7

Integer Programming

"God made the integers, rest is the work of man"

7:1. INTRODUCTION

A linear programming problem in which all or some of the decision variables are constrained to assume non-negative integer values is called an **Integer Programming Problem**. This type of problem is of particular importance in business and industry, where quite often, the fractional solutions are unrealistic because the units are not divisible. For example, it is absurd to speak of 2.3 men working on a project or 8.7 machines in a workshop. The integer solution to a problem can, however, be obtained by rounding off the optimum values of the variables to the nearest integer values. But, it is generally inaccurate to obtain an integer solution by rounding off in this manner, for there is no guarantee that the deviation from the 'exact' integer solution will not be too large to retain the feasibility.

The linear programming problem with the additional requirements that the variables can take on only, integer values may have the following mathematical form

$$\text{Maximize or Minimize } z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \text{ subject to the constraints :}$$

$$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n = b_i, \qquad i = 1, 2, \ldots, m$$

and

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n$$

where x_j are integer valued for $j = 1, 2, \ldots, p\ (p \leq n)$.

7:2. PURE AND MIXED INTEGER PROGRAMMING PROBLEMS

An integer programming problem in which all variables are required to be integers is called a *pure, or all-integer programming problem*. For example, the LPP

$$\text{Minimize } z = 9x_1 + 10x_2 \text{ subject to the constraints :}$$

$$4x_1 + 3x_2 \geq 4,\ x_1 \leq 8,\ x_2 \leq 8\,;$$

$$\text{where } x_1 \geq 0,\ x_2 \geq 0 \text{ and are integers}$$

is a pure integer linear programming problem.

On the other hand, an integer linear programming problem in which only some of the variables are required to be integers, is known as a **mixed integer programming problem**. For example, the LPP

$$\text{Maximize } z = -3x_1 + x_2 + 3x_3 \text{ subject to the constraints :}$$

$$-x_1 + 2x_2 + x_3 \leq 4,\ 4x_2 - 3x_3 \leq 2\,;$$

$$\text{where } x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \text{ and } x_1 \text{ as well as } x_3 \text{ are integers}$$

is a mixed-integer linear programming problem. Note that x_2 is not required to be an integer.

An integer programming problem in which all the variables must have integer values only zero or unity, is called the **zero-one integer programming problem**.

7:3. GOMORY'S ALL-I.P.P. METHOD

A systematic procedure for obtaining an optimum integer solution to an all-integer programming problem was first suggested by R.E. Gomory. His method starts without taking into consideration the integer requirements. If the solution so obtained is integral then the current solution is optimum. However if some of the basic variables are not integer valued then an additional linear constraint called *Gomory's constraint (fractional cut)* is generated. After having generated a linear constraint (cutting plane), it is added as the last row, of the optimum simplex table indicating that the solution is no longer feasible. The modified problem is then solved by using dual simplex method. An optimum integer solution is obtained if all the variables in the solution are integer valued, otherwise another Gomory constraint is added and the procedure is repeated. The optimum integer solution will be reached eventually after necessary new constraints have been added to drive away all the superior non-integer solutions.

7:4. CONSTRUCTION OF GOMORY'S CONSTRAINTS

To illustrate construction of a Gomory's constraint, let us consider a linear programming problem for which an optimum *non-integer* basic feasible solution has been attained as displayed in the simplex table below :

$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
$\mathbf{y}_2$	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}
$\mathbf{y}_3$	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}
z	y_{00}	y_{01}	y_{02}	y_{03}	y_{04}

Clearly, the optimum basic feasible solution is given by

$$\mathbf{x}_B = [x_2, x_3] = [y_{10}, y_{20}]\,;\ \text{max. } z = y_{00}.$$

Since $\mathbf{x}_B$ is a non-integer solution, we assume, for the sake of exposition only, that y_{10} is *fractional*.

Now, the constraint equation

$$y_{10} = y_{11}x_1 + y_{12}x_2 + y_{13}x_3 + y_{14}x_4$$

reduces to

$$y_{10} = y_{11}x_1 + x_2 + y_{14}x_4. \qquad \text{...(1)}$$

Because x_2 and x_3 are basic variables, we must have $\mathbf{B} = \mathbf{I}_2 = (\mathbf{y}_2\ \ \mathbf{y}_3)$. This implies that $y_{12} = 1$ and $y_{13} = 0$.

Now, since $y_{10} \geq 0$, the fractional part of y_{10} must also be non-negative. We split over each of the y_{1j} in (1) into an integral part I_{1j}, and a non-negative fractional part*, f_{1j}, for $j = 0, 1, 2, 3, 4$. After this decomposition, (1) may be written as

$$I_{10} + f_{10} = (I_{11} + f_{11})x_1 + x_2 + (I_{14} + f_{14})x_4$$

or

$$f_{10} - f_{11}x_1 - f_{14}x_4 = x_2 + I_{11}x_1 + I_{14}x_4 - I_{10} \qquad \text{...(2)}$$

A comparison between (1) and (2) suggests that if we add an additional constraint in such a way that the L.H.S. of (2) is an integer, then we shall be forcing the non-integer y_{10} towards an integer (and hence the corresponding non-integer solution towards an integer one). This is what we precisely want. The desired Gomory's constraint is $f_{10} - f_{11}x_1 - f_{14}x_4 \leq 0$. To see the truth, let it be possible to have $f_{10} - f_{11}x_1 - f_{14}x_4 = h$, where $h > 0$ is an integer. Then $f_{10} = h + f_{11}x_1 + f_{14}x_4$ is

*If a y_{1j} is negative, even then a non-negative fractional part of it may be separated. For example, $-5/8$ can be written as $-1 + 3/8$ thereby giving $3/8$ as a fractional part (non-negative) of it.

obviously greater than one. This contradicts the fact that $0 < f_{1j} < 1$ for $j = 0, 1, 2, 3, 4$. Thus, the fractional cut is

$$f_{11}x_1 + f_{14}x_4 \geq f_{10} \quad \text{or} \quad -f_{11}x_1 - f_{14}x_4 \leq -f_{10}$$

or

$$-f_{11}x_1 - f_{14}x_4 + G_1 = -f_{10},$$

where G_1 is a slack variable in the above first Gomory constraint or fractional cut.

This additional constraint is to be included in the given L.P.P. in order to move further towards obtaining an optimum all-integer solution. After the addition of this constraint, the optimum simplex table looks like as given below :

y_B	x_B	y_1	y_2	y_3	y_4	G_1
y_1	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	0
y_2	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}	0
G_1	$-f_{10}$	$-f_{11}$	0	0	$-f_{14}$	1
z	y_{00}	y_{01}	y_{02}	y_{03}	y_{04}	y_{05}

Since $-f_{10}$ is negative, the optimum solution is infeasible and thus the dual simplex method is to be applied for obtaining an optimum feasible solution. After obtaining this solution, the above referred procedure is applied for constructing second fractional cut. The process is to be continued so long as an all-integer solution has not been obtained.

7:5. FRACTIONAL CUT METHOD—ALL INTEGER LPP

An iterative procedure for the solution of an all integer linear programming problem by Gomory's fractional cut method (or cutting plane method) may be summarized in the following steps :

Step 1. Express the given linear programming problem into its standard form and determine an optimum solution by using simplex method ignoring the integer value restriction.

Step 2. Test the integrality of the optimum solution.

(*a*) If the optimum solution admits all-integer values, an optimum basic feasible solution is attained.

(*b*) If the optimum solution does not include all-integer values, then move on to next *step*.

Step 3. Choose the largest fractional value of the basic variables. Let it be f_{k0}.

Step 4. Express each of the negative fractions, if any, in the kth row of the optimum simplex table as the sum of a negative integer and a non-negative fraction.

Step 5. Generate the Gomorian constraint (fractional cut) in the form

$$G_1 = -f_{k0} + f_{k1}x_1 + f_{k2}x_2 + \ldots + f_{kn}x_n,$$

where $0 \leq f_{kj} < 1$ and $0 < f_{k0} < 1$.

Step 6. Add the Gomorian constraint generated in *step 5* at the bottom of the optimum simplex table. Use dual simplex method to find an improved optimum solution.

Step 7. Go to *step 2* and repeat the procedure until an optimum basic feasible all-integer solution is obtained.

SAMPLE PROBLEM

701. *Find the optimum integer solution to the following LPP :*

Maximize $z = x_1 + 4x_2$ *subject to the constraints :*

$$2x_1 + 4x_2 \leq 7, \quad 5x_1 + 3x_2 \leq 15;$$

$x_1, x_2 \geq 0$ *and are integers.* **[Annamalai M.B.A. (Nov.) 2010]**

Solution.

