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Waves in Fluids HW#2

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad \text{—— (1)}$$

$\rho(x,t)$ :: # of cars per unit dist (density of cars)
$q(x,t)$ :: # of " passing a given x-positⁿ. per unit time.

Closing the model · constitutive relⁿ → $q = Q(\rho)$. —— (1b)

$q = \rho v$ —— (1c) $\qquad v \equiv$ "flow" velocity (velo of individual cars)

$v = v_m(1 - \rho/\rho_m)$ —— (1d) $\qquad v_m \equiv$ max car velocity
$\rho_m \equiv$ max car density.

$$\frac{\partial q}{\partial x} = \frac{\partial Q(\rho)}{\partial x} = \frac{dQ}{d\rho}\frac{\partial \rho}{\partial x} = c(\rho)\frac{\partial \rho}{\partial x} \quad \ldots \quad c(\rho) = \frac{dQ}{d\rho} \quad \text{—— (1e)}$$

⇒ (1) becomes
$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \quad \text{—— (2)}$$

IC :- $(t=0) \rightarrow \quad \rho(x,0) = f(x) = R\exp\left[-\frac{x^2}{L^2}\right]$ —— (3)

$-\infty < x < \infty$, $L > 0$, $\rho_m \geq R > 0$ → const. parameters

- METHOD OF CHARACTERISTICS:

Seek $\rho = \rho[x(t), t]$ —— (4a)

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial t} \quad \text{—— (4b)}$$

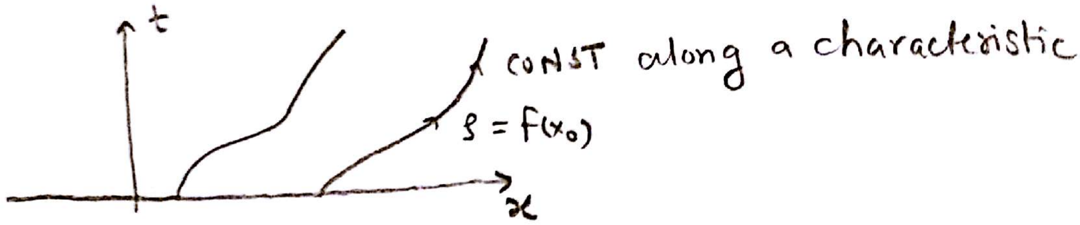
EQᴺ FOR CHARACTERISTICS.

Comparing (4b) with (2), If we choose $\boxed{\frac{dx}{dt} = c(\rho)}$ —— (4c)

$\boxed{\frac{d\rho}{dt} = 0}$ ON curves that satisfy (4c)

$\because \quad \frac{d\rho}{dt} = 0 \quad \Rightarrow \quad \boxed{\rho = \text{const. along characteristics}}$

$\rho = f(x_0) \equiv$ constant along characteristics.



