# HW #4: Boundary Layer Analysis of Large $Re$ Flows

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## 1 $2d$ Laminar Jet:

Governing equations are the $2d$ Navier-Stokes equations plus the continuity equation.

$$\cancelto{0 \because \text{steady}}{\frac{\partial \underline{u}^*}{\partial t^*}} + [\underline{u}^*.\nabla^*]\underline{u}^* = -\frac{1}{\rho^*}\cancelto{\frac{\partial p}{\partial y}\hat{e}_y \because \text{no imposed pressure gradient in x}}{\nabla^* p^*} + \nu^* {\nabla^*}^2 \underline{u}^*, \tag{1}$$

$$\nabla^* \cdot \underline{u}^* = 0, \tag{2}$$

where stars denote dimensional quantities. We define the dimensionless quantities as follows: $x = x^*/L, Y = y^*/\delta, u = u^*/U_0, V = v^*/V_0$, where $Y, V$ are the boundary layer variables. By continuity, $U_0/L \sim V_0/\delta$, or $\boxed{V_0 = \epsilon U_0}$, with $\epsilon = \delta/L$. We now write the dimensional momentum equations inside the boundary layer (BL) and the relative sizes of terms.

### 1.1 Laminar Jet - Scaling

$$\begin{array}{cccc} u^* \dfrac{\partial u^*}{\partial x^*} & + v^* \dfrac{\partial u^*}{\partial y^*} & = \nu^* \left[ \dfrac{\partial^2 u}{\partial x^2} \right. & + \left. \dfrac{\partial^2 u}{\partial Y^2} \right] \\ \dfrac{U_0^2}{L} & \dfrac{U_0^2}{L} & \dfrac{\nu^* U_0}{L^2} & \dfrac{\nu^* U_0}{\epsilon^2 L^2} \\ 1 & 1 & \dfrac{1}{Re} & \dfrac{1}{\epsilon^2 Re} \end{array} \tag{3}$$

Since physically, we know there is diffusion of vorticity in the $y$ direction and we can see that $\frac{\partial^2 u}{\partial Y^2} \gg \frac{\partial^2 u}{\partial x^2}$, for keeping the diffusion term at the leading order, we must have $\epsilon \sim O(\frac{1}{\sqrt{Re}})$. We choose $\boxed{\epsilon = \dfrac{1}{\sqrt{Re}}}$. Hence, the dimensionless $x$-momentum equation at the leading order takes the following form:

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = \frac{\partial^2 u}{\partial Y^2}. \tag{4}$$

Let us do a similar scaling analysis for the $y$-moementum equation:

$$
\begin{array}{ccccc}
u^* \dfrac{\partial v^*}{\partial x^*} + & v^* \dfrac{\partial v^*}{\partial y^*} = & -\dfrac{1}{\rho^*} \dfrac{\partial p^*}{\partial Y^*} + & \nu^* \left[ \dfrac{\partial^2 v}{\partial x^2} + \right. & \left. \dfrac{\partial^2 v}{\partial Y^2} \right] \\
\dfrac{\epsilon U_0^2}{L} & \dfrac{\epsilon U_0^2}{L} & \dfrac{1}{\rho^*} \dfrac{P}{\epsilon L} & \dfrac{\epsilon \nu^* U_0}{L^2} & \dfrac{\cancel{\epsilon} \nu^* U_0}{\epsilon^{\cancel{2}} L^2} \\
1 & 1 & \dfrac{P^*}{\epsilon^2 \rho^* U_0^2} & \dfrac{1}{Re} & \dfrac{1}{\epsilon^2 Re} \\
1 & 1 & \dfrac{1}{\epsilon^2} & \dfrac{1}{Re} & 1
\end{array}
\tag{5}
$$

choosing $P^* = \rho U_0^2$, we see that at the leading order, the dimensionless $y$-momentum equation reads:

$$
\frac{\partial p}{\partial y} = 0. \tag{6}
$$

This says that the outer pressure is simply impressed inside the BL, which is often the case in BL analyses. Hence, writing the dimensional governing equations:

$$
\begin{aligned}
u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial Y} &= \frac{\partial^2 u}{\partial Y^2} \\
\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} &= 0.
\end{aligned}
\tag{7}
$$

## 1.2 Laminar Jet - Conservation of momentum flux

Now, the LHS of the $x$-momentum equation (Eqn. 4) can be modified as follows:

$$
\frac{\partial (u^2)}{\partial x} - u \cancel{\frac{\partial u}{\partial x}} + \frac{\partial V u}{\partial Y} - u \cancel{\frac{\partial V}{\partial Y}} = \frac{\partial^2 u}{\partial Y^2} \tag{8}
$$

In modifying the LHS, we have used the incompressibility of the jet. Integrating Eqn.(8) and using the boundary and symmetry conditions, we obtain:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left[ \frac{\partial (u^2)}{\partial x} dY + \frac{\partial V u}{\partial Y} = \frac{\partial^2 u}{\partial Y^2} \right] dY, \\
&\frac{\partial}{\partial x} \left[ \int_{-\infty}^{\infty} u^2 dY \right] + \cancelto{0}{[Vu]_{-\infty}^{\infty}} = \cancelto{0 \because \text{ symmetry wrt } Y=0}{\left. \frac{\partial u}{\partial Y} \right|_{-\infty}^{\infty}}, \\
&\boxed{\int_{-\infty}^{\infty} u^2 dY = M},
\end{aligned}
\tag{9}
$$

where $M$ does not vary along the streamwise direction $x$.

