HW #2: Weakly Nonlinear Analysis of Porous Medium Convection

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1 Governing Equations, Problem Formulation:

Use Darcy's law as the momentum equation.

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\boldsymbol{u} = \nabla P - RaT\hat{e}_z, \tag{2}$$

$$\partial_t T + \boldsymbol{u}.\nabla T = \nabla^2 T. \tag{3}$$

Since $2D$, incompressible, use streamfunction ψ such that (s.t.) $u = \partial_z \psi, w = -\partial_x \psi$. The governing equations then become (derived in class):

$$\begin{aligned} \nabla^2 \psi &= -Ra\partial_x T, \\ \partial_t T + \partial_z \psi \partial_x T - \partial_x \psi \partial_z T &= \nabla^2 T. \end{aligned} \tag{4}$$

Summary of linear stability analysis done in class and exercise 6.13 in [Drazin, 2002]:

- Basic state $\psi_B = 0, T_B(z) = 1 - z$.
- Linear stability $T(x, z, t) = (1 - z) + \theta(x, z, t), \psi(x, z, t) = 0 + \psi(x, z, t)$.
- $$\begin{aligned} \partial_t \theta + \partial_z \psi \partial_x \theta - \partial_x \psi \partial_z \theta &= -\partial_x \psi + \nabla^2 \theta, \\ \nabla^2 \psi &= -Ra\partial_x \theta, \\ \theta = \psi = 0 \text{ at } z = 0, 1 &\text{ and periodic in } x \text{ with } L_x. \end{aligned} \tag{5}$$
- Linearized above equations, used normal mode ansatz:

$$\begin{bmatrix} \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \hat{\theta}(z) \\ \hat{\psi}(z) \end{bmatrix} e^{\sigma t} e^{ikx} + c.c. \tag{6}$$

- Obtained $\sigma = \frac{k^2 Ra}{k^2 + n^2\pi^2} - (k^2 + n^2\pi^2)$.
- Deduced $Ra_c = 4\pi^2$ and $k_c = \pi, n = 1$.

2 Weakly Nonlinear Analysis:

- Assume near-criticality: $(Ra - Ra_c/Ra_c \ll 1)$.
- In the following analysis $n = 1, k_c = \pi$ and $Ra_c = 4\pi^2$.
- Inspired by linear stability, introduce slow time $T = \epsilon^2 t$ and $X = \epsilon x$, where $\epsilon = \left((Ra - Ra_c)/\tilde{R}\right)^{1/2}$.
- Here $\tilde{R} \sim O(1)$ as $\epsilon \to 0$. We define $p = Ra_c \tilde{R}$.
- Write fast time and fast space variables as $\tau = t, \chi = x$.
- Immediately ignore variations with fast time, because we are interested in dynamics on long times T.
- The value of $\epsilon \sim (Ra - Ra_c)^{1/2}$ is deduced from dominant balance arguments (to be explained).

Chain rule immediately implies:

$$\begin{aligned}
\partial_x &= \partial_\chi + \epsilon \partial_X, \\
\partial_x^2 &= \partial_\chi^2 + 2\epsilon \partial_\chi \partial_X + \epsilon^2 \partial_X^2, \\
\partial_t &= \cancelto{0}{\partial_\tau} + \epsilon \partial_T.
\end{aligned} \tag{7}$$

Writing full nonlinear equations Eqns.(5) in terms of slow time (T) and slow space (X) variables, we obtain:

