HW #1: Basic Fluid Mechanics

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1 Q 1: Constitutive Relationships

Hypothesized:

- The deviotric stress tensor d_{ij} is only a function of gradients in velocities and not the velocities themselves (Galilean invariance): $d_{ij} \equiv d_{ij}(\frac{\partial u_k}{\partial x_l})$.
- Not just that, it is a linear function (details in the handout for the time-scales argument): $d_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l}$.
- Isotropic: $A_{ijkl} = \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk} + \mu''\delta_{ij}\delta_{kl}$.

(a): $\mu = \mu'$:

Since σ_{ij} is symmetric, so is d_{ij}.

$$\begin{aligned} d_{ij} &= (\mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk} + \mu''\delta_{ij}\delta_{kl})\frac{\partial u_k}{\partial x_l} \\ d_{ji} &= (\mu\delta_{jk}\delta_{il} + \mu'\delta_{jl}\delta_{ik} + \mu''\delta_{ij}\delta_{kl})\frac{\partial u_k}{\partial x_l} \\ \because d_{ij} &= d_{ji} \Rightarrow d_{ij} - d_{ji} = 0 \\ &\Rightarrow \boxed{\mu = \mu'}. \end{aligned} \tag{1}$$

(b): $A_{ijkl} = A_{ijlk}$:

$$A_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mu''\delta_{ij}\delta_{kl}$$

$$A_{ijlk} = \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \mu''\delta_{ij}\delta_{kl}$$

By inspection, $A_{ijkl} = A_{ijlk}$.

(c)

Substituting $\frac{\partial u_k}{\partial x_l} = e_{kl} + r_{kl} = e_{kl} - \frac{1}{2}\varepsilon_{klm}\omega_m$, we have:

$$
\begin{aligned}
d_{ij} &= A_{ijkl}(e_{kl} - \frac{1}{2}\varepsilon_{klm}\omega_m) \\
d_{ij} &= A_{ijlk}(e_{lk} - \frac{1}{2}\varepsilon_{lkm}\omega_m) \\
\text{adding the two equations and noting: } & \varepsilon_{klm} = -\varepsilon_{lkm}, e_{kl} = e_{lk}, A_{ijkl} = A_{ijlk} \\
& \boxed{d_{ij} = A_{ijkl}e_{kl}}.
\end{aligned}
\tag{2}
$$

(d)

Finally, $d_{ii} = 0$, since by construction, d_{ij} is deviotric.

$$
\begin{aligned}
d_{ii} &= [\mu(\delta_{ik}\delta_{il} + \delta_{il}\delta_{ik}) + \mu''\cancelto{3}{\delta_{ii}}\delta_{kl}]\frac{\partial u_k}{\partial x_l} \\
0 &= 2\mu\delta_{ik}\delta_{il} + 3\mu''\delta_{kl} \\
& \text{contracting with} \delta_{kl} \\
0 &= 2\mu\delta_{ik}\delta_{il}\delta_{kl} + 3\mu''\delta_{kl}\delta_{kl} \\
0 &= 2\mu\cancelto{\delta_{ll}=3}{\delta_{il}}\delta_{il} + 3\mu''\cancelto{3}{\delta_{kk}} \\
0 &= 2\mu + 3\mu'' \\
& \boxed{\mu'' = -\frac{2}{3}\mu}.
\end{aligned}
\tag{3}
$$

(e)

Putting it all together:

$$
\begin{aligned}
d_{ij} &= A_{ijkl}e_{kl}, \\
&= \mu\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)e_{kl}, \\
&= 2\mu e_{ij} - \frac{2}{3}e_{kk}\delta_{ij}, \\
&= 2\mu\left[e_{ij} - \frac{\Delta}{3}\delta_{ij}\right].
\end{aligned}
\tag{4}
$$

$$\boxed{d_{ij} = 2\mu\left[e_{ij} - \frac{\Delta}{3}\delta_{ij}\right]}.$$

2 Q 2: Energy Equation:

We want to apply the first law of thermodynamics to obtain the energy balance for a material volume of fluid.

(a)

$$de = \delta q + \delta w. \tag{5}$$

Here

$$e = \frac{1}{2}u_i u_i + \hat{u}, \tag{6}$$

where e is the total energy (kinetic + potential), $\hat{u}$ is the internal energy, de is a small change in e and $\delta q, \delta w$ are infinitesimal heat added to and work done on the system. An alternative form of 5 is

$$\frac{DE}{Dt} = \dot{Q} + \dot{W}, \tag{7}$$

where E is the total energy of the material volume and $\dot{Q}$ and $\dot{W}$ represent the rates of energy transferred to the system as heat or work.

To determine $\dot{Q}$, we assume that there are no heat sources or sinks and neglect radiative transfer. The only mode of heat transfer is conduction and we assume **Fourier's law** of heat conduction, which relates the heat flux per unit area per unit time ($\dot{q}_{ci}$ with conductivity (κ, a material property) and the local temperature T.

$$\dot{q}_{cj} = -\kappa \frac{dT}{dx_j}. \tag{8}$$

For a surface S, consider an infinitesimal surface dS. If the normal points out in the n_j direction,

$$\begin{aligned}\dot{Q} &= -\int_S \dot{q}_{cj} n_j dS \\ &= \int_S \kappa \frac{\partial T}{\partial x_j} n_j dS \\ &= \int_V \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) dV \text{ ...Gauss's divergence theorem.}\end{aligned} \tag{9}$$

The rate of work done involves surface and body (limited to gravity) forces:

$$\begin{aligned}\dot{W} &= \int_S \sigma_{ij} u_i n_j dS + \int_V \rho g_i u_i dV \\ &= \int_V \left[\frac{\partial}{\partial x_j}(\sigma_{ij} u_i) + \rho g_i u_i\right] dV.\end{aligned} \tag{10}$$

Therefore, we obtain the equation for the total energy:

$$\begin{aligned}
\rho \int_V \frac{De}{Dt} dV &= \int_V \left[\frac{d}{d_j} \left(\kappa \frac{dT}{dx_j} \right) + \frac{\partial}{\partial x_j} (\sigma_{ij} u_i) + \rho g_i u_i \right] dV \\
&\boxed{\rho \frac{de}{dt} = \frac{d}{d_j} \left(\kappa \frac{dT}{dx_j} \right) + \frac{\partial}{\partial x_j} (\sigma_{ij} u_i) + \rho g_i u_i} \text{ ... Since dV arbitrary.}
\end{aligned} \tag{11}$$

