Localized Structures and Homoclinic Snaking

Nonlinear Dynamics Final Presentation

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Motivation

Examples of Localization:

Figure: Localized structures on the surface of a ferrofluid, see [5], [3]

Examples of Localization:

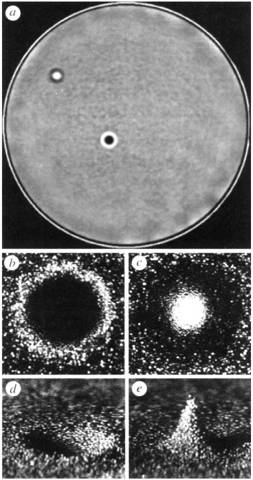


Figure: Localized structures in vertically vibrated granular layer, see [7]

Examples of Localization:

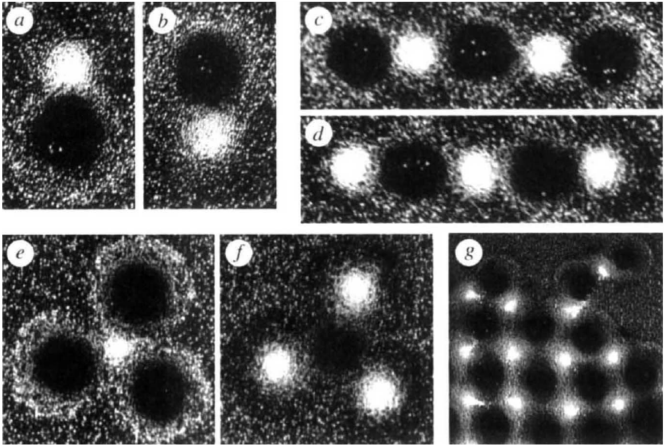


Figure: Oscillon 'molecules' (a), (b) are like dipoles, (c), (d) are like polymeric chains, (e), (f) are triangular tetramers and (g) is a lattice, see [7]

Examples of Localization:

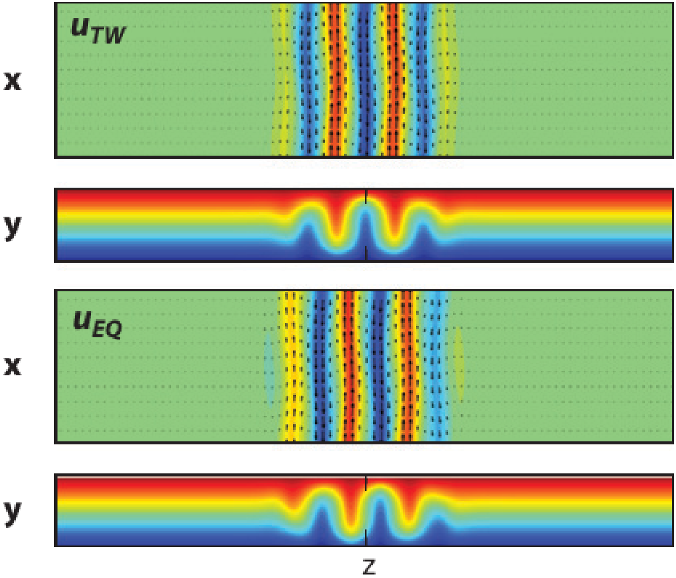


Figure: Localized structures in plane Couette flow, see [6]

Examples of Localization:

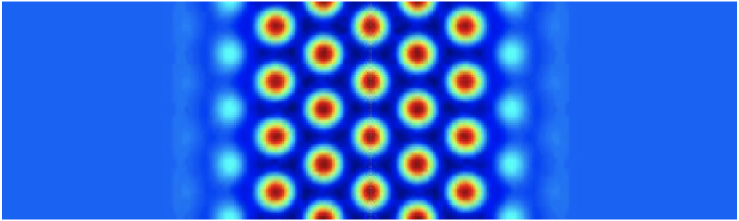
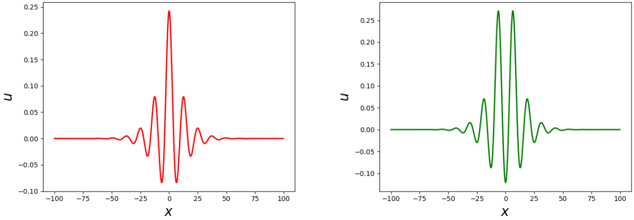


