3. INDEFINITE INTEGRATION

Let us Study

- Definition and Properties
- Different Techniques : **1.** by substitution **2.** by parts **3.** by partial fraction

Introduction :

In differential calculus, we studied differentiation or derivatives of some functions. We saw that derivatives are used for finding the slopes of tangents, maximum or minimum values of the function.

Now we will try to find the function whose derivative is known, or given $f(x)$. We will find $g(x)$ such that $g'(x) = f(x)$. Here the integration of $f(x)$ with respect to x is $g(x)$ or $g(x)$ is called the primitive of $f(x)$. For example, we know that the derivative of x^3 *w. r. t.* x is $3x^2$. So $\frac{d}{dx} x^3 = 3x^2$; and integral of $3x^2$ *w. r. t.* x is x^3. This is shown with the sign of integration namely '$\int$'. We write $\int 3x^2 \cdot dx = x^3$.

In this chapter we restrict ourselves only to study the methods of integration. The theory of integration is developed by Sir Isaac Newton and Gottfried Leibnitz.

$\int f(x) \cdot dx = g(x)$, read as an integral of $f(x)$ with respect to x, is $g(x)$. Since the derivative of constant function with respect to x is zero (0), we can also write

$\int f(x) \cdot dx = g(x) + c$, where c is an arbitarary constant and c can take infinitely many values.

For example :

$f(x) = x^2 + c$ represents familly of curves for different values of c.

$f'(x) = 2x$ gives the slope of the tangent to $f(x) = x^2 + c$.

In the figure we have shown the curves

$y = x^2$, $y = x^2 + 4$, $y = x^2 - 5$.

Note that at the points (2, 4), (2, 8) (2, −1) respectivelly on those curves, the slopes of tangents are 2 (2) = 4.

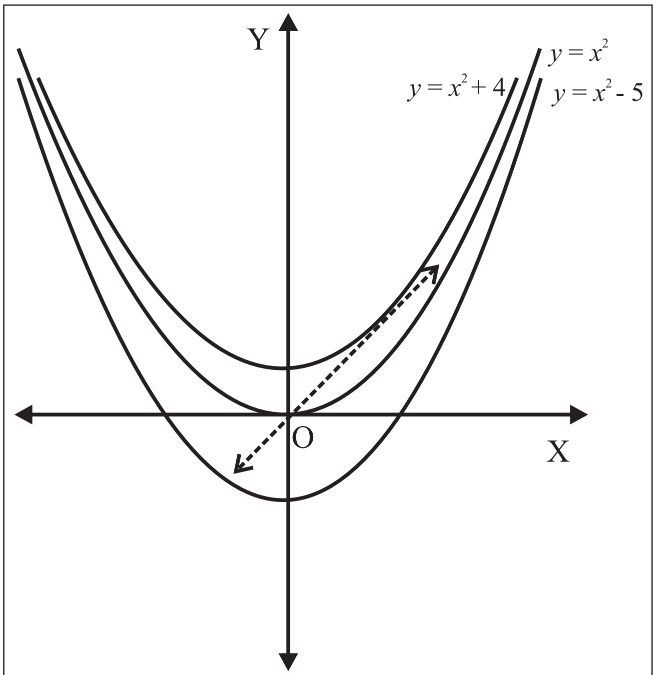

Fig. 3.1.1

3.1.1 Elementary Integration Formulae

(i) $$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)\,x^{n}}{(n+1)} \quad , n \neq -1$$
$$= x^{n} \quad \Rightarrow \quad \therefore \quad \int x^{n}\cdot dx = \frac{x^{n+1}}{n+1} + c$$
$$\frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{(n+1)\cdot a}\right) = \frac{(n+1)\,(ax+b)^{n}}{(n+1)}$$
$$= (ax+b)^{n} \quad \Rightarrow \quad \therefore \quad \int (ax+b)^{n}\cdot dx = \frac{(ax+b)^{n+1}}{n+1}\cdot\frac{1}{a} + c$$

This result can be extended for n replaced by any rational $\frac{p}{q}$.

(ii) $$\frac{d}{dx}\left(\frac{a^{x}}{\log a}\right) = a^{x},\ a>0 \quad \Rightarrow \quad \therefore \quad \int a^{x}\cdot dx = \frac{a^{x}}{\log a} + c$$
$$\int A^{ax+b}\cdot dx = \frac{A^{ax+b}}{\log A}\cdot\frac{1}{a} + c\ ,\ A>0$$

(iii) $$\frac{d}{dx}e^{x} = e^{x} \quad \Rightarrow \quad \int e^{x}\cdot dx = e^{x} + c$$
$$\int e^{ax+b}\cdot dx = e^{ax+b}\cdot\frac{1}{a} + c$$

(iv) $$\frac{d}{dx}\sin x = \cos x \quad \Rightarrow \quad \int \cos x\cdot dx = \sin x + c$$
$$\int \cos\,(ax+b)\cdot dx = \sin\,(ax+b)\cdot\frac{1}{a} + c$$

(v) $$\frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad \int \sin x\cdot dx = -\cos x + c$$
$$\int \sin\,(ax+b)\cdot dx = -\cos\,(ax+b)\cdot\frac{1}{a} + c$$

(vi) $$\frac{d}{dx}\tan x = \sec^{2} x \quad \Rightarrow \quad \int \sec^{2} x\cdot dx = \tan x + c$$
$$\int \sec^{2}\,(ax+b)\cdot dx = \tan\,(ax+b)\cdot\frac{1}{a} + c$$

(vii) $$\frac{d}{dx}\sec x = \sec x\cdot\tan x \quad \Rightarrow \quad \int \sec x\cdot\tan x\cdot dx = \sec x + c$$
$$\int \sec\,(ax+b)\cdot\tan\,(ax+b)\cdot dx = \sec\,(ax+b)\cdot\frac{1}{a} + c$$

(viii) $$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x\cdot\cot x \quad \Rightarrow \quad \int \operatorname{cosec} x\cdot\cot x\cdot dx = -\operatorname{cosec} x + c$$
$$\int \operatorname{cosec}\,(ax+b)\cdot\cot\,(ax+b)\cdot dx = -\operatorname{cosec}\,(ax+b)\cdot\frac{1}{a} + c$$

(ix) $$\frac{d}{dx}\cot x = -\operatorname{cosec}^{2} x \quad \Rightarrow \quad \int \operatorname{cosec}^{2} x\cdot dx = -\cot x + c$$
$$\int \operatorname{cosec}^{2}\,(ax+b)\cdot dx = -\cot\,(ax+b)\cdot\frac{1}{a} + c$$

(x) $$\frac{d}{dx}\log x = \frac{1}{x},\ x>0 \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \log x + c,\ x \neq 0.$$

$$\therefore \quad \text{also} \int \frac{1}{(ax+b)}\cdot dx = \log\,(ax+b)\cdot\frac{1}{a} + c$$

We assume that the trigonometric functions and logarithmic functions are defined on the respective domains.

3.1.2

Theorem 1 : If f and g are real valued integrable functions of x, then

$$\int [f(x) + g(x)] \cdot dx = \int f(x) \cdot dx + \int g(x) \cdot dx$$

Theorem 2 : If f and g are real valued integrable functions of x, then

$$\int [f(x) - g(x)] \cdot dx = \int f(x) \cdot dx - \int g(x) \cdot dx$$

Theorem 3 : If f and g are real valued integrable functions of x, and k is constant, then

$$\int k\,[f(x)] \cdot dx = k \int f(x) \cdot dx$$

Proof : 1. Let $\int f(x) \cdot dx = g_1(x) + c_1$ and $\int g(x) \cdot dx = g_2(x) + c_2$ then

$$\frac{d}{dx}[(g_1(x) + c_1)] = f(x) \quad \text{and} \quad \frac{d}{dx}[(g_2(x) + c_2)] = g(x)$$

$$\therefore \quad \frac{d}{dx}[(g_1(x) + c_1) + (g_2(x) + c_2)]$$

$$= \frac{d}{dx}[(g_1(x) + c_1)] + \frac{d}{dx}[(g_2(x) + c_2)]$$

$$= f(x) + g(x)$$

By definition of integration.

$$\int f(x) + g(x) = (g_1(x) + c_1) + (g_2(x) + c_2)$$

$$= \int f(x) \cdot dx + \int g(x) \cdot dx$$

Note : Students can construct the proofs of the other two theorems (Theorem 2 and Theorem 3).

SOLVED EXAMPLES

Ex. : Evaluate the following :

1. $\int (x^3 + 3^x) \cdot dx$

Solution :

$$\int (x^3 + 3^x) \cdot dx$$

$$= \int x^3 \cdot dx + \int 3^x \cdot dx$$

$$= \frac{x^4}{4} + \frac{3^x}{\log 3} + c$$

2. $$\int\left(\sin x+\frac{1}{x}+\frac{1}{\sqrt[3]{x}}\right)\cdot dx$$

Solution : $$\int\left(\sin x+\frac{1}{x}+\frac{1}{\sqrt[3]{x}}\right)\cdot dx$$

$$= \int\sin x\cdot dx+\int\frac{1}{x}\cdot dx+\int\frac{1}{\sqrt[3]{x}}\cdot dx$$

$$= \int\sin x\cdot dx+\int\frac{1}{x}\cdot dx+\int x^{-\frac{1}{3}}\cdot dx$$

$$= -\cos x+\log x+\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+c$$

$$= -\cos x+\log x+\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+c$$

3. $$\int(\tan x+\cot x)^2\cdot dx$$

Solution : $$\int(\tan x+\cot x)^2\cdot dx$$

$$= \int(\tan^2 x+2\tan x\cdot\cot x+\cot^2 x)\cdot dx$$

$$= \int(\tan^2 x+2+\cot^2 x)\cdot dx$$

$$= \int(\sec^2 x-1+2+\operatorname{cosec}^2 x-1)\cdot dx$$

$$= \int(\sec^2 x+\operatorname{cosec}^2 x)\cdot dx$$

$$= \int\sec^2 x\cdot dx+\int\operatorname{cosec}^2 x\cdot dx$$

$$= \tan x+(-\cot x)+c$$

$$= \tan x-\cot x+c$$

4. $$\int\frac{\sqrt{x}+1}{x+\sqrt{x}}\cdot dx$$

Solution : $$\int\frac{\sqrt{x}+1}{x+\sqrt{x}}\cdot dx$$

$$= \int\frac{\sqrt{x}+1}{\sqrt{x}(\sqrt{x}+1)}\cdot dx$$

$$= \int\frac{1}{\sqrt{x}}\cdot dx$$

$$= 2\cdot\int\frac{1}{2\sqrt{x}}\cdot dx$$

$$= 2\sqrt{x}+c$$

5. $$\int\frac{e^{4\log x}-e^{5\log x}}{x^5}\cdot dx$$

Solution : $$\int\frac{e^{4\log x}-e^{5\log x}}{x^5}\cdot dx$$

$$= \int\frac{e^{\log x^4}-e^{\log x^5}}{x^5}\cdot dx,\qquad \because a^{\log_a f(x)}=f(x)$$

$$= \int\left(\frac{x^4-x^5}{x^5}\right)\cdot dx$$

$$= \int\left(\frac{1}{x}-1\right)\cdot dx$$

$$= \log(x)-x+c$$

6. $$\int\frac{2x+3}{5x-1}\cdot dx$$

Solution : $$\frac{\mathrm{N}}{\mathrm{D}}=\mathrm{Q}+\frac{\mathrm{R}}{\mathrm{D}}$$

$$\begin{array}{r|l} & \quad\frac{2}{5} \\ (5x-1) & \quad 2x+3 \\ & -\quad 2x-\frac{2}{5} \\ & -\quad\quad + \\ \hline & \quad 3+\frac{2}{5}=\frac{17}{5} \end{array}$$

$$\therefore\quad 2x+3=\frac{2}{5}(5x-1)+3+\frac{2}{5}$$

$$\mathrm{I} = \int\left[\frac{2}{5}+\frac{\frac{17}{5}}{5x-1}\right]\cdot dx$$

$$= \frac{2}{5}x+\frac{17}{5}\log(5x-1)\cdot\frac{1}{5}+c$$

$$= \frac{2}{5}x+\frac{17}{25}\log(5x-1)+c$$

7. $$\int \frac{1}{\sqrt{3x+1}-\sqrt{3x-5}} \cdot dx$$

Solution : $$\int \frac{1}{\sqrt{3x+1}-\sqrt{3x-5}} \cdot dx$$

$$= \int \left(\frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\right) \cdot \left(\frac{\sqrt{3x+1}+\sqrt{3x-5}}{\sqrt{3x+1}+\sqrt{3x-5}}\right) \cdot dx$$

$$= \int \frac{\sqrt{3x+1}+\sqrt{3x-5}}{3x+1-3x+5} \cdot dx$$

$$= \int \frac{\sqrt{3x+1}+\sqrt{3x-5}}{6} \cdot dx$$

$$= \frac{1}{6} \cdot \int \left((3x+1)^{\frac{1}{2}} + (3x-5)^{\frac{1}{2}}\right) \cdot dx$$

$$= \frac{1}{6} \left\{ \int (3x+1)^{\frac{1}{2}} \cdot dx + \int (3x-5)^{\frac{1}{2}} \cdot dx \right\}$$

$$= \frac{1}{6} \cdot \left\{ \frac{(3x+1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) \cdot 3} + \frac{(3x-5)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) \cdot 3} \right\} + c$$

$$= \frac{1}{18} \cdot \left\{ \frac{2}{3}(3x+1)^{\frac{3}{2}} + \frac{2}{3}(3x-5)^{\frac{3}{2}} \right\} + c$$

$$= \frac{1}{27} \cdot \left\{ (3x+1)^{\frac{3}{2}} + (3x-5)^{\frac{3}{2}} \right\} + c$$

8. $$\int \frac{2x-7}{\sqrt{3x-2}} \cdot dx$$

Solution : Express $(2x-7)$ in terms of $(3x-2)$

$$2x-7 = \frac{2}{3}(3x-2) + \frac{4}{3} - 7$$

$$= \frac{2}{3}(3x-2) - \frac{17}{3}$$

$$I = \int \left[\frac{\frac{2}{3}(3x-2) - \frac{17}{3}}{\sqrt{3x-2}} \right] \cdot dx$$

$$= \int \left[\frac{\frac{2}{3}(3x-2)}{\sqrt{3x-2}} - \frac{\frac{17}{3}}{\sqrt{3x-2}} \right] \cdot dx$$

$$= \frac{2}{3} \int \sqrt{3x-2} \cdot dx - \frac{17}{3} \int \frac{1}{\sqrt{3x-2}} \cdot dx$$

$$= \frac{2}{3} \int (3x-2)^{\frac{1}{2}} \cdot dx - \frac{17}{3} \int \frac{1}{\sqrt{3x-2}} \cdot dx$$

$$= \frac{2}{3} \cdot \frac{(3x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \cdot \frac{1}{3} - \frac{17}{3} \cdot 2 \cdot \left(\sqrt{3x-2}\right) \cdot \frac{1}{3} + c$$

$$= \frac{4}{27} \cdot (3x-2)^{\frac{3}{2}} - \frac{34}{9} \cdot (3x-2)^{\frac{1}{2}} + c$$

9. $$\int \frac{x^3}{x-1} \cdot dx$$

Solution :

$$I = \int \frac{x^3-1+1}{x-1} \cdot dx$$

$$= \int \left(\frac{x^3-1}{x-1} + \frac{1}{x-1} \right) \cdot dx$$

$$= \int \left(\frac{(x-1)(x^2+x+1)}{(x-1)} + \frac{1}{x-1} \right) \cdot dx$$

$$= \int \left(x^2+x+1+\frac{1}{x-1} \right) \cdot dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + \log(x-1) + c$$

10. $$\int \frac{3^x - 4^x}{5^x} \cdot dx$$

Solution :

$$I = \int \left(\frac{3^x}{5^x} - \frac{4^x}{5^x} \right) \cdot dx$$

$$= \int \left[\left(\frac{3}{5}\right)^x - \left(\frac{4}{5}\right)^x \right] \cdot dx$$

$$= \frac{\left(\frac{3}{5}\right)^x}{\log\frac{3}{5}} - \frac{\left(\frac{4}{5}\right)^x}{\log\frac{4}{5}} + c$$

11. $\int \cos^3 x \cdot dx$

Solution : $\cos 3A = 4\cos^3 A - 3\cos A$

$$I = \int \frac{1}{4}(\cos 3x + 3\cos x)\cdot dx$$

$$= \frac{1}{4}\left(\sin 3x \cdot \frac{1}{3} + 3\cdot \sin x\right) + c$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$$

