We define some specific values which would be convenient for comparing two voltage or current waveforms.

a) Average or mean value of AC:

This is the average of all values of the voltage (or current) over one half cycle. As can be seen in Fig. 13.1, the average over a full cycle is always zero since the average value of $\sin\omega t$ over a cycle is zero. So the mean value of AC over a cycle has no significance and the mean value of AC is defined as the average over half cycle.

Average value of $\sin\theta$ in the range 0° to π^c

$$<\sin\theta> = \frac{\int_0^{\pi}\sin\theta\, d\theta}{\int_0^{\pi} d\theta} = \frac{[-\cos\theta]_0^{\pi}}{[\theta]_0^{\pi}}$$

$$= \frac{2}{\pi} = 0.637$$

Therefore, average value of current or emf = 0.637 × their peak value

i.e., $i_{av} = 0.637\ i_0$ and $e_{av} = 0.637\ e_0$

where i_{av} and e_{av} are the average values of alternating current and emf (voltage) respectively.

b) Root-mean-square (or rms) value:

A moving coil ammeter and voltmeter measure the average value of current and voltage applied across it respectively. It is obvious therefore that the moving coil instruments cannot be used to measure the alternating current and voltages. Hence in order to measure there quantities it is necessary to make use of a property which does not depend upon the changes in direction of alternating current or voltage. Heating effect depends upon the square of the current (the square of the current is always positive) and hence does not depend upon the direction of flow of current. Consider an alternating current of peak value i_0, flowing through a resistance R. Let H be the heat produced in time t. Now the same quantity of heat (H) can be produced in the same resistance (R) in the same time (t) by passing a steady current of constant magnitude through it. The value of such steady current is called the effective value or virtual value or rms value of the given alternating current and is denoted by i_{rms}. The relation between the rms value and peak value of alternating current is given by

$$i_{rms}^2 = \frac{\int_0^{2\pi} i^2 d\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} i_0^2 \sin^2\theta\, d\theta$$

$$= \frac{i_0^2}{2\pi}\int_0^{2\pi}\frac{(1-\cos 2\theta)}{2}\, d\theta$$

$$= \frac{i_0^2}{2\times 2\pi}\left[\left(\theta - \frac{\sin 2\theta)}{2}\right)\right]_0^{2\pi}$$

$$= \frac{i_0^2}{2}$$

$$\therefore i_{rms} = \frac{i_0}{\sqrt{2}} = 0.707\ i_0$$

Similarly it can be shown that

$$e_{rms} = \frac{e_0}{\sqrt{2}} = 0.707\ e_0$$

The heat produced by a sinusoidally varying AC over a complete cycle will be given by

$$H = \int_0^{2\pi/\omega} i^2(t) R dt$$

$$= \frac{R}{\omega}\int_0^{2\pi/\omega} i^2(\omega t)\, \mathrm{d}(\omega t)$$

$$= \frac{2\pi R}{\omega}\frac{i_0^2}{2}$$

$$H = R(i_{rms})^2 \cdot \frac{2\pi}{\omega}$$

It is the same as the heat produced by a DC current of magnitude i_{rms} for time $t = \frac{2\pi}{\omega}$.

Example 13.1: An alternating voltage is given by e = 6 sin 314 t. find (i) the peak value (ii) frequency (iii) time period and (iv) instantaneous value at time t = 2 ms

Solution:

$e = e_0 \sin\omega t$

e = 6 sin 314 *t*

(i) Comparing the two equations, the peak value of the alternating voltage is e_0 = 6 V

(ii) ω t = 314 *t* ∴ 2πft = 314 *t*

$$\text{Frequency } f = \frac{314}{2\pi} = 50 \text{ Hz}$$

(iii) Time period $T = \frac{1}{f} = \frac{1}{50} = 0.02\ s$

(iv) The instantaneous value of the voltage

At $t = 2 \times 10^{-3}$ s is $e = 6 \sin 314 \times 2 \times 10^{-3}$ = 6 V

13.4 Phasors:

The study of AC circuits is much simplified, if we represent alternating current and alternating emf as rotating vectors with the angle between them equal to the phase difference between the current and emf. These rotating vectors are called phasors.

