HW #1: Linear and Energy Stability Theory

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1 Thin Liquid Films on a Deformable Substrate:

Governing equations in dimensionless form:

$$\partial_t h^+ + \frac{1}{3}\partial_x \left[(h^+)^3(\partial_x^3 h^+ + \partial_x \kappa)\right] = 0, \tag{1}$$
$$\partial_t h^- + \frac{1}{3}\partial_x \left[(h^-)^3(\partial_x^3 h^- - \partial_x \kappa)\right] = 0, \tag{2}$$
$$B\partial_x^2 \kappa + [A(1-\Lambda) - 2]\kappa = \partial_x^2 h^+ - \partial_x^2 h^-. \tag{3}$$

Here, h^+, h^- are the thicknesses of the 'upper' and 'lower' liquid films, respectively, while κ is the curvature of the substrrate, which in the small curvature limit is related to the transverse deflection of the substrate $\eta(x,t)$ via the relation $\partial_x^2 \eta = \kappa$ (i.e. η is a displacement measured in the direction normal to the 'horizontal' x axis). A, B, and Λ are positive external (dimensionless) control parameters related to the axial stiffness, the flexural (bending) stiffness, and the imposed compression of the axial substrate, respectively. The base state is given by $\eta_B = \kappa_B = 0$ and $h_B^+ = h_B^- = 1$. Perturbing the base state fields and substituting $\phi = \phi_B + \phi$, linearizing, we obtain:

$$\partial_t h^+ + \frac{1}{3}\partial_x \left[(\partial_x^3 h^+ + \partial_x \kappa)\right] = 0, \tag{4}$$
$$\partial_t h^- + \frac{1}{3}\partial_x \left[(\partial_x^3 h^- - \partial_x \kappa)\right] = 0, \tag{5}$$
$$B\partial_x^2 \kappa + [A(1-\Lambda) - 2]\kappa = \partial_x^2 h^+ - \partial_x^2 h^-. \tag{6}$$

Substituting the normal mode ansatz $\phi = \phi_0 e^{\sigma t} e^{i\alpha x} + c.c.$, we get:

$$\begin{aligned} \sigma h_0^+ &= -\frac{1}{3}(i\alpha)^4 h_0^+ - \frac{1}{3}(i\alpha)^2 \kappa_0, \\ \sigma h_0^- &= -\frac{1}{3}(i\alpha)^4 h_0^- + \frac{1}{3}(i\alpha)^2 \kappa_0, \\ B(i\alpha)^2\kappa_0 + [A(1-\Lambda) - 2]\kappa_0 &= (i\alpha)^2(h_0^+ - h_0^-). \end{aligned} \tag{7}$$

Simplifying,

$$\begin{aligned} \sigma h_0^+ &= -\frac{1}{3}\alpha^4 h_0^+ + \frac{1}{3}\alpha^2\kappa_0, \\ \sigma h_0^- &= -\frac{1}{3}\alpha^4 h_0^- - \frac{1}{3}\alpha^2\kappa_0, \\ -B\alpha^2\kappa_0 + [A(1-\Lambda)-2]\kappa_0 &= -\alpha^2(h_0^+ - h_0^-). \end{aligned} \tag{8}$$

Subtracting the first two equations,

$$\sigma(h_0^+ - h_0^-) = \frac{1}{3}\{2\alpha^2\kappa_0 - \alpha^4(h_0^+ - h_0^-)\} \tag{9}$$

This gives

$$(\sigma + \frac{\alpha^4}{3})(h_0^+ - h_0^-) = \frac{2}{3}\alpha^2\kappa_0. \tag{10}$$

Substituting in the third equation,

$$-B\alpha^2\kappa_0 + [A(1-\Lambda)-2]\kappa_0 = -\alpha^2\frac{(2/3)\alpha^2\kappa_0}{(\sigma + \frac{\alpha^4}{3})} \tag{11}$$

This gives

$$\boxed{\sigma(\alpha) = -\frac{\alpha^4}{3}\left[1 + \frac{2}{([A(1-\Lambda)-2] - B\alpha^2)}\right]}. \tag{12}$$

If $[A(1-\Lambda)-2] \ll B\alpha^2$, it is the Type-IIs instability.

