2 & 3 & -1 & 0 \\ 5 & 4 & 2 & -2 \\ 1 & 3 & 8 & 2 \end{pmatrix} \end{array}$$

1766. In a well-known children's game, each player says 'stone' or 'scissors' or 'papers'. If one says 'store' and the other 'scissors' then the former wins a rupee. Similarly 'scissors' beats 'paper' and 'paper' beats 'stone', *i.e.*, the player calling the former word wins a rupee. If the two players name the same item, then there is a tie, *i.e.*, there is no payoff. Write down the payoff matrix. Find the value of the game and hence write down the optimal strategies of both players. ***[AIMA Dip. in Management (Dec.) 1995]***

[***Hint :*** Payoff matrix is :

$$\begin{array}{cc} & \textit{Player B} \\ \text{Player } A & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{array}{l} \text{Stone} \\ \text{Paper} \\ \text{Scissors} \end{array} \\ & \begin{array}{ccc} \text{Stone} & \text{Paper} & \text{Sciessors} \end{array} \end{array}$$

17:10. GAME AGAINST PASSIVITY

In many a situation, the opponent may be somewhat *passive* in character. For example, the game theory can also be applied against nature which, by itself, cannot take any decision. Only the opponent would guess what nature can thrust upon the system. Let us take a simple example :

A particular day was fixed to launch a new rocket. The probability of rain on that day was estimated to be 0.1. It was also found that being caught in rain without firing the rocket was 5 times more inconvenient as having to fire the rocket on such a bad weather.

We can easily describe the given situation in terms of the following payoff matrix :

$$\begin{array}{ccc} & & \textit{Nature} \\ & & \begin{array}{cc} \textit{Rain} & \textit{No Rain} \end{array} \\ \text{Man} & \begin{array}{l} \text{Fire Rocket} \\ \text{Fire No Rocket} \end{array} & \begin{pmatrix} 0 & -1 \\ -5 & 0 \end{pmatrix} \end{array}$$

We see that the man's expected gain if rocket is fired is –9/10 and the expected gain if rocket is not fired is –5/10. Therefore, firing the rocket won't be a good proposition. But this proposition itself would be flexible so long as the probability of rain changes.

17:11. LIMITATIONS AND EXTENSIONS

We have restricted our discussion to two-person zero-sum games only. However, there are practically no applications of these games to the real world situations, because of the underlying assumptions. The assumption that each player not acts rationally so that his preference of the outcomes is determined by their associated payoffs, but also he assumes that the opponent is also rational in that sense. Besides, it is also assumed that both the players are having complete knowledge about the

strategies available to them. However, in real world situations, it is only but rarely that each of the persons would have complete knowledge about all the strategies available to his competitor, as also of the *exact* payoff values associated with various combinations. Business managers do not always adopt conservative approach in their decisions and prefer to take risks.

Further in a given situation, there are likely to be more than two participants and the sum of gains and losses of the opponents may not be equal to zero. Several extensions of the two-person zero-sum games are possible, like the non-zero sum games, *n*-person games, games with uncertain payoffs, games involving an infinite number of pure strategies...and so on. A discussion of these is beyond the scope of this book.

MULTIPLE CHOICE QUESTIONS

1. A two person game is said to be zero-sum, if
(*a*) gain of one player is exactly matched by a loss to the other so that their sum is equal to zero.
(*b*) gain of one player does not match the loss to the other.
(*c*) both the players must have an equal number of strategies.
(*d*) diagonal entries of the pay-off matrix are zero.

2. For a two person zero-sum game, the value of game can be
(*a*) determined only if the pay-off matrix has a saddle point.
(*b*) positive, negative or zero.
(*c*) determined only if the game is fair.
(*d*) none of the above.

3. A game is said to be fair, if
(*a*) upper value is more than lower value of the game.
(*b*) upper and lower values of the game are not equal.
(*c*) upper and lower values of the game are same and zero.
(*d*) none of the above.

4. When maximin and minimax values of the game are same, then
(*a*) there is a saddle point. (*b*) solution does not exit.
(*c*) strategies are mixed. (*d*) none of the above.

5. The pay-off value for which each player in a game always selects the same strategy is called the
(*a*) equilibrium point, (*b*) saddle point,
(*c*) both (*a*) and (*b*), (*d*) none of the above.

6. For a two person game with *A* and *B*, the minimising and the maximising players, the optimum strategies are :
(*a*) Minimax for *A* and Maximin for *B*, (*b*) Maximax for *A* and Minimax for *B*,
(*c*) Minimin for *A* and Maximin for *B*, (*d*) Maximin for *A* and Minimax for *B*.

7. If the pay-off matrix of a game is transposed,
(*a*) value of game does not change.
(*b*) saddle point of the game, if exists, changes.
(*c*) optimum strategies of both players does not change.
(*d*) none of the above.

8. A mixed strategy game can be solved by
(*a*) matrix method, (*b*) algebraic method,
(*c*) graphical method, (*d*) all of the above.

9. The size of the pay-off matrix of a game can be reduced by using the principle of
(*a*) dominance, (*b*) rotation reduction,
(*c*) game inversion, (*d*) game transpose.

10. In a two person zero-sum game

(*a*) the solution is based on the assumption that player *A* will always play his strategy first and the other player *B*, would play his strategy thereafter.

(*b*) player *A*'s strategy is determined on the basis of maximum criterion, whereas player *B*'s on the basis of minimax criterion.

(*c*) game is said to be fair if both the players have equal number of strategies.

(*d*) the value of game for both the players need not be the same.

11. In a two-person zero-sum game

(*a*) mixed strategy for a player can involve no more than two moves.