(b)

To obtain the equation for the kinetic energy of the fluid, we take a dot product of the Cauchy's equations of motion with u_i and use the fact that $u_i \frac{Du_i}{Dt} = \frac{1}{2} \frac{Du_i u_i}{Dt}$.

$$\begin{aligned}
u_i \left[\rho \frac{Du_i}{Dt} = \frac{\partial}{\partial x_j} \sigma_{ij} + \rho g_i \right] \\
\rho \frac{1}{2} \frac{Du_i u_i}{Dt} = u_i \frac{\partial}{\partial x_j} \sigma_{ij} + \rho g_i u_i.
\end{aligned} \tag{12}$$

Subtracting Eqn.(12) from Eqn.(11), we obtain the equation for the internal energy of a fluid element:

$$\boxed{\rho \frac{D\hat{u}}{Dt} = \frac{\partial}{\partial x_j} \left(\kappa \frac{\partial T}{\partial x_j} \right) + \sigma_{ij} \frac{\partial u_i}{\partial x_j}}. \tag{13}$$

(c)

Writing $\frac{\partial u_i}{\partial x_j} = e_{ij} + r_{ij}$, where $e_{ij} = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$ and $r_{ij} = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right]$ are the symmetric strain rate tensor and the rotation tensor, respectively, we obtain:

$$\begin{aligned}
\sigma_{ij} \frac{\partial u_i}{\partial x_j} &= \sigma_{ij} (e_{ij} + r_{ij}) \\
&= \sigma_{ij} e_{ij}.
\end{aligned} \tag{14}$$

The last term vanishes because σ_{ij} is a symmetric tensor and its dot contraction with an anti-symmetric r_{ij} will be identically zero.

Substituting Eqn.(14) into the internal energy equation, Eqn. (13) and substituting the expression for the stress tensor,

$$\rho \frac{D\hat{u}}{Dt} = \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) + \sigma_{ij}e_{ij} \tag{15}$$

$$= \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) + \left[-p_t\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}\Delta\delta_{ij}\right)\right]e_{ij} \tag{16}$$

$$= \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right) + -p_t\Delta + 2\mu\left(e_{ij}e_{ij} - \frac{1}{3}\Delta^2\right) \tag{17}$$

(d): Writing the internal energy equation in terms of directly measurable quantities using the 2^{nd} law of thermodynamics:

The Gibbs equation is another way of writing the first law of thermodynamics:

$$Tds = d\hat{u} + p_t dv \tag{18}$$

Here, the specific entropy $s \equiv s(\hat{u}, v)$ and the goal is to transform it to $s \equiv s(T, p_t)$, where T and p_t are temperature and thermodynamic pressure, respectively.

We start by introducing the Gibbs potential $g = \hat{u} + p_t v - Ts$. This immediately implies the following:

$$\begin{aligned} dg &= \cancel{d\hat{u}} + \cancel{p_t dv} + v dp_t - \cancel{Tds} - sdT \\ dg &= v dp_t - sdT. \\ \Rightarrow \frac{\partial g}{\partial p_t}\bigg|_T &= v, \ \frac{\partial g}{\partial T}\bigg|_{p_t} = -s \end{aligned} \tag{19}$$

Now, considering $\frac{\partial^2 g}{\partial p_t \partial T} = \frac{\partial^2 g}{\partial T \partial p_t}$

$$\frac{\partial v}{\partial T}\bigg|_{p_t} = -\frac{\partial s}{\partial p}\bigg|_T \tag{20}$$

gives the required Maxwell relation.

Now, since we want $s \equiv s(T, p_t)$,

$$\begin{aligned} ds &= \frac{\partial s}{\partial T}\bigg|_{p_t} dT + \frac{\partial s}{\partial p_t}\bigg|_T dp_t \\ Tds &= T\frac{\partial s}{\partial T}\bigg|_{p_t} dT + T\frac{\partial s}{\partial p_t}\bigg|_T dp_t \\ Tds &= c_p dT - T\frac{\partial v}{\partial T}\bigg|_{p_t} dp_t \\ Tds &= c_p dT - (\beta v T) dp_t \end{aligned} \tag{21}$$

where $c_p = T\frac{\partial s}{\partial T}\Big|_{p_t}$ is the specific heat capacity at constant pressure and $\beta = \frac{1}{v}\frac{\partial v}{\partial T}\Big|_{p_t} dp_t$ is the coefficient of thermal expansion.

Finally, writing $v = 1/\rho$, we obtain the material derivative for entropy.

$$\begin{aligned} T\frac{Ds}{Dt} &= c_p\frac{DT}{Dt} - \frac{\beta T}{\rho}\frac{Dp_t}{Dt} \\ &= \frac{D\hat{u}}{Dt} - \frac{p_t}{\rho^2}\frac{D\rho}{Dt} \\ &= \Phi + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right). \end{aligned} \tag{22}$$

(e):

Assuming that bulk viscous effects are negligible $\lambda = 0$ and $p_t = p$, i.e., thermodynamic and mechanical pressures are equal.This yields the equation for the evolution of temperature.

$$c_p\frac{DT}{Dt} = \frac{\beta T}{\rho}\frac{Dp}{Dt} + \Phi + \frac{1}{\rho}\nabla.(\kappa\nabla T), \tag{23}$$

which must be supplemented by an equation of state $p \equiv p(\rho, T)$ in order to close the system.

Physical Interpretation: The two contributions to the term $\frac{\partial u_i \sigma_{ij}}{\partial x_j}$ are $\frac{\partial u_i \sigma_{ij}}{\partial x_j} = u_i\frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$. The first term arises from the difference in stresses at the two faces of an infinitesimal cube in the fluid and hence contributes to the accelerations of the fluid parcel. This is why it shows up in the expression for the kinetic energy. The other term, $\sigma_{ij}\frac{\partial u_i}{\partial x_j}$ arises from the gradient in velocity, which, as we know, contributes to the deformation of a fluid parcel. Hence it shows up in the internal energy equation.

