<u>INDIVIDUAL TASK-2</u>

Equations of Activation and Synaptic Dynamics in Neural Networks

1. Introduction:-

Neural networks, both biological and artificial, rely on **activation dynamics** and **synaptic interactions** to process information. In biological neurons, the membrane potential evolves according to ion channel dynamics, while synapses mediate communication between neurons. In artificial neural networks (ANNs), activation functions and synaptic weights mimic these behaviors. Understanding these dynamics is critical for modeling learning, stability, and computation in networks.

2. Neuronal Activation Dynamics:-

The **activation** of a neuron refers to its output in response to inputs. In biological neurons, this is governed by the membrane potential (V(t)). The canonical model is the **leaky integrate-and-fire (LIF) neuron**:

$$\tau_m \frac{dV(t)}{dt} = -(V(t) - V_{\text{rest}}) + R_m I_{\text{syn}}(t)$$

Where:

- $V(t)$ is the membrane potential
- V_{rest} is the resting potential
- τ_m is the membrane time constant
- R_m is the membrane resistance
- $I_{\text{syn}}(t)$ is the synaptic input current

When $V(t)$ reaches a threshold V_{th}, the neuron fires a spike and $V(t)$ resets.

2.1 Rate-Based Models

For networks where spiking is abstracted, neurons are often modeled by **firing rates** (r_i(t)):

$$\tau_r \frac{dr_i(t)}{dt} = -r_i(t) + F\left(\sum_j w_{ij} r_j(t) + I_i(t)\right)$$

This equation captures **temporal smoothing** of the neuron's response.

2.2 Activation Functions

Activation functions determine how input currents are converted to output firing rates:

- **Sigmoid:**
- **Hyperbolic tangent:**
- **Rectified Linear Unit (ReLU):** (F(x) = \max(0, x))

3. Synaptic Dynamics:-

Synapses transmit signals between neurons and often display **temporal dynamics**. The basic synaptic model is:

$$\tau_s \frac{ds_{ij}(t)}{dt} = -s_{ij}(t) + \sum_k \delta(t - t_j^k)$$

Where:

- $s_{ij}(t)$ is the synaptic gating variable for the synapse from neuron j to i
- τ_s is the synaptic time constant
- t_j^k are the spike times of neuron j

The **synaptic current** is then:

$$I_{\text{syn}}(t) = \sum_j w_{ij} s_{ij}(t)$$

This formulation accounts for **synaptic filtering**, i.e., the finite rise and decay of postsynaptic potentials.

3.1 Short-Term Synaptic Plasticity

Synapses exhibit dynamic changes depending on activity, modeled as **facilitation** and **depression**:

- **Depression:**

$$\frac{dR(t)}{dt} = \frac{1 - R(t)}{\tau_d} - UR(t) \sum_k \delta(t - t_k)$$

- **Facilitation:**

$$\frac{du(t)}{dt} = \frac{U - u(t)}{\tau_f} + U(1 - u(t)) \sum_k \delta(t - t_k)$$

Where (R(t)) is the fraction of available synaptic resources, (u(t)) is the utilization factor, and (U) is baseline utilization. The effective synaptic strength is (u(t) R(t)).

4. Coupled Neuronal-Synaptic Dynamics:-

Combining neuron and synapse models yields:

$$\tau_m \frac{dV_i}{dt} = -(V_i - V_{\text{rest}}) + \sum_j w_{ij} s_{ij}(t)$$

$$\tau_s \frac{ds_{ij}}{dt} = -s_{ij} + \sum_k \delta(t - t_j^k)$$

This forms the basis for **spiking neural networks (SNNs)**, which can capture temporal coding, oscillations, and network synchrony.

5. Learning and Synaptic Plasticity:-

For a network of N neurons:

$$\tau_r \frac{d\mathbf{r}}{dt} = -\mathbf{r} + F(\mathbf{W}\mathbf{r} + \mathbf{I}(t))$$

- $\mathbf{r}$ is the vector of firing rates
- $\mathbf{W}$ is the weight matrix
- $\mathbf{I}(t)$ is external input vector

Analysis of eigenvalues of $\mathbf{W}$ predicts stability, oscillations, and bifurcations.

