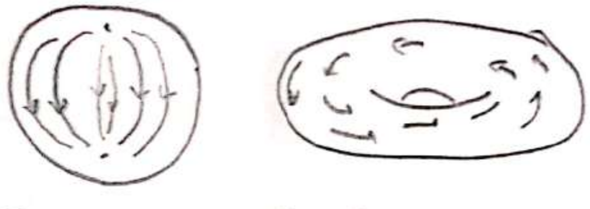
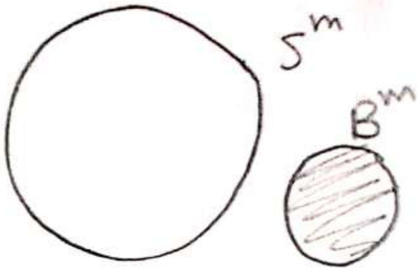
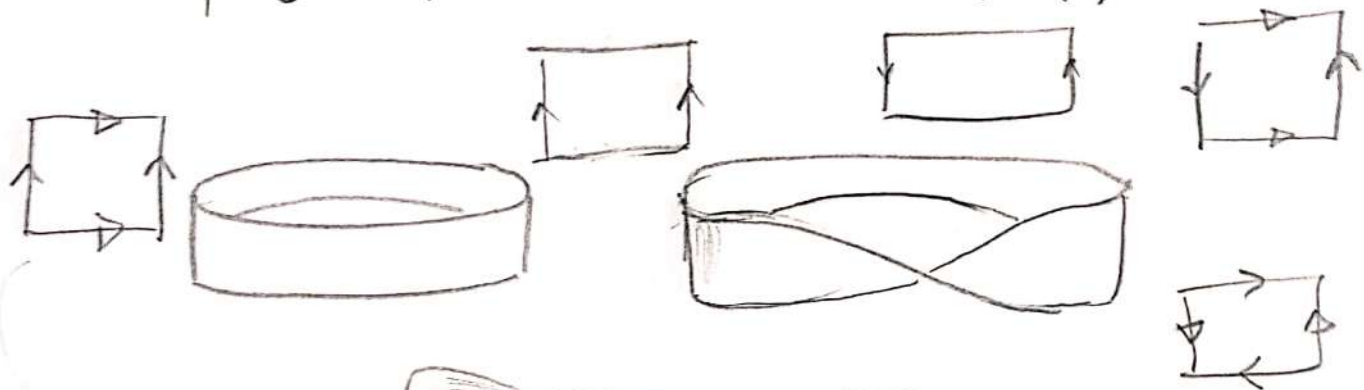
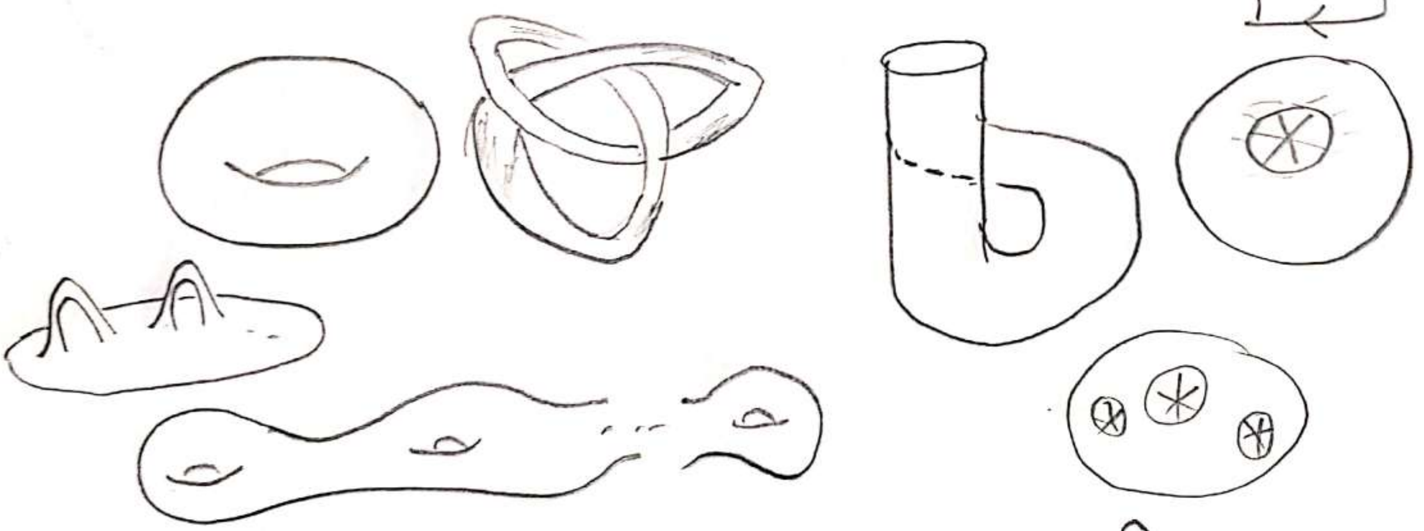


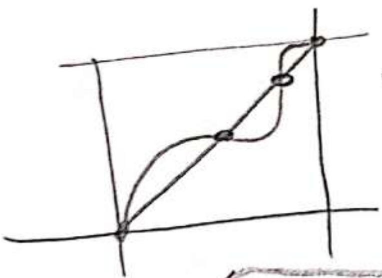


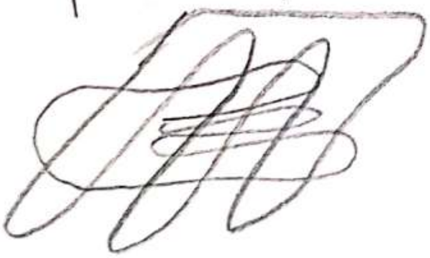
Topology & Geometry:

By Dr Tadashi Tokieda (T^2)









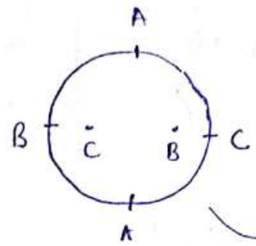
NOTES BY :- Pratik Aghor
(UNH, 2020)

TOPOLOGY

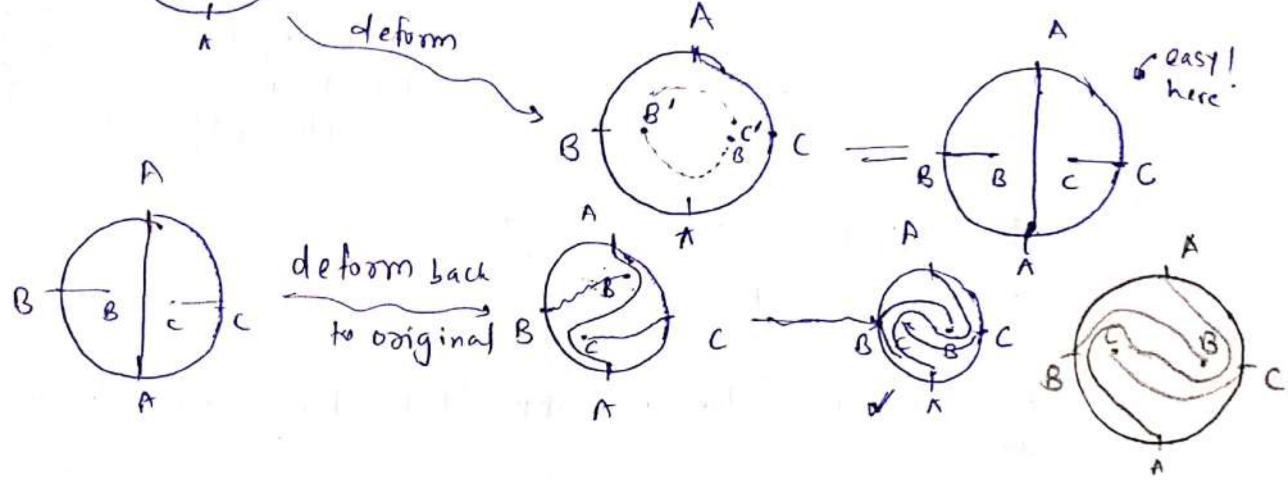
Topology & Geometry

By Tadashi Tokieda (T^2).

lec 1

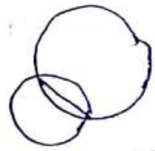


Connect all without intersections. (A-A, B-B, C-C)

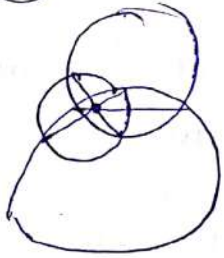


Example which can be solved by going to higher dimensions.

2 intersecting circles have a common chord



3 intersecting circles have a ~~ch~~ common pt.

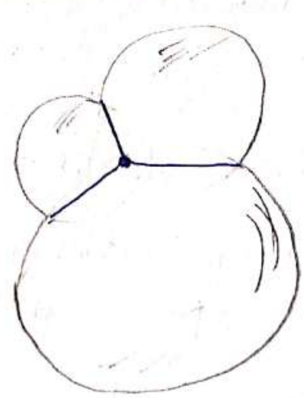


these thm:- 3 chords always meet in a point.

proof:- go to higher dim.

3 spheres

like bubbles →



Project in 2D.
single pt. intersectn

QED!

* Basic Strategy for Applying Topology:

- Problem complecated by degeneracies (e.g. det(matrix) = 0 (can't invert) or 2^{nd} derivative = 0 (can't determine max/min) etc.

deform → Generic situat^n (free from degeneracies) - simple sol^n

deform back → get the sol^n for the original problem!

NOTE: Sometimes, we can use the same approach to show some problems are unsolvable.

Example #3 :- P, Q :- matrices of size $n \times n$. , $PQ \neq QP$ in general, but show that $[PQ]$ & $[QP]$ have the same eigenvalues. — (*).

Sol. $Q(PQ)Q^{-1} = QP$ ... Notice. ⇒ PQ & QP are similar matrices or conj. matrices.

⇒ PQ, QP are matrices of the same linear transformation, just in different bases. ⇒ (*) holds.
And the change of basis - matrix is Q.

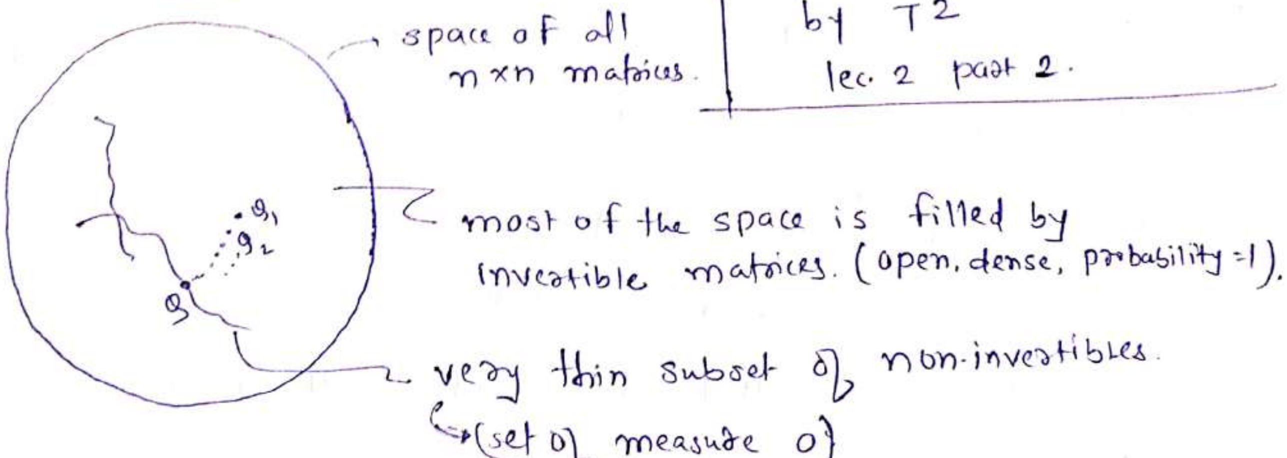
Works only when Q is invertible. (Q^{-1} exists).
i.e., only when $\det Q \neq 0$.

- But what are the chances of $\det(Q) = 0$? very rare.
- Suppose $\det(Q) = 0$, change entries of Q by $\epsilon \to 0$ & det. will no longer remain "0" exactly.

∴ Q is invertible in most situations, in the sense that (stable property)
if Q is invertible ⇒ small perturbat^{n}s of Q remain invertable.
if Q is non-invertible ⇒ " " " " & Q becomes invertible. (unstable property!)

Topology & Geometry
by T2
lec. 2 part 2.

let's draw a picture for this.



- we have proved that (*) holds in the generic case when g^{-1} exists. & most of the times, with prob. = 1, it is true.
- To prove (*) in the degenerate (really unlucky) situat^n, choose invertible $g_1, g_2, \dots$ s.t. $\{g_n\} \to g$ when $n \to \infty$.

  e.g. $g = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is non invertible.

  approx. it by $g_n = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$ as $n \to \infty$ $g_n \to g$.

- (*) depends continuously on g.
  $\Rightarrow$ if I change entries of g a little bit, eigenvalues will change by a little bit.

& $(*)_n$ true for Pg_n & g_nP.

$\lim_{n\to\infty} (*)_n \to (*)$ true also for g! & QED! □.

lemma :- Since $tr = \sum$ eigenvalues

We can now p.t. $PQ - QP = \hbar I$ can NEVER happen! ← from QM.

why? take the trace on both sides.

$$tr(PQ - QP) \overset{?}{=} \hbar\, tr(I)$$

$$0 \neq \hbar n. \implies QED!$$

But QM uses it all the time. What's the problem?

QM uses ∞-dim. matrices.

In finite dimensions, what we proved is okay. But in ∞-dim., it in fact is true. in the Hilbert spaces.

i.e. Even in ∞-dim., in case of bounded linear operators, the proof goes trough.

$\Rightarrow$ QM cannot avoid using unbounded operators in ∞ dimensions!

---

* Catalogue of the Most Important Manifolds: lec. 2. part 3.

$\approx$ Def :- A manifold (mfd henceforth) is a 'nice, smooth geometric fig.' that generalizes curves & surfaces to arbitrary dims.

Topologically, mflds can be built from:-

· Basic Building Blocks of Mflds:

(i) n-ball → $B^n = \{ (x_1, \dots x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1 \}$

dim. "n"	B^n.
1	-1 1 ← B^1
2	← 'disk' ← B^2
3	← sphere ← B^3. ball

- terminology:- $[0,1] = I$ ← unit interval.

(ii) (n-1)-sphere :- $S^{n-1} = \{ (x_1, \dots x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 = 1 \}$

n	S^{n-1}
1	S^0
2	← S^1
3	← S^2.

Boundaries of the balls = spheres.

Topology & Geometry
By T^2
lec. 2 Part 3.

disk
→ B^2

circle
→ S^1

3-ball
→ B^3

2-sphere
→ S^2.

Balls → B^n
spheres → S^{n-1}.

- <u>Useful symbols</u> :- (1) ∂ = "boundary of".

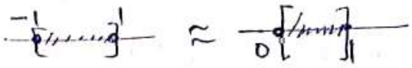
e.g. $\partial(B^n) = S^{n-1}$

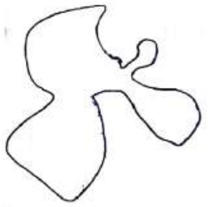
∂ ( ordinary of strip) $\equiv$  ← this union of 2 disjoint circles. (2 copies of S^1).

∂ ( Möbius strip) $\equiv$  $\equiv S^1$! single circle!

(2) $\approx$ ⇒ 'homeomorphic to', 'meaning topologically the same as'

e.g. $B^1 \approx I$.



 $\approx S^1$

- These building blocks generate more complicated (& interesting) mflds via operations of product, quotient (divide), connected sum or surgery (cut & paste).

Topology & Geometry
By T^2
lec. 3 part 1.

Operation I :- Product.

Ex. 2 :- m- cube.

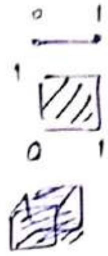
$$I^m = \underbrace{I \times \cdots \times I}_{m} \approx B^m.$$

unit interval $[0,1]$. homeomorphic to

e.g. $I^1 = [0,1]$

I^2 ← square

I^3 ← cube.



$$\boxed{\partial (I^m) \approx S^{m-1}}$$

Ex. 3 m-torus :- $T^m = \underbrace{S^1 \times \cdots \times S^1}_{m}$

T^1 T^2

≈ ≈

surface of a mug.

circle
1- torus

torus
2- torus.
surface
(NOT the Inside!).

hollow
entangled tube.

- $\boxed{\dim (M \times N) = \dim M + \dim N.}$

- How does ∂ act on products? $\partial(M \times N)$? M & N are manifolds.

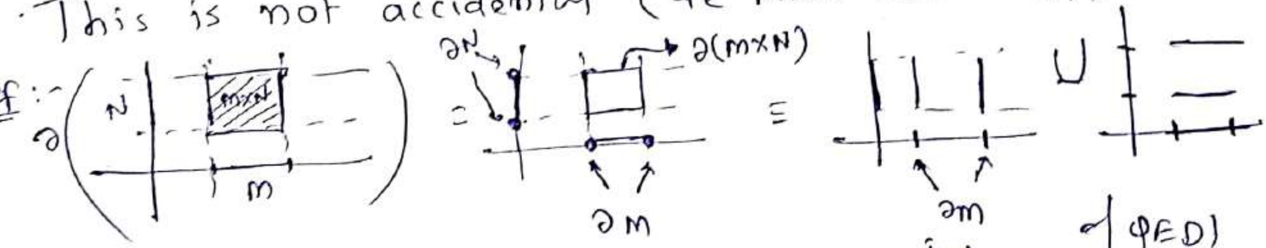
- $\boxed{\partial (M \times N) = ((\partial M) \times N) \cup (M \times (\partial N))}$ — (Theorem #4)

boundary of M; × ; the whole of N; union; the whole of M; boundary of N.

→ ∂ acts according to 'Leibniz law' for differentiat^n.

· This is not accidental (de Rham cohomology).

Pf :-



of φED)

CNTD:- • Q.E.D = Quod Erat Demonstrandum (This is what was to be demonstrated).

• Modern version → Quite Easily Done!

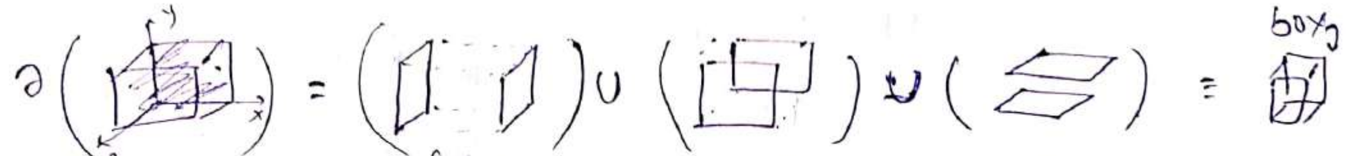
•

Topology & Geometry By T^2 lec. 3 part 1

Ex. 2 & 3:- revisited :- Cubes & tori:-

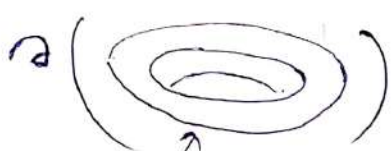
• $\partial(I^m) = \big(\partial(I) \times \underbrace{I \dots \times I}_{(m-1)}\big) \cup \big(I \times \partial I \times \partial I \times I \times \dots I\big) \cup \dots (I \times I \times \dots \partial I)$

let's look @ 3d :-



∂(x-dir^n) × (whole in y & z) ∂(z-dir^n) × (whole in x & y) ∂(y-dir^n) × (whole in x & z).