12. $\int \sqrt{1+\sin 3x}\cdot dx$

Solution :

$$I = \int \sqrt{\cos^2\frac{3x}{2} + \sin^2\frac{3x}{2} + 2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2}}\cdot dx$$

$$= \int \sqrt{\left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right)^2}\cdot dx$$

$$= \int \left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right)\cdot dx$$

$$= \sin\frac{3x}{2}\cdot\frac{1}{\frac{3}{2}} - \cos\frac{3x}{2}\cdot\frac{1}{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left(\sin\frac{3x}{2} - \cos\frac{3x}{2}\right) + c$$

13. $\int \sin^4 x \cdot dx$

Solution :

$$I = \int (\sin^2 x)^2 \cdot dx$$

$$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \cdot dx$$

$$= \frac{1}{4}\cdot\int (1 - 2\cos 2x + \cos^2 2x)\cdot dx$$

$$= \frac{1}{4}\cdot\int \left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right]\cdot dx$$

$$= \frac{1}{4}\cdot\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)\cdot dx$$

$$= \frac{1}{4}\cdot\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)\cdot dx$$

$$= \frac{1}{4}\cdot\left[\frac{3}{2}x - 2\sin 2x\cdot\frac{1}{2} + \frac{1}{2}\sin 4x\cdot\frac{1}{4}\right] + c$$

$$= \frac{1}{4}\cdot\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + c$$

14. $\int \sin 5x \cdot \cos 7x \cdot dx$

Solution : We know that

$$2\sin A\cdot\cos B = \sin(A+B) + \sin(A-B)$$

$$I = \frac{1}{2}\int 2\sin 5x\cdot\cos 7x\cdot dx$$

$$= \frac{1}{2}\int [\sin(5x+7x) + \sin(5x-7x)]\cdot dx$$

$$= \frac{1}{2}\int [\sin(12x) + \sin(-2x)]\cdot dx$$

$$= \frac{1}{2}\int (\sin 12x - \sin 2x)\cdot dx$$

$$= \frac{1}{2}\cdot\left[-\cos 12x\cdot\frac{1}{12} + \cos 2x\cdot\frac{1}{2}\right] + c$$

$$I = -\frac{1}{24}\cos 12x + \frac{1}{4}\cos 2x + c$$

15. $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x\cdot\cos^2 x}\cdot dx$

Solution : $I = \int \left(\frac{\sin^3 x}{\sin^2 x\cdot\cos^2 x} - \frac{\cos^3 x}{\sin^2 x\cdot\cos^2 x}\right)\cdot dx$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)\cdot dx$$

$$= \int \left(\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} - \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\right)\cdot dx$$

$$= \int (\sec x\cdot\tan x - \operatorname{cosec} x\cdot\cot x)\cdot dx$$

$$= \sec x - (-\operatorname{cosec} x) + c$$

$$I = \sec x + \operatorname{cosec} x + c$$

16. $\displaystyle\int \frac{1}{1-\sin x}\cdot dx$

Solution :

$$\begin{aligned}
\text{I} &= \int\left(\frac{1}{1-\sin x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)\cdot dx\\
&= \int \frac{1+\sin x}{1-\sin^2 x}\cdot dx\\
&= \int \frac{1+\sin x}{\cos^2 x}\cdot dx\\
&= \int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)\cdot dx\\
&= \int(\sec^2 x+\sec x\cdot\tan x)\cdot dx\\
&= \tan x+\sec x+c
\end{aligned}$$

17. $\displaystyle\int\left(\frac{\cos x}{1-\cos x}\right)\cdot dx$

Solution :

$$\begin{aligned}
\text{I} &= \int\left(\frac{\cos x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)\cdot dx\\
&= \int \frac{\cos x\,(1+\cos x)}{1-\cos^2 x}\cdot dx\\
&= \int\left(\frac{\cos x+\cos^2 x)}{\sin^2 x}\right)\cdot dx\\
&= \int\left(\frac{\cos x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\right)\cdot dx\\
&= \int(\operatorname{cosec} x\cdot\cot x+\cot^2 x)\cdot dx\\
&= \int(\operatorname{cosec} x\cdot\cot x+\operatorname{cosec}^2 x-1)\cdot dx\\
&= (-\operatorname{cosec} x)+(-\cot x)-x+c\\
&= -\operatorname{cosec} x-\cot x-x+c
\end{aligned}$$

Activity :

18. $\displaystyle\int \frac{\cos x-\cos 2x}{1-\cos x}\cdot dx$

Solution :

$$\begin{aligned}
&\int \frac{\cos x-\cos 2x}{1-\cos x}\cdot dx\\
&= \int \frac{\cos x-(\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,)}{1-\cos x}\cdot dx\\
&= \int \frac{\cos x-\;.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.}{1-\cos x}\cdot dx\\
&= \int \frac{\cos x\,(\,1-\cos x)+\;.\,.\,.\,.\,.\,.\,.\,.\,.\,.}{1-\cos x}\cdot dx\\
&= \int\left[\cos x+\frac{.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.}{1-\cos x}\right]\cdot dx\\
&= \int[\cos x+(1+\cos x)]\cdot dx\\
&= \int(1+2\cos x)\cdot dx\\
&= x+2\sin x+c
\end{aligned}$$

19. $\int \sin^{-1}(\cos 3x)\cdot dx$

Solution :

$$I = \int \sin^{-1}\left(\sin\frac{\pi}{2}-3x\right)\cdot dx$$

$$= \int\left(\frac{\pi}{2}-3x\right)\cdot dx$$

$$= \frac{\pi}{2}x - 3\frac{x^2}{2} + c$$

20. $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)\cdot dx$

Solution :

$$I = \int \cot^{-1}\left(\frac{1+\cos 2x}{\sin 2x}\right)\cdot dx$$

$$= \int \cot^{-1}\left(\frac{2\cos^2 x}{2\sin x\cdot\cos x}\right)\cdot dx$$

$$= \int \cot^{-1}(\cot x)\cdot dx$$

$$= \int x\cdot dx = \frac{x^2}{2} + c$$

21. $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\cdot dx$

Solution :

$$I = \int \tan^{-1}\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}\cdot dx$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\cdot dx$$

$$= \int \tan^{-1}\sqrt{\tan^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\cdot dx$$

$$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]\cdot dx$$

$$= \int\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot dx$$

$$= \frac{\pi}{4}x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + c$$

EXERCISE 3.1

I. Integrate the following functions *w. r. t. x* :

(i) $x^3 + x^2 - x + 1$ (ii) $x^2\left(1-\frac{2}{x}\right)^2$

(iii) $3\sec^2 x - \frac{4}{x} + \frac{1}{x\sqrt{x}} - 7$

(iv) $2x^3 - 5x + \frac{3}{x} + \frac{4}{x^5}$ (v) $\frac{3x^3-2x+5}{x\sqrt{x}}$

II. Evaluate :

(i) $\int \tan^2 x\cdot dx$ (ii) $\int \frac{\sin 2x}{\cos x}\cdot dx$

(iii) $\int \frac{\sin x}{\cos^2 x}\cdot dx$ (iv) $\int \frac{\cos 2x}{\sin^2 x}\cdot dx$

(v) $\int \frac{\cos 2x}{\sin^2 x\cdot\cos^2 x}\cdot dx$ (vi) $\int \frac{\sin x}{1+\sin x}\cdot dx$

(vii) $\int \frac{\tan x}{\sec x+\tan x}\cdot dx$ (viii) $\int \sqrt{1+\sin 2x}\cdot dx$

(ix) $\int \sqrt{1-\cos 2x}\cdot dx$ (x) $\int \sin 4x\cdot\cos 3x\cdot dx$

III. Evaluate :

(i) $\int \frac{x}{x+2}\cdot dx$ (ii) $\int \frac{4x+3}{2x+1}\cdot dx$

(iii) $\int \frac{5x+2}{3x-4}\cdot dx$ (iv) $\int \frac{x-2}{\sqrt{x+5}}\cdot dx$

(v) $\int \frac{2x-7}{\sqrt{4x-1}}\cdot dx$ (vi) $\int \frac{\sin 4x}{\cos 2x}\cdot dx$

(vii) $\int \sqrt{1+\sin 5x}\cdot dx$ (viii) $\int \cos^2 x\cdot dx$

(ix) $\int \frac{2}{\sqrt{x}-\sqrt{x+3}}\cdot dx$

(x) $\int \frac{3}{\sqrt{7x-2}-\sqrt{7x-5}}\cdot dx$

IV. $f'(x) = x - \frac{3}{x^3}$, $f(1) = \frac{11}{2}$ then find $f(x)$.

3.2 Methods of integration :

We have evaluated the integrals which can be reduced to standard forms by algebric or trigonometric simplifications. This year we are going to study three special methods of reducing an integral to a standard form, namely –

1. Integration by substitution
2. Integration by parts
3. Integration by partial fraction

3.2.1 Integration by substitution :

Theorem 1 : If $x = \phi(t)$ is a differentiable function of t, then $\int f(x) \cdot dx = \int f[\phi(t)] \cdot \phi'(t)\, dt$.

Proof : $x = \phi(t)$ is a differentiable function of t.

$\therefore \quad \dfrac{dx}{dt} = \phi'(t)$

Let $\int f(x)\, dx = g(x) \Rightarrow \dfrac{d}{dx}[g(x)] = f(x)$

By Chain rule,

$$\begin{aligned} \frac{d}{dt}[g(x)] &= \frac{d}{dx}[g(x)] \cdot \frac{dx}{dt} \\ &= f(x) \cdot \frac{dx}{dt} \\ &= f[\phi(t)] \cdot \phi'(t) \end{aligned}$$

By definition of integration,

$$g(x) = \int f[\phi(t)] \cdot \phi'(t) \cdot dt$$

$$\therefore \quad \int f(x) \cdot dx = \int f[\phi(t)] \cdot \phi'(t) \cdot dt$$

For example 1 : $\int 3x^2 \sin(x^3) \cdot dx$

Let $x^3 = t$

$\therefore \quad 3x^2\, dx = dt$

$= \int \sin t \cdot dt$

$= -\cos t + c$

$= -\cos(x^3) + c$

Corollary I :

If $\int f(x) \cdot dx = g(x) + c$

then $\int f(ax+b) \cdot dx = g(ax+b)\dfrac{1}{a} + c$

Proof : Let $I = \int f(ax+b) \cdot dx$

put $ax + b = t$

Differentiating both the sides

$a \cdot dx = 1 \cdot dt \Rightarrow dx = \dfrac{1}{a} dt$

$$\begin{aligned} I &= \int f(t) \cdot \frac{1}{a} \cdot dt \\ &= \frac{1}{a} \cdot \int f(t) \cdot dt \\ &= \frac{1}{a} \cdot g(t) + c \\ &= \frac{1}{a} \cdot g(ax+b) + c \end{aligned}$$

$\therefore \quad \int f(ax+b) \cdot dx = g(ax+b)\dfrac{1}{a} + c$

For example : $\int \sec^2(5x-4) \cdot dx$

$= \dfrac{1}{5} \tan(5x-4) + c$

Corollary II :

$$\int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$$

Proof : Let $I = \int [f(x)]^{n+1} \cdot f'(x) \cdot dx$

put $f(x) = t$

Differentiating both the sides

$f'(x) \cdot dx = dt$

$$\begin{aligned} I &= \int [t]^n \cdot dt \\ &= \frac{t^{n+1}}{n+1} + c, \qquad n \neq -1 \\ &= \frac{[f(x)]^{n+1}}{n+1} + c \end{aligned}$$

$\therefore \quad \int [f(x)]^n \cdot f'(x) \cdot dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$

For example : $\int \dfrac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} \cdot dx$

$= \int [(\sin^{-1} x)^3] \cdot \left(\dfrac{1}{\sqrt{1-x^2}}\right) \cdot dx$

$= \dfrac{(\sin^{-1} x)^4}{4} + c$

Corollary III :

$$\int \frac{f'(x)}{f(x)} \cdot dx = \log(f(x)) + c$$

Proof : Consider $\int \dfrac{f'(x)}{f(x)} \cdot dx$

put $f(x) = t$

Differentiating both the sides

$f'(x) \cdot dx = dt$

$$\begin{aligned} I &= \int \frac{1}{t} \cdot dt \\ &= \log(t) + c \\ &= \log\big(f(x)\big) + c \end{aligned}$$

$\therefore \quad \int \dfrac{f'(x)}{f(x)} \cdot dx = \log\big(f(x)\big) + c$

For example : $\int \cot x \cdot dx$

$= \int \dfrac{\cos x}{\sin x} \cdot dx$

$\dfrac{d}{dx} \sin x = \cos x$

$= \int \left(\dfrac{\frac{d}{dx}\sin x}{\sin x} \cdot dx\right)$

$= \log(\sin x) + c$

Corollary IV :

$$\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx = 2\sqrt{f(x)} + c$$

Proof : Consider $\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx$

put $f(x) = t$

Differentiating both the sides

$f'(x) \cdot dx = dt$

$$\text{I} = \int \frac{1}{\sqrt{t}} \cdot dt$$

$$= 2 \cdot \int \frac{1}{2\sqrt{t}} \cdot dt$$

$$= 2\sqrt{t} + c$$

$$= 2\sqrt{f(x)} + c$$

$$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx = 2\sqrt{f(x)} + c$$

For example : $\int \frac{1}{x\sqrt{\log x}} \cdot dx$

$$= \int \left(\frac{\frac{1}{x}}{\sqrt{\log x}} \right) \cdot dx$$

$$= \int \left(\frac{\frac{d}{dx} \log x}{\sqrt{\log x}} \cdot dx \right)$$

$$= 2\sqrt{\log x} + c$$

Using corollary III, $\int \frac{f'(x)}{f(x)} \cdot dx = \log\left(f(x)\right) + c$ we find the integrals of some trigonometric functions.

3.2.2 Integrals of trignometric functions :

1. $\int \tan x \cdot dx$

Solution :

$$\text{I} = \int \tan x \cdot dx$$

$$= \int \frac{\sin x}{\cos x} \cdot dx$$

$$= -\int \frac{-\sin x}{\cos x} \cdot dx$$

$$= -\log(\cos x) + c$$

$$= \log(\sec x) + c$$

Activity :

2. $\int \cot(5x - 4) \cdot dx$

Solution :

$$\text{I} = \int \frac{\text{............}}{\sin(5x - 4)} \cdot dx$$

$$= \frac{1}{\text{.....}} \int \frac{5\cos(5x - 4)}{\text{............}} \cdot dx$$

$$\frac{d}{dx}(\text{............}) = \text{............}$$

$$= \frac{1}{5} \log\left[\sec(5x - 4)\right] + c$$

3. $\int \sec x \cdot dx = \log(\sec x + \tan x) + c$

Solution : Let $I = \int \sec x \cdot dx$

$$= \int \frac{(\sec x)(\sec x + \tan x)}{\sec x + \tan x} \cdot dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \cdot dx$$

$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} \cdot dx$$

$\because \frac{d}{dx}(\sec x + \tan x) = \sec x \cdot \tan x + \sec^2 x$

$\therefore \int \sec x \cdot dx = \log(\sec x + \tan x) + c$

Also,

$$\int \sec x \cdot dx = \log\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right] + c$$

Activity :

4. $\int \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x) + c$

Solution : Let $I = \int \operatorname{cosec} x \cdot dx$

$$= \int \frac{(\operatorname{cosec} x)(\ldots\ldots\ldots\ldots)}{(\ldots\ldots\ldots\ldots)} \cdot dx$$

$$= \int \frac{\ldots\ldots\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots} \cdot dx$$

$$= \int \frac{-\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x}{\ldots\ldots\ldots} \cdot dx$$

$\frac{d}{dx}(\operatorname{cosec} x - \cot x)$

$= \ldots\ldots\ldots\ldots\ldots\ldots$

$= \log(\operatorname{cosec} x - \cot x) + c$

$\therefore \int \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x) + c$

Also,

$$\int \operatorname{cosec} x \cdot dx = \log\left(\tan\frac{x}{2}\right) + c$$

SOLVED EXAMPLES

Ex. : Evaluate the following functions :

1. $\int \frac{\cot(\log x)}{x} \cdot dx$

Solution : Let $I = \int \frac{\cot(\log x)}{x} \cdot dx$

put $\log x = t$

$\therefore \frac{1}{x} \cdot dx = 1 \cdot dt$

$= \int \cot t \cdot dt$

$= \log(\sin t) + c$

$= \log(\sin \log x) + c$

2. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot dx$

Solution : Let $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot dx$

put $\sqrt{x} = t$

$\therefore \frac{1}{2\sqrt{x}} \cdot dx = 1 \cdot dt$

$\therefore \frac{1}{\sqrt{x}} \cdot dx = 2 \cdot dt$

$= 2 \cdot \int \cos t \cdot dt$

$= 2 \cdot \sin t + c$

$= 2 \cdot \sin\sqrt{x} + c$

3. $\displaystyle\int \frac{\sec^8 x}{\operatorname{cosec} x} \cdot dx$

Solution : $\displaystyle I = \int \sec^7 x \cdot \sec x \cdot \frac{1}{\operatorname{cosec} x} \cdot dx$