## 1.3 Laminar Jet - Similarity Solution:

Since there are no imposed length or time-scales, there is a possibility that a similarity solution might exist. First, we introduce a stream-function $\psi$, such that $u = \psi_Y, v = -\psi_x$. The incompressibility is automatically satisfied and the $x$-momentum equation becomes:

$$\psi_Y \psi_{xY} - \psi_x \psi_{YY} = \psi_{YYY}. \tag{10}$$

We now introduce the similarity ansatz:

$$\psi(x, Y) = F(x) f(\eta), \tag{11}$$

with $\eta = Y/g(x)$. Substituting Eqn.(11) into Eqn.(9), we get a relationship between $F(x)$ and $g(x)$.

$$\begin{aligned} &\int_{-\infty}^{\infty} (\psi_Y)^2 dY = M \\ &\int_{-\infty}^{\infty} F^2 f'^2 \cdot \left(\frac{\partial \eta}{\partial Y}\right)^2 \cdot g d\eta = M \\ &\frac{F(x)^2}{g(x)} \int_{-\infty}^{\infty} f'^2 d\eta = M, \end{aligned} \tag{12}$$

here, primes denote differentiation wrt $\eta$. As suggested in the problem, setting $\int_{-\infty}^{\infty} f'^2 d\eta = 2/3$ gives $\boxed{F(x) = \left(\frac{3M}{2}\right)^{1/2} [g(x)]^{1/2}}$.

Now, we want to substitute Eqn.(11) into Eqn.(10), but before doing so, we evaluate the specific derivatives. Note, we treat $x$ and $\eta$ as independent variables. Dots represent derivatives wrt $x$ and primes represent differentiation wrt $\eta$.

$$\begin{aligned} \psi_X &= \left(\frac{3M}{2}\right)^{1/2} \cdot \frac{1}{2} g^{-1/2} \dot{g} \cdot f \\ \psi_Y &= \frac{F}{g} f' = \left(\frac{3M}{2}\right)^{1/2} \frac{f'}{g^{1/2}}, \\ \psi_{YY} &= \partial_Y \left( \left(\frac{3M}{2}\right)^{1/2} \frac{f'}{g^{1/2}} \right) = \left(\frac{3M}{2}\right)^{1/2} \frac{f''}{g^{3/2}}, \\ \psi_{YYY} &= \partial_Y \left( \left(\frac{3M}{2}\right)^{1/2} \frac{f''}{g^{3/2}} \right) = \left(\frac{3M}{2}\right)^{1/2} \frac{f'''}{g^{5/2}}. \end{aligned} \tag{13}$$

Substituting these in Eqn.(10) and canceling common factors, we obtain:

$$-\frac{1}{2} \left(\frac{3M}{2}\right)^{1/2} f'^2 \dot{g} - \frac{1}{2} \left(\frac{3M}{2}\right)^{1/2} \dot{g} f f' = f''' g^{-1/2}. \tag{14}$$

To make the above equation independent of $x$, we must have $\dot{g} \sim g^{-1/2}$, which can be easily integrated to yield $\boxed{g(x) \sim x^{2/3}}$. As suggested in the problem, choosing $\boxed{g(x) = \left(\frac{3M}{2}\right)^{-1/3}(3x)^{2/3}}$, the above equation reduces to:

$$f'^2 + ff'' + f''' = 0. \tag{15}$$

The BC $u = 0$ as $y \to \pm\infty$ reduces to $\boxed{f'(\infty) = 0}$. The symmetry condition at $Y = 0$, $\frac{du}{dY} = 0$ reduces to $\boxed{f''(0) = 0}$. Also, by symmetry, we set the $\psi = 0$ streamline at $Y = 0$, giving us another required BC $\boxed{f(0) = 0}$.