• for m-tori → let's try T^2 :-



only the surface.

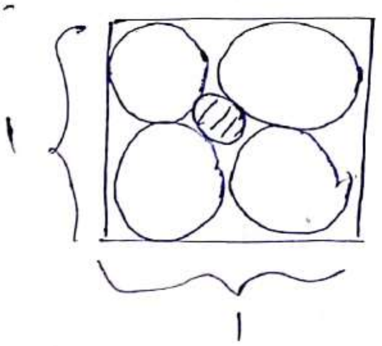
$= \partial(S^1 \times S^1)$ → S^1 is just 1-sphere. No boundary!

$= (\partial S^1 \times S^1) \cup (S^1 \times \partial S^1)$

$= (\phi \times S^1) \cup (S^1 \times \phi)$

$= \phi$ ← empty set! ∴ There's no boundary for T^2! or T^m in gen!

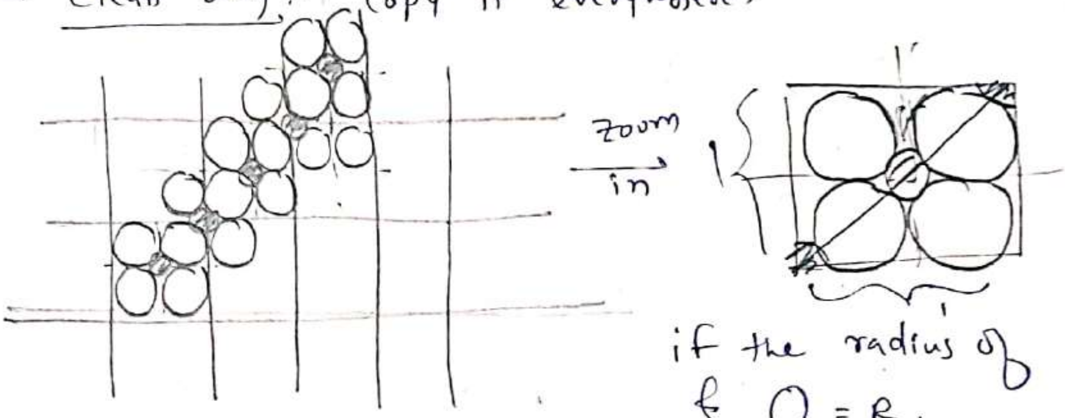
Ex. 5 :-



In I^2, pack 4 equal disks in the corners & fit a disk in the middle.

Q. What is the diameter of ⊘ ? if the side of the square = 1 ?

→ Clean way :- Copy it everywhere.



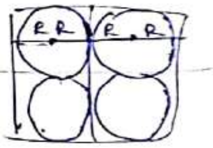
if the radius of ⊘ ≡ r
& ○ ≡ R.

$\Rightarrow$

$$r + 2R + 2r + 2R + r = \sqrt{2}.$$

$\Rightarrow$

$$4R + 4r = \sqrt{2}$$



but we know $4R$!

$4R = 1$

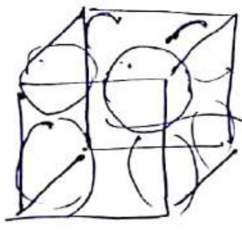
$\Rightarrow \quad 1 + 4r = \sqrt{2}$

$$\Rightarrow \boxed{r = \frac{\sqrt{2} - 1}{4}} \Rightarrow \boxed{d = \frac{\sqrt{2} - 1}{2}}$$

$$\Rightarrow d = \frac{1.4142\ldots - 1}{2} \sim \frac{0.41}{2} \approx 0.2 < 1.$$

---

in 3D :-



← we can put 8 ○ & put a sm ⊘ in the middle.

what's the diameter of ⊘ ?

same argument in 3D :- we'll get $d = \frac{\sqrt{3} - 1}{2}$

$$\text{in } m\text{-}d \rightarrow d(\oslash) = \frac{\sqrt{m} - 1}{2}$$

When $m \geq 10$, $d(\oslash) > 1$!

$\Rightarrow$ the middle ball becomes bigger than the corner ones! & it sticks outside the cube! Cartoon :-

← for $m \geq 10$!

Ex. 6 :- What is the volume of B^n? (radius 1.)

Topology & Geometry
By T^2
lec. 3 part 2.

dim m	vol (B^n)
1	2
2	π
3	$\frac{4}{3}\pi$

It turns out

$$\boxed{\mathrm{vol}(B^m) = \frac{\pi^{\frac{m}{2}}}{\left(\frac{m}{2}\right)!}}$$

where $s! = \Gamma(s+1) = \int_0^\infty x^s e^{-x} dx$

$\because$ factorial grows faster than exponential

$$\lim_{m\to\infty} \mathrm{vol}(B^m) = 0.$$

To find vol. (S^{n-1}), differentiate the vol. of m-ball of radius 'r' & then put $r=1$.

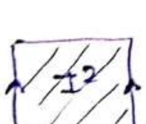
$$V_r = \frac{\pi^{m/2}}{m/2\,!} r^m \quad \& \quad V_r(S^{n-1}) = \frac{m\,\pi^{m/2}}{m/2\,!} r^{m-1}$$

$\Rightarrow$

$$\boxed{V(S^{n-1}) = \frac{2\pi^{m/2}}{\left(\frac{m}{2}-1\right)!}}$$

Operation III: Quotient :-

Ex. 7



identify $\downarrow$ vertical edges

Cylinder = $S^1 \times I$.

We write this as

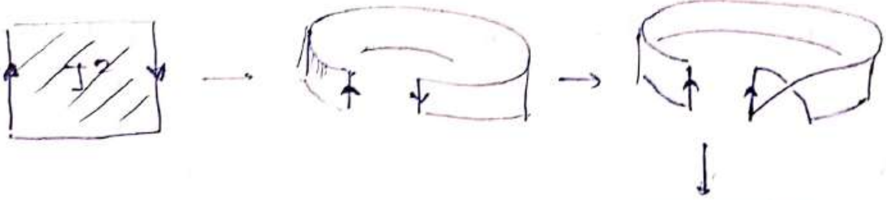
$$\rightarrow \quad I^2 / (0,y) \sim (1,y) \ \forall y$$

$(0,y)$ is identified as $(1,y)\ \forall y$.

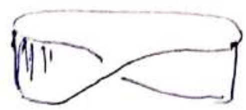
||

$S^1 \times I$ and say that the cylinder is the quotient of the square by the given equivalence / identificatn.

Ex. 8

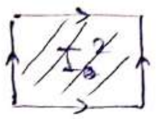


$I^2/(0,y)\sim(1,1-y)\ \forall y$ = Möbius strip.



Möbius strip.

Ex. 9



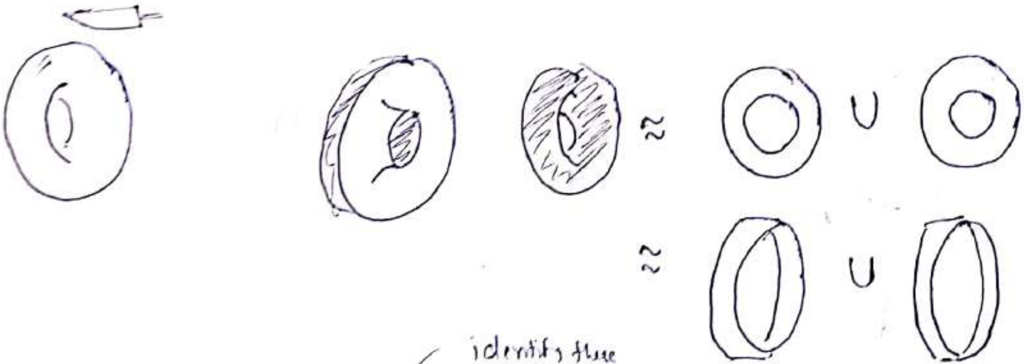
identify $\downarrow$ vertical edges

$$I^2/\begin{matrix}(0,y)\sim(1,y)\ \forall y\\(x,0)\sim(x,1)\ \forall x\end{matrix}$$

$$\| \\ T^2.$$

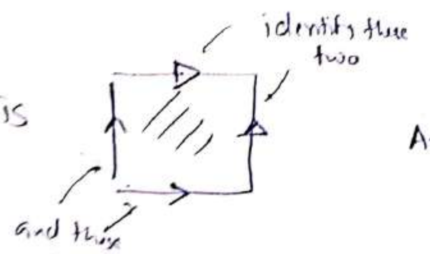
identify $\downarrow$ horizontal edges

- What do we get if we slice this torus in halves?



2 copies of ordinary strip ($S^1\times I$)

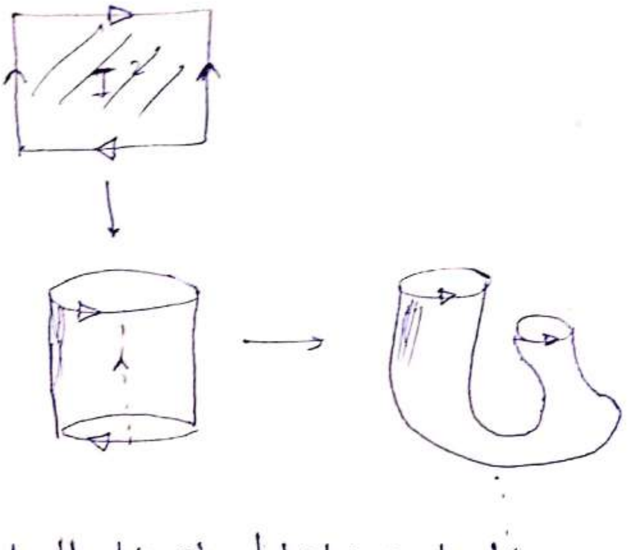
Ex. 10 What is



Answer is :- , S^2

Topology & Geometry
By T^2
lec 3 · part 3.

Ex. 11



in order to join them with the right orientatⁿ, we'll have to cut the surface in 3D.

∴ It's impossible in 3D without cutting the surface! "But that's okay! Let's relax!"

- It's impossible in 3D, & but it's possible in higher dimensions! (which we'll see later.)
- Imagine that the ~~material~~ surface is made of some super-material which can passthrough itself.

$$I^2 \Big/ \begin{matrix} (0,y) \sim (1,y) \; \forall y \\ (x,0) \sim (1-x,0) \; \forall x \end{matrix} =$$

Self-intersecting picture.

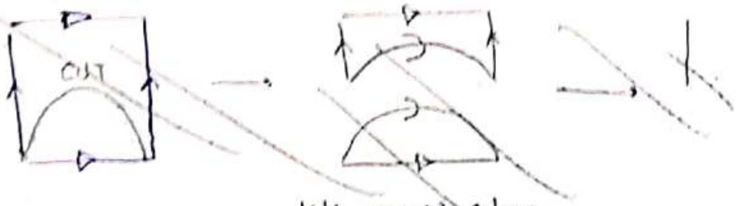
The famous Klein bottle! It's really a higher dimensional object!

- What do we get if we slice this klein bottles in halves?

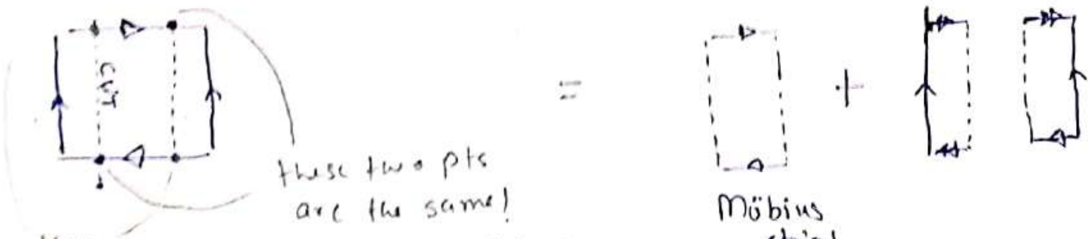
→ google it for better representations!

≈ 2 copies of Möbius strip!

- We can check this by <u>calculating with pictures</u>!



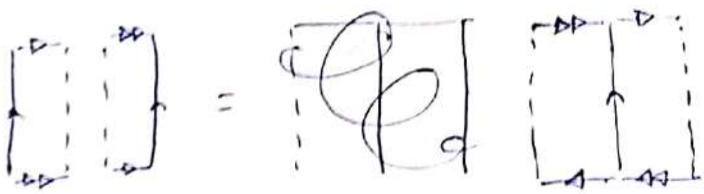
We must glue the cuts @ end.



these two pts are the same! So we must keep cutting

these two are the same & that completes the cut!

Möbius strip!



Möbius strip!

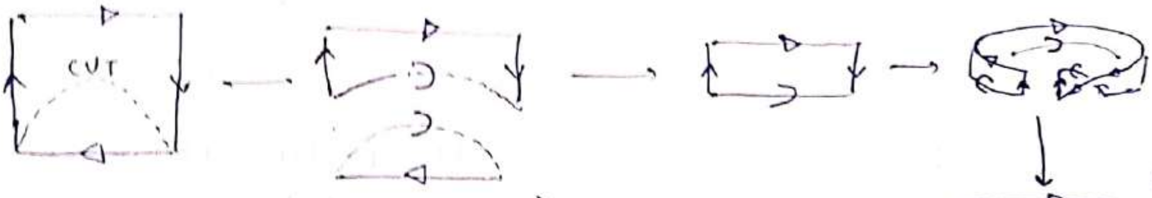
∴ a Klein bottle is obtained by gluing together 2 Möbius strips along their boundary circles.

This Ex. will be used in the theory of surgery later.

Topology & Geometry
By T^2
lec. 4 part 1.

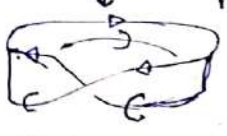
Ex. 12 :- What is  ? Difficult to visualize,

So we try calculating with pictures.



cuts must be glued in the end in the same dirn. ∴ represented as →

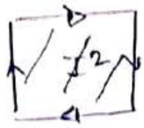
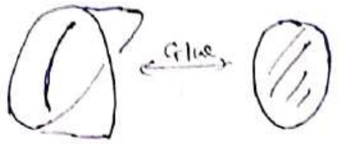
Sloid arrow, turns into a rounded arrow



Möbius strip → familiar

Solid arrow, turns into a rounded arrow.

→ same boundary! Glue them accordingly!

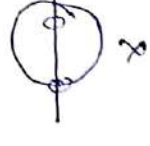
∴  ≈ 

~~take~~

take a Möbius strip & a disk & identify their boundary circles.

 ≈  ≈ 

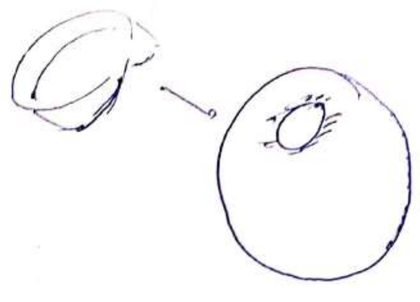
the hole does not go through the sphere but is inside.

I mean, it's <u>not</u> this → 

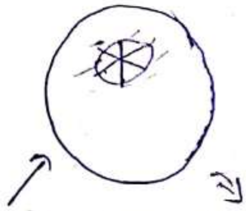
In other words if I put something in the hole, it <u>doesn't</u> come out the other side but stays inside.
e.g. जार !

Generally, 'a" way to close this hole is to put a circular lid on. However, instead of using a disk (having a circular boundary), we're going to put a Möbius strip as our lid!

$\approx$



We will draw / represent this with ______



This resulting closed surface is called the (real) (projective plane) :- Geometry on the ~~real~~ proj. plane — ~~મોબિયસ સ્ટ્રીપ~~ Möbius stripનો ઉપયોગ.

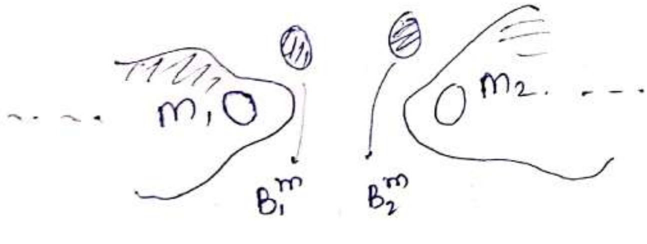
→ projective geometry!

→ NOTATN :- $\mathbb{R}P^2$

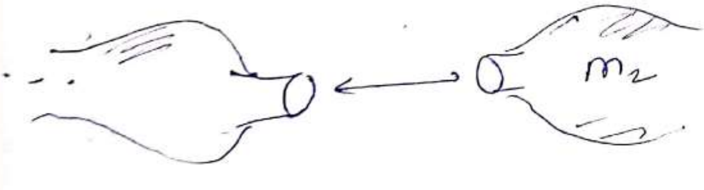
real projective plane in 2d!

Operation III :- Surgery

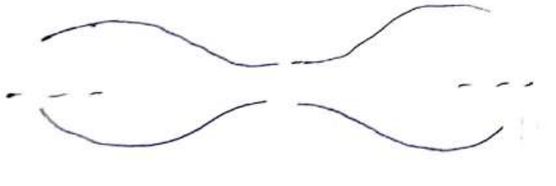
Def #13 :- Given two mflds M_1, M_2 of the same dim m.



remove an m-ball from each manifold.



identify the boundaries of the holes.



The result is

$$\frac{(M_1 \setminus B_1^m) \cup (M_2 \setminus B_2^m)}{\partial(M_1 \setminus B_1^m) \sim \partial(M_2 \setminus B_2^m)} = M_1 \# M_2.$$

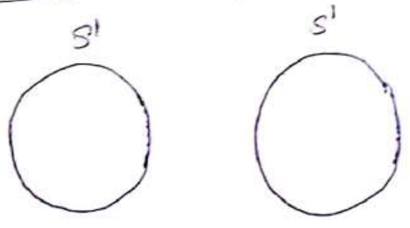
identifying/gluing along boundaries

$M_1 \# M_2$ = the connected sum of M_1 & M_2.

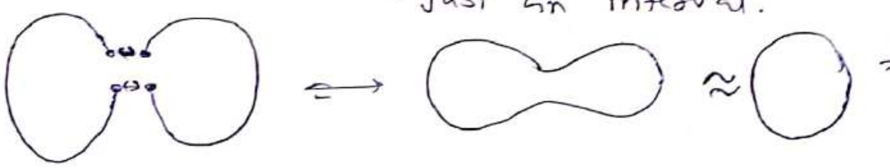
Cntd.

Topology & Geometry
By T^2
lec. 4 part 1

Ex 14: What is $S^1 \# S^1$



We take out B^1 from both circles.
↳ just an interval.



$\boxed{S^1 \# S^1 \approx S^1}$

in a way, S^1 behaved like "0". $0+0=0$.

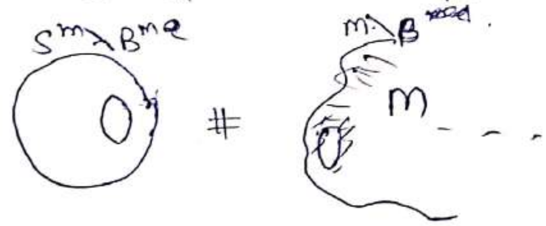
Q. What plays the role of zero for # ?

What mfld Z has the property $Z \# M = M$?

- Thm #15 :- $\forall$ mflds "M" of dim m

$$S^m \# M = M.$$

Pf :-



QED/ □.