$\displaystyle = \int \sec^7 x \cdot \frac{1}{\cos x} \cdot \sin x \cdot dx$

$= \int \sec^7 x \cdot \tan x \cdot dx$

$= \int \sec^6 x \cdot \sec x \cdot \tan x \cdot dx$

put $\sec x = t$

$\therefore \quad \sec x \cdot \tan x \cdot dx = dt$

$= \int t^6 \cdot dt$

$\displaystyle = \frac{t^7}{7} + c$

$\displaystyle = \frac{\sec^7 x}{7} + c$

4. $\displaystyle\int \frac{1}{x + \sqrt{x}} \cdot dx$

Solution : $\displaystyle I = \int \frac{1}{x + \sqrt{x}} \cdot dx$

$\displaystyle = \int \frac{1}{\sqrt{x}\,(\sqrt{x} + 1)} \cdot dx$

put $\sqrt{x} + 1 = t$

$\displaystyle \therefore \quad \frac{1}{2\sqrt{x}} \cdot dx = 1 \cdot dt$

$\displaystyle \therefore \quad \frac{1}{\sqrt{x}} \cdot dx = 2 \cdot dt$

$\displaystyle = \int \frac{1}{t} \cdot 2 \cdot dt$

$\displaystyle = 2 \cdot \int \frac{1}{t} \cdot dt$

$= 2 \cdot \log (t) + c$

$= 2 \cdot \log (\sqrt{x} + 1) + c$

5. $\int 5^{5^x} \cdot 5^x \cdot dx$

Solution : $I = \int 5^{5^x} \cdot 5^x \cdot dx$

put $5^x = t$

$\therefore \quad 5^x \cdot \log 5 \cdot dx = 1\ dt$

$\displaystyle 5^x \cdot dx = \frac{1}{\log 5} \cdot dt$

$\displaystyle = \int 5^t \cdot \frac{1}{\log 5} \cdot dt$

$\displaystyle I = \frac{1}{\log 5} \cdot \int 5^t \cdot dt$

$\displaystyle = \frac{1}{\log 5} \cdot 5^t \cdot \frac{1}{\log 5} + c$

$\displaystyle = \left(\frac{1}{\log 5}\right)^2 \cdot 5^{5^x} + c$

6. $\displaystyle\int \frac{1}{1 + e^{-x}} \cdot dx$

Solution : $\displaystyle I = \int \frac{1}{1 + e^{-x}} \cdot dx$

$\displaystyle = \int \frac{1}{1 + \frac{1}{e^x}} \cdot dx$

$\displaystyle = \int \frac{1}{\frac{e^x + 1}{e^x}} \cdot dx$

$\displaystyle = \int \frac{e^x}{e^x + 1} \cdot dx$

$\displaystyle \because \quad \frac{d}{dx}\left(e^x + 1 \right) \cdot dx = e^x$

$= \log [\, e^x + 1 \,] + c$

7. $\displaystyle\int \frac{e^x (1 + x)}{\cos (x \cdot e^x)} \cdot dx$

Solution : put $x \cdot e^x = t$

Differentiating both sides

$(x \cdot e^x + e^x \cdot 1) \cdot dx = 1\ dt$

$e^x (1 + x) \cdot dx = 1\ dt$

$\displaystyle I = \int \frac{1}{\cos t} \cdot dt$

$= \int \sec t \cdot dt$

$= \log (\sec t + \tan t) + c$

$= \log (\sec (xe^x) + \tan (xe^x)) + c$

8. $$\int \frac{1}{3x + 7x^{-n}} \cdot dx$$

Solution : Consider $\int \frac{1}{3x + 7x^{-n}} \cdot dx$

$$= \int \frac{1}{3x + \frac{7}{x^n}} \cdot dx = \int \frac{1}{\frac{3x^{n+1} + 7}{x^n}} \cdot dx$$

$$= \int \frac{x^n}{3x^{n+1} + 7} \cdot dx$$

put $3x^{n+1} + 7 = t$

Differentiate $w.\ r.\ t.\ x$

$3(n + 1)\, x^n \cdot dx = dt$

$\therefore \quad x^n \cdot dx = \frac{1}{3(n+1)}\, dt$

$$= \int \frac{\frac{1}{3(n+1)}}{t} \cdot dt$$

$$= \frac{1}{3(n+1)} \cdot \log(t) + c$$

$$= \frac{1}{3(n+1)} \cdot \log\left(3x^{n+1} + 7\right) + c$$

9. $$\int (3x + 2)\sqrt{x - 4} \cdot dx$$

Solution : put $x - 4 = t$

$\therefore \quad x = 4 + t$

Differentiate

$1 \cdot dx = 1 \cdot dt$

$$= \int [\,3(4 + t) + 2\,] \cdot \sqrt{t} \cdot dt$$

$$= \int (14 + 3t) \cdot t^{\frac{1}{2}} \cdot dt$$

$$= \int \left(14t^{\frac{1}{2}} + 3t^{\frac{3}{2}}\right) \cdot dt$$

$$= 14\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 3\,\frac{t^{\frac{5}{2}}}{\frac{5}{2}} \cdot dx$$

$$= \frac{28}{3}(x - 4)^{\frac{3}{2}} + \frac{6}{5}(x - 4)^{\frac{5}{2}} + c$$

10. $$\int \frac{\sin(x + a)}{\cos(x - b)} \cdot dx$$

Solution :

$$= \int \frac{\sin[(x - b) + (a + b)]}{\cos(x - b)} \cdot dx$$

$$= \int \frac{\sin(x - b) \cdot \cos(a + b) + \cos(x - b) \cdot \sin(a + b)}{\cos(x - b)} \cdot dx$$

$$= \int \left[\frac{\sin(x - b) \cdot \cos(a + b)}{\cos(x - b)} + \frac{\cos(x - b) \cdot \sin(a + b)}{\cos(x - b)}\right]$$

$$= \int [\cos(a + b) \cdot \tan(x - b) + \sin(a + b)] \cdot dx$$

$$= \cos(a + b) \cdot \log(\sec(x - b)) + x \cdot \sin(a + b) + c$$

11. $\displaystyle\int \frac{e^x+1}{e^x-1}\cdot dx$

Solution :

$$I = \int \frac{e^x-1+2}{e^x-1}\cdot dx$$

$$= \int \left(\frac{e^x-1}{e^x-1}+\frac{2}{e^x-1}\right)\cdot dx$$

$$= \int \left(1+\frac{2}{e^x-1}\right)\cdot dx$$

$$= \int dx + \int \frac{2}{e^x(1-e^{-x})}\cdot dx$$

$$= \int 1dx + 2\int \frac{e^{-x}}{1-e^{-x}}\cdot dx$$

put $(1-e^{-x}) = t$

Differentiate *w. r. t. x*

$-(e^{-x})\,(-1)\cdot dx = 1\,dt$

$e^{-x}\cdot dx = 1\,dt$

$$I = \int 1dx + 2\int \frac{1}{t}\cdot dt$$

$$= x + 2\cdot \log(t) + c$$

$$= x + 2\log(1-e^{-x}) + c$$

$$\therefore \int \frac{e^x+1}{e^x-1}\cdot dx = x + 2\log(1-e^{-x}) + c$$

12. $\displaystyle\int \frac{1}{1-\tan x}\cdot dx$

Solution :

$$I = \int \frac{1}{1-\dfrac{\sin x}{\cos x}}\cdot dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}$$

$$= \int \frac{\cos x}{\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos x - \dfrac{1}{\sqrt{2}}\cos x\right)}\cdot dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos x}{\cos\dfrac{\pi}{4}\cos x - \sin\dfrac{\pi}{4}\sin x}\cdot dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos x}{\cos\left(x+\dfrac{\pi}{4}\right)}\cdot dx$$

put $x+\dfrac{\pi}{4} = t \quad \therefore x = t-\dfrac{\pi}{4}$

Differentiating both sides

$1\cdot dx = 1\cdot dt$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos\left(t-\dfrac{\pi}{4}\right)}{\cos t}\cdot dt$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos t\cdot\cos\dfrac{\pi}{4} + \sin t\cdot\sin\dfrac{\pi}{4}}{\cos t}\cdot dt$$

$$= \frac{1}{\sqrt{2}}\int \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan t\right]\cdot dt$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\left[t + \log(\sec t)\right] + c$$

$$= \frac{1}{2}\left[x+\frac{\pi}{4} + \log\sec\left(x+\frac{\pi}{4}\right)\right] + c$$

To evaluate the integrals of type $\displaystyle\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\cdot dx$, express the Numerator as Nr = λ (Dr) + μ (Dr)' , find the constants λ & μ by compairing the co-efficients of like terms and then integrate the function.

EXERCISE 3.2 (A)

I. Integrate the following functions *w. r. t. x* :

1. $\dfrac{(\log x)^n}{x}$

2. $\dfrac{(\sin^{-1} x)^{\frac{3}{2}}}{\sqrt{1-x^2}}$

3. $\dfrac{1+x}{x \cdot \sin(x+\log x)}$

4. $\dfrac{x \cdot \sec^2(x^2)}{\sqrt{\tan^3(x^2)}}$

5. $\dfrac{e^{3x}}{e^{3x}+1}$

6. $\dfrac{(x^2+2)}{(x^2+1)} \cdot a^{x+\tan^{-1} x}$

7. $\dfrac{e^x \cdot \log(\sin e^x)}{\tan(e^x)}$

8. $\dfrac{e^{2x}+1}{e^{2x}-1}$

9. $\sin^4 x \cdot \cos^3 x$

10. $\dfrac{1}{4x+5x^{-11}}$

11. $x^9 \cdot \sec^2(x^{10})$

12. $e^{3\log x} \cdot (x^4+1)^{-1}$

13. $\dfrac{\sqrt{\tan x}}{\sin x \cdot \cos x}$

14. $\dfrac{(x-1)^2}{(x^2+1)^2}$

15. $\dfrac{2\sin x \cdot \cos x}{3\cos^2 x + 4\sin^2 x}$

16. $\dfrac{1}{\sqrt{x}+\sqrt{x^3}}$

17. $\dfrac{10x^9 + 10^x \cdot \log 10}{10^x + x^{10}}$

18. $\dfrac{x^{n-1}}{\sqrt{1+4x^n}}$

19. $(2x+1)\sqrt{x+2}$

20. $x^5 \cdot \sqrt{a^2+x^2}$

21. $(5-3x)(2-3x)^{-\frac{1}{2}}$

22. $\dfrac{7+4x+5x^2}{(2x+3)^{\frac{3}{2}}}$

23. $\dfrac{x^2}{\sqrt{9-x^6}}$

24. $\dfrac{1}{x(x^3-1)}$

25. $\dfrac{1}{x \cdot \log x \cdot \log(\log x)}$

II. Integrate the following functions w. r. t. x :

1. $\dfrac{\cos 3x - \cos 4x}{\sin 3x + \sin 4x}$

2. $\dfrac{\cos x}{\sin(x-a)}$

3. $\dfrac{\sin(x-a)}{\cos(x+b)}$

4. $\dfrac{1}{\sin x \cdot \cos x + 2\cos^2 x}$

5. $\dfrac{\sin x + 2\cos x}{3\sin x + 4\cos x}$

6. $\dfrac{1}{2+3\tan x}$

7. $\dfrac{4e^x - 25}{2e^x - 5}$

8. $\dfrac{20+12e^x}{3e^x+4}$

9. $\dfrac{3e^{2x}+5}{4e^{2x}-5}$

10. $\cos^8 x \cdot \cot x$

11. $\tan^5 x$

12. $\cos^7 x$

13. $\tan 3x \cdot \tan 2x \cdot \tan x$

14. $\sin^5 x \cdot \cos^8 x$

15. $3^{\cos^2 x} \cdot \sin 2x$

16. $\dfrac{\sin 6x}{\sin 10x \cdot \sin 4x}$

17. $\dfrac{\sin x \cdot \cos^3 x}{1+\cos^2 x}$

3.2.3 Some Special Integrals

1. $\displaystyle\int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

2. $\displaystyle\int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

3. $\displaystyle\int \frac{1}{a^2-x^2} \cdot dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

4. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

5. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left(x+\sqrt{x^2-a^2}\right) + c$

6. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}} \cdot dx = \log\left(x+\sqrt{x^2+a^2}\right) + c$

7. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

While evaluating an integral there is no unique substitution, we can use some standard substitutions and try.

No.	Function	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a\cdot\sin\theta$ ($x = a\cdot\cos\theta$ can also be used.)
2.	$\sqrt{a^2 + x^2}$	$x = a\cdot\tan\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a\cdot\sec\theta$
4.	$\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cdot\cos 2\theta$

1. $$\int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Proof :

Let $\quad I = \displaystyle\int \frac{1}{x^2 + a^2} \cdot dx$

put $x = a\cdot\tan\theta \Rightarrow \tan\theta = \dfrac{x}{a}$

i.e. $\theta = \tan^{-1}\left(\dfrac{x}{a}\right)$

$\therefore dx = a\cdot\sec^2\theta \cdot d\theta$

$$I = \int \frac{1}{a^2\cdot\tan^2\theta + a^2} \cdot a\cdot\sec^2\theta \cdot d\theta$$

$$= \int \frac{a\cdot\sec^2\theta}{a^2(\tan^2\theta + 1)} \cdot d\theta$$

$$= \int \frac{\sec^2\theta}{a\cdot\sec^2\theta} \cdot d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $$\int \frac{1}{x^2 + 5^2} \cdot dx = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$$

Alternatively

Cosider,

$$\frac{d}{dx}\left[\frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{d}{dx}\left[\frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right)\right] + \frac{d}{dx}c$$

$$= \frac{1}{a}\cdot\frac{1}{1 + \left(\frac{x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{a}\right) + 0$$

$$= \frac{1}{a}\cdot\frac{1}{1 + \frac{x^2}{a^2}}\cdot\frac{1}{a}$$

$$= \frac{1}{a^2}\cdot\frac{1}{\frac{a^2 + x^2}{a^2}}$$

$$= \frac{1}{x^2 + a^2}$$

Therefore,

by definition of integration

$$\therefore \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

2. $\displaystyle\int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

Proof :

Let $\displaystyle I = \int \frac{1}{x^2 - a^2} \cdot dx$

$\displaystyle = \int \frac{1}{(x+a)(x-a)} \cdot dx$

$\displaystyle = \int \frac{1}{2a} \cdot \left[\frac{1}{x-a} - \frac{1}{x+a}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot \int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot [\, \log (x-a) - \log (x+a) \,] + c$

$\displaystyle = \frac{1}{2a} \cdot \log\left(\frac{x-a}{x+a}\right) + c$

$\displaystyle \therefore \quad \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

e.g. $\displaystyle\int \frac{1}{x^2 - 9} \cdot dx = \frac{1}{2(3)} \log\left(\frac{x-3}{x+3}\right) + c$

Activity :

3. $\displaystyle\int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$

Proof : Consider,

$\displaystyle I = \int \frac{1}{a^2 - x^2} \cdot dx$

$\displaystyle = \int \frac{1}{(\ldots.)(\ldots.)} \cdot dx$

$\displaystyle = \int \frac{1}{2a} \cdot \left[\frac{1}{\ldots.} - \frac{1}{\ldots.}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot \int \left[\frac{\ldots.}{\ldots.} - \frac{1}{a+x}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot [\, \log (a+x) - \log (a-x) \,] + c$

$\displaystyle = \frac{1}{2a} \cdot \log\left(\frac{\ldots.}{\ldots.}\right) + c$

$\displaystyle \therefore \quad \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$

e.g. $\displaystyle\int \frac{1}{16 - x^2} \cdot dx = \frac{1}{2(4)} \log\left(\frac{4+x}{4-x}\right) + c$

4. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Proof :

Let $\displaystyle I = \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx$

put $x = a \sin\theta \Rightarrow \sin\theta = \dfrac{x}{a}$

$\displaystyle \therefore \quad \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a \cdot \cos\theta \, d\theta$

$\displaystyle I = \int \frac{1}{\sqrt{a^2 - a^2 \sin^2\theta}} \cdot a \cdot \cos\theta \, d\theta$

$\displaystyle I = \int \frac{a \cdot \cos\theta}{a\sqrt{1 - a^2 \sin^2\theta}} \cdot d\theta$