3 Q 3: $\tau_i = \sigma_{ij}n_j$

Here, I am basically reproducing the calculation in §1.3 from [Batchelor, 2000]. We consider a tetrahedron $OABC$ as shown in the Fig. such that $A(AOC) = \delta A_2, A(AOB) = \delta A_3.A(BOC) = \delta A_1$ and $A(ABC) = \delta A$, with the normals, $-\underline{a}, -\underline{b}, -\underline{c}$ and $\underline{n}$ respectively. Since this is an infinitesimal tetrahedron, we drop the dependence of Σ on $\underline{x}, t$, i.e., $\underline{\Sigma}(\underline{x}, t, \underline{n}) \equiv \underline{\Sigma}(\underline{n})$.

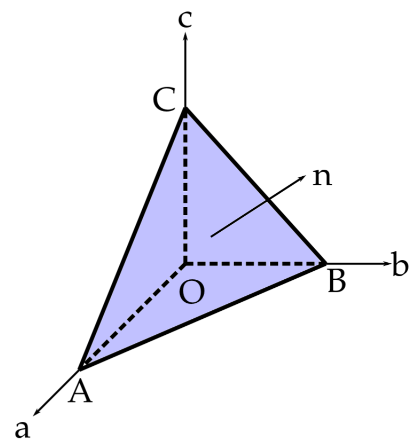

Figure 1: An infinitesimal tetrahedron

At the leading order, surface forces ($O(\delta l^2)$) dominate the body forces ($O(\delta l^3)$) since $\delta l \ll 1$. Therefore, the force balance gives:

$$\begin{aligned}&\underline{\Sigma}(\underline{n})\delta A+\underline{\Sigma}(-\underline{a})\delta A_1+\underline{\Sigma}(-\underline{b})\delta A_2+\underline{\Sigma}(-\underline{c})\delta A_3=0,\\&\underline{\Sigma}(\underline{n})\delta A-\underline{\Sigma}(\underline{a})\delta A_1-\underline{\Sigma}(\underline{b})\delta A_2-\underline{\Sigma}(\underline{c})\delta A_3=0,\\&\underline{\Sigma}(\underline{n})\delta A=\underline{\Sigma}(\underline{a})\delta A_1+\underline{\Sigma}(\underline{b})\delta A_2+\underline{\Sigma}(\underline{c})\delta A_3.\end{aligned}\tag{24}$$

We know that the volume of a tetrahedron is given by

$$\begin{aligned}dV&=\frac{1}{6}\underline{a}\cdot(\underline{b}\times\underline{c})\\&=\frac{1}{6}(\perp\text{ distance})(\text{area of the suface})\end{aligned}\tag{25}$$

Since it is the same tetrahedron, we can express the above formula in 4 different ways, one corresponding to each surface, and all of them must yield the same result. For example $\delta A_1\cdot\underline{a}=\delta A\cdot\underline{n}$, giving $\delta A_1=\delta A(\underline{a}\cdot\underline{n})$. Cancelling factors of 1/6, we obtain:

$$\begin{aligned}\delta A_1&=\delta A(\underline{a}\cdot\underline{n}),\\\delta A_2&=\delta A(\underline{b}\cdot\underline{n}),\\\delta A_3&=\delta A(\underline{c}\cdot\underline{n}),\end{aligned}\tag{26}$$

Substituting Eqn.(26) into Eqn. (24), we obtain

$$\begin{aligned}\underline{\Sigma}(\underline{n})&=[\underline{\Sigma}(\underline{a})\underline{a}+\underline{\Sigma}(\underline{b})\underline{b}+\underline{\Sigma}(\underline{c})\underline{c}]\cdot\underline{n}\\\Sigma_i(\underline{n})&=[\Sigma_i(\underline{a})a_j+\Sigma_i(\underline{b})b_j+\Sigma_i(\underline{c})c_j]\,n_j\end{aligned}\tag{27}$$

The quantity in the brackets on the RHS is a generalization of a second order tensor. Hence, $\boxed{\tau_i(\underline{n})=\Sigma_i(\underline{n})=\sigma_{ij}n_j}$. QED.

4 Q 4: Level Sets of Stream Function ψ:

(a)

First, we start by deriving an equation for a streamline. Streamline is defined as a line everywhere parallel to the velocity field. If $d\underline{s} = dx_i\underline{e_i}$ is the arclength along the streamline, we must have $d\underline{s} \times \underline{u} = 0$.

$$\begin{aligned}(d\underline{s} \times \underline{u})_k &= \varepsilon_{ijk} dx_i u_j \\ \underline{0} &= \varepsilon_{ijk} dx_i u_j \\ \underline{0} &= (w\, dy - v\, dz)e_1 + (w\, dx - u\, dz)e_2 + (v\, dx - u\, dy)e_3\end{aligned} \tag{28}$$

In $2D$, we have $v\, dx - u\, dy = 0$ or $dy/dx = v/u$ to be the equation of a streamline.

Now, let ψ be a stream function in $2D$, such that $\frac{\partial \psi}{\partial x} = -v$ and $\frac{\partial \psi}{\partial y} = u$. Differentiating, we obtain:

$$\begin{aligned}d\psi &= \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy, \\ &= -v\, dx + u\, dy.\end{aligned} \tag{29}$$

For a $\psi = constant$ surface, we have $d\psi = 0$, yielding $-v\, dx + u\, dy = 0$ or $dy/dx = v/u$, which is an equation of a streamline as derived above. Hence, the level sets of a stream function are the streamlines. QED.

(b)

Now, we consider two different streamlines with different values of ψ, say $\psi = \psi_1$ on one streamline and $\psi = \psi_2$ on the other.

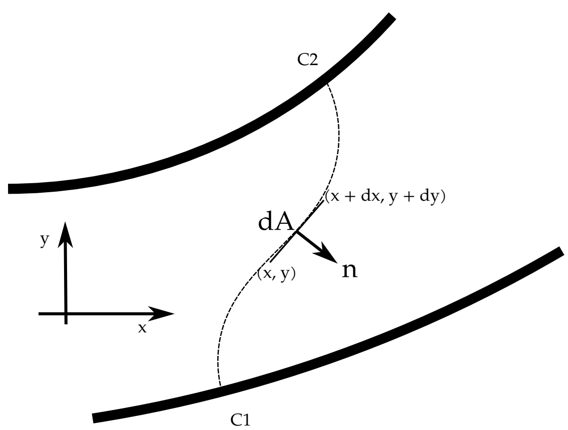

Figure 2

Let C_1C_2 be a curve spanning $\psi = \psi_1$ and $\psi = \psi_2$. Let dA be the infinitesimal area per unit depth (b) on this curve.