A rotating vector that represents a quality varying sinusoidally with time is called a **phasor** and the diagram representing it is called **phasor** diagram.

The phasor for alternating emf and alternating current are inclined to the horizontal axis at angle ωt or ωt + α. and rotate in anticlockwise direction. The length of the arrow represents the maximum value of the quantity (i_0 and e_0).

The projection of the vector on fixed axis gives the instantaneous value of alternating current and alternating emf. In sine form, ($i = i_0 \sin \omega t$) and ($e = e_0 \sin \omega t$) projection is taken on Y-axis as shown in Fig. 13.2 (a). In cosine form $i = i_0$ cost ω t and $e = e_0 \cos \omega t$ projection is taken X-axis on as shown in Fig. 13.2 (b).

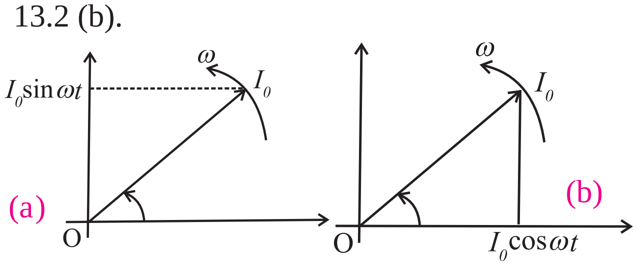


Fig. 13.2 (a) and (b): Phasor diagrams.

The representation of the harmonically varying quantities as rotating vectors enable us to use the laws of vector addition for adding these quantities.

13.5 Different Types of AC Circuits:

In this section we will derive voltage current relations for individual as well as combined circuit elements like resistors, inductors and capacitors, carrying a sinusoidal current. We assume the capacitor and inductor to be ideal unless otherwise specified.

(a) AC voltage applied to a resistor:

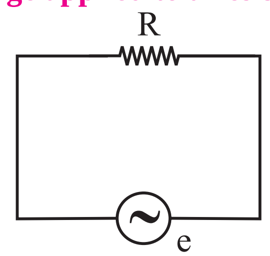


Fig.13.3 An AC voltage applied to a resistor.

Suppose a resistor of resistance *R* is connected to an AC source of emf with instantaneous value *e* given by

$$e = e_0 \sin \omega t \qquad \text{--- (13.2)}$$

Where e_0 is the peak value of the voltage and ω is its angular frequency. Let *e* be the potential drop across the resistance.

$$\therefore\ e = iR \qquad \text{--- (13.3)}$$

∵ instantaneous emf = instantaneous value of potential drop

From Eq (13.2) and Eq (13.3) we have,

$$iR = e = e_0 \sin \omega t$$

$$\therefore i = \frac{e}{R} = \frac{e_0 \sin \omega t}{R}$$

$$\therefore\ i = i_0 \sin \omega t \left(\because i_0 = \frac{e_0}{R}\right) \qquad \text{--- (13.4)}$$

Comparing $i_0 = \frac{e_0}{R}$ with Ohm's law, we find that resistors behave similarly for both AC and DC voltage. Hence the behaviour of *R* in DC and AC circuits is the same. *R* can reduce DC as well as AC equally effectively.

From Eq (13.2) and Eq (13.4) we know that for a resistor there is zero phase difference between instantaneous alternating current and instantaneous alternating emf, i.e., they are in phase. Both *e* and *i* reach zero, minimum and maximum values at the same time as shown in Fig. 13.4.