Figure: Localized hexagons in $2d$ Swift-Hohenberg equation, see [4]

Examples of Localization:



(a) (b)

Figure: Localized solutions of $1d$ Swift-Hohenberg equation

Linear Stability Analysis:

Linear Stability:

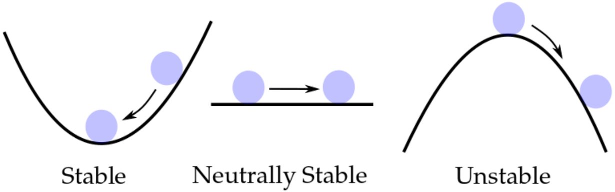


Figure: Intuitive linear stability

Linear Stability:

Let $\underline{x}_0$ be an equilibrium of $\dot{\underline{x}} = \underline{f}(\underline{x})$.
$\Rightarrow \underline{f}(\underline{x}_0) = 0$.
Small perturbation about $\underline{x}_0 \Rightarrow \underline{x} = \underline{x}_0 + \underline{\delta x}$
Substituting in the governing equation and using Taylor expansion:

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

$$\cancelto{0}{\dot{\underline{x}}_0} + \dot{\underline{\delta x}} = \cancelto{0}{\underline{f}(\underline{x}_0)} + [DF]|_{x_0}\underline{\delta x} + h.o.t.(O(\underline{\delta x}^2))$$

$$\dot{\underline{\delta x}} = \underbrace{[DF]|_{x_0}}_{\text{Stability Matrix } A} \underline{\delta x}$$

We obtained a "linearized" system around $\underline{x}_0$ where the stability matrix $A_{ij} = \frac{\partial f_i}{\partial x_j}$, evaluated at $\underline{x}_0$.

Linear Stability of $1d$ Swift-Hohenberg Equation with quadratic-cubic nonlinearity ($SH23$):

$SH23$ is given by:

$$\frac{\partial u}{\partial t} = ru - (\partial_x^2 + q_c^2)^2 u + vu^2 - gu^3 \tag{1}$$

Here, we choose $q_c = 0.5, v = 0.41, g = 1$, in accordance with [1]
Let's find the stationary solutions of Eqn. (1)

$$0 = (r - q_c^4)u + vu^2 - gu^3 \tag{2}$$

Equilibria:

$$u = 0, \quad u_{\pm} = \frac{1}{2g}[v \pm \sqrt{v^2 + 4g(r - q_c^4)}] \tag{3}$$

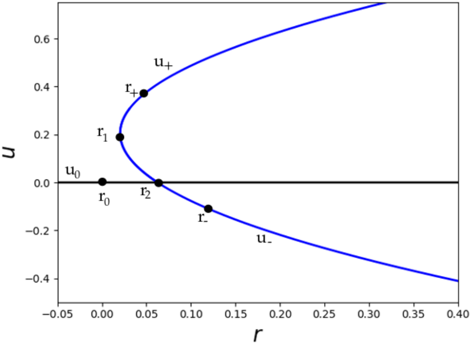


Figure: Analytically obtained steady solutions of $SH23$.

Equilibria:Linear Stability

For a stationary solution u_s, to find stability: substitute

$$u = u_s + \epsilon \tilde{u} e^{\beta t}, \tag{4}$$

into Eqn.(1) and linearize

$$\beta \tilde{u} = \mathcal{L}[u_s]\tilde{u}, \quad \tilde{u}(x+L) = \tilde{u}(x), \tag{5}$$

with $\mathcal{L} = [r - (\partial_x^2 + q_c^2)^2 + 2vu_s(x) - 3gu_s^2(x)]$.