By construction, $\hat{n} \perp d\underline{A}$. Since $d\underline{A} \parallel (dx\,\hat{\imath}+dy\,\hat{\jmath})$, we have $\hat{n}dA = dy\,\hat{\imath}-dx\,\hat{\jmath}$. The volume flow rate per unit depth (b) across C_1C_2 is given by

$$
\begin{aligned}
Q = \dot{m}/\rho &= \int_{C_1}^{C_2} \underline{u}\cdot\underline{n}dA \\
&= \int_{C_1}^{C_2} \underline{u}\cdot(dy\,\hat{\imath} - dx\,\hat{\jmath}) \\
&= \int_{C_1}^{C_2} (u\,dy - v\,dx) \\
&= \int_{C_1}^{C_2} d\psi \\
&= \psi_2 - \psi_1.
\end{aligned} \tag{30}
$$

Hence the volume flow rate per unit depth across level surfaces of ψ is given by the difference between the values of ψ on the two level surfaces. QED.

5 Q 5: Decomposing Shear Flow: $\underline{u} = \beta y \underline{e}_x$

We will work in only $2D$ - $x - y$ plane. For this velocity field, the writing decomposing the velocity gradient tensor ($\nabla\underline{u}$) into symmetric (e_{ij}) and anti-symmetric(r_{ij}) parts in the matrix form:

$$
\begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta/2 \\ \beta/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta/2 \\ -\beta/2 & 0 \end{bmatrix}. \tag{31}
$$

Let us focus on the symmetric part e_{ij}. Since e_{ij} is symmetric, it has real eigenvalues and it is diagonalizable.

Finding eigenvalues of e_{ij}:

$$
\begin{aligned}
\det|e_{ij} - \lambda\delta_{ij}| &= 0 \\
\lambda^2 - \beta^2/4 &= 0 \\
\lambda_{1,2} &= \pm\beta/2.
\end{aligned} \tag{32}
$$

The corresponding eigenvectors can be easily calculated $v_1 = [1/\sqrt{2}, 1/\sqrt{2}]$ corresponding to $\lambda_1 = \beta/2$ and $v_2 = [1/\sqrt{2}, -1/\sqrt{2}]$ corresponding to $\lambda_2 = -\beta/2$. In the co-ordinate system of (v_1, v_2), e_{ij} will only have normal components as the eigenvalues.

In other words if $V = [v_1|v_2]$,i.e., if $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, then

$$
V^{-1}eV = \begin{bmatrix} \beta/2 & 0 \\ 0 & -\beta/2 \end{bmatrix}. \tag{33}
$$

This constitues the pure straining components shown with the red arrows in Fig. 3.

To find out the rotation, we use the relation $\omega_i = \frac{1}{2}\varepsilon_{ijk}r_{kj} = -\frac{1}{2}\varepsilon_{ijk}r_{jk}$. This immediately gives $\omega_3 = -\frac{\beta}{2}$, giving the sense of rotation of the flow, which will be clockwise, as indicated by the blue arrows in Fig. 3. The superposition of the two flows recovers back the simple shear flow velocity profile.

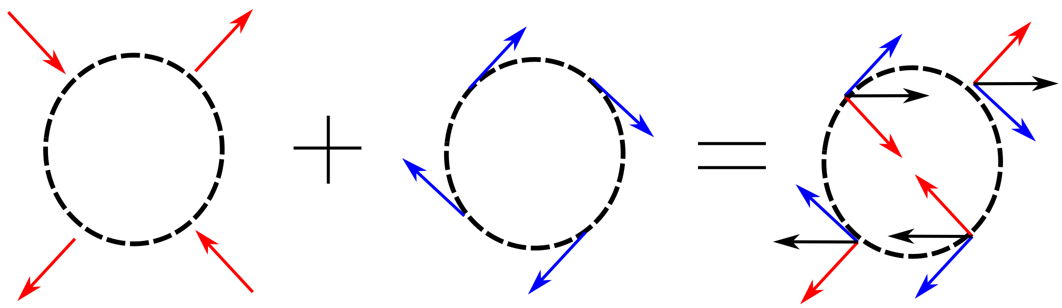

Figure 3: Superposition of pure straining (red) and rotation (blue) yields the simple shear flow.

6 Q 6: Stagnation-point Flow in Eulerian and Lagrangian Coordinates:

Planar stagnation point flow Eulerian representation $\underline{u} = \alpha x \underline{e}_x - \alpha y \underline{e}_y$ for some constant $\alpha > 0$.

(a) Stream function:

Computing the divergence of the flow $\nabla \cdot \underline{u} = \alpha - \alpha = 0$. The flow is also $2D$. Hence a stream function will exist for this flow.

Calculating the stream function ψ: ψ must satisfy: $\frac{\partial \psi}{\partial x} = -\alpha y, \frac{\partial \psi}{\partial y} = -(\alpha x) = -\alpha x$.

Integrating the x-equation, we obtain $\psi(x,y) = \alpha xy + g(y)$ for some unknown $g(y)$. Substituting this into the y-equation gives

$$\begin{aligned} &\alpha x + g'(y) = \alpha x, \\ &\Rightarrow g(y) = c, \\ &\Rightarrow \psi(x,y) = \alpha xy + c, \end{aligned} \tag{34}$$

where c is some arbitrary constant, which we set to 0 without loss of generality, since the physically important fields such as velocities, depend on the gradients of ψ and not ψ itself. Therefore, $\boxed{\psi(x,y) = \alpha xy}$.