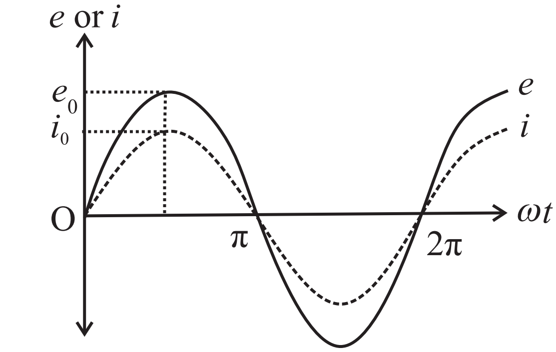


Fig. 13.4 Graph of *e* and *i* versus ωt.

Phasor diagram:

In the AC circuit containing *R* only, current and voltage are in the same phase, hence both phasors for *i* and for *e* are in the same direction making an angle ωt with OX. Their projections on vertical axis give their instantaneous values. The phase angle between alternating current and alternating voltage through *R* is zero as shown in Fig. 13.5.

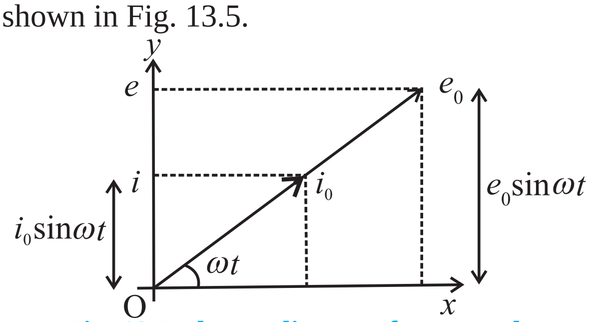


Fig. 13.5 Phasor diagram for a purely resistive circuit.

Example 13.2: An alternating voltage given by e = 140 sin 3142 t is connected across a pure resistor of 50 Ω. Find (i) the frequency of the source (ii) the rms current through the resistor.

Solution: Given

$e = 140 \sin 3142\, t$

$R = 50\ \Omega$

i) On comparing with $e = e_0 \sin \omega t$

We get

$\omega = 3142,\ e_0 = 140$ V

$\omega = 2\pi f \ \therefore f = \dfrac{\omega}{2\pi} = \dfrac{3142}{2\times 3.142} = 50$ Hz

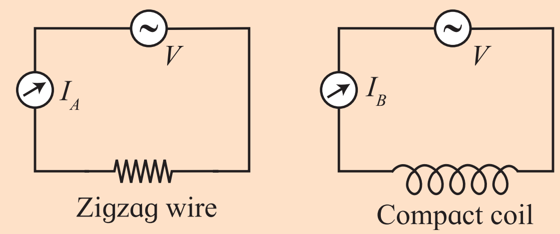
(ii) $e_0 = 140$ V

$e_{rms} = \dfrac{e_0}{\sqrt{2}} = \dfrac{140}{\sqrt{2}} = 99.29$ V

$\therefore i_{rms} = \dfrac{e_{rms}}{R} = \dfrac{99.29}{50} = 1.98$ A

Activity

Take 2 identical thin insulated copper wires about 10 cm long, imagine one of them in a zigzag form (called A) and the other in the form of a compact coil of average diameter not more than 5 cm (called B). Connect the two independently to 1.5 V cell or to a similar DC voltage and record the respective current passing through them as I_A and I_B. You will notice that the two are the same.



(b) AC voltage applied to an Inductor:

Let us now connect the source of alternating emf to a circuit containing pure inductor (*L*) only as shown in Fig. 13.6. Let us assume that the inductor has negligible resistance. The circuit is therefore a purely inductive circuit. Suppose the alternating emf supplied is represented by

$$e = e_0 \sin \omega t \qquad \text{--- (13.5)}$$

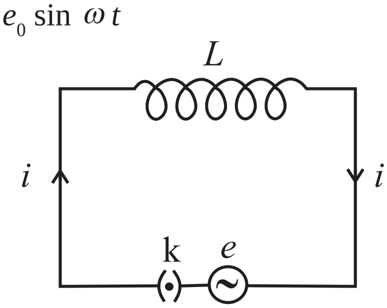


Fig. 13.6: An AC source connected to an Inductor.