- Common mistake :-

Beware :- in # (connected sum) you remove only one ball from each side.

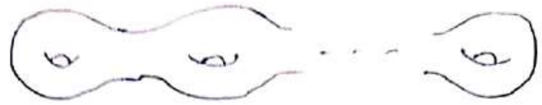
instead of removing 1 ball
I remove 2 balls

Euler's rule of g = const. is not followed.

Sphere did not behave as the zero! This is NOT how you make $S^2 \# T^2$. In fact, this resulting surface is $T^2 \# T^2 \approx$ 

Def. #16: $\Sigma_g = \underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g \text{ copies}}$ is called the "Riemann surface of genus g".

genus :- latin origin · plural → genera → meaning kind/ species.

$\Sigma_g =$  .

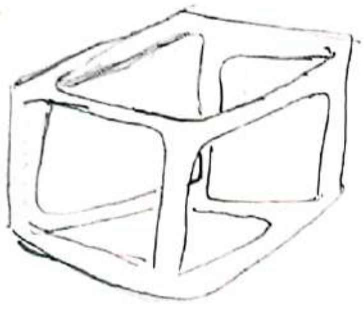
We put $\Sigma_0 = S^2$.
Note that $\Sigma_1 = T^2$.

An alternative picture of Σ_g :- attaching handles to a sphere.

---

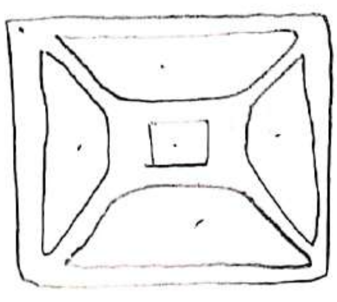
lec. 4 part 2.

Ex. 17



Which Σ_g is this?

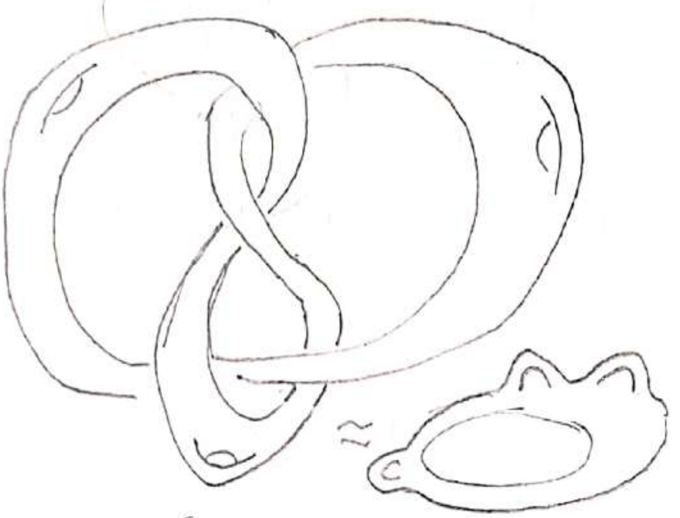
View from top :-



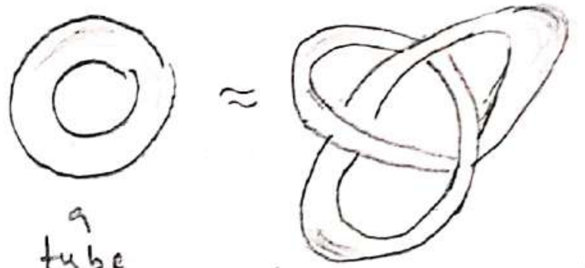
5 holes!
Ans :- Σ_5!

Topology & Geometry
By T^2
lec. 4 part 2.

How about this?



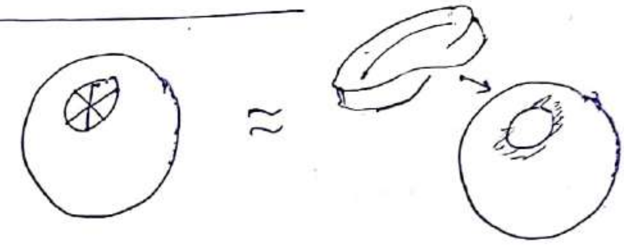
$\hookrightarrow \approx \Sigma_4$



a tube

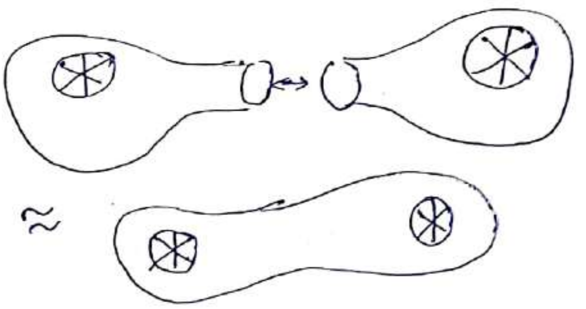
just an entangled more complicated tube

- Recall that $\mathbb{R}P^2$



- We construct connected sums of copies of $\mathbb{R}P^2$

2 copies $\to$



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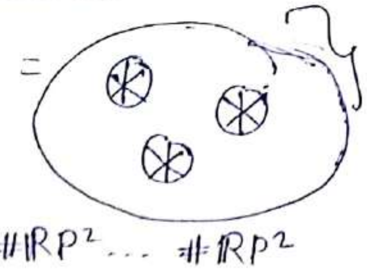
* Def. #18:- $N_g =$



$\to$ g lids of Möbius strips on a sphere.

This type., close the lid with Möbius strip.

$N_g = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \dots \# \mathbb{R}P^2$

The surface has no standard name.

We put $N_0 = S^2$.

Note $N_1 = \mathbb{R}P^2$.

* Theorem #19 :- [Proved by Möbius]

- Σ_g's are orientable (having 2 sides).
- N_g's " non-orientable (have only 1 side).

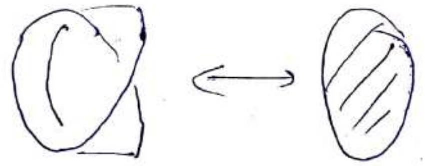
- Every 2-dim compact mfld (surface) without boundary is homeomorphic to one of Σ_g or N_g.

(Thus we have a complete classification of mflds in dim = 2).

We state this w/o proof.

Ex. 20 :- Which of Σ_g or N_g is the Klein bottle?

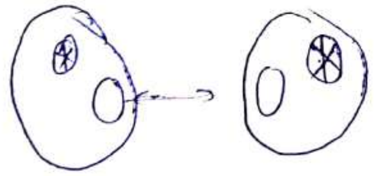
→ It is non-orientable ⊣ one of the N_g's.

Now,  is $\mathbb{R}P^2$. 

Möbius glued to disk

⊣  is a Möbius strip !

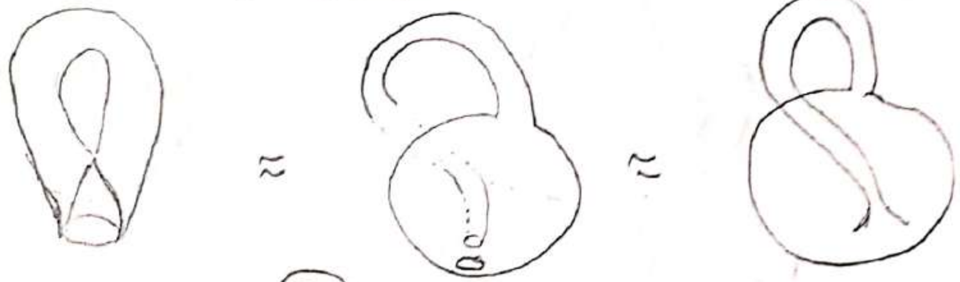
$\mathbb{R}P^2$ - a disk

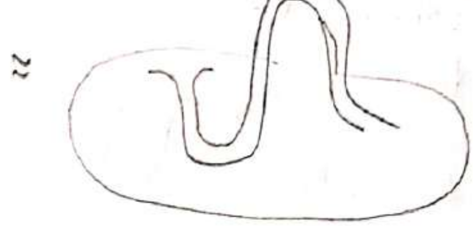
But  $\approx N_2$.

Möbius strip + Möbius strip

Klein bottle $\approx N_2$!

Another picture of Klein bottle, which we'll often use is as follows:-





← The picture we will use in calculat^{n}s with pictures

Remember this

is T^2.

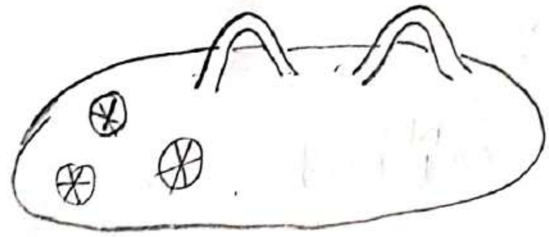
but this is N_2.

Topology & Geometry
By T^2.
Lec. 5 part 1.

Thm#19 :- Classificat^n of Surfaces (due to Möbius)

All surfaces in the universe are either Σ_g's or N_g's.

- How about



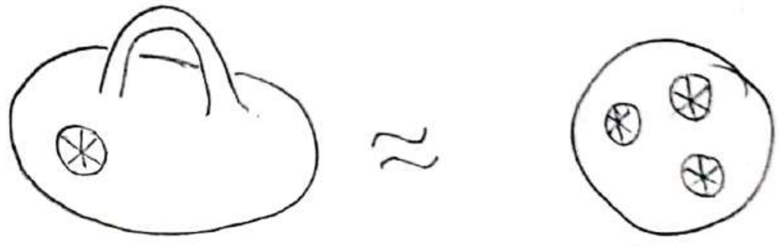
?

- looks like a mixture of Σ & N

Q. Where does it fit in the classificat^n of surfaces?

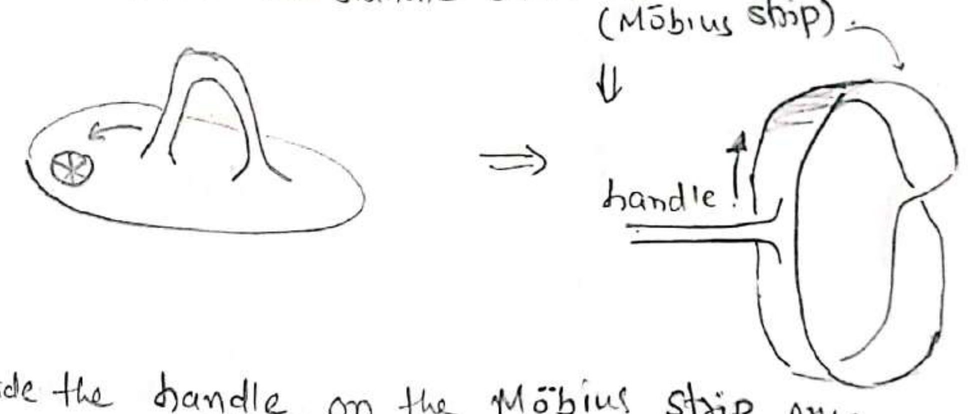
To ans., need technique – handle sliding

Lemma #21:

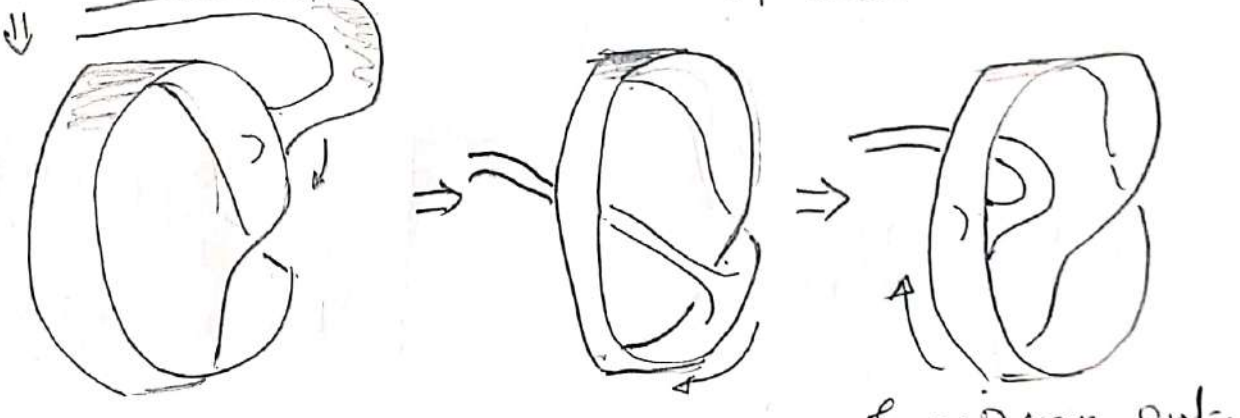


N_3

Pf:– Slide the handle onto ⊛

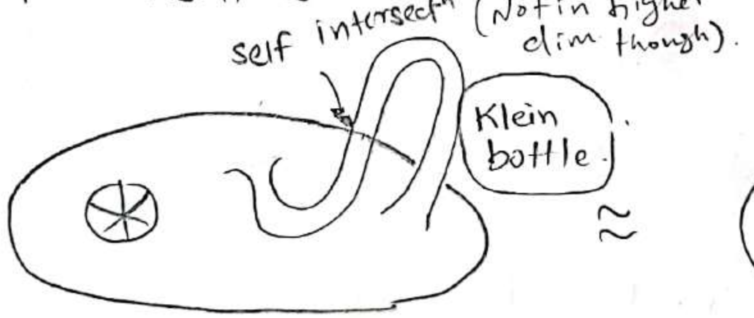


Slide the handle on the Möbius strip once.

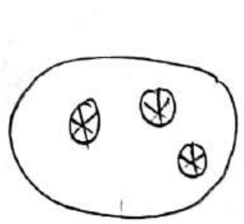


& now come out.

The result is



≈



QED.

... By Ex. #20.

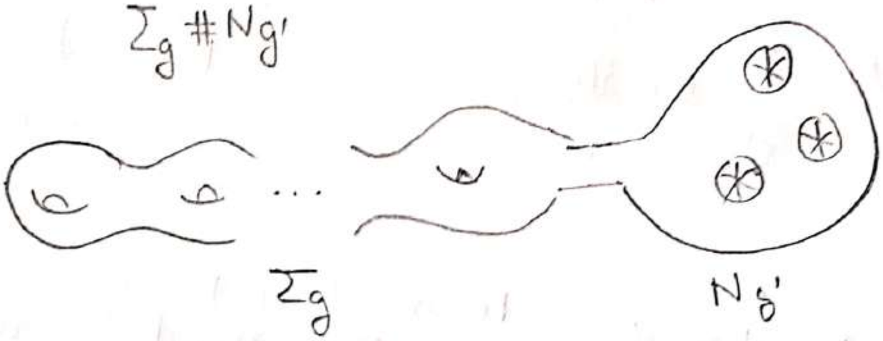
Topology & Geometry
By T^2.
lec #5-1 Cntd.

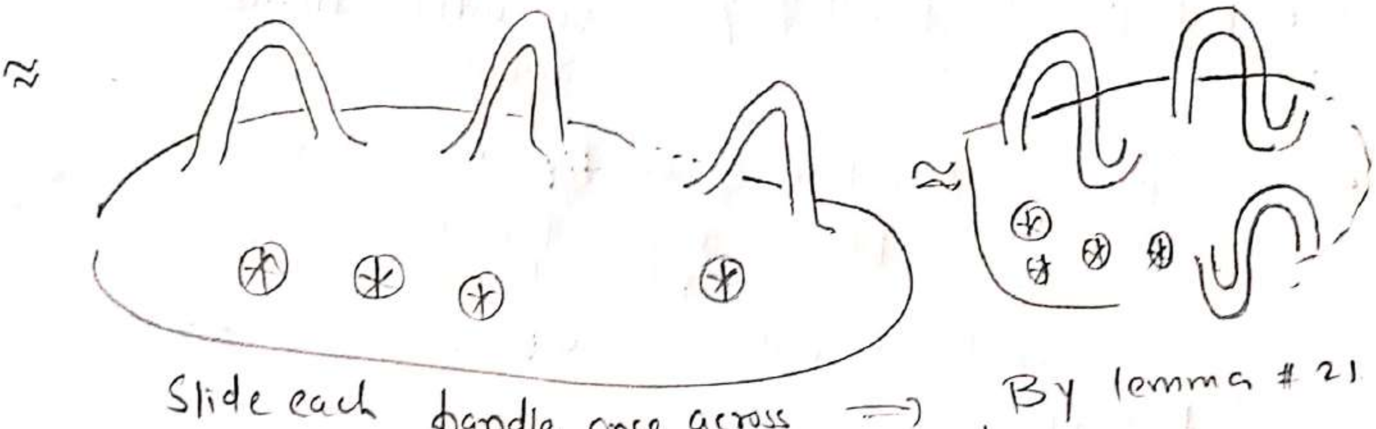
Thm #22 :- Table of Connected Sum

$\#$	$\Sigma_{g'}$	$N_{g'}$
Σ_g	$\Sigma_{g+g'}$	$N_{g'+2g}$
N_g	$N_{g+2g'}$	$N_{g+g'}$

Pf :- Diagonal entries → trivial. Off diag →

$\Sigma_g \# N_{g'}$





Slide each handle once across any Möbius strip. → By lemma #21 ⊣ $2g + g'$ ⊕

⊣ $\Sigma_g \# N_{g'} \approx N_{2g+g'}$ QED

Remark #23: Making the # of Möbius strips as small as possible →

- If have 2 $\circledast$ ⇄ Klein bottle
  if have a 3rd $\circledast$ → Slide the Klein bottle across the 3rd Möbius strip once to make a handle!

ⓐ Σ_g ⓑ N_{odd}. ⓒ N_{even}.

If have odd $\circledast$ By sliding across & making a handle, get ⓑ
— " — even $\circledast$ — " — get ⓒ.

Chap. #2. Isotopy:

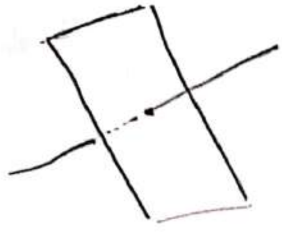
learn how to move submflds inside an ambient mfld.

- **In $\mathbb{R}^2$** → two lines generically intersect in a pt.
  lines → dim = 1
  pt → dim = 0.

$\exists$ cases that do not intersect (∥ lines),

but these cases are degenerate (exceptional, unstable, unlikely, probability = 0).

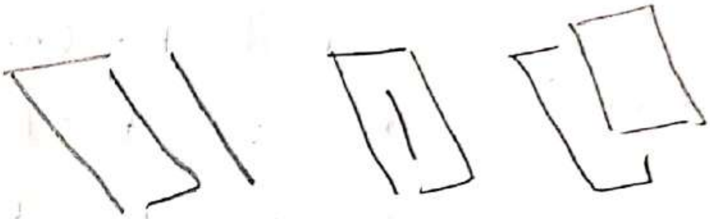
- **In $\mathbb{R}^3$**: a line & a plane intersect in a pt (dim = 0).
  (dim = 1) (dim = 2)

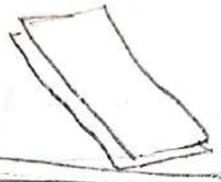


Geometry & Topology
By T^2, lec #15-1

· In $\mathbb{R}^3$, → 2 planes generically intersect
(dim 2, dim 2)
in a line. (dim (1))

Degenerate situat^{n}s :-





But these are unstable.

Theorem #1

lec #5.2 .

let $K, L \subseteq M$

Static case

call $\dim K + \dim L - \dim M$ = "overflow".