$\displaystyle = \int \frac{\cos\theta}{\cos\theta} \cdot d\theta$

$\displaystyle = \int 1 \cdot d\theta$

$= \theta + c$

$\displaystyle \therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

e.g. $\displaystyle\int \frac{1}{\sqrt{81 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{9}\right) + c$

5. $$\int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left(x + \sqrt{x^2 - a^2}\right) + c$$

Proof : Let $\quad I = \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx$

put $x = a\sec\theta \Rightarrow \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\therefore \quad dx = a\cdot\sec\theta\cdot\tan\theta\cdot d\theta$

$$\begin{aligned}
I &= \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}} \cdot a\cdot\sec\theta\cdot\tan\theta\cdot d\theta \\
&= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2(\sec^2\theta - 1)}} \cdot d\theta \\
&= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2\cdot\tan^2\theta}} \cdot d\theta \\
&= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{a\cdot\tan\theta} \cdot d\theta \\
&= \int \sec\theta\cdot d\theta \\
&= \log(\sec\theta + \tan\theta) + c \\
&= \log(\sec\theta + \sqrt{\sec^2\theta - 1}) + c \\
&= \log\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) + c_1 \\
&= \log\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) + c_1 \\
&= \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c_1 \\
&= \log\left(x + \sqrt{x^2 - a^2}\right) - \log a + c_1 \\
&= \log\left(x + \sqrt{x^2 - a^2}\right) + c
\end{aligned}$$

where $c = c_1 - \log a$

$$\therefore \quad \int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \log\left(x + \sqrt{x^2 - a^2}\right) + c$$

e.g. $$\int \frac{1}{\sqrt{x^2 - 16}} \cdot dx = \log\left(x + \sqrt{x^2 - 16}\right) + c$$

Activity :

6. $$\int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \log\left(x + \sqrt{a^2 - x^2}\right) + c$$

Proof : use substitution $x = a \cdot \tan\theta$

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Activity :

7. $$\int \frac{1}{x\sqrt{x^2 - a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$$

Proof : Let $I = \int \frac{1}{x\sqrt{x^2 - a^2}} \cdot dx$

put $x = a\sec\theta \Rightarrow \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\therefore \quad dx = a\cdot\sec\theta\cdot\tan\theta\cdot d\theta$

$$\begin{aligned}
I &= \int \frac{1}{a\sec\theta\sqrt{.....- a^2}} \cdot \;.........\\
&= \int \frac{\tan\theta}{\sqrt{a^2(........)}} \cdot d\theta \\
&= \frac{1}{a}\int 1\cdot d\theta \\
&= \frac{1}{a}\cdot\theta + c \\
&= \frac{1}{a}\cdot\sec^{-1}\left(\frac{x}{a}\right) + c
\end{aligned}$$

$$\therefore \quad \int \frac{1}{x\sqrt{x^2 - a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $$\int \frac{1}{x\sqrt{x^2 - 64}} \cdot dx = \frac{1}{8}\sec^{-1}\left(\frac{x}{8}\right) + c$$

3.2.4

In order to evaluate the integrals of type $\int \frac{1}{ax^2+bx+c}\cdot dx$ and $\int \frac{1}{\sqrt{ax^2+bx+c}}\cdot dx$ we can use the following steps.

(1) Write ax^2+bx+c as, $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$, $a>0$ and take a or $\sqrt{a}$ out of the integral sign.

(2) $\left(x^2+\frac{b}{a}x\right)$ or $\left(\frac{b}{a}x-x^2\right)$ is expressed by the method of completing square by adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2$.

(3) Express the quadractic expression as a sum or difference of two squares i.e. $\left((x+\beta)^2 \pm \alpha^2\right)$ or $\left(\alpha^2-(x+\beta)^2\right)$

(4) We know that $\int f(x)\,dx = g(x)+c \quad \Rightarrow \quad \int f(x+\beta)\,dx = g(x+\beta)+c$

$$\int f(\alpha x+\beta)\,dx = \frac{1}{\alpha}g(\alpha x+\beta)+c$$

(5) Use the standard integral formula and express the result in terms of x.

3.2.5

In order to evaluate the integral of type $\int \frac{1}{a\sin^2 x+b\cos^2 x+c}\cdot dx$ we can use the following steps.

(1) Divide the numerator and denominator by $\cos^2 x$ or $\sin^2 x$.

(2) In denominator replace $\sec^2 x$ by $1+\tan^2 x$ and /or $\operatorname{cosec}^2 x$ by $1+\cot^2 x$, if exists.

(3) Put $\tan x = t$ or $\cot x = t$ so that the integral reduces to the form $\int \frac{1}{at^2+bt+c}\cdot dt$

(4) Use the standard integral formula and express the result in terms of x.

3.2.6

To evaluate the integral of the form $\int \frac{1}{a\sin x+b\cos x+c}\cdot dx$, we use the standard substitution $\tan\frac{x}{2}=t$.

If $\tan\frac{x}{2}=t$ then (i) $\sec^2\frac{x}{2}\cdot\frac{1}{2}\cdot dx = 1\cdot dt$

$$\text{i.e. } dx = \frac{2}{\sec^2\frac{x}{2}}\cdot dt = \frac{2}{1+\tan^2\frac{x}{2}}\cdot dt = \frac{2\,dt}{1+t^2}$$

(ii) $$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\text{(iii)}\quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

We put $\tan x = t$ for the integral of the type $\int \frac{1}{a\sin 2x + b\cos 2x + c} \cdot dx$

therefore $\quad dx = \frac{1}{1+t^2} \cdot dt$

$$\sin 2x = \frac{2t}{1+t^2} \cdot dt$$

and $\quad \cos 2x = \frac{1-t^2}{1+t^2} \cdot dt$

With this substitution the integral reduces to the form $\int \frac{1}{ax^2+bx+c} \cdot dx$. Now use the standard integral formula and express the result in terms of x.

SOLVED EXAMPLES

Ex. : Evaluate :

1. $\int \frac{1}{4x^2+11} \cdot dx$

Solution : $I = \int \frac{1}{4\left(x^2+\frac{11}{4}\right)} \cdot dx$

$$= \frac{1}{4} \cdot \int \frac{1}{x^2 + \left(\frac{\sqrt{11}}{2}\right)^2} \cdot dx$$

$$\because \int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$I = \frac{1}{4} \cdot \left[\frac{1}{\left(\frac{\sqrt{11}}{2}\right)}\right] \cdot \tan^{-1}\left[\frac{x}{\left(\frac{\sqrt{11}}{2}\right)}\right] + c$$

$$= \frac{1}{2\sqrt{11}}\tan^{-1}\left(\frac{2x}{\sqrt{11}}\right) + c$$

2. $\int \frac{1}{a^2-b^2x^2} \cdot dx$

Solution : $I = \int \frac{1}{b^2\left(\frac{a^2}{b^2}-x^2\right)} \cdot dx$

$$= \frac{1}{b^2} \cdot \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2} \cdot dx$$

$$\because \int \frac{1}{a^2-x^2} \cdot dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$$

$$I = \frac{1}{b^2} \cdot \frac{1}{2\left(\frac{a}{b}\right)} \cdot \log\left(\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right) + c$$

$$= \frac{1}{b^2} \cdot \frac{1}{2\left(\frac{a}{b}\right)} \cdot \log\left(\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right) + c$$

$$= \frac{1}{2ab} \cdot \log\left(\frac{a+bx}{a-bx}\right) + c$$

3. $$\int \frac{1}{\sqrt{3x^2-7}} \cdot dx$$

Solution : $$I = \int \frac{1}{\sqrt{3\left(x^2-\frac{7}{3}\right)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{3} \cdot \sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} \cdot dx$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{1}{\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} \cdot dx$$

$$\because \int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$I = \frac{1}{\sqrt{3}} \cdot \log\left(x+\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}\right) + c$$

$$= \frac{1}{\sqrt{3}} \cdot \log\left(x+\sqrt{x^2-\frac{7}{3}}\right) + c$$

4. $$\int \frac{1}{x^2+8x+12} \cdot dx$$

Solution : $$I = \int \frac{1}{x^2+8x+16-4} \cdot dx$$

$$= \int \frac{1}{(x+4)^2-(2)^2} \cdot dx$$

$$\because \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$$

$$I = \frac{1}{2(2)} \cdot \log\left(\frac{(x+4)-2}{(x+4)+2}\right) + c$$

$$= \frac{1}{4} \cdot \log\left(\frac{x+2}{x+6}\right) + c$$

$$\therefore \int \frac{1}{x^2+8x+12} \cdot dx = \frac{1}{4} \cdot \log\left(\frac{x+2}{x+6}\right) + c$$

5. $$\int \frac{1}{\sqrt{3x^2-4x+2}} \cdot dx$$

Solution : $$= \int \frac{1}{\sqrt{3\left(x^2-\frac{4}{3}x+\frac{2}{3}\right)}} \cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}\left(-\frac{4}{3}\right)\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$= \int \frac{1}{\sqrt{3} \cdot \sqrt{x^2-\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{2}{3}}} \cdot dx$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{1}{\sqrt{\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+\left(\frac{2}{3}-\frac{4}{9}\right)}} \cdot dx$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{1}{\sqrt{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}} \cdot dx$$

$$\because \int \frac{1}{\sqrt{x^2+a^2}} \cdot dx = \log\left|x+\sqrt{x^2+a^2}\right| + c$$

$$= \frac{1}{\sqrt{3}} \cdot \log\left(\left(x-\frac{2}{3}\right)+\sqrt{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}\right) + c$$

$$= \frac{1}{\sqrt{3}} \cdot \log\left(\left(x-\frac{2}{3}\right)+\sqrt{x^2-\frac{4}{3}x+\frac{2}{3}}\right) + c$$

6. $$\int \frac{1}{3-10x-25x^2} \cdot dx$$

Solution :

$$I = \int \frac{1}{25\left(\frac{3}{25}-\frac{10}{25}x-x^2\right)} \cdot dx$$

$$= \int \frac{1}{25\left[\frac{3}{25}-\left(x^2+\frac{2}{5}x\right)\right]} \cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2\right.$$

$$\left.= \left(\frac{1}{2}\left(\frac{2}{5}\right)\right)^2 = \left(\frac{1}{5}\right)^2 = \frac{1}{25}\right\}$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{3}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}-\frac{1}{25}\right)} \cdot dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{3}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}\right)+\frac{1}{25}} \cdot dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{4}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}\right)} \cdot dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\left(\frac{2}{5}\right)^2-\left(x-\frac{1}{5}\right)^2} \cdot dx$$

$$\because \int \frac{1}{a^2-x^2} \cdot dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$$

$$I = \frac{1}{25}\cdot\frac{1}{2\left(\frac{2}{5}\right)}\cdot\log\left(\frac{\frac{2}{5}+\left(x-\frac{1}{5}\right)}{\frac{2}{5}-\left(x-\frac{1}{5}\right)}\right)+c$$

$$= \frac{1}{5}\cdot\log\left(\frac{1+5x}{3-5x}\right)+c$$

Activity :

7. $$\int \frac{1}{\sqrt{1+x-x^2}} \cdot dx$$

Solution : $$I = \int \frac{1}{\sqrt{1-\left(\text{. . . . .}\right)}} \cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2\right.$$

$$\left.= \left(\frac{1}{2}(-1)\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$$

$$= \int \frac{1}{\sqrt{1-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{1-\left(\text{. . . . .}\right)}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} \cdot dx$$

$$\because \int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right)+c$$

I = ...

= . . . .

8. $$\int \frac{\sin 2x}{3\sin^4 x - 4\sin^2 x + 1} \cdot dx$$

Solution : $$I = \int \frac{\sin 2x}{3(\sin^2 x)^2 - 4(\sin^2 x) + 1} \cdot dx$$

put $\sin^2 x = t$ $\quad \therefore\ 2\sin x \cdot \cos x \cdot dx = 1 \cdot dt$

$\therefore\ \sin 2x \cdot dx = 1 \cdot dt$

$$= \int \frac{1}{3t^2 - 4t + 1} \cdot dt$$

$$= \int \frac{1}{3\left(t^2 - \frac{4}{3}t + \frac{1}{3}\right)} \cdot dt$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2 = \left(\frac{1}{2}\left(-\frac{4}{3}\right)\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$I = \frac{1}{3} \cdot \int \frac{1}{t^2 - \frac{4}{3}t + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}} \cdot dt$$

$$= \frac{1}{3} \cdot \int \frac{1}{\left(t^2 - \frac{4}{3}t + \frac{4}{9}\right) - \frac{1}{9}} \cdot dt$$

$$= \frac{1}{3} \cdot \int \frac{1}{\left(t - \frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2} \cdot dt$$

$$= \frac{1}{3} \cdot \frac{1}{2\left(\frac{1}{3}\right)} \cdot \log\left(\frac{\left(t - \frac{2}{3}\right) - \frac{1}{3}}{\left(t - \frac{2}{3}\right) + \frac{1}{3}}\right) + c$$

$$= \frac{1}{2} \cdot \log\left(\frac{3t - 3}{3t - 1}\right) + c$$

$$= \frac{1}{2} \cdot \log\left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

$$\therefore \int \frac{\sin 2x}{3\sin^4 x - 4\sin^2 x + 1} \cdot dx$$

$$= \frac{1}{2} \cdot \log\left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

9. $$\int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}} \cdot dx$$

Solution :

$$I = \int \frac{\sqrt{e^x}}{\sqrt{\frac{1}{e^x} - e^x}} \cdot dx$$

$$= \int \frac{\sqrt{e^x}}{\sqrt{\frac{1 - (e^x)^2}{e^x}}} \cdot dx$$

$$= \int \frac{\sqrt{e^x}}{\frac{\sqrt{1 - (e^x)^2}}{\sqrt{e^x}}} \cdot dx$$

$$= \int \frac{\sqrt{e^x} \cdot \sqrt{e^x}}{\sqrt{1 - (e^x)^2}} \cdot dx$$

$$= \int \frac{e^x}{\sqrt{1 - (e^x)^2}} \cdot dx$$

put $e^x = t$

$\therefore\ e^x \cdot dx = 1 \cdot dt$

$$I = \int \frac{1}{\sqrt{1 - t^2}} \cdot dt$$

$$= \sin^{-1}(t) + c$$

$$= \sin^{-1}(e^x) + c$$

$$\therefore \int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}} \cdot dx = \sin^{-1}(e^x) + c$$

10. $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) \cdot dx$

Solution : $\mathrm{I} = \int \left(\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}}\right) \cdot dx$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}} \cdot dx$$

put $\sqrt{\tan x} = t \quad \therefore \tan x = t^2 \;\therefore\; x = \tan^{-1} t^2$

$\therefore \quad 1 \cdot dx = \frac{1}{1+(t^2)^2} \cdot 2t \cdot dt$

$\because \quad \sec^2 x \cdot dx = 2t \cdot dt$

$\therefore \quad dx = \frac{2t}{\sec^2 x} \cdot dx = \frac{2t}{1+\tan^2 x} \cdot dx = \frac{2t}{1+t^4} \cdot dt$

$$= \int \frac{t^2+1}{t} \cdot \frac{2t}{1+t^4} \cdot dt = 2\int \frac{t^2+1}{t^4+1} \cdot dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} \cdot dt = 2\int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2} \cdot dt$$

put $t - \frac{1}{t} = u \qquad \because \left[\frac{d}{dt}\left(t-\frac{1}{t}\right) = 1 + \frac{1}{t^2}\right]$

$$\therefore \left(t - \left(-\frac{1}{t^2}\right)\right) dt = 1 \cdot du$$

$$\therefore \left(1 + \frac{1}{t^2}\right) dt = 1 \cdot du$$

$$\mathrm{I} = 2\int \frac{1}{u^2+2} \cdot du$$

$$= 2\int \frac{1}{u^2+(\sqrt{2})^2} \cdot du$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= \sqrt{2} \cdot \tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) + c$$

$$= \sqrt{2} \cdot \tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + c$$

$$= \sqrt{2} \cdot \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2} \cdot \sqrt{\tan x}}\right) + c$$