When the key k is closed, current *i* begins to grow in the inductor because magnetic flux linked with it changes and induced emf is produced which opposes the applied emf (Faraday's law).

According to Lenz's law

$$e = -L\frac{di}{dt} \qquad \text{--- (13.6)}$$

Where e is the induced emf and $\frac{di}{dt}$ is the rate of change of current.

To maintain the flow of current in the circuit, applied emf (e) must be equal and opposite to the induced emf (e'). According to Kirchhoff's voltage law as there is no resistance in the circuit,

$$e = - e'$$

$$\therefore e = -\left(-L\frac{di}{dt}\right) = L\frac{di}{dt} \quad \text{(from Eq. (13.6))}$$

$$\therefore di = \frac{e}{L}dt$$

Integrating the above equation on both the sides, we get,

$$\int di = \int \frac{e}{L}dt$$

$$i = \int \frac{e_0 \sin \omega t}{L}dt \qquad (\because e = e_0 \sin \omega t)$$

$$i = \frac{e_0}{L}\left[\frac{-\cos \omega t}{\omega}\right] + \text{constant}$$

Constant of integration is time independent and has the dimensions of *i*. As the emf oscillates about zero, *i* also oscillates about zero so that there cannot be any component of current which is time independent.

Thus, the integration constant is zero

$$\therefore i = \frac{-e_0}{\omega L}\sin\left(\frac{\pi}{2} - \omega t\right) \qquad \left(\because \sin\left(\frac{\pi}{2} - \omega t\right) = \cos \omega t\right)$$

$$\therefore i = \frac{e_0}{\omega L}\sin\left[\omega t - \frac{\pi}{2}\right]$$

$$i = i_0 \sin\left[\omega t - \frac{\pi}{2}\right] \qquad \text{--- (13.7)}$$

where $i_0 = \frac{e_0}{\omega L}$ --- (13.8)

where i_0 is the peak value of current. Eq. (13.7) gives the alternating current developed in a purely inductive circuit when connected to a source of alternating emf.

Comparing Eq. (13.5) and (13.7) we find that the alternating current *i* lags behind the alternating voltage emf *e* by a phase angle of π/2 radians (90°) or the voltage across *L* leads the current by a phase angle of π/2 radians (90°) as shown in Fig. 13.7.

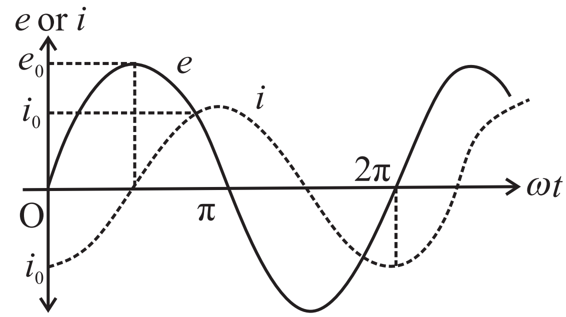


Fig 13.7: Graph of *e* and *i* versus ωt.

Phasor diagram:

The phasor representing peak emf e_0 makes an angle ωt in anticlockwise direction from horizontal axis. As current lags behind the voltage by 90°, so the phasor representing i_0 is turned clockwise with the direction of e_0 as shown in Fig. 13.8.

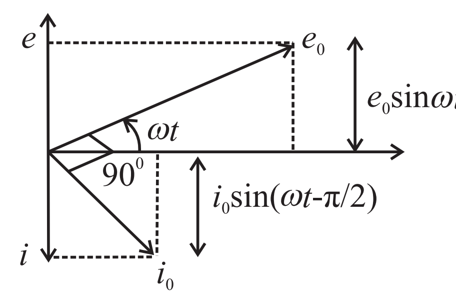


Fig. 13.8 Phasor diagram for purely inductive.

Inductive Reactance (X_L):

The opposing nature of inductor to the flow of alternating current is called inductive reactance.