Combining $ff'' + f'^2 = (ff')'$, we get

$$f''' + (ff')' = 0 \tag{16}$$

Integrating once wrt $\eta$, obtain $f'' + ff' = c_1$. Since $f''(0) = f(0) = 0$, $\boxed{c_1 = 0}$. Rewriting $f'' + ff' = 0$ as $f'' + \left(\frac{f^2}{2}\right)' = 0$ and integrating once more in $\eta$,

$$f' + \frac{f^2}{2} = c_2. \tag{17}$$

Solving this

$$f(\eta) = 2A \tanh A(\eta + k) \tag{18}$$

Since $f(0) = 0$, $\tanh A(k) = 0$, giving $\boxed{k = 0} \Rightarrow \boxed{f(\eta) = 2A \tanh(A\eta)}$.
We had set $\int_{-\infty}^{\infty} f'^2 d\eta = 2/3$. This yields,

$$\begin{aligned}
&\int_{-\infty}^{\infty} f'^2 d\eta = 2/3, \\
&4A^4 \int_{-\infty}^{\infty} \operatorname{sech}^4(A\eta) d\eta = 2/3, \\
&\frac{2A^4}{A} \int_{-\infty}^{\infty} \operatorname{sech}^4 \zeta d\zeta = 1/3, \\
&2A^3 \cdot 4/3 = 1/3 \\
&\boxed{A = 1/2}.
\end{aligned} \tag{19}$$

Now, we can obtain $u(x, Y)$ as follows:

$$\begin{aligned}
&u = \psi_Y = \left(\frac{3M}{2}\right)^{1/2} \frac{f'}{g^{1/2}}, \\
&\boxed{u = \frac{1}{2}\left(\frac{3M^2}{4x}\right)^{1/3} \operatorname{sech}^2(\eta/2)}.
\end{aligned} \tag{20}$$

# 2 Quasi-Geostrophic (QG) Vorticity Equation, Ekman Boundary Layer (BL) in the $\beta$-Plane:

From our discussion of Ekman boundary layers that the leading-order interior geostrophic flow ($u_0$ and $v_0$) is not constrained at leading order; that is, $u_0$ and $v_0$ are in geostrophic balance with the leading order interior pressure $\pi_0$, but these variables are otherwise unknown. This indeterminacy can be remedied by going to the next order in the expansion for the interior flow and making use of our leading-order Ekman boundary layer analysis.

## 2.1 The $\beta$-Plane:

The "Coriolis parameter" $f$ is given by (twice) the component of the planetary angular velocity in the local $z$-direction. We measure $\theta$ from the equator, so $\theta = 0$ corresponds to the equator while at the North pole, $\theta = \pi/2$.

$$\begin{aligned}
&f = 2\Omega \sin\theta, \\
&f = 2\Omega \sin\theta_0 + (\Delta\theta)2\Omega\cos\theta_0 + \text{h.o.t.}, \\
&\text{noting } r_0\Delta\theta = \tilde{y}, \\
&f \approx 2\Omega\sin\theta_0 + \frac{\tilde{y}}{r_0}2\Omega\cos\theta_0, \\
&\boxed{f \approx f_0 + \beta_0\tilde{y}},
\end{aligned} \tag{21}$$

where $f_0 = 2\Omega\sin\theta_0, \beta_0 = \frac{2\Omega\cos\theta_0}{r_0}$ are dimensional parameters and $\tilde{y}$ is the local cross-flow (northward) co-ordinate (we assume that the wind is blowing in the $x$-direction).

## 2.2 Governing Equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial \pi}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right], \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial \pi}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right], \\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial \pi}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right], \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned} \tag{22}$$

First, let us look at the Coriolis terms. For example, take $fu$.

$$fu = (f_0 + \beta_0\tilde{y})u = f_0 U_0\left(1 + \frac{\beta_0 L}{f_0}y\right)u \tag{23}$$

where $y$ and $U$ on the RHS are dimensionless terms. Simplifying further and taking only the dimensionless version now,

$$\left(1+\frac{\beta_0 L}{f_0}y\right)u=\left(1+\underbrace{\frac{\beta_0 L^2}{U_0}}_{\beta}\underbrace{\frac{U_0}{Lf_0}}_{\epsilon}y\right)u=(1+\epsilon\beta y)u, \tag{24}$$

where $\beta=\frac{\beta_0 L^2}{U_0}$ is a new parameter and $\epsilon=\frac{U_0}{Lf_0}$ is the familiar Rossby number. We scale $x,y\sim L, z\sim D, f\sim f_0, t\sim L/U_0, u,v\sim U_0, w\sim W_0$. From continuity, it is immediately clear that $\frac{W_0}{D}\sim\frac{U_0}{L}$ or $W_0\sim\Gamma U_0$, where $\Gamma=\frac{D}{L}$ is the aspect ratio.
Let $p\sim P$. For geostrophic balance at the leading order, $\frac{1}{\rho}\frac{\partial\pi}{\partial x}\sim fv$, yielding $\boxed{P=\rho f_0 U_0 L}$. Using these scalings, let us non-dimensionalize the $x$-momentum equation:

$$\begin{array}{cccccccccc}
\dfrac{\partial u}{\partial t} & +u\dfrac{\partial u}{\partial x} & +v\dfrac{\partial u}{\partial y} & +w\dfrac{\partial u}{\partial z} & -fv & =-\dfrac{1}{\rho}\dfrac{\partial\pi}{\partial x} & +\nu\left[\dfrac{\partial^2 u}{\partial x^2}\right. & +\dfrac{\partial^2 u}{\partial y^2} & \left.+\dfrac{\partial^2 u}{\partial z^2}\right], \\
\dfrac{U_0^2}{L} & \dfrac{U_0^2}{L} & \dfrac{U_0^2}{L} & \dfrac{U_0^2}{L} & f_0U_0 & f_0U_0 & \dfrac{\nu\Gamma^2U_0}{D^2} & \dfrac{\nu\Gamma^2U_0}{D^2} & \dfrac{\nu U_0}{D^2} \\
\epsilon & \epsilon & \epsilon & \epsilon & 1 & 1 & \Gamma^2E & \Gamma^2E & E
\end{array} \tag{25}$$

Here, we divided throughout by $f_0U_0$ to get the coefficients. In the above $\epsilon=\frac{U_0}{Lf_0}$ is the Rossby number, while $E=\frac{\nu}{f_0D_0^2}$ is the Ekman number. We can do similar analyses for other momenum equations. The governing dimensionless equations now become:

$$\begin{aligned}
\epsilon\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]-(1+\beta\epsilon y)v&=-\frac{\partial\pi}{\partial x}+\Gamma^2E\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+E\frac{\partial^2 u}{\partial z^2},\\
\epsilon\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]+(1+\beta\epsilon y)u&=-\frac{\partial\pi}{\partial y}+\Gamma^2E\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]+E\frac{\partial^2 v}{\partial z^2},\\
\Gamma^2\epsilon\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]&=-\frac{\partial\pi}{\partial z}+\Gamma^4E\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]+\Gamma^2E\frac{\partial^2 w}{\partial z^2},\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0.
\end{aligned} \tag{26}$$

### 2.3 "Outer" Interior Solution:

As suggested in the problem writing $\underline{u} = \underline{u}_0 + \epsilon \underline{u}_1 + \ldots$ and $\pi = \pi_0 + \epsilon\pi_1 + \ldots$, substituting in Eqn. (26), collecting terms at different powers of $\epsilon$, we get:

$$\begin{aligned}
O(1) :& \text{Geostrophic balance:}\\
&\frac{\partial \pi_0}{\partial x} = v_0,\\
&\frac{\partial \pi_0}{\partial y} = -u_0,\\
&\frac{\partial \pi_0}{\partial z} = 0,\\
&\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0. \qquad (27)\\
\text{BCs :}& u_0 = v_0 = w_0 = 0 \quad \text{at} \quad z = -1,\\
& w_0 = 0 \quad \text{at} \quad z = 0\\
&\frac{\partial u_0}{\partial z} = \frac{\tau}{\sqrt{E}} \quad \text{at} \quad z = 0,\\
&\frac{\partial v_0}{\partial z} = 0 \quad \text{at} \quad z = 0,
\end{aligned}$$

where $\tau$ is the dimensionless wind-shear at the free surface. The system is not closed, since there is no way of determining $\pi_0$ from the current equations. However, we can still draw some conclusions from the leading order geostrophic balance. Substituting the $x$- and $y$- geostrophic balance equations into continuity, we see $-\cancel{\frac{\partial^2 \pi_0}{\partial x \partial y}} + \cancel{\frac{\partial^2 \pi_0}{\partial y \partial x}} + \frac{\partial w_0}{\partial z} = 0$. This shows that $w_0$ is independent of $z$ or $\boxed{w_0 \equiv w_0(x, y, t)}$.

Taking the $z$-derivative of the $x$- and $y$-geostrophic balance equations and using the fact that $\frac{\partial \pi_0}{\partial z} = 0$, we obtain $\frac{\partial u_0}{\partial z} = \frac{\partial v_0}{\partial z} = 0$. That is, $u_0$ and $v_0$ are also independent of $z$ or $\boxed{[u_0, v_0] \equiv [u_0, v_0](x, y, t)}$. Now, we go to the next order. Using the distinguished limit, we had related in the two small parameters $\epsilon$ and $E$, such that $r = \frac{\sqrt{E}}{\epsilon} = O(1)$ as $\epsilon \to 0$. Keeping this in mind, we write the $O(\epsilon)$

equations as follows:

$$
\begin{aligned}
&O(\epsilon): \\
&\qquad \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + w_0\frac{\partial u_0}{\partial z} - v_1 - \beta y v_0 = -\frac{\partial \pi_1}{\partial x}, \\
&\qquad \frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + w_0\frac{\partial v_0}{\partial z} + u_1 + \beta y u_0 = -\frac{\partial \pi_1}{\partial y}, \\
&\qquad \Gamma^2\left[\frac{\partial w_0}{\partial t} + u_0\frac{\partial w_0}{\partial x} + v_0\frac{\partial w_0}{\partial y} + w_0\frac{\partial w_0}{\partial z}\right] = -\frac{\partial \pi_1}{\partial z}, \\
&\qquad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0. \\
&\text{BCs}: u_1 = v_1 = w_1 = 0 \quad \text{at} \quad z = -1, \\
&\qquad w_1 = 0 \quad \text{at} \quad z = 0 \\
&\qquad \frac{\partial u_1}{\partial z} = 0 \quad \text{at} \quad z = 0, \\
&\qquad \frac{\partial v_1}{\partial z} = 0 \quad \text{at} \quad z = 0.
\end{aligned}
\tag{28}
$$