$$\begin{aligned}
&\frac{\partial^2 \theta}{\partial \chi^2} + 2\epsilon \frac{\partial^2 \theta}{\partial \chi \partial X} + \epsilon^2 \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial z^2} - \frac{\partial \psi}{\partial \chi} - \epsilon \frac{\partial \psi}{\partial X} \\
&\qquad = \epsilon^2 \frac{\partial \theta}{\partial T} + \frac{\partial \psi}{\partial z}\left(\frac{\partial \theta}{\partial \chi} + \epsilon \frac{\partial \theta}{\partial X}\right) - \left(\frac{\partial \psi}{\partial \chi} + \epsilon \frac{\partial \psi}{\partial X}\right)\frac{\partial \theta}{\partial z}, \\
&\frac{\partial^2 \psi}{\partial \chi^2} + 2\epsilon \frac{\partial^2 \psi}{\partial \chi \partial X} + \epsilon^2 \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial z^2} + Ra_c \frac{\partial \theta}{\partial \chi} + Ra_c \epsilon \frac{\partial \theta}{\partial X} \\
&\qquad = -\epsilon^2 p \left(\frac{\partial \theta}{\partial \chi} + \epsilon \frac{\partial \theta}{\partial X}\right).
\end{aligned} \tag{8}$$

where p is a parameter that we introduced earlier to unfold bifurcations, if needed. We now perform the standard multiple scales analysis, by substituting

$$\begin{aligned}
\theta &= \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \dots, \\
\psi &= \epsilon \psi_1 + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 + \dots,
\end{aligned} \tag{9}$$

and collect terms at different orders of ϵ:

2.1 $O(\epsilon)$:

$$\begin{aligned}\nabla^2\theta_1 - \frac{\partial\psi_1}{\partial\chi} &= 0,\\ Ra_c\frac{\partial\theta_1}{\partial\chi} + \nabla^2\psi_1 &= 0.\end{aligned} \tag{10}$$

Or, writing it in matrix form,

$$\underbrace{\begin{bmatrix}\nabla^2 & -\partial_\chi\\ Ra_c\partial_\chi & \nabla^2\end{bmatrix}}_{\mathcal{L}}\underbrace{\begin{bmatrix}\theta_1\\ \psi_1\end{bmatrix}}_{w_1} = \begin{bmatrix}0\\ 0\end{bmatrix}, \tag{11}$$

and $\theta_1 = \psi_1 = 0$ at $z = 0, 1$. Since we have assumed near criticality, we look for solutions of the form

$$\begin{bmatrix}\theta_1\\ \psi_1\end{bmatrix} = \begin{bmatrix}\hat{\theta}_1(z)\\ \hat{\psi}_1(z)\end{bmatrix}e^{i\pi x} + \text{c.c.}, \tag{12}$$

where c.c. stands for complex conjugate. Substituting into Eqn.(11), we get

$$\begin{aligned}&\frac{d^2\hat{\theta}_1}{dz^2} - k_c^2\hat{\theta}_1 - ik_c\hat{\theta}_1 = 0\\ &\frac{d^2\hat{\psi}_1}{dz^2} - k_c^2\hat{\psi}_1 + iRa_ck_c\hat{\theta}_1 = 0\end{aligned} \tag{13}$$

Furthermore, due to homogeneous Dirichlet B.C.s, we can guess the form $\hat{\theta}_1(z), \hat{\theta}_1(z) \sim \sin n\pi z$, but again, we use $n = 1$ due to near-criticality assumption. Showing this dependence on sine modes is straightforward. For example, assume $\hat{F}(z) = Fe^{i\pi z} + F^*e^{-i\pi z}$.

$$\begin{aligned}\hat{F}(z) &= Fe^{i\pi z} + F^*e^{-i\pi z}\\ &= (f + ig)(\cos\pi z + i\sin\pi z) + (f - ig)(\cos\pi z - i\sin\pi z)\\ &= 2f\cos\pi z - 2g\sin\pi z\end{aligned} \tag{14}$$

Using BCs, $\hat{F}(z = 0) = 0 \Rightarrow f = 0$ and $\hat{F}(z = 1) = 0$ is trivially satisfied. Therefore, we get $\hat{F}(z) = \beta\sin\pi z$, with $\beta = 2g$. Therefore, we assume $\hat{\theta}_1 = \alpha\sin\pi z$ and $\hat{\psi}_1 = \beta\sin\pi z$. Therefore, our ansatz now takes the form

$$\begin{bmatrix}\theta_1\\ \psi_1\end{bmatrix} = A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\sin(\pi z)e^{i\pi\chi} + \text{c.c.}, \tag{15}$$