6. Network-Level Dynamics:-

For a network of N neurons:

$$\tau_r \frac{d\mathbf{r}}{dt} = -\mathbf{r} + F(\mathbf{W}\mathbf{r} + \mathbf{I}(t))$$

- $\mathbf{r}$ is the vector of firing rates
- $\mathbf{W}$ is the weight matrix
- $\mathbf{I}(t)$ is external input vector

Analysis of eigenvalues of $\mathbf{W}$ predicts stability, oscillations, and bifurcations.

7. Continuous-Time Recurrent Neural Networks (CTRNNs):-

CTRNNs use:

$$\tau_i \frac{dx_i}{dt} = -x_i + \sum_j w_{ij}\sigma(x_j + \theta_j) + I_i(t)$$

Where σ is a nonlinear activation, and θ_j is a bias. This system can emulate biological circuits and serve as models for robotics and cognitive computation.

8. Example: Excitatory-Inhibitory Network:-

Consider an excitatory (E) and inhibitory (I) population:

$$\tau_E \frac{dr_E}{dt} = -r_E + F_E(w_{EE}r_E - w_{EI}r_I + I_E)$$

$$\tau_I \frac{dr_I}{dt} = -r_I + F_I(w_{IE}r_E - w_{II}r_I + I_I)$$

Such networks generate oscillations, balance excitation and inhibition, and exhibit realistic cortical dynamics.

9. Numerical Integration:-

Equations are typically solved using **Euler or Runge-Kutta methods**:

$$V(t + \Delta t) = V(t) + \Delta t \frac{dV}{dt}$$

$$s(t + \Delta t) = s(t) + \Delta t \frac{ds}{dt}$$

Time step Δt must satisfy $\Delta t \ll \min(\tau_m, \tau_s)$ for stability.

Absolutely — we can expand this into an additional **5 pages** by going deeper into **advanced activation dynamics, synaptic plasticity mechanisms, network phenomena, stochastic effects, and computational implementations**. I'll continue seamlessly from the previous draft.

11. Conductance-Based Synaptic Models:-

While current-based synapses ((I_\text{syn} = w s(t))) are simple, biological synapses are **conductance-based**, meaning the synaptic current depends on the membrane potential:

$$I_{\text{syn}}(t) = g_{\text{syn}}(t)\big(V(t) - E_{\text{rev}}\big)$$

Where:

- $g_{\text{syn}}(t)$ is the synaptic conductance
- E_{rev} is the reversal potential ($0\,\mathrm{mV}$ for excitatory, $-70\,\mathrm{mV}$ for inhibitory)
- $V(t)$ is the postsynaptic membrane potential

The conductance dynamics are often modeled as:

$$\tau_{\text{syn}} \frac{dg_{\text{syn}}}{dt} = -g_{\text{syn}} + \sum_k w\delta(t - t_k)$$

This model captures **shunting effects**, voltage dependence of postsynaptic responses, and more realistic postsynaptic potentials.

12. Nonlinear Activation Dynamics:-

Neurons often exhibit **nonlinearities beyond simple rate functions**:

12.1 Sigmoidal Saturation

For high inputs, firing rates saturate:

$$r(t) = \frac{r_{\max}}{1 + e^{-\beta(I_{\text{syn}} - \theta)}}$$

- $r_{\max}$ is the maximal firing rate
- β controls slope
- θ is threshold

This allows networks to implement **soft thresholding**, avoiding unbounded activity.

12.2 Adaptation

Neurons show **spike-frequency adaptation**, modeled by an additional variable (a(t)):

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + I_{\text{syn}} - a$$

$$\tau_a \frac{da}{dt} = -a + br(t)$$

Where b scales adaptation. This mechanism supports **temporal pattern detection** and prevents runaway excitation.

13. Advanced Synaptic Plasticity:-

Beyond STDP, **synaptic dynamics** include:

$$\frac{dw_{ij}}{dt} = \eta(r_i r_j - \alpha w_{ij})$$

- η is the learning rate
- α is a decay term
- Ensures weights stabilize

13.1 Long-Term Plasticity

Long-term potentiation (LTP) and depression (LTD) can be modeled as continuous weight changes:

$$\frac{dw_{ij}}{dt} = \eta(r_i r_j - \alpha w_{ij})$$

- η is the learning rate
- α is a decay term
- Ensures weights stabilize

13.2 Triplet STDP

Captures frequency-dependent learning:

$$\Delta w \propto A_3 r_{\text{pre}} r_{\text{post}}^2 - A_2 r_{\text{pre}} r_{\text{post}}$$

This reflects the **interaction of multiple spikes** and matches experimental data better than pair-based STDP.