Equilibria:Linear Stability

The eigenfunctions of Eqn.(5) are $\sin kx, \cos kx$ for $u_0, u_\pm$ and we can find their corresponding growth rates $\beta_0, \beta_\pm$.

$$\beta_0 = r - (q_c^2 - k^2)^2 \tag{6}$$

$$\beta_\pm = 3q_c^4 - (q_c^2 - k^2)^2 - 2r - \frac{v}{2g}\left[v \pm \sqrt{v^2 + 4g(r - q_c^4)}\right]. \tag{7}$$

Equilibria: Linear Stability

$u_\pm$ branches are generated at $r = r_1 = q^4 - v^2/4g$ in a saddle-node bifurcation. u_- branch bifurcates from the trivial solution u_0 at $r = r_2 = q_c^4$ in a transcritical bifurcation.

Trivial solution: perturbation theory, Method of multiple scales

Define small parameter $r = -\epsilon^2 \mu_2$, ($\mu_2 > 0$) and look for stationary solutions of Eqn. 1 of the form:

$$u_s(x) = \epsilon u_1(x, X) + \epsilon^2 u_2(x, X) + \dots \tag{8}$$

where $X = \epsilon x$ is the slow space-scale.

Trivial solution: perturbation theory, Method of multiple scales

$$\frac{d}{dx} = \frac{\partial}{\partial x_0} + \epsilon \frac{\partial}{\partial x_1} \tag{9a}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}(\frac{d}{dx}) = \frac{\partial^2}{\partial x_0^2} + 2\epsilon \frac{\partial}{\partial x_0 \partial x_1} + \epsilon^2 \frac{\partial^2}{\partial x_1^2} \tag{9b}$$

$$\frac{d^3}{dx^3} = \frac{\partial^3}{\partial x_0^3} + 3\epsilon \frac{\partial^3}{\partial x_0^2 \partial x_1} + 3\epsilon^2 \frac{\partial^3}{\partial x_1^2 \partial x_0} + \epsilon^3 \frac{\partial^3}{\partial x_1^3} \tag{9c}$$

$$\frac{d^4}{dx^4} = \frac{\partial^4}{\partial x_0^4} + 4\epsilon \frac{\partial^4}{\partial x_0^3 \partial x_1} + 6\epsilon^2 \frac{\partial^4}{\partial x_0^2 \partial x_1^2} + 4\epsilon^3 \frac{\partial^4}{\partial x_0^1 \partial x_0} + \epsilon^4 \frac{\partial^4}{\partial x_1^4} \tag{9d}$$

Trivial solution: Weakly nonlinear perturbation theory

$x_0 = x$, $x_1 = X$

$$O(\epsilon) : (\partial_x^2 + q_c^2)u_1 = 0 \tag{10a}$$

$$O(\epsilon^2) : -(\partial_x^2 + q_c^2)u_2 = 4\frac{\partial^4}{\partial x_0^3 \partial x_1} u_1 + 4q_c^2 \frac{\partial}{\partial x_0 \partial x_1} u_1 - v u_1^2 \tag{10b}$$

We can solve the leading order equation: $\Rightarrow$

$$u_1(x, X) = Z_1(X) \exp(iq_c x) + h.o.t. \tag{11}$$

Trivial solution: Weakly nonlinear perturbation theory

Where the slowly varying envelope amplitude $Z(X,\epsilon) = Z_1(X) + \epsilon Z_2(X)$ satisfies:

$$4q_c^2 Z_{1XX} = \mu_2 Z_1 - \gamma_3 Z_1 |Z_1|^2 \tag{12}$$

at the leading order (comes from Fredholm's alternative, see [1], [3] for details). Here $\gamma_3 = \frac{38v^2}{9q_c^4} - 3g$. Substituting values for our parameters $\gamma_3 \approx 8.35 > 0 \Rightarrow$ subcritical bifurcation at origin!

Trivial solution: Weakly nonlinear perturbation theory

The simplest trivial solution of Eqn.(12) is

$$Z(X) = (\mu_2/\gamma_3)^{1/2} e^{i\phi} + O(\epsilon) \tag{13}$$

Spatially periodic solutions of period L_c near origin.

$$u_P(x) = 2\left(\frac{-r}{\gamma_3}\right)^{1/2} \cos\left(q_c x + \phi\right) + O(r) \tag{14}$$

where ϕ is an arbitrary phase and $-r > 0 \quad \because \mu_2 > 0$.

