6. DIFFERENTIAL EQUATIONS

Let us Study

- Differential Equation
- Formation of differential equation
- Types of differential equation.
- Order and degree of differential equation
- Solution of differential equation
- Application of differential equation.

Let us Recall

- The differentiation and integration of functions and the properties of differentiation and integration.

Let us Learn

6.1.1 Introduction :

In physics, chemistry and other sciences we often have to build mathematical models which involves differential equations. We need to find functions which satisfy those differential equations.

6.1.2 Differential Equation :

Equation which contains the derivative of a function is called a **differential euqation**. The following are differential equations.

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + k^2y = 0$ (iii) $\left(\frac{d^2w}{dx^2}\right) - x^2\frac{dw}{dx} + w = 0$

(iv) $\frac{d^2y}{dt^2} + \frac{d^2x}{dt^2} = x$, here x and y are functions of 't'.

(v) $\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 4xy = 0$, here x is a function of y. (vi) $r\frac{dr}{d\theta} + \cos\theta = 5$

6.2 Order and degree of the differential equation :

The order of a differential equation is the highest order of the derivative appearing in the equation.

The degree of differential equation is the power of the highest ordered derivative present in the equation. To find the degree of the differential equation, we need to have a positive integer as the index of each derivative.

SOLVED EXAMPLES

Ex. 1 : Find order and degree of the following differential equations.

(i) $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 4y = 0$

Solution : It's order is 2 and degree is 1.

(ii) $\left(\dfrac{d^3y}{dx^3}\right)^2 + xy\dfrac{dy}{dx} - 2x + 3y + 7 = 0$

Solution : It's order is 3 and degree is 2.

(iii) $r\dfrac{dr}{d\theta} + \cos\theta = 5$

Solution : It's order is 1 and degree is 1.

(iv) $\left(\dfrac{d^2y}{dx^2}\right)^2 + \left(\dfrac{dy}{dx}\right)^2 = e^x$

Solution : It's order is 2 and degree is 2.

(v) $\dfrac{dy}{dx} + \dfrac{3xy}{\dfrac{dy}{dx}} = \cos x$

Solution : This equation expressed as

$$\left(\frac{dy}{dx}\right)^2 + 3xy = \cos x\left(\frac{dy}{dx}\right)$$

It's order is 1 and degree is 2.

(vi) $\sqrt{1 + \dfrac{1}{\left(\dfrac{dy}{dx}\right)^2}} = \left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}}$

Solution : This equation can be expressed as

$$1 + \frac{1}{\left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^3$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 + 1 = \left(\frac{d^2y}{dx^2}\right)^3\left(\frac{dy}{dx}\right)^2$$

It's order is 2 and degree is 3.

(vii) $\dfrac{d^4y}{dx^4} = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3$

Solution : It's order is 4 and degree is 1.

(viii) $e^{\frac{dy}{dx}} + \dfrac{dy}{dx} = x$

Solution : It's order is 1, but equation can not be expressed as a polynomial differential equation.

$\therefore$ The degree is not defined.

(ix) $\begin{vmatrix} x^3 & y^3 & 3 \\ 2x^2 & 3y\dfrac{dy}{dx} & 0 \\ 5x & 2\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right] & 0 \end{vmatrix} = 0$

Solution : $\therefore \quad x^3[0-0] - y^2[0-0] + 3\left\{4x^2\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right] - 15xy\dfrac{dy}{dx}\right\} = 0$

$$\therefore \quad 4x^2 y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 15xy\frac{dy}{dx} = 0$$

$\therefore$ Its order is 2 and degree is 1.

Notes : (1) $\dfrac{dy}{dx}$ is also denoted by y', $\dfrac{d^2y}{dx^2}$ is also denoted by y'', $\dfrac{d^3y}{dx^3}$ is also by y''' and so-on.

(2) The order and degree of a differential equation are always positive integers.

EXERCISE 6.1

(1) Determine the order and degree of each of the following differential equations.

(i) $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) + y = 2 \sin x$

(ii) $\sqrt[3]{1 + \left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$

(iii) $\frac{dy}{dx} = \frac{2 \sin x + 3}{\frac{dy}{dx}}$

(iv) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$

(v) $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + 7x + 5 = 0$

(vi) $(y''')^2 + 3y'' + 3xy' + 5y = 0$

(vii) $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

(viii) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\frac{d^2y}{dx^2}$

(ix) $\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}} - \left(\frac{dy}{dx}\right)^{\frac{1}{3}} = 20$

(x) $x + \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$

6.3 Formation of Differential Equation :

From the given information, we can form the differential equation. Sometimes we need to eliminate the arbitrary constants from a given relation. It may be done by differentiation.

SOLVED EXAMPLES

Ex. 1 : Obtain the differential euqation by eliminating the arbitrary constants from the following :

(i) $y^2 = 4ax$ (ii) $y = Ae^{3x} + Be^{-3x}$ (iii) $y = (c_1 + c_2x)\, e^x$

(iv) $y = c^2 + \frac{c}{x}$ (v) $y = c_1e^{3x} + c_2e^{2x}$

Solution :

(i) $y^2 = 4ax \ldots (1)$

Here a is the arbitraty constant, we differentiate $w.\ r.\ t.\ x$,

$\therefore \frac{dy}{dx} = 4a$

then eq. (1) gives

$y^2 = \left(\frac{dy}{dx}\right)x$ is required differential equation.

(ii) $y = Ae^{3x} + Be^{-3x} \quad \ldots (1)$

Here A and B are arbitrary constants.

Differentiate $w.\ r.\ t.\ x$, we get

$\therefore \frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$

again Differentiate $w.\ r.\ t.\ x$, we get

$\frac{d^2y}{dx^2} = 3 \times 3Ae^{3x} - 3 \times 3Be^{-3x}$

$= 9\left(Ae^{3x} + Be^{-3x}\right) = 9y \quad \ldots$ from eq.(1)

$\therefore \frac{d^2y}{dx^2} = 9y$

(iii) $y = (c_1 + c_2 x)\, e^x \ldots (1)$

Here c_1 and c_2 are arbitrary constants.

Differentiate *w. r. t. x*, we get

$$\therefore \quad \frac{dy}{dx} = (c_1 + c_2 x)\, e^x + c_2 e^x$$

$$\therefore \quad \frac{dy}{dx} = y + c_2 e^x \quad \ldots (2) \qquad \ldots \text{from eq.(1)}$$

Again differentiate *w. r. t. x*, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + c_2 e^x$$

$$\therefore \quad c_2 e^x = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

put in eq.(2)

$$\frac{dy}{dx} = y + \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$\therefore \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

(iv) $y = c^2 + \dfrac{c}{x} \qquad \ldots (1)$

Differentiate *w. r. t. x*, we get

$$\therefore \quad \frac{dy}{dx} = -\frac{c}{x^2}$$

$$\therefore \quad c = -x^2 \frac{dy}{dx}$$

then eq.(1) gives

$$y = \left[-x^2 \left(\frac{dy}{dx}\right)^2\right]^2 - x^2 \frac{dy}{dx} \times \frac{1}{x}$$

$$\therefore \quad y = x^4 \left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx}$$

$$\therefore \quad x^4 \left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y = 0$$

(v) $y = c_1 e^{3x} + c_2 e^{2x} \qquad \ldots (I)$

Differentiate *w. r. t. x*, we get

$$\therefore \quad \frac{dy}{dx} = 3c_1 e^{3x} + 2c_2 e^{2x} \qquad \ldots (II)$$

Again differentiate *w. r. t. x*, we get

$$\frac{d^2y}{dx^2} = 9c_1 e^{3x} + 4c_2 e^{2x} \qquad \ldots (III)$$

As equations (I), (II) and (III) in $c_1 e^{3x}$ and $c_2 e^{2x}$ are consistent

$$\therefore \quad \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 3 & 2 \\ \frac{d^2y}{dx^2} & 9 & 4 \end{vmatrix} = 0$$

$$\therefore \quad y(12-18) - 1\left(4\frac{dy}{dx} - 2\frac{d^2y}{dx^2}\right) + 1\left(9\frac{dy}{dx} - 3\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore \quad -6y - 4\frac{dy}{dx} + 2\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 3\frac{d^2y}{dx^2} = 0$$

$$\therefore \quad -\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$$

$$\therefore \quad \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Ex. 2 : The rate of decay of the mass of a radioactive substance any time is k times its mass at that time, form the differential equation satisafied by the mass of the substance.