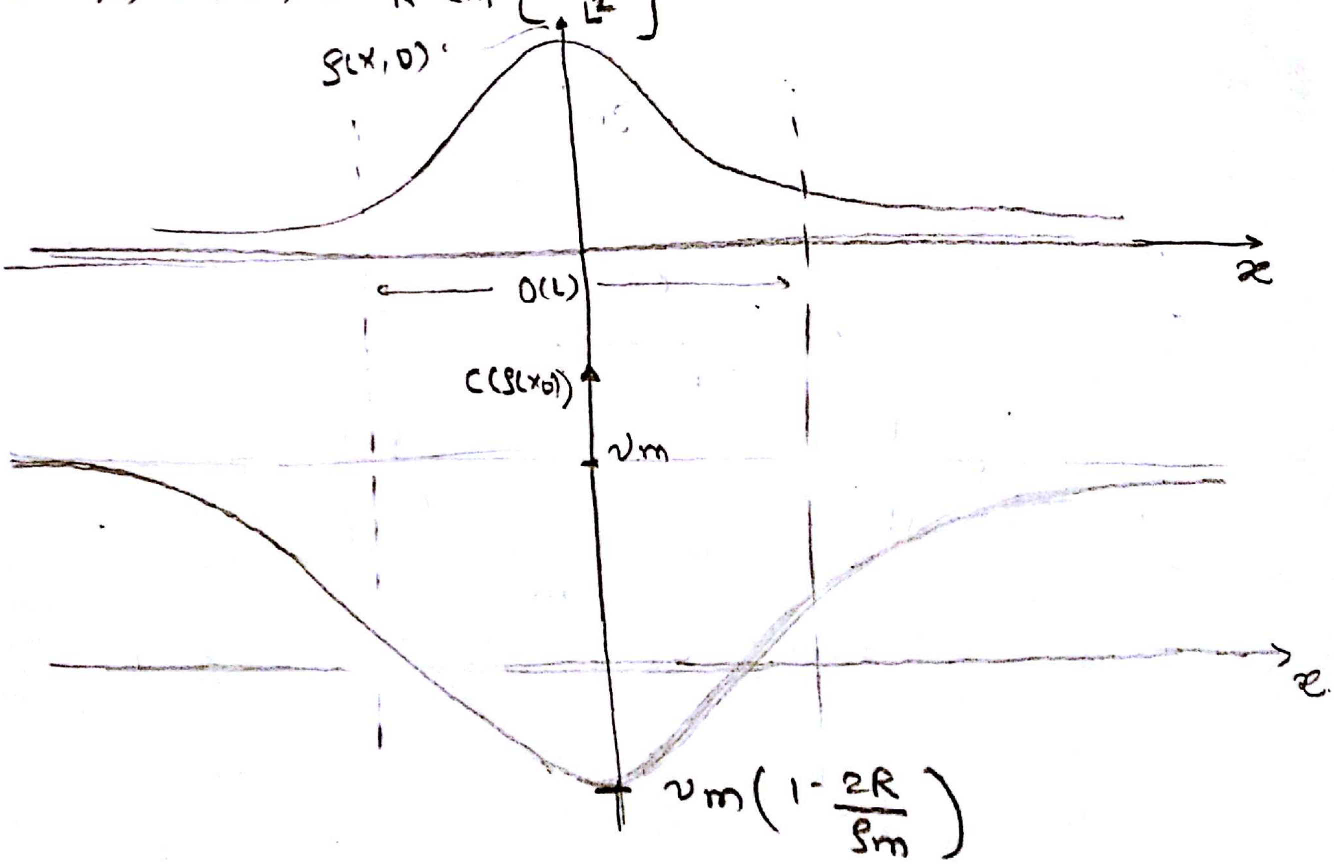
$\Rightarrow \quad c(\rho) \equiv c(f(x_0)) =$ const. along characteristics.

$\Rightarrow \quad \boxed{\frac{dx}{dt} = c(f(x_0))} \Rightarrow \boxed{x = x_0 + c(f(x_0))\,t}$

CHAR. ARE STRAIGHT LINES.

$\boxed{t\text{-}x \text{ plane slope} \equiv \frac{1}{c(f(x_0))}}$

$\rho(x,0) = f(x) = R \exp\left[-\frac{x^2}{L^2}\right]$



$$q = Q(\rho) = v_m\left[\rho - \frac{\rho^2}{\rho_m}\right]$$
(ρv)