Step 1. Introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$, an initial basic feasible solution is $s_1 = 7$ and $s_2 = 15$. Using simplex method an optimum non-integer solution is obtained and is given in the following simplex table :

Initial Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4
4	y_2	7/4	1/2	1	1/4	0
0	y_4	39/4	7/2	0	–3/4	1
		z (= 7)	1	0	1	0

Step 2. Since the optimum solution is not integer valued, we consider only the fractional parts of $x_{B1} = \frac{7}{4}\left(= 1 + \frac{3}{4}\right)$ and $x_{B2} = \frac{39}{4}\left(= 9 + \frac{3}{4}\right)$.

Step 3. Maximum $\{f_1, f_2\}$ = maximum $\left\{\frac{3}{4}, \frac{3}{4}\right\} = \frac{3}{4}$, *i.e.*, both f_1 and f_2 are equal. So, we arbitrarily select any one of these. Let us choose f_2.

Step 4. In the second row, since $y_{23} = -3/4$, we write $y_{23} = -1 + 1/4$.

Step 5. Let G_1 be the first Gomorian slack. Then, we write

$$G_1 = -f_{20} + f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 = -\frac{3}{4} + \frac{1}{2}x_1 + 0 \cdot x_2 + \frac{1}{4}x_3 + 0 \cdot x_4$$

Step 6. Adding this additional constraint in the optimum simplex table, we have

First Iteration. Drop G_1 and introduce y_1.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1
4	y_2	7/4	1/2	1	1/4	0	0
0	y_4	39/4	7/2	0	–3/4	1	0
0	G_1	–3/4	–1/2	0	–1/4	0	1
		z (= 7)	1	0	1	0	0

As $x_{B3} = -3/4$ only is negative, this basic variable leaves the basis. Further, since max. $\left\{\frac{(z_j - c_j)}{y_{3j}}, y_{3j} < 0\right\}$ = max. $\left\{\frac{1}{-1/2}, \frac{1}{-1/4}\right\} = -2$, y_1 enters the basis, *i.e.*, x_1 becomes basic variable in place of G_1.

Second Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1
4	y_2	1	0	1	0	0	1
0	y_4	9/2	0	0	–5/2	1	7
1	y_1	3/2	1	0	1/2	0	–2
		z (= 11/2)	0	0	1/2	0	2

Since, the optimum solution is still non-integral, we introduce the second Gomorian constraint.

Now, $x_{B2} = \frac{9}{2}\left(= 4 + \frac{1}{2}\right)$ and $x_{B3} = \frac{3}{2}\left(= 1 + \frac{1}{2}\right)$

$\therefore$ maximum $\{f_2, f_3\}$ = maximum $\left\{\frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}$, *i.e.*, both f_2 and f_3 are equal. So, let us choose $f_2 = 1/2$ and write $y_{23} = -6 + 1/2$.

$$\therefore \quad G_2 = -f_{20} + f_{21}x_1 + f_{22}x_2 + f_{23}x_3 + f_{24}x_4 = -\frac{1}{2} + 0 \cdot x_1 + 0 \cdot x_2 + \frac{1}{2}x_3 + 0 \cdot x_4$$

Adding this additional constraint in the second iterative table, we have

Third Iteration. Drop G_2 and introduce y_3.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1	G_2
4	y_2	1	0	1	0	0	1	0
0	y_4	9/2	0	0	–5/2	1	7	0
1	y_1	3/2	1	0	1/2	0	–2	0
0	G_2	–1/2	0	0	–1/2*	0	0	1
	z	11/2	0	0	1/2	0	2	0

Final Iteration. Optimum Solution in Integers.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	G_1	G_2
4	y_2	1	0	1	0	0	1	0
0	y_4	7	0	0	0	1	7	–5
1	y_1	1	1	0	0	0	–2	1
0	y_3	1	0	0	1	0	0	–2
	z	5	0	0	0	0	2	1

The table shows that an optimum basic feasible integer solution has been obtained. Hence, the optimum solution is

$$x_1 = 1,\ x_2 = 1 \text{ and maximum } z = 5.$$

702. *Solve the following integer linear programming problem using the cutting-plane algorithm :*

Maximize $z = 3x_1 + x_2 + 3x_3$ *subject to the constraints :*

$$-x_1 + 2x_2 + x_3 \le 4,\ \ 4x_2 - 3x_3 \le 2,\ \ x_1 - 3x_2 + 2x_3 \le 3\ ;$$

x_1, x_2 *and* x_3 *all are non-negative integers.*

Solution. Introducing slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective constraints, an initial basic feasible solution is $s_1 = 4$, $s_2 = 2$ and $s_3 = 3$. Using, now, simplex method an optimum non-integer solution is obtained as given in the following simplex table :

Initial Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
3	y_3	10/3	0	0	1	4/9	1/9	4/9
1	y_2	3	0	1	0	1/3	1/3	1/3
3	y_1	16/3	1	0	0	1/9	7/9	10/9
		z (= 87/3)	0	0	0	2	3	5

Since, $x_{B1} = \frac{10}{3}\left(= 3 + \frac{1}{3}\right)$ and $x_{B3} = \frac{16}{3}\left(= 5 + \frac{1}{3}\right)$, we choose maximum $\{f_1, f_3\} = \frac{1}{3}$, *i.e.*, both f_1 and f_3 are equal. So, let us choose $f_1 = \frac{1}{3}$.

Let G_1 be the first Gomorian slack. Then, we write

$$G_1 = -f_{10} + f_{11}x_1 + f_{12}x_2 + f_{13}x_3 + f_{14}s_1 + f_{15}s_2 + f_{15}s_3$$

$$= -\frac{1}{3} + 0.x_1 + 0.x_2 + 0.x_3 + \frac{4}{9}s_1 + \frac{1}{9}s_2 + \frac{4}{9}s_3.$$

By the addition of this constraint in the above optimum simplex table, we get

First Iteration. Drop G_1 and introduce $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
3	$\mathbf{y}_3$	(3 + 1/3)	0	0	0	4/9	1/9	4/9	0
1	$\mathbf{y}_2$	3	0	1	0	1/3	1/3	1/3	0
3	$\mathbf{y}_1$	(5 + 1/3)	1	0	1	1/9	7/9	10/9	0
0	$\mathbf{G}_1$	−1/3	0	0	0	−4/9	−1/9	−4/9	1
		z (= 87/3)	0	0	0	2	3	5	0

Since, $\mathbf{x}_{B4}\left(=-\frac{1}{3}\right)$ only is negative, G_1 leaves the basis. Further, max. $\left\{\frac{2}{-4/9}, \frac{3}{-1/9}, \frac{5}{-4/9}\right\} = -\frac{9}{2}$. Therefore, $\mathbf{y}_4$ enters the basis, thereby, giving $y_{44}\left(=-\frac{4}{9}\right)$ as the leading element.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
3	$\mathbf{y}_3$	3	0	0	1	0	0	0	1
1	$\mathbf{y}_2$	2 + 3/4	0	1	0	0	1/4	0	3/4
3	$\mathbf{y}_1$	5 + 1/4	1	0	0	0	3/4	1	1/4
0	$\mathbf{y}_4$	3/4	0	0	0	1	1/4	1	−9/4
		z (= 55/2)	0	0	0	0	5/2	3	9/2

Here, max. $\left\{\frac{3}{4}, \frac{1}{4}\right\} = \frac{3}{4}$. This suggests that the second fractional cut (Gomorian constraint) will be :

$$\mathbf{G}_2 = -\frac{3}{4} + \frac{1}{4}s_1 + \frac{3}{4}G_1$$

By adding this constraint in the optimum simplex table, we have

Second Iteration. Drop G_2 and introduce G_1.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$	$\mathbf{G}_2$
3	$\mathbf{y}_3$	3	0	0	1	0	0	0	1	0
1	$\mathbf{y}_2$	11/4	0	1	0	0	1/4	0	3/4	0
3	$\mathbf{y}_1$	21/4	1	0	0	0	3/4	1	1/4	0
0	$\mathbf{y}_4$	3/4	0	0	0	1	1/4	1	−9/4	0
0	$\mathbf{G}_2$	−3/4	0	0	0	0	−1/4	0	−3/4	1
		z (= 55/2)	0	0	0	0	5/2	3	9/2	0

Since $y_{57}\left(=-\frac{3}{4}\right)$ is the leading element, the next iterative simplex table is :