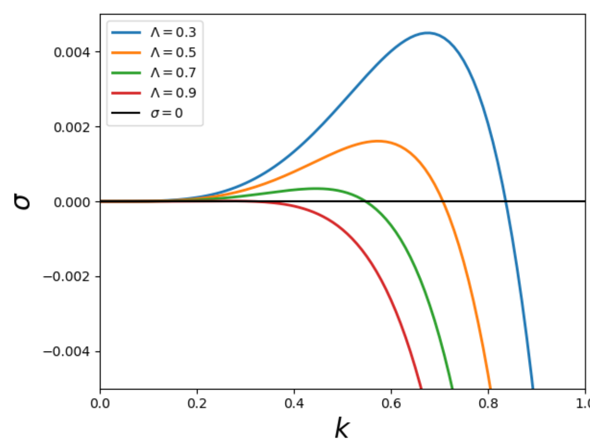


Figure 1: As Λ decreases, we see a Type-IIs type instability. Here, $A = B = 0.5$.

For marginal stability curve, we substitute $\sigma = 0$ in Eqn.(12). This yields:

$$\frac{[A(1-\Lambda)]}{B} = \alpha^2 \tag{13}$$

Assuming Λ to be our control parameter,

$$\Lambda = 1 - B\alpha^2/A \tag{14}$$

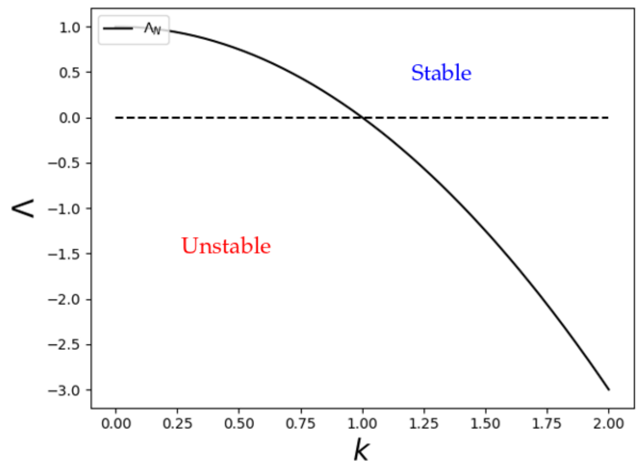


Figure 2: Marginal stability curve. Here, $A = B = 0.5$.

Fastest growing mode can be found by finding where $\frac{\partial\sigma}{\partial\alpha} = 0$. However, since $\sigma \equiv \sigma(\alpha^2)$, we can equivalently find $\frac{\partial\sigma}{\partial(\alpha^2)} = 0$ to find the fastest growing mode. We represent $[A(1-\Lambda)-2] \equiv \#$ for simplicity.

$$\begin{aligned}
\frac{\partial\sigma}{\partial(\alpha^2)} &= \frac{-2\alpha^2}{3} - \frac{2}{3}\frac{\left[(\# - B\alpha^2)(2\alpha^2) - \alpha^4(-B)\right]}{(\# - B\alpha^2)^2} \\
0 &= -\alpha^2 - \frac{-B\alpha^4 + 2\#\alpha^2}{(\# - B\alpha^2)^2} \\
0 &= -1 + \frac{B\alpha^2 - 2\#}{(\# - B\alpha^2)^2} \\
(\# - B\alpha^2)^2 &= B\alpha^2 - 2\# \\
\#^2 + B^2\alpha^4 - 2\#B\alpha^2 &= B\alpha^2 - 2\# \\
0 &= B^2\alpha^4 - (2\# + 1)B\alpha^2 + (\#^2 + 2\#) \\
\alpha^2 &= \frac{(2\# + 1) \pm \sqrt{(1 - 4\#)}}{2B}
\end{aligned} \tag{15}$$

Therefore, the fastest growing modes will be $\boxed{\alpha_f = \pm\sqrt{\frac{(2\# + 1) \pm \sqrt{(1 - 4\#)}}{2B}}}$.

(Have to check which of the inner $\pm$ modes will be fastest. Can be done by substituting into the dispersion relation and varying parameters.)

2 $2D$ pattern forming system (one confined direction): Two-component porous medium convection:

2D convection in a fluid-saturated porous layer confined between plane parallel rigid boundaries coincident with $z = 0$ and $z = 1$ but now with two (dimensionless) scalar fields, namely the temperature $T(x, z, t)$ and the salt concentration $S(x, z, t)$, that determine the density of the fluid.