Substituting Eqn.(15) into Eqn.(13), get:

$$\begin{aligned}&-\pi^2\alpha - \pi^2\alpha - i\pi\beta = 0\\ &-\pi^2\beta - \pi^2\beta + 4i\pi^3\alpha = 0.\end{aligned} \tag{16}$$

Both equations give the same relationship between α and β, i.e., $\beta = 2\pi i\alpha$. We choose $\alpha = 1$ so $\beta = 2\pi i$ (normalization). Multiplying by slow spatially and temporally varying amplitude $A(X,T)$, get

$$\begin{bmatrix} \theta_1 \\ \psi_1 \end{bmatrix} = A(X,T) \begin{bmatrix} 1 \\ 2\pi i \end{bmatrix} \sin(\pi z) e^{i\pi\chi} + \text{c.c.}, \tag{17}$$

i.e.

$$\boxed{\theta_1 = \sin(\pi z)\left[A e^{i\pi\chi} + A^* e^{-i\pi\chi}\right],} \tag{18}$$

$$\boxed{\psi_1 = 2\pi \sin(\pi z)\left[iA e^{i\pi\chi} - iA^* e^{-i\pi\chi}\right].} \tag{19}$$

2.2 $O(\epsilon^2)$:

$$\begin{aligned}
&\nabla^2\theta_2 - \frac{\partial\psi_2}{\partial\chi} \\
&= -2\frac{\partial^2\theta_1}{\partial\chi\partial X} + \frac{\partial\psi_1}{\partial X} + \frac{\partial\psi_1}{\partial z}\frac{\partial\theta_1}{\partial\chi} - \frac{\partial\psi_1}{\partial\chi}\frac{\partial\theta_1}{\partial z}, \\
&Ra_c\frac{\partial\theta_2}{\partial\chi} + \nabla^2\psi_2 \\
&= -2\frac{\partial^2\psi_1}{\partial\chi\partial X} - Ra_c\frac{\partial\theta_1}{\partial X}.
\end{aligned} \tag{20}$$

Or, writing it in matrix form,

$$\underbrace{\begin{bmatrix} \nabla^2 & -\partial_\chi \\ Ra_c\partial_\chi & \nabla^2 \end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} \theta_2 \\ \psi_2 \end{bmatrix}}_{w_2} = \underbrace{\begin{bmatrix} -2\frac{\partial^2\theta_1}{\partial\chi\partial X} + \frac{\partial\psi_1}{\partial X} + \frac{\partial\psi_1}{\partial z}\frac{\partial\theta_1}{\partial\chi} - \frac{\partial\psi_1}{\partial\chi}\frac{\partial\theta_1}{\partial z} \\ -2\frac{\partial^2\psi_1}{\partial\chi\partial X} - Ra_c\frac{\partial\theta_1}{\partial X} \end{bmatrix}}_{f_2}, \tag{21}$$

and $\theta_2 = \psi_2 = 0$ at $z = 0, 1$. Let us start by evaluating f_2 on the RHS.