Final Iteration. Optimum Solution in Integers.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$	$\mathbf{G}_2$
3	$\mathbf{y}_3$	2	0	0	1	0	−1/3	0	0	4/3
1	$\mathbf{y}_2$	2	0	1	0	0	0	0	0	1
3	$\mathbf{y}_1$	5	1	0	0	0	2/3	1/9	0	1/3
0	$\mathbf{y}_4$	3	0	0	0	1	1	1	0	−3
0	$\mathbf{G}_1$	1	0	0	0	0	1/3	0	1	−4/3
		z (= 23)	0	0	0	0	1	1/3	0	6

This table shows that an optimum basic feasible integer solution has been reached. Hence, the optimum integer valued solution is :

$$x_1 = 5, \ x_2 = 2 \text{ and } x_3 = 2, \text{ with maximum } z = 23.$$

703. *A company manufactures three products P_1, P_2 and P_3 which yield per unit profit of Rs. 200, Rs. 400 and Rs. 300 respectively. Each of these products is processed on three different machines. The time required on each machine per unit of the product is given below :*

Product	Time required (hours per unit)		
	Machine 1	*Machine 2*	*Machine 3*
P_1	30	20	10
P_2	40	10	30
P_3	20	20	20
Time available (hours)	600	400	800

How many products of each type should be produced to maximise profit ?

Fractional units of product are not permissible.

Solution. Let x_1, x_2 and x_3 be the number of units of products P_1, P_2 and P_3 respectively. Then, the appropriate mathematical formulation of the given problem is :

$$\text{Maximize } z = 200x_1 + 400x_2 + 300x_3 \text{ subject to the constraints :}$$

$$30x_1 + 40x_2 + 20x_3 \le 600 \qquad \textit{(Time constraint on } M_1\textit{)}$$

$$20x_1 + 10x_2 + 20x_3 \le 400 \qquad \textit{(Time constraint on } M_2\textit{)}$$

$$10x_1 + 30x_2 + 20x_3 \le 800 \qquad \textit{(Time constraint on } M_3\textit{)}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \text{ and are integers.}$$

Introducing the slack variables $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$ in the respective constraints and then using simplex method, a non-integer optimum solution is displayed below :

Initial Iteration. Non-integer Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
400	y_2	20/3	1/3	1	0	1/3	–1/3	0
300	y_3	50/3	5/6	0	1	–1/6	2/3	0
0	y_6	80/3	–5/3	0	0	–2/3	–1/3	1
		z (= 23,000/3)	550/3	0	0	250/3	200/3	0

Here, $x_{B_1} = \frac{20}{3} = 6 + \frac{2}{3}$, $x_{B_2} = \frac{50}{3} = 16 + \frac{2}{3}$ and $x_{B_3} = \frac{80}{3} = 26 + \frac{2}{3}$. Thus, all the three fractional parts are same, *i.e.*, $f_1 = f_2 = f_3 = \frac{2}{3}$. So, let us choose $f_2 = \frac{2}{3}$ for the construction of Gomorian constraint. Therefore, from the above table, second row is written as :

$$16 + \frac{2}{3} = \frac{5}{6}x_1 + x_3 - \left(\frac{1}{6}\right)s_1 + \frac{2}{3}s_2$$

or

$$\left(16 + \frac{2}{3}\right) = \left(0 + \frac{5}{6}\right)x_1 + x_3 + \left(-1 + \frac{5}{6}\right)s_1 + \left(0 + \frac{2}{3}\right)s_2.$$

Let G_1 be the first Gomorian slack. Then, the corresponding cut or Gomorian constraint is :

$$-\frac{2}{3} = G_1 - \frac{5}{6}x_1 - \frac{5}{6}s_1 - \frac{2}{3}s_2.$$

On the addition of this constraint in the optimum simplex table (given above), we get

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	G_1
400	y_2	20/3	1/3	1	0	1/3	–1/3	0	0
300	y_3	50/3	5/6	0	1	–1/6	2/3	0	0
0	y_6	80/3	–5/3	0	0	–2/3	–1/3	1	0
0	G_1	–2/3	–5/6	0	0	–5/6	–2/3	0	1
		z (= 2000/3)	550/3	0	0	250/3	200/3	0	0

Since, $\mathbf{x}_{B_4}\left(=-\frac{2}{3}\right)$ is only negative, G_1 leaves the current basis. Further, maximum $\left\{\frac{550/3}{-5/6}, \frac{250/3}{-5/6}, \frac{200/3}{-2/3}\right\} = -100$. This indicate that $\mathbf{y}_5$ enter the basis and $y_{45}\left(=-\frac{2}{3}\right)$ becomes the leading element. The next iterative simplex table is :

Final Iteration. Optimum Solution in Integers.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	G_1
400	y_2	7	3/4	1	0	3/4	0	0	−1/2
300	y_3	16	0	0	1	−1	0	0	1
0	y_6	27	−5/12	0	0	−1/4	0	1	−1/2
0	y_5	1	5/4	0	0	5/4	1	0	−3/2
		z (= 7600)	100	0	0	0	0	0	100

Hence, the all integer optimum solution is $x_1 = 0$, $x_2 = 7$, $x_3 = 16$ with maximum $z = 7{,}600$, *i.e.*, the company should manufacture 7 units of product P_2 and 16 units of product P_3 only to get the maximum profit of Rs. 7,600.

PROBLEMS

704. Find the optimum solution to the I.P.P. :

Maximize $z = x_1 - 2x_2$ subject to the constraints :

$4x_1 + 2x_2 \le 15,\ x_1, x_2 \ge 0$ and are integers.

705. Solve the following I.P.P. :

Maximize $z = 3x_2$ subject to the constraints :

$3x_1 + 2x_2 \le 7,\ -x_1 + x_2 \le 2;\ x_1 \ge 0,\ x_2 \ge 0$ and are integers. *[Annamalai M.B.A. (Nov.) 2009]*

706. Find the optimal solution to the following integer programming problem :

Maximize $z = x_1 - x_2$ subject to the constraints :

$x_1 + 2x_2 \le 4,\ 6x_1 + 2x_2 \le 9,\ x_1, x_2 \ge 0;\ x_1$ and x_2 are integers.

707. Solve the following I.P.P. :

Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$-3x_1 + 7x_2 \le 14,\ 7x_1 - 3x_2 \le 14;\ x_1, x_2 \ge 0$ and are integers.

[Meerut M.Sc. (Math.) 1996]

708. Describe any method of solving an integer programming problem. Use it to solve the problem :

Maximize $z = 2x_1 + 2x_2$ subject to the constraints :

$5x_1 + 3x_2 \le 8,\ x_1 + 2x_2 \le 4;\ x_1, x_2$ non-negative integers.

709. Solve the following I.P.P. :

Minimize $z = 9x_1 + 10x_2$ subject to the constraints :

$x_1 \le 9,\ x_2 \le 8,\ 4x_1 + 3x_2 \ge 40;\ x_1, x_2 \ge 0$ and are integers.

710. Solve the following I.P.P. :

Maximize $z = 11x_1 + 4x_2$ subject to the constraints :

$-x_1 + 2x_2 \le 4,\ 5x_1 + 2x_2 \le 16,\ 2x_1 - x_2 \le 4,\ x_1$ and x_2 are non-negative integers.

711. Find the optimum integer solution to the following all-I.P.P. :

Maximize $z = x_1 + 2x_2$ subject to the constraints :

$x_1 + x_2 \le 7,\ 2x_1 \le 11,\ 2x_2 \le 7,\ x_1, x_2 \ge 0$ and are integers.

712. An exporter of ready-made garments makes two types of shirts : X and Y. He makes a profit of Rs. 10 and Rs. 40 per shirt on X and Y shirts respectively. He has two tailors, A and B, at his disposal to stitch these shirts. Tailors A and B can devote at the most 7 hours and 15 hours per day respectively. Both these shirts are to be stitched by both the tailors. Tailor A and tailor B spend two hours and five hours respectively in stitching a X shirt, and four hours and three hours respectively in stitching a Y shirt. How many shirts of both the types should be stitched in order to maximise daily profits? (A non-integer solution for this problem will not be accepted.)

[Delhi M.B.A. (Nov.) 1995, (PT) 2008]

713. A manufacturer of toys makes two types of toys, say A and B. Processing of these toys is done on two machines X and Y. Toy A requires two hours on machine X and six hours on machine Y. Toy B requires four hours on machine X and five hours on machine Y. There are 16 hours of time per day available on machine X and 30 hours on machine Y. The profit obtained on both the toys is same, that is Rs. 5 per toy. What should be the daily production of each type of the two toys? (A non-integer solution for this problem will not be accepted.)