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} &= 0, \\
\boldsymbol{u} &= -\nabla P + (Ra_T T - Ra_S S)\hat{\boldsymbol{e}}_{\boldsymbol{z}}, \\
\partial_t T + \boldsymbol{u} \cdot \nabla T &= \nabla^2 T, \\
\partial_t S + \boldsymbol{u} \cdot \nabla S &= \tau \nabla^2 S.
\end{aligned}
\tag{16}
$$

where Ra_T and Ra_S are thermal and solutal Rayleigh numbers and $0 < \tau < 1$ is the diffusivity ratio. We define a streamfunction ψ such that $u = \partial_z \psi$ and $w = -\partial_x \psi$. Eliminating pressure by taking the curl of the Darcy's law (momentum equation) and taking the y component, we obtain:

$$
\begin{aligned}
\cancel{-}\left[\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right] &= \cancel{-}\frac{\partial}{\partial x}\left[Ra_T T - Ra_S S\right] \\
-\psi_{xx} - \psi_{zz} &= Ra_T T_x - Ra_S S_x, \\
\Rightarrow \nabla^2 \psi &= Ra_S S_x - Ra_T T_x.
\end{aligned}
\tag{17}
$$

2.1 Linear Stability Analysis:

Base state $T_B(z) = C_B(z) = 1 - z$ and $\psi_B = 0$. Substituting $T = T_B + \theta$,$S = C_B + c$, $\psi = \psi_B + \psi$ and linearizing, obtain:

$$
\begin{aligned}
\psi_{xx} + \psi_{zz} &= Ra_S S_x - Ra_T \theta_x \\
\partial_t \theta - \psi_x(-1) &= \theta_{xx} + \theta_{zz} \\
\partial_t c - \psi_x(-1) &= c_{xx} + c_{zz}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\psi_{xx} + \psi_{zz} + (Ra_T \theta_x - Ra_S c_x) &= 0 \\
-\psi_x + \theta_{xx} + \theta_{zz} &= \partial_t \theta \\
-\psi_x + c_{xx} + c_{zz} &= \partial_t c
\end{aligned}
\tag{19}
$$

Substituting the normal mode ansatz $\phi = \hat{\phi}(z)e^{ikx} + \text{c.c.}$, where c.c. is the complex conjugate, we obtain the following linear eigenvalue problem:

$$
\begin{bmatrix} D^2 - k^2 & (ik)Ra_T & -(ik)Ra_S \\ -(ik) & D^2 - k^2 & 0 \\ -(ik) & 0 & \tau(D^2 - k^2) \end{bmatrix}
\begin{bmatrix} \hat{\psi} \\ \hat{\theta} \\ \hat{c} \end{bmatrix}
= \sigma
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \hat{\psi} \\ \hat{\theta} \\ \hat{c} \end{bmatrix}
\tag{20}
$$

However, for this particular problem, we can do better than this. We have Dirichlet BC's on θ, c and w, i.e., $\theta = c = \partial_x \psi = 0$ at $z = 0, 1$. The last condition, however, can be interpreted as follows: At the boundaries ($z = 0, 1$), ψ is constant in x. Since ψ is independent of x, we can set the value of $\psi = 0$ at the boundaries, without loss of generality. Anyway, we are interested in the velocity fields and those can be obtained from the derivatives of ψ, not ψ itself. Hence, the boundary conditions can be taken to be $\theta = c = \psi = 0$ at $z = 0, 1$. With this information, we can guess the z-dependence of the perturbation fields to be $\sim \sin(m\pi z)$. Writing $\hat{\phi} = \phi_0 \sin(m\pi z)$, we further simplify the above system as:

$$\begin{bmatrix} -(k^2+m^2\pi^2) & (ik)Ra_T & -(ik)Ra_S \\ -(ik) & -(k^2+m^2\pi^2) & 0 \\ -(ik) & 0 & -\tau(k^2+m^2\pi^2) \end{bmatrix} \begin{bmatrix} \psi_0 \\ \theta_0 \\ c_0 \end{bmatrix} = \sigma \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \theta_0 \\ c_0 \end{bmatrix} \tag{21}$$

We can solve this system analytically for each k, yielding a quadratic in σ, giving two roots for each k. However, I found the expression to be too cumbersome and resorted to numerics at this point. For $Ra_S = \tau = 0$, I was able to recover the dispersion relation for the one-component porous medium convection that we derived in class, with $k_c = \pi$ and $Ra_{Tc} = 4\pi^2$.