Comparing Eq (13.8) with Ohm's law, $i_0 = \frac{e_0}{R}$ we find that ωL represents the effective resistance offered by the inductance *L*, it is called the inductive reactance and denoted by X_L.

$\therefore X_L = \omega L = 2\pi f L$. ($\because \omega = 2\pi/T = 2\pi f$)

where *f* is the frequency of the AC supply.

The function of the inductive reactance is similar to that of the resistance in a purely resistive circuit. It is directly proportional to the inductance (*L*) and the frequency (*f*) of the alternating current.

The dimensions of inductive reactance is the same as those of resistance and its SI unit is ohm (Ω).

In DC circuits $f = 0 \therefore X_L = 0$

It implies that a pure inductor offers zero resistance to DC, i.e., it cannot reduce DC. Thus, its passes DC and blocks AC of very high frequency.

In an inductive circuit, the self induced emf opposes the growth as well as decay of current.

Example 13.3: An inductor of inductance 200 mH is connected to an AC source of peak emf 210 V and frequency 50 Hz. Calculate the peak current. What is the instantaneous voltage of the source when the current is at its peak value?

Solution: Given

L = 200 mH =0.2 H

e_0 = 210 V

f = 50 Hz

Peak Current $i_0 = \dfrac{e_0}{X_L} = \dfrac{e_0}{2\pi fL}$

$= \dfrac{210}{2\times 3.142\times 50\times 0.2}$

$\therefore i_0$ = 3.342 A

As in an inductive AC circuit, current lags behind the emf by $\dfrac{\pi}{2}$, so the voltage is zero when the current is at its peak value.

c) AC voltage applied to a capacitor:

Let us consider a capacitor with capacitance C connected to an AC source with an emf having instantaneous value

$$e = e_0 \sin \omega t \qquad \text{--- (13.9)}$$

This is shown in Fig. 13.9

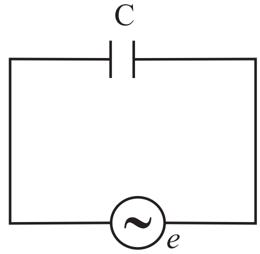


Fig: 13.9 An AC source connected to a capacitor.

The current flowing in the circuit transfers charge to the plates of the capacitor which produces a potential difference between the plates. As the current reverses its direction in each half cycle, the capacitor is alternately charged and discharged.

Suppose q is the charge on the capacitor at any given instant t. The potential difference across the plates of the capacitor is

$$V = \frac{q}{C} \text{ or } q = CV \qquad \text{--- (13.10)}$$

The instantaneous value of current (i) in the circuit is

$$i = \frac{dq}{dt} = \frac{d}{dt}(Ce) \quad (\because V = e \text{ at every instant})$$

$$= \frac{d}{dt}(Ce_0 \sin\omega t) \quad (\because e = e_0 \sin\omega t)$$

$$= Ce_0 \cos\omega t \cdot \omega$$

$$= \frac{e_0}{1/\omega C}\cos\omega t$$

$$\therefore i = \frac{e_0}{1/\omega C}\sin\left(\omega t + \frac{\pi}{2}\right) \left(\because \cos\omega t = \sin\left(\frac{\pi}{2} + \omega t\right)\right)$$

--- (13.11)

The current will be maximum when sin ($\omega t + \pi/2$) = 1, so that $i = i_0$ where, peak value of current is

$$i_0 = \frac{e_0}{1/\omega C} \qquad \text{--- (13.12)}$$

$$\therefore i = i_0 \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \text{--- (13.13)}$$

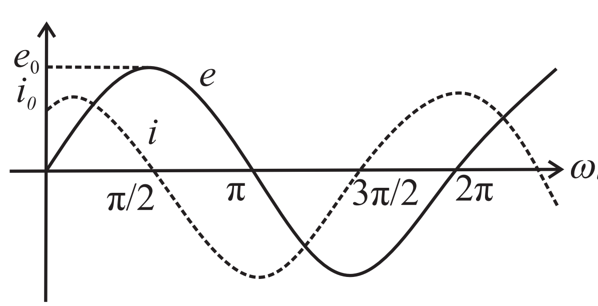


Fig. 13.10 Graph of e and i versus ωt.

