HW #3: Thin-film Flows and Inertia-less Convection

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Note: All the codes used for plotting are available here.

1 Q 1: Adhesive force in a 'squeeze film':

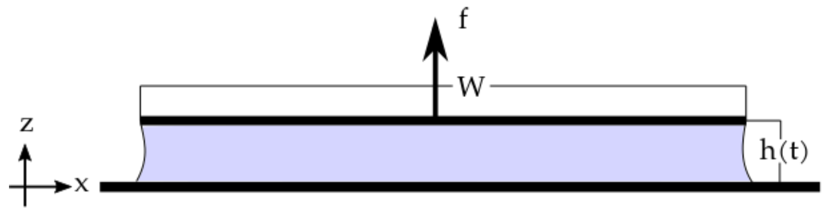


Figure 1: Thin film beneath a knife

Thin-film equations are valid here.

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial z^2}, \\
\frac{\partial p}{\partial z} &= \mu \frac{\partial^2 w}{\partial z^2}, \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.
\end{aligned} \tag{1}
$$

Let $x \sim W, z \sim h_0, u \sim U, p \sim P$. The continuity equation demands $\frac{\partial u}{\partial x}$ and $\frac{\partial w}{\partial z}$ to balance each other, hence $U/L \sim \tilde{W}/h$, giving a scale for w in terms of U, i.e., $\tilde{W} \sim U h_0/L = U\epsilon$.

The x-momentum equation gives P in terms of U. $\frac{P}{W} \sim \frac{\mu U}{h_0^2}$, giving $P \sim \frac{\mu U W}{h_0^2} = \mu U/(\epsilon^2 W)$.

The dimensionless equations then become:

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2}, \\
\frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2}, \\
\frac{\partial u}{\partial x} &+ \frac{\partial w}{\partial z} = 0,
\end{aligned}
\tag{2}
$$

where all the terms are now dimensionless. The boundary conditions (BCs) are:

$$
\begin{aligned}
&u = w = 0 \quad \text{at } z = 0, \\
&u = 0 \quad \text{at } z = h, \\
&w = \partial_t h + u \partial_x h \quad \text{at } z = h \\
&w = \partial_t h \quad \text{using } u = 0 \quad \text{at } z = h, \\
&p = p_0 \quad \text{at } x = 0, 1.
\end{aligned}
\tag{3}
$$

The leading order z-momentum equation ($\partial_z p = 0$) tells us that p is not a function of z, i.e., $p \equiv p(x,t)$. Integrating the x-momentum equation wrt z, obtain

$$
\begin{aligned}
&\frac{\partial u}{\partial z} = \frac{\partial p}{\partial x} \int_0^z dz \\
&\frac{\partial u}{\partial z} = \frac{\partial p}{\partial x} z + c_1(x,t) \\
\Rightarrow \quad &u = \frac{\partial p}{\partial x} \frac{z^2}{2} + c_1(x,t) z + c_2(x,t) \\
&u = 0 \quad \text{at z} = 0, h, \\
&c_2 = 0 \\
&c_1 = -\frac{\partial p}{\partial x} \frac{h}{2} \\
&\boxed{u = \frac{1}{2} \frac{\partial p}{\partial x} \left[z^2 - hz \right]}.
\end{aligned}
\tag{4}
$$

Integrating the continuity equation across the domain wrt z:

$$
\begin{aligned}
&\int_{z=0}^{h(x)} [\partial_x u + \partial_z w_z = 0] dz, \\
&w\Big|_0^h + \int_{z=0}^{h(x)} \partial_x u dz = 0, \\
&\partial_t h + u|_h \partial_x h - 0 + \int_{z=0}^{h(x)} (\partial_x u) dz = 0 \quad \ldots \text{using BCs for } w, \\
&\partial_t h + \cancel{u|_h \partial_x h} + \partial_x \int_{z=0}^{h(x)} u dz - \cancel{u|_h \partial_x h} = 0 \quad \ldots \text{Leibniz rule}, \\
&\partial_t h + \partial_x \left[\int_{z=0}^{h(x)} u dz \right] = 0 \\
&\frac{dh}{dt} + \partial_x \left[\frac{1}{2} \frac{\partial p}{\partial x} \left[z^3/3 - hz^2/2 \right]_0^h \right] = 0 \\
&\frac{dh}{dt} - \partial_x \left[\frac{1}{2} \frac{\partial p}{\partial x} \frac{h^3}{6} \right] = 0 \\
&\frac{dh}{dt} - \frac{h^3}{12} \frac{\partial^2 p}{\partial x^2} = 0 \quad \ldots h \equiv h(t) \text{ only.}
\end{aligned} \tag{5}
$$