[Delhi M.B.A. (Nov.) 2008]

714. A company manufactures two products P_1 and P_2, which require time on three different machines M_1, M_2 and M_3. Product P_1 requires 2 hours on machine M_1 and 1 hour on machine M_2. Product P_2 requires 4 hours on machine M_2 and 2 hours on machine M_3. There are 50 hours available on machine M_1, 16 hours on machine M_2, and 20 hours on machine M_3 per week. The per unit profit on products P_1 and P_2 are Rs. 30 and Rs. 50 respectively. Determine the output mix that will maximize the total profit per week, when only integer solution is acceptable.

715. The ABC company requires an output of at least 200 units of a product per day and to accomplish this target, it can buy machine A and B or both. Machine A costs Rs. 20,000 while machine B costs Rs. 15,000 and the company has a budget of Rs. 2,00,000 for the same. Machines A and B will produce 24 and 20 units respectively of this product per day. However, machine A will require a floor space of 12 square feet while machine B will require 18 square feet and the company has a total floor space of 180 square feet only. Determine the minimum number of machines that should be purchased. (A non-integer solution for this problem will not be accepted.)

[Delhi M.B.A. (Nov.) 2005]

716. Anna furniture firm manufactures tables (A) and chairs (B). The processing times are 3 hours and 4 hours per unit for A on operations one and two respectively, while 4 hours and 5 hours per unit for B on operations one and two respectively. The available time is 18 hours and 21 hours for operations one and two respectively. The product A can be sold at Rs. 3 profit per unit and B at Rs. 8 profit per unit. Solve the LPP. (A non-integer solution for this problem will not be accepted.) ***[Madras M.B.A. 2000]***

717. The ABC Electric Appliances Company produces two products : refrigerators and ranges. Production takes place in two separate departments. Refrigerators are produced in department I and ranges in department II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators in department I and 35 ranges in department II, because of the limited available facilities in these two departments. The company regularly employs a total of 50 workers in the departments. A refrigerator requires two man-weeks of labour, while a range requires one man-week of labour. A refrigerator contributes a profit of Rs. 300 and a range gives profit of Rs. 200. How many units of refrigerators and ranges should the company produce to realize a maximum profit? (A non-integer solution for this problem will not be accepted.)

[Delhi M.B.A. (PT) Nov. 2003]

718. A company produces two products, each of which requires stamping, assembly and painting operations. Total productive capacity by operation, if it were solely devoted to one product or the other is shown below:

Operation	*Productive capacity (units/week)*	
	Product A	*Product B*
Stamping	50	75
Assembly	40	80
Painting	90	45

Pro-rata allocation of productive capacity is permissible and also combination of production of the two products. Demand for the two products is unlimited. The profit on product A and B is Rs. 150 and Rs. 120 respectively. Determine the optimal product-mix. (A non-integer solution will not be accepted). ***[Delhi M.B.A. (Nov.) 2009]***

719. The dietician at the local hospital is planning the breakfast menu for the maternity ward patients. She is primarily concerned with Vitamin E and iron requirements in planning the breakfast. According to the State Medical Association (SMA), new mothers must get at least 12 milligrams of Vitamin E and 24 milligrams of iron from breakfast. The SMA handbook reports that a scoop of scrambled egg contains 2 milligrams of Vitamin E and 8 milligrams of iron. The SMA handbook recommends that new mothers should eat at least two scoops of cottage cheese for their breakfast. The dietician considers this as one of the model constraints. The hospital accounting

department estimate that a scoop of cottage cheese costs Rs. 2 and scoop of scrambled egg also costs Rs. 2. The dietician is attempting to determine the optimum breakfast menu that satisfies all the requirements and minimizes total cost. The cook insists that he can serve foods by only full scoop, thus necessitating an integer solution. *[Delhi M.B.A. (HCA) 2000]*
Determine the optimum integer solution to the problem.

7:6. FRACTIONAL CUT METHOD—MIXED INTEGER LPP

In mixed integer linear programming problem, like the pure integer linear programming problem, all the coefficients and constants should be integers. Only some of the decision variables are restricted to integers, while the others may take integer or continuous values.

The iterative procedure for the solution of such problems is as follows :

Step 1. Determine an optimum solution to the given linear programming problem by using simplex method ignoring the restriction of integer value.

Step 2. Test the integrality of the optimum solution.

(*a*) If all $x_{Bi} \geq 0$ ($i = 1, 2, \ldots, m$) and are integers, the current solution is an optimum one.

(*b*) If all $x_{Bi} \geq 0$ ($i = 1, 2, \ldots, m$) but the integer restricted variables are not integers, go to the next step.

Step 3. Choose the largest fraction value among the basic variables which are restricted to integers. Let it be x_{Bk} (= f_{ko}, say).

Step 4. Generate the Gomorian constraint in the form

$$\sum_{j \in R_+} f_{kj} x_j + \left(\frac{f_{ko}}{f_{ko} - 1}\right) \sum_{j \in R_-} f_{kj} x_j \geq f_{ko},$$

where, $0 < f_{ko} < 1$ and $R_+ = \{ j : f_{kj} > 0 \}$, $R_- = \{ j : f_{kj} < 0 \}$

i.e.,
$$G_1 = -f_{ko} + \sum_{j \in R_+} f_{kj} x_j + \left(\frac{f_{ko}}{f_{ko} - 1}\right) \sum_{j \in R_-} f_{kj} x_j,$$

where G_1 is known as the Gomorian slack variable and this constraint as the Gomory's cut.

Step 5. Add the Gomorian constraint generated in *step 4* at the bottom of the optimum simplex table. Use dual simplex method to find an improved optimum solution.

Step 6. Go to *step 2* and repeat the procedure until all $x_{Bi} \geq 0$ ($i = 1, 2, \ldots, m$) and all restricted variables are integers.

SAMPLE PROBLEM

720. *Solve the following mixed integer programming problem :*

Maximize $z = 4x_1 + 6x_2 + 2x_3$ *subject to the constraints :*

$$4x_1 - 4x_2 \leq 5, \quad -x_1 + 6x_2 \leq 5, \quad -x_1 + x_2 + x_3 \leq 5$$

$$x_1, x_2, x_3 \geq 0; \quad x_1 \text{ and } x_3 \text{ are integers.}$$

Solution. Introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$; an initial basic feasible solution is $s_1 = 5$, $s_2 = 5$ and $s_3 = 5$.

Ignoring the integer restrictions, the optimum solution of the given L.P.P. is displayed in the table below :

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
4	y_1	5/2	1	0	0	3/10	1/5	0
6	y_2	5/4	0	1	0	1/20	1/5	0
2	y_3	25/4	0	0	1	1/4	0	1
		z (= 30)	0	0	0	2	2	2

Since, x_1 and x_3 both are not integers and max. $\{f_1, f_3\}$ = max. $\left\{\frac{1}{2}, \frac{1}{4}\right\} = \frac{1}{2}$; from the first row, we have

$$\left(2 + \frac{1}{2}\right) = x_1 + \frac{3}{10} s_1 + \frac{1}{5} s_2$$

Here, both s_1 and s_2 have positive coefficients, therefore the mixed fractional cut is :

$$G_1 = \frac{-1}{2} + \frac{3}{10} s_1 + \frac{1}{5} s_2$$

First Iteration. Drop G_1 and enter $\mathbf{y}_4$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
4	$\mathbf{y}_1$	5/2	1	0	0	3/10	1/5	0	0
6	$\mathbf{y}_2$	5/4	0	1	0	1/20	1/5	0	0
2	$\mathbf{y}_3$	25/4	0	0	1	1/4	0	1	0
0	$\mathbf{G}_1$	−1/2	0	0	0	−3/10	−1/5	0	1
		z (= 30)	0	0	0	2	2	2	0

Here, since $x_{B4} < 0$, $\mathbf{G}_1$ leaves the basis and since max. $\left\{\frac{2}{-3/10}, \frac{2}{-1/5}\right\}$ = max. $\left\{\frac{-20}{3}, -10\right\}$, *i.e.*, $-\frac{20}{3}$; $\mathbf{y}_4$ *enters the basis.*

