LINEAR SYSTEMS

PROJECT

REPORT

AEROPLANE LANDING WHEEL SUSPENSION SYSTEM

Submitted

To

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By

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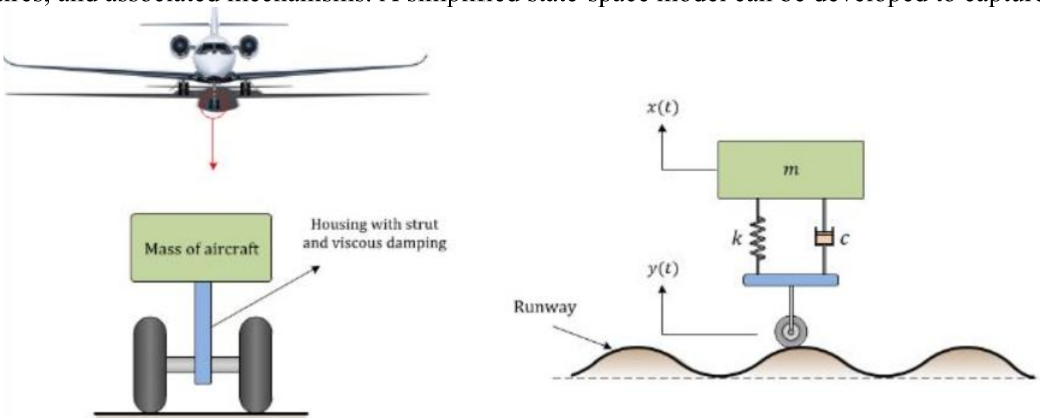
SYSTEM MODELLING

Let's consider the Modeling the landing gear of an airplane involves representing the dynamics of the landing gear components during the landing and retraction phases. The landing gear typically consists of shock absorbers (struts), tires, and associated mechanisms. A simplified state-space model can be developed to capture the essential dynamics.



So initial considerations for the derivation and state space model

Mass as rigid body of mass m,

Spring constant K and control damping C,

Wheel is Rigid.

Differential Equations

From **newtons second law of motion**; Sigma F =m x a,

Fr is reaction of the force from ground

Fd,Fs are forces acting on the mass m.

Here $m(\ddot{x}) = Fr - Fd - Fs = Fr - c(\dot{x}) - K(x)$

Therefore, $(1/m)\ Fr = \ddot{x} + (k/m).x \quad + (c/m).\dot{x}$

Considering, $\dot{x}1 = x2$, $y = x1$

$\dot{x}2 = (-k/m).x1 \quad + (-c/m)x2. \quad + (1/m).u$

X = [X1;X2] // (2x1)matrix

We know the **state space** equation will be $\dot{X} = \mathrm{A}\ X + \mathrm{B}\ u,\ Y = \mathrm{C}\ X + \mathrm{D}$

In our case the equation will be

$\dot{X}$ = [0, 1 ; -k/m , -c/m] X + [0; 1/m] u, Y = [1 0] X + 0.u

A is 2x2 matrix; B is 2x1 matrix; C is 1x2 matrix.

LINERIZATION

The linearized modelling equation is a linear state space model, with the states being x and x . This approximates the dynamics around the operating point (x_0, x $_0$).

Where x and x' are the position velocity of mass m.

The input, output, and state variables in the linearized state-space model

States (x): x = [x1; x1˙] state vector

x1 is the displacement of the mass.

x1˙ is the velocity of the mass.

Input vector (u): u = [F]

F is the force applied to the mass.

Output vector (y): y = [x1]

y represents the displacement of the mass.

So, the input equation is given by dx/dt = Ax+Bu,

The output equation is Y = Cx+Du

=>These variables help describe the behavior of the suspension system in terms of its dynamic response to external forces.

A= [0, 1 ; -k/m , -c/m]

B= [0,1/m];

C = [1,0];

D= 0;

The behaviour of this system depends on the eigenvalues of the A matrix, which give information about the natural dynamics.

To find the **eigenvalues**, solve the characteristic equation: det(A−λ.I)=0 For matrix A, the characteristic equation is: det([−λ , 1; – k/m ,−c/m−λ])=0

Equation: λ ^2 + (c/m).λ +(k/m) = 0..