From Eq. (13.9) and Eq. (13.13) we find that in an AC circuit containing a capacitor only, the alternating current i leads the alternating emf e by phase angle of $\pi/2$ radian as shown in Fig. 13.10.

Phasor diagram:

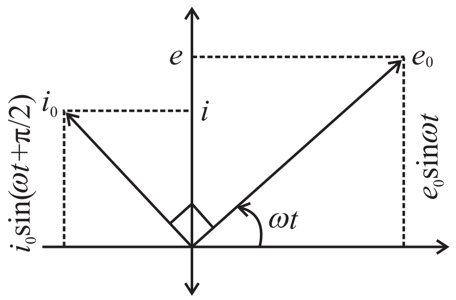


Fig.13.11: Phasor diagram for purely capacitive circuit.

The phasor representing peak emf makes an angle ωt in anticlockwise direction with respect to horizontal axis. As current leads the voltage by 90°, the phasor representing i_0 current is turned 90° anticlockwise with respect to the phasor representing emf e_0. The projections of these phasors on the vertical axis gives instantaneous values of e and i.

Capacitive Reactance: The instantaneous value of alternating current through a capacitor is given by

$$i = \frac{e_0}{(1/\omega C)}\sin\left(\omega t+\frac{\pi}{2}\right)$$

$$= i_0\sin\left(\omega t+\frac{\pi}{2}\right)$$

Comparing Eq. (13.12) with Ohm's law, $i_0 = \frac{e_0}{R}$ we find that $(1/\omega C)$ represents effective resistance offered by the capacitor called the capacitive reactance denoted by X_C.

$\therefore X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}$ where f is the frequency of AC supply.

The function of capacitive reactance in a purely capacitive circuit is to limit the amplitude of the current similar to the resistance in a purely resistive circuit.

X_C varies inversely as the frequency of AC and also as the capacitance of the condenser.

In a DC circuit, $f = 0 \therefore X_C = \infty$

Thus, capacitor blocks DC and acts as open circuit while it passes AC of high frequency.

The dimensions of capacitive reactance are the same as that of resistance and its SI unit is ohm (Ω).

Table 13.1: Comparison between resistance and reactance.

Resistance	Reactance
Equally effective for AC and DC	Current is affected (reduced) but energy is not consumed (heat is not generated). The energy consumption by a coil is due to its resistive component.
Its value is independent of frequency of the AC	Inductive reactance ($X_L = 2\pi fL$) is directly proportional and capacitive reactance $\left(Xc = \frac{1}{2\pi fC}\right)$ is inversely proportional to the frequency of the AC.
Current opposed by a resistor is in phase with the voltage.	Current opposed by a pure inductor lags in phase while that opposed by a pure capacitor leads is phase by π^c over the voltage.

Example 13.4: 4. A Capacitor of 2 μF is connected to an AC source of emf $e = 250 \sin 100\pi t$. Write an equation for instantaneous current through the circuit and give reading of AC ammeter connected in the circuit.

Solution: Given

$C = 2\mu F = 2\times10^{-6} F$

$e_0 = 250$ V

$\omega = 100\pi$ rad/sec

The instantaneous current through the circuit

$i = i_0\sin(\omega t + \frac{\pi}{2})$

$= \omega C e_0 \sin(\omega t + \frac{\pi}{2})$

$= 3.142 \times 2 \times 10^{-4} \times 250 \sin(100\pi t + \frac{\pi}{2})$

$= 0.1571 \sin (100\pi t + \frac{\pi}{2})$

Reading of the AC ammeter is

$i_{rms} = 0.707\, i_0$

$= 0.707 \times 0.1571$

$i_{ms} = 0.111$A

(d) AC circuit containing resistance inductance and capacitance in series (LCR circuit):

Above we have studied the opposition offered by a resistor, pure inductor and capacitor to the flow of AC current independently.