Trivial solution: Weakly nonlinear perturbation theory

Localized solutions - elliptic equations, see [1].

$$Z(X) = \left(\frac{2\mu_2}{\gamma_3}\right)^{1/2} \operatorname{sech}\left(\frac{X\sqrt{\mu_2}}{2q_c}\right) e^{i\phi} + O(\epsilon) \tag{15}$$

and the localized solution corresponds to

$$u_l(x) = 2\left(\frac{-2r}{\gamma_3}\right)^{1/2} \operatorname{sech}\left(\frac{X\sqrt{\mu_2}}{2q_c}\right) \cos\left(q_c x + \phi\right) + O(r) \tag{16}$$

Phases $\phi = 0, \pi$ ‘selected’ for complicated reasons!

Trivial solution: Weakly nonlinear perturbation theory

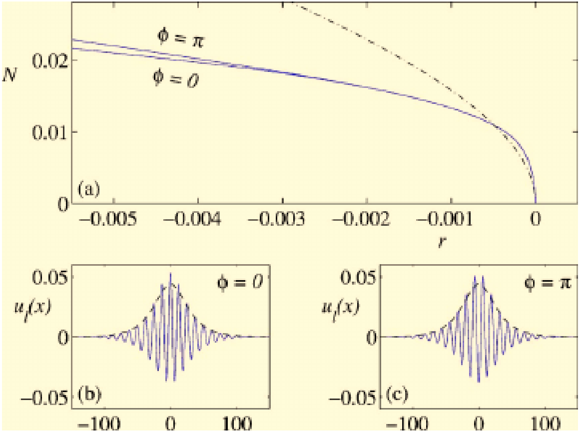


Figure: Phase selection of localized solutions, figure reproduced from[1].

Equilibria: Numerical Continuation

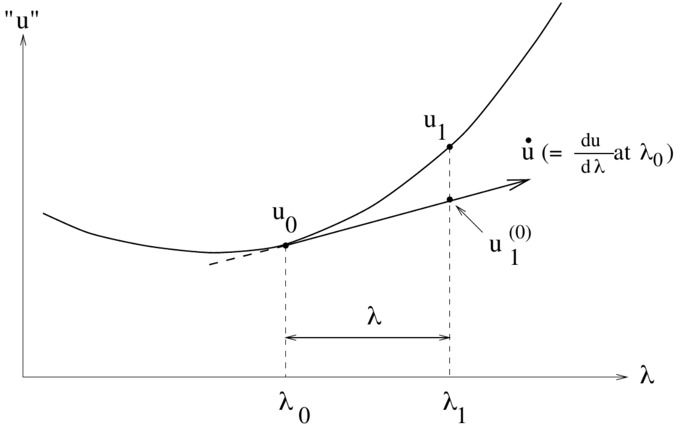


Figure: Parametric Continuation, see [2].

Equilibria: Numerical Continuation

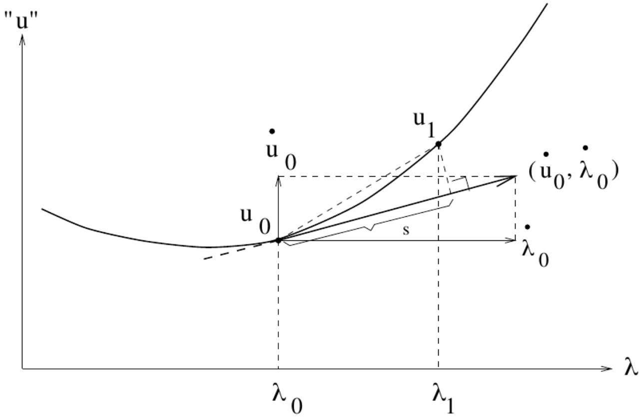


Figure: Pseudo arc-length continuation, see [2]. The dots are now wrt the arc-length s.