$$\begin{aligned}
&-2\frac{\partial^2\theta_1}{\partial\chi\partial X} + \frac{\partial\psi_1}{\partial X} + \frac{\partial\psi_1}{\partial z}\frac{\partial\theta_1}{\partial\chi} - \frac{\partial\psi_1}{\partial\chi}\frac{\partial\theta_1}{\partial z} \\
&= \cancel{-2\sin(\pi z)\left[\frac{\partial A}{\partial X}(i\pi)e^{i\pi\chi} + \text{c.c.}\right]} + \cancel{2\pi\sin(\pi z)\left[i\frac{\partial A}{\partial X}e^{i\pi\chi} + \text{c.c.}\right]} \\
&+ \left\{2\pi^2\cos(\pi z)\left[iAe^{i\pi\chi} - iA^*e^{-i\pi\chi}\right]\right\}\left\{\sin(\pi z)\left[i\pi Ae^{i\pi\chi} - i\pi A^*e^{-i\pi\chi}\right]\right\} \\
&- \left\{2\pi\sin(\pi z)\left[-\pi Ae^{i\pi\chi} - \pi A^*e^{-i\pi\chi}\right]\right\}\left\{\pi\cos(\pi z)\left[Ae^{i\pi\chi} + A^*e^{-i\pi\chi}\right]\right\} \\
&= 2\pi^3\sin(\pi z)\cos(\pi z)\left\{-\left[Ae^{i\pi\chi} - A^*e^{-i\pi\chi}\right]^2 + \left[Ae^{i\pi\chi} + A^*e^{-i\pi\chi}\right]^2\right\} \\
&= 2\pi^3\sin(\pi z)\cos(\pi z)(4|A|^2) \\
&= 8\pi^3|A|^2\sin(\pi z)\cos(\pi z)
\end{aligned} \tag{22}$$

$$
\begin{aligned}
&-2\frac{\partial^2\psi_1}{\partial\chi\partial X} - Ra_c\frac{\partial\theta_1}{\partial X} \\
&= \cancel{-4\pi\sin(\pi z)\left[-\pi\frac{\partial A}{\partial X}e^{i\pi\chi} + \text{c.c.}\right]} - \cancel{4\pi^2\sin(\pi z)\left[\frac{\partial A}{\partial X}e^{i\pi\chi} + \text{c.c.}\right]} \\
&= 0.
\end{aligned} \tag{23}
$$

Therefore,

$$
f_2 = \begin{bmatrix} 4\pi^3|A|^2\sin(2\pi z) \\ 0 \end{bmatrix}, \tag{24}
$$

i.e.

$$
\underbrace{\begin{bmatrix} \nabla^2 & -\partial_\chi \\ Ra_c\partial_\chi & \nabla^2 \end{bmatrix}}_{\mathcal{L}}\underbrace{\begin{bmatrix} \theta_2 \\ \psi_2 \end{bmatrix}}_{w_2} = \underbrace{\begin{bmatrix} 4\pi^3|A|^2\sin(2\pi z) \\ 0 \end{bmatrix}}_{f_2}, \tag{25}
$$

To check for solvability of Eqn. (25), we must use the Fredholm alternative. To be able to use the Fredholm alternative, we must get our hands on the adjoint of the linear operator $\mathcal{L}$, denoted by $\mathcal{L}^\dagger$. We start by defining the adjoint operator as follows:

$$
< u, \mathcal{L}v > = < \mathcal{L}^\dagger u, v > . \tag{26}
$$

We define the inner product $< a, b > = \int_0^1 \int_0^{L_x} a^* b dx dz$.

Define $u = \begin{bmatrix} \theta^\dagger \\ \psi^\dagger \end{bmatrix}$ and $v = \begin{bmatrix} \theta_2 \\ \psi_2 \end{bmatrix}$. Integrating the first derivative-terms by parts once and Laplacians twice, using periodic BCs in $\chi = x$ direction and Dirichlet BCs in z:

$$
\begin{aligned}
< u, \mathcal{L}v > &= \int_0^1 \int_0^{L_x} \left[(\theta^\dagger)^*\nabla^2\theta_2 - (\theta^\dagger)^*\frac{\partial\psi_2}{\partial\chi}\right] dx dz \\
&+ \int_0^1 \int_0^{L_x} \left[(\psi^\dagger)^* Ra_c\frac{\partial\theta_2}{\partial\chi} + (\psi^\dagger)^*\nabla^2\psi_2\right] dx dz \\
&= \int_0^1 \int_0^{L_x} \left[\nabla^2[(\theta^\dagger)^*]\theta_2 + \psi_2\frac{\partial(\theta^\dagger)^*}{\partial\chi}\right] dx dz \\
&+ \int_0^1 \int_0^{L_x} \left[-Ra_c\theta_2\frac{\partial(\psi^\dagger)^*}{\partial\chi} + \psi_2\nabla^2[(\psi^\dagger)^*]\right] dx dz \\
&= < \mathcal{L}^\dagger u, v > .
\end{aligned} \tag{27}
$$