To eliminate pressure, cross differentiating the $x$- and $y$- moementum equations and subtracting, we obtain:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[\frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + w_0\frac{\partial v_0}{\partial z} + u_1 + \beta y u_0 = -\frac{\partial \pi_1}{\partial y}\right] \\
&-\frac{\partial}{\partial y}\left[\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + w_0\frac{\partial u_0}{\partial z} - v_1 - \beta y v_0 = -\frac{\partial \pi_1}{\partial x}\right] \\
&-------------------------- \\
&\frac{\partial \omega_{z0}}{\partial t} + u_0\frac{\partial \omega_{z0}}{\partial x} + v_0\frac{\partial \omega_{z0}}{\partial y} + w_0\frac{\partial \omega_{z0}}{\partial z} + \underbrace{\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right)}_{-\frac{\partial w_1}{\partial z}} + \beta y \cancel{\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)} + \beta v_0 = 0, \\
&\boxed{\frac{D\omega_{z0}}{Dt} + \beta v_0 = \frac{\partial \omega_{z0}}{\partial t} + u_0\frac{\partial \omega_{z0}}{\partial x} + v_0\frac{\partial \omega_{z0}}{\partial y} + w_0\frac{\partial \omega_{z0}}{\partial z} + \beta v_0 = \frac{\partial w_1}{\partial z}}.
\end{aligned}
\tag{29}
$$

Here $\boxed{\omega_{z0} = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}}$ is the leading-order (*relative*) vertical vorticity. Copying the comment from the assignment here for completion. "Physically, $\omega_{z0}$ is the leading-order relative z-vorticity: the total z-vorticity equals the sum of $\omega_{z0}$ plus $\epsilon^{-1}$ , where in dimensional terms "$\epsilon^{-1}$" corresponds to $f_0 \equiv 2\Omega \sin\theta_0$, i.e. the $z$-vorticity fluid particles acquire simply because they are rotating with the Earth. Thus, in terms of vorticity dynamics, the right-hand-side of Eqn.(29) is a vortex-stretching term; more specifically, stretching of planetary vortex tubes by the difference in vertical velocities at the ends of these vertically-oriented

tubes. (The term involving $\beta$ physically represents the advection of planetary vortex tubes.)"
To close Eqn.(29), we integrate it across the basin. Noting that $u_0, v_0$(and hence, $\omega_{z0}$) as well as $w_0$ are independent of $z$, we get:

### 2.4 "Inner" Solution at the Upper Free Surface:

Redefining boundary layer variables as $\hat{u}, \hat{v}, \hat{w}, \hat{\pi}$ and the co-ordinates as $\hat{x} = x, \hat{y} = y, \hat{z} = z/h(\epsilon)$, where $h(\epsilon) = h^*/D$ is the dimensionless thickness of the BL, we obtain:

$$
\begin{aligned}
\epsilon\left[\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\hat{w}}{h}\frac{\partial \hat{u}}{\partial \hat{z}}\right] - (1+\beta\epsilon\hat{y})\hat{v} &= -\frac{\partial \hat{\pi}}{\partial \hat{x}} + \Gamma^2 E\left[\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right] + \frac{E}{h^2}\frac{\partial^2 \hat{u}}{\partial \hat{z}^2}, \\
\epsilon\left[\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\hat{w}}{h}\frac{\partial \hat{v}}{\partial \hat{z}}\right] + (1+\beta\epsilon\hat{y})\hat{u} &= -\frac{\partial \hat{\pi}}{\partial \hat{y}} + \Gamma^2 E\left[\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right] + \frac{E}{h^2}\frac{\partial^2 \hat{v}}{\partial \hat{z}^2}, \\
\Gamma^2\epsilon\left[\frac{\partial \hat{w}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{w}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{w}}{\partial \hat{y}} + \hat{w}\frac{\partial \hat{w}}{\partial \hat{z}}\right] &= -\frac{1}{h}\frac{\partial \hat{\pi}}{\partial \hat{z}} + \Gamma^4 E\left[\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{y}^2}\right] + \Gamma^2\frac{E}{h^2}\frac{\partial^2 \hat{w}}{\partial \hat{z}^2}, \\
\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{1}{h}\frac{\partial \hat{w}}{\partial \hat{z}} &= 0.
\end{aligned}
\tag{30}
$$