Integrating twice wrt x, we get p.

$$
\begin{aligned}
p &= \frac{12}{h^3} \frac{dh}{dt} \frac{x^2}{2} + c_1 x + c_2 \\
p &= p_0 \quad \text{at } x = 0, 1, \\
c_2 &= p_0 \\
c_1 &= -\frac{6}{h^3} \frac{dh}{dt} \\
p - p_0 &= \frac{6}{h^3} \frac{dh}{dt} (x^2 - x)
\end{aligned} \tag{6}
$$

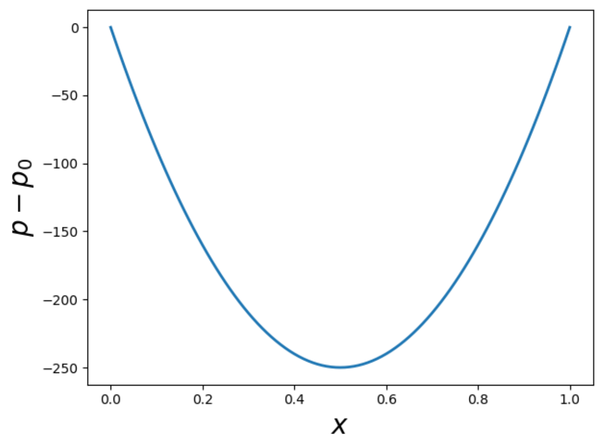


Figure 2: Gauge pressure $p - p_0$ in the film

Force per unit length (into the paper) exerted by the fluid on the knife is

$$
\begin{aligned}
f_1 &= \int_0^1 (p - p_0) dx, \\
f_1 &= \frac{6}{h^3} \frac{dh}{dt} \int_0^1 (x^2 - x) dx \\
f_1 &= \frac{6}{h^3} \frac{dh}{dt} \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 \\
f_1 &= -\frac{1}{h^3} \frac{dh}{dt}
\end{aligned} \tag{7}
$$

Therefore, the force (per unit length into the plane of paper) needed to pull the knife upward is $f = -f_1 = \frac{1}{h^3} \frac{dh}{dt}$, which is huge if dimensionless $h \sim \epsilon$ is small.

2 Q 2: Static shape of a pendant droplet with uniform surface tension and gravity:

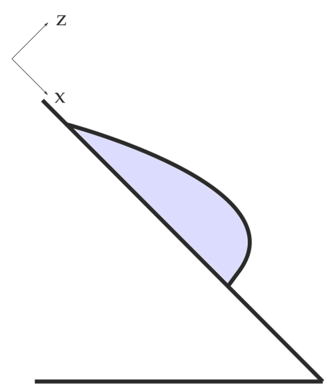


Figure 3: Pendant droplet on an incline

The dimensional governing equations can be written as:

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial z^2} + \rho g \sin\alpha \\
\frac{\partial p}{\partial z} &= \mu \frac{\partial^2 w}{\partial z^2} - \rho g \cos\alpha \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}
\tag{8}
$$

and the dimensional boundary conditions (BCs) are:

$u = w = 0 \quad \text{at} \quad z = 0,$

kinematic BC: $\dfrac{D(z-h)}{Dt} = 0 \Rightarrow \quad w = \cancelto{0 \because \text{steady}}{\partial_t h} + u\partial_x h \quad \text{at} \quad z = h(x),$

Dynamic BC (tangential): $t_i \sigma_{ij} n_j - \cancelto{0}{t_i \sigma_{aij} n_j} = \cancelto{0}{\dfrac{\partial \gamma}{\partial s}} \Rightarrow (\partial_z u + \partial_x w)(1 - (\partial_x)h^2) - 4\partial_x h \partial_x u = 0$

at $\quad z = h(x),$

Dynamic BC (normal):$n_i \sigma_{ij} n_j - \cancelto{p_0}{n_i \sigma_{aij} n_j} = \gamma K \quad \text{at} \quad z = h(x),$

$$\tag{9}$$

Let us non-dimensionalize the governing equations and BCs.
The scalings used are as follows:

$$x \sim L, z \sim h, u \sim U, w \sim W, p \sim P \tag{10}$$

From the continuity equation $O(\partial_x u) \sim O(\partial_z w)$ for balancing each other.

