Discontinuous Galerkin Methods

Study and Application to PDEs

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Outline

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Motivation

- Variants of DGM used to effectively solve diffusion (e.g. the heat eqaution) and pure convection (e.g. in a convection transport equation) problems
- Heat eqn: $\frac{\partial u}{\partial t} - \alpha \nabla^2 u$
- Convection Transport Eqn: $\frac{\partial u}{\partial t} + \nabla \cdot (\vec{v} u)$

Outline

1. Introduction
   - Motivation
   - How is it different from Galerkin FEM
2. Notations and Preliminaries
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3. The Hello World of PDEs !- Poissons equation
   - DGM formulation

Difference between Galerkin and DGM

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Difference between Galerkin and DGM

- Element-wise conservative
- Can support high order local approximation which varies over the mesh
- Leads to block diagonal mass matrices, even for high order polynomial approximation in time dependent problem

Outline

Notations used

- Domain is Ω which is a bounded, open set in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$
- Γ_D on $\partial\Omega$ is the Dirichlet condition prescribed boundary
- Γ_N on $\partial\Omega$ is the Neumann condition prescribed boundary
- $\Gamma_N \cup \Gamma_D = \partial\Omega$ and $\Gamma_N \cap \Gamma_D = \phi$
- P_h is a partition of domain Ω, it numbers N_e partitions in this domain
- $\Omega = \bigcup_{K_i \in P_h} K_i,\ K_i \cap K_j = \phi,\ i \neq j$
- Set of edges E_h =set of$\gamma_l, l = 1, 2 ... N_\gamma$
- $E_h = E_{h,D} \cup E_{h,N} \cup E_{h,int}$

Outline

DGM Formulation - Poisson's Equation

Poisson's Equation

$-\Delta u + cf = u$ in Ω

Boundary conditions

- $u = u_o$ on Γ_D
- $\vec{n} \cdot \nabla u = g$ on Γ_N

Weak Formulation

- Multiply PDE by test function v and integrate over Ω
- $\int_\Omega (\nabla \cdot \nabla u + cu)vdx = \int_\Omega fvdx$
- Decomposing the above integrals into element contributions (unlike classical FEM approach) and integrating by parts :

$$\sum_{K \in P_h} \int_K (\nabla u \cdot \nabla v + cuv)dx - \sum_{K \in P_h} \int_{\partial K} ((\vec{n}) \cdot \nabla u)vds = \sum_{K \in P_h} \int_K fvdx$$

- Boundary integral is defined on each boundary element as follows:

$$\sum_{K \in P_h} \int_{\partial K} (\vec{n} \cdot \nabla u)vds = \int_{\Gamma_D} (\vec{n} \cdot \nabla u)vds + \int_{\Gamma_N} (\vec{n} \cdot \nabla u)vds +$$

$$\sum_{\gamma_{ij} \in E_{h,int}} \int_{\gamma_{ij}} (\vec{n} \cdot \nabla u)_i v_i + (\vec{n} \cdot \nabla u)_j v_j ds$$

Simplifying

- Using $ac - bd = 1/2(a+b)(c-d) + 1/2(a-b)(c+d)$
- $$\vec{n} \cdot (\nabla u)_i v_i - \vec{n} \cdot (\nabla)_j v_j = \langle \vec{n} \cdot \nabla u \rangle [v] + [vecn \cdot \nabla u] \langle v \rangle$$
- Where:Jump is
$$[v] = v_i - v_j$$
and average is
$$\langle v \rangle = \frac{v_i + v_j}{2}$$

Simplifying (Continued)

- An edge lying on Γ_D has

$$[v] = v = v$$

  and

$$\langle v \rangle = v$$

-

- Allowing us to combine interior and Dirichlet boundary conditions in one term :

$$\int_{\Gamma_{int} \cup \Gamma_D} \langle \vec{n} \cdot \nabla u \rangle [v] + [\vec{n} \cdot \nabla u] \langle v \rangle ds$$

Variational Form

- $$\sum_{K\in P_h}\int_K(\nabla u\cdot\nabla v+cuv)dx-\int_{\Gamma_int\cup\Gamma_D}\langle\vec{n}\cdot\nabla u\rangle ds=\sum_{K\in P_h}\int_K fvdx+\int_{\Gamma_N} gvds$$

Variational Form

- $$\sum_{K \in P_h} \int_K (\nabla u \cdot \nabla v + cuv)dx - \int_{\Gamma_int \cup \Gamma_D} \langle \vec{n} \cdot \nabla u \rangle ds = \sum_{K \in P_h} \int_K fvdx + \int_{\Gamma_N} gvds$$