Solution : Let m be the mass of a radioactive substance time 't'

$\therefore$ The rate of of decay of mass is $\dfrac{dm}{dt}$

Here $\dfrac{dm}{dt} \propto m$

$\therefore$ $\dfrac{dm}{dt} = mk$, where $k < 0$

is the required differential equation.

Ex. 3 : Form the differential equation of family of circles above the X-axis and touching the X-axis at the origin.

Solution : Let c (a, b) be the centre of the circle touching X-axis at the origin $(b < 0)$.

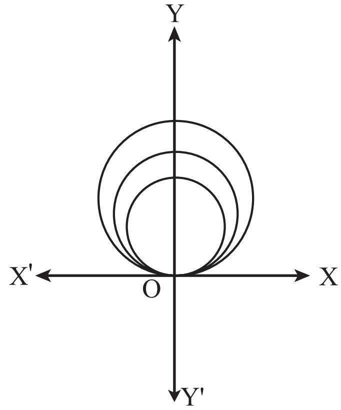

Fig. 6.1

The radius of the circle of b.

The equation of the circle is

$(x - 0)^2 + (y - b)^2 = b^2$

$\therefore$ $x^2 + y^2 - 2by + b^2 = b^2$

$\therefore$ $x^2 + y^2 - 2by = 0$ $\quad \ldots$ (I)

Differentiate $w.$ $r.$ $t.$ x, we get

$$2x + 2y\left(\frac{dy}{dx}\right) - 2b\left(\frac{dy}{dx}\right) = 0$$

$\therefore$ $x + (y - b)\dfrac{dy}{dx} = 0$

$\therefore$ $\dfrac{x}{\left(\frac{dy}{dx}\right)} + (y - b) = 0$ $\qquad$ $\therefore$ $b = y + \dfrac{x}{\left(\frac{dy}{dx}\right)}$ $\quad \ldots$ (II)

From eq. (I) and eq. (II)

$\therefore$ $x^2 + y^2 - 2\left[y + \dfrac{x}{\left(\frac{dy}{dx}\right)}\right]y = 0$

$\therefore$ $x^2 + y^2 - 2y^2 - \dfrac{2xy}{\left(\frac{dy}{dx}\right)} = 0$

$\therefore$ $x^2 - y^2 = \dfrac{2xy}{\left(\frac{dy}{dx}\right)}$

$\therefore$ $(x^2 - y^2)\dfrac{dy}{dx} = 2xy$

is the required differential equation.

Activity : Form the differential equation of family of circles touching Y-axis at the origin and having their centres on the X-axis.

Ex. 4 : A particle is moving along the X-axis. Its acceleration at time t is proportional to its velocity at that time. Find the differential equation of the motion of the particle.

Solution : Let s be the displacement of the particle at time 't'.

Its velocity and acceleration are $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ respectively.

Here $\frac{d^2s}{dt^2} \propto \frac{ds}{dt}$

$$\therefore \quad \frac{d^2s}{dt^2} = k\frac{ds}{dt}, \qquad \text{(where } k \text{ is constant} \neq 0\text{)}$$

is the required differential equation.

EXERCISE 6.2

(1) Obtain the differential equations by elliminating arbitrary constants c_1 and c_2.

(i) $x^3 + y^3 = 4ax$

(ii) $Ax^2 + By^2 = 1$

(iii) $y = A\cos(\log x) + B\sin(\log x)$

(iv) $y^2 = (x + c)^3$

(v) $y = Ae^{5x} + Be^{-5x}$

(vi) $(y - a)^2 = 4(x - b)$

(vii) $y = a + \frac{a}{x}$

(viii) $y = c_1 e^{2x} + c_2 e^{5x}$

(ix) $c_1 x^3 + c_2 y^2 = 5$

(x) $y = e^{-2x}(A\cos x + B\sin x)$

(2) Form the differential equation of family of lines having intercepts a and b on the co-ordicate ares respectively.

(3) Find the differential equation of all parabolas having length of latus rectum $4a$ and axis is parallel to the X-axis.

(4) Find the differential euqation of an ellipse whose major axis is twise its minor axis.

(5) Form the differential equation of family of lines parallel to the line $2x + 3y + 4 = 0$

(6) Find the differential equations of all circles having radius 9 and centre at point $A(h, k)$.

(7) Form the differential equation of all parabolas whose axis is the X-axis.

6.4 Solution of differential equation :

Verify that

$$y = a \sin x \text{ and } y = b \cos x$$

are solutions of the differential equation, where a and b are any constants.

Also $y = a \sin x + b \cos x$ is a solution of the equation.

Here $\sin x$ and $\cos x$ are particular solutions where as $a \sin x + b \cos x$ is the general solution which describes all possible solutions.

A solution which can be otained from the general solution by giving particular values to the arbitarary constants is called a **particular solution**.

Therefore the differential equation has infinitely many solutions.

SOLVED EXAMPLES

Ex. 1 : Verify that : $y \sec x = \tan x + c$

is a solution of the differential equation

$$\frac{dy}{dx} + y \tan x = \sec x.$$

Solution : Here $y \sec x = \tan x + c$

Differentiate $w.\ r.\ t.\ x$, we get

$$y \sec x \tan x + \sec x \frac{dy}{dx} = \sec^2 x$$

$$\therefore \quad \frac{dy}{dx} + y \tan x = \sec x$$

Hence $y \sec x = \tan x + c$

is a solution of the differential equation

$$\frac{dy}{dx} + y \tan x = \sec x$$

Ex. 2 : Verify that : $y = \log x + c$

is a solution of the differential equation

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

Solution : Here $y = \log x + c$

Differentiate $w.\ r.\ t.\ x$, we get

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \quad x \frac{dy}{dx} = 1$$

Differentiate $w.\ r.\ t.\ x$, we get

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 = 0$$

$$\therefore \quad x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$y = \log x + c$ is the solution of

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

Consider the example :

$$\frac{dy}{dx} = x^2y + y$$

$$\therefore \qquad \frac{1}{y} \cdot \frac{dy}{dx} = x^2 + 1$$

We can consider x and y both as variables and write this as

$$\frac{dy}{y} = (x^2 + 1) \cdot dx$$

Now we can integrate L.H.S. *w. r. t.* y and R.H.S. *w. r. t.* x, then we get

$$\therefore \qquad \log y = \frac{x^3}{3} + x + c$$

This integration is obtained by separating the variables.

It helps to examine the equation and find out if such a separation is possible.

The above method is known as the method of separation of variables.

In general, if the given differential equation can be written as

$$f(x)\, dx = g\,(y)\, dy$$

then this method is applicable.