This immediately yields the scaling for W in terms of U, i.e.

$$\begin{aligned} \frac{U}{L} &\sim \frac{W}{h_0}, \\ \Rightarrow W &\sim \frac{Uh_0}{L} = \epsilon U. \end{aligned} \tag{11}$$

where $\epsilon = h_0/L \ll 1$ is the thin-film approximation. We now turn to the x-momentum equation to obtain the scale for pressure in terms of U. Balancing the pressure gradient and the viscous terms, we obtain

$$\begin{aligned} \frac{P}{L} &\sim \frac{\mu U}{h^2}, \\ P &\sim \frac{\mu UL}{h^2} = \frac{\mu U}{\epsilon^2 L}. \end{aligned} \tag{12}$$

We now turn to the normal stress boundary condition at $z = h(x)$. As shown in class, the normal stress boundary condition at the leading order reduces to the so called Young-Laplace equation $p - p_0 = -\epsilon^3 \bar{c}^{-1} \partial_x^2 h$, where all the variables are dimensionless and $\bar{c} = \frac{\mu U}{\gamma}$ is the capillary number. In order to retain the effects of surface tension at the leading order, we demand $\epsilon^3 \bar{c}^{-1} = O(1)$. Specifically, re-scaling $\epsilon^{-3}\bar{c} = C$, where $C = O(1)$ is the new capillary number. This yields the scaling for u,

$$\begin{aligned} C &= \bar{c}/\epsilon^3, \\ &= \mu U/\gamma\epsilon^3, \\ U &\sim \gamma\epsilon^3/\mu. \end{aligned} \tag{13}$$

Using these scales, the dimensionless governing equations and boundary conditions become:

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2} + (\rho g L^2 \sin\alpha/\gamma\epsilon) \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2} - (\rho g L^2 \cos\alpha/\gamma) \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \tag{14}$$

Defining $G = \rho g L^2/\gamma$ to be the "gravity" number, we obtain

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2} + \frac{G\sin\alpha}{\epsilon} \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2} - G\cos\alpha \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \tag{15}$$

And the BCs become:

$$\begin{aligned}
&u = w = 0 \quad \text{at} \quad z = 0, \\
&w = uh_x \quad \text{at} \quad z = h(x), \\
&u_z = 0 \quad \text{at} \quad z = h(x), \\
&p - p_0 = -C^{-1}h_{xx} \quad \text{at} \quad z = h(x), \\
&h = 0, \quad \text{at} \quad x = 0, 1.
\end{aligned} \tag{16}$$

At the leading order, the z-momentum equation becomes $p_z = -G\cos\alpha$, which is just hydrostatic balance. Integrating wrt z, we get:

$$p = -[G\cos\alpha]z + \tilde{p}(x), \tag{17}$$

where $\tilde{p}(x)$ is a constant of integration. Applying the normal-stress BC (the Young-Laplace condition), we obtain, $p|_h = p_0 - C^{-1}h_{xx}$.

$$\begin{aligned}
p_0 - h_{xx} &= -[G\cos\alpha]h + \tilde{p}(x), \\
\Rightarrow \tilde{p}(x) &= p_0 + [G\cos\alpha]h - C^{-1}h_{xx} \\
\Rightarrow p &= p_0 + [G\cos\alpha](h - z) - C^{-1}h_{xx}.
\end{aligned} \tag{18}$$

Therefore, $\boxed{p = p_0 + [G\cos\alpha](h - z) - C^{-1}h_{xx}}$.

Substituting in the x-momentum equation and integrating wrt z twice, obtain u:

$$\begin{aligned}
&\int_{z=0}^{z}\left[(G\cos\alpha)h_x - C^{-1}h_{xxx} = u_{zz} + \frac{G\sin\alpha}{\epsilon}\right]dz, \\
&u_z = \left[(G\cos\alpha)h_x - C^{-1}h_{xxx} - \frac{G\sin\alpha}{\epsilon}\right]z + \tilde{u_z}(x), \\
&\because u_z = 0 \quad \text{at} \quad z = h(x), \\
&\tilde{u_z}(x) = -\left[(G\cos\alpha)h_x - C^{-1}h_{xxx} - \frac{G\sin\alpha}{\epsilon}\right]h, \\
&\boxed{u_z = \left[(G\cos\alpha)h_x - C^{-1}h_{xxx} - \frac{G\sin\alpha}{\epsilon}\right](z - h)}, \\
&\Rightarrow u = \left[(G\cos\alpha)h_x - C^{-1}h_{xxx} - \frac{G\sin\alpha}{\epsilon}\right]\left(\frac{z^2}{2} - hz\right) + \tilde{u}(x), \\
&\because u = 0 \quad \text{at} \quad z = 0, \\
&\tilde{u} = 0, \\
&\Rightarrow \boxed{u = \left[(G\cos\alpha)h_x - C^{-1}h_{xxx} - \frac{G\sin\alpha}{\epsilon}\right]\left(\frac{z^2}{2} - hz\right)}.
\end{aligned} \tag{19}$$