Reading off the adjoint operator

$$
\boxed{\mathcal{L}^\dagger = \begin{bmatrix} \nabla^2 & -Ra_c\partial_\chi \\ \partial_\chi & \nabla^2 \end{bmatrix}} \tag{28}
$$

Fredholm alternative dictates that if $< f_2, v > = 0 \forall v$ in the null space of $\mathcal{L}^\dagger$, i.e., $\forall v$ that satisfy $\mathcal{L}^\dagger v = \mathbf{0}$, then system $\mathcal{L}w_2 = f_2$ is solvable. It can be shown that the null space of $\mathcal{L}^\dagger$ is spanned by $v = \begin{bmatrix} -\psi_1 \\ \theta_1 \end{bmatrix}$. This can be verified easily by checking $\mathcal{L}^\dagger v = \mathbf{0}$. Therefore, for solvability, we need to check if $< f_2, v >= 0$, i.e.,

$$\begin{aligned} < f_2, v > &= \int_0^1 \int_0^{L_x} [4\pi^3 |A|^2 \sin(2\pi z)](-\psi_1) dx dz \\ &= 4\pi^3 |A|^2 \int_0^1 \int_0^{L_x} \sin(2\pi z) \sin(\pi z) e^{i\pi x} dx dz \\ &= 0, \end{aligned} \tag{29}$$

since $e^{i\pi x}$ is periodic in L_x and $\int_0^1 \sin(2\pi z)\sin(\pi z) dz = 0$. Therefore we verified that $< f_2, v >= 0$ and the system given by Eqn.(25) is solvable. Let us find a particular solution of Eqn.(25) by the method of undetermined coefficients. Guess

$$\begin{bmatrix} \theta_{2p} \\ \psi_{2p} \end{bmatrix} = \begin{bmatrix} \theta_{21} \\ \psi_{21} \end{bmatrix} \sin(2\pi z) + \begin{bmatrix} \theta_{22} \\ \psi_{22} \end{bmatrix} \cos(2\pi z) \tag{30}$$

Substituting into Eqns.(25), obtain:

$$\begin{aligned} &-4\pi^2 \theta_{21} \sin(2\pi z) - -4\pi^2 \theta_{22} \cos(2\pi z) + 0 = 4\pi^3 |A|^2 \sin(2\pi z), \\ &0 + (-4\pi^2)\psi_{21} \sin(2\pi z) - 4\pi^2 \psi_{22} \cos(2\pi z) = 0, \end{aligned} \tag{31}$$

yeilding $\theta_{22} = -\pi |A|^2$ and $\psi_{21} = \theta_{22} = \psi_{22} = 0$. Note that we can always add $c_1 \begin{bmatrix} \theta_1 \\ \psi_1 \end{bmatrix}$ to the particular solution and it will still satisfy Eqn. (25), since $c_1 \begin{bmatrix} \theta_1 \\ \psi_1 \end{bmatrix}$ satisfies the homogeneous part. We set $c_1 = 0$ since it can always be included in the $O(\epsilon)$ solution as a higher order correction. Therefore, we conclude,

$$\begin{aligned} &\boxed{\theta_2 = -\pi |A|^2 \sin(2\pi z)}, \\ &\boxed{\psi_2 = 0}. \end{aligned} \tag{32}$$

We still have not obtained the evolution equation for the amplitude $A(X,T)$. We will see that we must go to the $O(\epsilon^3)$ to obtain the Ginzburg-Landau equation, since the slow time T enters into the picture only at $O(\epsilon^3)$.