In what follows, since $\hat{x} = x, \hat{y} = y$, we will drop the hats on the $x$ and $y$, but will keep the hat on $\hat{z}$ to remind ourselves that $\hat{z}$ is a BL variable. Before posing an asymptotic expansion and collecting terms, some observations are important. We first choose $h = O(\sqrt{E})$, specifically $h = \sqrt{E} = r\epsilon$, in order to keep the $\hat{z}$-diffusion at the leading order. This is the crucial physics that our interior, outer solution lacks. If we multiply through by $h = \epsilon$ the continuity equation, at leading order, we would obtain: $\boxed{\frac{\partial \hat{w}_0}{\partial \hat{z}} = 0}$, i.e., $\hat{w}_0$ is independent of $\hat{z}$. Using the upper surface BC, $\hat{w} = 0$ at $\hat{z} = 0$, we get $\hat{w}_0 = 0$. Hence, the asymptotic sequence for $\boxed{\hat{w} \sim \epsilon\hat{w}_1 + \ldots}$.
Now, posing $\hat{u} \sim \hat{u}_0 + \epsilon\hat{u}_1 + \ldots$, $\hat{v} \sim \hat{v}_0 + \epsilon\hat{v}_1 + \ldots$ and $\hat{\pi} \sim \hat{\pi}_0 + \epsilon\hat{\pi}_1 + \ldots$, collecting terms at leading order, we get the following (note: since the leading order in the $\hat{z}$-momentum equation is $O(1/h)$, multiply through by $h = r\epsilon$ and collect terms):

$$
\begin{aligned}
-\hat{v}_0 &= -\frac{\partial \hat{\pi}_0}{\partial x} + \frac{\partial^2 \hat{u}_0}{\partial \hat{z}^2}, \\
\hat{u}_0 &= -\frac{\partial \hat{\pi}_0}{\partial y} + \frac{\partial^2 \hat{v}_0}{\partial \hat{z}^2}, \\
0 &= \frac{\partial \hat{\pi}}{\partial \hat{z}}.
\end{aligned}
\tag{31}
$$

We solve for $\hat{u}_0, \hat{v}_0$ by obtaining the homogeneous solution (with inhomogeneous BCs: $\frac{1}{\cancel{h}}\frac{\partial \hat{u}_H}{\partial \hat{z}} = \frac{\tau}{\cancel{\sqrt{E}}}$) and the inhomogeneous solution (with homogeneous BCs:

$\frac{\partial \hat{u}_I}{\partial \hat{z}} = 0$). Defining:

$$\begin{aligned} \hat{u}_0 &= \hat{u}_H + \hat{u}_I \\ \hat{v}_0 &= \hat{v}_H + \hat{v}_I \end{aligned} \tag{32}$$

### 2.4.1 Homogeneous Solution:

The homogeneous part of Eqn.(31) can be written as:

$$\begin{aligned} -\hat{v}_H &= \frac{\partial^2 \hat{u}_H}{\partial \hat{z}^2}, \\ \hat{u}_H &= \frac{\partial^2 \hat{v}_H}{\partial \hat{z}^2}. \end{aligned} \tag{33}$$

The two equations can be combined into the form

$$\frac{\partial^2 \mathcal{U}_H}{\partial \hat{z}^2} - i\mathcal{U}_H = 0, \tag{34}$$

where $\mathcal{U}_H = \hat{u}_H + i\hat{v}_H$. The solutions of Eqn.(34) are

$$\mathcal{U}_H = Ae^{\lambda \hat{z}}. \tag{35}$$

Substituting into Eqn.(34), we get $\lambda^2 = i$, or $\lambda = \pm\frac{(1+i)}{\sqrt{2}}$. The general solution then becomes:

$$\mathcal{U}_H = Ae^{\frac{(1+i)}{\sqrt{2}}\hat{z}} + Be^{-\frac{(1+i)}{\sqrt{2}}\hat{z}}. \tag{36}$$

When matching with, we'd need to take the outer limit $\hat{z} \to -\infty$ of this inner solution. Hence, for consistency, we must have $\boxed{B = 0}$, giving

$$\mathcal{U}_H = Ae^{\frac{(1+i)}{\sqrt{2}}\hat{z}}. \tag{37}$$

As we wrote earlier, we'd impose the inhomogeneous BCs on the homogeneous solution. The BC for $\mathcal{U}_H$ at $\hat{z} = 0$ is obtained by combining BCs for $\hat{u}_H$ and $\hat{v}_H$, namely,

$$\begin{aligned} &\frac{1}{\cancel{k}}\frac{\partial \hat{u}_H}{\partial \hat{z}}\bigg|_{\hat{z}=0} = \frac{\tau}{\cancel{\sqrt{E}}}, \\ &\frac{\partial \hat{u}_H}{\partial \hat{z}}\bigg|_{\hat{z}=0} = \tau, \\ &\frac{\partial \hat{v}_H}{\partial \hat{z}}\bigg|_{\hat{z}=0} = 0, \\ &\boxed{\frac{\partial \mathcal{U}_H}{\partial \hat{z}}\bigg|_{\hat{z}=0} = \tau + i(0)}. \end{aligned} \tag{38}$$