14. Network Phenomena:-

14.1 Oscillations

Neural networks can generate **rhythmic activity**:

$$\tau_E \frac{dr_E}{dt} = -r_E + F_E(w_{EE} r_E - w_{EI} r_I + I_E)$$
$$\tau_I \frac{dr_I}{dt} = -r_I + F_I(w_{IE} r_E - w_{II} r_I + I_I)$$

Analysis of eigenvalues of the connectivity matrix predicts **gamma, beta, or theta oscillations**, critical for sensory processing and memory.

14.2 Winner-Take-All (WTA) Dynamics

Networks with **mutual inhibition** implement selective responses:

$$r_i(t) = F\big(I_i - \sum_{j \neq i} w_{ij} r_j\big)$$

This leads to **competition between neurons**, forming the basis of decision-making circuits.

15. Stochastic Effects:-

Neurons and synapses are inherently noisy:

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + I_{\text{syn}} + \sigma \xi(t)$$

- $\xi(t)$ is Gaussian white noise
- σ scales noise amplitude

Noise can **enhance information transmission** (stochastic resonance), induce variability, and prevent deterministic network traps.

16. Synaptic Filtering and Temporal Integration:-

Synapses act as **low-pass filters**:

$$s(t) = \int_0^\infty e^{-\tau/\tau_s} \sum_k \delta(t - t_k) d\tau$$

- Longer τ_s increases temporal integration
- Important for working memory and temporal sequence learning

Multiple timescales in synapses allow networks to **store short-term patterns** and implement **eligibility traces** for learning.

17. Computational Implementation:-

17.1 Discrete-Time Approximation

Neuron and synapse equations are discretized for simulation:

$$V(t + \Delta t) = V(t) + \frac{\Delta t}{\tau_m}\big(-(V(t) - V_{\text{rest}}) + I_{\text{syn}}(t)\big)$$

$$s(t + \Delta t) = s(t) + \frac{\Delta t}{\tau_s}\big(-s(t) + \sum_k \delta(t - t_k)\big)$$

17.2 Efficient Simulation Techniques

- **Event-driven methods:** Only update synapses at spike times
- **Vectorized updates:** For large networks in Python, MATLAB, or C++
- **GPU acceleration:** Parallelize neuron and synapse dynamics for deep SNNs

18. Multi-Scale Models

Neural dynamics can be modeled at multiple scales:

1. **Single neuron level:** Membrane potentials, ion channels
2. **Synapse level:** Conductance, plasticity, short-term dynamics
3. **Network level:** Population rates, oscillations, attractors
4. **System level:** Cortical areas, feedback loops, cognitive computation

These models connect microscopic biophysics with macroscopic behaviors like **memory, decision-making, and perception**.

19. Example: Cortical Microcircuit Model

A microcircuit with E/I populations and dynamic synapses can be modeled as:

$$\tau_E \frac{dr_E}{dt} = -r_E + F_E(w_{EE} r_E - w_{EI} r_I + I_E)$$

$$\tau_I \frac{dr_I}{dt} = -r_I + F_I(w_{IE} r_E - w_{II} r_I + I_I)$$

$$\tau_s \frac{ds_{ij}}{dt} = -s_{ij} + \sum_k u(t) R(t) \delta(t - t_k)$$

- Combines **synaptic dynamics, short-term plasticity, and population activity**
- Used in modeling **working memory, attention, and sensory processing**

20. Conclusion:-

Activation and synaptic dynamics form the fundamental basis of both biological and artificial neural networks. Neuronal activation, governed by membrane potentials or firing rates, determines how neurons process incoming information over time, while synaptic dynamics, including conductance changes, short- and long-term plasticity, and filtering, mediate communication and learning between neurons.

By combining these mechanisms, neural networks can exhibit a rich repertoire of behaviors, including oscillations, pattern recognition, decision-making, and memory storage. Advanced models incorporating stochasticity, adaptation, and multi-scale interactions bridge the gap between simplified artificial networks and realistic biological circuits.

Understanding these dynamics not only provides insight into brain function but also guides the design of more efficient and robust computational models, including spiking neural networks and continuous-time recurrent networks. Future developments integrating dendritic computation, neuromodulation, and network connectivity will further enhance the fidelity of neural modeling and expand applications in artificial intelligence, robotics, and neuroscience research.

21. References:-

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