- Bilnear form :

$$B(u, v) = \sum_{K \in P_h} \int_K (\nabla u \cdot \nabla v + cuv)dx$$

and

$$F(v) = \sum_{K \in P_h} \int_K fvdx + \int_{\Gamma_N} gvds$$

Variational Form

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- Also Bilinear form for Boundaries Γ_D and $\Gamma_i nt$ is :

$$J(u, v) = \int_{\Gamma_D \cup \Gamma_{int}} \langle \vec{n} \cdot \nabla u \rangle [v] ds$$

Variational formulation

- A general discontinuous weak formulation of the Poisson Equation hence reads :

$$B(u, v) - J(u, v) = F(v), \forall v \in H^2(P_h)$$

Introduction of a new linear form

- Observation : $u \in H^1(\Omega) \cap H^2(P_h)$, the jump $[u]$ vanishes on each γ_{ij} :

$$\int_{\gamma_{ij}} v[u]ds = 0, \forall v \in L^2(\gamma_{ij})$$

- Follows that :

$$\int_{\Gamma_{int}} \langle \vec{n} \cdot \nabla v \rangle [u]ds = 0, \forall v \in H^2(P_h)$$

- Dirichlet B.C. applied will give :

$$\int_{\Gamma_D} (\vec{n} \cdot \nabla v)uds = \int_{\Gamma_D} (\vec{n} \cdot \nabla v)u_0 ds, \forall v \in H^2(P_h)$$

Continued

- The new linear form defined as :

$$J_0(v) = \int_{\Gamma_D} (\vec{n} \cdot \nabla v) u_0 ds, \forall v \in H^2(P_h)$$

- We observe $u = u_0$ on Γ_D,

$$J(u, v) = J_0(v), \forall v \in H^2(P_h)$$

IMPORTANT

- We will, hence forth, only discuss the discrete formulation of different methods (and assume that the continuous form is similar, with the exception of the domains where the answer for u is searched in)

Global Element Method

- $B_{-}(u,v) = B(u,v) - J(u,v) - J(v,u)$
- $F_{-}(v) = F(v) - J_0(v)$
- GEM consists of finding $u_h \in V^{hp}$ such that :

$$B_{-}(u,v) = F_{-}(v), \forall v \in V^{hp}$$

Symmetric Interior Penalty Galerkin Method

- Penalty terms added to ensure continuity of solution at the interface of elements
- Let σ be penalty parameter depending on length of edges γ_{ij} and γ and the polynomial degree used in elements i.e $\sigma = \sigma(h, p)$
- Introducing Penalty terms

$$J^{\sigma}(u, v) = \int_{\Gamma_i nt \cup \Gamma_D} \sigma [u][v] ds$$

and

$$J_0^{\sigma}(v) = \int_{\Gamma_D} \sigma u_0 v ds$$

- $B_{-}(u, v)^{\sigma} = B(u, v) - J(u, v) - J(v, u) + J^{\sigma}(u, v)$
- $F_{-}^{\sigma}(v) = F(v) - J_0(v) + J_0(v)^{\sigma}$

Continued

- SIPG Problem is find $u_h \in V^{hp}$ such that :

$$B_{-}(u,v)^{\sigma} = F_{-}^{\sigma}(v), \forall v \in V^{hp}$$

Discontinuous hp Galerkin FEM - DGM

- $B_+(u,v) = B(u,v) - J(u,v) + J(v,u)$
- $F_+(v) = F(v) + J_0(v)$
- DGM consists of finding $u_h \in V^{hp}$ such that :

$$B_+(u,v) = F_+(v), \forall v \in V^{hp}$$

Non-Symmetric Interior Penalty Galerkin Method (NIPG)

- $B_{-}(u,v)^{\sigma} = B(u,v) - J(u,v) + J(v,u) + J^{\sigma}(u,v)$
- $F_{+}^{\sigma}(v) = F(v) + J_0(v) + J_0(v)^{\sigma}$
- NIPG Problem is find $u_h \in V^{hp}$ such that :

$$B_{+}(u,v)^{\sigma} = F_{+}^{\sigma}(v), \forall v \in V^{hp}$$

References

- Bahriawati, C., and C. Carstensen. "Three MATLAB implementations of the lowest-order Raviart-Thomas MFEM with a posteriori error control." Comput. Methods Appl. Math. 5, no. 4 (2005): 333-361.
- Prudhomme, S., F. Pascal, J. T. Oden, and A. Romkes. "Review of a priori error estimation for discontinuous Galerkin methods." (2000).
- Hecht, Frdric, Olivier Pironneau, A. Le Hyaric, and K. Ohtsuka. "FreeFem++ manual." (2005).

Numerical Implementation

Open the report.