(*b*) mixed strategies are adopted by the players when they are doubtful about the optimum strategies for them.

(*c*) strategies of the players are played in random order in the ratio determined by the optimum mix.

(*d*) if the optimum solution requires one player to use a pure strategy, the other player must also do the same.

12. Games which involve more than two players are called

(*a*) biased games, (*b*) negotiable games,

(*c*) conflicting games, (*d*) *n*-person games.

13. Which of the following is not correct ?

(*a*) The graphic approach to the solution of games can be applied when one of the players has two strategies available.

(*b*) The dominance principle suggests that strategies of one player are dominating the strategies of the other.

(*c*) It is possible that a convex linear combination of the two strategies dominates a third strategy of the same player.

(*d*) If the largest entry in the pay-off matrix is negative, the game should favour player *B* irrespective of what the optimum strategies of the players are.

14. Which of the following is not correct ?

(*a*) Every two-person zero-sum game cannot be represented by a pair of LPP's with primal dual relationship.

(*b*) The constant value, if added to each element of the pay-off matrix to formulate the given problem as an LPP, should be subtracted from the value of the game, determined from the solution to the LPP.

(*c*) Any two person game can be formulated and solved as an LPP.

(*d*) In a $2 \times n$ or $m \times 2$ game, each of the players can mix at most two strategies if the multiple optimum solutions do not exist.

15. Which of the following is not correct ?

(*a*) A two-person zero-sum game wherein the two players have equal number of strategies can be solved by 'Arithmetic Method', if the sum of row oddments equals the sum of column oddments.

(*b*) In a two person zero-sum game, Row/Column player is said to dominate the other player if he has more strategies to play.

(*c*) Bidding problems can be classified into open bids and closed bids.

(*d*) In real world, the applications of game theory is very restrictive because of the nature of assumptions make.

ANSWERS

1. (*a*)	**2.** (*b*)	**3.** (*c*)	**4.** (*a*)	**5.** (*c*)
6. (*a*)	**7.** (*b*)	**8.** (*d*)	**9.** (*a*)	**10.** (*b*)
11. (*c*)	**12.** (*d*)	**13.** (*b*)	**14.** (*a*)	**15.** (*b*).

REVIEW QUESTIONS

1. What is game theory? Explain its advantages and disadvantages. *[Panjab Tech. Univ. M.B.A. (Dec.) 2008]*
2. What are the characteristics of a game? *[Panjab Tech. Univ. M.B.A. (Dec.) 2011]*
3. Explain two-person zero-sum game by suitable example. *[Panjab Tech. Univ. M.B.A. (Dec.) 2012]*
4. (*a*) Explain the terms : Pure strategy, Mixed strategy, Saddle point, Competitive games, Payoff matrix, and Rectangular games. *[Madras M.B.A. 1999; Panjab Tech. Univ. M.B.A. (Dec.) 2010]*
   (*b*) Define mixed strategy in a game. *[Annamalai M.B.A. 2010]*
5. Define the term 'strategy' and 'optimal strategy' with reference to Game Theory. *[Meerut M.Sc. (Math.) 1999]*
6. Construct maximax and maximin decision strategies. Under what circumstances is each appropriate? *[Panjab Tech. Univ. M.B.A. (June) 2012]*
7. What are the properties of a game? Explain the "best strategy" on the basis of minimax criterion of optimality. *[Delhi M.B.A. 2001]*
8. Explain the method of finding optimal solution of two-person zero-sum games. *[Madurai M.Com. 2003]*
9. Explain the concept of value of a game. *[Madurai M.Com. (Nov.) 2002]*
10. Let (a_{ij}) be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then show that $\overline{v} \geq \underline{v}$. *[Delhi B.Sc. (Stat.) 1995]*
11. Define saddle point. Is it necessary that a game should always possess a saddle point? *[Madras M.B.A., 2004]*
12. What do you mean by the term 'saddle point'? How do you determine the saddle point in case of pure strategy game? *[Delhi MIB 2011]*
13. Show that every two-person 'zero-sum' game with mixed strategies has a solution. *[Calicut M.Sc. (Stat.) 1994]*
14. Show that for any zero-sum two-person game where there is no saddle point and for which A's payoff matrix is
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$
the optimal strategies (x_1, x_2) and (y_1, y_2) for A and B respectively are determined by
$$\frac{x_1}{x_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{y_1}{y_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}.$$
What is the value of the game to A?
15. For the game $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, where a, b, c, d are all non-negative, prove that the optimal strategies are :
$$\left[\frac{c+d}{a+b+c+d}, \frac{a+b}{a+b+c+d}\right], \left[\frac{b+d}{a+b+c+d}, \frac{a+c}{a+b+c+d}\right] \text{ and } v = \frac{ad - bc}{a+b+c+d}.$$
[Delhi B.Sc. (Stat.) 1998; Meerut M.Sc. (Math.) 1996]
16. Discuss dominance with reference to game theory. *[Annamalai M.B.A. 2009]*
17. Explain the algebraic method for solving a rectangular game. *[Delhi B.Sc. (Stat.) 2002]*
18. Show how a two-person zero-sum game problem can be reduced to a linear programming problem. *[Delhi B.Sc. (Stat.) 1996; Calicut M.Sc. (Stat.) 1994]*
19. Explain the method of solving a zero-sum two-person game as a linear programming problem. *[Delhi B.Sc. (Math.) 1995]*
20. Explain the graphical method of solving $2 \times n$ and $m \times 2$ games.
21. What are the major limitations and applications of the Game theory? *[Delhi M.Com. 1997]*
22. State the assumptions underlying game theory. Discuss its importance in business decisions.