Second Iteration. Improved Solution.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$
4	$\mathbf{y}_1$	2	1	0	0	0	0	0	1
6	$\mathbf{y}_2$	7/6	0	1	0	0	1/6	0	1/6
2	$\mathbf{y}_3$	35/6	0	0	1	0	−1/6	1	5/6
0	$\mathbf{y}_4$	5/3	0	0	0	1	2/3	0	−10/3
		z (= 80/3)	0	0	0	0	2/3	2	20/3

Since x_3 is still not an integer, we write from the third row of second iteration

$$\left(5 + \frac{5}{6}\right) = x_3 - \frac{1}{6} s_2 + s_3 + \frac{5}{6} G_1$$

Here, since s_2 has negative coefficient, the mixed fractional cut is :

$$G_2 = -\frac{5}{6} + \left(\frac{5/6}{-1 + 5/6}\right)\left(-\frac{1}{6}\right) s_2 + \frac{5}{6} G_1$$

$$= -\frac{5}{6} + \frac{5}{6} s_2 + \frac{5}{6} G_1$$

Third Iteration. Drop G_2 and enter $\mathbf{y}_5$.

c_B	$\mathbf{y}_B$	$\mathbf{x}_B$	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{G}_1$	$\mathbf{G}_2$
4	$\mathbf{y}_1$	2	1	0	0	0	0	0	1	0
6	$\mathbf{y}_2$	7/6	0	1	0	0	1/6	0	1/6	0
2	$\mathbf{y}_3$	35/6	0	0	1	0	−1/6	1	5/6	0
0	$\mathbf{y}_4$	5/3	0	0	0	1	2/3	0	−10/3	0
0	$\mathbf{G}_2$	−5/6	0	0	0	0	−5/6	0	−5/6	1
		z (= 80/3)	0	0	0	0	2/3	2	20/3	0

Here, $x_{B5} < 0$ implies $\mathbf{G}_2$ leaves the basis and since y_{55} is the only negative in G_2-row, $\mathbf{y}_5$ enters the basis.

Final Iteration. Optimum Integer Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	G_1	G_2
4	y_1	2	1	0	0	0	0	0	1	0
6	y_2	1	0	1	0	0	0	0	1/6	1/5
2	y_3	6	0	0	1	0	0	1	5/6	−1/5
0	y_4	1	0	0	0	1	0	0	−10/3	4/5
0	y_5	1	0	0	0	0	1	0	1	−6/5
		$z(=26)$	0	0	0	0	0	2	20/3	4/5

Solution obtained above is optimum and satisfies the condition that x_1 and x_3 are integers. Hence, the required solution is :

$$x_1 = 2,\ x_2 = 1,\ x_3 = 6, \text{ and maximum } z = 26.$$

PROBLEMS

Solve the following mixed-integer programming problems, using Gomory's cutting plane method :

721. Maximize $z = x_1 + x_2$ subject to the constraints :

$$3x_1 + 2x_2 \le 5,\ x_2 \le 2;\ x_1, x_2 \ge 0 \text{ and } x_1 \text{ is an integer.}$$

[Annamalai M.B.A. (June) 2007]

722. Minimize $z = x_1 - 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 5,\ -2x_1 + 4x_2 \le 11;\ x_1, x_2 \ge 0 \text{ and } x_2 \text{ is an integer.}$$

723. Maximize $z = 7x_1 + 9x_2$ subject to the constraints :

$$-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35;\ x_1, x_2 \ge 0 \text{ and } x_1 \text{ is an integer.}$$ *[Madurai M.B.A. 2008]*

724. Maximize $z = 3x_1 + x_2 + 3x_3$ subject to the constraints :

$$-x_1 + 2x_2 + x_3 \le 4,\ 4x_2 - 3x_3 \le 2,\ x_1 - 3x_2 + 2x_3 \le 3;$$
$$x_j \ge 0\ (j = 1, 2, 3) \text{ with } x_1 \text{ and } x_3 \text{ as integers.}$$

725. Maximize $z = 1.5x_1 + 3x_2 + 4x_3$ subject to the constraints :

$$2.5x_1 + 2x_2 + 4x_3 \le 12,\ 2x_1 + 4x_2 - x_3 \le 7;\ x_1, x_2, x_3 \ge 0 \text{ and } x_3 \text{ is an integer.}$$

7:7. BRANCH AND BOUND METHOD

The concept behind this method is to divide the entire feasible solution space of LP problem into smaller parts called sub-problems and then search each of them for an optimal solution. This approach is useful in those cases where there is large number of feasible solutions and enumeration of those becomes economically impractical or impossible.

The branch and bound method starts by imposing feasible and infeasible upper and/or lower bounds for the decision variables in each sub-problem. This helps in reducing the number of simplex method iterations to arrive at the optimal solution, because each sub-problem worse than the current feasible bound is discarded and only the remaining sub-problems are examined. At a point where no more sub-problems can be created, we will find an optimal solution.

The iterative procedure is summarized below :

Step 1. Obtain an optimum solution of the given L.P.P. ignoring the integer restriction.

Step 2. Test the integrability of the optimum solution obtained in *step 1.* There are two cases :

(*i*) If the solution is in integers, the current solution is optimum to the given integer programming problem.

(*ii*) If the solution is not in integers, go to next step.

Step 3. Considering the value of objective function as upper bound, obtain the lower bound by rounding off to integral values of the decision variables.

Step 4. Let the optimum value x_j^* of the variable x_j be not an integer. Then subdivide (branch) the given L.P.P. into two problems :

SUB-PROBLEM *1.* Given L.P.P. with an additional constraint $x_j \le [x_j^*]$.

SUB-PROBLEM 2. Given L.P.P. with an additional constraint $x_j \ge [x_j^*] + 1$.

where $[x_j^*]$ is the largest integer contained in x_j^*.

Step 5. Solve the two sub-problems obtained in *step 4.* There may arise three cases :

(*i*) If the optimum solutions of the two sub-problems are integral, then the required solution is one that gives larger value of z.

(*ii*) If the optimum solution of one sub-problem is integral and the other sub-problem has no feasible optimum solution, the required solution is same as that of the sub-problem having integer valued solution.

(*iii*) If the optimum solution of one sub-problem is integral while that of the other is not integer valued then record the integer valued solution and repeat *steps 3* and *4* for the non-integer valued sub-problem.

Step 6. Repeat *steps 3* to *5*, until all integer valued solutions are recorded.

Step 7. Choose the solution amongst the recorded integer valued solutions that yields an optimum value of z.

SAMPLE PROBLEM

726. *Use branch and bound method to solve the following L.P.P. :*

$$\text{Maximize } z = 7x_1 + 9x_2 \text{ subject to the constraints :}$$

$$-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35,\ x_2 \le 7,\ x_1, x_2 \ge 0 \text{ and are integers.}$$

Solution. *Step 1.* Ignoring the integer restriction, the optimum solution to the given LPP can easily be obtained as :

$$x_1 = \frac{9}{2},\ x_2 = \frac{7}{2} \text{ and maximum } z = 63.$$

Step 2. Since the solution is not in integers, let us choose x_1, *i.e.*, $x_1^* = 9/2$ being the largest fractional value.

Step 3. Considering the value of z as initial upper bound, *i.e.*, $z = 63$; the lower bound is obtained by rounding off the values of x_1 and x_2 to the nearest integers, *i.e.*, $x_1 = 4$ and $x_2 = 3$. Then the lower bound is $z_1 = 55$.

Step 4. Since $[x_1^*] = [9/2] = 4$; we have

SUB-PROBLEM *1.* Maximize $z = 7x_1 + 9x_2$ subject to the constraints :

$$-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35,\ 0 \le x_1 \le 4 \text{ and } 0 \le x_2 \le 7.$$

SUB-PROBLEM 2. Maximize $z = 7x_1 + 9x_2$ subject to the constraints :

$$-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35,\ 0 \le x_1 \ge 5,\ 0 \le x_2 \le 7.$$

Step 5. Optimum solutions to the sub-problems are obtained as follows :

SUB-PROBLEM *1.* $x_1 = 4$, $x_2 = 10/3$ and maximum $z = 58$.

SUB-PROBLEM 2. $x_1 = 5$, $x_2 = 0$ and maximum $z = 35$.

Since the solution to *sub-problem 1* is not in integers, we subdivide it into following two sub-problems :

SUB-PROBLEM *3.* Maximize $z = 7x_1 + 9x_2$ subject to the constraints :

$$-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35,\ 0 \le x_1 \le 4 \text{ and } 0 \le x_2 \le 3.$$

SUB-PROBLEM *4*. Maximize $z = 7x_1 + 9x_2$ subject to the constraints :

$$-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35,\ 0 \le x_1 \le 4 \text{ and } 0 \le x_2 \ge 4.$$

Step 6. The optimum solutions to the *sub-problems 3* and *4* are :

SUB-PROBLEM *3*. $x_1 = 4,\ x_2 = 3$ and maximum $z = 55$.