SOLVED EXAMPLES

Ex. 1 : Find the general solution of the following differential equations :

(i) $\frac{dy}{dx} = x\sqrt{25 - x^2}$ (ii) $\frac{dx}{dt} = \frac{x \log x}{t}$

Solution :

(i) $\frac{dy}{dx} = x\sqrt{25 - x^2}$

$\therefore \quad dy = x\sqrt{25 - x^2} \cdot dx$

Integrating both sides, we get

$\int dy = \int \sqrt{25 - x^2} \cdot x \cdot dx \qquad \ldots \text{(I)}$

Put $25 - x^2 = t$

$\therefore \quad -2x \cdot dx = dt$

$\therefore \quad x \cdot dx = -\frac{dt}{2}$

Eq. (I) becomes, $\int dy = \int \sqrt{t}\left(-\frac{dt}{2}\right)$

$\therefore \quad 2\int dy = -\int \sqrt{t} \cdot dt$

$\therefore \quad 2\int dy + \int t^{\frac{1}{2}} \cdot dt = 0$

$\therefore \quad 2y + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} = c_1$

$\therefore \quad 2y + \frac{2}{3}t^{\frac{3}{2}} = c_1$

$\therefore \quad 6y + 2t^{\frac{3}{2}} = 3c_1$

$\therefore \quad 6y + 2\,(25 - x^2)^{\frac{3}{2}} = c \qquad \ldots [c = 3c_1]$

(ii) $\dfrac{dx}{dt} = \dfrac{x \log x}{t}$

$\therefore \quad \dfrac{dx}{x \log x} = \dfrac{dt}{t}$

Integrating both sides, we get

$$\int \frac{dx}{x \log x} = \int \frac{dt}{t}$$

$\therefore \quad \log(\log x) = \log(t) + \log c$

$\therefore \quad \log(\log x) = \log(tc)$

$\therefore \quad \log x = ct$

$\therefore \quad e^{ct} = x$

Ex. 2 : Find the particular solution with given initial conditions :

(i) $\dfrac{dy}{dx} = e^{2y}\cos x$ when $x = \dfrac{\pi}{6}, y = 0$

(ii) $\dfrac{y-1}{y+1} + \dfrac{x-1}{x+1} \cdot \dfrac{dy}{dx} = 0$, when $x = y = 2$

Solution :

(i) $\dfrac{dy}{dx} = e^{2y}\cos x$

$\therefore \quad \dfrac{dy}{e^{2y}} = \cos x \cdot dx$

$\therefore \quad e^{-2y} \cdot dy = \cos x \cdot dx$

Integrating both sides, we get

$\therefore \quad \int e^{-2y} \cdot dy = \int \cos x \cdot dx$

$\therefore \quad \dfrac{e^{-2y}}{-2} = \sin x + c \qquad \ldots \text{(I)}$

When $x = \dfrac{\pi}{6}, y = 0$. So eq. (1), becomes

$\therefore \quad \dfrac{e^{0}}{-2} = \sin\dfrac{\pi}{6} + c \quad \therefore -\dfrac{1}{2} = \dfrac{1}{2} + c$

$\therefore \quad -\dfrac{1}{2} - \dfrac{1}{2} = c \qquad \therefore c = -1$

(Given initial condition determines the value of c)

Put in eq. (1), we get

$\therefore \quad \dfrac{e^{-2y}}{-2} = \sin x - 1$

$\therefore \quad -e^{-2y} = 2\sin x - 2$

$\therefore \quad e^{2y}(2\sin x - 2) + 1 = 0$ is the required particular solution.

(ii) $\dfrac{y-1}{y+1} + \dfrac{x-1}{x+1} \cdot \dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{x+1}{x-1} \cdot dx + \dfrac{y+1}{y-1} \cdot dy = 0$

$\therefore \quad \dfrac{(x-1)+2}{x-1} \cdot dx + \dfrac{(y-1)+2}{y-1} \cdot dy = 0$

$\therefore \quad \left(1 + \dfrac{2}{x-1}\right) \cdot dx + \left(1 + \dfrac{2}{y-1}\right) \cdot dy = 0$

Integrating , we get

$$\int dx + 2\int \frac{dx}{x-1} + \int dy + 2\int \frac{dy}{y-1} = 0$$

$\therefore \quad x + 2\log(x-1) + y + 2\log(y-1) = c$

$\therefore \quad x + y + 2\log[(x-1)(y-1)] = c \qquad \ldots \text{(I)}$

When $x = 2, y = 2$. So eq. (I), becomes

$\therefore \quad 2 + 2 + 2\log[(2-1)(2-1)] = c$

$\therefore \quad 4 + 2\log(1 \times 1) = c$

$\therefore \quad 4 + 2\log 1 = c$

$\therefore \quad 4 + 2(0) = c$

$\therefore \quad c = 4$ Put in eq. (I), we get

$\therefore \quad x + y + 2\log[(x-1)(y-1)] = 4$ is required particular solution.

Ex. 3 : Reduce each of the following differential equations to the separted variable form and hence find the general solution.

(i) $1 + \frac{dy}{dx} = \text{cosec}\,(x + y)$

(ii) $\frac{dy}{dx} = (4x + y + 1)^2$

Solution :

(i) $1 + \frac{dy}{dx} = \text{cosec}\,(x + y) \quad \ldots \text{(I)}$

Put $x + y = u$

$\therefore \quad 1 + \frac{dy}{dx} = \frac{du}{dx}$

Given differential equation becomes

$\frac{du}{dx} = \text{cosec}\, u$

$\therefore \quad \frac{du}{\text{cosec}\, u} = dx$

$\therefore \quad \sin u \cdot du = dx$

Integrating both sides, we get

$\therefore \quad \int \sin u \cdot du = \int dx$

$\therefore \quad -\cos u = x + c$

$\therefore \quad x + \cos u + c = 0$

$\therefore \quad x + \cos\,(x + y) + c = 0 \quad \ldots (\because x + y = u)$

(ii) $\frac{dy}{dx} = (4x + y + 1)^2 \quad \ldots \text{(I)}$

Put $4x + y + 1 = u$

$\therefore \quad 4 + \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} - 4$

Given differential equation becomes

$\frac{du}{dx} - 4 = u^2$

$\therefore \quad \frac{du}{dx} = u^2 + 4$

$\therefore \quad \frac{du}{u^2 + 4} = dx$

Integrating both sides, we get

$\therefore \quad \int \frac{du}{u^2 + 4} = \int dx$

$\therefore \quad \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) = x + c_1$

$\therefore \quad \tan^{-1}\left(\frac{u}{2}\right) = 2x + 2c_1$

$\therefore \quad \tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + c \quad \ldots [2c_1 = c]$

EXERCISE 6.3

(1) In each of the following examples verity that the given expression is a solution of the corresponding differential equation.

(i) $xy = \log y + c\, ;\ \frac{dy}{dx} = \frac{y^2}{1 - xy}$

(ii) $y = (\sin^{-1} x)^2 + c\, ;\ (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

(iii) $y = e^{-x} + Ax + B\, ;\ e^x \frac{d^2y}{dx^2} = 1$

(iv) $y = x^m\, ;\ x^2 \frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my = 0$

(v) $y = a + \frac{b}{x}\, ;\ x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

(vi) $y = e^{ax}\, ;\ x\frac{dy}{dx} = y \log y$

(2) Solve the following differential equations.

(i) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

(ii) $\log\left(\frac{dy}{dx}\right) = 2x + 3y$

(iii) $y - x\frac{dy}{dx} = 0$

(iv) $\sec^2 x \cdot \tan y \cdot dx + \sec^2 y \cdot \tan x \cdot dy = 0$

(v) $\cos x \cdot \cos y \cdot dy - \sin x \cdot \sin y \cdot dx = 0$

(vi) $\frac{dy}{dx} = -k$, where k = constant.

(vii) $\frac{\cos^2 y \cdot dy}{x} + \frac{\cos^2 x \cdot dx}{y} = 0$

(viii) $y^3 - \frac{dy}{dx} = x^2\frac{dy}{dx}$

(ix) $2e^{x+2y} \cdot dx - 3dy = 0$

(x) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

(3) For each of the following differential equations find the particular solution satisfying the given condition.

(i) $3e^x \tan y \cdot dx + (1 + e^x) \sec^2 y \cdot dy = 0$, when $x = 0$, $y = \pi$.

(ii) $(x - y^2 x) \cdot dx - (y + x^2 y) \cdot dy = 0$, when $x = 2$, $y = 0$.

(iii) $y(1 + \log x)\frac{dx}{dy} - x \log x = 0$, $y = e^2$, when $x = e$.

(iv) $(e^y + 1)\cos x + e^y \sin x\frac{dy}{dx} = 0$, when $x = \frac{\pi}{6}$, $y = 0$.

(v) $(x+1)\frac{dy}{dx} - 1 = 2e^{-y}$, $y = 0$, $x = 1$

(vi) $\cos\left(\frac{dy}{dx}\right) = a$, $a \in 1^2$, $y(0) = 2$

(4) Reduce each of the following differential to the variable separable form and hence solve.