This gives

$$
\begin{aligned}
&A\frac{(1+i)}{\sqrt{2}} = \tau, \\
&\boxed{A = \tau\frac{(1-i)}{\sqrt{2}}}, \\
&\boxed{\mathcal{U}_H = \left[\frac{(\tau - i\tau)}{\sqrt{2}}\right]\exp\left(\frac{(1+i)}{\sqrt{2}}\hat{z}\right)}.
\end{aligned}
\tag{39}
$$

### 2.4.2 Inomogeneous Solution:

We can just read-off the inhomogeneous solution.

$$
\begin{aligned}
&\hat{u}_I = -\frac{\partial\hat{\pi}_0}{\partial y}, \\
&\hat{v}_I = \frac{\partial\hat{\pi}_0}{\partial x}, \\
&\boxed{\mathcal{U}_I = -\frac{\partial\hat{\pi}_0}{\partial y} + i\frac{\partial\hat{\pi}_0}{\partial x}}.
\end{aligned}
\tag{40}
$$

This is consistent since $\frac{\partial\hat{\pi}_0}{\partial z} = 0$. Consistency can be checked by substituting the inhomogeneous solution into Eqn.(31). The inhomogeneous solution satisfies homogeneous BCs $\left.\frac{\partial\hat{u}_I}{\partial\hat{z}}\right|_{\hat{z}=0} = \left.\frac{\partial\hat{v}_I}{\partial\hat{z}}\right|_{\hat{z}=0} = 0$.

### 2.4.3 Matching: Inner(Outer) = Outer(Inner)

- Inner limit of the outer geostrophic flow (Eqn. 27):

$$
\lim_{z\to 0} u_0 + iv_0 = -\frac{\partial\pi_0}{\partial y} + i\frac{\partial\pi_0}{\partial x}. \tag{41}
$$

- Outer limit of the inner BL flow:

$$
\lim_{\hat{z}\to-\infty} \mathcal{U} = 0 - \frac{\partial\hat{\pi}_0}{\partial y} + i\frac{\partial\hat{\pi}_0}{\partial x}. \tag{42}
$$

Matching the two limits, we obtain

$$
\begin{aligned}
\frac{\partial \pi_0}{\partial x} &= \frac{\partial \hat{\pi}_0}{\partial x}, \\
\frac{\partial \pi_0}{\partial y} &= \frac{\partial \hat{\pi}_0}{\partial y}. \\
&\Rightarrow \boxed{\frac{\partial \hat{\pi}_0}{\partial x} = v_0} \text{ and } \\
&\boxed{-\frac{\partial \hat{\pi}_0}{\partial y} = u_0}.
\end{aligned} \tag{43}
$$

Substituting these into the BL solution and writing in terms of components, we get

$$
\hat{u}_0 \hat{e}_x + \hat{v}_0 \hat{e}_y = \left\{ u_0 + \frac{\tau}{\sqrt{2}} \exp\left(\frac{\hat{z}}{\sqrt{2}}\right) \left[\cos\left(\frac{\hat{z}}{\sqrt{2}}\right) + \sin\left(\frac{\hat{z}}{\sqrt{2}}\right)\right]\right\} \hat{e}_x \tag{44}
$$

$$
+ \left\{ v_0 + \frac{\tau}{\sqrt{2}} \exp\left(\frac{\hat{z}}{\sqrt{2}}\right) \left[\sin\left(\frac{\hat{z}}{\sqrt{2}}\right) - \cos\left(\frac{\hat{z}}{\sqrt{2}}\right)\right]\right\} \hat{e}_y \tag{45}
$$

$$
\equiv \underline{u}_G + \underline{u}_E. \tag{46}
$$

where $\underline{u}_G = u_0 \hat{e}_x + v_0 \hat{e}_y$ is the interior geostrophic flow and $\underline{u}_E = (\hat{u}_0 - u_0)\hat{e}_x + (\hat{v}_0 - v_0)\hat{e}_y$ is the frictionally driven Ekman velocity (Ekman spiral!) confined to the BL. In order to obtain $w_1$, we must go to the next order.