→ if overflow < 0 then generically $K \cap L = \phi$

if overflow ≥ 0 then generically $K \cap L \neq \phi$

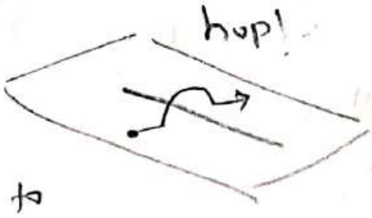
& $\dim(K \cap L)$ = overflow

Pf:- trivial □

MOVING SUB-MFLD'S IN AN AMBIENT MFLD.

· In $\mathbb{R}^2$: a pt. (dim 0) moving from one side of the line (dim 1) to the other side must make a crossing.

· In $\mathbb{R}^3$:- it can avoid the crossing.



· In $\mathbb{R}^3$:- a line (dim 1) moving from one side to the other of another line (dim 1) must make a crossing.

However, in $\mathbb{R}^4$, it can avoid a crossing.
as the line can jump into the 4th dimension!

Theorem #2: let $K, L \subseteq M$ be s.t. the overflow $(= \dim K + \dim L - \dim M) < 0$

Dynamic case

so that (by thm #1) initially $K \cap L = \emptyset$.

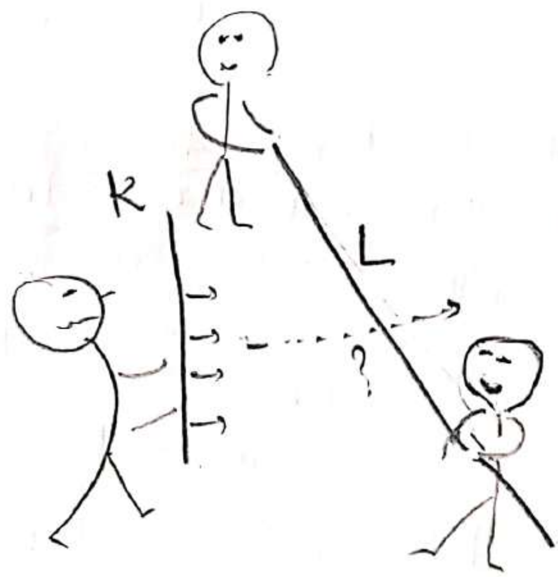
Suppose K moves from 1 side to the other side of L, & tries to "step around" to avoid crossing L.

if overflow $= -1$ then it fails (crossing inevitable).
if — " — < -1 then succeeds (crossing avoided)

See examples above.

As K moves, it sweeps out a surface of $(\dim K + 1)$. (sheet)

"Stepping around" would be successful if there is enough ambient dim. left



By thm #1:-
$\text{sheet} \cap L = \emptyset$
if $\dashv (\dim K + 1) + \dim L - \dim M < 0 \dashv 1 + \text{overflow} < 0$

$\dashv$ overflow < -1 QED.

Corollary #1: Every knot (made of curve) can be unknotted in $\mathbb{R}^m$, $m \geq 4$.

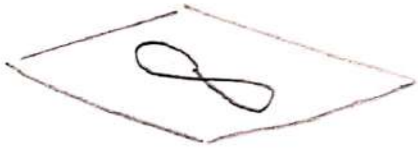
Topology & Geometry
Lec #5.2 Cntd
By T^2

Pf:-  If I can do this, I can unknot everything.

this move is possible w/o crossing if, by thm #2

$$1+1-m < -1 \Rightarrow m > 3 \Rightarrow m \geq 4. \quad \square$$

On the other hand, in $\mathbb{R}^2$ a loop may be generic self-intersect^{n}s (by thm #1).



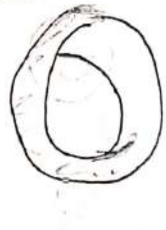
$\mathbb{R}^4$ & above $\rightarrow$ knot theory completely trivial.

$\mathbb{R}^2$ $\rightarrow$ generic self-intersect^{n}s.

$\Rightarrow$ knot theory interesting only in $\mathbb{R}^3$.

Topology & Geometry
By T^2:- Lec #6.1

Ex #4: In what $\mathbb{R}^m$ can the tube knot be unknotted

 to  a T^2?

(surface).

from last lec, if string $\rightarrow$  to  can be done in $\mathbb{R}^4$ & above

To unknot, we need to pass the 2d surface (K) past itself (2d surface, F acts as L) By last lec, to avoid crossing,

$$\dim K + \dim L - m < -1. \Rightarrow 2+2-m < -1$$

$\dashv$ $m > 5$ $\dashv$ $\boxed{m \geq 6}$ in $\mathbb{R}^6$ & above, can unknot tube knots.

Def #5. Imagine we're moving a submfld K in an ambient Mfld M.

K_t = posit^n of K @ time $t \in [0,1]$.

If K_t NEVER has self. intersect^n $\forall t$, we say K_0 is isotopic to K_1 & denote as $K_0 \underset{M}{\sim} K_1$. (in M).

T^2's wisdom :- To be a good mathematician :

def $\ll$ # thms $\ll$ # examples !

Ex. #6

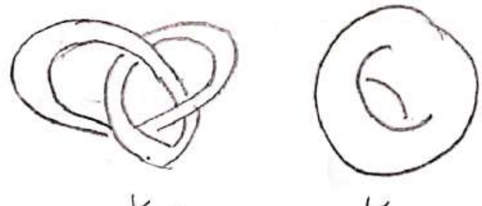
In $\mathbb{R}^6$ they're isotopic. (By ex. #4).

isotopic in $\mathbb{R}^3$? $\boxed{\text{YES}}$!

Ans.:-

26.

In contrast



$\subset \mathbb{R}^3 \subset \mathbb{R}^4 \subset \mathbb{R}^5 \subset \mathbb{R}^6 \subset \ldots$

$K_0 \underset{\mathbb{R}^m}{\sim} K_1$ for $m \geq 6$

$K_0 \approx K_1$ but $K_0 \underset{\mathbb{R}^m}{\not\sim} K_1$ for $m = 3, 4, 5$.
(homeomorphic) (see ex. #4)
same shape → bijectn from $K_0 \rightleftarrows K_1$

In general $\boxed{\text{Thm \#7:}}$ Isotopic $\Longrightarrow$ homeomorphic.

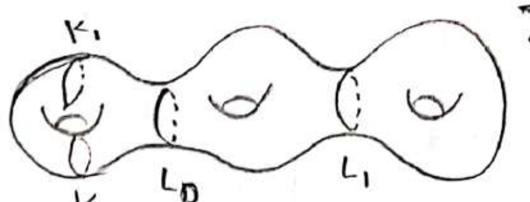
But $\Leftarrow$ is not necessarily true.

To know which submflds are isotopic to which, it is indispensible to specify in what ambient mfld we're trying to isotope.

'Homeomorphism' is an intrinsic property (doesn't depend on ambient mfld) whereas 'isotopic' is an extrinsic prop.

The larger (smaller) the ambient dim, the easier (harder) it becomes to isotope.

$\boxed{\text{Ex. \#8}}$

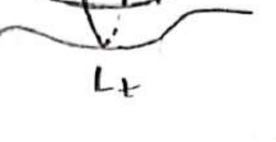


which is isotopic to which in Σ_3?

- Easy cases

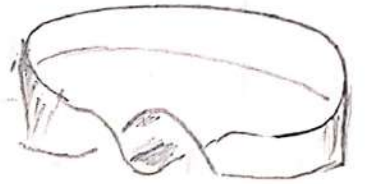
$K_0 \underset{\Sigma_3}{\sim} K_1$

Q. $L_0 \underset{\Sigma_3}{\overset{?}{\sim}} L_1 \rightarrow \boxed{\text{NO}}$ why?



L_t not in Σ_3 as the loop isn't touching the surface everywhere!

Proj:-



Möbius Strip with (2 twists)

Ordinary strip (0 twists)

Homeomorphic ✓.

Isotopic in $\mathbb{R}^m$?

$m=3$ No.

$m = 4, 5, 6, \ldots$?

Homeomorphic as both are 2d & orientable

If isotopic → give min m. If not, explain why.

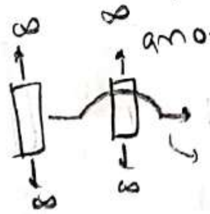
ANS:→ After discussing with Mayank Toprani:-

In 1d:- pt cannot go across to the other side of a pt w

In 2d:- a line can't go to the other side of a line w/o crossing.

But a chopped line (segment) can go to the other side of another chopped line (seg.) in 2D! ∴ They behave like pts.

In 3d:- A plane cannot go to the other side of another plane w/o crossing. But a finite width strip can go across another of its kind. (in 3d). ↳ they behave like lines.



↳ lift in 3d & transport.

In fact 2 chopped planes from both sides (2 A4 sheets) can go to the other side of each other even in 2D.

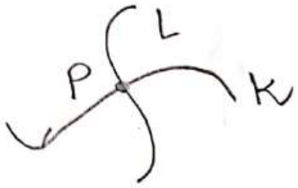
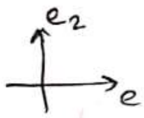
∴ Chopping reduces the eff. dim. in the formula $\boxed{\dim K + \dim L - \dim M < -1}$

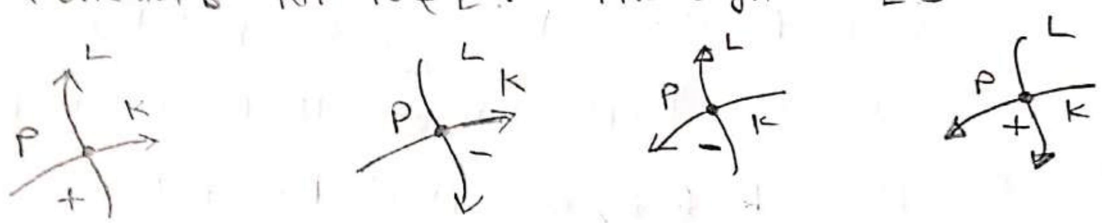
ANS:- Strip ≡ Chopping in 1 direct^n. of a 2d surface → possible in $\mathbb{R}^4$ & above.

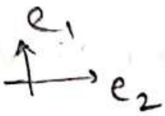
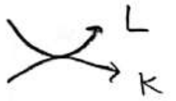
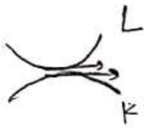
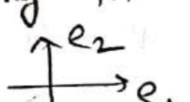
$\Rightarrow (2-1)+(2-1)-m < -1 \Rightarrow m > 3 \Rightarrow \boxed{m \geq 4}$

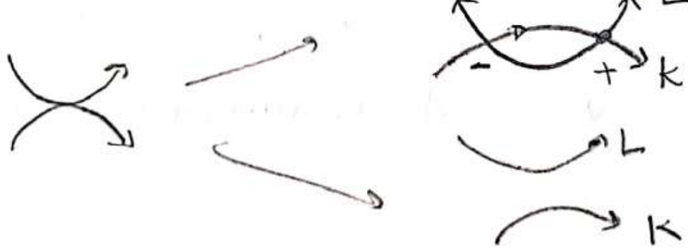
Chap. 3 :- Intersect^n Number (Index)

Topology & Geometry
By T^2 :- lec #6.2

- We'll start counting intersect^{n}s with signs.
- Consider 2 curves  in $\mathbb{R}^2$. intersecting @ P.
- Choose an orientat^n for $\mathbb{R}^2$. e.g.  K 1st L 2nd.
  & orientat^{n}s for K & L. The sign of $\{P\} = K \cap L$ is



- If had chosen  for $\mathbb{R}^2$ then opp. signs.
- What about degenerat cases like  ?
- The tangent vectors  are linearly dependent
  so cannot decide $\pm$.
- But let's deform to generic (like "unfolding" in non-linear dynamics). 

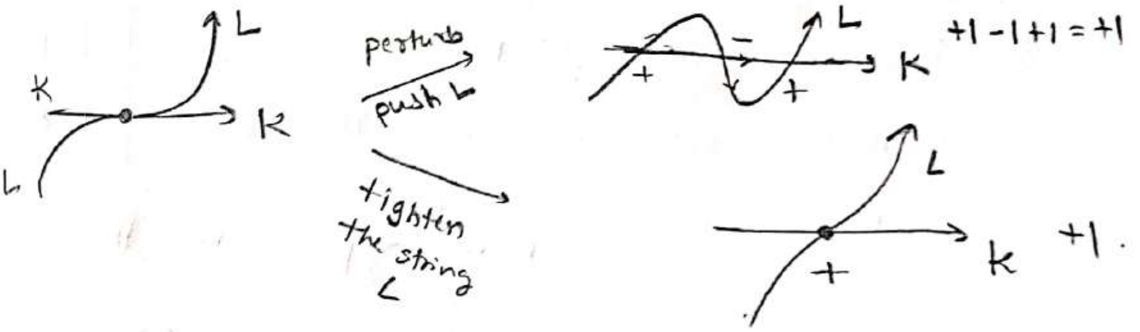


$\Rightarrow +1-1=0$

0 intersect^n.

- get the same sum of signs independently of how we deform.

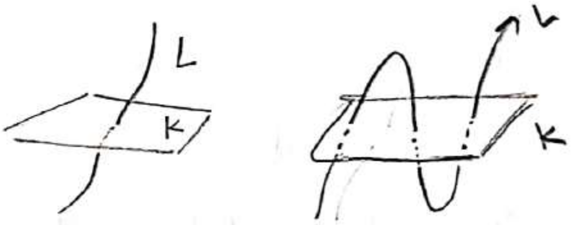
Likewise



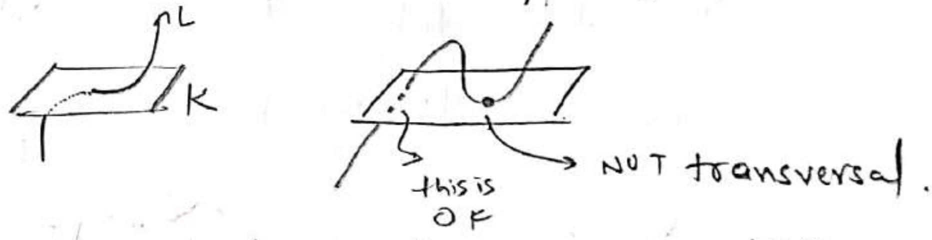
Def #10 :- let $K, L \subseteq M$ be s.t. $\dim K + \dim L = \dim M$ ('overflow' = 0) ∃ chap. 2 thm 1. ∃ K, L generically intersect in discrete pts. Assume that at all these intersectn pts., K & L are NOT tangent (generic case) i.e., they ∩ @ nonzero angles.

Then we say that K & L are transversal & write $K \pitchfork_M L$

Ex. #11 transversal



Not transversal :-



- Transversality is a robust, stable, nice, generic conditn.

Def #12. In an oriented mfld M, let K, L = oriented submflds

Topology & Geometry By T^2 - lec # 7.)

- If $K \pitchfork_M L$, the intersect^n number of K & L (in this order) $\boxed{K \circ L = \sum_{p \in K \cap L} \text{sign} @ p.}$

where sign $= \pm 1$ accordingly as the joint orientat^n of K & L (in this order) matches or not with the ambient orientat^n of M.

- If $K \not\pitchfork_M L$, 1st deform (perturb) $K, L \Rightarrow K \pitchfork_M L \rightarrow$ calculate $K \circ L$

- Intersect^n # is independent of how we perturb in corollary #18.

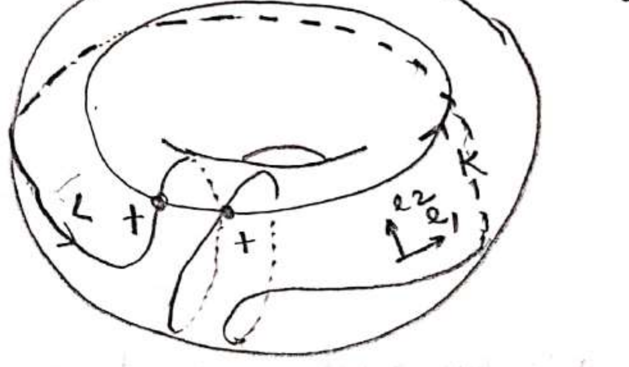
Generalizat^n #13 In non-orientable situat^{n}s (e.g. N_g) we can do the theory mod 2.

$0+0=0$
$1+0=1$
$0+1=1$
$1+1=0$

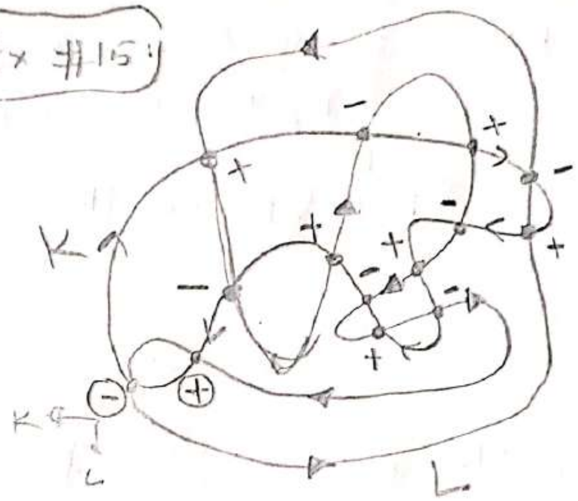
$+1=-1$.

Ex. #14 $K \circ L = +1+1 = +2$

$T^2 = \Sigma_2$



Ex #15:

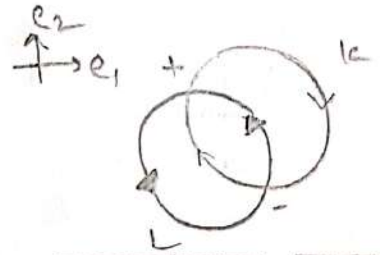


let's take a non-std orientat^n for $\mathbb{R}^2$ $\uparrow^{e_1} \rightarrow e_2$

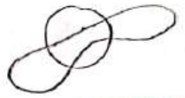
$K \circ L = -1+1-1+1\ldots$ 14 times $= 0$

Q. Is Intersect^n # of 2 loops in $\mathbb{R}^2$ is $= 0$?

$\uparrow^{e_2} \rightarrow e_1$

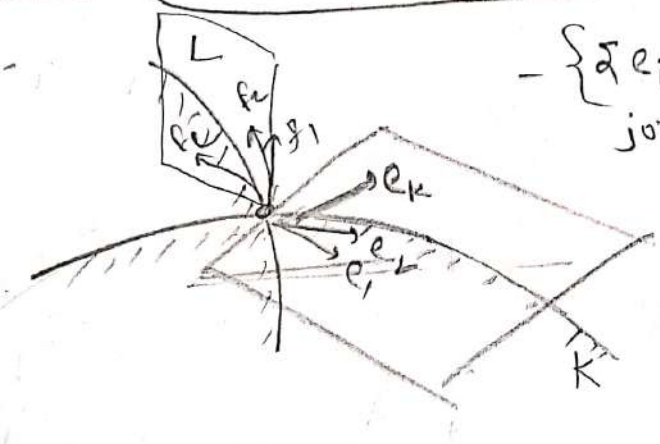


Experimentat^n $\Rightarrow$ Yes.



Thm #16 $\boxed{L \circ K = (-1)^{kl} K \circ L \; ; \; k \equiv \dim K \quad l = \dim L}$.