(i) $\frac{dy}{dx} = \cos(x + y)$

(ii) $(x - y)^2\frac{dy}{dx} = a^2$

(iii) $x + y\frac{dy}{dx} = \sec(x^2 + y^2)$

(iv) $\cos^2(x - 2y) = 1 - 2\frac{dy}{dx}$

(v) $(2x - 2y + 3)\,dx - (x - y + 1)\,dy = 0$, when $x = 0$, $y = 1$.

6.4.1 Homogeneous differential :

Recall that the degree of a term is the sum of the degrees in all variables in the equation, eg. : degree of $3x^2y^2z$ is 5. If all terms have the same degree, the equation is called **homogeneous differential equation.**

For example : (i) $x + y\frac{dy}{dx} = 0$ is a homogeneous differential equation of degree 1.

(ii) $x^3y + xy^3 + x^2y^2\frac{dy}{dx} = 0$ is a homogeneous differential equation of degree 4.

(iii) $x\frac{dy}{dx} + x^2y = 0$

(iv) $xy\frac{dy}{dx} + y^2 + 2x = 0$

(iii) and (iv) are not homogeneous differential equations.

To solve the homogeneous differential equation, we use the substitution $y = vx$ or $u = vy$.

SOLVED EXAMPLES

Ex. 1 : Solve the following differential equations :

(i) $x^2y\cdot dx - (x^3+y^3)\cdot dy = 0$ (ii) $x\dfrac{dy}{dx} = x\tan\left(\dfrac{y}{x}\right)+y$ (iii) $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$

Solution :

(i) $x^2y\cdot dx - (x^3+y^3)\cdot dy = 0$

$\therefore \quad x^2y - (x^3+y^3)\dfrac{dy}{dx} = 0 \qquad \ldots \text{(I)}$

This is homogeneous Differential equation.

Put $y = vx \qquad \ldots \text{(II)}$

Differentiate *w. r. t.* x, we get

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \qquad \ldots \text{(III)}$

Put (II) and (III) in Eq. (I), it becomes,

$$x^2\cdot vx - (x^3+v^3x^3)\left(v + x\dfrac{dv}{dx}\right) = 0$$

divide by x^3, we get

$$v - (1+v^3)\left(v + x\dfrac{dv}{dx}\right) = 0$$

$\therefore \quad \cancel{v} - \cancel{v} - x\dfrac{dv}{dx} - v^4 - v^3x\dfrac{dv}{dx} = 0$

$\therefore \quad -x(1+v^3)\dfrac{dv}{dx} = v^4$

$\therefore \quad \dfrac{1+v^3}{v^4}\cdot dv = -\dfrac{dx}{x}$

$\therefore \quad \dfrac{1+v^3}{v^4}\cdot dv = -\dfrac{dx}{x}$

$\therefore \quad \left(\dfrac{1}{v^4} - \dfrac{v^3}{v^4}\right)dv + \dfrac{dx}{x} = 0$

integrating eq., we get

$\therefore \quad \int v^{-4}\,dv + \int\dfrac{dv}{v} + \int\dfrac{dx}{x} = c_1$

$\therefore \quad \dfrac{v^{-3}}{-3} + \log(v) + \log(x) = c_1$

$\therefore \quad \log(vx) = c_1 + \dfrac{v^{-3}}{3} \qquad \therefore \log(y) = c_1 + \dfrac{1}{3}\cdot\dfrac{v^{-3}}{x^{-3}}$

$\therefore \quad 3\log(y) = 3c_1 + \dfrac{x^3}{y^3}$

$\therefore \quad 3\log y = \dfrac{x^3}{y^3} + c \qquad \ldots \text{where } c = 3c_1$

(ii) $x\dfrac{dy}{dx} = x\tan\left(\dfrac{y}{x}\right) + y \qquad \ldots \text{(I)}$

This is homogeneous Differential equation.

Put $y = vx \qquad \ldots \text{(II)}$

Differentiate *w. r. t.* x, we get

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \qquad \ldots \text{(III)}$

Put (II) and (III) in Eq. (I), it becomes,

$$x\left(v + x\dfrac{dv}{dx}\right) = x\tan\left(\dfrac{vx}{x}\right) + vx$$

divide by x, we get

$\therefore \quad v + x\dfrac{dv}{dx} = \tan v + v$

$\therefore \quad x\dfrac{dv}{dx} = \tan v$

$\therefore \quad \dfrac{dv}{\tan v} = \dfrac{dx}{x}$

integrating eq., we get

$\therefore \quad \int\cot v\,dv = \int\dfrac{dx}{x}$

$\therefore \quad \log(\sin v) = \log(x) + \log c$

$\therefore \quad \log(\sin v) = \log(x\times c)$

$\therefore \quad \sin v = cx$

$\therefore \quad \sin\left(\dfrac{y}{x}\right) = cx$ is the solution.

(iii) $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$ . . . (I)

Solution : It is homogeneous differential equation.

Put $y = vx$ . . . (II)

Differentiate *w. r. t. x*, we get

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ . . . (III)

Put (II) and (III) in Eq. (I), it becomes,

$v + x\dfrac{dv}{dx} = \dfrac{vx + \sqrt{x^2 + v^2x^2}}{x}$

$\therefore \quad \cancel{v} + x\dfrac{dv}{dx} = \cancel{v} + \sqrt{1 + v^2}$

$\therefore \quad x\dfrac{dv}{dx} = \sqrt{1 + v^2}$

$\therefore \quad \dfrac{dv}{\sqrt{1 + v^2}} = \dfrac{dx}{x}$ . . . (IV)

integrating eq. (IV), we get

$\therefore \quad \displaystyle\int \dfrac{dv}{\sqrt{1 + v^2}} = \int \dfrac{dx}{x}$

$\therefore \quad \log\left(v + \sqrt{1 + v^2}\right) = \log(x) + \log c$

$\therefore \quad \log\left(v + \sqrt{1 + v^2}\right) = \log(cx)$

$\therefore \quad v + \sqrt{1 + v^2} = cx$

$\therefore \quad \dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}} = cx$

$\therefore \quad y + \sqrt{x^2 + y^2} = cx^2$ is the solution.

EXERCISE 6.4

I. Solve the following differential equations :

(1) $x \sin\left(\dfrac{y}{x}\right) dy = \left[y \sin\left(\dfrac{y}{x}\right) - x\right] dx$

(2) $(x^2 - y^2)\, dx - 2xy\cdot dy = 0$

(3) $\left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)\dfrac{dy}{dx} = 0$

(4) $y^2\cdot dx + (xy + x^2)\, dy = 0$

(5) $(x^2 - y^2)\, dx + 2xy\cdot dy = 0$

(6) $\dfrac{dy}{dx} + \dfrac{x - 2y}{2x - y} = 0$

(7) $x\dfrac{dy}{dx} - y + x \sin\left(\dfrac{y}{x}\right) = 0$

(8) $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right) dy = 0$

(9) $y^2 - x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

(10) $xy\dfrac{dy}{dx} = x^2 + 2y^2$, $y(1)=0$

(11) $x\, dy + 2y\cdot dx = 0$, when $x = 2$, $y = 1$

(12) $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$

(13) $(9x + 5y)\, dy + (15x + 11y)\, dx = 0$

(14) $(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$

(15) $(x^2 + y^2)\, dx - 2xy\cdot dy = 0$

6.4.2 Linear Differential Equation :

The differential equation of the type, $\frac{dy}{dx} + Py = Q$ (where P, Q are functions of x.)

is called **linear differential equation.**

To get the solution of equation, multiply the equation by $e^{\int Pdx}$, which is helping factor here.