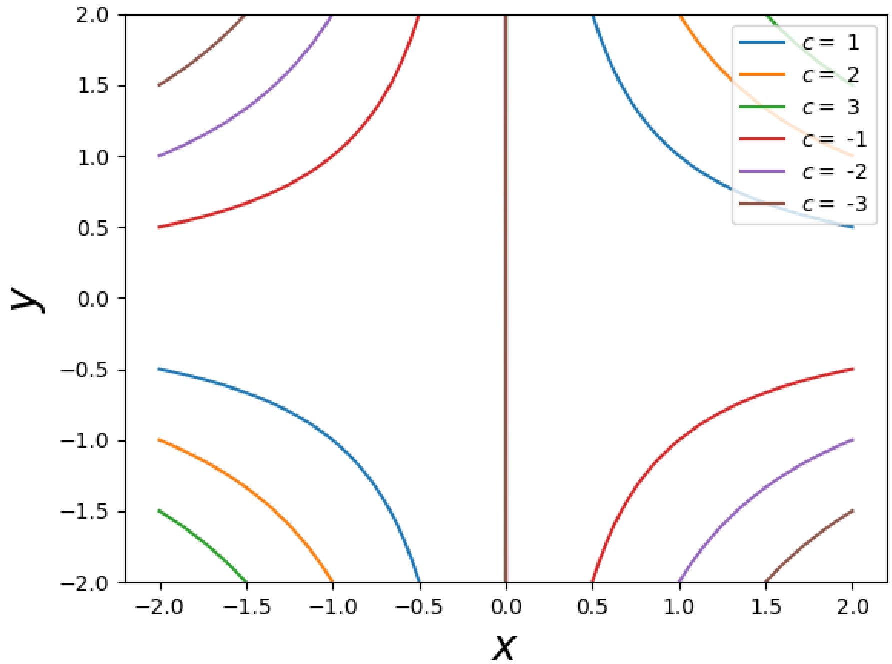

Figure 4: Stagnation-point flow streamlines for $\alpha = 1$.

(b) Incompressible and Irrotational:

1. Incompressible - proved in above subsection $\nabla \cdot \underline{u} = 0$.
2. Irrotational?: Calculating vorticity $\nabla \times \underline{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \underline{e}_z = \underline{0}$. Hence the flow is irrotational.

(c) Material derivative of pollutant concentration:

Given: $c(x, y, t) = \beta x^2 y e^{-\alpha t}$. Rate of change of pollutant concentration of a fluid parcel over time

$$\begin{aligned} \frac{Dc}{Dt} &= \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} \\ &= e^{-\alpha t}\left[\beta x^2 y(-\alpha) + \alpha x \cdot 2\beta x y - \alpha y \cdot \beta x^2\right] \\ &= 0. \end{aligned} \tag{35}$$

Hence the concentration of pollutant for a given fluid parcel remains constant (does not change) over time.

(d) Path lines:

The equation of the path lines will be given by

$$\begin{aligned} \frac{dx}{dt} &= \alpha x \\ \frac{dy}{dt} &= -\alpha y. \end{aligned} \tag{36}$$

(e) Lagrangian representation:

We define $\underline{X}(\underline{x}_0, t)$ to be the Lagrangian co-ordinate of a fluid particle that was at $\underline{x}_0$ at $t = 0$. At time $t = t$, let the particle be at (x, y), the velocity of the particle at that instant must match the Eulerian velocity $\underline{u}(x, y)$ at that spatial location. Therefore, we must have,

$$\begin{aligned} \left.\frac{\partial X}{dt}\right|_{\underline{x}_0} &= \alpha X \Rightarrow X = x_0 e^{\alpha t} \\ \left.\frac{\partial Y}{dt}\right|_{\underline{x}_0} &= -\alpha Y \Rightarrow Y = y_0 e^{-\alpha t} \end{aligned} \tag{37}$$

We have calculated the above derivatives at a spatial point (x, y) or in the Lagrangian terms, when the particle is at (X, Y). We also stress that this is done for a **fixed** (x_0, y_0), meaning we are talking about a specific fluid parcel that was at (x_0, y_0) at $t = 0$ and now (at $t = t$) is at (X, Y). TO obtain the Lagrangian velocity $\underline{U}(X, Y, t)$ at (X, Y), we must differentiate $(X(x_0, y_0, t), Y(x_0, y_0, t))$ twice with respect to (w.r.t.) time.

$$\begin{aligned} U &= \left.\frac{\partial X}{dt}\right|_{\underline{x}_0} = \alpha x_0 e^{\alpha t} \\ V &= \left.\frac{\partial Y}{dt}\right|_{\underline{x}_0} = -\alpha y_0 e^{-\alpha t} \end{aligned} \tag{38}$$

Hence $\boxed{\underline{U}(X, Y, t) = (\alpha x_0 e^{\alpha t}, -\alpha y_0 e^{-\alpha t})}$.

This immediately implies

$$\begin{aligned} \left.\frac{\partial U}{dt}\right|_{\underline{x}_0} &= \alpha^2 x_0 e^{\alpha t} = \alpha^2 X \\ \left.\frac{\partial V}{dt}\right|_{\underline{x}_0} &= \alpha^2 y_0 e^{-\alpha t} = \alpha^2 Y \end{aligned} \tag{39}$$

in terms of (X, Y), $\boxed{\left.\frac{\partial \underline{U}}{\partial t}\right|_{\text{at}(X,Y)} = (\alpha^2 X, \alpha^2 Y)}$.

On the other hand, we can find the material derivative of the Eulerian velocity as follows:

$$\begin{aligned}
\frac{Du}{Dt} &= \overbrace{\cancel{\frac{\partial u}{\partial t}}}^{0} + u\frac{\partial u}{\partial x} + v\overbrace{\cancel{\frac{\partial u}{\partial y}}}^{0}, \\
\frac{Du}{Dt} &= \alpha^2 x \\
\frac{Dv}{Dt} &= \overbrace{\cancel{\frac{\partial v}{\partial t}}}^{0} + u\overbrace{\cancel{\frac{\partial v}{\partial x}}}^{0} + v\frac{\partial v}{\partial y}, \\
\frac{Dv}{Dt} &= \alpha^2 y
\end{aligned} \tag{40}$$

Comparing at $(X,Y) = (x,y)$, we obtain $\boxed{\frac{\partial \underline{U}}{\partial t}\Big|_{(X,Y)} = \frac{D\underline{u}}{Dt}}$.

For the concentration of the pollutant, the expression in part (c) is an Eulerian expression, in that it gives the concentration as a function of (x,y,t). If, however, we are following the same fluid particle which was at $\underline{x}_0$ at $t=0$ and is now at (X,Y), we can write a Lagrangian version for that fluid parcel at the point (X,Y).

$$\begin{aligned}
C(X,Y,t) &= \beta X^2 Y e^{-\alpha t} \\
&= \beta x_0^2 \cancel{e^{2\alpha t}} \cdot y_0 \cancel{e^{-\alpha t}} \cdot \cancel{e^{-\alpha t}} \\
&= \beta x_0^2 y_0.
\end{aligned} \tag{41}$$

This immediately implies that $\frac{\partial C}{\partial t}\big|_{X,Y} = 0$, confirming that the concentration of a particular blob of fluid does not change with time!