11. $\int \frac{1}{5 - 4\cos x} \cdot dx$

Solution : put $\tan \frac{x}{2} = t$

$\therefore \quad dx = \frac{2}{1+t^2} \cdot dt \quad$ and $\quad \cos x = \frac{1-t^2}{1+t^2}$

$$\mathrm{I} = \int \frac{1\left(\frac{2}{1+t^2}\right)}{5 - 4\left(\frac{1-t^2}{1+t^2}\right)} \cdot dt$$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{5(1+t^2) - 4(1-t^2)}{1+t^2}} \cdot dt$$

$$= \int \frac{2}{5 - 5t^2 - 4 - 4t^2} \cdot dt$$

$$= \int \frac{2}{9t^2+1} \cdot dt$$

$$= \int \frac{1}{9\left(t^2+\frac{1}{9}\right)} \cdot dt$$

$$= \frac{2}{9} \cdot \int \frac{1}{t^2 + \left(\frac{1}{3}\right)^2} \cdot dt$$

$$= \frac{2}{9} \cdot \frac{1}{\left(\frac{1}{3}\right)} \cdot \tan^{-1}\left(\frac{t}{\left(\frac{1}{3}\right)}\right) + c$$

$$= \frac{2}{3} \cdot \tan^{-1}(2t) + c$$

$$= \frac{2}{3} \cdot \tan^{-1}\left(2\tan\frac{x}{2}\right) + c$$

$$\therefore \int \frac{1}{5 - 4\cos x} \cdot dx = \frac{2}{3} \cdot \tan^{-1}\left(2\tan\frac{x}{2}\right) + c$$

12. $\displaystyle\int \frac{1}{2-3\sin 2x}\cdot dx$

Solution : put $\tan x = t$

$\therefore \quad dx = \dfrac{1}{1+t^2}\cdot dt$ and $\sin 2x = \dfrac{2}{1+t^2}$

$$\text{I} = \int \frac{1\left(\dfrac{1}{1+t^2}\right)}{2-3\left(\dfrac{2}{1+t^2}\right)}\cdot dt$$

$$= \int \frac{\dfrac{1}{1+t^2}}{\dfrac{2(1+t^2)-3(2t)}{1+t^2}}\cdot dt$$

$$= \int \frac{1}{2+2t^2-6t}\cdot dt = \int \frac{1}{2(t^2-3t+1)}\cdot dt$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2 = \left(\frac{1}{2}(-3)\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}\right\}$$

$$= \frac{1}{2}\cdot\int \frac{1}{t^2-3t+\dfrac{9}{4}-\dfrac{9}{4}+1}\cdot dt$$

$$= \frac{1}{2}\cdot\int \frac{1}{\left(t^2-3t+\dfrac{9}{4}\right)-\dfrac{5}{4}}\cdot dt$$

$$= \frac{1}{2}\cdot\int \frac{1}{\left(t-\dfrac{3}{2}\right)^2-\left(\dfrac{\sqrt{5}}{2}\right)^2}\cdot dt$$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\dfrac{\sqrt{5}}{2}\right)}\cdot\log\left(\frac{\left(t-\dfrac{3}{2}\right)-\dfrac{\sqrt{5}}{2}}{\left(t-\dfrac{3}{2}\right)+\dfrac{\sqrt{5}}{2}}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\cdot\log\left(\frac{2t-3-\sqrt{5}}{2t-3+\sqrt{5}}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\cdot\log\left(\frac{2\tan x-3-\sqrt{5}}{2\tan x-3+\sqrt{5}}\right)+c$$

13. $\displaystyle\int \frac{1}{3-2\sin x+5\cos x}\cdot dx$

Solution : put $\tan\dfrac{x}{2} = t$

$\therefore \quad dx = \dfrac{2}{1+t^2}$

$\therefore \quad \sin x = \dfrac{2}{1+t^2}\cdot dt$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

$$\text{I} = \int \frac{1\left(\dfrac{2}{1+t^2}\right)}{3-2\left(\dfrac{2}{1+t^2}\right)+5\left(\dfrac{1-t^2}{1+t^2}\right)}\cdot dt$$

$$= \int \frac{\dfrac{2}{1+t^2}}{\dfrac{3(1+t^2)-2(2t)+5(1-t^2)}{1+t^2}}\cdot dt$$

$$= \int \frac{2}{3+3t^2-4t+5-5t^2}\cdot dt$$

$$= \int \frac{2}{8-4t-2t^2}\cdot dt$$

$$= \int \frac{1}{4-2t-t^2}\cdot dt$$

$$= \int \frac{1}{4-(t^2+2t)}\cdot dt$$

$$= \int \frac{1}{4-(t^2+2t+1-1)}\cdot dt$$

$$= \int \frac{1}{5-(t^2+2t+1)}\cdot dt$$

$$= \int \frac{1}{(\sqrt{5})^2-(t+1)^2}\cdot dt$$

$$= \frac{1}{2(\sqrt{5})}\cdot\log\left(\frac{\sqrt{5}+(t+1)}{\sqrt{5}-(t+1)}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\cdot\log\left(\frac{\sqrt{5}+1+\tan\dfrac{x}{2}}{\sqrt{5}-1-\tan\dfrac{x}{2}}\right)+c$$

Activity : 14. $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\cdot dx$

Solution : put $\tan\frac{x}{2} = t$ $\quad \therefore\ dx = \ldots\ldots\ldots$

$\therefore\ \sin x = \ldots\ldots\ldots$ and $\cos x = \ldots\ldots\ldots$

$$\mathrm{I} = \int \frac{1\left(\frac{2}{1+t^2}\right)}{\ldots\ldots\ldots + \sqrt{3}\ \ldots\ldots\ldots}\cdot dt$$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{\ldots\ldots}{1+t^2}}\cdot dt$$

$$= \int \frac{2}{\ldots\ldots\ldots}\cdot dt$$

$$= \int \frac{2}{\sqrt{3}\left(1-\left(t^2-\frac{2}{\sqrt{3}}t\right)\right)}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{1-\left(t^2-\frac{2}{\sqrt{3}}t+\frac{1}{3}-\frac{1}{3}\right)}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{1-\left(t^2-\frac{2}{\sqrt{3}}t+\frac{1}{3}\right)+\frac{1}{3}}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{\left(\ldots\ldots\right)-\left(t-\frac{1}{\sqrt{3}}\right)^2}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{(\ldots)^2-(\ldots)^2}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{1}{2\,(\ldots)}\cdot\log\left(\frac{\quad}{\quad}\right)+c$$

$$= \frac{\quad}{\quad}\cdot\log\left(\frac{\quad}{\quad}\right)+c$$

$$= \frac{1}{2}\cdot\log\left(\frac{1+\sqrt{3}\tan\frac{x}{2}}{3-\sqrt{3}\tan\frac{x}{2}}\right)+c$$

Alternative method :

14. $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\cdot dx$

Solution : For any two positive numbers a and b, we can find an angle θ, such that

$\therefore\ \sin\theta = \dfrac{a}{\sqrt{a^2-b^2}}$ and $\cos\theta = \dfrac{b}{\sqrt{a^2-b^2}}$

Using this we express $\sin x - \sqrt{3}\cos x$

$= \sqrt{1+3}\,(\cos\theta\cdot\sin x - \sin\theta\cdot\cos x)$

$= 2\cdot\sin(x-\theta)$

$= 2\cdot\sin\left(x-\frac{\pi}{3}\right)$

$$\therefore\ \mathrm{I} = \int \frac{1}{2\cdot\sin\left(x-\frac{\pi}{3}\right)}\cdot dx$$

$$= \frac{1}{2}\cdot\int \operatorname{cosec}\left(x-\frac{\pi}{3}\right)\cdot dx$$

$$= \frac{1}{2}\cdot\log\left(\operatorname{cosec}\left(x-\frac{\pi}{3}\right)-\cot\left(x-\frac{\pi}{3}\right)\right)+c$$

$$= \frac{1}{2}\cdot\log\left(\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right)+c$$

15. $\displaystyle\int \frac{1}{3 + 2\sin^2 x + 5\cos^2 x} \cdot dx$

Solution : Divide Numerator and Denominator by $\cos^2 x$

$$I = \int \frac{\frac{1}{\cos^2 x}}{\frac{3 + 2\sin^2 x + 5\cos^2 x}{\cos^2 x}} \cdot dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x + 2\tan^2 x + 5} \cdot dx$$

$$= \int \frac{\sec^2 x}{3(1 + \tan^2 x) + 2\tan^2 x + 5} \cdot dx$$

$$= \int \frac{\sec^2 x}{5\tan^2 x + 8} \cdot dx$$

$$= \frac{1}{5} \int \frac{\sec^2 x}{\tan^2 x + \frac{8}{5}} \cdot dx$$

put $\tan x = t \therefore \sec^2 x \cdot dx = 1 \cdot dt$

$$I = \frac{1}{5} \int \frac{1}{t^2 + \frac{8}{5}} \cdot dt$$

$$= \frac{1}{5} \int \frac{1}{t^2 + \left(\frac{\sqrt{8}}{\sqrt{5}}\right)^2} \cdot dt$$

$$= \frac{1}{5} \cdot \frac{1}{\frac{\sqrt{8}}{\sqrt{5}}} \cdot \tan^{-1}\left(\frac{t}{\frac{\sqrt{8}}{\sqrt{5}}}\right) + c$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{2\sqrt{2}} \cdot \tan^{-1}\left(\frac{\sqrt{5}\,t}{2\sqrt{2}}\right) + c$$

$$= \frac{1}{2\sqrt{10}} \cdot \tan^{-1}\left(\frac{\sqrt{5}\tan x}{2\sqrt{2}}\right) + c$$

$$\therefore \int \frac{1}{3 + 2\sin^2 x + 5\cos^2 x} \cdot dx = \frac{1}{2\sqrt{10}} \cdot \tan^{-1}\left(\frac{\sqrt{5}\tan x}{2\sqrt{2}}\right) + c$$

16. $\displaystyle\int \frac{\cos\theta}{\cos 3\theta} \cdot d\theta$

Solution : $I = \displaystyle\int \frac{\cos\theta}{4\cos^3\theta - 3\cos\theta} \cdot d\theta$

$$= \int \frac{1}{4\cos^2\theta - 3} \cdot d\theta$$

Divide Numerator and Denominator by $\cos^2\theta$

$$I = \int \frac{\frac{1}{\cos^2\theta}}{\frac{4\cos^2\theta - 3}{\cos^2\theta}} \cdot d\theta$$

$$= \int \frac{\sec^2\theta}{4 - 3\sec^2\theta} \cdot d\theta$$

$$= \int \frac{\sec^2\theta}{4 - 3(1 + \tan^2\theta)} \cdot d\theta$$

$$= \int \frac{\sec^2\theta}{1 - 3\tan^2\theta} \cdot d\theta$$

put $\tan\theta = t \therefore \sec^2\theta \cdot d\theta = 1 \cdot dt$

$$I = \int \frac{1}{1 - 3t^2} \cdot dt$$

$$= \frac{1}{3} \cdot \int \frac{1}{\frac{1}{3} - t^2} \cdot dt$$

$$= \frac{1}{3} \cdot \int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2} \cdot dt$$

$$= \frac{1}{3} \cdot \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)} \cdot \log\left(\frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t}\right) + c$$

$$= \frac{1}{2\sqrt{3}} \cdot \log\left(\frac{1 + \sqrt{3}\,t}{1 - \sqrt{3}\,t}\right) + c$$

$$= \frac{1}{2\sqrt{3}} \cdot \log\left(\frac{1 + \sqrt{3}\tan\theta}{1 - \sqrt{3}\tan\theta}\right) + c$$

$$\therefore \int \frac{\cos\theta}{\cos 3\theta} \cdot d\theta = \frac{1}{2\sqrt{3}} \cdot \log\left(\frac{1 + \sqrt{3}\tan\theta}{1 - \sqrt{3}\tan\theta}\right) + c$$

EXERCISE 3.2 (B)

I. Evaluate the following :

1. $\int \frac{1}{4x^2-3} \cdot dx$
2. $\int \frac{1}{25-9x^2} \cdot dx$
3. $\int \frac{1}{7+2x^2} \cdot dx$
4. $\int \frac{1}{\sqrt{3x^2+8}} \cdot dx$
5. $\int \frac{1}{\sqrt{11-4x^2}} \cdot dx$
6. $\int \frac{1}{\sqrt{2x^2-5}} \cdot dx$
7. $\int \sqrt{\frac{9+x}{9-x}} \cdot dx$
8. $\int \sqrt{\frac{2+x}{2-x}} \cdot dx$
9. $\int \sqrt{\frac{10+x}{10-x}} \cdot dx$
10. $\int \frac{1}{x^2+8x+12} \cdot dx$
11. $\int \frac{1}{1+x-x^2} \cdot dx$
12. $\int \frac{1}{4x^2-20x+17} \cdot dx$
13. $\int \frac{1}{5-4x-3x^2} \cdot dx$
14. $\int \frac{1}{\sqrt{3x^2+5x+7}} \cdot dx$
15. $\int \frac{1}{\sqrt{x^2+8x-20}} \cdot dx$
16. $\int \frac{1}{\sqrt{8-3x+2x^2}} \cdot dx$
17. $\int \frac{1}{\sqrt{(x-3)(x+2)}} \cdot dx$
18. $\int \frac{1}{4+3\cos^2 x} \cdot dx$
19. $\int \frac{1}{\cos 2x+3\sin^2 x} \cdot dx$
20. $\int \frac{\sin x}{\sin 3x} \cdot dx$

II. Integrate the following functions *w. r. t. x* :

1. $\int \frac{1}{3+2\sin x} \cdot dx$
2. $\int \frac{1}{4-5\cos x} \cdot dx$
3. $\int \frac{1}{2+\cos x-\sin x} \cdot dx$
4. $\int \frac{1}{3+2\sin x-\cos x} \cdot dx$
5. $\int \frac{1}{3-2\cos 2x} \cdot dx$
6. $\int \frac{1}{2\sin 2x-3} \cdot dx$
7. $\int \frac{1}{3+2\sin 2x+4\cos 2x} \cdot dx$
8. $\int \frac{1}{\cos x-\sin x} \cdot dx$
9. $\int \frac{1}{\cos x-\sqrt{3}\sin x} \cdot dx$

3.2.6 Integral of the form $\int \frac{px+q}{ax^2+bx+c} \cdot dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}} \cdot dx$

The integral of the form $\int \frac{px+q}{ax^2+bx+c} \cdot dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \cdot \frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} \cdot dx + \int \frac{B}{ax^2+bx+c} \cdot dx \text{ for some constants } A \text{ and } B.$$

The numerator, $px+q = A \cdot \frac{d}{dx}(ax^2+bx+c) + B$

i.e. $\quad \mathrm{Nr} = A \cdot \frac{d}{dx}\mathrm{Dr} + B$

The first integral is evaluated by putting $ax^2+bx+c=t$

The Second integral is evaluated by expressing the integrand in the form either

$\frac{1}{A^2+t^2}$ or $\frac{1}{t^2-A^2}$ or $\frac{1}{A^2-t^2}$ and applying the methods discussed previously.

The integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} \cdot dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \cdot \frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} \cdot dx + \int \frac{B}{\sqrt{ax^2+bx+c}} \cdot dx \quad \text{for constants } A \text{ and } B.$$

The numerator, $px+q = A \cdot \frac{d}{dx}(ax^2+bx+c) + B$

The first integral is evaluated by putting $ax^2+bx+c=t$

The second integral is evaluated by expressing the integrand in the form either

$\frac{1}{\sqrt{A^2+t^2}}$ or $\frac{1}{\sqrt{t^2-A^2}}$ or $\frac{1}{\sqrt{A^2-t^2}}$ and applying the methods which discussed previously.