SUB-PROBLEM *4*. No feasible solution.

Step 7. Among the recorded integer valued solutions, since the largest value of z is 55; the required optimum solution is :

$$x_1 = 4,\ x_2 = 3 \text{ and maximum } z = 55.$$

The entire branch and bound procedure for the given problem is shown below :

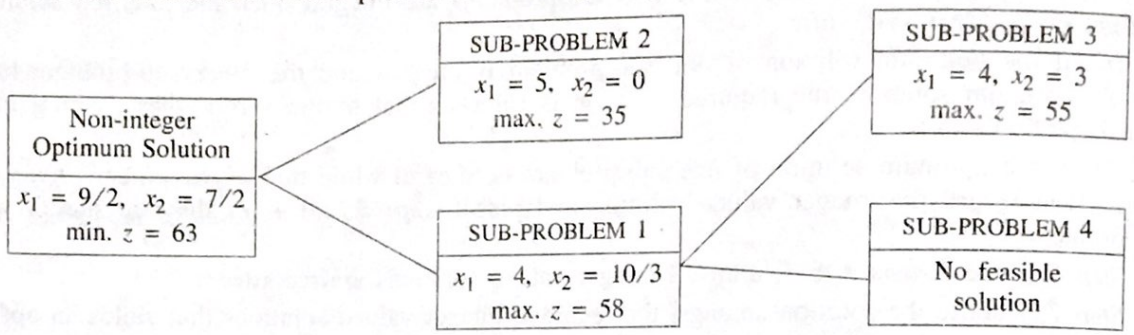

Fig. 7.1.

727. *Use branch and bound method to solve the following L.P.P. :*

Minimize $z = 4x_1 + 3x_2$ *subject to the constraints :*

$$5x_1 + 3x_2 \ge 30,\ x_1 \le 4,\ x_2 \le 6,\ x_1, x_2 \ge 0 \text{ and are integers.}$$

[Pondicherry B.Sc. (Comp. Sc.) 2001]

Solution. Ignoring the integer restriction, the optimum solution to the L.P.P. can easily be obtained as :

$$x_1 = 4,\ x_2 = \frac{10}{3} \text{ and minimum } z = 26$$

Since the value of x_2 is not an integer, we branch on this variable. The two branches are $x_2 \le 3$ and $x_2 \ge 4$, since $[x_2] = [10/3] = 3$. Thus, we have

SUB-PROBLEM *1*. Minimize $z = 4x_1 + 3x_2$ subject to the constraints :

$$5x_1 + 3x_2 \ge 30,\ x_1 \le 4,\ x_2 \le 6 \text{ and } x_2 \le 3;\ x_1, x_2 \ge 0.$$

SUB-PROBLEM *2*. Minimize $z = 4x_1 + 3x_2$ subject to the constraints :

$$5x_1 + 3x_2 \ge 30,\ x_1 \le 4,\ x_2 \le 6 \text{ and } x_2 \ge 4;\ x_1, x_2 \ge 0.$$

The optimum solution to the sub-problems can easily be obtained as :

SUB-PROBLEM *1*. No feasible solution.

SUB-PROBLEM *2*. $x_1 = 18/5,\ x_2 = 4$ and minium $z = 132/5$.

Since the value of x_1 in sub-problem 2 is not an integer, we branch on this variable. The two branches are $x_1 \le 3$ and $x_1 \ge 4$, since $[x_1] = [18/5] = 3$. Thus, we have

SUB-PROBLEM *3*. Minimize $z = 4x_1 + 3x_2$ subject to the constraints :

$$5x_1 + 3x_2 \ge 30,\ 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ x_2 \ge 4 \text{ and } x_1 \le 3.$$

SUB-PROBLEM *4*. Minimize $z = 4x_1 + 3x_2$ subject to the constraints :

$$5x_1 + 3x_2 \ge 30,\ 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ x_2 \ge 4 \text{ and } x_1 \ge 4.$$

The optimum solutions to the sub-problems are obtained as :

SUB-PROBLEM 3. $x_1 = 3,\ x_2 = 5$ and minimum $z = 27$.

SUB-PROBLEM 4. $x_1 = 4,\ x_2 = 4$ and minimum $z = 28$.

Among the feasible solutions to the integer programming problem, since the minimum value of z is 27; the required optimum solution is

$$x_1 = 3,\ x_2 = 5 \text{ and minimum } z = 27.$$

The complete branch and bound procedure for the given LPP is shown below :

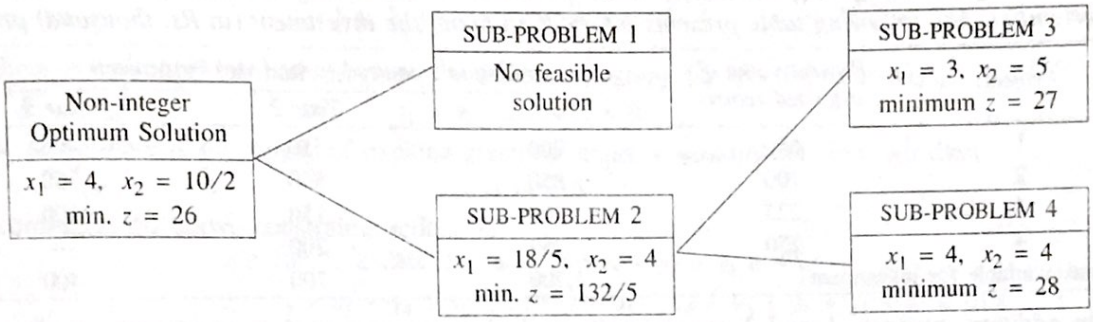

Fig. 7.2.

PROBLEMS

Use branch and bound method to solve the following integer linear programming problems :

728. Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$$5x_1 + 7x_2 \le 35,\ 4x_1 + 9x_2 \le 36;\ x_1, x_2 \ge 0 \text{ and are integers.}$$

729. Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 7,\ 0 \le x_1 \le 5,\ 0 \le x_2 \le 4;\ x_1 \text{ and } x_2 \text{ are integers.}$$

730. Maximize $z = x_1 + 2x_2$ subject to the constraints :

$$x_1 + 2x_2 \le 12,\ 4x_1 + 3x_2 \le 14;\ x_1 \ge 0,\ x_2 \ge 0 \text{ and are integers.}$$

731. Maximize $z = 2x_1 + 3x_2$ subject to the constraints :

$$6x_1 + 5x_2 \le 25,\ x_1 + 3x_2 \le 10;\ x_1 \ge 0,\ x_2 \ge 0 \text{ and are integers.}$$

732. Maximize $z = 2x_1 + x_2$ subject to the constraints :

$$x_1 \le 3/2,\ x_2 \le 3/2;\ x_1, x_2 \ge 0 \text{ and are integers.}$$ *[Meerut M.Sc. (Math.) 1996]*

733. Maximize $z = 3x_1 + 3x_2 + 13x_3$ subject to the constraints :

$$-3x_1 + 6x_2 + 7x_3 \le 8,\ 6x_1 - 3x_2 + 7x_3 \le 8,\ 0 \le x_j \le 5\ (j = 1, 2, 3)$$

$$x_1, x_2, x_3 \text{ are integers.}$$

7:8. APPLICATIONS OF INTEGER PROGRAMMING

So far we have discussed integer programming as a tool to solve linear programming problems where units are non-divisible. However, there are many practical problems that can realistically be structured only as integer programming problems. Such problems revolve around the introduction of 1 or 0 integer variable representing a yes/no choice and are commonly known as *zero-one programming problems.*

The zero-one integer programming problem, in general, can be stated as :

Maximize or Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the constraints :

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ \{\le \text{ or } \ge \text{ or } =\}\ b_i,\ i = 1, 2, \ldots, n \text{ and } x_j = 0 \text{ or } 1.$$

Formulation Possibilities through Integer Programming

In the real-world situations, a large number of problems such as : capital budgeting, transportation, assignment, scheduling, fixed cost or fixed charge, etc. require that its decision variables are either 0 or 1. This section deals with the formulation of a few such problems.