7 Q 7: Free Surface of a Rotation Bucket of Water:

(a) Failure of the Bernoulli Argument:

The Bernoulli argument fails because although the flow is inviscid, it is not irrotational. Hence expecting the value of C to be the same across streamlines will lead to erroneous conclusions.

(b) Correct pressure distribution from Euler's Equations:

The velocity field is given by $\underline{u} = -\Omega y\hat{e}_x + \Omega x\hat{e}_y + 0\hat{e}_z$.Writing Euler's equations in x and y directions:

$$
\begin{aligned}
\rho \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} \\
\rho \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} \\
\rho \frac{Dw}{Dt} &= -\frac{\partial p}{\partial z} + \rho g
\end{aligned} \tag{42}
$$

In the context of the given velocity field, Euler's equations take the following form:

$$
\begin{aligned}
-\rho \Omega^2 x &= -\frac{\partial p}{\partial x} \\
-\rho \Omega^2 y &= -\frac{\partial p}{\partial y} \\
0 &= -\frac{\partial p}{\partial z} + \rho g
\end{aligned} \tag{43}
$$

Integrating the x equation, we obtain $p(x, y, z) = \rho \Omega^2 x^2/2 + f(y, z)$. Substituting this into the y equation, we get:

$\frac{\partial f}{\partial y} = \rho \Omega^2 y$, giving, $f(y, z) = \rho \Omega^2 y^2/2 + h(z)$, so $p(x, y, z) = \rho \Omega^2 (x^2 + y^2)/2 + h(z)$. Finally, substituting this into the z equation, we obtain: $p(x, y, z) = \rho \Omega^2 (x^2 + y^2)/2 + \rho g z + c$ We set $c = 0$ without loss of generality.

$\boxed{p(x, y, z) = \rho \Omega^2 (x^2 + y^2)/2 + \rho g z}$. Hence, contours of constant pressure are paraboloids.

8 Q 8: Rayleigh Flat Plate Problem with Oscillating Boundary:

We consider a semi-infinite fluid lying above a plane solid boundary at $y = 0$ and initially at rest. At time $t = 0$, the boundary begins to oscillate periodically in its plane (x direction) with $U(t) = U_0 \sin \Omega t$.

We look for a unidirectional flow assuming:

$$
\begin{aligned}
&\boldsymbol{u} = u(y, t)\boldsymbol{e_x}, \text{ where} \\
&u(y, t) = U_0 f(y) \sin \Omega t + U_0 g(y) \cos \Omega t.
\end{aligned} \tag{44}
$$

The Navier-Stokes equations reduce to the diffusion equation:

$$
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial t^2}. \tag{45}
$$

We solve the problem, noting that the governing equation is linear and real. Define $U(y, t)$ to be some complex velocity field. Express the boundary condition for U as $U(0, t) = U_0 \exp i\Omega t$. Assume $U(y, t) = U_0 h(y) \exp i\Omega t$, then the BC at $y = 0$ gives $h(0) = 1$.

The the solution to the problem will be given by $u(y,t) = \mathrm{Im}(U(y,t))$. Substituting $U(y,t) = U_0 h(y) \exp i\Omega t$ in Eqn.(45):

$$\begin{aligned}
U_t &= \nu U_{yy} \\
i\Omega h &= \nu h_{yy} \\
h_{yy} - \frac{i\Omega}{\nu} h &= 0 \\
h(y) &= A \exp\left(\sqrt{\frac{i\Omega}{\nu}}\right) y + B \exp\left(\sqrt{\frac{-i\Omega}{\nu}}\right) y \\
\because \sqrt{i} &= [\exp(i\pi/2)]^{1/2} = \exp(i\pi/4) = \frac{1+i}{\sqrt{2}} \\
&= A \exp\left(\frac{1+i}{\sqrt{2}} \sqrt{\frac{\Omega}{\nu}} y\right) + B \exp\left(-\frac{1+i}{\sqrt{2}} \sqrt{\frac{\Omega}{\nu}} y\right) \\
\because h(0) &= 1, h(\infty) = 0, \\
h(y) &= \exp\left(-\frac{1+i}{\sqrt{2}} \sqrt{\frac{\Omega}{\nu}} y\right).
\end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned}
U(y,t) &= U_0 \exp\left(-\frac{1+i}{\sqrt{2}} \sqrt{\frac{\Omega}{\nu}} y\right) \exp i\Omega t \\
&= U_0 \exp\left(-\sqrt{\frac{\Omega}{2\nu}} y\right) \exp\left[i\left(-\sqrt{\frac{\Omega}{2\nu}} y + \Omega t\right)\right] \\
u(y,t) &= U_0 \exp\left(-\sqrt{\frac{\Omega}{2\nu}} y\right) \sin\left(-\sqrt{\frac{\Omega}{2\nu}} y + \Omega t\right).
\end{aligned} \tag{47}$$

I wrote a small python code to plot the solution at different instances during one oscillation of the plate.