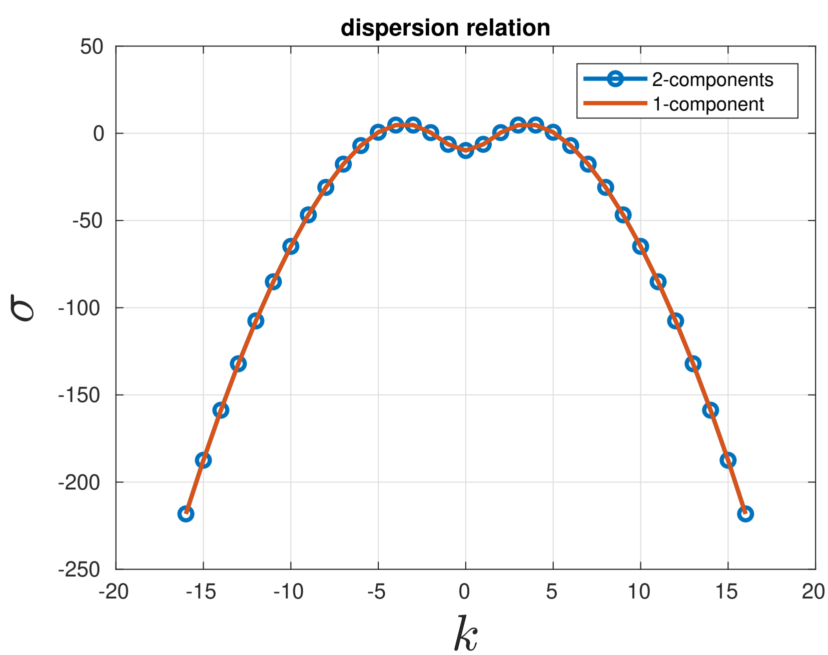


Figure 3: Comparison between dispersion relations obtained for $Ra_S = \tau = 0$ (2-components) and for the one-component porous medium convection. $Ra_T = 4\pi^2 + 10$.

For marginal stability we substitute $\sigma = 0$ and the condition for marginal stability then becomes $\det(A) = 0$, where A is the LHS matrix in Eqn.(21). It is easy to calculate the determinant using third row and expansion in minors.

$$k^2(\tau Ra_T - Ra_S) = \tau(k^2 + m^2\pi^2)^2. \tag{22}$$

2.2 Energy Stability Analysis:

To perform energy stability analysis, we first need to create an energy-like quantity. We do not linearize equations at any point and deal with full nonlinear evolution equations in this approach. Multiplying θ-equation by θ, c-equation by c and adding them, we create a quadratic, positive definite energy-like quantity. Let us first focus on the θ-equation.

$$\int_0^1 \int_0^{L_x} \theta \left[\partial_t \theta + \boldsymbol{u}.\nabla\theta = w + \nabla^2\theta\right] dxdz$$

$$\int_0^1 \int_0^{L_x} \theta \left[\partial_t \theta + \nabla \cdot \left(\boldsymbol{u}\frac{\theta^2}{2}\right) - \frac{\theta^2}{2}\cancelto{0}{\nabla \cdot \boldsymbol{u}} = w + \nabla^2\theta\right] dxdz$$

$$\text{NOTE:}\nabla \cdot \left(\boldsymbol{u}\frac{\theta^2}{2}\right) = \int_0^1 \cancelto{0\ \because\ \text{periodic BCs}}{\left[\int_0^{L_x} \partial_x(u\theta^2/2)dx\right]} dz + \int_0^{L_x} \cancelto{0\ \because\ \text{z-BCs}}{\left[\int_0^1 \partial_z(w\theta^2/2)dz\right]} dx$$

$$\partial_t \left[\int_0^1 \int_0^{L_x} \frac{\theta^2}{2} dxdz\right] = \int_0^1 \int_0^{L_x} \left[w\theta + \theta\nabla^2\theta\right] dxdz$$

using integration by parts (IBP) and BCs . . .

$$\partial_t \left[\int_0^1 \int_0^{L_x} \frac{\theta^2}{2} dxdz\right] = \int_0^1 \int_0^{L_x} \left[w\theta + \cancelto{0\ \because\ \theta = 0 \text{ at } z=0,1}{\nabla \cdot \theta\nabla\theta} - |\nabla\theta|^2\right] dxdz$$

$$\partial_t \left[\int_0^1 \int_0^{L_x} \frac{\theta^2}{2} dxdz\right] = \int_0^1 \int_0^{L_x} \left[w\theta - |\nabla\theta|^2\right] dxdz \tag{23}$$

Similarly, we can obtain another equation by multiplying c to the c-equation and integrating over the domain.

$$\partial_t \left[\int_0^1 \int_0^{L_x} \frac{c^2}{2} dxdz\right] = \int_0^1 \int_0^{L_x} \left[wc - \tau|\nabla c|^2\right] dxdz \tag{24}$$

