1 Harmonic Function

A real valued function $h = \phi(x, y)$ of two variables is said to be *Harmonic function* in a certain domain of xy-plane it it has continuous partial derivatives of the first and second order and satisfies the Laplace's equation $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$

Theorem 1:
If a function f(z) = u(x,y) + iv(x,y) is analytic in a domain D, then its component functions u and v are harmonic in D.
Proof:

Given that $f(z)$ is analytic so that two functions u(x,y) and v(x,y) have continuous partial derivatives of all orfers at that point and satisfying the relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

Differentiating eq^n (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial x\, \partial y} \tag{3}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial^2 v}{\partial y\, \partial x}$$

Since v is continuous, partial derivatives exists. Hence,

$$\frac{\partial^2 v}{\partial x\, \partial y} = \frac{\partial^2 v}{\partial y\, \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$