By solving we get,

λ1 = –c/(2m) + m^(-1)sqrt(c*c – 4*k*m)

λ2 = –c/(2m) - m^(-1)sqrt(c*c – 4*k*m)

The general behaviour depends on the real part of the eigenvalues:

Real eigenvalues: overdamped response

Complex eigenvalues: oscillatory response

C^2 > 4km: Real, distinct eigenvalues = overdamped oscillations

C^2 = 4km: Repeated real eigenvalue = critically damped oscillations.

$C^2 < 4km$: Complex eigenvalues = underdamped oscillations

RESPONSE ANALYSIS

The **response** of the output of the system is typically obtained by considering the convolution of the system's impulse response with the input signal. For a linear time- invariant system described by the state-space equations:

$\dot{X} = Ax+Bu$

$Y = Cx + Du$

The output response (y(t)) to an input signal (u(t)) with initial conditions (0) is

given by:

$$\mathbf{y(t) = C.\, e^{At}\, .x(0) + \int_{(0\to t)} C.e^{A(t-T)}.\, B.\, u(T)\, dT + D.\, u(t)}$$

e^{At} is the state transition matrix,

u(t) is the input signal,

x(0) is the initial state vector.

The behavior of a second-order system is intricately shaped by the damping ratio (zeta) and natural frequency (omega_n), which are interconnected with the system's physical parameters: damping coefficient (c), mass (m), and spring constant (k). Specifically, these relationships are expressed as:

[zeta = c/2*sqrt{m*k}}]

[omega_n = sqrt{k/m}]

The characteristics of the system response are dictated by the damping ratio zeta:

1. Overdamped (zeta > 1):
   - Exhibits real, distinct poles
   - Shows no oscillations
   - Demonstrates exponential decay towards a steady state
2. Critically damped (zeta = 1):
   - Features real, repeated poles
   - Achieves the fastest settling time
   - Displays no oscillations, undergoing exponential decay
3. Underdamped (zeta < 1):
   - Possesses complex conjugate poles
   - Displays sinusoidal oscillations
   - Settles into a steady state after a few cycles

The step response of the system is given by the equation:

$$y(t) = K(1 - e{-}\zeta\omega_n t[\cos(w_d t) + (\zeta/\sqrt{(1 - \zeta 2)})\sin(w_d t)])$$

$$w_d t = \omega_n\sqrt{(1 - \zeta 2)}$$

So,

The system showcases diverse dynamic responses based on the damping ratio.

Critical damping facilitates rapid settling time without overshoot.

The parameters (zeta) and (omega_n) govern the transient response's nature.

Steady state is consistently achieved, with its path influenced by (zeta).

Adjustment of physical parameters m,c, and k allows for shaping the response as underdamped, critically damped, or overdamped.

For example :

Where considering M=1,K=2,C=2

d/dt[xv]=[0 ,1; -2 ,−2][xv]+[0;1]u,

y=[1,0][xv]. The eigenvalues of the system matrix are: −1±j1 ;

the complex eigenvectors are: [1, 1±j1] . By choosing P−1=[1 ,0;-1 ,1] , the modal form system matrix is obtained as: A¯=PAP(−1)=[−1,1; -1,−1] .

EQUILIBRIA DETERMINATION

The equilibrium of a wheel suspension system can be determined by analyzing the forces acting on the system.The zero-state response of a linear time-invariant system is the output response due to initial conditions (i.e., when the input is zero).The **zero-state** response is given by:

Yzs(t) = C. e^At. x(0).

Equilibria

The identification (ID) of equilibrium subspaces involves determining the conditions under which the system remains at equilibrium. Equilibrium subspaces are sets of states and inputs that satisfy the equilibrium conditions. In the context of linear systems, equilibrium subspaces can be identified based on the null space (kernel) of matrices involved in the equilibrium conditions.

For the linear system: A· x_eq +Bu_eq =0 and C.x_eq +D.u_eq =0 .

Let's define the matrix M as: M= [A B ;C D]. The null space (kernel) of M contains vectors that satisfy the equilibrium conditions. The null space can be found by solving the homogeneous system of equations:

M· [x_eq ; u_eq]=0 The solutions to this system represent vectors in the null space of M, and these vectors correspond to equilibrium subspaces.