#### 2.4.4 $w_1$ at the Upper Layer:

The continuity equation at the next order ($O(1)$ formally, if we multiply through by $\epsilon$, then $O(\epsilon)$) becomes:

$$
\begin{aligned}
\frac{\partial \hat{w}_1}{\partial \hat{z}} &= -\left(\frac{\partial \hat{u}_0}{\partial x} + \frac{\partial \hat{v}_0}{\partial y}\right) \\
&= -\cancel{\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)} - \frac{1}{\sqrt{2}} \frac{\partial \tau}{\partial x} \exp\left(\frac{\hat{z}}{\sqrt{2}}\right) \left[\cos\left(\frac{\hat{z}}{\sqrt{2}}\right) + \sin\left(\frac{\hat{z}}{\sqrt{2}}\right)\right] \\
&- \frac{1}{\sqrt{2}} \frac{\partial \tau}{\partial y} \exp\left(\frac{\hat{z}}{\sqrt{2}}\right) \left[\sin\left(\frac{\hat{z}}{\sqrt{2}}\right) - \cos\left(\frac{\hat{z}}{\sqrt{2}}\right)\right]
\end{aligned} \tag{47}
$$

We need to integrate this from $\hat{z} = 0$ to $\hat{z} = \hat{z}$ to obtain $\hat{w}_1 \equiv \hat{w}_1(\hat{z})$. It is best to go back to complex representation and realize that $\exp\left(\frac{z}{\sqrt{2}}\right)\left[\cos\left(\frac{z}{\sqrt{2}}\right) + \sin\left(\frac{z}{\sqrt{2}}\right)\right] = \mathbf{Re}\left\{\frac{1-i}{\sqrt{2}} \exp\left(\frac{(1+i)}{\sqrt{2}} z\right)\right\}$ and $\exp\left(\frac{z}{\sqrt{2}}\right)\left[\sin\left(\frac{z}{\sqrt{2}}\right) - \cos\left(\frac{z}{\sqrt{2}}\right)\right] = \mathbf{Im}\left\{\frac{1-i}{\sqrt{2}} \exp\left(\frac{(1+i)}{\sqrt{2}} z\right)\right\}$.

In the notes, we are given:

$$
\begin{aligned}
\int_0^{\hat z} e^{\frac{1+i}{\sqrt{2}}s} ds &= \frac{1}{\sqrt{2}}\left[-1 + e^{\hat z/\sqrt{2}}\left(\cos\left(\frac{\hat z}{\sqrt{2}}\right) + \sin\left(\frac{\hat z}{\sqrt{2}}\right)\right)\right] \\
&\quad + \frac{i}{\sqrt{2}}\left[1 - e^{\hat z/\sqrt{2}}\left(\cos\left(\frac{\hat z}{\sqrt{2}}\right) - \sin\left(\frac{\hat z}{\sqrt{2}}\right)\right)\right], \\
\left(\frac{1-i}{\sqrt{2}}\right)\int_0^{\hat z} e^{\frac{1+i}{\sqrt{2}}s} ds &= \left[e^{\hat z/\sqrt{2}}\sin\left(\frac{\hat z}{\sqrt{2}}\right)\right] \\
&\quad + i\left[1 - e^{\hat z/\sqrt{2}}\cos\left(\frac{\hat z}{\sqrt{2}}\right)\right],
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
\int_0^{\hat z} \frac{\partial \hat w_1}{\partial \hat s} d\hat s &= -\left(\frac{\partial \tau}{\partial x}\right) \mathbf{Re} \int_0^{\hat z} \left\{\frac{1-i}{\sqrt{2}} \exp\left(\frac{(1+i)}{\sqrt{2}} s\right)\right\} ds \\
&\quad - \left(\frac{\partial \tau}{\partial y}\right) \mathbf{Im} \int_0^{\hat z} \left\{\frac{1-i}{\sqrt{2}} \exp\left(\frac{(1+i)}{\sqrt{2}} s\right)\right\} ds, \\
\hat w_1(\hat z) - 0 &= -\left(\frac{\partial \tau}{\partial x}\right)\left[e^{\hat z/\sqrt{2}}\sin\left(\frac{\hat z}{\sqrt{2}}\right)\right] \\
&\quad - \left(\frac{\partial \tau}{\partial y}\right)\left[1 - e^{\hat z/\sqrt{2}}\cos\left(\frac{\hat z}{\sqrt{2}}\right)\right]
\end{aligned}
\tag{49}
$$

Now, to get the value $\boxed{w_1|_{z=0} = \lim_{\hat z \to -\infty} \hat w_1 = -\frac{\partial \tau}{\partial y} = \hat e_z \cdot (\nabla \times \tau)}$.

### 2.5 "Inner" Solution at the Bottom Surface:

The analysis is exactly same until Eqn.(36). However, in this case, when matching, we will have to take the limit $\hat z \to \infty$, hence $\boxed{A = 0}$. This gives:

#### 2.5.1 Homogeneous Solution:

$$
\mathcal{U}_H = Be^{-\frac{(1+i)}{\sqrt{2}}z}. \tag{50}
$$

The boundary conditions are all homogeneous here.