Pf:-



- $\{\{e_i\}_{i=1}^{k}, \{f_j\}_{j=1}^{l}\}$ define the joint orientat^n of K & L. (in this order).
- the joint orientat^n of L & K $L \circ K$ (in this order) is defined by

$$\{\{f_j\}_{j=1}^{l}, \{e_i\}_{i=1}^{k}\}$$

We can rearrange the former into the latter by a sequence of transpositions (swaps).

How many transpositions? :- Ans.: kl

Case $k=3$, $l=2$. $[e_1\ e_2\ e_3\ f_1\ f_2]$

$\xrightarrow{1} [e_1\ e_2\ f_1\ e_3\ f_2] \to [e_1\ e_2\ f_1\ f_2\ e_3] \to [e_1\ f_1\ e_2\ f_2\ e_3]$

$\to [e_1\ f_1\ f_2\ e_2\ e_3] \to [f_1\ e_1\ f_2\ e_2\ e_3] \to [f_1\ f_2\ e_1\ e_2\ e_3]$

6 swaps $= 3 \cdot 2 = kl$.

$\Rightarrow$ sign $\equiv (-1)^{kl}$. ... each transposit^n gives a factor of -1.

$\therefore$ $L \circ K = (-1)^{kl}\, K \circ L$ $\square$.

In ex. #14, $k=1$, $l=1$ $\Rightarrow$ $L \circ K = (-1)^{1\cdot 1} \cdot K \circ L = -K \circ L$.

Important Thm #17: How intersect^n # behaves when we deform things.

Let K, L be oriented, closed (compact w/o boundary) submflds in an oriented mfld M. Their intersect^n # is isotopy-invariant.

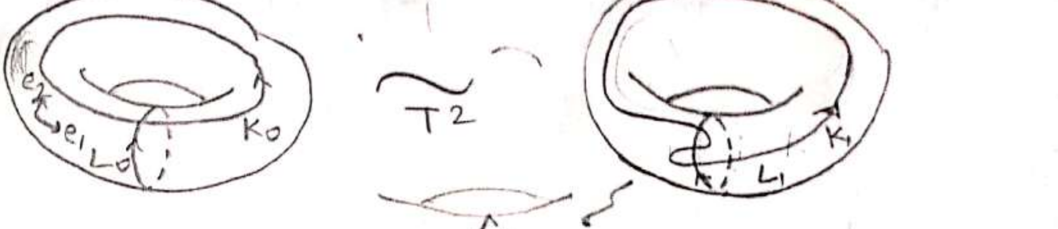
IF $K_0 \underset{M}{\sim} K_1$ and $L_0 \underset{M}{\sim} L_1$

then $K_0 \circ L_0 = K_1 \circ L_1$.

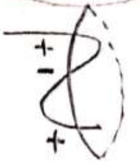
($\therefore$ if $K_0 \circ L \neq K_1 \circ L$ for some L then $K_0 \underset{M}{\not\sim} K_1$).

Ex #18:

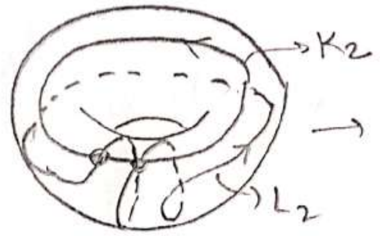
$K_0 \circ L_0 = 1$



$K_1 \circ L_1 = +1-1+1 = +1 \Leftarrow$



isotopic $\overset{?}{\sim}$



$\rightarrow$ $K_2 \circ L_2 = +1+1 = +2$ (Ex #14)

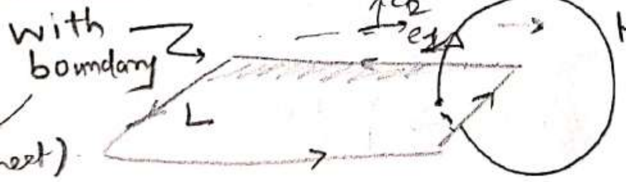
$\therefore$ $K_0 \circ L_0 \neq K_2 \circ L_2$

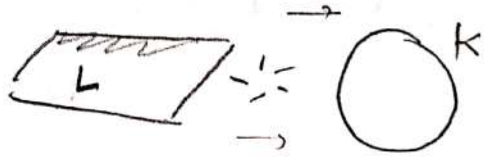
& $K_0 = K_2$, $\quad L_0 \underset{T^2}{\not\sim} L_1$ by thm #17.

Remark #19: The closedness hypothesis in thm #17 is essential. For submflds with boundary, the intersectn # may <u>not</u> be isotopy-invarient.

T²'s wisdom: "One good example is worth a thousand theorems!"

<u>Ex.</u> with boundary (A4 Sheet)





Initially $K \circ L = 1$ $\qquad$ $K \circ L = 0$

But it was a perfectly good isotopy.

ok $\because$ a pt. fell-off the boundary.

While applying thm #17. must make sure that boundaries are abscent.

Corollary #20: In the non-transversal situat^{n}s of def #12 ($K \not\pitchfork_M L$), still $K \circ L$ is well-defined independently of how we deform/perturb non-transversality.

Pf.

non-transversal sit. (A) $\overset{\text{perturb}}{\sim}$ transversal posit^n #B

$\underset{\text{perturb}}{\sim}$ —"— #C.

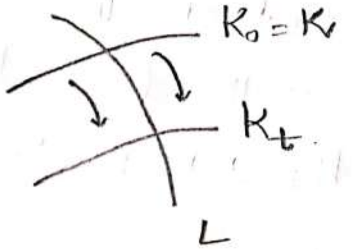
$A \sim B$, $A \sim C$ $\Rightarrow$ take $B \sim A \sim C \Rightarrow B \sim C$.

$\Rightarrow$ intersect^n # for B & C must be the same! QED.

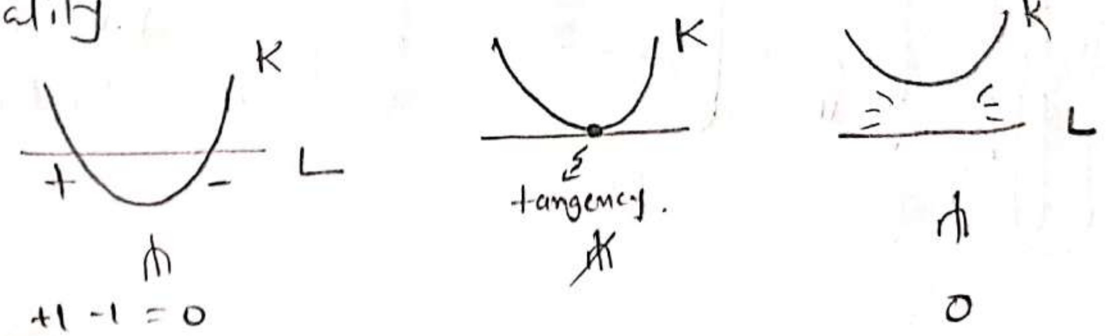
Pf of thm #17: It is enough to prove this when we just isotope K. (keep L the same).

① while $K_t \pitchfork_M L$ $\quad K_t \circ L$ does not change.

:



② But when we pass through a moment of non-transversality.



... We either create or annihilate a pair of intersect^n pts. of opp. signs. (when passing through a moment of $\not\pitchfork$).

ex: particle + anti-particle $\longleftrightarrow$ vacuum (+energy).

$\therefore$ sign of $K_t \circ L$ remains unchanged. $\square$.

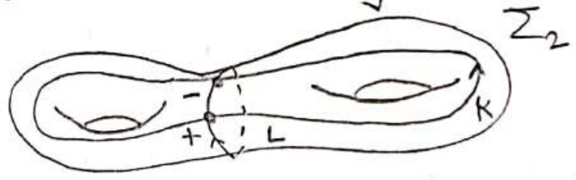
Topology & Geometry
By T^2:- lec. # 8.2

It follows from thm #17.

$\boxed{\text{Thm \#21:}}$ If closed K, L can be disjointed by isotopy, then $K \circ L = 0$.

The converse is not necessarily true.

$\boxed{\text{Ex. \#22}}$



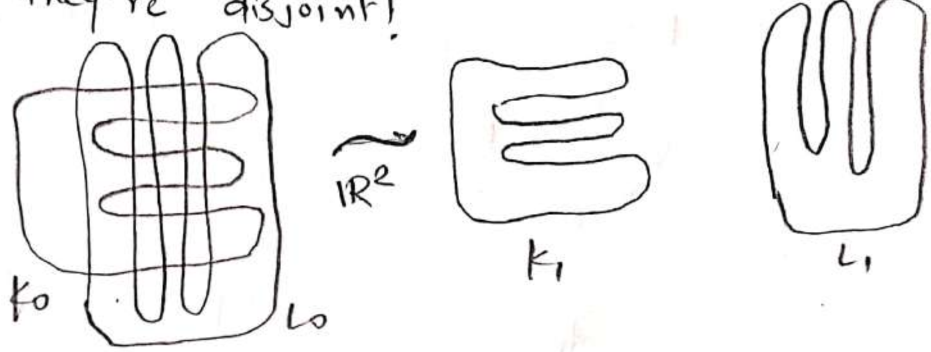
$K \circ L = +1 - 1 = 0$

but we cannot disjoint K & L by isotopy (on Σ_2 ambient).

$\boxed{\text{Corollary \#23:}}$ In $\mathbb{R}^m$, any 2 closed submflds have intersect^n # $= 0$. (see Ex. #15)

Pf:- We'll work in $\mathbb{R}^2$, but works for $\mathbb{R}^m$.

Translat^n is an isotopy in $\mathbb{R}^2$ (not always, e.g. on Σ_2, it's not.) We'll translate the 2 loops so far that they're disjoint!



$K_1 \circ L_1 = 0$

$\Rightarrow K_0 \circ L_0 = 0$

By thm #17.

$\square$.

Corollary #24: [Generalizat^n of Jordan Curve Theorem]

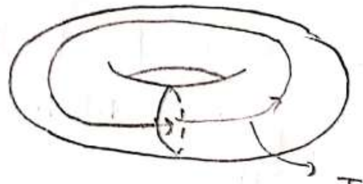
- In $\mathbb{R}^m$, every closed (compact w/o boundaries) hypersurface (of dim $m-1$) separates $\mathbb{R}^m$ into two parts.

- This is not obvious, even in $\mathbb{R}^2$.

does this divide $\mathbb{R}^2$ in inside & outside?

In gen. spaces, it's not even true. e.g. T^2.

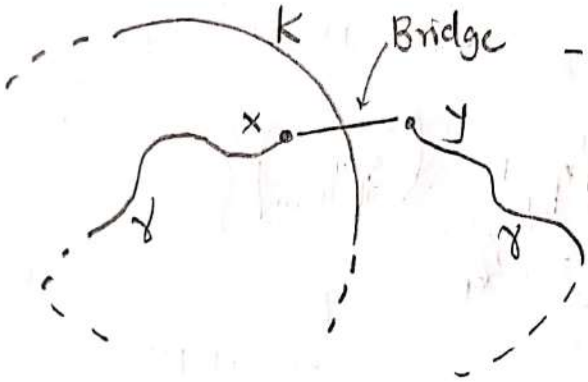
e.g.



It's a loop, but doesn't separate the ambient in 2 parts!

I can just come back!

PF:- By contradict^n. Assume not. let K be a hypersurface (of dim $m-1$) that does not separate $\mathbb{R}^m$ in 2 parts.



- We can take 2 pts x, y opp. each other across K s.t. x & y can be joined by a curve γ that does NOT cross K.

- Add a bridge b/w x & y. $\{\text{Bridge}\} \cup \gamma$ is a closed loop. K is also a closed hypersurf. $\Rightarrow$ By corr. #23 $K \circ (\text{bridge} \cup \gamma)$ should be $= 0$. But they intersect at most once (on the bridges). $\Rightarrow$ $K \circ (\text{bridge} \cup \gamma) = \pm 1$! Contradict^n. QED.

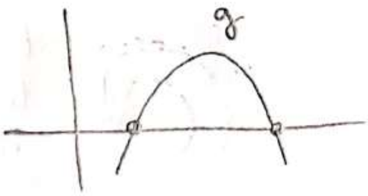
Chap #4: Fixed Point Theorems:

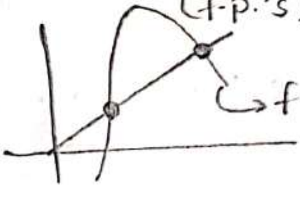
Topology & Geometry
By T^2; lec. # 8.2

Suppose, need to solve eq^n $g(x)=0$.. Can be algebraic, ode, pde,.

put $f(x)=g(x)+x$. Then $g(x)=0 \Leftrightarrow f(x)=x$.

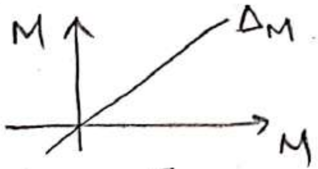
Solving the eq^n $\equiv$ Finding the fixed pts of some map $f(x)$ (f.p.'s)





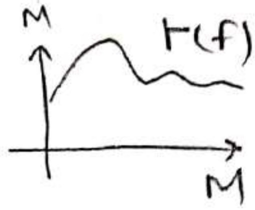
Conventions #1: Given a mfld M oriented locally by a basis of tangent vectors $\{e_1, e_2, \dots, e_m\}$ & a map $f: M \to M$,

- the diagonal $\Delta_M = \{(\underline{x}, \underline{x}) \in M \times M\}$



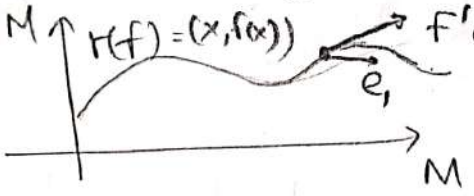
We orient the diag. Δ_M by $\left\{\begin{pmatrix} e_1 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} e_m \\ e_m \end{pmatrix}\right\}$

- the graph $\Gamma(f) = \{(\underline{x}, f(\underline{x})) \in M \times M\}$ by



We orient $\Gamma(f)$ by :- $\left\{\begin{pmatrix} e_1 \\ f'e_1 \end{pmatrix} \cdots \begin{pmatrix} e_m \\ f'e_m \end{pmatrix}\right\}$

Why?



(e_1, tangent) specify the orientat^n.

- We orient $M \times M$ by $\left\{\begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} e_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ e_m \end{pmatrix}\right\}$

2m basis vectors $\because$ $M \times M$ has dim 2m.

Remark #2: $\dim \Delta_M = \dim \Gamma(f) = \dim M = m$; $\dim(M \times M) = 2m$.

- We can play intersect^n theory on Δ_m & $\Gamma(f)$ inside $(M\times M)$!

Consider $z = x+iy$

$\frac{1}{2}z^2 = \frac{x^2-y^2}{2} + ixy$.

map is actualy $z \to z^2$
But in $\mathbb{R}^2$

$\boxed{\text{Ex. \#3:}}$ let $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto \left(\frac{x^2-y^2}{2}, xy\right)$$

we have $f'_{ij} = \dfrac{\partial f_i}{\partial x_j}$

$$f' = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \quad \text{'Jacobian'.}$$

If $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$ orient $\mathbb{R}^2$, then we orient

Δ_M by $\left\{\begin{pmatrix}1\\2\\1\\2\end{pmatrix}, \begin{pmatrix}3\\4\\3\\4\end{pmatrix}\right\}$

$\Gamma(f)$ by $\left\{\begin{pmatrix}1\\2\\ \begin{pmatrix}x & -y\\ y & x\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\end{pmatrix}, \begin{pmatrix}3\\4\\ \begin{pmatrix}x & -y\\ y & x\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}\end{pmatrix}\right\} = \left\{\begin{pmatrix}1\\2\\x-2y\\y+2x\end{pmatrix}\begin{pmatrix}3\\4\\3x-4y\\3y+4x\end{pmatrix}\right\}$

e.g. @ pt. $(x,y) = (5,6)$ these are $\left\{\begin{pmatrix}1\\2\\-7\\16\end{pmatrix}, \begin{pmatrix}3\\4\\-9\\38\end{pmatrix}\right\}$

& $\underset{\mathbb{R}^2\times\mathbb{R}^2}{M\times M}$ by $\left\{\begin{pmatrix}1\\2\\0\\0\end{pmatrix}, \begin{pmatrix}3\\4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\3\\4\end{pmatrix}\right\}$

Def. #4: let M be an oriented mfld & $f: M \to M$ be a map.

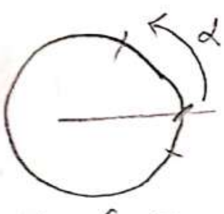
The Lefschetz number of f is

$$\boxed{\Lambda(f) = \Delta_M \circ \Gamma(f)} \text{ in } M \times M.$$

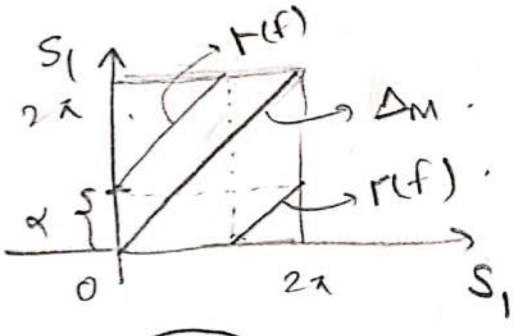
Thm. #5: If M is closed then $\Lambda(f)$ is invariant under deformat^{n}s of f (i.e., under isotopy of $\Gamma(f)$ in M).

PF:- Intersect^n # invarient under isotopy. $\Lambda(f)$ is just a type of ~~I.N~~ - $\Rightarrow$ trivial. (Obvious from Chap. #3. thm #17)

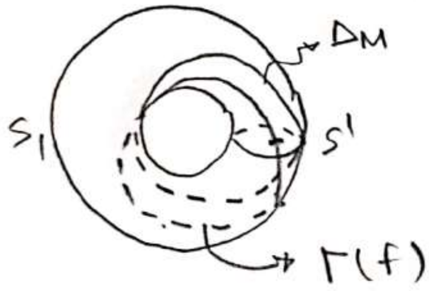
Ex. #6: Rotat^n of S^1 by angle α. $f: \theta \mapsto \theta + \alpha \pmod{2\pi}$



No f.p. $\Rightarrow \Lambda(f)$ should be 0.
(assume $\alpha \notin 2\pi\mathbb{Z}$).



No intersect^{n}s! $\Lambda(f)$ is indeed "0".

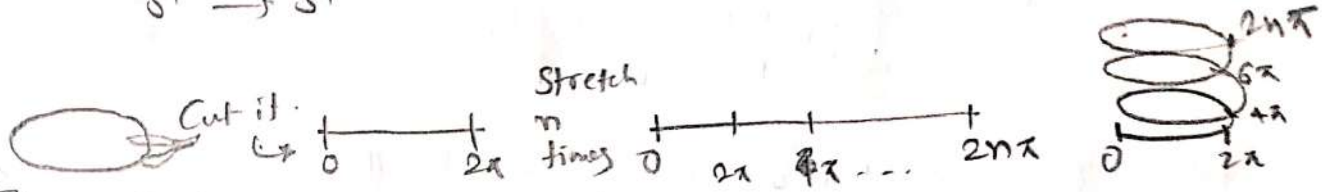


Next, given $n \in \mathbb{Z}$, consider

$f: \theta \mapsto n\theta \pmod{2\pi}$

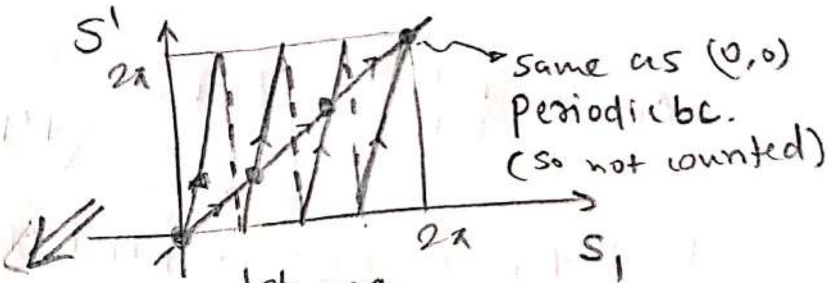
$S^1 \to S^1$

Topology & Geometry
By T^2:- lec. #8.2.