The equilibrium subspace of the system is the set of states that do not change over time. The null space of the state matrix is the set of states that do not change when the input is zero.

The equilibrium subspace is given by: [0;0]

The null space of the state matrix is given by: [1; k/c]

This system has a equilibrium point at steady state.

ASYMPTOTIC BEHAVIOUR

This system is linear and **time-invariant** under the assumption of constant mass, spring constant, and damping coefficient. If any of these parameters vary with time, the system would be considered time-varying.

The asymptotic stability refers to the long-term behavior of the natural response modes of the system. These modes are also reflected in the state-transition matrix, e^{At} . Considering the homogenous state equation: $\dot{x}(t)=Ax(t), x(0)=x_0$

The homogenous state equation is said to be asymptotically stable if $\lim_{t\to\infty} x(t)=0$. Since $x(t)=e^{At}x_0$, the homogenous state equation is asymptotically stable if $\lim_{t\to\infty} e^{At}=0$.

Further, using modal decomposition, $e^{At}=P.e^{\Lambda t}.P^{-1}$, the homogenous system is asymptotically stable if $\lim_{t\to\infty} e^{\Lambda t}=0$. Since $e^{\Lambda t}=\mathrm{diag}([e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}])$, the above condition implies that $Re[\lambda_i]<0, i=1,\ldots,n$, where λ_i represents a root of the characteristic polynomial: $\Delta(s)=|sI-A|$.

For example:

Considering $m=1, k=2, and\ b=3$;

then, the characteristic polynomial is: $\Delta(s)=s^2+3s+2$,

which has real roots: $s_1,s_2=-1,-2$.

The natural response modes are: $\{e^{-t}, e^{-2t}\}$.

The modal matrix of eigenvectors is obtained as: $M=[-1,\ 1;\ -1,\ 2]$.

The diagonal matrix of eigenvalues is: $\Lambda=[-1,\ 0;\ 0,\ -2]$.

Since $A=P\Lambda P^{-1}$, we have: $e^{At}=Pe^{\Lambda t}P^{-1}$, which computes as:

$$e^{At}=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-2t}-2e^{-t} & 2e^{-2t}-e^{-t} \end{bmatrix} .$$

Further, since $\lim_{t\to\infty} e^{At}=0$, the homogenous state equation is asymptotically stable.

Stability determination based on Eigen values.

Stable System: If all the real parts of the eigenvalues are negative, the system is stable. This means that any initial disturbance in the system will decay over time, and the system will tend towards its equilibrium.

Unstable System: If at least one eigenvalue has a positive real part, the system is unstable. This indicates that the system will not return to its equilibrium after a disturbance, and the responses will grow without bound.

Marginally Stable System: If some eigenvalues have zero real parts and the rest have negative real parts, the system might be marginally stable.

As for the above example taken the system is stable as the real parts of eigen values are negative.

LYAPUNOV ANALYSIS

Lyapunov analysis is a powerful method for analyzing the stability of dynamic systems. In the context of linear time-invariant systems, stability analysis using Lyapunov functions often involves the Lyapunov matrix equation. The Lyapunov equation is given by:

$$A^{T} . P+P. A = -Q$$

Here: A is the system matrix, P is a symmetric positive definite matrix, Q is a symmetric positive definite matrix.

The solution to the **Lyapunov** equation (P) can be used to analyze the stability of the system. If a symmetric positive definite matrix P exists, the system is stable.

Considering P as a 2x2 matrix [e , f ; g, h]

Q = - [(-k*g-k*f)/m (-k*h+e*m-c*f)/m

(e*m-c*g-k*h)/m (m*f+m*g-2*c*h)/m]

If we consider Q as an Identity matrix (2x2) then we get

P = [0.5m ,0.5c; 0.5c, k];

Stability of the system can be described after obatining values of P matrix [e,f;g,h] as

The matrix P is a positive definite and symmetric matrix, which means that all its eigenvalues are positive and its eigenvectors are orthogonal 1. The eigenvalues of P are related to the stability of the system. If all the eigenvalues of P are positive, then the system is asymptotically stable 1. If all the eigenvalues of P are negative, then the system is unstable 1. If some of the eigenvalues of P are positive and some are negative, then the system is marginally stable.