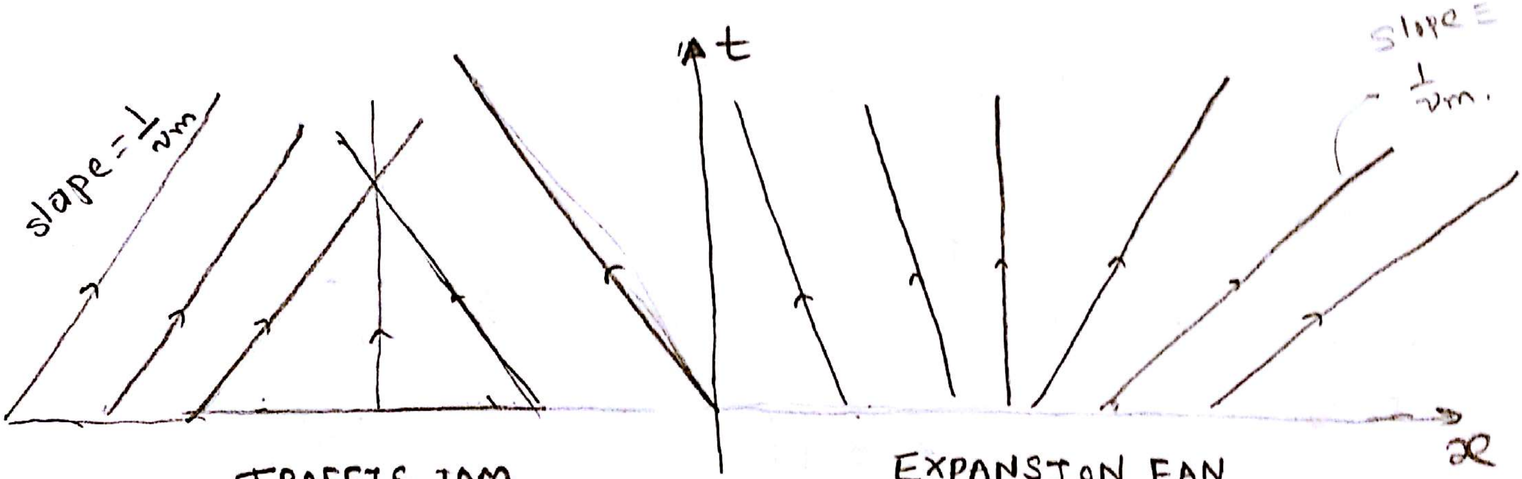
$\dashv$ $$c(\rho) = \frac{dQ}{d\rho} = v_m\left[1 - \frac{2\rho}{\rho_m}\right]$$

$\dashv$ $$c(\rho(x_0)) = v_m\left[1 - \frac{2\rho(x_0,0)}{\rho_m}\right]$$

$$= v_m\left[1 - \frac{2R}{\rho_m}\exp\left(-\frac{x^2}{L^2}\right)\right]$$

@ $x = 0 \rightarrow$ $c = v_m\left[1 - \frac{2R}{\rho_m}\right]$ ... assume $\frac{2R}{\rho_m} > 1$.

@ $x \rightarrow \pm\infty$ $c = v_m$



TRAFFIC JAM EXPANSION FAN

$\rightarrow$ TIME OF INCIPIENT BREAKING (τ_B)

$\tau_B \equiv$ earliest time @ which $\left|\frac{\partial\rho}{\partial x}\right| \rightarrow \infty$

$\rightarrow$ $$x_0 = x - c(\rho)t$$

$$\frac{\partial\rho}{\partial x} = \frac{\partial x_0}{\partial x} f'(x_0) \quad \ldots \rho(x,t) = f(x - c(\rho)t)$$

$$= \left[1 - \frac{dc}{d\rho}\cdot\frac{\partial\rho}{\partial x}t\right]\cdot f'(x_0) \quad \ldots x_0 = x - c(\rho)t$$

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$$\rightarrow \quad \frac{\partial \rho}{\partial x} = \frac{f'(x_0)}{1 + f'(x_0)\,\frac{dc}{d\rho}\,t}$$

$$\left|\frac{\partial \rho}{\partial x}\right| \to \infty \quad \text{as} \quad t \to t_B$$

$$\rightarrow \quad 1 + f'(x_0)\,\frac{dc}{d\rho}\,t_B = 0$$

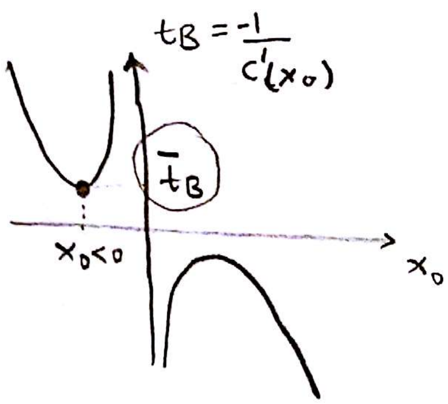
$$\rightarrow \quad \boxed{t_B = \frac{-1}{f'(x_0)\left(\frac{dc}{d\rho}\right)} = \frac{-1}{c'(x_0)} \quad \dots \quad \rho \equiv f.}$$

