Lecture 2: Preliminaries and One-sample problem

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Examples

Example 1.7 (Spatial Ability Scores of Students)

- Data on a student's spatial ability using four tests of visualization.
- For each student, a single score representing their overall measure of spatial ability.
- The spatial ability scores for 68 female and 82 male high school students enrolled in advanced placement calculus classes in Florida.
  - What is the distribution of spatial ability scores for the population represented by this sample of data?
  - Does the distribution for the male students appear to possess different characteristics than that of the female students?
- These questions are problems in density estimation

Example 1.8 (Sunspots)

- Data on mean monthly sunspot observations collected at the Swiss Federal Observatory in Zurich and the Tokyo Astronomical Observatory from the years 1749 to 1983.
- Excessive variability over time, obscuring any underlying trend in the cycle of sunspot appearances.
- No apparent analytical form or simple parametric model.
- Powerful method for obtaining the trend from a noise in this case is wavelet estimation and thresholding.

Preliminaries

Notations

- X: random variable
- x: realizations (observed random variables)
- $f(x)$: probability density function (pdf)
- $F_X(x) = P(X \leq x)$: cumulative distribution function (cdf)
- $X_1, \cdots X_n$: random sample (independent and identically distributed)

Distribution-free test statistic

- Test statistic: $T(\cdot) = T(X_1, \cdots X_n)$, function of the data.
  - Example: $T = \dfrac{\bar{X} - \mu}{s/\sqrt{n}}$, where $\bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ and $s^2 = \dfrac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1}$, μ is known under H_0.
- Distribution-free test statistic
  - Example: $\mathcal{U} = \mathrm{MVN}\left(\boldsymbol{\mu} = (\mu, \cdots, \mu), \boldsymbol{\Sigma} = \sigma^2\mathbf{I}\right)$
    - $T_1 = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$.
    - $T_2 = \dfrac{\bar{X} - \mu}{s/\sqrt{n}} \sim \mathrm{t}_{n-1}$.
- Nonparametric distribution-free test statistic
  - The class $\mathcal{U}$, $T(\cdot)$ is distribution free over contains more than one distributional forms.

- Distribution-free confidence interval, distribution-free multiple comparison procedure, distribution-free confidence band, asymptotically distribution-free test statistic, asymptotically distribution-free multiple comparison procedure, and asymptotically distribution-free confidence band.

Rank statistic

- Absolute rank: For any random variable $Z_1, \cdots Z_n$, the absolute rank of Z_i, denoted by R_i is the rank of $|Z_i|$ among $|Z_1|, \cdots, |Z_n|$.
- Rank statistic: A statistic $T(\boldsymbol{R})$ based only on the ranks of a sample is rank statistic.
  - $T(\boldsymbol{R})$ is distribution-free over iid joint continuous distribution.
- Signed rank: The signed rank of Z_i is $R_i \psi_i$, where

$$\psi_i = \begin{cases} 1, & \text{if} \qquad Z_i > 0, \\ 0, & \text{if} \qquad Z_i < 0. \end{cases} \tag{1}$$

- Signed rank statistic: A statistic $T(\boldsymbol{\psi}, \boldsymbol{R}) = T(R_1\psi_1, \cdots, R_n\psi_n)$ that is a function of $Z_1, \cdots Z_n$ only through the signed ranks is the signed rank statistic.
  - $T(\boldsymbol{\psi}, \boldsymbol{R})$ is ditribution-free over iid joint continuous distribution symmetric about 0.

Sign test (Fisher) - paired replicates data/one-sample data

Sign test

- $Z_1, \cdots Z_n$ random sample from a continuous population that has a common median θ.
  - If $Z_i \sim F_i$, $F_i(\theta) = F_i(Z_i \leq \theta) = F_i(Z_i > \theta) = 1 - F_i(\theta)$.
- Hypothesis testing:
  - $\mathrm{H}_0 : \theta = 0$ versus $\mathrm{H}_A : \theta \neq 0$.

Sign test (Cont.)

- Sign test statistic: $B = \sum_{i=1}^{n} \psi_i$.
- Motivation:
  - When θ is larger than 0, there will be larger number of positive Z_is $->$ big B value $->$ reject H_0 in favor of $\theta > 0$.
- Under H_0, $B \sim (n, 1/2)$
- Significance level α: probability of rejecting H_0 when it is true.
- Note
  - choices of α are limited to possible values of the $B \sim (n, 1/2)$ cdf.
  - compare the distribution of B under H_0 and the observed test statistic value.

Sign test (Cont.)

- Rejection regions
  - $\mathrm{H}_A : \theta > 0$, Reject H_0 if $B \geq b_{\alpha;n,1/2}$.
  - $\mathrm{H}_A : \theta < 0$, Reject H_0 if $B \leq n - b_{\alpha;n,1/2}$.
  - $\mathrm{H}_A : \theta \neq 0$, Reject H_0 if $B \geq b_{\alpha/2;n,1/2}$ or $B \leq n - b_{\alpha/2;n,1/2}$.

Large-Sample Approximation (Sign test)

- $B^* = \dfrac{B - \mathbb{E}_0(B)}{\mathbb{V}_0(B)^{1/2}} \sim \mathrm{N}(0,1) \quad \text{as} \quad n \to \infty \quad$, where
- $\mathbb{E}_0(B) = \dfrac{n}{2}$ and $\mathbb{V}_0(B) = \dfrac{n}{4}$
- Rejection regions
  - $\mathrm{H}_A : \theta > 0$, Reject H_0 if $B^* \geq z_\alpha$.
  - $\mathrm{H}_A : \theta < 0$, Reject H_0 if $B^* \leq -z_\alpha$.
  - $\mathrm{H}_A : \theta \neq 0$, Reject H_0 if $B^* \geq z_{\alpha/2}$ or $B \leq -z_{\alpha/2}$.

Ties (Sign test)

- Discard zero Z values and redefine n.
- If too many zeros, choose alternative statistical procedure (Chapter 10)

References

References for this lecture

HWC: Chapter 1.2

HWC: Chapter 1.3

HWC: Chapter 3.4-3.6