SAMPLE PROBLEMS

734. (Capital Budgeting Problem). *A Corporation is considering the four possible investment opportunities. The following table presents information about the investment (in Rs. thousand) profits :*

Project	*Present value of expected return*	*Capital required (year-wise) by projects*		
		Year 1	*Year 2*	*Year 3*
1	650	700	550	400
2	700	850	550	350
3	225	300	150	100
4	250	350	200	—
Capital available for investment		1,200	700	400

In addition, projects 1 and 2 are mutually exclusive and project 4 is contingent on the prior acceptance of project 3. Formulate an integer programming model to determine which projects should be accepted and which should be rejected to maximise the present value from accepted project.

[Delhi M.B.A. 2008]

Integer Programming Formulation

Decision Variables : $x_j = \begin{cases} 1, \text{ if project } j \text{ is accepted} \\ 0, \text{ if project } j \text{ is rejected} \end{cases}$

Constraints on Expenditure :

$$700x_1 + 850x_2 + 300x_3 + 350x_4 \le 1{,}200$$
$$550x_1 + 550x_2 + 150x_3 + 200x_4 \le 700$$
$$400x_1 + 350x_2 + 100x_3 \le 400$$
$$x_1 + x_2 \le 1, \; x_3 - x_4 \ge 0 \text{ and } x_j = 1, 0$$

Objective Function :

$$\text{Maximize (Total Profit)} = 650x_1 + 700x_2 + 225x_3 + 250x_4.$$

735. (Caterer Problem). *A caterer is to organise garden parties for a week. He needs a total of 160, 120, 60, 80, 90, 100 and 105 fresh napkins during the seven days of the week. Each new napkin costs Rs. 2. He can also use soiled napkins after getting them washed from a laundry. Ordinarily, washing charges are Re. 0.35 per napkin and they are returned after 4 days. However, the laundry also provides express service at a cost of Re. 0.60 per napkin, in which case they are returned after 2 days. Determine the planning schedule the caterer should adopt to buy or send napkins to the laundry so as to minimize the cost if he has no napkins at the beginning of the week.*

Integer Programming Formulation

Let z denote the number of purchased napkins and x_j, y_j denote the number of washed napkins obtained after ordinary washing and express washing respectively on any day. Then the total supply S_j on any day will be

$$S_j = z + \sum_{j=1}^{7} (x_j + y_j).$$

Therefore, the requirement for each day must be met can be expressed as

$$S_j \ge D_j; \quad j = 1, 2, 3, 4, 5, 6, 7$$

where D_j denotes the cumulative demand of napkins.

Thus, various constraints are :

$$S_1 \geq D_1, \quad S_2 \geq D_2, \ldots, \quad S_7 \geq D_7.$$

i.e.,

$$z + x_1 + y_1 \geq 160, \quad z + (x_1 + x_2) + (y_1 + y_2) \geq 280,$$

$$z + (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) \geq 340,$$

$$z + \sum_{j=1}^{4} (x_j + y_j) \geq 420, \quad z + \sum_{j=1}^{5} (x_j + y_j) \geq 510,$$

$$z + \sum_{j=1}^{6} (x_j + y_j) \geq 610, \quad z + \sum_{j=1}^{7} (x_j + y_j) \geq 715.$$

Since, there is no supply of napkins given for ordinary washing for the first four days,

$$x_1 = x_2 = x_3 = x_4 = 0.$$

Likewise since there is no supply of napkins given for express washing for first two days,

$$y_1 = y_2 = 0.$$

Accordingly, the above constraints reduce to

$$z \geq 160, \quad z \geq 280, \quad z + y_3 \geq 340, \quad z + y_3 + y_4 \geq 420,$$

$$z + x_5 + y_3 + y_4 + y_5 \geq 510, \quad z + x_5 + x_6 + y_3 + y_4 + y_5 + y_6 \geq 610,$$

$$z + x_5 + x_6 + x_7 + y_3 + y_4 + y_5 + y_6 + y_7 \geq 715.$$

Further, out of these first constraint becomes redundant and can be deleted.

The objective is to minimize the cost, *viz.*,

$$\text{Minimize (total cost)} = 2z + 0.35 \sum_{j=1}^{7} x_j + 0.60 \sum_{j=1}^{7} y_j$$

$$= 2z + 0.35 (x_5 + x_6 + x_7) + 0.60 (y_3 + y_4 + y_5 + y_6 + y_7).$$

Thus, the problem is to find non-negative integer values of z, x_j and y_j that minimize the above objective function, while satisfying the six constraints given above.

Fixed Cost (or Charge) Problem

Consider a production planning problem wherein it is required to have n items. Let the production of the jth item involve two types of set-up cost—the fixed cost k_j ($k_j \geq 0$) independent of the quantity produced and a variable cost c_j per unit. If x_j is the number of units of the jth item produced, the production cost function for the jth item may be written as

$$c_j(x_j) = \begin{cases} K_j + c_j x_j, & x_j > 0 \\ 0, & x_j = 0 \end{cases}$$

The objective function then becomes

$$\text{Minimize } z = \sum_{j=1}^{n} (K_j + c_j x_j)$$

The corresponding constraint is

$$\sum_{j=1}^{n} x_j \geq n \quad \text{and} \quad x_j \geq 0 \text{ (integer)}.$$

It may be noted that because of fixed cost K_j, the objective function is non-linear. In order to overcome this difficulty, we reformulate the given problem as integer pro- gramming problem with the introduction of auxiliary binary variables y_j, given by

$$y_j = 1 \text{ or } 0 \text{ according as } x_j > 0 \text{ or } x_j = 0 \text{ resp.}$$

These conditions can be expressed as a single linear constraint : $x_j \leq My_j$,

where the symbol M denotes a number that is so large that whenever x_j is feasible, $x_j \leq M$.

The general fixed cost problem can be stated as to

$$\text{Minimize } z = \sum_{j=1}^{n} (c_j x_j + K_j y_j) \text{ subject to the constraints :}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i ; \quad x_j - M y_j \le 0 ; \qquad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n$$

$$x_j \ge 0 \text{ and } y_j = 0 \text{ or } 1, \qquad \text{for all } j.$$

SAMPLE PROBLEMS

736. *Consider the following production data :*

Product	*Profit per unit (Rs.)*	*Direct labour requirement (hours)*
1	6	12
2	8	11
3	5	14

Fixed cost (Rs.)	*Direct labour requirement*
8,000	*upto 10,000 hours*
16,000	*10,000—20,000 hours*
22,000	*20,000—30,000 hours*

Formulate an integer programming problem to determine the production schedule so as to maximize the total net profit. **[Delhi M.B.A. (Nov.) 2005]**

Integer Programming Formulation

Let x_1, x_2, x_3 = number of units of product *1, 2* and *3* respectively to be produced, and

y_j = fixed cost in rupees; $j = 1, 2, 3$.

Objective function :

$$\text{Maximize } z = 6x_1 + 8x_2 + 5x_3 - 8{,}000y_1 - 16{,}000y_2 - 22{,}000y_3.$$

Constraints :

$$12x_1 + 11x_2 + 14x_3 \le 10{,}000y_1 + 20{,}000y_2 + 30{,}000y_3$$

$$y_1 + y_2 + y_3 = 1 \text{ and } y_j = 0 \text{ or } 1 ; \; x_j \ge 0 \; (j = 1, 2, 3).$$

737. (Knapsack Problem). *Four items are considered for loading on an aeroplane, which has a capacity to load upto 25 tonnes. The weights and values of the items are as follows :*

Item :	*A*	*B*	*C*	*D*
Weight (tonnes) :	2	7	5	3
Value (per unit) :	10	36	25	14

Which items, and what quantities should be loaded on the plane so as to maximize the value of the cargo transported? Formulate this as an IPP.

Integer Programming Formulation

Let x_1, x_2, x_3 and x_4 be the number of items *A, B, C* and *D* respectively, that are loaded on the plane. Accordingly, the mathematical formulation of the problem is :

$$\text{Maximize } z = 10x_1 + 36x_2 + 25x_3 + 14x_4 \text{ subject to the constraints :}$$

$$2x_1 + 7x_2 + 5x_3 + 3x_4 \le 25$$

$$x_j = 0 \text{ or } 1 \qquad (j = 1, 2, 3, 4)$$

Here the decision variables are : 0 (do not load) and 1 (load).

