Matrix Assignment - Circle

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CONTENTS

I. PROBLEM

For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum.

II. SOLUTION

The equation of a circle is given as,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The circle given in the question can be written in the above form as follows

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f = 0$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}^\top = \begin{pmatrix} 0 & 0 \end{pmatrix}, f = -r^2 \tag{2}$$

The points of contact of the two tangents drawn from the given external point P to the circle can be found from the following equation

$$\boxed{\mathbf{q}_i = \mathbf{V^{-1}}(k_i \mathbf{n_i} - \mathbf{u})} ....(i = 1, 2) \tag{3}$$

here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n^T V^{-1} n}}} \tag{4}$$

$$f_0 = \mathbf{u^T V^{-1} u} - f \tag{5}$$

$$n_1 = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \tag{6}$$

$$n_2 = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{7}$$

$$\mathbf{P} = (\mathbf{Vh} + \mathbf{u})(\mathbf{Vh} + \mathbf{u})^T - \mathbf{V}(\mathbf{h^T V h} + \mathbf{2u^T h} + f) \tag{8}$$

On solving the above equations by substituting the values in (2), we get two point of contacts of tangents to the circle in terms of 'r',

$$\mathbf{q_1} = -r/10 \begin{pmatrix} -0.8\sqrt{100 - r^2} - 0.6r \\ 0.6\sqrt{100 - r^2} - 0.8r \end{pmatrix} \tag{9}$$

$$\mathbf{q_2} = r/10 \begin{pmatrix} -0.8\sqrt{100 - r^2} + 0.6r \\ 0.6\sqrt{100 - r^2} + 0.8r \end{pmatrix} \tag{10}$$

Let these two points be $\mathbf{A}$ and $\mathbf{B}$. Then the area of the triangle with vertices $\mathbf{P}, \mathbf{A}, \mathbf{B}$ is given by

$$\frac{1}{2}(||(\mathbf{P} - \mathbf{A}) \times (\mathbf{P} - \mathbf{B})||) \tag{11}$$

where, $\mathbf{P} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

By substituting the values in (11) and solving, we get the area of the triangle in terms of 'r' as follows,

$$f(r) = \frac{r}{100}(100 - r^2)^{\frac{3}{2}} \tag{12}$$

A. *Gradient ascent*

Using gradient ascent method we can find the value of 'r' for which the area of the triangle with vertices $\mathbf{P}, \mathbf{A}, \mathbf{B}$ is maximum,

$$\implies r_{n+1} = r_n + \alpha \nabla f(r_n)$$

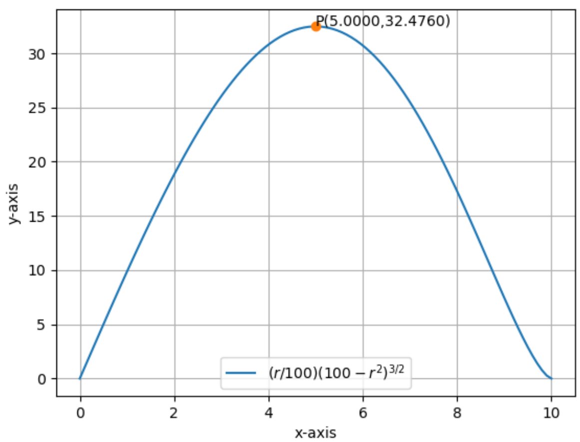
$$\implies r_{n+1} = r_n + \alpha \nabla \left(\frac{(100 - r^2)^{\frac{3}{2}} - 3r^2(100 - r^2)^{\frac{1}{2}}}{100}\right)$$

Taking $r_0 = 0.5, \alpha = 0.0001$ and precision = 0.000000001, values obtained using python are:

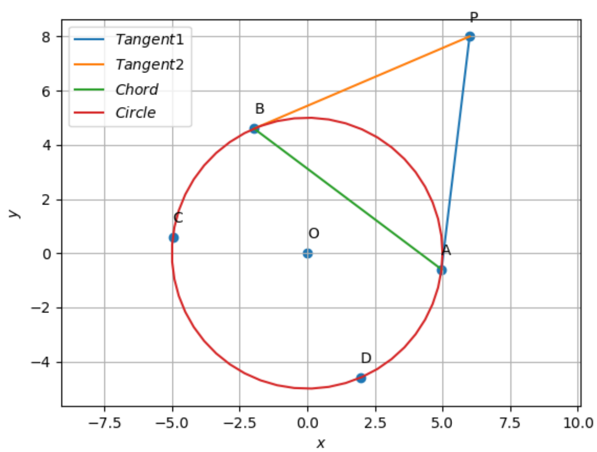
$$\boxed{\text{Maxima} = 32.47595264190203} \tag{13}$$

$$\boxed{\text{Maxima Point} = 4.9999971149068765} \tag{14}$$

The maxima point can be rounded to 5.0.
So, the value of 'r' for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum is 5.



III. Figure



IV. Code Link

https://github.com/PratheekDarla/FWC–IITH/blob/main/Matrix/Circle/circle–assignment.py

Execute the code by using the command
python3 circle-assignment.py