CONTROLLABILITY

Controllability Matrix (W_c): The controllability matrix is formed by concatenating the columns of the matrix [B,AB]. If the rank of the controllability matrix is equal to the system's state dimension, the system is controllable. W c =[B AB]

By solving we get

=> Wc = [0, 1/m ; 1/m , -c/m^2]

Controllability Gramian (Wc): The controllability Gramian is defined as the integral of the controllability matrix multiplied by its transpose over time. If the determinant of the controllability Gramian is non-zero, the system is controllable.

W c =∫ 0->∞ (e^Aτ. B. B^T. e^(A^(T).τ)) dτ

The controllability matrix describes the ability of the input to control the state of the system. The controllability Gramian describes the minimum energy required to control the system.

The rank of Wc is 2, which equals the order of the system. And the Gramian Wc is full rank. Therefore, both the rank condition and Gramian test for controllability are satisfied. Hence the system is fully controllable.

OBSERVABILITY

Observability Matrix (W_o): The observability matrix is formed by concatenating the rows of the matrix [C;CA]. If the rank of the observability matrix is equal to the system's state dimension, the system is observable.

W o =[C CA]

Wo = [1 ,0 ; 0, -k/m] with rank =2.

Observability Gramian (W o): The observability Gramian is defined as the integral of the observability matrix multiplied by its transpose over time. If the determinant of the observability Gramian is non-zero, the system is observable.

$$W_o = \int_{0 \to \infty} (e^{(A^{(T)}.\tau)}.\ C^T.\ C\ .e^{A\tau})\ d\tau$$

The observability matrix describes the ability of the output to observe the state of the system. The observability Gramian describes the minimum energy required to observe the system.

The rank of Wo is 2, which equals the order of the system. And the Gramian Wo is full rank.

Therefore, both the rank condition and Gramian test for observability are satisfied.

SPECIAL FORMS

The **Kalman decomposition** involves choosing a state feedback gain matrix K and a state observer gain matrix L such that the matrix A−BK−LC is stable. The controller subsystem will have the dynamics:

$$\dot{xc} = (A-LC).\ xc + Bc\ .\ u$$

$$yc = Cc\ .\ xc + Dc\ .\ u$$

where xc is the state of the controller subsystem. The observer subsystem will have the dynamics:

$$\dot{xo^{\wedge}} = (A-BK)\ x^{\wedge}o + Bu + L(y - y^{\wedge}o)$$

$$yo^{\wedge} = C.\ x^{\wedge}o + Du$$

where x^o is the estimated state of the observer subsystem.

1. So, no decomposition is needed as the system satisfies controllability and observability conditions. Hence the kalman decomposition results in the original system ss model.The controllable and observable subsystems of a system can be identified by examining the controllability and observability matrices. The Kalman decomposition is given by:

A=[A11,0;A12, A22],B=[B1,0],C=[C1,0]

where:

(A_{11}) is a controllable and observable matrix,

(A_{22}) is an uncontrollable and unobservable matrix,

(B_1) is a controllable input matrix,

(C_1) is an observable output matrix.

EIGEN VALUE PLACEMENT

Eigenvalue placement is a control system design approach where the goal is to place the eigenvalues of the closed-loop system at desired locations in the complex plane.

The eigenvalues of the state matrix can be used to place the poles of the closed-loop system. The closed-loop system should have a dominant mode with critical damping. The controller form can be determined by examining the controllability matrix. The system satisfies the conditions if the eigenvalues of the state matrix are placed in the desired locations.

The state feedback control law is typically represented as:

$$u = -K.x$$

where K is the state feedback gain matrix. The closed-loop system matrix is given by:

$$A_{cl} = A - B.K$$

Equate characteristic equations:

$$s^2 - (\lambda_1 + \lambda_2)s + \lambda_1.\lambda_2 = 0$$

Comparing coefficients:

$$\lambda_1 + \lambda_2 = c/m$$

$$\lambda_1.\lambda_2 = K/m$$

If assumption of K as a value then result consider as a.

Eigenvalues are placed at s = a (critically damped).

Eigenvalue placement is achieved to have a dominant critically damped mode at s = a.