Now let us consider the total opposition offered by a resistor, pure inductor and capacitor connected in series with the alternating source of emf as shown in Fig. 13.12.

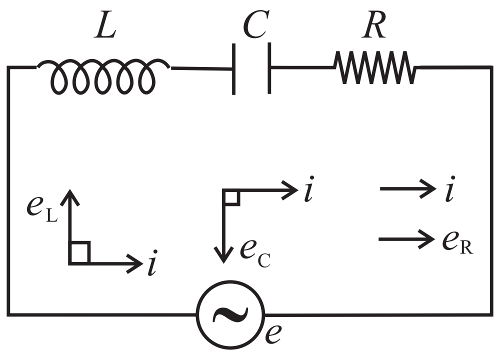


Fig. 13.12: Series LCR circuit.

Let a pure resistor R, a pure inductance L and an ideal capacitor of capactance C be connected in series to a source of alternative emf. As R, L and C are in series, the current at any instant through the three elements has the same amplitude and phase. Let it be represented by

$i = i_0 \sin \omega t.$

The voltage across each element bears a different phase relationship with the current. The voltages e_L, e_C and e_R are given by

$e_R = iR$, $e_L = iX_L$ and $e_C = iX_C$

As the voltage across the capacitor lags behind the alternating current by 90°, it is represented by $\overrightarrow{OC}$, rotated clockwise through 90° from the direction of $\vec{i}_0$. $\overrightarrow{OC}$ is along OY′ in the phasor diagram shown in the phasor diagrams in Fig. 13.13.

As e_R is in phase with current i_0 the vector e_R is drawn in the same direction as that of i, along the positive direction of X-axis represented by $\overrightarrow{OA}$. The voltage across L and C have a phase different of 180° hence the net reactive voltage is $(e_L - e_C)$.

Assuming $e_L > e_C$ represented by OB′ in the figure.

The resultant of $\overrightarrow{OA}$ and $\overrightarrow{OB'}$ is the diagonal OK of the rectangle OAKB'

$$\therefore OK = \sqrt{OA^2 + OB^2}$$

$$e_0 = \sqrt{e_R^{\,2} + (e_L - e_C)^2}$$

$$= \sqrt{(i_0 R)^2 + (i_0 X_L - i_0 X_C)^2}$$

$$e_0 = i_0 \sqrt{R^2 + (X_L - X_C)^2}$$

$$\therefore \frac{e_0}{i_0} = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\frac{e_0}{i_0} = Z$$

Comparing the above equation with the relation $\frac{V}{i} = R$, the quantity $\sqrt{R^2 + (X_L - X_C)}$ represents the effective opposition offered by the inductor, capacitor and resistor connected in series to the flow of AC current. This total effective resistance of LCR circuit is called the impedance of the circuit and is represented by Z. The reciprocal of impedance of an AC circuit is called admittance. Its SI unit is ohm^{-1} or siemens.

It can be defined as the ratio of rms voltage to the rms value of current Impedance is expressed in ohm (Ω).

Phasor diagram:

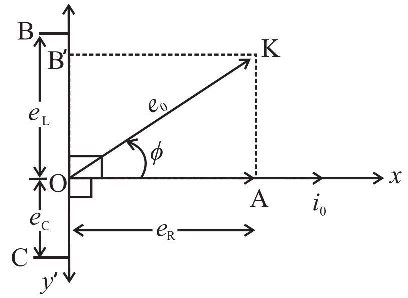


Fig. 13.13: Phasor diagram for an LCR circuit.