Now, integrating continuity equation across the domain wrt z, we obtain:

$$\begin{aligned}
\int_{z=0}^{h(x)} [\partial_x u + \partial_z w_z = 0] dz, \\
w\Big|_0^h + \int_{z=0}^{h(x)} \partial_x u dz = 0, \\
\overset{0 \,\because\, \text{steady}}{\cancel{\partial_t h}} + u|_h \partial_x h - 0 + \int_{z=0}^{h(x)} (\partial_x u) dz = 0 \quad \ldots \text{using BCs for } w, \\
\cancel{u|_h \partial_x h} + \partial_x \int_{z=0}^{h(x)} u dz - \cancel{u|_h \partial_x h} = 0 \quad \ldots \text{Leibniz rule}, \\
\int_{z=0}^{h(x)} u dz = c
\end{aligned} \tag{20}$$

However, $Q = \int_{z=0}^{h(x)} udz$ corresponds to the volume flux and there is no volume flux here. So $c = 0$. We then get the equation for h.

$$\begin{aligned}
\int_0^h \left[(G\cos\alpha) h_x - C^{-1} h_{xxx} - \frac{G\sin\alpha}{\epsilon} \right] \left(\frac{z^2}{2} - hz \right) dz = 0 \\
\left[(G\cos\alpha) h_x - C^{-1} h_{xxx} - \frac{G\sin\alpha}{\epsilon} \right] \left(\frac{z^3}{6} - h\frac{z^2}{2} \right) \Big|_0^h = 0 \\
(G\cos\alpha) h_x - C^{-1} h_{xxx} - \frac{G\sin\alpha}{\epsilon} = 0.
\end{aligned} \tag{21}$$

For gravity to do anything, it must have an $O(1)$ effect in the x-direction. Redefining $\tilde{G} = G/\epsilon$ and demanding $\tilde{G} \sim O(1)$, we get

$$(\epsilon \tilde{G} \cos\alpha) h_x - C^{-1} h_{xxx} - \tilde{G} \sin\alpha = 0. \tag{22}$$

Neglecting the $O(\epsilon)$ term at the leading order, we obtain

$$C^{-1} h_{xxx} + \tilde{G} \sin\alpha = 0 \tag{23}$$

Finally defining the Bond number to be $B = \tilde{G}C = \rho g L^2 C/\epsilon\gamma$, and integrating thrice in x, we get:

$$h_{xxx} = (-B\sin\alpha) \tag{24}$$

$$h_{xx} = (-B\sin\alpha)x + c_1 \tag{25}$$

$$h_x = (-B\sin\alpha)\frac{x^2}{2} + c_1 x + c_2 \tag{26}$$

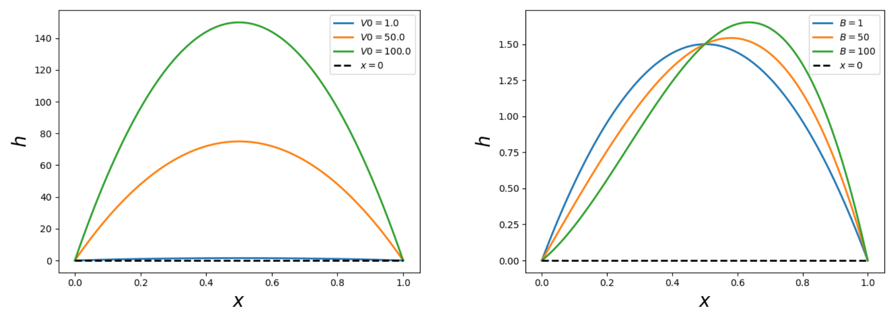
$$h = (-B\sin\alpha)\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3 \tag{27}$$

Since $h = 0$ at $x = 0, 1$, $c_3 = 0$ and $c_1/2 + c_2 = B\sin\alpha/6$. Hence, $h = (-B\sin\alpha)\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x$.