Adding these two equations and defining $\boxed{E(t) = \int_0^1 \int_0^{L_x} \left[\frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2}\right] dxdz}$, we obtain:

$$\frac{dE}{dt} = \int_0^1 \int_0^{L_x} \left[\tilde{w}(\tilde{\theta} + \tilde{c}) - |\nabla\tilde{\theta}|^2 - \tau|\nabla\tilde{c}|^2\right] dxdz \tag{25}$$

We define a quantity $\lambda = \min\{\int_0^1 \int_0^{L_x} \left[|\nabla\tilde{\theta}|^2 + \tau|\nabla\tilde{c}|^2 - \tilde{w}(\tilde{\theta}+\tilde{c}) \right] dxdz\}$, where $\tilde{w}$ is slaved to $\tilde{\theta}, \tilde{c}$ in the same way as w is slaved to θ and c, i.e. $\nabla^2 w = -Ra_s c_{xx} + Ra_T \theta_{xx}$ or $w = \mathcal{L}(\theta, c)$. If we can prove $\lambda > 0$, this would imply $dE/dt < 0$ and that the system is energy stable.

2.2.1 Variational Formulation:

Let

$$J[\tilde{\theta}, \tilde{c}] = \int_0^1 \int_0^{L_x} \left[|\nabla\tilde{\theta}|^2 + \tau|\nabla\tilde{c}|^2 - \tilde{w}(\tilde{\theta}+\tilde{c}) \right] dxdz \tag{26}$$

, such that $\tilde{w} = \mathcal{L}(\tilde{\theta}, \tilde{c})$. Also, we impose a normalization constraint on the energy-like quantity to keep book-keeping simple (The idea of normalizing is similar to matrix norms $||A|| = \max_{x\neq 0} \frac{||Ax||}{||x||} = \max_{x\neq 0, ||x||=1} ||Ax||$). We demand, $\int_0^1 \int_0^{L_x} \left[\frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2} \right] dxdz = 1$. We now have a variational problem at hand, with the 'cost functional' $J(\tilde{\theta}, \tilde{c})$ (to be minimized) subject to constraints $\tilde{w} = \mathcal{L}(\tilde{\theta}, \tilde{c})$ and $\int_0^1 \int_0^{L_x} \left[\frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2} \right] dxdz = 1$. This leads us to define the Lagrangian/ Lagrange functional L:

$$\begin{aligned} L(\tilde{\theta}, \tilde{c}, \tilde{w}, v, \Lambda) \equiv & \int_0^1 \int_0^{L_x} \left[|\nabla\tilde{\theta}|^2 + \tau|\nabla\tilde{c}|^2 - \tilde{w}(\tilde{\theta}+\tilde{c}) \right] dxdz \\ & - \int_0^1 \int_0^{L_x} v(\nabla^2\tilde{w} + Ra_s c_{xx} - Ra_T \theta_{xx}) dxdz \\ & - \Lambda \left[\int_0^1 \int_0^{L_x} \frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2} dxdz - 1 \right], \end{aligned} \tag{27}$$

where v is a Lagrange multiplier field and Λ is a scalar Lagrange multiplier. Now, we must set the first variations of L with respect to each of its arguments. We start with $\frac{\delta L}{\delta \tilde{\theta}} = 0$. From variational calculus, we know this is equivalent to $\frac{d}{d\epsilon} L(\tilde{\theta} + \epsilon\eta, \tilde{c}, \tilde{w}, v, \Lambda)\Big|_{\epsilon=0} = 0$, with η satisfying $\eta = 0$ at $z = 0, 1$.

$$
\begin{aligned}
L(\tilde{\theta}+\epsilon\eta,\tilde{c},\tilde{w},v,\Lambda) = &\int_0^1\int_0^{L_x}\left[|\nabla\tilde{\theta}+\epsilon\eta|^2+\tau|\nabla\tilde{c}|^2-\tilde{w}(\tilde{\theta}+\epsilon\eta+\tilde{c})\right]dxdz \\
&-\int_0^1\int_0^{L_x} v(\nabla^2\tilde{w}+Ra_s c_{xx}-Ra_T\theta_{xx}-\epsilon Ra_T\eta_{xx})dxdz \\
&-\Lambda\left[\int_0^1\int_0^{L_x}\left(\frac{(\tilde{\theta}+\epsilon\eta)^2}{2}+\frac{\tilde{c}^2}{2}\right)dxdz-1\right], \\
=&\int_0^1\int_0^{L_x}\left[|\nabla\tilde{\theta}|^2+2\epsilon\nabla\tilde{\theta}\cdot\nabla\eta+\epsilon^2|\nabla\eta|^2+\tau|\nabla\tilde{c}|^2-\tilde{w}(\tilde{\theta}+\epsilon\eta+\tilde{c})\right]dxdz \\
&-\int_0^1\int_0^{L_x} v(\nabla^2\tilde{w}+Ra_s c_{xx}-Ra_T\theta_{xx}-\epsilon Ra_T\eta_{xx})dxdz \\
&-\Lambda\left[\int_0^1\int_0^{L_x}\left(\frac{\tilde{\theta}^2+2\epsilon\tilde{\theta}\eta+\epsilon^2\eta^2)}{2}+\frac{\tilde{c}^2}{2}\right)dxdz-1\right]
\end{aligned} \tag{28}
$$