2.3 $O(\epsilon^3)$:

$$
\begin{aligned}
&\nabla^2\theta_3 - \frac{\partial \psi_3}{\partial \chi} \\
&= -2\cancelto{0}{\frac{\partial^2 \theta_2}{\partial \chi \partial X}} - \frac{\partial^2 \theta_1}{\partial X^2} + \cancelto{0}{\frac{\partial \psi_2}{\partial X}} + \frac{\partial \theta_1}{\partial T} + \frac{\partial \psi_1}{\partial z}\cancelto{0}{\frac{\partial \theta_2}{\partial \chi}}, \\
&+ \cancelto{0}{\frac{\partial \psi_2}{\partial z}}\frac{\partial \theta_1}{\partial \chi} + \frac{\partial \psi_1}{\partial z}\frac{\partial \theta_1}{\partial X} - \frac{\partial \psi_1}{\partial \chi}\frac{\partial \theta_2}{\partial z} - \cancelto{0}{\frac{\partial \psi_2}{\partial \chi}}\frac{\partial \theta_1}{\partial z} - \frac{\partial \psi_1}{\partial X}\frac{\partial \theta_1}{\partial z} \\
&Ra_c\frac{\partial \theta_3}{\partial \chi} + \nabla^2\psi_3 \\
&= -2\cancelto{0}{\frac{\partial^2 \psi_2}{\partial \chi \partial X}} - \frac{\partial^2 \psi_1}{\partial X^2} - Ra_c\frac{\partial \theta_2}{\partial X} - p\frac{\partial \theta_1}{\partial \chi}.
\end{aligned}
\tag{33}
$$

Also, from our analysis at $O(\epsilon^2)$, we know $\frac{\partial \psi_1}{\partial z}\frac{\partial \theta_1}{\partial X} - \frac{\partial \psi_1}{\partial X}\frac{\partial \theta_1}{\partial z} = 0$, in f_{31}. Writing it in matrix form,

$$
\underbrace{\begin{bmatrix} \nabla^2 & -\partial_\chi \\ Ra_c\partial_\chi & \nabla^2 \end{bmatrix}}_{\mathcal{L}}\underbrace{\begin{bmatrix} \theta_3 \\ \psi_3 \end{bmatrix}}_{w_3} = \underbrace{\begin{bmatrix} -\frac{\partial^2 \theta_1}{\partial X^2} + \frac{\partial \theta_1}{\partial T} - \frac{\partial \psi_1}{\partial \chi}\frac{\partial \theta_2}{\partial z} \\ -\frac{\partial^2 \psi_1}{\partial X^2} - Ra_c\frac{\partial \theta_2}{\partial X} - p\frac{\partial \theta_1}{\partial \chi} \end{bmatrix}}_{f_3}, \tag{34}
$$

and $\theta_3 = \psi_3 = 0$ at $z = 0, 1$. Let us now start by evaluating f_3.

$$
\begin{aligned}
f_{3_1} &= -\frac{\partial^2 \theta_1}{\partial X^2} + \frac{\partial \theta_1}{\partial T} - \frac{\partial \psi_1}{\partial \chi}\frac{\partial \theta_2}{\partial z} \\
&= -\sin(\pi z)\left[A_{XX}e^{i\pi\chi} + A^*_{XX}e^{-i\pi\chi}\right] + \sin(\pi z)\left[A_T e^{i\pi\chi} + A^*_T e^{-i\pi\chi}\right] \\
&\quad - 4\pi^4\sin(\pi z)\cos(2\pi z)|A|^2\left[Ae^{i\pi\chi} + A^*e^{-i\pi\chi}\right] \\
f_{3_2} &= -\frac{\partial^2 \psi_1}{\partial X^2} - Ra_c\frac{\partial \theta_2}{\partial X} - p\frac{\partial \theta_1}{\partial \chi} \\
&= -2\pi\sin(\pi z)\left[iA_{XX}e^{i\pi\chi} - iA^*_{XX}e^{-i\pi\chi}\right] - \cancelto{4\pi^3}{Ra_c\pi}\sin(2\pi z)\left(A_XA^* + A^*A_X\right) \\
&\quad - p\pi\sin(\pi z)\left[iAe^{i\pi\chi} - iA^*e^{-i\pi\chi}\right]
\end{aligned}
\tag{35}
$$