Take a circle

squash →



let n=4.

$(n-1)$ intersect^{n}s.

all with sign $+1$. $\Rightarrow$ $\Lambda(f) = n-1$

$+1$ if $n>0$

-1 if $n\leq 0$.

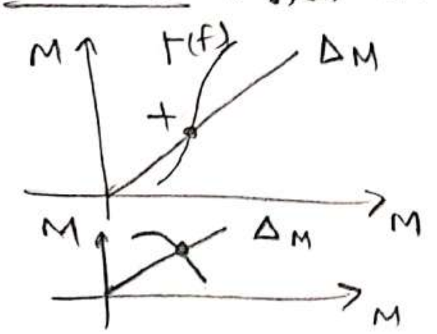
Formula #8:- To calculate $\Lambda(f)$.

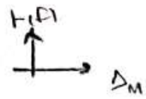
Topology & Geometry
By T^2: lec. #9.1

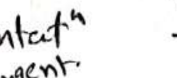
If f is locally expressed in co-ords. as $\begin{pmatrix} f_1(x_1, x_2, \ldots x_m) \\ \vdots \\ f_m(x_1, x_2, \ldots x_m) \end{pmatrix}$, then

$$\Lambda(f) = \sum_{\substack{x_0 \text{ s.t.} \\ \underline{x}_0 = \underline{f}(\underline{x}_0) \\ (\text{f.p.'s of } f)}} \text{sign of } \det(f'(x) - \underbrace{I}_{m \times m \text{ identity}})$$

Picture:- when $m=1$.

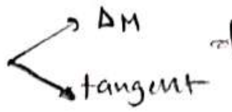


ambient orientat^n 

at f.p. → 

$\Rightarrow$ sign $= +1$.

This corr. to $f'(x) - 1 > 0$. $\Leftarrow$ $\boxed{\Lambda = +1}$



$\Rightarrow$ sign $= -1$ $\boxed{\Lambda = -1}$ $\Rightarrow$ $f'(x) - 1 < 0$

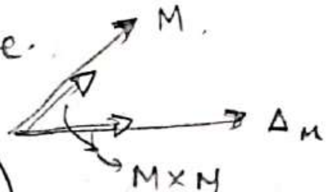
Pf:- let $\{e_1, \dots e_m\}$ orient M. Denote

$$E = \left(e_1 \,\middle|\, e_2 \,\middle|\, \cdots \,\middle|\, e_m \right),$$ E is $m \times m$ matrix.

Then the orientat^{n}s are (convent^n #1). $m \times m$ "O" mat.

$$\Delta_M : \begin{pmatrix} E \\ E \end{pmatrix}, \quad \Gamma(f) : \begin{pmatrix} E \\ f'E \end{pmatrix}, \quad M \times M : \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

Each f.p. $(x, x) \in \Delta_M \cap \Gamma(f)$ contributes $+1$ or -1 accordingly as $\det \begin{pmatrix} E & E \\ E & f'(E) \end{pmatrix}$ & $\det \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$ have the same sign or the opposite.



$$\det \left(\begin{array}{c|c} E & E \\ \hline E & f'(E) \end{array} \right) = \det \left(\begin{array}{c|c} E & E \\ \hline 0 & f'E - E \end{array} \right)$$

$$= \det \boxed{E} \cdot \det (f'E - E) = (\det E)^2 \cdot \det (f' - I) \quad \text{—①}$$

& $\det \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = (\det E)^2$ —②

① & ② have the same or opp sign of $\det (F' - I)$.

$\therefore$ Sign of $\det(f' - I)$ decides ± 1 □.

Ex. #3 revisited: $f : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto \left(\frac{x^2-y^2}{2}, xy\right)$

Topology & Geometry
By T^2:- lec. # 9.1

has 2 f.p. $x,y = \begin{cases} (0,0) \\ (2,0) \end{cases}$ and we shall

compute the signs @ these f.p.'s.

$f' = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ $\quad f'\big|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad f'\big|_{(2,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(f' - I)\big|_{(0,0)} = \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = +1$

$\det(f' - I)\big|_{(2,0)} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = +1$

$\det(f' - I) = \det\begin{pmatrix} x-1 & -y \\ y & x-1 \end{pmatrix} = (x-1)^2 + y^2$, always +ve!

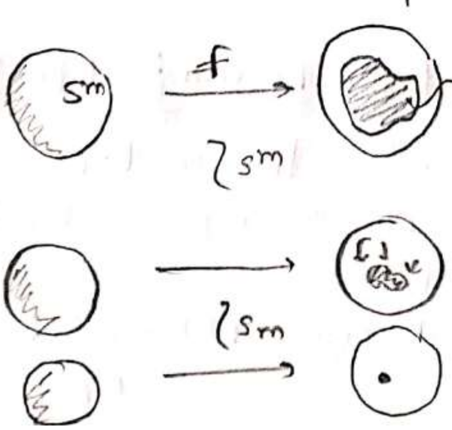
$\Rightarrow \boxed{\Lambda(f) = +1+1 = +2}$

(This makes sense:- $\because \frac{1}{2}z^2 = z$ has 2 simple roots $z = \{0, 2\}$)

Def. #9: A map: $M \to M$ that sends all pts to a single image is called a <u>const. map</u>.

Topology & Geometry
By T^2: lec #9.2

Ex. #10: A non-surjective map $f : S^m \to S^m$ is deformable to a const. map.



$f(S^m)$:- some region which does not fill the whole of S^m.

shrink the image

... down to a single pt.

NOTE: if f is surjective, we cannot start shrinking without tearing the image.

Thm #11: Suppose M closed. If $f: M \to M$ is deformable to some const. map $f_c: M \to c (\in M)$.
Then $\boxed{\Lambda(f) = (-1)^{\dim M} \neq 0}$, so f has a f.p.

Pf: $\Lambda(f) = \Lambda(f_c)$ by Thm #5.
$= \sum_{\substack{x \text{ f.p.} \\ \text{of } f_c}} \text{sign of } \det(f_c' - I)$. by formula #8.

But "c" is the only f.p. of f_c. & $f_c' = 0$ @ c.

$$\therefore \boxed{\begin{aligned} \Lambda(f) &= \text{sign of } \det(-I) \\ &= (-1)^{\dim M} \neq 0 \quad \text{QED.} \end{aligned}}$$

Topology & Geometry By T^2: lec. # 9-3

if $A \Rightarrow B$ then not $A \Rightarrow$ not B.
contrapositive statements.

This thm #11 also shows $\Lambda(f) \neq \pm 1$
$\Rightarrow f$ not deformable to const. map; See Ex #6 Case $n \neq 0$.

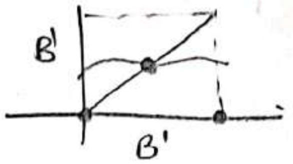
Corollary #12: [Brouwer fixed pt. theorem] (1910).
Every continuous map $f: B^m \to B^m$ has a f.p.

B^m can be replaced by any set $\approx B^m$.

∞-d version: PDE's: Schauder f.p thm.

ex.: $m = 1$

$B^1 =$



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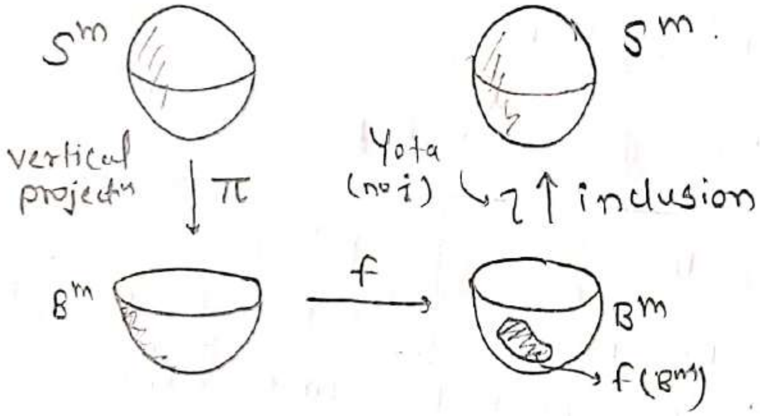
[Brouwer f.p. thm 1910] cntd.

Pf:- We cannot apply Lefschetz directly as $\partial B^m \neq \phi$.
We resort to a trick.
Regard B^m as the southern hemisphere of S^m & consider



By Ex. # 10. The non-surjective composition

$\iota \circ f \circ \pi : S^m \to S^m$

"Now we're talking!"
B^m had a boundary, S^m doesn't!

$\iota \circ f \circ \pi : S^m \to S^m$ is deformable to a const map
$\because \partial S^m = \phi$, so we can apply thm # 11. and find a f.p. $x \in S^m$.

$(*)$ — $x = \iota(f(\pi(x)))$ ... not yet the f.p. we want

But $\iota \circ f \circ \pi$ moves every pt. in the northen hemi.
$\Rightarrow$ x must lie on the southern hemisphere!
$\Rightarrow$ $\pi(x) = x$ $\Rightarrow$ $(*)$ simplifies to

$x = \iota(f(x))$

$\because$ $f(x)$ is in Southern hemi. $\iota(f(x)) = f(x)$!
$\Rightarrow$ $x = f(x)$ $\Rightarrow$ x is a f.p. of "f": $B^m \to B^m$! $\square$.

Topology & Geometry By T^2:- lec. # 10·1

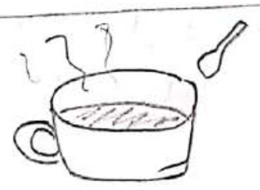
Warning #13: We defined B^m to be $\{(x_1, \dots, x_m) \in \mathbb{R}^m \mid \sum_i x_i^2 \leq 1\}$. The conclusion of Brouwer <u>fails</u> for $\{(x_1, \dots x_m) \in \mathbb{R}^m \mid \sum_i x_i^2 < 1\}$.

e.g. on $(0, 1) = B^1 \setminus \{0, 1\}$.

Consider $x \longmapsto \frac{1}{2}x$ has no f.p.

Applicatⁿs of Brouwer's f.p. thm:

Ex #14:



The surface of coffee consists of many molecules. Stir the coffee.

Motⁿ fairly continuous:- 2 neighbouring molecules stay close. (stir slowly!)

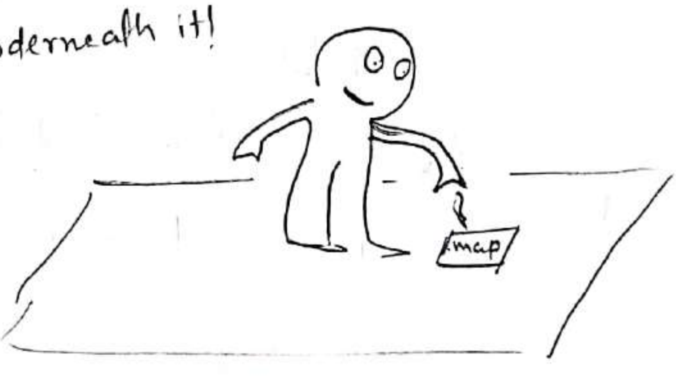
The molecules move to a new positⁿ.

<u>Thm</u>:- (a consequence of Brouwer Cor. #12): $\exists$ molecule which finds itself in the original positⁿ after stirring.

$\because$ it's a conti map from $B^1 \to B^1$!

Ex. #15: Drop a map of Muizenberg (in lec.) the world (in real life) anywhere in Muizenberg / real world.

Thm: $\exists$ a pt on the map that represents <u>exactly</u> the pt on the ground directly underneath it!



Ex. #16: The Brouwer f.p. thm was used in game theory to prove the existence of a Nash equilibrium [Nash, 1950, Nobel prize econ. 1994].

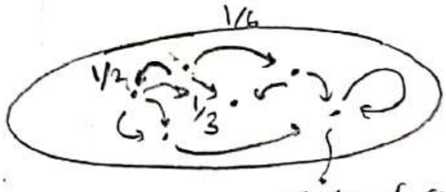
T^2's claim: Assume every zero-sum game has a Nash eq. Then we can prove Brouwer's f.p. thm as a consequence.

∴ Brouwer's f.p. thm ≡ every zero-sum game has a Nash eq.

Application #17: To probability, finite Markov chains.

T^2: "Markov chains are the best kind of stochastic processes."

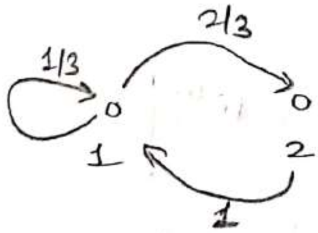
- $i = 1, 2, \dots, m$ label diff. states.
- $P_i(t)$ = prob of being in state "i" at time $t \in \mathbb{Z}$. (discrete time).



State space.

state of a sys. (dynamical sys. pt of view)

- G generator (or transitⁿ) matrix $\equiv m \times m \to G_{ij}$ = prob. of jumping from state "j" to "i"



$$G = \begin{bmatrix} 1/3 & 1 \\ 2/3 & 0 \end{bmatrix}$$

- $\begin{bmatrix} P_1(1) \\ \vdots \\ P_m(1) \end{bmatrix} = G \begin{bmatrix} P_1(0) \\ \vdots \\ P_m(0) \end{bmatrix}$; $\begin{bmatrix} P_1(2) \\ \vdots \\ P_m(2) \end{bmatrix} = G^2 \begin{bmatrix} P_1(0) \\ \vdots \\ P_m(0) \end{bmatrix}, \dots, \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} = G^t \underbrace{\begin{bmatrix} P_1(0) \\ \vdots \\ P_m(0) \end{bmatrix}}_{\text{initial distributⁿ of states}}$

The Ergodic Thm: It turns out that

$\forall$ initial distrib, the chain converges to a unique limit distrib. ("statistical eqm") as $t \to \infty$.

Pf: The limit distrib. is a f.p of mapping G.

[Convergence is not difficult to prove. Inequalities, analysis]

$$\begin{bmatrix} p_1(t+1) \\ \vdots \\ p_m(t+1) \end{bmatrix} = G \begin{bmatrix} p_1(t) \\ \vdots \\ p_m(t) \end{bmatrix}$$

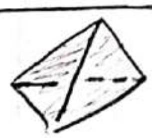
then, $\lim_{t\to\infty} \Rightarrow$ $$\begin{bmatrix} p_1(\infty) \\ \vdots \\ p_m(\infty) \end{bmatrix} = G \begin{bmatrix} p_1(\infty) \\ \vdots \\ p_m(\infty) \end{bmatrix}$$ f.p. of G.

All that is required is to prove existence of this f.p.
or ——"—— of an eigvec. with an eigenvalue 1. ... $G\,p(\infty) = 1\,p(\infty)$ s.t $p_i \geq 0\ \forall i$ & $\sum p_i = 1$

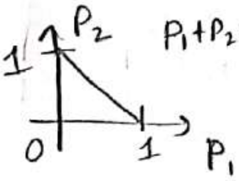
[Perron · Frobenius thm
1907 1912]

The space of all prob. distrib. $\left\{ \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \in \mathbb{R}^m \;\middle|\; p_i \geq 0\ \forall i,\ \sum_i p_i = 1 \right\}$

(can be repres. by a picture!) is a 'tetrahedron' of dim $m-1$.

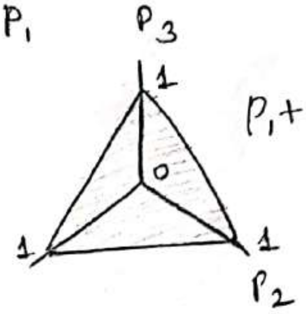


e.g. $m=2$

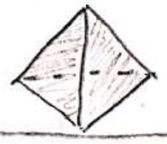


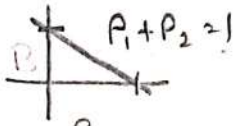
$p_1 + p_2 = 1$.

$m=3$

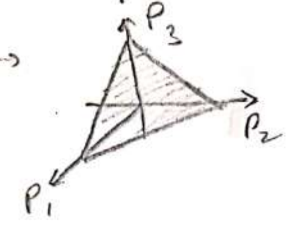


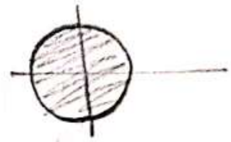
$p_1 + p_2 + p_3 = 1$.

- $\boxed{G \text{ acts on }}$  $\approx B^{m-1}$

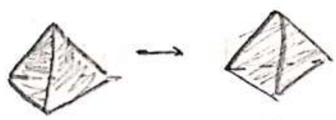
e.g. $m=2 \Rightarrow$ 

$\approx$ $B^1 =$ 

$m=3 \rightarrow$ 

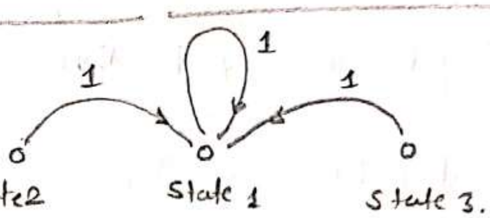
$\approx$ $B^2 =$  & so on.

Apply G to $\underline{p}$ & stay on the surface.

$\Rightarrow$ G :  $\Rightarrow$ $\boxed{G : B^{m-1} \rightarrow B^{m-1}}$

By Brouwer's f.p. thm, $\exists$ a fixed pt! $\square$.

$\boxed{\text{Ex. \#18}}$ $m=3$.



eq^{m} :- everything @ State 1.

$$G = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G^2 = G^3 = \cdots = G^t = G.$$

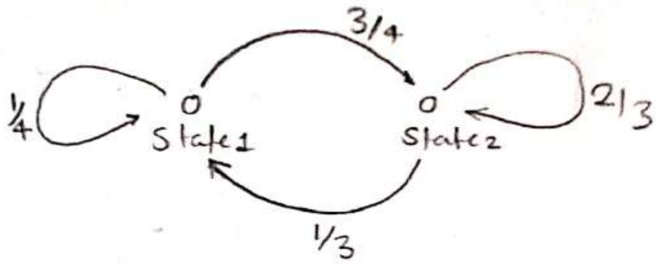
note:-

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a f.p. of G.

$\Rightarrow$ $\forall$ distrib $\begin{pmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{pmatrix}$, $\lim_{t\to\infty} G^t \begin{bmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{bmatrix} = G \begin{bmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{bmatrix} = \begin{bmatrix} P_1(0)+P_2(0)+P_3(0) \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{Ex. \#19:}}$ $m=2$ Two state sys.



$$G = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{pmatrix}$$

easy calc $\begin{pmatrix} 4/13 \\ 9/13 \end{pmatrix}$ is a f.p. of G. $\therefore \forall$ init distrib. $\begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}$

$$\lim_{t\to\infty} G^t \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix} = \begin{bmatrix} 4/13 \\ 9/13 \end{bmatrix}$$

NOTE: The kind of stuff Predrag Cvitanovic keeps emphasizing. We get something "global", like the lim distribut^n, from a local info of how to go to the next step. (G). Cool!