Taking the first derivative of the above expression wrt ϵ and evaluating at $\epsilon=0$, we get:

$$
\begin{aligned}
\frac{\delta L}{\delta\tilde{\theta}} =& \int_0^1\int_0^{L_x}\left[2\nabla\tilde{\theta}\cdot\nabla\eta-\tilde{w}\eta\right]dxdz \\
&-\int_0^1\int_0^{L_x} v(-Ra_T\eta_{xx})dxdz \\
&-\Lambda\left[\int_0^1\int_0^{L_x}\tilde{\theta}\eta dxdz\right], \\
0 =& \int_0^1\int_0^{L_x}\left[2\nabla\tilde{\theta}\cdot\nabla\eta-\tilde{w}\eta+Ra_T v\eta_{xx}-\Lambda\tilde{\theta}\eta\right]
\end{aligned} \tag{29}
$$

We want everything to be proportional to η, so using IBP to factor out η from each term - For example,

$$
\begin{aligned}
\int_0^1\int_0^{L_x} 2\nabla\tilde{\theta}\cdot\nabla\eta dxdz =& \int_0^1\int_0^{L_x} 2\nabla\cdot\eta\nabla\tilde{\theta}-2\eta\nabla^2\tilde{\theta}dxdz \\
=& \int_0^1\cancelto{0}{\left(\int_0^{L_x}\partial_x[\eta\partial_x\tilde{\theta}]dx\right)}dz+\int_0^{Lx}\cancelto{0}{\left(\int_0^1\partial_z[\eta\partial_z\tilde{\theta}]dz\right)}dx \\
&-\int_0^1\int_0^{L_x} 2\eta\nabla^2\tilde{\theta}dxdz,
\end{aligned} \tag{30}
$$

Both the boundary terms vanish since $\eta(0)=\eta(1)=0$ in z and periodic BCs in x. Similarly for other terms, we obtain

$$
0=\int_0^1\int_0^{L_x}\eta\left[-2\nabla^2\tilde{\theta}-\tilde{w}+Ra_T\partial_{xx}v-\Lambda\tilde{\theta}\right]dxdz. \tag{31}
$$

Since this must be true for all η (satisfying appropriate BCs), we must have

$$\boxed{-2\nabla^2\tilde{\theta} - \tilde{w} + Ra_T\partial_{xx}v = \Lambda\tilde{\theta}}. \tag{32}$$

Similarly, setting variation of L wrt $\tilde{c}$ to be zero ($\frac{\delta L}{\delta \tilde{c}} = 0$), we get

$$\boxed{-2\tau\nabla^2\tilde{c} - \tilde{w} - Ra_S\partial_{xx}\tilde{v} = \Lambda\tilde{c}}. \tag{33}$$

$\frac{\delta L}{\delta \tilde{w}} = 0$ gives

$$\boxed{\nabla^2\tilde{v} + \tilde{\theta} + \tilde{c} = 0}. \tag{34}$$

$\frac{\delta L}{\delta \tilde{v}} = 0$ gives

$$\boxed{\nabla^2\tilde{w} + (Ra_S\tilde{c}_{xx} - Ra_T\tilde{\theta}_{xx}) = 0.} \tag{35}$$

And finally, $\frac{\delta L}{\delta \Lambda} = 0$ gives back the normalization condition. Substituting $\tilde{\phi} = \phi_E \sin(m\pi z)e^{ikx}$, we get:

$$\begin{aligned}
&2(m^2\pi^2 + k^2)\theta_E - w_E - k^2 Ra_T v_E = \Lambda\theta_E,\\
&2\tau(m^2\pi^2 + k^2)c_E - w_E + k^2 Ra_S v_E = \Lambda c_E,\\
&-(m^2\pi^2 + k^2)v_E + \theta_E + c_E = 0,\\
&-(m^2\pi^2 + k^2)w_E + k^2(Ra_T\theta_E - Ra_S c_E) = 0.
\end{aligned} \tag{36}$$