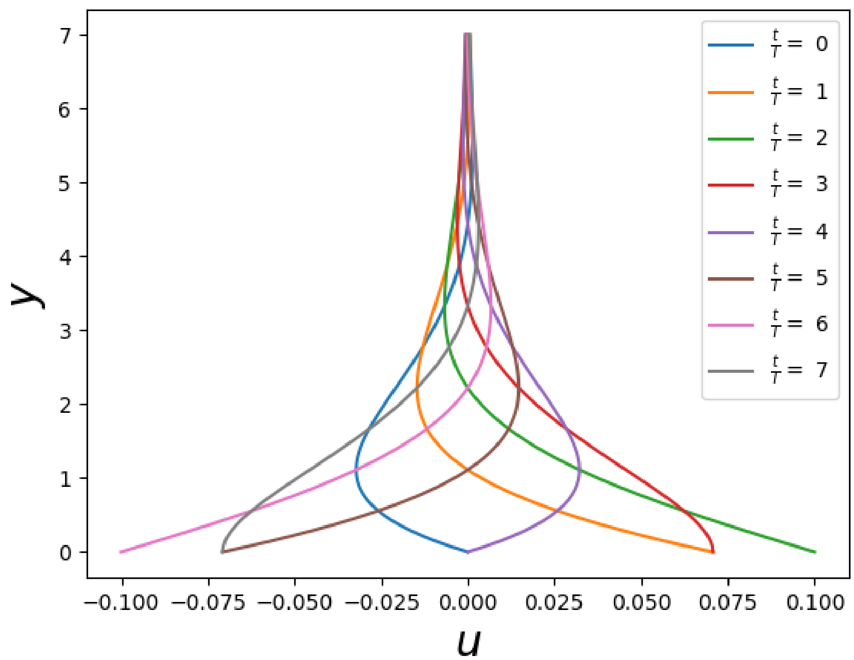

Figure 5: $T = \frac{2\pi}{\Omega}$. Sample velocity profiles for 8 different instances of time, for $\Omega = 1$, $\nu = 1$, $U_0 = 0.1$.

We can calculate the vorticity as follows:

$$\omega_z = \cancelto{0}{\frac{\partial v}{\partial x}} - \frac{du}{dy}$$

$$\omega_z = U_0 \left(\sqrt{\frac{\Omega}{2\nu}} \right) \exp\left(-\sqrt{\frac{\Omega}{2\nu}} y \right) \left[\sin\left(-\sqrt{\frac{\Omega}{2\nu}} y + \Omega t \right) + \cos\left(-\sqrt{\frac{\Omega}{2\nu}} y + \Omega t \right) \right] \tag{48}$$

So for all times, since the damping factor $\exp\left(-\sqrt{\frac{\Omega}{2\nu}} y\right)$ is independent of time, the vorticity is constrained to a region of $y \sim O\left(\sqrt{\frac{2\nu}{\Omega}}\right)$.

9 Q 9: Gravity Driven Film Flow:

(a) BCs:

1. **No slip** at $y = 0 \Rightarrow u(x, y = 0, t) = 0$.

2. **No normal-flow** at $y = 0 \Rightarrow v(x, y = 0, t) = 0$.

3. **Kinematic BC:** Consider a material parcel of fluid at the liquid-air interface. Represent the interface by $\mathcal{F}(x, y, t) \equiv y - h(x, t) = 0$. Since the material parcel moves along $mathcalF = 0$ surface, we have

$$\begin{aligned} \frac{D\mathcal{F}}{Dt} &= \frac{\partial \mathcal{F}}{\partial t} + u\frac{\partial \mathcal{F}}{\partial x} + v\frac{\partial \mathcal{F}}{\partial y} \\ 0 &= \cancelto{-1}{\frac{\partial \mathcal{F}}{\partial h}}\frac{\partial h}{\partial t} + u\cancelto{-1}{\frac{\partial \mathcal{F}}{\partial h}}\frac{\partial h}{\partial x} + v\cancelto{1}{\frac{\partial \mathcal{F}}{\partial y}} \\ v(x, h, t) &= \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}. \end{aligned} \tag{49}$$

This holds for both right below and above the surface, yielding 2 kinematic bcs.

$$\begin{aligned} v_a\Big|_{(x,h,t)} &= \frac{\partial h}{\partial t} + u_a|_{(x,h,t)}\frac{\partial h}{\partial x} \\ v_l\Big|_{(x,h,t)} &= \frac{\partial h}{\partial t} + u_l|_{(x,h,t)}\frac{\partial h}{\partial x} \end{aligned} \tag{50}$$

Another kinematic boundary condition that we require (because we are modeling air as a viscous fluid) is the continuity of the tangential velocity.

$$\underline{u_l} \cdot \hat{t}\big|_{y=h^-} = \underline{u_a} \cdot \hat{t}\big|_{y=h^+} \tag{51}$$

4. **Dynamic BC:** We neglect surface tension. First, we derive the expressions for the normal ($\hat{n}$) and tangent ($\hat{t}$) vectors at the surface. Since the surface is given by $\mathcal{F} = 0$, the normal will be parallel to $\nabla\mathcal{F}$, i.e.,

$$\begin{aligned} \hat{n} &\parallel \left[\frac{\partial \mathcal{F}}{\partial x}, \frac{\partial \mathcal{F}}{\partial y}\right] \\ &\parallel \left[-\frac{\partial h}{\partial x}, 1\right] \\ \hat{n} &= \frac{-\partial h/\partial x}{\sqrt{(\partial h/\partial x)^2 + 1}}\hat{e}_x + \frac{1}{\sqrt{(\partial h/\partial x)^2 + 1}}\hat{e}_y. \end{aligned} \tag{52}$$

To construct the unit tangent vector $\hat{t}$, we parametrize the surface $\mathcal{F}$ by $[x, h(x, t)]$. Then $\hat{t} \parallel \left[1, \frac{\partial h}{\partial x}\right]$. Also, the tangent vector will be perpendicular to $\hat{n}$. Normalizing:

$$\hat{t} = \frac{1}{\sqrt{(\partial h/\partial x)^2 + 1}}\hat{e}_x + \frac{\partial h/\partial x}{\sqrt{(\partial h/\partial x)^2 + 1}}\hat{e}_y. \tag{53}$$

Writing the stress balance in the normal direction

$$
\begin{aligned}
&n_i(\sigma_{l_{ij}} n_j) = n_i(\sigma_{a_{ij}} n_j) \\
&n_1\sigma_{l_{11}} n_1 + n_1\sigma_{l_{12}} n_2 + n_2\sigma_{l_{21}} n_1 + n_2\sigma_{l_{22}} n_2 = n_1\sigma_{a_{11}} n_1 + n_1\sigma_{a_{12}} n_2 + n_2\sigma_{a_{21}} n_1 + n_2\sigma_{a_{22}} n_2 \\
&\sigma_{l_{11}}(\partial h/\partial x)^2 + 2\sigma_{l_{12}}(-\partial h/\partial x) - \sigma_{l_{22}} = \sigma_{a_{11}}(\partial h/\partial x)^2 + 2\sigma_{a_{12}}(-\partial h/\partial x) - \sigma_{a_{22}} \\
&- p_l(\partial h/\partial x)^2 + 2\mu_l e_{l_{11}}(\partial h/\partial x)^2 + 4\mu_l e_{l_{12}}(-\partial h/\partial x) - p_l + 2\mu_l e_{l_{22}} \\
&= -p_a(\partial h/\partial x)^2 + 2\mu_a e_{a_{11}}(\partial h/\partial x)^2 + 4\mu_a e_{a_{12}}(-\partial h/\partial x) - p_a + 2\mu_a e_{a_{22}}.
\end{aligned}
\tag{54}
$$