Again, since we have the same $\mathcal{L}$, the null space of $\mathcal{L}^\dagger$ would be spanned by $v = \begin{bmatrix} -\psi_1 \\ \theta_1 \end{bmatrix}$. From Fredholm's alternative, the solvability condition for Eqn.(34) is:

$$
\begin{aligned}
0 &= < v, f_3 > \\
0 &= \int_0^1 \int_0^{L_x} \left\{(-\psi_1)^* f_{3_1} + (\theta_1)^* f_{3_2}\right\} d\chi dz
\end{aligned}
\tag{36}
$$

Remember, ψ_1, θ_1 are actually real $((-\psi_1)^* = (-\psi_1)$ and $(\theta_1)^* = (\theta_1))$. Hence the solvability condition becomes $\int_0^1 \int_0^{L_x} \{(-\psi_1)f_{3_1} + (\theta_1)f_{3_2}\} \, d\chi dz = 0$. Let us begin by evaluating the integrand

$$(-\psi_1)f_{3_1} + (\theta_1)f_{3_2} = (\#_1)e^{i\pi\chi} + (\#_1^*)e^{-i\pi\chi} + (\#_3) + (\#_4)e^{2i\pi\chi} + (\#_4^*)e^{-2i\pi\chi}, \tag{37}$$

where

$$\begin{aligned} \#_3 &= (A_T - A_{XX} - 4\pi^4|A|^2 A\cos(2\pi z))\sin(\pi z)\cdot 2\pi i A^*\sin(\pi z) \\ &\quad + (-2iA_{XX} - ipA)(\pi\sin(\pi z))(A^*\sin(\pi z)) + \text{c. c.}. \end{aligned} \tag{38}$$

All the other terms in the integrand vanish due to periodic BCs in χ upon integration. Therefore, solvability condition reduces to

$$\begin{aligned} 0 &= \int_0^1 \int_0^{L_x} \#_3 d\chi dz \\ &= \int_0^1 \int_0^{L_x} \Bigg(2\pi i A^* A_T - 2\pi i A^* A_{XX} - 8i\pi^5|A|^3\cos(2\pi z) \\ &\quad - 2\pi i A^* A_{XX} - ip|A|^2\Bigg)\sin^2(\pi z) d\chi dz \\ &= \int_0^1 \int_0^{L_x} \Bigg(2\pi i A^* A_T - 4\pi i A^* A_{XX} - 8i\pi^5|A|^3\cos(2\pi z) \\ &\quad - ip\pi|A|^2\Bigg)\sin^2(\pi z) d\chi dz \\ 0 &= \cancel{L_x}\int_0^1 \left(2\pi i A^* A_T - 4\pi i A^* A_{XX} - 8i\pi^5|A|^3\cos(2\pi z) - ip\pi|A|^2\right)\sin^2(\pi z)dz \\ 0 &= 2\pi i A^* A_T - 4\pi i A^* A_{XX} - 8i\pi^5|A|^3\left(\frac{-1}{4}\right) - ip\pi|A|^2 \\ 0 &= A_T - 2A_{XX} + 2\pi^4|A|^2 A - \frac{p}{2}A \end{aligned} \tag{39}$$

Therefore, the Ginzburg-Landau equation for the evolution of the amplitude on slow time and spatial scales is:

$$\boxed{A_T + 2\pi^4|A|^2 A - 2A_{XX} - \frac{p}{2}A = 0}. \tag{40}$$

References

[Drazin, 2002] Drazin, P. G. (2002). *Introduction to hydrodynamic stability*, volume 32. Cambridge university press.