Equilibria: Numerical Continuation

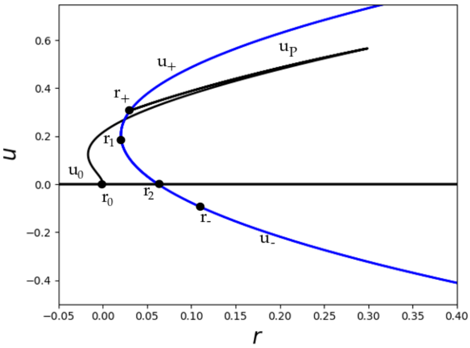


Figure: A new branch emanating from r_0, obtained via numerical continuation

Equilibria: Numerical Continuation

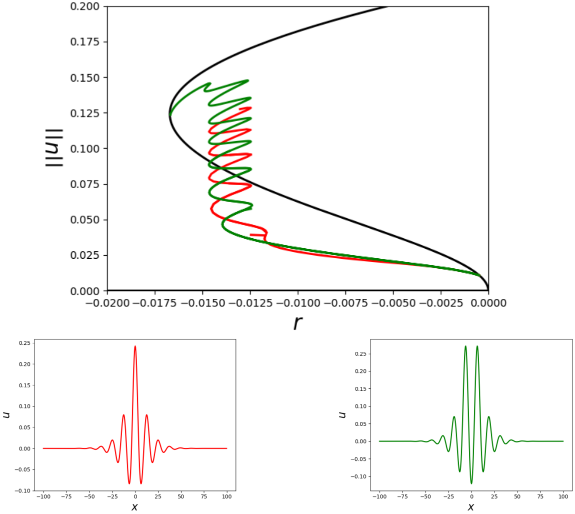


Figure: Snaking, level:noob.

Equilibria: Numerical Continuation

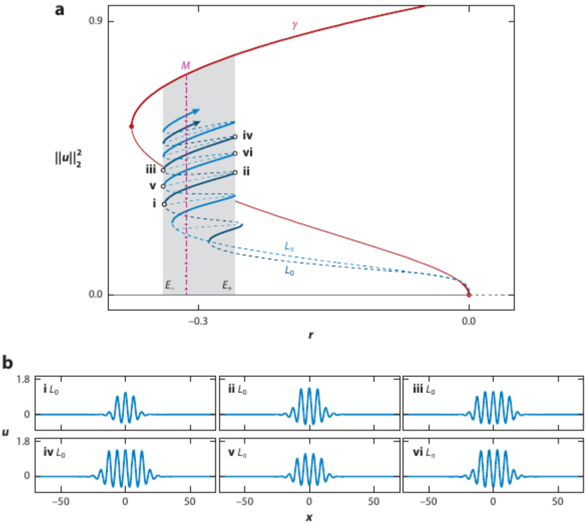


Figure: Snakes and ladders, obtained from [3], level:pro.

Geometric/Dynamical Systems Explaination:

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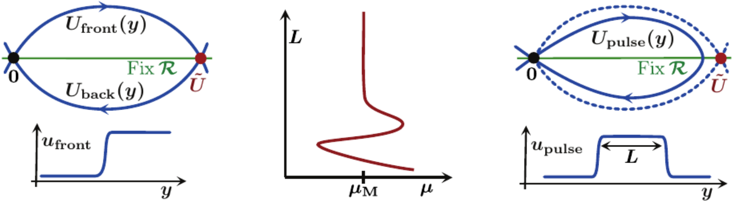
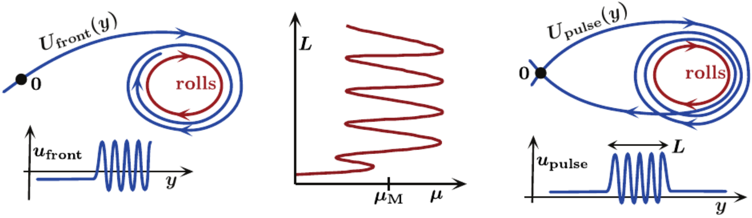


(a) Non-snaking scenario.



(b) Snaking scenario.

Figure: Non-snaking vs snaking scenario, see [4].

Take-home messages:

- Localized patterns are ubiquitous in nature.
- Linear stability analysis comes in handy to produce the skeleton of the bifurcation diagram and provides insights.
- Weakly nonlinear analysis can tell the nature of bifurcations near equilibria.
- The dynamics can be understood as a (spatial) homoclinic orbit of the trivial solution that visits the neighborhood of the periodic pattern (a spatial periodic orbit).

References

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