SOLVED EXAMPLES

1. $\displaystyle\int \frac{2x-3}{3x^2+4x+5}\cdot dx$

Solution : $2x-3 = A\cdot\frac{d}{dx}\left(3x^2+4x+5\right)+B$

$2x-3 = A(6x+4)+B$

$= (6A)x+(4A+B)$

compairing the sides/ the co–efficients of like variables and constants

$6A = 2$ and $4A+B = -3$

$\Rightarrow A = \frac{1}{3}$ and $B = -\frac{13}{3}$

$$= \int \frac{\frac{1}{3}\cdot\frac{d}{dx}(3x^2+4x+5)+\left(-\frac{13}{3}\right)}{3x^2+4x+5}\cdot dx$$

$$= \frac{1}{3}\cdot\int \frac{\frac{d}{dx}(3x^2+4x+5)}{3x^2+4x+5}\cdot dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5}\cdot dx$$

$$= \frac{1}{3}\cdot\int \frac{6x+4}{3x^2+4x+5}\cdot dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5}\cdot dx$$

$= I_1 - I_2$ ..... (i)

$\therefore\ I_1 = \frac{1}{3}\cdot\int \frac{6x+4}{3x^2+4x+5}\cdot dx$

put $3x^2+4x+5 = t$

$\therefore\ (6x+4)\cdot dx = 1\cdot dt$

$I_1 = \frac{1}{3}\cdot\int \frac{1}{t}\cdot dt$

$= \frac{1}{3}\cdot\log(t)+c_1$

$= \frac{1}{3}\cdot\log\left(3x^2+4x+5\right)+c_1$ ..... (ii)

$$\therefore\ I_2 = \frac{13}{3}\cdot\int \frac{1}{3x^2+4x+5}\cdot dx$$

$$= \frac{13}{3}\cdot\frac{1}{3}\cdot\int \frac{1}{x^2+\frac{4}{3}x+\frac{5}{3}}\cdot dx$$

$$\because\ \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2 = \left(\frac{1}{2}\left(\frac{4}{3}\right)\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$I_2 = \frac{13}{9}\cdot\int \frac{1}{x^2+\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{5}{3}}\cdot dx$$

$$= \frac{13}{9}\cdot\int \frac{1}{x^2+\frac{4}{3}x+\frac{4}{9}+\frac{11}{9}}\cdot dx$$

$$= \frac{13}{9}\cdot\int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}\cdot dx$$

$$\because\ \int \frac{1}{X^2+A^2}\cdot dx = \frac{1}{A}\tan^{-1}\left(\frac{X}{A}\right)+c$$

$$= \frac{13}{9}\cdot\frac{1}{\frac{\sqrt{11}}{3}}\cdot\tan^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{11}}{3}}\right)+c_1$$

$$I_2 = \frac{13}{3\sqrt{11}}\cdot\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)+c_2 \quad \text{..... (iii)}$$

thus, from (i), (ii) and (iii)

$$\therefore\ \int \frac{2x-3}{3x^2+4x+5}\cdot dx$$

$$= \frac{1}{3}\cdot\log\left(3x^2+4x+5\right)-\frac{13}{3\sqrt{11}}\cdot\tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right)+c$$

$(\because c_1+c_2 = c)$

2. $\displaystyle\int\sqrt{\frac{x-5}{x-7}}\cdot dx$

Solution : $\displaystyle I = \int\sqrt{\frac{(x-5)\cdot(x-5)}{(x-7)\cdot(x-5)}}\cdot dx = \int\sqrt{\frac{(x-5)^2}{x^2-12x+35}}\cdot dx$

$\because \quad x-5 = A\cdot\frac{d}{dx}\left(x^2-12x+35\right)+B$

$x-5 = A(2x-12)+B$

$= (2A)x+(-12A+B)$

compairing, the co−efficients of like variables and constants

$2A = 1$ and $-12A+B = -5$

$\Rightarrow A = \frac{1}{2}$ and $B = 1$

$$I = \int\frac{\frac{1}{2}\cdot\frac{d}{dx}(x^2-12x+35)+(1)}{\sqrt{x^2-12x+35}}\cdot dx$$

$$= \frac{1}{2}\cdot\int\frac{\frac{d}{dx}(x^2-12x+35)}{\sqrt{x^2-12x+35}}\cdot dx+\int\frac{1}{\sqrt{x^2-12x+35}}\cdot dx$$

$= I_1 + I_2 \qquad \ldots\ldots \text{(i)}$

$\therefore \quad I_1 = \frac{1}{2}\cdot\int\frac{2x-12}{\sqrt{x^2-12x+35}}\cdot dx$

put $x^2-12x+35 = t$

$\therefore \quad (2x-12)\cdot dx = 1\cdot dt$

$I_1 = \frac{1}{2}\cdot\int\frac{1}{\sqrt{t}}\cdot dt$

$= \int\frac{1}{2\sqrt{t}}\cdot dt$

$= \sqrt{t}+c_1$

$= \sqrt{x^2-12x+35}+c_1 \qquad \ldots\ldots \text{(ii)}$

$\therefore \quad I_2 = \int\frac{1}{\sqrt{x^2-12x+35}}\cdot dx$

$= \int\frac{1}{\sqrt{x^2-12x+36-1}}\cdot dx$

$= \int\frac{1}{\sqrt{(x-6)^2-(1)^2}}\cdot dx$

$\because \quad \int\frac{1}{\sqrt{X^2-A^2}}\cdot dx = \log\left(X+\sqrt{X^2-A^2}\right)+c$

$I_2 \quad = \log\left((x-6)+\sqrt{(x-6)^2-1}\right)+c_2$

$= \log\left((x-6)+\sqrt{x^2-12x+35}\right)+c_2$

$\ldots\ldots \text{(iii)}$

Thus, from (i), (ii) and (iii)

$\int\sqrt{\frac{x-5}{x-7}}\cdot dx$

$= \sqrt{x^2-12x+35}+\log\left((x-6)+\sqrt{x^2-12x+35}\right)+c$

$(c_1+c_2 = c)$

Activity :

3. $\displaystyle\int\sqrt{\frac{8-x}{x}}\cdot dx$

Solution : $\displaystyle = \int\sqrt{\frac{(8-x)}{x}\times\frac{(\quad)}{(\quad)}}\cdot dx \qquad = \int\sqrt{\frac{(\qquad)^2}{8x-x^2}}\cdot dx \qquad = \int\frac{(8-x)}{\sqrt{8x-x^2}}\cdot dx$

$$\because \qquad 8-x = A\cdot\frac{d}{dx}\left(8x-x^2\right)+B$$

$$8-x = A\,(\ldots\ldots\ldots\ldots)+B$$

$$= (8A+B)-2Ax$$

compairing, the co−efficients of like variables and constants

$$8A+B = \ldots\ldots \qquad \text{and} \quad -2A = -1$$

$$\Rightarrow \quad A = \frac{\ldots}{\ldots} \quad \text{and} \quad B = \ldots\ldots$$

$$= \int\frac{\frac{1}{2}\cdot\frac{d}{dx}(8x-x^2)+(4)}{\sqrt{8x-x^2}}\cdot dx$$

$$= \frac{1}{2}\cdot\int\frac{\frac{d}{dx}(8x-x^2)}{\sqrt{8x-x^2}}\cdot dx + 4\cdot\int\frac{1}{\sqrt{8x-x^2}}\cdot dx$$

$$= \frac{1}{2}\cdot\int\frac{8-2x}{\sqrt{8x-x^2}}\cdot dx + 4\cdot\int\frac{1}{\sqrt{8x-x^2}}\cdot dx$$

$$= I_1 + I_2 \qquad \ldots\ldots \text{(i)}$$

$$\therefore \quad I_1 = \frac{1}{2}\cdot\int\frac{8-2x}{\sqrt{8x-x^2}}\cdot dx$$

put .... $= t$

$$\therefore \quad (\ldots\ldots\ldots\ldots)\cdot dx = 1\cdot dt$$

$$= \frac{1}{2}\cdot\int\frac{1}{\sqrt{t}}\cdot dt$$

$$= \int\frac{1}{2\sqrt{t}}\cdot dt$$

$$= \sqrt{t}+c_1$$

$$= \sqrt{8x-x^2}+c_1 \quad \ldots\ldots \text{(ii)}$$

$$\therefore \quad I_2 = 4\cdot\int\frac{1}{\sqrt{8x-x^2}}\cdot dx$$

$$= 4\cdot\int\frac{1}{\sqrt{-(\ldots\ldots\ldots\ldots\ldots\ldots)}}\cdot dx$$

$$= 4\cdot\int\frac{1}{\sqrt{\ldots\ldots\ldots-(\ldots\ldots\ldots)}}\cdot dx$$

$$= 4\cdot\int\frac{1}{\sqrt{\ldots\ldots\ldots-(x-4)^2}}\cdot dx$$

$$\because \quad \int\frac{1}{\sqrt{A^2-X^2}}\cdot dx = \sin^{-1}\left(\frac{X}{A}\right)+c$$

$$I_2 = 4\cdot\sin^{-1}\left(\frac{x-4}{4}\right)+c_2 \qquad \ldots\ldots \text{(iii)}$$

thus, from (i), (ii) and (iii)

$$\therefore \quad \int\sqrt{\frac{8-x}{x}}\cdot dx$$

$$= \sqrt{8x-x^2}+4\cdot\sin^{-1}\left(\frac{x-4}{4}\right)+c$$

$$(\because c_1+c_2 = c)$$

EXERCISE 3.2 (C)

I. Evaluate :

1. $\displaystyle\int \frac{3x+4}{x^2+6x+5}\cdot dx$

2. $\displaystyle\int \frac{2x+1}{x^2+4x-5}\cdot dx$

3. $\displaystyle\int \frac{2x+3}{2x^2+3x-1}\cdot dx$

4. $\displaystyle\int \frac{3x+4}{\sqrt{2x^2+2x+1}}\cdot dx$

5. $\displaystyle\int \frac{7x+3}{\sqrt{3+2x-x^2}}\cdot dx$

6. $\displaystyle\int \sqrt{\frac{x-7}{x-9}}\cdot dx$

7. $\displaystyle\int \sqrt{\frac{9-x}{x}}\cdot dx$

8. $\displaystyle\int \frac{3\cos x}{4\sin^2 x+4\sin x-1}\cdot dx$

9. $\displaystyle\int \sqrt{\frac{e^{3x}-e^{2x}}{e^x+1}}\cdot dx$

3.3 Integration by parts :

This method is useful when the integrand is expressed as a product of two different types of functions; one of which can be differentiated and the other can be integrated conveniently.

The following theorem gives the rule of integration by parts.

3.3.1 Theorem : If u and v are two differentiable functions of x then

$$\int u\cdot v\cdot dx = u\cdot\int v\cdot dx - \int\left(\frac{d}{dx}\cdot u\right)\left(\int v\cdot dx\right)\cdot dx$$

Proof : Let $\int v\cdot dx = w \quad \ldots\text{(i)} \quad \Rightarrow \quad v = \frac{dw}{dx} \quad \ldots\text{(ii)}$

Consider, $\dfrac{d}{dx}(u\cdot w) = u\cdot\dfrac{d}{dx}w + w\cdot\dfrac{d}{dx}u$

$$= u\cdot v + w\cdot\frac{du}{dx}$$

By definition of integration

$$u\cdot w = \int\left[u\cdot v + w\cdot\frac{du}{dx}\right]\cdot dx$$

$$= \int u\cdot v\cdot dx + \int w\cdot\frac{du}{dx}\cdot dx$$

$$= \int u\cdot v\cdot dx + \int \frac{du}{dx}\cdot w\cdot dx$$

$$\therefore \quad u\cdot\int v\cdot dx = \int u\cdot v\cdot dx + \int\frac{du}{dx}\cdot\int v\cdot dx\cdot dx$$

$$\therefore \quad \int u\cdot v\cdot dx = u\cdot\int v\cdot dx - \int\left(\frac{d}{dx}\cdot u\right)\left(\int v\cdot dx\right)\cdot dx$$

In short, $\int u\cdot v = u\cdot\int v - \int\left(u'\int v\right)$

For example : $\int x \cdot e^x \cdot dx = x \int e^x \cdot dx - \int \left(\frac{d(x)}{dx} \cdot \int e^x \cdot dx \right) \cdot dx$

$$= x \cdot e^x - \int (1) \cdot e^x \cdot dx$$

$$= x \cdot e^x - \int e^x \cdot dx$$

$$= x \cdot e^x - e^x + c$$

now let us reverse the choise of u and v

$$\therefore \quad \int e^x \cdot x \cdot dx = e^x \cdot \int x^1 \cdot dx - \int \frac{d}{dx} \cdot e^x \int x \cdot dx \cdot dx$$

$$= e^x \cdot \frac{x^2}{2} - \int e^x \cdot \frac{x^2}{2} \cdot dx$$

$$= \frac{1}{2} \cdot e^x \cdot x^2 - \frac{1}{2} \cdot \int e^x \cdot x^2 \cdot dx$$

We arrive at an integral $\int e^x \cdot x^2 \cdot dx$ which is more difficult, but it helps to get $\int e^x \cdot x^2 \cdot dx$

Thus it is essential to make a proper choise of the first function and the second function. The first function to be selected will be the one, which comes first in the order of **L I A T E**.

L	**Logarithmic function.**
I	**Inverse trigonometric function.**
A	**Algebric function.**
T	**Trigonometric function.**
E	**Exponential function.**

For example : $\int \sin x \cdot x \cdot dx$

$= \int x \cdot \sin x \cdot dx$ . . . . . by LIATE

$$= x \cdot \int \sin x \cdot dx - \int \frac{d}{dx} \cdot x \cdot \int \sin x \cdot dx \cdot dx$$

$$= x \cdot (-\cos x) - \int (1)\,(-\cos x) \cdot dx$$

$$= -x \cdot \cos x + \int \cos x \cdot dx$$

$$= -x \cdot \cos x + \sin x + c$$

SOLVED EXAMPLES

1. $\int x^2 \cdot 5^x \cdot dx$

Solution : I $= x^2 \cdot \int 5^x \cdot dx - \int \frac{d}{dx} \cdot x^2 \cdot \int 5^x \cdot dx \cdot dx$

$$= x^2 \cdot 5^x \cdot \frac{1}{\log 5} - \int 2x \cdot 5^x \cdot \frac{1}{\log 5} \cdot dx$$

$$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5}\left\{x \cdot \int 5^x \cdot dx - \int \frac{d}{dx} \cdot x \cdot \int 5^x \cdot dx \cdot dx\right\}$$

$$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5}\left\{x \cdot 5^x \cdot \frac{1}{\log 5} - \int (1)\left(5^x \cdot \frac{1}{\log 5}\right) \cdot dx\right\}$$

$$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5}\left\{\frac{1}{\log 5} \cdot x \cdot 5^x \cdot - \int \frac{1}{\log 5} \cdot 5^x \cdot dx\right\}$$

$$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5}\left\{\frac{1}{\log 5} \cdot x \cdot 5^x \cdot - \frac{1}{\log 5} \cdot 5^x \cdot \frac{1}{\log 5}\right\} + c$$

$$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{(\log 5)^2} \cdot x \cdot 5^x \cdot + \frac{2}{(\log 5)^3} \cdot 5^x + c$$

$$\therefore \quad \int x^2 \cdot 5^x \cdot dx = \frac{5^x}{\log 5} \cdot \left\{x^2 - \frac{2x}{\log 5} + \frac{2}{(\log 5)^2}\right\} + c$$

2. $\int x \cdot \tan^{-1} x \cdot dx$

Solution : I $= \int (\tan^{-1} x \cdot) x \cdot dx$ ..... by LIATE

$$= \tan^{-1} x \cdot \int x \cdot dx - \int \frac{d}{dx} \cdot \tan^{-1} x \cdot \int x \cdot dx \cdot dx$$

$$= \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \left[1 - \frac{1}{1+x^2}\right] \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + c$$

$$\therefore \quad \int x \cdot \tan^{-1} x \cdot dx = \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

3. $\displaystyle\int \frac{x}{1-\sin x}\cdot dx$

Solution : $\displaystyle I = \int \frac{x}{1-\sin x}\cdot\frac{(1+\sin x)}{(1+\sin x)}\cdot dx$

$$= \int \frac{x\,(1+\sin x)}{1-\sin^2 x}\cdot dx = \int \frac{x\,(1+\sin x)}{\cos^2 x}\cdot dx = \int x\cdot\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)\cdot dx$$

$$= \int x\cdot(\sec^2 x + \sec x\cdot\tan x)\cdot dx$$

$$= \int x\cdot\sec^2 x\cdot dx + \int x\cdot\sec x\cdot\tan x\cdot dx$$

$$= \left(x\cdot\int\sec^2 x\cdot dx - \int\frac{d}{dx}x\cdot\int\sec^2 x\cdot dx\cdot dx\right) + \left(x\cdot\int\sec x\cdot\tan x\cdot dx - \int\frac{d}{dx}\cdot x\cdot\int\sec x\cdot\tan x\cdot dx\cdot dx\right)$$

$$= x\cdot\tan x - \int(1)\cdot\tan x\cdot dx + x\cdot\sec x - \int(1)\cdot\sec x\cdot dx$$

$$= x\cdot\tan x - \log\,(\sec x) + x\cdot\sec x - \log\,(\sec x + \tan x) + c$$

$$= x\cdot(\sec x + \tan x) - \log\,(\sec x) - \log\,(\sec x + \tan x) + c$$

$$\therefore \int \frac{x}{1-\sin x}\cdot dx = x\cdot(\sec x + \tan x) - \log\,[(\sec x)\,(\sec x + \tan x)] + c$$

4. $\int e^{2x}\cdot\sin 3x\cdot dx$

Solution : $I = \int e^{2x}\cdot\sin 3x\cdot dx$

Here we use repeated integration by parts.