Also, the volume V_0 is preserved. In dimensionless terms $V_0 = \int_0^1 h dx$. Therefore, we get,

$$V_0 = -B \sin\alpha/24 + c_1/6 + c_2/2 \tag{28}$$

Solving $c_1/2 + c_2 = B \sin\alpha/6$ and Eqn.(28) simultaneously, we obtain: $c_1 = \frac{-24V_0 + B\sin\alpha}{2}$ and $c_2 = \frac{72V_0 - B\sin\alpha}{12}$



(a) (b)

Figure 4: $h(x)$ vs x for $\alpha = \pi/4$. (a) Fix $B = 1$, vary V_0 and (b) Fix $V_0 = 1$, vary B.

3 Q 3: Linear stability of a liquid film with non-uniform surface tension and destabilizing gravity.

Governing Equations and BCs:

The governing equations, as before, can be written as

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2} - G \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \tag{29}$$

These can be easily retrieved from Eqns.(15) by setting $\alpha = 0$. Here, $G = \rho g L^2/\gamma_0$ as defined in the previous problem.

The tangential stress boundary condition can be written as follows:

$$t_i\sigma_{ij}n_j - t_i\cancelto{0}{\sigma_{aij}n_j} = \partial_s\gamma \quad s\text{is the arclength along the surface,}$$

$$\hat{t} = \frac{\hat{e}_x + \partial_x h\hat{e}^z}{\sqrt{1+(\partial_x h)^2}}, \quad \hat{n} = \frac{-\partial_x h\hat{e}_x + \hat{e}^z}{\sqrt{1+(\partial_x h)^2}}$$

$$\therefore t_1\sigma_{11}n_1 + t_1\sigma_{12}n_2 + t_1\sigma_{12}n_2 + t_2\sigma_{22}n_2 = \partial_s\gamma$$

$$\mu(\partial_z u + \partial_x w)(1-(\partial_x h)^2) - 4\mu\partial_x h(\partial_x u - \partial_z w) = \partial_s\gamma. \tag{30}$$

But $ds \approx \sqrt{dx^2+dy^2} = dx(\sqrt{1+(\partial_x h)^2})$ at $y=h$. This gives $\partial_s = \frac{\partial_x}{\sqrt{1+(\partial_x h)^2}}$. In dimensionless terms, using the scaling for u from Eqn. (13), $U \sim \gamma_0\epsilon^3/\mu$ we get:

$$\frac{1}{1+\epsilon^2(\partial_x h)^2}\left[\mu(\partial_z u + \partial_x w)(1-\epsilon^2(\partial_x h)^2)U/(L\epsilon) - 4\mu\partial_x h(\partial_x u - \partial_z w)\epsilon U/L\right]$$
$$= (\gamma_0/L)\frac{\partial_x\gamma}{\sqrt{1+(\partial_x h)^2}},$$
$$\frac{1}{\sqrt{1+\epsilon^2(\partial_x h)^2}}\left[\mu(\partial_z u + \partial_x w)(1-\epsilon^2(\partial_x h)^2)(\gamma_0\epsilon^3/\mu)/(\epsilon) - 4\mu\partial_x h(\partial_x u - \partial_z w)\epsilon(\gamma_0\epsilon^3/\mu)\right]$$
$$= \gamma_0\partial_x\gamma,$$
$$\boxed{\epsilon^2\partial_z u = \partial_x\gamma} \quad \text{at the leading order.} \tag{31}$$

Writing $\gamma = 1+\epsilon^2\gamma$ and equating terms of the same order in ϵ, the tangential BC at the leading order reduces to $\boxed{\partial_z u = \partial_x\gamma_1}$. Also, the dynamic boundary condition (in the normal direction), in dimensional terms is the Young-Lapace equation $p - p_0 = -\gamma\partial_x^2$. In dimensionless terms, we remember that γ is no longer a constant. The dimensionless version will read $p - p_0 = C^{-1}\gamma\partial_x^2 h$, where all the quantities are now dimensionless.