We get,

$$e^{\int Pdx}\left[\frac{dy}{dx} + Py\right] = Q\cdot e^{\int Pdx}$$

Note that, $\frac{d}{dx}\left[y\cdot e^{\int Pdx}\right] = \left[\frac{dy}{dx} + y\cdot P\right]\cdot e^{\int Pdx}$

$\therefore$ $\frac{d}{dx}\left[y\cdot e^{\int Pdx}\right] = Q\cdot e^{\int Pdx}$

$\therefore$ $\int Q\cdot e^{\int Pdx}\cdot dx = y\cdot e^{\int Pdx}$

Hence, $y\cdot e^{\int Pdx} = \int Q\cdot\left(e^{\int Pdx}\right)\, dx + c$ is the solution of the given equation

Here $e^{\int Pdx}$ is called the integrating factor. (I.F.)

Note : For the linear differential equation.

$\frac{dx}{dy} + py = Q$ (where P, Q are constants or functions of y) the general solution is

$x\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\, dy + c$, where I.F. (integrating factor) $= e^{\int Pdy}$

SOLVED EXAMPLES

Ex. 1: Solve the following differential equations :

(i) $\frac{dy}{dx} + y = e^{-x}$

(ii) $x \sin \frac{dy}{dx} + (x \cos x + \sin y) = \sin x$

(iii) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

Solution :

(i) $\frac{dy}{dx} + y = e^{-x}$ . . . (I)

This is linear differential equation of the form

$\frac{dy}{dx} + Py = \text{Q}$ where $P = 1$, $Q = e^{-x}$

It's Solution is

$y\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\, dx + c$ . . . (II)

where I.F. $= e^{\int Pdx} = e^{\int dx} = e^{x}$

eq. (II) becomes,

$y\cdot e^{x} = \int e^{-x} \times e^{x}\cdot dx + c$

$\therefore$ $y\cdot e^{x} = \int e^{-x+x}\cdot dx + c$

$\therefore$ $y\cdot e^{x} = \int e^{0}\cdot dx + c$

$\therefore$ $y\cdot e^{x} = \int dx + c$

$\therefore$ $y\cdot e^{x} = x + c$ is the general solution.

(ii) $x \sin x \dfrac{dy}{dx} + (x \cos x + \sin x)\, y = \sin x$

divide by $x \sin x$, we get

$$\frac{dy}{dx} + \left(\cot x + \frac{1}{x}\right) y = \frac{1}{x} \qquad \ldots \text{(I)}$$

It is the linear differential equation of the type

$$\frac{dy}{dx} + Py = Q \qquad \text{where} \qquad P = \cot x + \frac{1}{x},$$

$$Q = \frac{1}{x}$$

Its solution is

$$y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c \qquad \ldots \text{(II)}$$

where $\quad \text{I.F.} = e^{\int P dx} = e^{\int \left(\cot x + \frac{1}{x}\right) dx}$

$$\text{I.F.} = e^{\int \cot x\, dx + \int \frac{dx}{x}}$$

$$\text{I.F.} = e^{\log|\sin x| + \log x}$$

$$\text{I.F.} = x \sin x$$

eq. (II) becomes,

$$y \cdot x \sin x = \int \frac{1}{x} \times x \sin x \cdot dx + c$$

$\therefore \quad xy \cdot \sin x = -\cos x + c$

$\therefore \quad xy \cdot \sin x + \cos x = c$ is the general solution.

(iii) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

$$\therefore \quad \frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{(1 + y^2)}$$

$$\therefore \quad \frac{dx}{dy} + \left(\frac{1}{1 + y^2}\right) x = \frac{\tan^{-1} y}{1 + y^2}$$

This is linear differential equation of the type

$$\frac{dx}{dy} + Px = Q \text{ where } P = \frac{1}{1 + y^2},\ Q = \frac{\tan^{-1} y}{1 + y^2}$$

Its solution is

$$x\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dy + c \qquad \ldots \text{(II)}$$

where $\quad \text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy}$

$$\text{I.F.} = e^{\tan^{-1} y}$$

eq. (II) becomes,

$$x \cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2} \cdot e^{\tan^{-1} y} \cdot dy + c \ldots \text{(III)}$$

in R.H.S. Put $\quad \tan^{-1} y = t$

differentiate *w. r. t. x*, we get

$$\therefore \qquad \frac{dy}{1 + y^2} = dt$$

eq. (III) becomes

$$x \cdot e^{\tan^{-1} y} = \int t \cdot e^t \cdot dt + c$$

$$= t \int e^t \cdot dt - \int [1 \times e^t]\, dt + c$$

$$= t \cdot e^t - \int e^t \cdot dt + c$$

$$= t \cdot e^t - e^t + c$$

$$x \cdot e^{\tan^{-1} y} = \tan^{-1} y \cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + \text{c}$$

$$\therefore \quad x = \tan^{-1} y - 1 + \frac{c}{e^{\tan^{-1} y}}$$

$\therefore \quad x + 1 - \tan^{-1} y = c \cdot e^{-\tan^{-1} y}$ is the solution.

Ex. 2: The slope of the targent to the curve at any point is equal to $y + 2x$. Find the equation of the curve passing through the origin.

Solution : Let P (x, y) be any point on the curve $y = f(x)$

The slope of the tangent at point P (x, y) is $\frac{dy}{dx}$.

$\therefore \quad \frac{dy}{dx} = y + 2x \qquad \therefore \quad \frac{dy}{dx} - y = 2x$

This is linear differential equation of the type

$\frac{dy}{dx} + Py = Q$ where $P = -1$, $Q = 2x$

Its solution is

$y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c \qquad \ldots \text{(I)}$

where $\quad \text{I.F.} = e^{\int P dx} = e^{\int -dx}$

$\text{I.F.} = e^{-x}$

eq. (I) becomes,

$y \cdot e^{-x} = \int 2x \times e^{-x} \cdot dx + c$

$y \cdot e^{-x} = 2\int x \cdot e^{-x} \cdot dx + c \qquad \ldots \text{(II)}$

Consider, $\int x \cdot e^{-x} \cdot dx$

$= x\int e^{-x} \cdot dx - \int \left(1 \times \frac{e^{-x}}{-1}\right) dx$

$= \frac{x \cdot e^{-x}}{-1} + \int e^{-x} \cdot dx$

$= -xe^{-x} \cdot dx + \int e^{-x} \cdot dx$

$= -xe^{-x} - e^{-x}$

(II) becomes

$y \cdot e^{-x} = 2\,[-xe^{-x} - e^{-x}] + c$

$\therefore \quad y = -2x - 2 + ce^{-x} \qquad \ldots \text{(III)}$

The curve passes through the origin (0, 0)

$\therefore \quad 0 = -2\,(0) - 2 + ce^{-0}$

$\therefore \quad 0 = -2 + c$

$\therefore \quad 2 = c$ Put in (III)

$\therefore \quad y = -2x - 2 + 2e^{-x}$

$\therefore \quad 2x + y + 2 = 2e^{-x}$

is the equation of the curve.

EXERCISE 6.5

(1) Solve the following differential equations :

(i) $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$

(ii) $\cos^2 x \frac{dy}{dx} + y = \tan x$

(iii) $(x + 2y^3)\frac{dy}{dx} = y$

(iv) $\frac{dy}{dx} + y \sec x = \tan x$

(v) $x\frac{dy}{dx} + 2y = x^2 \log x$

(vi) $(x + y)\frac{dy}{dx} = 1$

(vii) $(x + a)\frac{dy}{dx} - 3y = (x + a)^5$

(viii) $dr + (2r \cot \theta + \sin 2\theta)\, d\theta = 0$

(ix) $ydx + (x - y^2)\, dy = 0$

(x) $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\,(1 - x^2)^{\frac{1}{2}}$

(xi) $(1 + x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

(2) Find the equation of the curve which passes through the origin and has slope $x + 3y - 1$ at any point (x, y) on it.

(3) Find the equation of the curve passing through the point $\left(\dfrac{3}{\sqrt{2}}, \sqrt{2}\right)$ having slope of the tangent to the curve at any point (x, y) is $-\dfrac{4x}{9y}$.

(4) The curve passes through the point (0, 2). The sum of the co-ordinates of any point on the curve exceeds the slope of the tangent to the curve at that point by 5. Find the equation of the curve.