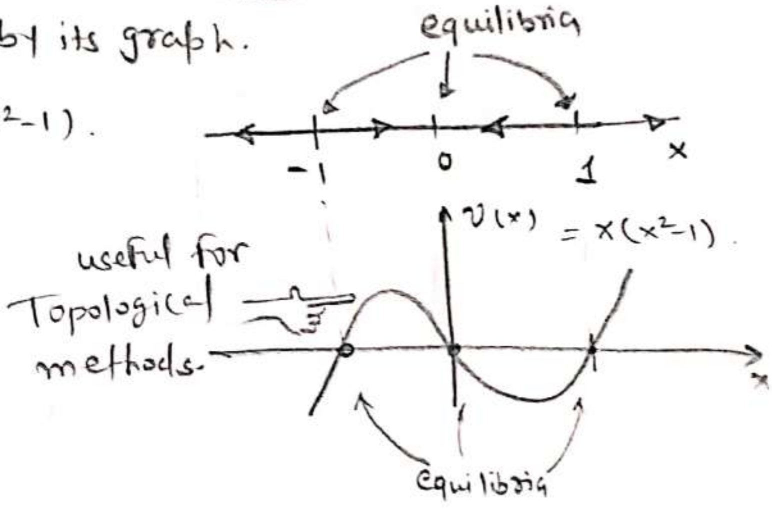
Applicat^n of Topology to Dynamical Sys:

Topology & Geometry By T^2: lec. # 11·1

Chap. # 5: Equilibria in Dynamical Sys:

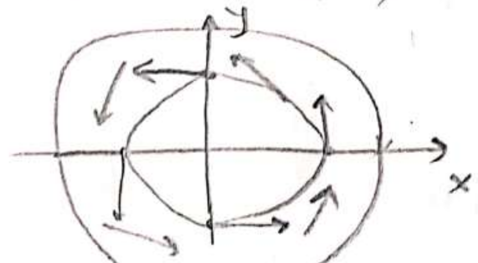
Visualizing a vector field by its graph.

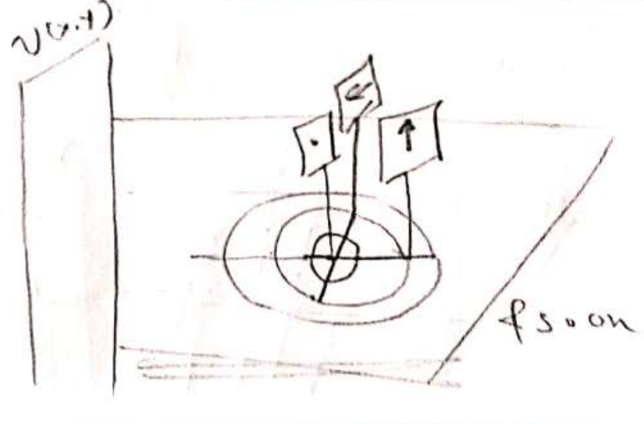
Ex. #1. $\frac{d}{dt} x = v(x) = x(x^2-1)$.



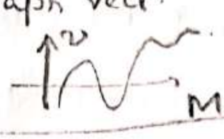
[Ex. #2] In dim > 1, can be tricky.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = v(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$



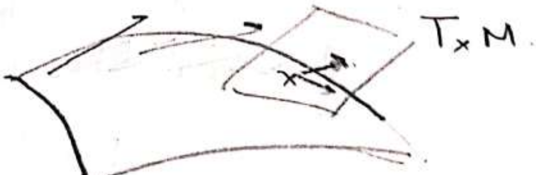


[Pic. #3] Even in dim > 1, we'll graph vect. fields methaphorically like



T^2's wisdom: " well, mathematics is a metaphor."

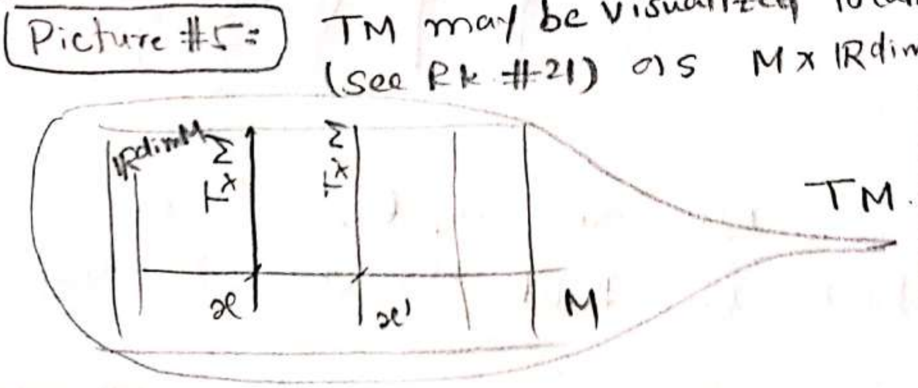
[Def. #4:] M = a mfld & $x \in M$. The collectⁿ of all vectors tangent to M at x. forms "the tangent space of M @ x" denoted $T_x M$.



The <u>tangent bundle</u> of M, denoted TM is the collectⁿ of all the tangent spaces.

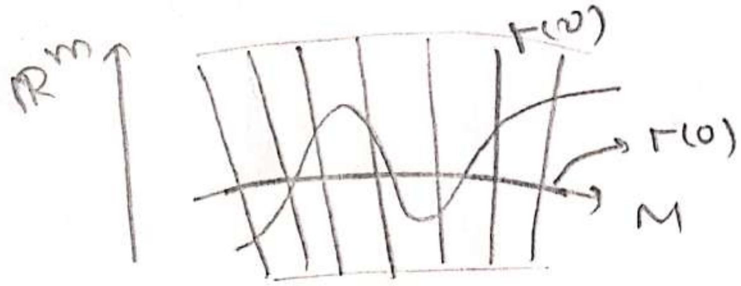
$$\boxed{TM = \bigcup_{x \in M} T_x M}$$

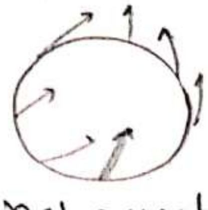
[Picture #5:] TM may be visualized locally (but not nec. globally) (see Rk #21) as $M \times \mathbb{R}^{\dim M}$.



Each $T_x M$ is drawn verti; though in reality $T_x M$ is along the dirⁿ of the tangents @ x.

Recall that by a 'vector field on M' we always mean a vect. field tangent to M. A vect. field v on M can be visualized locally as a graph $\Gamma(v)$ like



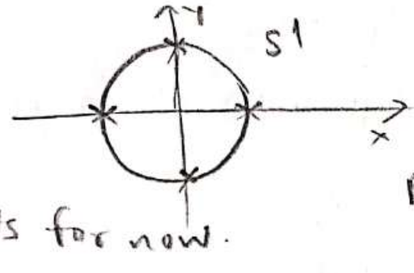


not a vect. field on S^1.

The graph $\Gamma(0)$ $[\Gamma(v=0)]$ of the everywhere "0" vect. field is called the "zero sectⁿ". We have $\Gamma(0) \approx M$. The equilibria of v are the pts of $\Gamma(0) \cap \Gamma(v)$, i.e., the pts x where $v(x) = 0$.

Puzzle #6 :- P.T. generically, a vect. field on S^1 always has an even # of equilibria.

Solⁿ:-



First open up S^1 with periodic BC's.

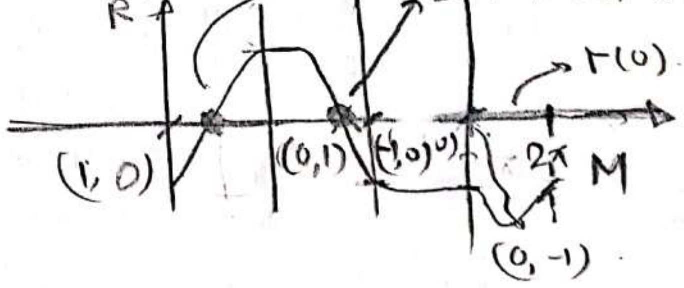
Sample @ 4 pts for now.

let's say

$v(0,1)$ +ve

$v(-1,0)$ −ve

$v(0,-1)$ −ve.



then, in order to satisfy continuity & periodic BC $\Gamma(v)$ will have to be even # of intersectⁿs with $\Gamma(0)$!

Def #7: [compare chap. #4 def. #4]. The <u>index</u> of v @ an eq^m x is ± 1 accordingly as the sign of $\cap$ b/w $\Gamma(0)$ & $\Gamma(v)$ (in this order) at x is ± 1.

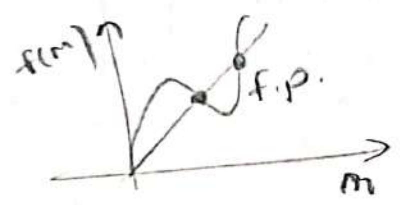
$$\boxed{\sum_{x \text{ eq}^m \text{ of } v} \text{index of } v \text{ at } x = \Gamma(0) \circ \Gamma(v)}$$

As usual, if $\Gamma(0) \underset{TM}{\not\pitchfork} \Gamma(v)$, then perturb vect. field into transversality & then count the index.

Formula #8: [compare chap. #8, formula #8] If v is expressed locally in components as $\begin{bmatrix} v_1(x_1, \cdots x_m) \\ \vdots \\ v_m(x_1, \cdots x_m) \end{bmatrix}$

$$\boxed{\text{index of } v \text{ @ } x = \text{sign of } \det v'}$$

In fixed pt. theory, we're comparing the graph with $\underbrace{y = x}_{\Delta_M}$.



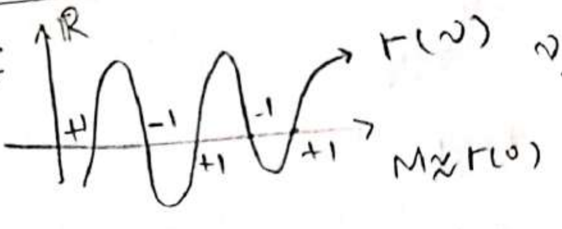
Δ_M has slope 1, so the identity comes there.

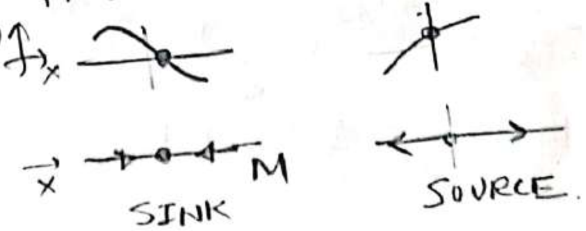
here, we're comparing with slope 0. Hence no id.

---

index $= \pm 1$ corresp. to $v'(x) > 0$ or < 0.

<u>Ex.</u>





(We'll see that a source always has index +1, however sink has index $(-1)^m$). $\to m \equiv \dim M$

<u>Pf:</u> Say M is oriented by a local basis of tangent vectors $e_1, \dots, e_m$. Write $\left(e_1 \mid e_2 \mid \cdots \mid e_m \right) = E$.

Then, the orientat's are given

$$\Gamma(0) : \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad \Gamma(v) = \begin{bmatrix} E \\ v'(x) E \end{bmatrix}, \quad TM : \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

@ each $x \in \Gamma(0) \cap \Gamma(v)$ (eqm), v has index $+1$ or -1 acc. as

$$\det \begin{bmatrix} E & E \\ 0 & v'_{(x)} E \end{bmatrix} = (\det E)^2 (\det v'(x))$$

$\uparrow \Gamma(0)$ $\uparrow \Gamma(v)$

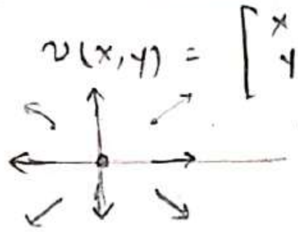
$$\det \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = (\det E)^2$$

if these 2 have the same or opp. signs.

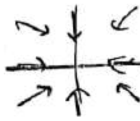
$\boxed{\det v'(x) = +1 \Rightarrow idx = +1 \quad \text{—''—} = -1 \Rightarrow idx = -1}$

$\therefore$ m-d sink $\Rightarrow \det(v') = (-1)^m$ QED. $\square$

$$v(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \det v' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = +1 > 0$$

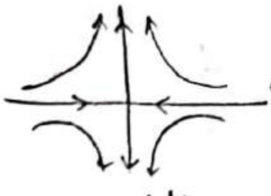


$$v(x,y) = \begin{bmatrix} -x \\ -y \end{bmatrix}$$



$$\det v' = \left|\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right| = +1 > 0 \, !$$

$$v(x,y) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

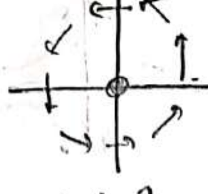


Saddle

$$\det v' = \det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} < 0 \Rightarrow idx = -1$$

$\Rightarrow \boxed{idx = -1}$

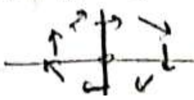
$$v(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$



$$\det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = +1 > 0.$$

$\boxed{idx = +1}$

$$v(x,y) = \begin{bmatrix} y \\ -x \end{bmatrix}$$



$$\det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = +1 > 0$$

$\Rightarrow \boxed{idx = +1}$

Thm #10: [Compare Chap#4 Thm #5].

If M is closed (compact w/o boundary)

then $\left(\sum_{x \text{ eq}^{\underline{m}} \text{ of } v} \text{index of } v \text{ @} x \right)$ is invariant under

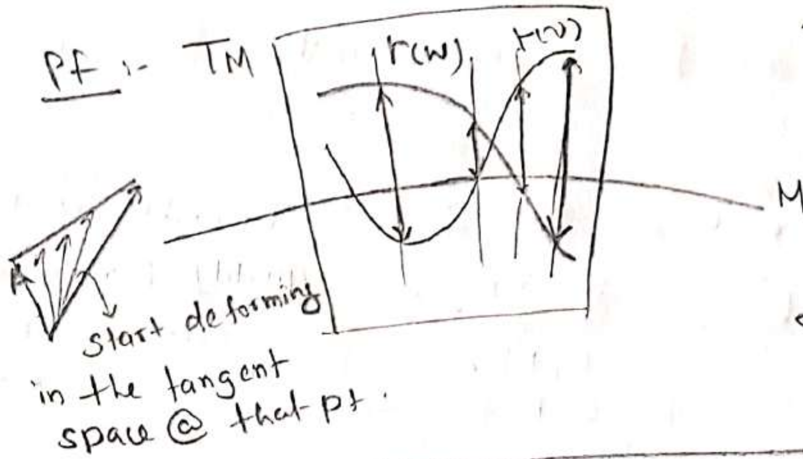
deformat^{n}s (isotopies of $\Gamma(v)$ in TM) of v.

Pf: Obvious from Chap. #3 Thm# 17 □

In fact,

Thm #11: Any two vector fields v & w on mfld M can be deformed into each other. (i.e. $\Gamma(v) \underset{TM}{\simeq} \Gamma(w)$)

Pf:-



Take $(1-s)\Gamma(v) + s\,\Gamma(w)$

$0 \leq s \leq 1$.

continuous deformat^{n} in TM of $\Gamma(v) \underset{TM}{\simeq} \Gamma(w)$ □

(Thm. #10 + Thm #11) ⇒ (Corollary #12:)

If M is closed (compact w/o boundary) then $\left(\sum_{x \text{ eq}^{\underline{m}} \text{ of } v} \text{index of } v \text{ @} x \right)$ does NOT depend on the choice of v, but rather is defined by M itself!

Well then, can we express this Σ directly in terms of information of M?

$\boxed{\text{Def. \#13}}$ **Euler Characteristic:-**

M

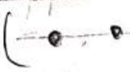
Given a mfld M of dim m,
partition it into a 'network' of

β_0 vertices
β_1 edges
β_2 faces
$\vdots$
β_k = # k-dim 'faces' or cells $\quad 0 \le k \le m$

Require each k-dim cell to be homeomorphic to B^k. The Euler char. of M is

$$\boxed{\chi(M) = \sum_k (-1)^k \beta_k}$$

(We'll prove in thm #15 that $\chi(M)$ is independent of the partition of M).

Ex. #14: $\chi(S^0) = \chi($  $) = \beta_0 = 2.$

$\chi(S^1) = \chi($  $) = \beta_0 - \beta_1 = 3 - 3 = 0$

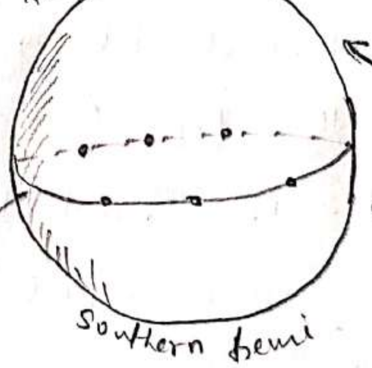
$\chi(S^2) = \chi($  $) = \beta_0 - \beta_1 + \beta_2 = 4 - 6 + 4 = 2$

$= \chi($  $) = 8 - 12 + 6 = 2$

Tadashi's wisdom:
quality of a mathematician
$\propto$ size of pictures they draw!

$\chi(S^m) = ?$

$$\chi(S^m) = \chi(S^{m-1}) + (-1)^m \underbrace{2}_{(2\beta_m)}$$



m	0	1	2	3	4	5	...
$\chi(S^m)$	2	0	2	0	2	0	...

$$\boxed{\chi(S^{\text{even}}) = 2 \quad \chi(S^{\text{odd}}) = 0.}$$

Q. What about S^2

$$\begin{array}{ccccccc} \beta_0 & & \beta_1 & & \beta_2 & & \\ 0 & - & 2 & + & 3 & = & 1 \end{array}$$?! Wrong ans.

as this 'partition' is illegal $\because$ the 'edges'  $\not\cong B^1$

the 'face'  $\not\cong B^2$.

Project: P.T. for mflds M_1, M_2 of the same dim m,

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^{m-1}) - (-1)^m \cdot 2$$

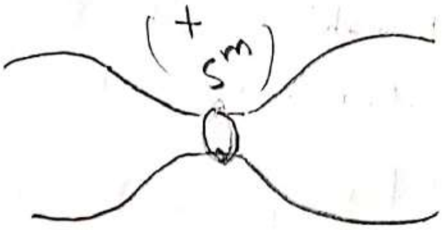
Soln.:- $\rightarrow$

NOTE:- NOT SURE ABOUT THIS. BUT THE PF WILL BE ALONG THESE LINES. OK FOR NOW.

take out m-balls

$\chi(M_1) - \chi(S^m)$ $\quad\quad$ $\chi(M_2) - \chi(S^m)$ $\leftarrow$ we're only concerned about the resulting surface

join.



$$\Rightarrow \begin{aligned} &\chi(M_1 \# M_2) \\ &= \chi(M_1) - \chi(S^m) + \chi(M_2) \\ &\quad - \cancel{\chi(S^m)} + \cancel{\chi(S^m)} \end{aligned}$$

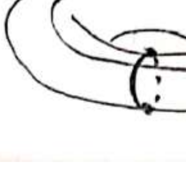
But $\chi(S^m) = \chi(S^{m-1}) + 2(-1)^m$

CAN ALWAYS EXPLICITLY TRIANGULATE & CALCULATE.
Ex. $S^1 \# S^1 = S^1$ ◯ # ◯ calc. $\beta_0, \beta_1, \beta_2$, etc.

$$\Rightarrow \chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^{m-1}) - 2(-1)^m$$

$\square$

Project #2:- $\chi(T^2) = 0$



$$\begin{array}{ccccccc} \beta_0 & - & \beta_1 & + & \beta_2 & & \\ 1 & - & 2 & + & 1 & = & 0 \end{array}$$

Project P.T. $\chi(\mathbb{R}P^2) = 1$

$\beta_0 - \beta_1 + \beta_2$
$2 - 2 + 1 = 1$ $\square$

$\therefore \chi(\Sigma_g) = 2 - 2g \qquad \chi(N_g) = 2 - g$

Project :- P.T. for mflds M_1, M_2, not necessarily of same dim

$$\chi(M_1 \times M_2) = \chi(M_1) \cdot \chi(M_2)$$

$\Rightarrow \chi(T) = 0$ in all dimensions

Internet
NOT DIFFICULT.