$\min(t_B) = \bar{t}_B \equiv$ time of incipient breaking.

$$c(x_0) = v_m\left[1 - \frac{2R}{\rho_m}\exp\left(-\frac{x_0^2}{L^2}\right)\right]$$

$$c'(x_0) = v_m\left(-\frac{2R}{\rho_m}\right)\cdot\left(-\frac{2x_0}{L^2}\right)\exp\left(-\frac{x_0^2}{L^2}\right)$$

$$c'(x_0) = \frac{4v_m R}{\rho_m L^2}\,x_0\exp\left(-\frac{x_0^2}{L^2}\right)$$



$$\rightarrow t_B = -\frac{1}{c'(x_0)} = \# \frac{1}{x_0}\exp\left(\frac{x_0^2}{L^2}\right) \rightarrow$$

$\rightarrow \quad \bar{t}_B = \min\{t_B\} \quad , x_0 < 0.$

$$\rightarrow \quad \frac{dt_B}{dx_0} = 0 \quad \rightarrow \quad -\frac{1}{x_0^2}\exp\left(\frac{x_0^2}{L^2}\right) + \frac{1}{x_0}\cdot\frac{2x_0}{L^2}\exp\left(\frac{x_0^2}{L^2}\right) = 0$$

$$\Rightarrow \quad +\frac{1}{x_0} = \frac{2x_0}{L^2} \quad \Rightarrow \quad x_0^2 = \frac{L^2}{2} \quad \rightarrow \quad x_0 = \pm\frac{L}{\sqrt{2}}$$

$$\because x_0 < 0 \ @\ \bar{t}_B \Rightarrow \boxed{x_0 = -\frac{L}{\sqrt{2}}}$$

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$\bar{t}_B$ occurs at $x_0 = -\frac{L}{\sqrt{2}}$

$$c'\left(x_0 = -\frac{L}{\sqrt{2}}\right) = v_m\left(\frac{-2R}{\rho_m}\right)\cdot\left(-\frac{2L}{\sqrt{2}\cdot L^2}\right)\exp\left(-\frac{\cancel{L^2}}{2\cancel{L^2}}\right)$$

$$\Rightarrow c'\left(x_0 = -\frac{L}{\sqrt{2}}\right) = 2\sqrt{2}\,\frac{R v_m}{\rho_m L}\exp\left(\frac{-1}{2}\right)$$

$$\Rightarrow \bar{t}_B = \frac{1}{c'\left(x_0 = -\frac{L}{\sqrt{2}}\right)}$$

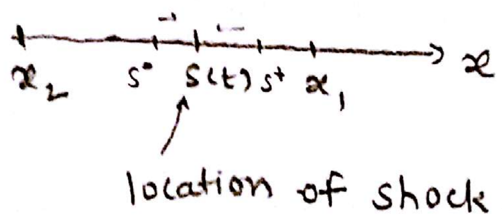
$$\Rightarrow \bar{t}_B = \left(\frac{1}{2\sqrt{2}}\right)\left(\frac{L}{R}\right)\left(\frac{\rho_m}{v_m}\right)\exp(1/2)$$

$$\Rightarrow \boxed{\bar{t}_B = \left(\frac{\sqrt{e}}{2\sqrt{2}}\right)\left(\frac{L}{R}\right)\left(\frac{\rho_m}{v_m}\right)}$$