Note: An IPP which has only one constraint is known as *Knapsack Problem.*

PROBLEMS

738. An airline has to arrange flights from the city *B* to the city *D* during the tourist season. It has three types of aeroplanes and a crew of 45. On the basis of the past experience, it expects a maximum of 1,000 passengers per day. Further, the airline is entitled to use the airport facilities for not more than 10 flights per day. For the data given in the table, find the number of aeroplanes of each type the airline should use so as to maximize its profit per day :

FLIGHT REQUIREMENT

	Aeroplane type		
	I	*II*	*III*
Crew	5	9	15
Passenger capacity	125	200	250
Profit per flight (thousand rupees)	6	10	15

739. *ABC* Construction Company is faced with the problem of determining which projects it should undertake over the next four years. The following table gives information about each project :

Project	*Estimated present value*	*Capital requirements (in thousand rupees)*			
		Year 1	*Year 2*	*Year 3*	*Year 4*
A	1,800,000	300,000	400,000	400,000	300,000
B	200,000	120,000	80,000	—	40,000
C	720,000	300,000	200,000	200,000	200,000
D	800,000	200,000	400,000	400,000	100,000
Funds available		650,000	800,000	800,000	500,000

Formulate a zero-one programming model to maximise the estimated present value. ***[Delhi M.B.A. 2000]***

740. A company manufacturing chemicals has 4 independent investment projects and must allocate a fixed capital budget to one or more of them so that the company's total assets are maximized. The estimated investments and the anticipated cash outflows associated with these projects are given in the table below :

Project	*Investment (Rs. lakhs)*	*Cash outflows (Rs. lakhs)*	
		1st year	*2nd year*
1	105	60	160
2	140	108	40
3	80	200	150
4	100	90	70

The company has earmarked Rs. 600 lakhs for investment in first year and Rs. 700 lakhs in the second year. If projects 1 and 3 are mutually exclusive, how should the investment be made so that the company's total assets are maximized?

Formulate the problem as integer programming model.

741. Three machines are available to make 5,000 units which are required for the internal use of a company as components to another product. If the production is to be made by any one of these machines, a set-up cost is incurred apart from the cost of making each unit on different machines. The cost data are given below :

Machine	*Set-up cost (Rs.)*	*Cost per unit (Rs.)*	*Maximum production*
1	8,000	5	4,000
2	5,000	4	3,000
3	4,000	8	1,000

What is the best production strategy? Formulate only. ***[Delhi M.B.A. 1994]***

742. *ABC* Associates is considering four investment projects. The net present value and resource requirement for capital and labour for each project are as follows :

Project	*Capital (Rs. lakh)*	*Labour power (Rs. lakh)*	*Net present value (Rs. lakh)*
1	15	5	50
2	3	4	5
3	10	15	15
4	4	2	8

The resources available for capital and labour power are Rs. 100 million and Rs. 150 million respectively. For administrative reasons, the projects *1* and *4* are mutually exclusive and the project *2* can be selected only if project *3* is selected. The Associates want to find a mix of projects having the greatest total value. Formulate this problem as an integer-programming problem. *[Delhi M.B.A. (Nov.) 2009]*

743. A jeweller and her apprentice make silver pins and necklaces by hand. Each week they have 80 hours of labour and 360 grams of silver available. It requires 8 hours of labour and 20 grams of silver to make a pin, and 10 hours of labour and 60 grams of silver to make a necklace. Each pin also contains a small gem of some kind. The demand for pins is no more than six per week. A pin earns the jeweller Rs. 4,000 in profit and a necklace earns Rs. 1,000. The jeweller wants to know how many of each item to make each week in order to maximize profit.

(*i*) Formulate an integer linear programming model for this problem.

(*ii*) Solve this model using the branch and bound method. Compare this solution with the solution without integer restrictions and indicate if the rounded down solution would have been optimal.

[Delhi M.B.A. 2008]

744. *XYZ* Construction Company has an opportunity to build five shopping malls during the next year. The expected net profit and expected cost for each of the shopping malls are shown in the following table :

Expected Net Profit and Expected Net Cost

Shopping mall	*Expected net profit (Rs. '000)*	*Expected cost (Rs. '000)*
1	200	150
2	150	90
3	140	50
4	125	80
5	180	100

The company has budgeted Rs. 3,00,000 for the construction of shopping malls during the next year. Also, due to various legal restrictions and marketing considerations, the following relationships among the projects must be met :

(*i*) Exactly one of the shopping malls 1, 2, and 5 must be constructed.

(*ii*) At most only one of the two shopping malls 1 and 4 may be constructed.

(*iii*) If shopping mall 4 is constructed, then shopping mall 5 must also be constructed.

Formulate the problem as an integer programming problem. *[Delhi M.B.A. (Nov.) 2009*

MULTIPLE CHOICE QUESTIONS

1. Which of the following is correct ?

(*a*) A linear programming problem with only one decision variable restricted to integer value is not an integer programming problem.

(*b*) An integer programming problem is an LPP with decision variables restricted to integer values.

(*c*) A mixed IPP is one where mixed constraints are involved.

(*d*) A pure IPP is one where all the decision variables are either zero or unity.

2. In cutting plane algorithm, each cut involves the introduction of

(*a*) an equality constraint

(*b*) less than or equal to constraint

(*c*) greater than or equal to constraint

(*d*) an artificial variable.

3. The use of cutting plane method
   (*a*) require the use of standard linear programming approach between each cutting plane application.
   (*b*) yields better value of the objective function.
   (*c*) reduces the number of constraints in the given problem.
   (*d*) both (*b*) and (*c*).
4. In a mixed integer programming problem
   (*a*) different objective functions are mixed together.
   (*b*) all of the decision variables require integer solutions.
   (*c*) only few of the decision variables require integer solutions.
   (*d*) none of the above.
5. In an IPP, rounding off solution values of decision variables in LPP may not be acceptable, because
   (*a*) it may violates non-negative conditions.
   (*b*) it does not satisfy the constraints.
   (*c*) objective function value of the IPP is more than (less than) the objective function value of the LPP.
   (*d*) none of the above.
6. While solving an IPP, any non-integer variable in the solution is picked up to
   (*a*) enter the solution (*b*) leave the solution
   (*c*) obtain the cut constraint (*d*) all of the above.
7. Branch and Bound method divides the feasible solution space into smaller parts by
   (*a*) enumerating (*b*) branching
   (*c*) bounding (*d*) all of the above.
8. The 0 – 1 integer programming problem
   (*a*) requires the decision variables to have values either 0 or 1.
   (*b*) requires that the decision variables have coefficients between zero and one.
   (*c*) requires that all the constraints have coefficients between zero and one.
   (*d*) all of the above.
9. Which the following is not correct ?
   (*a*) An IPP where all the variables must be equal to zero or one is called a "0 – 1" IPP.
   (*b*) An LPP in which all the decision variables are non-negative integers is called a pure IPP.
   (*c*) Variables in an IPP that are not integer constrained are called discontinuous variables.
   (*d*) Variables in an IPP that are not integer constrained are called continuous variables.
10. Which of the following is not correct ?
   (*a*) An IPP that has no constraint is known as a Knapsack Problem.
   (*b*) An IPP that have only one constraint is called Knapsack Problem.
   (*c*) Capital budgeting problems may be handled as a "0 – 1" type IPP.
   (*d*) A travelling salesman problem can be solved using Branch and Bound method.

ANSWERS

1. (*b*)	**2.** (*b*)	**3.** (*a*)	**4.** (*c*)	**5.** (*d*)
6. (*c*)	**7.** (*b*)	**8.** (*a*)	**9.** (*c*)	**10.** (*a*)

REVIEW QUESTIONS

1. What is integer linear programming? Explain the merits and demerits of "rounding off" a continuous optimum solution to an LPP to obtain an integer solution. *[Delhi M.B.A. 1998; Osmania M.B.A. 2001]*
2. What is the need for integer programming ? *[Annamalai M.B.A. (Nov.) 2002]*
3. Distinguish between pure and mixed integer programming problems.
4. Addition of a cut makes the previous non-integer optimum solution infeasible. Explain.

5. Describe Gomory's method of solving an all-integer linear programming problem. *[Meerut M.Sc. (Math.) 2007]*
6. What is meant by mixed integer programming problem? *[Annamalai M.B.A. 2010]*
7. Describe any one method of solving a mixed- integer programming problem.
8. Describe the branch and bound method for the solution of an integer programming problem. *[Annamalai M.B.A. 2009]*
9. Explain the branch and bound method, and cutting plane algorithm for pure and mixed integer programming problems. *[Annamalai M.B.A. 2010]*
10. Discuss the advantages of the branch and bound method.
11. Distinguish between linear programming problem and integer linear programming problem. *[Annamalai M.B.A. 2009]*
12. What is meant by zero-one programming problem ? Explain it with the help of an appropriate example. *[Delhi M.B.A. 2009]*