(5) If the slope of the tangent to the curve at each of its point is equal to the sum of abscissa and the product of the abscissa and ordinate of the point. Also the curve passes through the point (0, 1). Find the equation of the curve.

6.5 Application of differential Equations :

There are many situations where the relation in the rate of change of a function is known. This gives a differential equation of the function and we may be able to solve it.

6.5.1 Population Growth and Growth of Bacteria :

It is known that a number of bacteria in a culture increase with time. It means there is growth in the number of bacteria. It the population P increases at time t then the rate of change of P is proportional to the population present at that time.

$$\therefore \quad \frac{dP}{dt} \propto P$$

$$\therefore \quad \frac{dP}{dt} = k \cdot P, \quad (k > 0)$$

$$\therefore \quad \frac{dP}{P} = kdt$$

on integrating

$$\therefore \quad \int \frac{dP}{P} = \int kdt$$

$$\therefore \quad \log P = kt + c_1$$

$$\therefore \quad P = c \cdot e^{kt} \qquad \text{where } c = e^{c_1}$$

which gives the population at any time t.

SOLVED EXAMPLES

Ex. 1 : The population of a town increasing at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years.

$\left(\text{Given } \sqrt{\dfrac{3}{2}} = 1{\cdot}2247\right)$.

Solution : Let P be the population at time t. Since rate of increase of P is a proportional to P itself, we have,

$$\frac{dP}{dt} = k\cdot P \qquad \ldots (1)$$

where k is constant of proportionality.

Solving this differential equation, we get

$P = a\cdot e^{kt}$, where $a = e^{c}$ ... (2)

Initially $P = 40{,}000$ when $t = 0$

$\therefore$ From equation (2), we have

$40{,}000 = a\cdot 1 \qquad \therefore a = 40{,}000$

eq. (2) becomes

$\therefore \quad P = 40{,}000\cdot e^{kt} \qquad \ldots (3)$

Again given that $P = 60{,}000$ when $t = 40$

$\therefore$ From equation (3),

$60{,}000 = 40{,}000\cdot e^{40k}$

$$e^{40k} = \frac{3}{2} \qquad \ldots (4)$$

Now we have to find P when $t = 40 + 20$
$= 60$ years

$\therefore$ From equation (3), we have

$$P = 40{,}000\cdot e^{60k} = 40{,}000\,(e^{40k})^{\frac{3}{2}}$$

$$= 40{,}000\left(\frac{3}{2}\right)^{\frac{3}{2}} = 73482$$

$\therefore$ Required population will be 73482.

Ex. 2 : Bacteria increase at the rate proporational to the number of bacteria present. If the original number N doubles in 3 hours, find in how many hours the number of bacteria will be 4N?

Solution : Let x be the number of bacteria at time t. Since the rate of increase of x is proporational x, the differential equation can be written as :

$$\frac{dx}{dt} = kx$$

where k is constant of proportionality.

Solving this differential equation we have

$x = c_1\cdot e^{kt}$, where $c_1 = e^{c}$ ... (1)

Given that $x = \text{N}$ when $t = 0$

$\therefore$ From equation (1) we get

$\text{N} = c_1\cdot 1$

$\therefore \quad c_1 = \text{N}$

$\therefore \quad x = \text{N}\cdot e^{kt} \qquad \ldots (2)$

Again given that $x = 2\text{N}$ when $t = 3$

$\therefore$ From equation (2), we have

$2\text{N} = \text{N}\cdot e^{3k} \qquad \ldots (3)$

$e^{3k} = 2$

Now we have to find t, when $x = 4\text{ N}$

$\therefore$ From equation (2), we have

$4\text{ N} = \text{N}\cdot e^{kt}$

i.e. $4 = e^{kt} = (e^{3k})^{\frac{t}{3}}$

$\therefore \quad 2^2 = 2^{\frac{t}{3}} \qquad \ldots$ by eq. (3)

$\therefore \quad \dfrac{t}{3} = 2$

$\therefore \quad t = 6$

Therefore, the number of bacteria will be 4N in 6 hours.

6.5.2 Radio Active Decay :

We know that the radio active substances (elements) like radium, cesium etc. disintegrate with time. It means the mass of the substance decreases with time.

The rate of disintegration of such elements is always proportional to the amount present at that time.

If x is the amount of any material present at time t then

$$\frac{dx}{dt} = -k \cdot x$$

where k is the constant of proportionality and $k > 0$. The negative sign appears because x decreases as t increases.

Solving this differential equation we get

$$x = a \cdot e^{kt} \qquad \text{where } a = e^{c} \text{ (check!)} \quad \ldots (1)$$

If x_0 is the initial amount of radio active substance at time $t = 0$, then from equation (1)

$$x_0 = a \cdot 1$$

$$\therefore \quad a = x_0$$

$$\therefore \quad x = x_0\, e^{-kt} \qquad \ldots (2)$$

This expression gives the amount of radio active substance at any time t.

Half Life Period :

Half life period of a radio active substance is defined as the time it takes for half the amount/mass of the substance to disintegrate.

Ex. 3 : Bismath has half life of 5 days. A sample originally has a mass of 800 mg. Find the mass remaining after 30 days.

Solution : Let x be the mass of the Bismath present at time t.

Then $\frac{dx}{dt} = -k \cdot x$ where $k > 0$

Solving the differential equation, we get

$x = c \cdot e^{-kt}$ . . . (1)

where c is constant of proporationality.

Given that $x = 800$, when $t = 0$

using these values in euqation (1), we get

$800 = c \cdot 1 = c$

$\therefore$ $x = 800\, e^{-kt}$ . . . (2)

Since half life is 5 days, we have $x = 400$ when $t = 5$,

$\therefore$ From equation (2), we have

$400 = 800\, e^{-5k}$

$\therefore$ $e^{-5k} = \frac{400}{800} = \frac{1}{2}$ . . .(3)

Now we have determine x, when t = 30,

$\therefore$ From equation (2), we have

$x = 800\, e^{-30k} = 12 \cdot 5$ (verify !)

$\therefore$ The mass after 30 days will be $12 \cdot 5$ mg.

Now let us discuss another application of differential equation.

6.5.3 Newton's Law of Cooling :

Newton's law of cooling states that the rate of change of cooling heated body at any time is proporational to the difference between the temperature of a body and that of its surrounding medium.

Let θ be the temperature of a body at time t and θ_0 be the temperature of the medium.

Then $\frac{d\theta}{dt}$ is the rate of change of temperature with respect to time t and $\theta - \theta_0$ is the difference of temperature at time t. According to Newton's law of cooling.

$$\therefore \quad \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\therefore \quad \frac{d\theta}{dt} = -k(\theta - \theta_0) \qquad \ldots (1)$$

where k is constant of proportionality and negative sign indicates that difference of temperature is decreasing.

$$\text{Now} \quad \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\therefore \quad \frac{d\theta}{(\theta - \theta_0)} = -k\, dt$$

$\therefore$ Integrating and using the initial condition viz.

$$\therefore \quad \theta = \theta_1 \qquad \text{when } t = 0, \text{ we get}$$

$$\therefore \quad \theta = \theta_0 + (\theta_1 - \theta_0)\, e^{-kt} \text{ (verify)} \qquad \ldots (2)$$

Thus equation (2) gives the temperature of a body at any time t.

Ex. 4 : Water at 100°c cools in 10 minutes to 88°c in a room temperature of 25°c. Find the temperature of water after 20 minutes.

Solution : Let θ be the temperature of water at time t. Room temperature is given to be 25°c. Then according to Newton's law of cooling. we have

$$\frac{d\theta}{dt} \propto (\theta - 25)$$

$$\frac{d\theta}{dt} = -k(\theta - 25), \qquad \text{where } k > 0$$

After integrating and using initial condition.