- Extending the solⁿ using jump condition to times $t > \bar{t}_B$

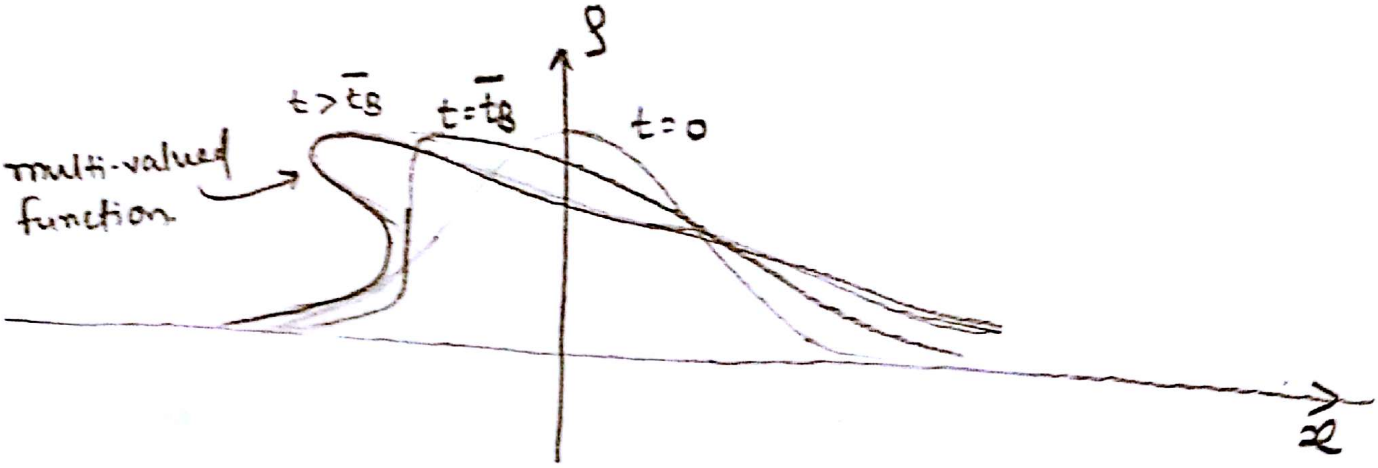
As derived in class, shock propagation velocity

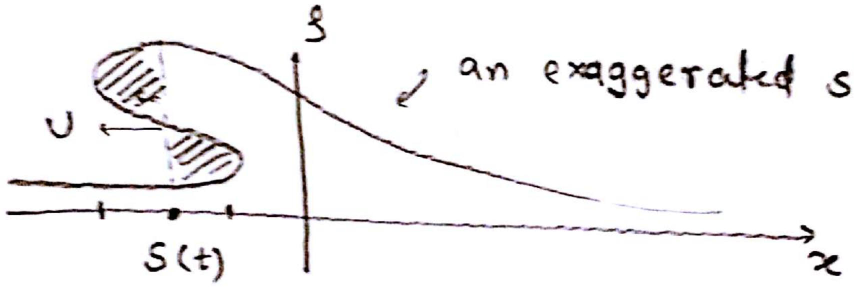
$$U = \frac{[q]}{[\rho]} \quad \text{where } [q] = q(s^+, t) - q(s^-, t)$$
$$[\rho] = \rho(s^+, t) - \rho(s^-, t)$$



I'll start by trying to draw the evolution of the given Gaussian I.C.

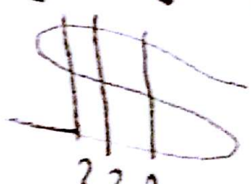
We know that the right half of the lobe does NOT form a shock (expansion fan). The shock formation happens @ $x < 0$ @ $t = \bar{t}_B$.





an exaggerated shock profile.