To evaluate $\int e^{ax}\cdot\sin\,(bx+c)\cdot dx$; $\int e^{ax}\cdot\cos\,(bx+c)\cdot dx$ any function can be taken as a first function.

$$I = e^{2x}\cdot\int\sin 3x\cdot dx - \int\frac{d}{dx}\cdot e^{2x}\cdot\int\sin 3x\cdot dx\cdot dx$$

$$= e^{2x}\cdot\left(-\cos 3x\cdot\frac{1}{3}\right) - \int e^{2x}\cdot 2\left(-\cos 3x\cdot\frac{1}{3}\right)\cdot dx$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{3}\int e^{2x}\cdot\cos 3x\cdot dx$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{3}\left(e^{2x}\cdot\int\cos 3x\cdot dx - \int\frac{d}{dx}\cdot e^{2x}\cdot\int\cos 3x\cdot dx\cdot dx\right)$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{3}\left[e^{2x}\cdot\left(\sin 3x\cdot\frac{1}{3}\right) - \int e^{2x}\cdot 2\cdot\left(\sin 3x\cdot\frac{1}{3}\right)\cdot dx\right]$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{9}\cdot e^{2x}\cdot\sin 3x - \frac{4}{9}\cdot\int e^{2x}\cdot\sin 3x\cdot dx$$

$$I = -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{9}\cdot e^{2x}\cdot\sin 3x - \frac{4}{9}\cdot I$$

$$I + \frac{4}{9}\cdot I = \frac{e^{2x}}{9}[-3\cos 3x + 2\sin 3x] + c$$

$$\frac{13}{9}\cdot I = \frac{e^{2x}}{9}[2\sin 3x - 3\cos 3x\,] + c$$

$$= \frac{e^{2x}}{13}[2\sin 3x - 3\cos 3x\,] + c$$

$$\therefore \int e^{2x}\cdot\sin 3x\cdot dx = \frac{e^{2x}}{13}[2\sin 3x - 3\cos 3x\,] + c$$

Activity :

Prove the following results.

(i) $\int e^{ax}\cdot \sin(bx+c)\cdot dx = \dfrac{e^{ax}}{a^2+b^2}\cdot[a\sin(bx+c)+b\cos(bx+c)]+c$

(ii) $\int e^{ax}\cdot \cos(bx+c)\cdot dx = \dfrac{e^{ax}}{a^2+b^2}\cdot[a\sin(bx+c)-b\cos(bx+c)]+c$

5. $\displaystyle\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]\cdot dx$

Solution :

$$\begin{aligned}
\mathrm{I} &= \int \log(\log x)\cdot 1\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot\int 1\cdot dx - \int \frac{d}{dx}\cdot \log(\log x)\int 1\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot x - \int \frac{1}{\log x}\cdot\frac{1}{x}\cdot(x)\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot x - \int \frac{1}{\log x}\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot x - \int (\log x)^{-1}\cdot 1\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot x - \left\{(\log x)^{-1}\cdot\int 1\cdot dx + \int \frac{d}{dx}\cdot (\log x)^{-1}\cdot\int 1\cdot dx\cdot dx\right\} + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot x - \left\{(\log x)^{-1}\cdot x - \int -1(\log x)^{-2}\cdot\frac{1}{x}\cdot x\cdot dx\right\} + \int \frac{1}{(\log x)^2}\cdot dx \\
&= \log(\log x)\cdot x - (\log x)^{-1}\cdot x - \int (\log x)^{-2}\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx \\
&= x\cdot\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\cdot dx + \int \frac{1}{(\log x)^2}\cdot dx
\end{aligned}$$

$$\therefore \int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]\cdot dx = x\cdot\log(\log x) - \frac{x}{\log x} + c$$

Note that :

To evaluate the integrals of type $\int \sin^{-1}x\cdot dx$; $\int \tan^{-1}x\cdot dx$; $\int \sec^{-1}x\cdot dx$; $\int \log x\cdot dx$, take the second function (v) to be 1 and then apply integration by parts.

$\int\sqrt{a^2-x^2}\cdot dx$; $\int\sqrt{a^2+x^2}\cdot dx$; $\int\sqrt{x^2-a^2}\cdot dx$

6. $\int\sqrt{a^2-x^2}\cdot dx$

Solution : Let $I = \int\sqrt{a^2-x^2}\cdot 1\cdot dx$

$$= \sqrt{a^2-x^2}\cdot\int 1\cdot dx - \int \frac{d}{dx}\cdot\sqrt{a^2-x^2}\cdot\int 1\cdot dx\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x - \int \frac{1}{2\sqrt{a^2-x^2}}(-2x)\cdot(x)\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x + \int \frac{x^2}{\sqrt{a^2-x^2}}\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x + \int\left[\frac{a^2}{\sqrt{a^2-x^2}} - \frac{(a^2-x^2)}{\sqrt{a^2-x^2}}\right]\cdot dx$$

$$= x\cdot\sqrt{a^2-x^2} + a^2\int\frac{1}{\sqrt{a^2-x^2}}\cdot dx - \int\sqrt{a^2-x^2}\cdot dx$$

$$I = x\cdot\sqrt{a^2-x^2} + a^2\int\frac{1}{\sqrt{a^2-x^2}}\cdot dx - I$$

$$\therefore \quad I + I = x\cdot\sqrt{a^2-x^2} + a^2\cdot\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \quad I = \frac{x}{2}\cdot\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \quad \int\sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $\int\sqrt{9-x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{9-x^2} + \frac{9}{2}\cdot\sin^{-1}\left(\frac{x}{3}\right) + c$

with reference to the above example solve these :

7. $\int\sqrt{a^2+x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot\log\left(x+\sqrt{x^2+a^2}\right) + c$

8. $\int\sqrt{x^2-a^2}\cdot dx = \frac{x}{2}\cdot\sqrt{x^2-a^2} - \frac{a^2}{2}\cdot\log\left(x+\sqrt{x^2+a^2}\right) + c$

9. $\int x \cdot \sin^{-1} x \cdot dx$

Solution : $I = \int \sin^{-1} x \cdot x \cdot dx$ . . . . . by LIATE

$$= \sin^{-1} x \cdot \int x \cdot dx - \int \frac{d}{dx} \cdot \sin^{-1} x \cdot \int x \cdot dx \cdot dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \frac{1-(1-x^2)}{\sqrt{1-x^2}} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \left[\frac{1}{\sqrt{1-x^2}} - \frac{(1-x^2)}{\sqrt{1-x^2}} \right] \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \int \sqrt{1-x^2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) \right] + c$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c$$

$$\therefore \quad \int x \cdot \sin^{-1} x \cdot dx = \frac{1}{2} x^2 \cdot \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c$$

Activity :

10. $\int \cos^{-1} \sqrt{x} \cdot dx$

Solution : put $\sqrt{x} = t$

$\therefore \quad x = t^2$

differentiating *w.r.t.* x

$\therefore \quad 1 \cdot dx = 2t \cdot dt$

$I = \int \cos^{-1} t \cdot 2t \cdot dt$

refer previous (example no. 9) example and solve it.

11. $\int \sqrt{4+3x-2x^2}\cdot dx$

Solution : $I = \int \sqrt{4-2x^2+3x}\cdot dx$

$$= \int \sqrt{4-2\left(x^2-\frac{3}{2}x\right)}\cdot dx$$

$$= \int \sqrt{2}\cdot\sqrt{2-\left(x^2-\frac{3}{2}x\right)}\cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left[\frac{1}{2}\left(-\frac{3}{2}\right)\right]^2 = \left(-\frac{3}{4}\right)^2 = \frac{9}{16}\right\}$$

$$I = \sqrt{2}\cdot\int \sqrt{2-\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}\right)}\cdot dx$$

$$= \sqrt{2}\cdot\int \sqrt{2-\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)+\frac{9}{16}}\cdot dx$$

$$= \sqrt{2}\cdot\int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}\cdot dx$$

$$\because \int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$= \sqrt{2}\cdot\left[\frac{\left(x-\frac{3}{4}\right)}{2}\cdot\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}+\frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\cdot\sin^{-1}\left(\frac{x-\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right)\right]+c$$

$$= \sqrt{2}\left[\frac{4x-3}{8}\cdot\sqrt{2+\frac{3}{2}x-x^2}+\frac{41}{32}\cdot\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)\right]+c$$

$$\therefore \int \sqrt{4+3x-2x^2}\cdot dx = \frac{4x-3}{8}\cdot\sqrt{4+3x-2x^2}+\frac{41}{16\sqrt{2}}\cdot\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c$$

Note that :

3.3.2 :

To evaluate the integral of type $\int (px+q)\sqrt{ax^2+bx+c}\cdot dx$

we express the term $px+q = A\cdot\frac{d}{dx}(ax^2+bx+c)+B$ . . . for constants A, B.

Then the integral will be evaluated by the useual known methods.

3.3.3 Integral of the type $\int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$

Let $e^x \cdot f(x) = t$

Differentiating *w. r. t. x*

$$\left[e^x [f'(x) + f(x)]\right] = \frac{dt}{dx}$$

$$e^x [f(x) + f'(x)] = \frac{dt}{dx}$$

By definition of integration,

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = t + c$$

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

e.g. $\int e^x [\tan x + \sec^2 x] \cdot dx = e^x \cdot \tan x + c$

$$\left(\because \frac{d}{dx} \tan x = \sec^2 x\right)$$

SOLVED EXAMPLES

1. $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) \cdot dx$

Solution :

$$I = \int e^x \left(\frac{2 + 2\sin x \cdot \cos x}{2 \cdot \cos^2 x}\right) \cdot dx$$

$$= \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cdot \cos x}{\cos^2 x}\right) \cdot dx$$

$$= \int e^x [\sec^2 x + \tan x] \cdot dx$$

$$= \int e^x [\tan x + \sec^2 x] \cdot dx$$

$$\therefore f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

$$I = e^x \cdot \tan x + c$$

$$\therefore \int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) \cdot dx = e^x \cdot \tan x + c$$

2. $\int e^x \left[\frac{x + 2}{(x + 3)^2}\right] \cdot dx$

Solution :

$$I = \int e^x \left[\frac{x + 3 - 1}{(x + 3)^2}\right] \cdot dx$$

$$= \int e^x \left[\frac{x + 3}{(x + 3)^2} + \frac{-1}{(x + 3)^2}\right] \cdot dx$$

$$= \int e^x \left[\frac{1}{x + 3} + \frac{-1}{(x + 3)^2}\right] \cdot dx$$

$$\therefore f(x) = \frac{1}{x + 3} \Rightarrow f'(x) = \frac{-1}{(x + 3)^2}$$

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

$$= e^x \cdot \left(\frac{1}{x + 3}\right) + c$$

$$= \frac{e^x}{x + 3} + c$$

$$\therefore \int e^x \left[\frac{x + 2}{(x + 3)^2}\right] \cdot dx = \frac{e^x}{x + 3} + c$$

3. $\int e^{\tan^{-1}x}\cdot\left(\dfrac{1+x+x^2}{1+x^2}\right)\cdot dx$

Solution : put $\tan^{-1}x = t$

$\therefore\quad x = \tan t$

differentiating *w. r. t. x*

$\therefore\quad \dfrac{1}{1+x^2}\cdot dx = 1\cdot dt$

$$\text{I} = \int e^t\cdot[\,1+\tan t+\tan^2 t\,]\cdot dt$$

$$= \int e^t\cdot[\,\tan t+(1+\tan^2 t)]\cdot dt$$

$$= \int e^t\cdot[\,\tan t+\sec^2 t\,]\cdot dt$$

Here $\quad f(t) = \tan t$

$\Rightarrow\quad f'(t) = \sec^2 t$

$$\text{I} = e^t\cdot f(t)+c$$

$$= e^t\cdot\tan t+c$$

$$= e^{\tan^{-1}x}\cdot x+c$$

$$\therefore \int e^{\tan^{-1}x}\cdot\left(\frac{1+x+x^2}{1+x^2}\right)\cdot dx = e^{\tan^{-1}x}\cdot x+c$$

4. $\int \dfrac{(x^2+1)\cdot e^x}{(x+1)^2}\cdot dx$

Solution :

$$\text{I} = \int e^x\left[\frac{x^2+1}{(x+1)^2}\right]\cdot dx$$

$$= \int e^x\left[\frac{x^2-1+2}{(x+1)^2}\right]\cdot dx$$

$$= \int e^x\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right]\cdot dx$$

$$= \int e^x\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]\cdot dx$$

Here $\quad f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow\quad f'(x) = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\because\quad \int[f(x)+f'(x)]\cdot dx = e^x\cdot f(x)+c$$

$$\text{I} = e^x\cdot\left(\frac{x-1}{x+1}\right)+c$$

$$\therefore\quad \int\frac{(x^2+1)\cdot e^x}{(x+1)^2}\cdot dx = e^x\cdot\left(\frac{x-1}{x+1}\right)+c$$

EXERCISE 3.3

I. Evaluate the following :

1. $\int x^2\cdot\log x\cdot dx$
2. $\int x^2\cdot\sin 3x\cdot dx$
3. $\int x\cdot\tan^{-1}x\cdot dx$
4. $\int x^2\cdot\tan^{-1}x\cdot dx$
5. $\int x^3\cdot\tan^{-1}x\cdot dx$
6. $\int(\log x)^2\cdot dx$
7. $\int \sec^3 x\cdot dx$
8. $\int x\cdot\sin^2 x\cdot dx$
9. $\int x^3\cdot\log x\cdot dx$
10. $\int e^{2x}\cdot\cos 3x\cdot dx$
11. $\int x\cdot\sin^{-1}x\cdot dx$
12. $\int x^2\cdot\cos^{-1}x\cdot dx$
13. $\int\dfrac{\log(\log x)}{x}\cdot dx$
14. $\int\dfrac{t\cdot\sin^{-1}t}{\sqrt{1-t^2}}\cdot dt$
15. $\int\cos\sqrt{x}\cdot dx$
16. $\int\sin\theta\cdot\log(\cos\theta)\cdot d\theta$
17. $\int x\cdot\cos^3 x\cdot dx$
18. $\int\dfrac{\sin(\log x)^2}{x}\cdot\log\cdot x\cdot dx$
19. $\int\dfrac{\log x}{x}\cdot dx$
20. $\int x\cdot\sin 2x\cdot\cos 5x\cdot dx$
21. $\int\cos(\sqrt[3]{x})\cdot dx$

II. Integrate the following functions *w. r. t. x* :

1. $e^{2x}\cdot\sin 3x$
2. $e^{-x}\cdot\cos 2x$
3. $\sin(\log x)$
4. $\sqrt{5x^2+3}$
5. $x^2\cdot\sqrt{a^2-x^6}$
6. $\sqrt{(x-3)(7-x)}$
7. $\sqrt{4^x(4^x+4)}$
8. $(x+1)\sqrt{2x^2+3}$
9. $x\sqrt{5-4x-x^2}$
10. $\sec^2 x\cdot\sqrt{\tan^2 x+\tan x-7}$
11. $\sqrt{x^2+2x+5}$
12. $\sqrt{2x^2+3x+4}$

III. Integrate the following functions *w. r. t. x* :

1. $(2+\cot x-\operatorname{cosec}^2 x)\cdot e^x$
2. $\left(\dfrac{1+\sin x}{1+\cos x}\right)\cdot e^x$
3. $e^x\cdot\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$
4. $\left(\dfrac{x}{(x+1)^2}\right)\cdot e^x$
5. $\dfrac{e^x}{x}\,[x(\log x)^2+2(\log x)]$
6. $e^{5x}\cdot\left(\dfrac{5x\cdot\log x+1}{x}\right)$
7. $e^{\sin^{-1}x}\cdot\left(\dfrac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$
8. $\log(1+x)^{(1+x)}$
9. $\operatorname{cosec}(\log x)\,[1-\cot(\log x)]$

3.4 Integration by partial fraction :

If $f(x)$ and $g(x)$ are two polynomials then $\dfrac{f(x)}{g(x)}$, $g(x)\neq 0$ is called a rational algebric function. $\dfrac{f(x)}{g(x)}$ is called a proper rational function provided degree of $f(x)$ < degree of $g(x)$; otherwise it is called **improper rational function**.

If degree of $f(x)\geq$ degree of $g(x)$ i.e. $\dfrac{f(x)}{g(x)}$ is an improper rational function then express it as in the form Quotient + $\dfrac{\text{Remainder}}{g(x)}$, $g(x)\neq 0$ where $\dfrac{\text{Remainder}}{g(x)}$ is proper rational function.

Lets see the three different types of the proper rational function $\dfrac{f(x)}{g(x)}$, $g(x)\neq 0$ where the denominator $g(x)$ is expressed as

(i) a non-repeated linear factors

(ii) repeated Linear factors and

(iii) product of Linear factor and non-repeated quadratic factor.

No.	Rational form	Partial form
(i)	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-b)}+\dfrac{C}{(x-c)}$
(ii)	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-c)}$
(iii)	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{(x-a)}+\dfrac{Bx+C}{x^2+bx+c}$

Type (i) : $\displaystyle\int \frac{px^2+qx+r}{(x-a)(x-b)(x-c)}\cdot dx$ i.e. denominator is expressed as non-repeated Linear factors.