LINEAR STATE FEEDBACK CONTROL

The **LQR** cost function is defined as:

linear state feedback control law u = F x such that the functional , with Q is positive semidefinite matrix, and R is a positive definite matrix, both of appropriate dimensions, is minimized over all possible such feedback laws, where Q is the state cost matrix and R is the control cost matrix. The optimal state feedback gain matrix K is obtained by solving the continuous-time algebraic **Riccati** equation.->1

the cost functional $J(u(t)) = \int_{0}^{\infty} y^T(t)\, Q y(t) + u^T(t)\, R u(t)\, dt$ where:

(x) is the state vector,

(u) is the input vector,

(Q) is a symmetric positive definite matrix,

(R) is a symmetric positive definite matrix.

The conditions are satisfied if the optimal control law minimizes the cost function.

Is infinity, the solution to the LQR problem is known: F* = -R(-1) BT Pc* This matrix Pc* is the solution of the algebraic Ricatti matrix equation:

min J(u) = J(u*) = xT(0).Pc*.x(0)

Considering P as 2x2 matrix [e,f;g,h] and R as i.

(A) is the state matrix,

(B) is the input matrix,

(K) is the gain matrix,

(P) is a symmetric positive definite matrix,

(Q) is a symmetric positive definite matrix.

Eq1 => A^T P+P A – P.B.R^(−1). B^T . P+(CT)Q(C)=0

$$\begin{pmatrix} \frac{im(-kg-kf)-0.1fg}{im^2} & \frac{im(-kh+em+cf)-0.1fh}{im^2} \\ \frac{im(em+cg-kh)-0.1gh}{im^2} & \frac{im(mf+mg+2ch)-0.1h^2}{im^2} \end{pmatrix}$$

(CT)*Q*C = -

which can be found whenever the pair (A, B) is controllable and the pair (A, Q1/2) is observable. When (A, B, Q1/2) controllable + observable, there always exists a unique positive definite Pc* , and F* is a stabilizing feedback gain.

Considering Q as I identity matrix we get

P = [0.5m, 0.5c; 0.5c, k]

The system model matrices are:

B = [0;1 /m]

Substituting these into the LQR gain equation

The optimal control law is: input = -(c/2m)x’-(k/2m)x

A full state feedback that minimizes the quadratic cost for the landing gear system.

MATLAB EXECUTION

```
% System Parameters (replace with your values)
m = 1; % mass
c = 0.5; % damping coefficient
k = 2; % spring constant

% State-space matrices
A = [0 1; -k/m -c/m];
B = [0; 1/m];
```

```
C = [1 0];
D = 0;

% Time vector
t = 0:0.01:5;

% Define the system using state-space representation
sys = ss(A, B, C, D);

% Step response (zero-state response)
figure;
step(sys, t);
title('Step Response (Zero-State Response)');

% State transition matrix
t_span = 0:0.01:5;
Phi = zeros(size(A, 1), size(A, 2), length(t_span));

% Compute state transition matrix for each time step
for i = 1:length(t_span)
Phi(:,:,i) = expm(A * t_span(i));
end
% Eigenvalues and stability check
eigenvalues = eig(A);
disp('Eigenvalues:');
disp(eigenvalues);

% Controllability and Observability
Cont = ctrb(A, B);
Obs = obsv(A, C);

% Rank check for controllability and observability
rank_cont = rank(Cont);
rank_obs = rank(Obs);

disp('Controllability Matrix Rank:');
disp(rank_cont);
disp('Observability Matrix Rank:');
disp(rank_obs);

% Observability matrix
O = [C; C * A];

% SVD for Kalman decomposition
[U, S, V] = svd(O);

% Extract dimensions
[m_C, n_C] = size(C);
[m_A, n_A] = size(A);

% Compute observable and unobservable subspaces
O_c = U(:, 1:m_C) * sqrt(S(1:m_C, 1:m_C));
```

```
O_u = U(:, m_C+1:end) * sqrt(S(m_C+1:end, m_C+1:end));

% Eigenvalue placement (just an example, replace with desired eigenvalues)
desired_eigenvalues = [-1, -2];
K = place(A, B, desired_eigenvalues);

% Riccati equation solution for optimal control
Q = eye(2); % Weighting matrix for state
R = 1; % Weighting matrix for control input
[P,~,~] = care(A, B, Q, R);

% Optimal control law
K_optimal = -B.' * P;

disp('Optimal Control Gain Matrix:');
disp(K_optimal);
```