Multiplying the first two equations by $(m^2\pi^2 + k^2)$ and substituting the third and fourth equations into them, we get

$$\begin{aligned}
&2(m^2\pi^2 + k^2)^2\theta_E - k^2(Ra_T\theta_E - Ra_S c_E) - k^2 Ra_T(\theta_E + c_E) = \Lambda(m^2\pi^2 + k^2)\theta_E,\\
&2\tau(m^2\pi^2 + k^2)^2 c_E - k^2(Ra_T\theta_E - Ra_S c_E) + k^2 Ra_S(\theta_E + c_E) = \Lambda(m^2\pi^2 + k^2)c_E.
\end{aligned} \tag{37}$$

Separating coefficients of θ_E, c_E, get:

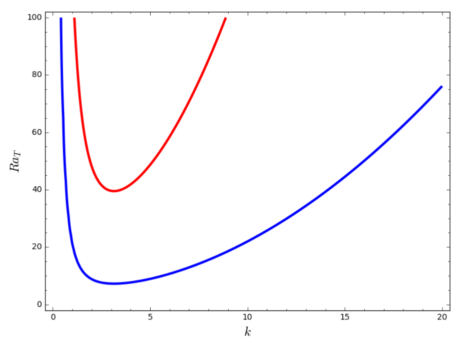
$$\begin{aligned}
[2(m^2\pi^2 + k^2)^2 - 2k^2 Ra_T]\theta_E + k^2(Ra_S - Ra_T)c_E &= \Lambda(m^2\pi^2 + k^2)\theta_E,\\
k^2(Ra_S - Ra_T)\theta_E + [2\tau(m^2\pi^2 + k^2)^2 + 2k^2 Ra_S]c_E &= \Lambda(m^2\pi^2 + k^2)c_E.
\end{aligned} \tag{38}$$

Writing into a matrix-form

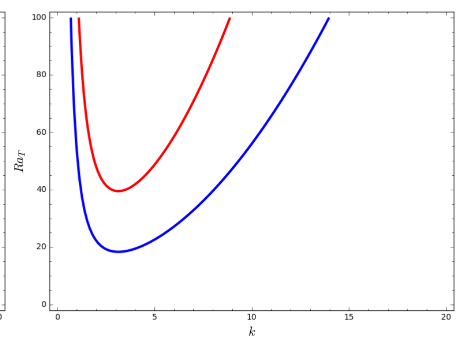
$$\begin{bmatrix} (2(m^2\pi^2 + k^2)^2 - 2k^2 Ra_T) & k^2(Ra_S - Ra_T) \\ k^2(Ra_S - Ra_T) & (2\tau(m^2\pi^2 + k^2)^2 + 2k^2 Ra_S) \end{bmatrix} \begin{bmatrix} \theta_E \\ c_E \end{bmatrix} = \Lambda(m^2\pi^2 + k^2) \begin{bmatrix} \theta_E \\ c_E \end{bmatrix}. \tag{39}$$

For energy stability, want $\Lambda > 0$, Minimum $\Lambda = 0$. To get the marginal energy-stability curve, we solve for $\Lambda = 0$. The determinant of the LHS matrix must vanish for $Ax = 0$ to have a nontrivial solution. Hence, the condition becomes:

$$
\begin{aligned}
&4\tau(m^2\pi^2 + k^2)^4 + 4k^2 Ra_S(m^2\pi^2 + k^2)^2 - 4\tau k^2 Ra_T(m^2\pi^2 + k^2)^2 - 4k^4 Ra_T Ra_S \\
&- k^4(Ra_S^2 + Ra_T^2 - 2Ra_T Ra_S) = 0, \\
\Rightarrow &4\tau(m^2\pi^2 + k^2)^4 + 4k^2 Ra_S(m^2\pi^2 + k^2)^2 - 4\tau k^2 Ra_T(m^2\pi^2 + k^2)^2 \\
&- 2k^4 Ra_T Ra_S - k^4(Ra_S^2 + Ra_T^2) = 0, \\
\Rightarrow &\boxed{4\tau(m^2\pi^2 + k^2)^4 + 4k^2(m^2\pi^2 + k^2)^2(Ra_S - \tau Ra_T) - k^4(Ra_S + Ra_T)^2 = 0}.
\end{aligned}
\tag{40}
$$



(a)