Theorem #15: [Poincaré 1881, H. Hopf 1926]
(2d) (md)

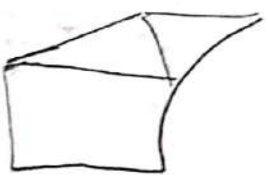
If M is closed, then $\forall$ vect field v on M,

$$\boxed{\sum_{\substack{x \text{ eq}^{\underline{m}} \\ \text{of } v}} \text{index of } v \text{ at } x = \chi(M)}$$

In particular, $\chi(M)$ is well-def^{ned} independently of how we partition M into a 'network' (See Def #13).

Pf: (We'll prove this in 2d; other dim analogous.)

Cor-#12. LHS can be calculated using any v we like.



draw a network

& place
@ each vertex - a source
@ — " — mid-edge - a saddle
@ — " — mid-face - a sink.
Fill the rest of the pic. with a suitable pic

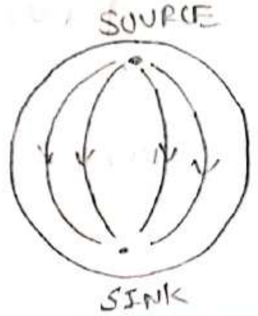
sources & sinks contribute +1
saddles −1

$$\text{LHS} = \underbrace{(+1)\beta_0}_{\substack{\#\text{ sources} \\ = \beta_0}} + \underbrace{(-1)\beta_1}_{\substack{\#\text{ saddles} \\ = \beta_1}} + \underbrace{(+1)\beta_2}_{\substack{\#\text{ sinks} \\ = \beta_2}} = \chi(M) \text{ ! QED.}$$

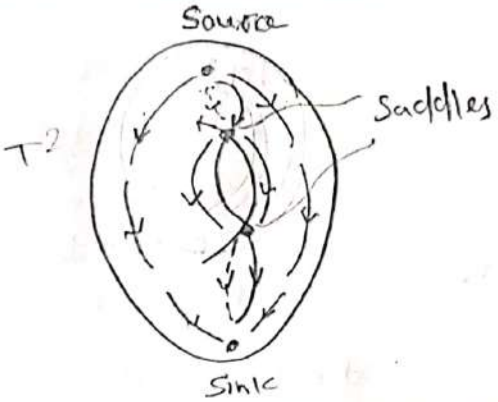
$$\sum_{\substack{x\ \text{eq}^n \\ \text{of } v}} \text{index of } v \ @\ x = \chi(M)$$

LHS → RHS Examples

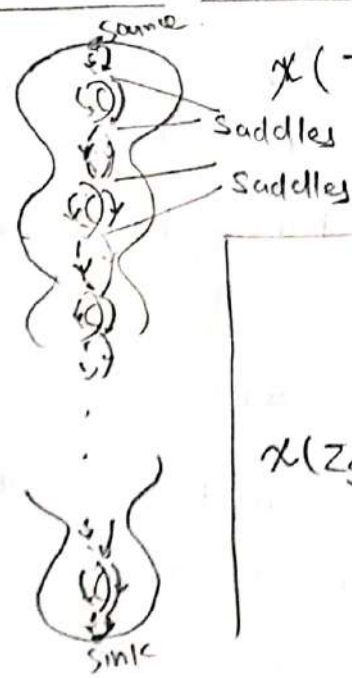
Ex #16: S^2  ($\chi = 2$ → we should deduce this). Consider downward gradient flows (water flowing downhill along surfaces) allow us to find χ.



$$\chi(S^2) = \sum_{\text{eq}^n} \text{index} = +1+1 = 2 \ ✓.$$



$$\chi(T^2) = \sum_{\text{eq}^n} \text{index} = \underset{\text{Source}}{+1} \underset{\text{sink}}{+1} \underset{\text{Saddles}}{-2} = 0$$



$$\chi(\Sigma_g) = +1+1-2g = 2-2g.$$

We can also posit^n Σ_g like this

$$\chi(\Sigma_g) = \underset{\text{sources}}{+g} \underset{\text{sinks}}{+g} \underset{\substack{\text{saddles} \\ \text{due to}}}{-2g} \underset{\substack{\text{saddles} \\ \text{due to}}}{-2(g-1)} = 2-2g\ !$$

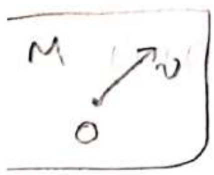
 

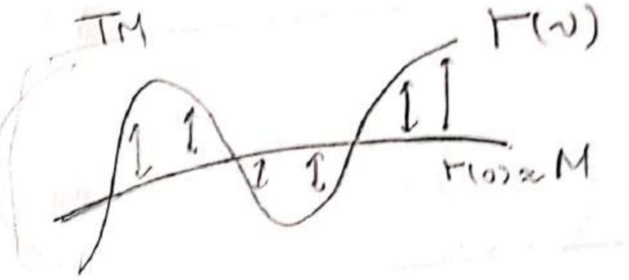
Lemma #17: If M is closed (compact w/o boundary) & $\dim M =$ odd then $\sum_{x \in q^{m}, f \\ v} \text{index of } v \text{ at } x = 0$

$\forall$ vector fields v on M!

<u>Pf:</u> Deform v to everywhere zero vect. field (zero sect^n) & vice versa.

In $M \to$





$s\Gamma(0) + (1-s)\Gamma(v) \quad 0 \le s \le 1.$

$\Gamma(0) \circ \Gamma(v) = \Gamma(v) \circ \Gamma(0)$ ... ① $\because$ can deform v to 0 & 0 to v. & intersect^n # remains the same $\because$ isotopy-invariant (by thm #10) $\because \Gamma(0) \sim \Gamma(v)$ (by thm #11)

But $\Gamma(0) \circ \Gamma(v) = (-1)^{m^2} \Gamma(v) \circ \Gamma(0)$ ... ② $\because \dim \Gamma(v) = m$ & $\dim \Gamma(0) = m$.

m & m^2 have the same parity.

$\Rightarrow$ m even $\Rightarrow m^2$ even $\Rightarrow \Gamma(0) \circ \Gamma(v) = \Gamma(v) \circ \Gamma(0)$.

m odd $\Rightarrow m^2$ odd $\Rightarrow \Gamma(0) \circ \Gamma(v) = (-1)\, \Gamma(v) \circ \Gamma(0)$ ③

from ① & ③ if m odd

$\Rightarrow \Gamma(0) \circ \Gamma(v) = 0$ QED.

Thm #18: for every closed (compact w/o boundary) mfld of odd dim, $\chi(M) = 0$.

<u>Pf:-</u> Combinat^n of lemma #17 & thm #15 (Poincaré-Hopf). $\square$

Generalizes $\chi(S^{odd}) = 0$.

Next, we do LHS ↔ RHS. $\sum \text{index} = \chi(M)$

Thm #19 $v \equiv$ vect. field on a closed (compact w/o boundary), orientable mfld M.

Then the number of equilibria of $v \geq |\chi(M)|$.

(On this LHS, all eqm are counted +vely unlike $r(o)$ or $r(v)$.)
(w/o sign ∵ this is just the # of equilibria).

In particular, every vect. field on S^{even} must have at least 2 equilibria (∵ $\chi(S^{even}) = 2$)

<u>Pf:</u> # of equilibria $= \sum_{\substack{x \text{ eq}^m \text{ of} \\ v}} +1 = \sum_{\substack{x \text{ eq}^m \text{ of} \\ v}} |\text{index of } v \text{ at } x|$

$\geq \left| \sum_{\substack{x \text{ eq}^m \text{ of} \\ v}} \text{index of } v @ x \right|$ ... Δ inequality

$\geq |\chi(M)|$ ... By Poincaré-Hopf.

QED.

Thm #19 is often called "the hairy ball thm" or the impossibility of combing a hedgehog →

comb. →

By thm 19.
have at least 2 equilibria.
can't have combed hair everywhere.

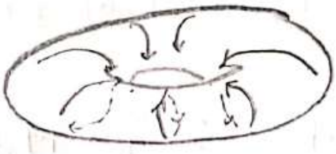
Rk #20: Can be shown that if $\chi(0) = 0$, then M admits an everywhere non-zero vect. field with <u>NO</u> eqm.
[due to Hopf, 1920's].

Examples → next page.

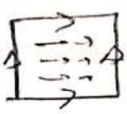
- $\chi(T^2) = 0$

just keep going, no eq^{m}.

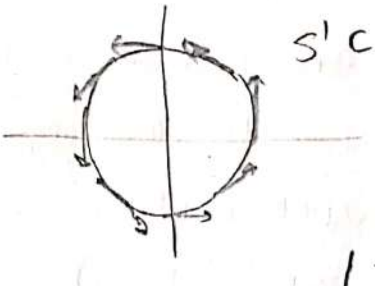
or



same actually



- $\chi(S^{odd}) = 0$



$S^1 \subset \mathbb{R}^2$.

$$v(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

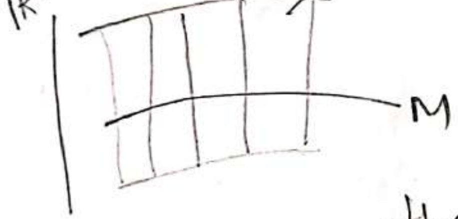
- In general $S^{2m-1} \subset \mathbb{R}^{2m}$

$$v(x_1, y_1, x_2, y_2, \dots, x_m, y_m) = \begin{pmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \\ \vdots \\ -y_m \\ x_m \end{pmatrix}$$

On S^{2m-1}, $\underline{v} \perp \hat{r}$ everywhere $\Rightarrow$ its is tangent to S^{2m-1};

Moreover $v \neq 0$ on S^{2m-1} $\because$ $|v|^2 = \sum_{k=1}^{m} \left((-y_k)^2 + x_k^2\right) = 1 \neq 0$.

$\boxed{\text{Rk \#21:}}$ In pic. #6, we visualized TM locally as $M \times \mathbb{R}^m$ ($m = \dim M$)



but globally, this picture as a direct prod. may be wrong. More precisely, $TU \approx U \times \mathbb{R}^m$ for every small enough open set on M, but it may happen that $TM \not\approx M \times \mathbb{R}^m$.

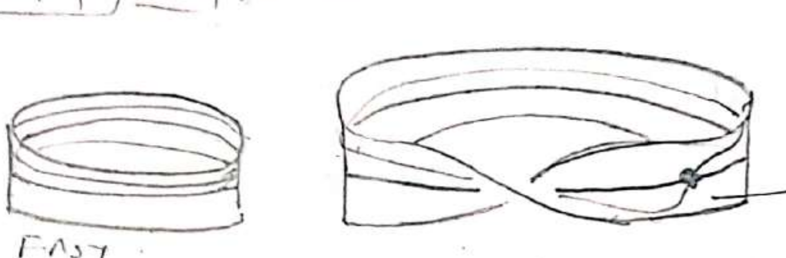
Indeed if $TM \approx M \times \mathbb{R}^m \Rightarrow$ a const vect. field $v(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ would be everywhere non-zero on M. But by thm. #19, we know, such a field v can only exist if $\chi(M) = 0$. But $\exists$ M (plenty) s.t. $\chi(M) \neq 0$!

In fact 'TM' is usually 'twisted.'

How 'twisted' TM is can be measured by things called char. classes, whose most elementary manifestat^n is χ ...

(Rk #21:) Cuttg :- draw a line in the middle.

Topology & Geometry
By T^2:- lec 13.1



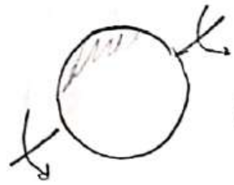
→ have to cross! The 'twisted'.

Easy

draw a line that does not intersect the previous line.

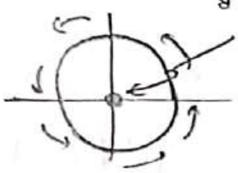
(Applicatⁿ #22:) Every rotatⁿ in $\mathbb{R}^{odd}$ fixes some axis.

Topology & Geometry
By T^2 - lec #14.1



(this statement false in $\mathbb{R}^{even}$).

e.g. in $\mathbb{R}^2$

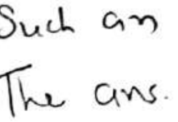


only thing fixed, not an axis.

— Our proof is such a rotatⁿ that induces a vect. field on $S^{even} = S^{odd-1} \subset \mathbb{R}^{odd}$, which, by thm #19, mus thave at least 2 f.p.'s.

, & then the axis joining them is also fixed ∵ rotatⁿ is a Euclidean motⁿ.(rigid).

|Applicatⁿ #23:| Can we design a conti. algorithm A that assigns to each vect field $u \in \mathbb{R}^3$ a non-zero vector $A(u)$ $\perp u$?



→ In $\mathbb{R}^2$ → easy :- rotate by $\pi/2$!

Such an algo. will be useful in comp. graphics.

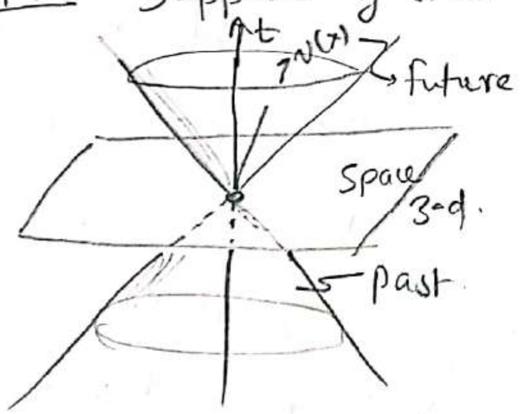
The ans. is NO. If yes, $A(u)|_{u \in S^2}$ would be a conti. everywhere non-zero vect field on S^2; contradicting thm #19 [compare Rk #20].

(Application #24:) Theoretical metereology: At any moment, $\exists$ some pt. on the surface of the Earth ($\approx S^2$) where there is no wind (lull). (in fact, at least 2 pts.). (assuming the atmosphere is S^2)

(Applicatⁿ #25:) In general relativity, a Lorentzian metric g (4×4 symm. matrix of signature $+++-$ that varies conti. from pt to pt & time to time.) defined over a 4d mfld M called space-time.

Claim:- If $\chi(M) \neq 0$ (e.g. $M \approx S^4$), then g must become singular somewhere ('black holes')

Pf:- Suppose g has no singularity, then $\forall x \in M$, we can draw $v(x) \in T_x M$ so that $v(x) \neq 0$ points along the 'avg. future dirⁿ'. But thm #19 forces v to be 0 somewhere on M, a contradictⁿ.



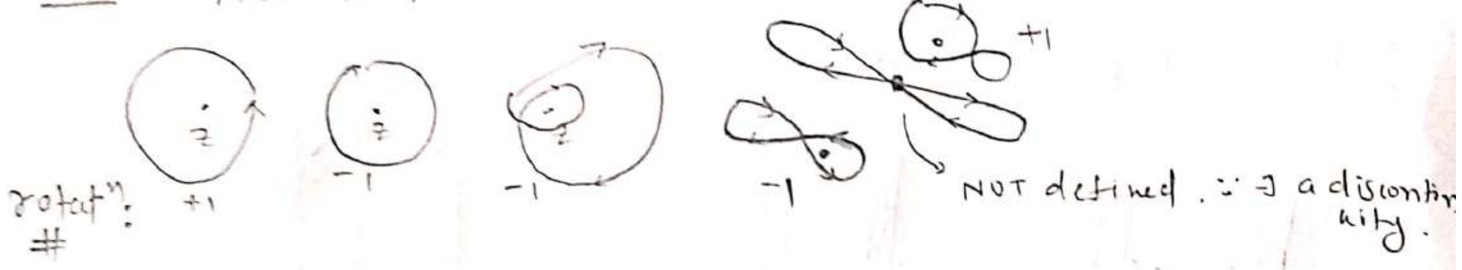
$\square$.

#6 :- Appendix :- The 'Fundamental thm of algebra.'

Topology & Geometry By T^2 - lec #14-2

Thm :- Let $f(z) = c_d z^d + c_{d-1} z^{d-1} + c_1 z + c_0$ be a complex poly of degree $d \geq 1$. Then $\exists\ z \in \mathbb{C}$ s.t. $f(z) = 0$.

Pf :- Assume for contradict^n, $\forall z$, $f(z) \neq 0$.



As $z = re^{i\theta}$ revolves around the origin 0 once

$z^d = r^d e^{id\theta}$ —n—————————— d times, and

$f(z)$ revolves a certain number N_r times around 0.

∵ $f(z)$ doesn't go through origin ($f(z) \neq 0$) we can count N_r.

If $f(z) = 0$, we cannot define N_r. (see above).

N_r well-defined ∵ $f(z) \neq 0$ by assumpt^n.

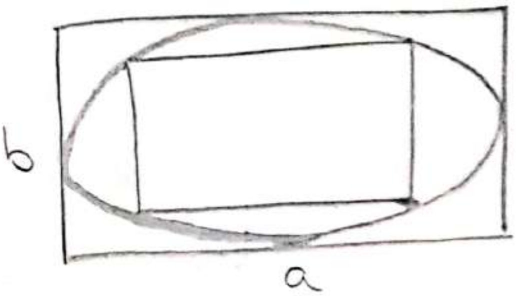
For $r = 0$, $f(z)$ reduces to a single pt. $f(0) \to N_0 = 0$.

for $r \gg 1$, $f(z) \sim c_d z^d \Rightarrow N_r = d$.

But N_r depends continuously on r: it cannot jump from 0 to $d \geq 1$. (an integer), as r varies., a contradict^n.

(∵ the only f^n that takes integer values at integers & is continuous everywhere is a const. f^n.)

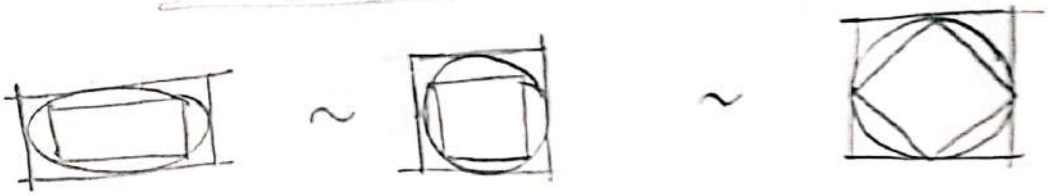
To finish, another puzzle :-



Inside a rect., inscribe an ellipse.
Inside the ellipse, inscribe rect.

Q. What's the $\dfrac{\text{area } \square^{\,b}_{\,a}}{\text{area } \square}$?

Sol^n :- Topological sol^n :-



$$\frac{\text{area } \square}{\text{area } \square} = \frac{\text{area } \square}{\text{area } \square} = \frac{\text{area } \square}{\text{area } \diamond} = 2 \ !$$

Lec. #15 is mainly summary of the course & it's better when T^2 recapitulates the course.

NOTES BY - Pratik Aglor (UNH, 2020).

$\square$