From the phasor diagram (Fig. 13.13) it can be seen that in an AC circuit containing L, C and R, the voltage leads the current by a phase angle ϕ,

$$\tan\phi = \frac{AK}{OA} = \frac{OB'}{OA} = \frac{e_L - e_C}{e_R} = \frac{i_o X_L - i_o X_C}{i_o R}$$

$$\tan\phi = \frac{X_L - X_C}{R} \quad \therefore \phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

$\therefore$ The alternating current in LCR circuit would be represented by

$$i = i_0 \sin(\omega t + \phi)$$

and $e = e_0 \sin(\omega t + \phi)$

We can now discuss three cases based on the above discussion.

(i) When $X_L = X_C$ then $\tan\phi = 0$.
Hence voltage and current are in phase. Thus the AC circuit is non inductive.

(ii) When $X_L > X_C$, $\tan\phi$ is positive $\therefore \phi$ is positive.
Hence voltage leads the current by a phase angle ϕ The AC circuit is inductance dominated circuit.

(iii) When $X_L < X_C$, $\tan\phi$ is negative $\therefore \phi$ is negative.
Hence voltage leads the current by a phase angle ϕ The AC circuit is capacitance dominated circuit.

Impedance triangle:

From the three phasors

$$\vec{e}_R = \vec{i}_0 R,\ \vec{e}_L = \vec{i}_0 X_L,\ \vec{e}_C = \vec{i}_0 X_C$$

we obtain the impedance triangle as shown in Fig 13.14.

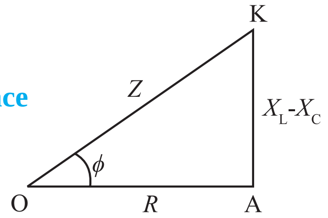


Fig. 13.14: Impedance triangle.

The diagonal OK represents the impedance Z of the AC circuit.

$Z = \sqrt{R^2 + (X_L - X_C)^2}$, the base OA represents the Ohmic resistance R and the perpendicular AK represents reactance (X_L-X_C). $\angle AOK = \phi$, is the phase angle by which the voltage leads the current is the circuit, where $\tan\phi = \frac{X_L - X_C}{R}$

Example 13.5: A 100mH inductor, a 25 µF capacitor and a 15 Ω resistor are connected in series to a 120 V, 50 Hz AC source. Calculate

(i) impedance of the circuit at resonance

(ii) current at resonance

(iii) Resonant frequency

Solution: Given

$L = 100$ mH $= 10^{-1}$H

$C = 25$ µF $= 25 \times 10^{-6}$F

$R = 15\Omega$

$e_{rms} = 120$ V

$f = 50$ Hz

(i) At resonance, $Z = R = 15\Omega$

(ii) $i_{rms} = \frac{e_{rms}}{R} = \frac{120}{15} = 8A$

$$\text{(iii)}\ f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\times3.142\sqrt{10^{-1}\times25\times10^{-6}}} = \frac{1}{9.9356\times10^{-3}}$$

$\therefore f = 100.65$ Hz

Example 13.6: A coil of 0.01H inductance and 1Ω resistance is connected to 200V, 50Hz AC supply. Find the impedance of the circuit and time lag between maximum alternating voltage and current.

Solution: Given

Inductance $L = 0.01$H

Resistance $R = 1\Omega$

$e_0 = 200$ V

Frequency $f = 50$ Hz

Impedance of the circuit $Z = \sqrt{R^2 + X_L^2}$

$$= \sqrt{R^2 + (2\pi fL)^2}$$

$$= \sqrt{(1)^2 + (2\times3.142\times50\times0.01)^2}$$

$$= \sqrt{10.872} = 3.297\Omega$$

$$\tan\phi = \frac{\omega L}{R} = \frac{2\pi fL}{R} = \frac{2\times3.142\times50\times0.01}{1} = 3.142$$

$$\phi = tan^{-1}(3.142) = 72.35°$$

Phase difference, $\phi = \frac{72.35\times\pi}{180}$ rad

Time lag between maximum alternating voltage and current

$$\Delta t = \frac{\phi}{\omega} = \frac{72.35\times\pi}{180\times2\pi\times50} = 0.004\ s$$