SOLVED EXAMPLES

1. $\displaystyle\int \frac{3x^2+4x-5}{(x^2-1)(x+2)}\cdot dx$

Solution : $\displaystyle I = \int \frac{3x^2+4x-5}{(x-1)(x+1)(x+2)}\cdot dx$

Consider, $$\frac{3x^2+4x-5}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{C}{(x+2)}$$
$$= \frac{A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$\therefore\quad 3x^2+4x-5 = A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)$

at $x=1$, $\quad 3(1)^2+4(1)-5 = A(2)(3)+B(0)+C(0)$

$2 = 6A \quad\Rightarrow\quad A = \dfrac{1}{3}$

at $x=-1$, $\quad 3(-1)^2+4(-1)-5 = A(0)+B(-2)(1)+C(0)$

$-6=-2B \quad\Rightarrow\quad B = 3$

at $x=-2$, $\quad 3(-2)^2+4(-2)-5 = A(0)+B(0)+C(-3)(-1)$

$-1 = 3C \quad\Rightarrow\quad C = -\dfrac{1}{3}$

Thus, $$\frac{3x^2+4x-5}{(x-1)(x+1)(x+2)} = \frac{\left(\frac{1}{3}\right)}{(x-1)}+\frac{3}{(x+1)}+\frac{\left(-\frac{1}{3}\right)}{(x+2)}$$

$$\therefore\ I = \int\left[\frac{\left(\frac{1}{3}\right)}{(x-1)}+\frac{3}{(x+1)}+\frac{\left(-\frac{1}{3}\right)}{(x+2)}\right]\cdot dx \quad = \frac{1}{3}\log(x-1)+3\log(x+1)-\frac{1}{3}\log(x+2)+c$$

$$= \frac{1}{3}\log\left[\frac{(x-1)(x+1)^9}{(x+2)}\right]+c \qquad \therefore \int\frac{3x^2+4x-5}{(x^2-1)(x+2)}\cdot dx = \frac{1}{3}\log\left[\frac{(x-1)(x+1)^9}{(x+2)}\right]+c$$

2. $\displaystyle\int \frac{2x^2-3}{(x^2-5)(x^2+4)}\cdot dx$

Solution : Consider, $\displaystyle\frac{2x^2-3}{(x^2-5)(x^2+4)}$

Let $x^2 = m$

$\therefore \quad = \dfrac{2m-3}{(m-5)(m+4)}$ . . . proper rational function.

Now, $\displaystyle\frac{2m-3}{(m-5)(m+4)} = \frac{A}{(m-5)} + \frac{B}{(m+4)} = \frac{A(m+4)+B(m-5)}{(m-5)(m+4)}$

$\therefore \quad 2m-3 = A(m+4) + B(m-5)$

at $m = 5$, $\quad 2(5)-3 = A(9) + B(0)$

$7 = 9A \Rightarrow A = \dfrac{7}{9}$

at $m = -4$, $\quad 2(-4)-3 = A(0) + B(-9)$

$-11 = -9B \Rightarrow B = \dfrac{11}{9}$

Thus, $\displaystyle\frac{2m-3}{(m-5)(m+4)} = \frac{\left(\frac{7}{9}\right)}{(m-5)} + \frac{\left(\frac{11}{9}\right)}{(m+4)}$ i.e. $\displaystyle\frac{2x^2-3}{(x^2-5)(x^2+4)} = \frac{\left(\frac{7}{9}\right)}{x^2-5} + \frac{\left(\frac{11}{9}\right)}{x^2+4}$

$$\therefore \text{I} = \int \left[\frac{\left(\frac{7}{9}\right)}{x^2-5} + \frac{\left(\frac{11}{9}\right)}{x^2+4}\right]\cdot dx$$

$$= \frac{7}{9}\cdot\int \frac{1}{x^2-(\sqrt{5})^2}\cdot dx + \frac{11}{9}\cdot\int\frac{1}{x^2+(2)^2}\cdot dx$$

$$= \frac{7}{9}\cdot\frac{1}{2(\sqrt{5})}\cdot\log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{9}\cdot\frac{1}{2}\cdot\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \text{I} = \frac{7}{18(\sqrt{5})}\cdot\log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{18}\cdot\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \int\frac{2x^2-3}{(x^2-5)(x^2+4)}\cdot dx = \frac{7}{18(\sqrt{5})}\cdot\log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{18}\cdot\tan^{-1}\left(\frac{x}{2}\right) + c$$

3. $\displaystyle\int \frac{1}{(\sin\theta)(3+2\cos\theta)}\cdot d\theta$

Solution : $\displaystyle I = \int \frac{1}{(\sin\theta)(3+2\cos\theta)}\cdot d\theta = \int \frac{\sin\theta}{(1-\cos^2\theta)(3+2\cos\theta)}\cdot d\theta$

$$= \int \frac{\sin\theta}{(1-\cos\theta)(1+\cos\theta)(3+2\cos\theta)}\cdot d\theta$$

put $\cos\theta = t$ $\quad\therefore\quad -\sin\theta\cdot d\theta = 1\cdot dt$

$\therefore\quad \sin\theta\cdot d\theta = -1\cdot dt$

Consider, $\displaystyle\frac{-1}{(1-t)(1+t)(3+2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(3+2t)}$

$$= \frac{A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)}{(1-t)(1+t)(3+2t)}$$

$\therefore\quad -1 = A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)$

at $t=1$, $\quad -1 = A(2)(5) + B(0) + C(0)$

$\displaystyle -1 = 10A \quad\Rightarrow\quad A = -\frac{1}{10}$

at $t=-1$, $\quad -1 = A(0) + B(2)(1) + C(0)$

$\displaystyle -1 = 2B \quad\Rightarrow\quad B = -\frac{1}{2}$

at $t = -\dfrac{3}{2}$, $\quad \displaystyle -1 = A(0) + B(0) + C\left(+\frac{5}{2}\right)\left(-\frac{1}{2}\right)$

$\displaystyle -1 = -\frac{5}{4}C \quad\Rightarrow C = \frac{4}{5}$

Thus, $\displaystyle\frac{-1}{(1-t)(1+t)(3+2t)} = \frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}$

$$\therefore\ I = \int\left[\frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}\right]\cdot dt$$

$$= -\frac{1}{10}\log(1-t)\cdot\frac{1}{(-1)} - \frac{1}{2}\log(1+t) + \frac{4}{5}\log(3+2t)\cdot\frac{1}{2} + c$$

$$= \frac{1}{10}\log(1-\cos\theta) - \frac{1}{2}\log(1+\cos\theta) + \frac{4}{10}\log(3+2\cos\theta) + c$$

$$= \frac{1}{10}\left(\log\frac{(1-\cos\theta)(3+2\cos\theta)^4}{(1+\cos\theta)^5}\right) + c \qquad \because \log a^m = m\cdot\log a$$

4. $\int \frac{1}{2\cos x + \sin 2x} \cdot dx$

Solution : $I = \int \frac{1}{2\cos x + \sin 2x} \cdot dx \quad = \int \frac{1}{2\cos x + 2\sin x \cdot \cos x} \cdot dx \quad = \int \frac{1}{2(\cos x)(1+\sin x)} \cdot dx$

$$= \frac{1}{2} \cdot \int \frac{\cos x}{\cos^2 x\,(1+\sin x)} \cdot dx = \frac{1}{2} \cdot \int \frac{\cos x}{(1-\sin^2 x)(1+\sin x)} \cdot dx$$

put $\sin x = t \qquad \therefore \quad \cos x \cdot dx = 1 \cdot dt$

$$= \frac{1}{2} \cdot \int \frac{1}{(1-t^2)(1+t)} \cdot dt \quad = \frac{1}{2} \cdot \int \frac{1}{(1-t)(1+t)(1+t)} \cdot dt = \frac{1}{2} \cdot \int \frac{1}{(1-t)(1+t)^2} \cdot dt$$

Consider, $\frac{1}{(1-t)(1+t)^2} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2} = \frac{A(1+t)^2 + B(1-t)(1+t) + C(1-t)}{(1-t)(1+t)^2}$

$$\therefore \quad 1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$$

at $t = 1$, $\quad 1 = A(2)^2 + B(0) + C(0)$

$1 = 4A \qquad \Rightarrow \quad A = \frac{1}{4}$

at $t = -1$, $\quad 1 = A(0) + B(0) + C(2)$

$1 = 2C \qquad \Rightarrow \quad C = \frac{1}{2}$

at $t = 0$, $\quad 1 = A(1)^2 + B(1)(1) + C(1)$

$1 = A + B + C$

$1 = \frac{1}{4} + B + \frac{1}{2} \qquad \Rightarrow \quad B = \frac{1}{4}$

Thus, $\quad \frac{1}{(1-t)(1+t)^2} = \frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2}$

$$\therefore I = \int \left[\frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2} \right] \cdot dt = \frac{1}{2}\left[\frac{1}{4}\log(1-t) \cdot \frac{1}{(-1)} + \frac{1}{4}\log(1+t) + \frac{1}{2} \cdot \frac{(-1)}{(1+t)} \right] + c$$

$$= \frac{1}{2}\left[\frac{1}{4}\log(1-t) \cdot \frac{1}{(-1)} + \frac{1}{4}\log(1+t) + \frac{1}{2} \cdot \frac{-1}{1+t} \right] + c$$

$$= \frac{1}{8}\left[-\log(1-\sin x) + \log(1+\sin x) - \frac{2}{1+\sin x} \right] + c \quad = \frac{1}{8}\left[\log\left(\frac{1+\sin x}{1-\sin x}\right) - \frac{2}{1+\sin x} \right] + c$$

$$\therefore \quad \int \frac{1}{2\cos x + \sin 2x} \cdot dx = \frac{1}{8}\left[\log\left(\frac{1+\sin x}{1-\sin x}\right) - \frac{2}{1+\sin x} \right] + c$$

5. $\displaystyle\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\cdot d\theta$

Solution : $\displaystyle I = \int \frac{(\tan\theta)\,(1+\tan^2\theta)}{1+\tan^3\theta}\cdot d\theta \quad = \int \frac{(\tan\theta)\cdot(1+\tan^2\theta)}{1+\tan^3\theta}\cdot d\theta = \int \frac{\tan\theta\cdot\sec^2\theta}{1+\tan^3\theta}\cdot d\theta$

put $\tan\theta = x$ $\quad\therefore\quad \sec^2\theta\cdot d\theta = 1\cdot dx$

$$= \int \frac{x}{1+x^3}\cdot dx = \int \frac{x}{(1+x)\,(1-x+x^2)}\cdot dx$$

Consider, $\displaystyle \frac{x}{(1+x)\,(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{(1-x+x^2)}$

$$= \frac{A\,(1-x+x^2)+Bx+C\,(1+x)}{(1+x)\,(1-x+x^2)}$$

$$\therefore x \;= A\,(1-x+x^2)+(Bx+C)\,(1+x) = A - Ax + Ax^2 + Bx + Bx^2 + C + Cx$$

$$0\,x^2 + 1\cdot x + 0 = (A+B)\,x^2 + (-A+B+C)\,x + (A+C)$$

compairing the co-efficients of like powers of variables.

$0 = A + B$ **. . . (I)**

$1 = -A + B + C$ **. . . (II)** and

$0 = A + C$ **. . . (III)**

Solving these equations, we get $A = -\dfrac{1}{3}$; $B = \dfrac{1}{3}$ and $C = \dfrac{1}{3}$

Thus, $\displaystyle \frac{x}{(1+x)\,(1-x+x^2)} = \frac{\left(-\frac{1}{3}\right)}{1+x} + \frac{\left(\frac{1}{3}x+\frac{1}{3}\right)}{(1-x+x^2)}$

$$\therefore\ I = \int \left[\frac{\left(-\frac{1}{3}\right)}{1+x} + \frac{\left(\frac{1}{3}x+\frac{1}{3}\right)}{(1-x+x^2)}\right]\cdot dx \quad = -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{3}\cdot\int \frac{x+1}{1-x+x^2}\cdot dx$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{3}\cdot\frac{1}{(2)}\int \frac{2x-1+3}{x^2-x+1}\cdot dx \qquad \because \quad \frac{d}{dx}\,x^2 - x + 1 = 2x - 1$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{3}\cdot\frac{1}{2}\cdot\int \frac{2x-1+3}{x^2-x+1}\cdot dx$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\cdot dx + \frac{1}{6}\cdot\int \frac{3}{x^2-x+1}\cdot dx$$

$= \; I_1 + I_2 + I_3$ **. . . (IV)**

$$\therefore \quad I_1 = -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx \qquad = -\frac{1}{3}[\log(1+x)]$$

$$= -\frac{1}{3}\log\left(1+\tan\theta\right) \qquad \ldots \textbf{(V)}$$

$$\therefore \quad I_2 = \frac{1}{6}\cdot\int \frac{2x-1}{x^2-x+1}\cdot dx \quad = \frac{1}{6}[\log(x^2-x+1)]$$

$$= \frac{1}{6}\log\left(\tan^2\theta - \tan\theta + 1\right) \qquad \ldots \textbf{(VI)}$$

$$\therefore \quad I_3 = \frac{1}{6}\cdot\int \frac{3}{x^2-x+1}\cdot dx$$

$$= \frac{1}{2}\cdot\int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}\cdot dx \qquad \because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(-1)\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$$

$$= \frac{1}{2}\cdot\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\cdot dx$$

$$= \frac{1}{2}\cdot\left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\right]\cdot\tan^{-1}\left[\frac{x-\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)}\right]+c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$$

$$\therefore \quad I_3 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)+c \qquad \ldots \textbf{(VII)}$$

$$\therefore \int \frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta}\cdot d\theta = -\frac{1}{3}\log\left(1+\tan\theta\right)+\frac{1}{6}\log\left(\tan^2\theta-\tan\theta+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)+c$$

EXERCISE 3.4

I. Integrate the following *w. r. t. x* :

1. $\dfrac{x^2+2}{(x-1)(x+2)(x+3)}$

2. $\dfrac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$

3. $\dfrac{12x+3}{6x^2+13x-63}$

4. $\dfrac{2x}{4-3x-x^2}$ 5. $\dfrac{x^2+x-1}{x^2+x-6}$ 6. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

7. $\dfrac{12x^2-2x-9}{(4x^2-1)(x+3)}$ 8. $\dfrac{1}{x(x^5+1)}$ 9. $\dfrac{2x^2-1}{x^4+9x^2+20}$

10. $\dfrac{x^2+3}{(x^2-1)(x^2-2)}$ 11. $\dfrac{2x}{(2+x^2)(3+x^2)}$ 12. $\dfrac{2^x}{4^x-3\cdot 2^x-4}$

13. $\dfrac{3x-2}{(x+1)^2(x+3)}$ 14. $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$ 15. $\dfrac{1}{x(1+4x^3+3x^6)}$

16. $\dfrac{1}{x^3-1}$ 17. $\dfrac{(3\sin x-2)\cdot\cos x}{5-4\sin x-\cos^2 x}$ 18. $\dfrac{1}{\sin x+\sin 2x}$

19. $\dfrac{1}{2\sin x+\sin 2x}$ 20. $\dfrac{1}{\sin 2x+\cos x}$ 21. $\dfrac{1}{\sin x\cdot(3+2\cos x)}$

22. $\dfrac{5\cdot e^x}{(e^x+1)(e^{2x}+9)}$ 23. $\dfrac{2\log x+3}{x(3\log x+2)[(\log x)^2+1]}$

3.5 Something Interesting :

Students/ now familier with the integration by parts.

The result is $\int u\cdot v\cdot dx = u\cdot\int v\cdot dx - \int\left(\dfrac{d}{dx}\cdot u\right)\left(\int v\cdot dx\right)\cdot dx$,

u and v are differentiable functions of x and $u\cdot v$ follows L I A T E order.

This result can be extended to the generalisation as -

$\int u\cdot v\cdot dx = u\cdot v_1 - u'\cdot v_2 + u''\cdot v_3 - u'''\cdot v_4 + \ldots$

($'$) dash indicates the derivative.

($_1$) subscript indicates the integration.

This result is more useful where the first function (u) is a polynomial, because $\dfrac{d^n u}{dx^n} = 0$ for some n.

For example : $\int x^2\cdot\cos 3x\cdot dx$

$$= x^2\cdot\left(\sin 3x\cdot\frac{1}{3}\right) - (2x)\left(-\cos 3x\cdot\frac{1}{3}\cdot\frac{1}{3}\right) + (2)\left(-\sin 3x\cdot\frac{1}{3}\cdot\frac{1}{9}\right) - (0)$$

$$= \frac{1}{3}x^2\cdot\sin 3x + \frac{2}{9}x\cdot\cos 3x - \frac{2}{27}\sin 3x + c$$

verify this example with usual rule of integration by parts.