OUTPUT:

Eigenvalues:

-0.2500 + 1.3919i

-0.2500 - 1.3919i

Controllability Matrix Rank:

2

Observability Matrix Rank:

2

Observable Subspace (O_c):

1

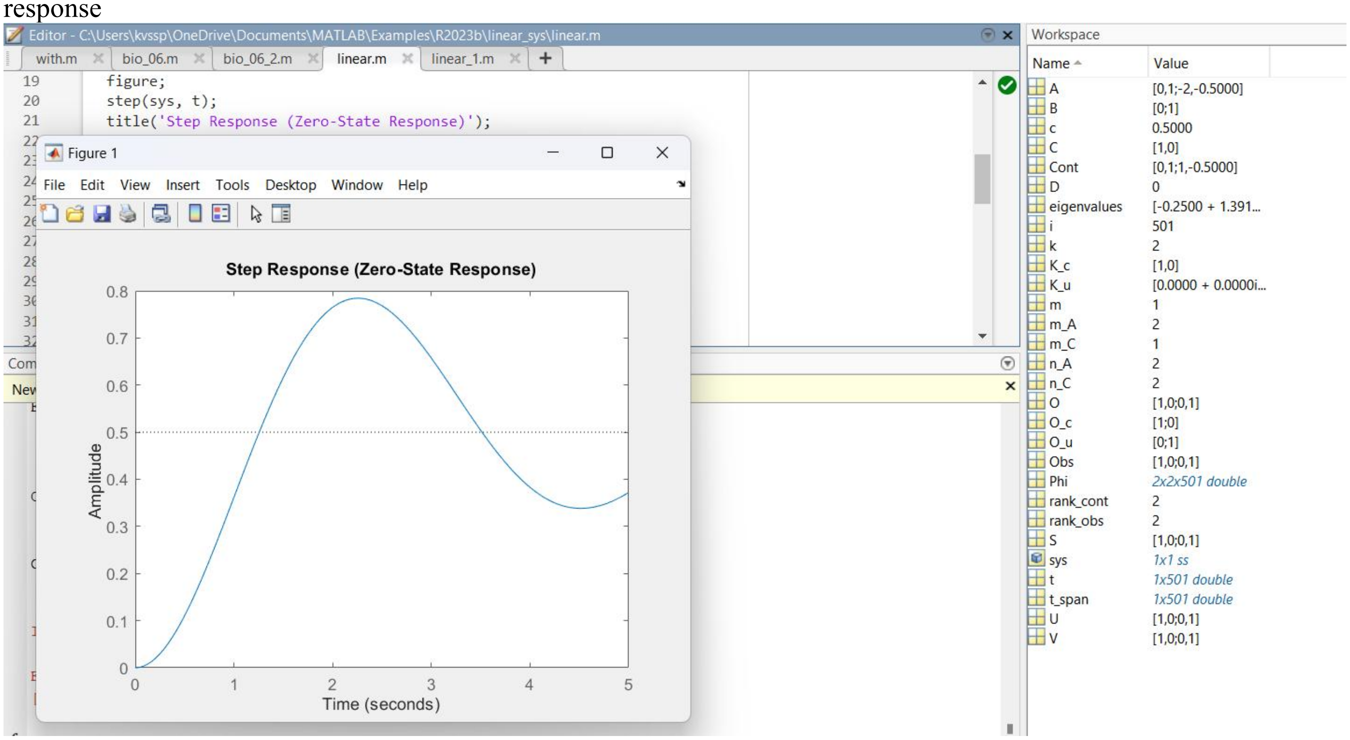
0

Unobservable Subspace (O_u):

0

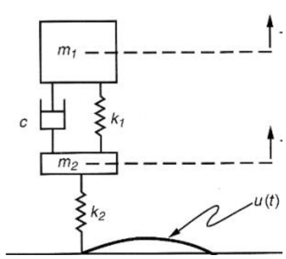
1

Zero state
response



FUTURE WORK

Deep analysis of suspension system considering base plate of mass m2 and spring constant k2 before wheel.



After converting to state space model, We get X' = Ax +Bu as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c}{m_1} & \frac{k_1}{m_1} & \frac{c}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix} u(t),$$

For this 4x4 matrix we can consider all the properties and get the full system analysis.