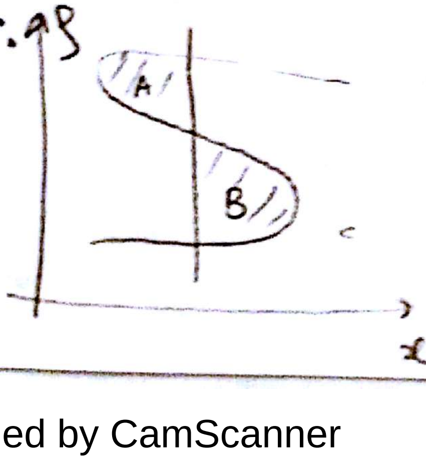
- We want to fit the multi-valued part by a jump.
- Question is - where to put the straight line?



- The answer lies in a simple argument.

We want the straight-line jump to also satisfy the conservation (we know that the multivalued part does, by construction.) → $\boxed{\text{area under the st line} = 0}$

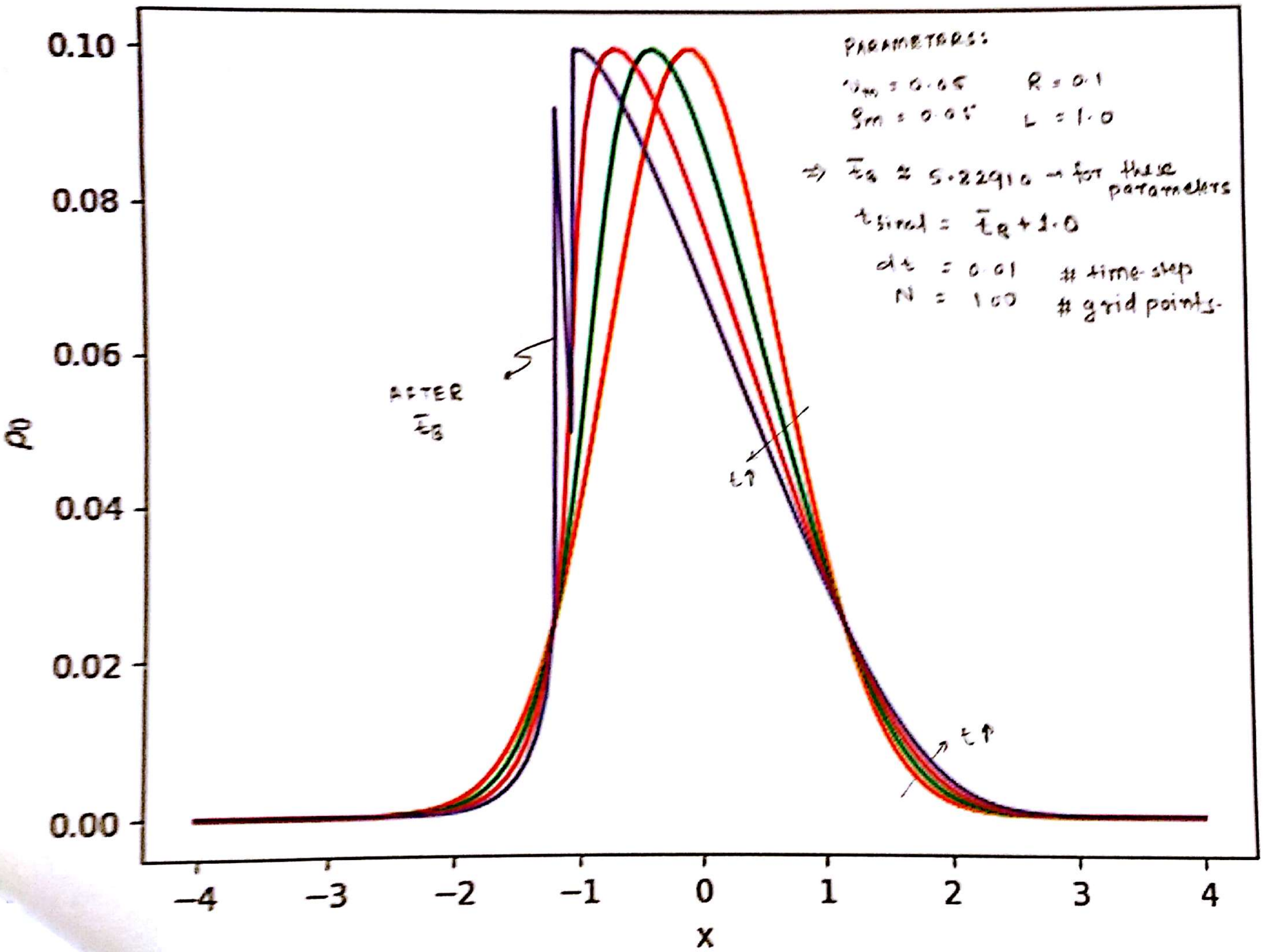
∴



$$\therefore \boxed{\int_A \rho \, dx = \int_B \rho \, dx}$$

∴ we construct the jump @ such an x-value where the areas of A & B are "EQUAL".

Equal area construction!


PARAMETERS:
$v_m = 0.05$ $R = 0.1$
$\rho_m = 0.05$ $L = 1.0$
$\Rightarrow \bar{t}_B \approx 5.22910$ → for these parameters
$t_{final} = \bar{t}_B + 1.0$
$dt = 0.01$ # time-step
$N = 100$ # grid points.
AFTER $\bar{t}_B$
t↑
t↑
0.10
0.08
0.06
0.04
0.02
0.00
ρ_0
−4
−3
−2
−1
0
1
2
3
4
x

WAVES IN FLUIDS – HW 2: TRAFFIC FLOW

Consider as a simple model of one-way, one-lane traffic flow (without off- or on-ramps) the following first-order conservation law:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{1}$$

where $\rho(x,t)$ is the number of cars per unit mile (i.e. the density of cars) and $q(x,t)$ is the number of cars crossing a given x position (a given position along the highway) per unit time (i.e. the flux of cars). The model is closed by specifying an algebraic relation between the flux q and the density ρ: $q = Q(\rho)$. Noting that $q \equiv \rho v$, where v is the "flow" velocity (i.e. the velocity of individual cars), assume that $v = v_m(1 - \rho/\rho_m)$, where v_m is the maximum car velocity and ρ_m is the maximum car density. With $Q(\rho)$ now specified, equation (1) becomes