We get $\theta = 25 + 75 \cdot e^{-kt}$ $\qquad \ldots (1)$

But given that $\theta = 88$°c when $t = 10$

$\therefore$ From equation (1) we get

$$88 = 25 + 75 \cdot e^{-10k}$$

$$\therefore \quad 63 = 75 \cdot e^{-10k} \quad \therefore \quad e^{-10k} = \frac{63}{75} = \frac{21}{25} \qquad \ldots (2)$$

Now we have to find θ, when $t = 20$,

$\therefore$ From equation (1) we have

$$\begin{aligned} \theta &= 25 + 75 \cdot e^{-20k} \\ &= 25 + 75\,(e^{-20k})^2 \\ &= 25 + 75\left(\frac{dy}{dx}\right)^2 \qquad \ldots \text{by (2)} \\ &= 25 + \frac{75 \times 21 \times 21}{25 \times 25} \\ &= 25 + \frac{1323}{25} = 77 \cdot 92 \end{aligned}$$

Therefore temperature of water after 20 minutes will be 77·92°c.

6.5.4 Surface Area :

Knowledge of a differential equation is also used to solve problems related to the surface area. We consider the following examples :

Ex. 5 : Water is being poured into a vessel in the form of an inverted right circular cone of semi vertical angle 45°c in such a way that the rate of change of volume at any moment is proporational to the area of the curved surfaces which is wet at that moment. Initially, the vessel is full to a height of 2 cms. And after 2 seconds the height becomes 10 cm. Show that after 3.5 seconds from that start, the height of water will be 16 cms.

Solution : Let the height of water at time t seconds be h cms.

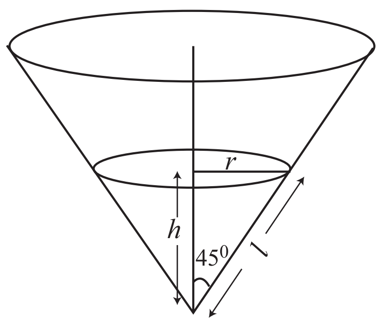

Fig. 6.2

We are given that initial height is 2 cms. and after 2 seconds, the height is 10 cms.

$$\therefore \quad h = 2 \text{ when } t = 0 \qquad \ldots (1)$$

$$\text{and } \quad h = 10 \text{ when } t = 2 \qquad \ldots (2)$$

Let v be the volume, r be the radius of the water surface and l be that slant height at time t seconds.

$\therefore$ Area of the curved surface at this moment is πrl.

But the semi vertical angle is 45°.

$$\therefore \quad \tan 45° = \frac{r}{h} = 1$$

$$\therefore \quad r = h$$

$$\text{and} \quad l^2 = r^2 + h^2 = 2h^2$$

$$\therefore \quad l = \sqrt{2}\, h$$

$$\therefore \quad \text{Area of the curved surface} = \pi rl = \pi \cdot h \cdot \sqrt{2}\, h$$

$$= \sqrt{2}\, \pi h^2$$

Since rate of change of volume is proporational to this area, we get

$$\frac{dv}{dt} \propto \sqrt{2}\,\pi h^2$$

$$\therefore \quad \frac{dv}{dt} = c\cdot\sqrt{2}\,\pi h^2$$

where c is constant of proportionality.

Let $c\sqrt{2}\,\pi = k$

$$\therefore \quad \frac{dv}{dt} = kh^2 \qquad \ldots (3)$$

$\therefore$ where k is constant

Now $v = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi r^2\cdot h = \frac{1}{3}\pi h^3, \text{ (since } r = h)$$

Differentiating with respect to t, we get

$$\therefore \quad \frac{dv}{dt} = \pi h^2 \frac{dh}{dt} \qquad \ldots (4)$$

Equating $\frac{dv}{dt}$ from (3) and (4) we get

$$\pi h^2 \frac{dh}{dt} = kh^2$$

$$\therefore \quad \frac{dh}{dt} = \frac{k}{\pi} = a \text{ (say)}$$

where a is constant.

integrating we get

$$h = at + b \qquad \ldots (5)$$

using (1) we have $2 = a.0 + b \quad \therefore \quad b = 2$

$\therefore$ Equation (5) becomes

$$h = at + 2$$

Now using (2) we get

$$10 = 2a + 2 \quad \therefore \quad a = 4$$

using the values of a and b in equation (5), we have

$$\therefore \quad h = 4t + 2$$

Now put $t = 3{\cdot}5$

$$\therefore \quad h = 4 \times 3{\cdot}5 + 2$$

$$= 14 + 2 = 16 \text{ cm}$$

Therefore, height of water after 3·5 seconds will be 16 cms.

EXERCISE 6.6

1. In a certain culture of bacteria the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.
2. If the population of a country doubles in 60 years, in how many years will it be triple (treble) under the assumption that the rate of increase is proporational to the number of inhabitants?
[Given log 2 = 0·6912, log 3 = 1·0986]
3. If a body cools from 80°c to 50°c at room temperature of 25°c in 30 minutes, find the temperature of the body after 1 hour.
4. The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number double in 1 hour, find the number of bacteria after $2\frac{1}{2}$ hours.
[Take $\sqrt{2} = 1{\cdot}414$]
5. The rate of disintegration of a radio active element at any time t is proportational to its mass at that time. Find the time during which the original mass of 1·5 gm. will disintegrate into its mass of 0·5 gm.
6. The rate of decay of certain substance is directly proportional to the amount present at that instant. Initially, there are 25 gms of certain substance and two hours later it is found that 9 gms are left. Find the amount left after one more hour.
7. Find the population of a city at any time t, given that the rate of increase of population is proporational to the population at the instant and that in a period of 40 years the population increased from 30,000 to 40,000.
8. A body cools according to Newton's law from 100°c to 60°c in 20 minutes. The temperature of the surrounding being 20°c how long will it take to cool down to 30°c?
9. A right circular cone has height 9 cms and radius of the base 5 cms. It is inverted and water is poured into it. If at any instant the water level rises at the rate of $\frac{\pi}{A}$ cms/ sec. where A is the area of water surface at that instant, show that the vessel will the full in 75 seconds.
10. Assume that a spherical raindrop evaporates at a rate proporational to its surface area. If its radius originally is 3mm and 1 hour later has been reduced to 2mm, find an expression for the radius of the raindrop at any time *t*.
11. The rate of growth of the population of a city at any time t is proportional to the size of the population. For a certain city it is found that the constant of proportionality is 0.04. Find the population of the city after 25 years if the initial population is 10,000. [Take $e = 2{\cdot}7182$]
12. Radium decomposes at the rate proportional to the amount present at any time. If *p* percent of amount disappears in one year, what percent of amount of radium will be left after 2 years ?

Let us Remember

- Equation which contains the derivative of a function is called a **differential equation.**
- The order of a differential equation is the highest order of the derivative appearing in the equation.
- The degree of the differential equation is the power of the highest ordered derivative present in the equation.
- Order and degree of a differential equation are always positive integers.
- Solution of a differential equation in which number of arbitary constants is equal to the order of a differential equation is the **general solution** of the differential equation.
- Solution obtained from the general solution by giving particular values to the arbitrary constants is the particular solution of the differential equation.
- The most general form of a **linear differential equation** of the first order is : $\frac{dy}{dx} + Py = Q$

  where P and Q are functions of x or constant.

  Its solution is given by : $y\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,dx + c$, where I.F. (integrating factor) $= e^{\int P dx}$
- Solution of a differential equation $\frac{dx}{dt} = kx$ is in the form $x = a\cdot e^{kt}$ where a is initial value of x.

  Further, $k > 0$ represents growth and $k > 0$, represents decay.
- Newton's law of cooling is $\theta = \theta_0 + (\theta_1 - \theta_0)\,e^{-kt}$.

MISCELLANEOUS EXERCISE 6

(I) Choose the correct option from the given alternatives :

(1) The order and degree of the differential equation $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$ are respectively . . .

(A) 2, 1 (B) 1, 2 (C) 3, 2 (D) 2, 3

(2) The differential equation of $y = c^2 + \frac{c}{x}$ is . . .

(A) $x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$

(B) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

(C) $x^3\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = y$

(D) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

(3) $x^2 + y^2 = a^2$ is a solution of ...