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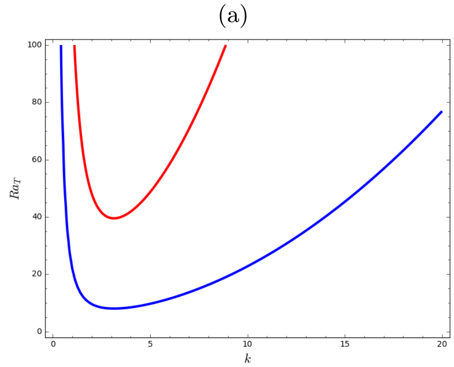
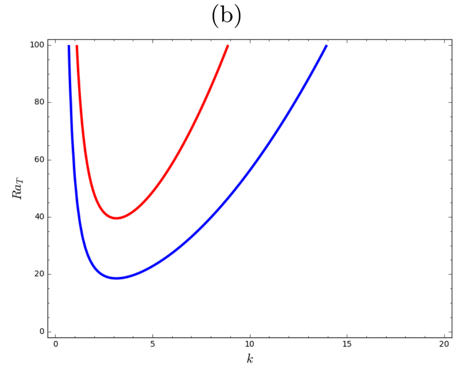


(c)



(d)

Figure 4: (a) $Ra_S = 10^{-2}, \tau = 10^{-2}$, (b)$Ra_S = 10^{-2}, \tau = 10^{-1}$, (c) $Ra_S = 10^{-1}, \tau = 10^{-2}$, (d) $Ra_S = 10^{-1}, \tau = 10^{-1}$. Here, the red curve is the marginal energy stability expression we derived in class for one-component porous medium convection: $Ra_T k^2 = (m^2\pi^2 + k^2)^2$. The blue curve is for the two-component case with different Ra_S and τ-values. Here, $m = 1$ in all the calculations. We can see that the marginal stability curves are more sensitive to τ as $\tau, Ra_S \to 0$.

Plots in Figs.(4) were generated in sagemath. For now, we fix $\tau = 0.1$, vary $Ra_S = [100, 150, 200, 250, 300]$ and obtain the marginal energy-stability curves.

When the curves go below $Ra_T = 0$, we see that even though $Ra_T < 0$, i.e., in a stably-stratified regime, the system might become unstable. Since we can only comment on the stability using energy stability analysis, we are not sure if the system *will* become unstable.

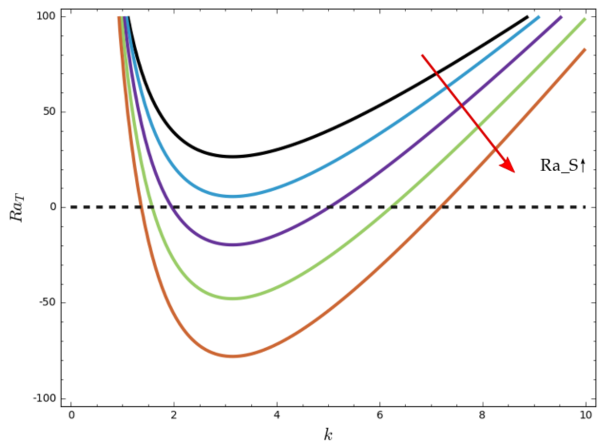


Figure 5: Marginal energy-stability curve for $\tau = 0.1$, increasing $Ra_S = [100, 150, 200, 250, 300]$.

Finally we compare the energy vs linear stability thresholds.Comparing Eqns.(22) and (40), we see that there is a gap between the linear and energy stability thresholds.

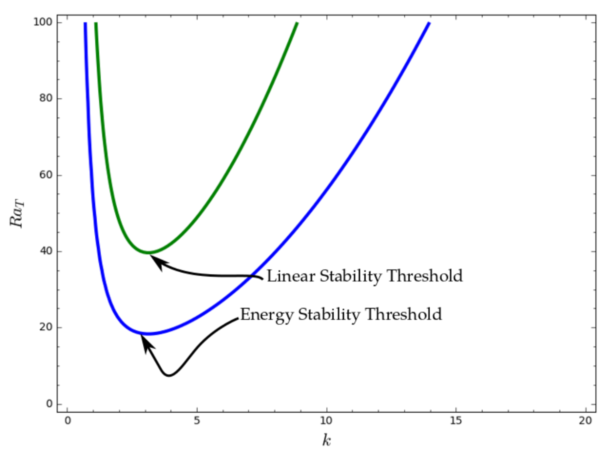


Figure 6: Comparing linear- vs energy-marginal stability curves. $m = 1$, $Ra_S = 10^{-2}, \tau = 10^{-1}$. We see a clear gap between the thresholds.