Rank Statistics

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4/7/2019

Joint distribution of ranks

Let $\boldsymbol{R} = (R_1, \cdots, R_n)$ be the rank of $\boldsymbol{Z} = (Z_1, \cdots, Z_n)$.

The number of permutation of ranks $\boldsymbol{r}$ from n objects is $n!$.

Thus, $P\left(\boldsymbol{R} = \boldsymbol{r}\right) = \dfrac{1}{n!}$.

Marginal distribution of ranks

The probability that rank R_i takes any value between $1, \cdots, n$ is $\dfrac{1}{n}$. Thus, $P\left(R_i = r\right) = \dfrac{1}{n}$ for $r = 1, \cdots, n$.

Let us prove it for $P\left(R_1 = 1\right) = \dfrac{1}{n}$.

Proof:

$$
\begin{aligned}
P\left(R_1 = 1\right) =& \sum_{R_n = n}^{n} \sum_{R_{n-1} = n-1}^{n} \sum_{R_{n-2} = n-2}^{n} \cdots \sum_{R_3 = 3}^{n} \sum_{R_2 = 2}^{n} \frac{1}{n!} \\
=& \sum_{R_n = n}^{n} \sum_{R_{n-1} = n-1}^{n} \sum_{R_{n-2} = n-2}^{n} \cdots \sum_{R_3 = 3}^{n} (n-1) \frac{1}{n!} \\
=& \sum_{R_n = n}^{n} \sum_{R_{n-1} = n-1}^{n} \sum_{R_{n-2} = n-2}^{n} \cdots (n-2)(n-1) \frac{1}{n!} \\
=& \sum_{R_n = n}^{n} \sum_{R_{n-1} = n-1}^{n} (3) \cdots (n-2)(n-1) \frac{1}{n!} \\
=& \sum_{R_n = n}^{n} (2)(3) \cdots (n-2)(n-1) \frac{1}{n!} \\
=& (1)(2)(3) \cdots (n-2)(n-1) \frac{1}{n!} \\
=& \frac{(n-1)!}{n!} \\
=& \frac{1}{n}.
\end{aligned} \tag{1}
$$

Moreover, $P\left(R_i = r, R_j = s\right) = \dfrac{1}{n(n-1)}$. Thus, $P\left(R_i = r, R_j = s\right) \neq P\left(R_i = r\right) P\left(R_j = s\right)$. This implies R_i's are dependent.

We can also show that

$$
\mathbb{E}\left(R_i\right) = \sum_{i=1}^{n} i \frac{1}{n} = \frac{(n+1)}{2}, \qquad i = 1, \cdots, n. \tag{2}
$$

$$\mathbb{V}\left(R_i\right)=\mathbb{E}\left(R_i^2\right)-\left(\mathbb{E}\left(R_i\right)\right)^2=\frac{(n+1)(n-1)}{12}, \qquad i=1,\cdots,n. \tag{3}$$

$$\operatorname{Cov}\left(R_i,R_j\right)=\mathbb{E}\left(R_iR_j\right)-\mathbb{E}\left(R_i\right)\mathbb{E}\left(R_j\right)=\frac{-(n+1)}{12}, \qquad i,j=1,\cdots,n \quad \text{and} \quad i\neq j. \tag{4}$$