(A) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

(B) $y = x\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} + a^2y$

(C) $y = x\dfrac{dy}{dx} + a\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

(D) $\dfrac{d^2y}{dx^2} = (x + 1)\dfrac{dy}{dx}$

(4) The differential equation of all circles having their centers on the line $y = 5$ and touching the X-axis is

(A) $y^2\left(1 + \dfrac{dy}{dx}\right) = 25$

(B) $(y - 5)^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = 25$

(C) $(y - 5)^2 + \left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = 25$

(D) $(y - 5)^2\left[1 - \left(\dfrac{dy}{dx}\right)^2\right] = 25$

(5) The differential equation $y\dfrac{dy}{dx} + x = 0$ represents family of . . .

(A) circles (B) parabolas (C) ellipses (D) hyper bolas

(6) The solution of $\dfrac{1}{x}\cdot\dfrac{dy}{dx} = \tan^{-1} x$ is ...

(A) $\dfrac{x^2 \tan^{-1} x}{2} + c = 0$

(B) $x \tan^{-1} x + c = 0$

(C) $x - \tan^{-1} x = c$

(D) $y = \dfrac{x^2 \tan^{-1} x}{2} - \dfrac{1}{2}(x - \tan^{-1} x) + c$

(7) The solution of $(x + y)^2\dfrac{dy}{dx} = 1$ is ...

(A) $x = \tan^{-1}(x + y) + c$

(B) $y \tan^{-1}\left(\dfrac{x}{y}\right) = c$

(C) $y = \tan^{-1}(x + y) + c$

(D) $y + \tan^{-1}(x + y) = c$

(8) The solution of $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 - y^2}}{2}$ is . . .

(A) $\sin^{-1}\left(\dfrac{y}{x}\right) = 2 \log |x| + c$

(B) $\sin^{-1}\left(\dfrac{y}{x}\right) = \log |x| + c$

(C) $\sin\left(\dfrac{y}{x}\right) = \log |x| + c$

(D) $\sin\left(\dfrac{y}{x}\right) = \log |x| + c$

(9) The solution of $\dfrac{dy}{dx} + y = \cos x - \sin x$ is . . .

(A) $y e^x = \cos x + c$

(B) $y e^x + e^x \cos x = c$

(C) $y e^x = e^x \cos x + c$

(D) $y^2 e^x = e^x \cos x + c$

(10) The integrating factor of linear differential equation $x\dfrac{dy}{dx} + 2y = x^2 \log x$ is . . .

(A) $\dfrac{1}{x}$ (B) k (C) $\dfrac{1}{n^2}$ (D) x^2

(11) The solution of the differential equation $\dfrac{dy}{dx} = \sec x - y \tan x$ is

(A) $y \sec x + \tan x = c$ (B) $y \sec x = \tan x + c$

(C) $\sec x + y \tan x = c$ (D) $\sec x = y \tan x + c$

(12) The particular solution of $\dfrac{dy}{dx} = xe^{y-x}$, when $x = y = 0$ is . . .

(A) $e^{x-y} = x + 1$ (B) $e^{x+y} = x + 1$ (C) $e^x + e^y = x + 1$ (D) $e^{y-x} = x - 1$

(13) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a solution of . . .

(A) $\dfrac{d^2y}{dx^2} + yx + \left(\dfrac{dy}{dx}\right)^2 = 0$ (B) $xy\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

(C) $y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + y = 0$ (D) $xy\dfrac{dy}{dx} + y\dfrac{d^2y}{dx^2} = 0$

(14) The decay rate of certain substance is directly proporational to the amount present at that instant. Initially there are 27 grams of substance and 3 hours later it is found that 8 grams left. The amount left after one more hour is...

(A) $5\dfrac{2}{3}$ grams (B) $5\dfrac{1}{3}$ grams (C) $5{\cdot}1$ grams (D) 5 grams

(15) If the surrounding air is kept at 20°c and a body cools from 80°c to 70°c in 5 minutes, the temparature of the body after 15 minutes will be...

(A) 51·7°c (B) 54·7°c (C) 52·7°c (D) 50·7°c

(II) Solve the following :

(1) Determine the order and degree of the following differential equations :

(i) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + y = x^3$ (ii) $\left(\dfrac{d^3y}{dx^3}\right)^2 = \sqrt[5]{1 + \dfrac{dy}{dx}}$

(iii) $\sqrt[3]{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{d^2y}{dx^2}$ (iv) $\dfrac{dy}{dx} = 3y + \sqrt[4]{1 + 5\left(\dfrac{dy}{dx}\right)^2}$

(v) $\dfrac{d^4y}{dx^4} + \sin\left(\dfrac{dy}{dx}\right) = 0$

(2) In each of the following examples, verify that the given function is a solution of the differential equation.

(i) $x^2 + y^2 = r^2,\ x\dfrac{dy}{dx} + r\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = y$

(ii) $y = e^{ax}\sin bx\ ,\ \dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2 + b^2)\,y = 0$

(iii) $y = 3\cos(\log x) + 4\sin(\log x),\ x\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

(iv) $y = ae^{x} + be^{-x} + x^2,\ x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + x^3 = xy + 2$

(v) $x^2 = 2y^2\log y,\ x^2 + y^2 = xy\dfrac{dx}{dy}$

(3) Obtain the differential equation by eliminating the arbitrary constants from the following equations.

(i) $y^2 = a(b - x)(b + x)$

(ii) $y = a\sin(x + b)$

(iii) $(y - a)^2 = b(x + 4)$

(iv) $y = \sqrt{a\cos(\log x) + b\sin(\log x)}$

(v) $y = Ae^{3x+1} + Be^{-3x+1}$

(4) Form the differential equation of :

(i) all circles which pass through the origin and whose centres lie on X−axis.

(ii) all parabolas which have $4b$ as latus rectum and whose axes is parallel to Y−axis.

(iii) an ellipse whose minor axis is twice its major axis.

(iv) all the lines which are normal to the line $3x - 2y + 7 = 0$.

(v) the hyperbola whose length of transverse and conjugate axes are half of that of the given hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{36} = k$.

(5) Solve the following differential equations :

(i) $\log\left(\dfrac{dy}{dx}\right) = 2x + 3y$

(ii) $\dfrac{dy}{dx} = x^2y + y$

(iii) $\dfrac{dy}{dx} = \dfrac{2y - x}{2y + x}$

(iv) $x\,dy = (x + y + 1)\,dx$

(v) $\dfrac{dy}{dx} + y\cot x = x^2\cot x + 2x$

(vi) $y\log y = (\log y^2 - x)\dfrac{dy}{dx}$

(vii) $4\dfrac{dx}{dy} + 8x = 5e^{-3y}$

(6) Find the particular solution of the following differential equations :

(1) $y\,(1 + \log x) = (\log x^x)\,\dfrac{dy}{dx}$, when $y\,(e) = e^2$

(2) $(x + 2y^2)\,\dfrac{dy}{dx} = y$, when $x = 2,\ y = 1$

(3) $\dfrac{dy}{dx} - 3y \cot x = \sin 2x$, when $y\left(\dfrac{\pi}{2}\right) = 2$

(4) $(x + y)\,dy + (x - y)\,dx = 0$, when $x = 1 = y$

(5) $2e^{\frac{x}{y}}\,dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$, when $y\,(0) = 1$

(7) Show that the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y^2 + y + 1}{x^2 + x + 1}$ is given by $(x + y + 1) = c\,(1 - x - y - 2xy)$

(8) The normal lines to a given curve at each point (x, y) on the curve pass through $(2, 0)$. The curve passes through $(2, 3)$. Find the equation of the curve.

(9) The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t second.

(10) A person's assets start reducing in such a way that the rate of reduction of assets is proportional to the square root of the assets existing at that moment. If the assets at the begining are ₹ 10 lakhs and they dwindle down to ₹ 10,000 after 2 years, show that the person will be bankrupt in $2\dfrac{2}{9}$ years from the start.