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{2}$$

where the *wave* (not car) speed $c(\rho) \equiv dQ/d\rho$. At some initial instant ($t = 0$), imagine that the distribution of cars may be approximated by a Gaussian, i.e.

$$\rho(x,0) = f(x) = R\exp\left(-\frac{x^2}{L^2}\right) \tag{3}$$

for $-\infty < x < \infty$, where $L > 0$ and $\rho_m \geq R > 0$ are constant parameters.

1. Does a traffic jam (i.e. wave breaking) occur? If so, determine the earliest time $\bar{t}_B$ and the x-location x_B at which braking occurs as a function of the model and initial-condition parameters.

2. Using the method of characteristics, write a short computer (e.g. Matlab) program to determine the solution for $\rho(x,t)$ at any given (input) time t such that $0 < t < \bar{t}_B$. Make plots showing the evolution of ρ versus x for various increasing times t. (You will need to pick some reasonable values for the various parameters.)

3. **EXTRA CREDIT** Extend your solution using an appropriate jump condition to times $t > \bar{t}_B$.

2. ON THE CHARACTERISTICS → $\frac{dx}{dt} = c(\rho)$ ∵ ρ = const. on charact.

$$\frac{dx}{dt} = c(x_0)$$

$x(t) = c(x_0)\,t + x_0$ → think of this as an eqn for x_0.

⇒ $\boxed{x_0 \equiv x_0(x;t)}$ → I give you x, t you tell me where it came from (i.e., x_0)

Use Newton-Raphson algorithm to find x_0.

INPUT t = some time

x	x0
⋮	⋮

→ plot this.

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Assign the value ~~@~~ of $\rho(x_0)$ to $\rho(x,t)$ → as the "ρ" is const. on that characteristic!

∴ we've found $\rho(x,t)$ without any numerical time-stepping!