Writing the stress balance in the tangential direction, we obtain:

$$
\begin{aligned}
&t_i(\sigma_{l_{ij}} n_j) = t_i(\sigma_{a_{ij}} n_j) \\
&t_1\sigma_{l_{11}} n_1 + t_1\sigma_{l_{12}} n_2 + t_2\sigma_{l_{21}} n_1 + t_2\sigma_{l_{22}} n_2 = t_1\sigma_{a_{11}} n_1 + t_1\sigma_{a_{12}} n_2 + t_2\sigma_{a_{21}} n_1 + t_2\sigma_{a_{22}} n_2 \\
&\sigma_{l_{11}}(-\partial h/\partial x) + 2\sigma_{l_{12}} - \sigma_{l_{22}}(\partial h/\partial x)^2 = \sigma_{a_{11}}(-\partial h/\partial x) + 2\sigma_{a_{12}} - \sigma_{a_{22}}(\partial h/\partial x)^2 \\
&p_l(\partial h/\partial x) + 2\mu_l e_{l_{11}}(-\partial h/\partial x) + 4\mu_l e_{l_{12}} + p_l(\partial h/\partial x)^2 + 2\mu_l e_{l_{22}}(\partial h/\partial x)^2 \\
&= p_a(\partial h/\partial x) + 2\mu_a e_{a_{11}}(-\partial h/\partial x) + 4\mu_a e_{a_{12}} + p_a(\partial h/\partial x)^2 + 2\mu_a e_{a_{22}}(\partial h/\partial x)^2
\end{aligned}
\tag{55}
$$

All quantities in the above balance are evaluated at the interface (liquid quantities at $y = h^-$ and air at $y = h^+$).

(b) Flat Surface BCs:

We simplify the BCs assuming the surface is flat, i.e., $\partial h/\partial x = 0$.

The tangential stress balance Eqn.(55) reduces to:

$$
\begin{aligned}
\mu_l e_{l_{12}} &= \mu_a e_{a_{12}} \\
\mu_l \left.\frac{\partial u_l}{\partial y}\right|_{y=h} &= \mu_a \left.\frac{\partial u_a}{\partial y}\right|_{y=h},
\end{aligned}
\tag{56}
$$

showing that there is a jump in the normal derivative of the tangential velocity across the interface.

(c) Flat Surface BCs with Free Surface:

Assuming $\partial h/\partial x = 0$ (flat surface) and $\mu_a \ll \mu_l$, the normal component of the dynamic boundary condition, Eqn. (54) reduces to:

$$
p_l - 2\mu_l \left.\frac{\partial v_l}{\partial y}\right|_{y=h} = P_0 \tag{57}
$$

where P_0 is the ambient pressure. The tangential stress balance Eqn.(56) further reduces to

$$
\left.\frac{\partial u_l}{\partial y}\right|_{y=h} = 0, \tag{58}
$$

(d) Unidirectional Solution:

We look for a steady, unidirectional solution $\underline{u} = u(x,y)\hat{e}_x$. First, we write the governing Stokes equations:

$$\frac{\partial u}{\partial x} + \frac{\partial \cancelto{0}{v}}{\cancel{\partial y}} = 0, \tag{59}$$

$$\frac{\partial u}{\partial x} = 0 \tag{60}$$

$$\Rightarrow u \equiv u(y). \tag{61}$$

$$\cancelto{0}{\frac{1}{\rho}\frac{\partial p}{\partial x}} = g\sin\theta + \nu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right] \tag{62}$$

$$\mu_l \frac{\partial^2 u}{\partial y^2} = -\rho g \sin\theta. \tag{63}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\cos\theta + \nu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}}\right] \tag{64}$$

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta. \tag{65}$$

Hydrostatic balance in the y direction gives $p = -\rho g y \cos\theta + c$. Using the boundary condition Eqn.(57), we require $p = P_0$ at $y = h$, giving $c = P_0 + \rho g h \cos\theta$. Hence the pressure field is given by $\boxed{p = P_0 + \rho g(h-y)\cos\theta}$.

Solving Eqn.(62), we obtain:

$$\begin{aligned}
&u = -\left(\frac{\rho g \sin\theta}{\mu_l}\right)\frac{y^2}{2} + c_1 y + c_2,\\
&u|_{y=0} = 0 \Rightarrow c_2 = 0,\\
&\mu_l \frac{\partial u}{\partial y}\bigg|_{y=h} = 0 \Rightarrow c_1 = \frac{\rho g \sin\theta}{\mu_l},\\
&\boxed{u(y) = \frac{\rho g \sin\theta}{\mu_l}\left[hy - \frac{y^2}{2}\right]}.
\end{aligned} \tag{66}$$

The volume flow rate is given by

$$
\begin{aligned}
q &= \int_0^h u(y)dy \\
&= \frac{\rho g \sin\theta}{\mu_l} \int_0^h hy - \frac{y^2}{2} dy \\
&= \frac{\rho g \sin\theta}{\mu_l} \frac{h^3}{3}
\end{aligned} \tag{67}
$$

Drag D is given by (b is the depth into the paper)

$$
\begin{aligned}
D &= b \int_0^L \tau_1 dx \\
&= b \int_0^L \sigma_{11} \cancelto{0}{n_1} + \sigma_{12} \cancelto{1}{n_2} dx \\
&= \int_0^L \sigma_{12} dx \\
&= b\mu_l \int_0^L \left. \frac{du}{dy} \right|_{y=0} dx \\
&= \mu_l L b \left. \frac{du}{dy} \right|_{y=0} \\
&= \mu_l L b h \frac{\rho g \sin\theta}{\mu_l} \\
&= (\rho g \sin\theta) L b h.
\end{aligned} \tag{68}
$$

References

[Batchelor, 2000] Batchelor, G. K. (2000). *An introduction to fluid dynamics.* Cambridge university press.