Hence the BCs become:

$$\begin{aligned}
&u = w = 0 \quad \text{at} \quad z = 0,\\
&w = \partial_t h + u\partial_x x \quad \text{at} \quad z = h(x),\\
&u_z = (\partial_x\gamma_1) \quad \text{at} \quad z = h(x),\\
&p - p_0 = -C^{-1}(1+\epsilon^2\gamma_1)\partial_x^2 h \quad \text{at} \quad z = h(x),\\
&h = 0, \quad \text{at} \quad x = 0, 1.
\end{aligned} \tag{32}$$

As before, integrating the z-momentum equation at the leading order is just the hydrostatic balance. Integrating the z-momentum equation in z, we obtain the pressure distribution. This is similar to the previous question and we directly write p, by setting $\alpha = 0$ in Eqn.(18).

$$p = p_0 + G(h-z) - C^{-1}(1+\epsilon^2\gamma_1)h_{xx}. \tag{33}$$

Let us define $\pi = p - p_0$ to be the gauge pressure. Therefore, $\pi = G(h-z) - C^{-1}h_{xx}$.
Substituting in the x-momentum equation and integrating twice wrt z, we get:

$$
\begin{aligned}
& \partial_z^2 u = \partial_x \pi \quad \ldots \because \partial_x p = \partial_x \pi \\
\Rightarrow \quad & \partial_z u = (\partial_x \pi) z + c_1(x,t) \\
& \partial_z u = (\partial_x \gamma_1) \quad \text{at} \quad z = h(x), \\
\Rightarrow \quad & u_z = (\partial_x \pi)(z-h) + (\partial_x \gamma_1) \\
\Rightarrow \quad & u = (\partial_x \pi)\left(\frac{z^2}{2} - hz\right) + (\partial_x \gamma_1) z + c_2(x,t) \\
& u = 0 \quad \text{at} \quad z = 0 \Rightarrow c_2(x,t) = 0. \\
\Rightarrow \quad & \boxed{u = (\partial_x \pi)\left(\frac{z^2}{2} - hz\right) + (\partial_x \gamma_1) z}.
\end{aligned} \tag{34}
$$

Substituting into conservation of mass Eqn.(20),

$$
\begin{aligned}
& \partial_t h + \partial_x \int_0^{h(x)} \partial_x \pi \left(\frac{z^2}{2} - hz\right) + (\partial_x \gamma_1) z dz = 0 \\
& \partial_t h + \partial_x \left[\partial_x \pi \left(\frac{z^3}{6} - \frac{hz^2}{2}\right) + (\partial_x \gamma_1)\frac{z^2}{2}\right]_0^h = 0 \\
& \partial_t h + \partial_x \left[(\partial_x \gamma_1)\frac{h^2}{2} - (\partial_x \pi)\frac{h^3}{3}\right] = 0 \\
& \partial_t h + \partial_x \left[(\partial_x \gamma_1)\frac{h^2}{2} - \left(G\partial_x h - C^{-1}(1+\epsilon^2\gamma_1)\partial_x^3 h - \epsilon^2 C^{-1}(\partial_x \gamma_1)(\partial_x^2 h)\right)\frac{h^3}{3}\right] = 0.
\end{aligned} \tag{35}
$$

Rescaling time to $T = t/C$, old time scale U_0/L changes to $T = t'/(\mu L/\epsilon^3 \gamma_0)$, where t' is the dimensional time. Also, defining $CG = C\rho g L^2/\gamma_0$ to be the "Bond number", we obtain:

$$
\partial_T h + \partial_x \left[C(\partial_x \gamma_1)\frac{h^2}{2} - \left(B\partial_x h - (1+\epsilon^2\gamma_1)\partial_x^3 h - \epsilon^2(\partial_x \gamma_1)(\partial_x^2 h)\right)\frac{h^3}{3}\right] = 0.
$$

at the leading oredr $O(1)$

$$
\boxed{\partial_T h + \partial_x \left[C(\partial_x \gamma_1)\frac{h^2}{2} - \left(B\partial_x h - \partial_x^3 h\right)\frac{h^3}{3}\right] = 0}. \tag{36}
$$

Linear stability of a uniformly thick film lining the underside of a rigid flat horizontal substrate

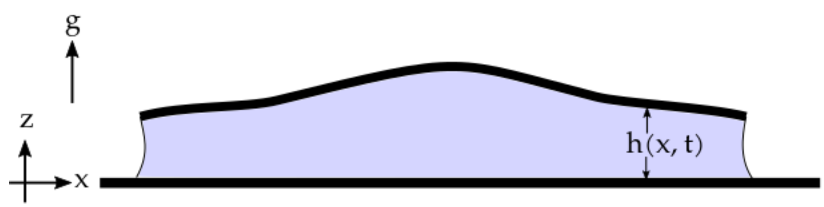


Figure 5: A film lining the underside of a rigid flat horizontal substrate.

We say that a film lining underside of a rigid horizontal substrate is equivalent to the case of regular film on a horizontal substrate, with gravity pointing upwards. So we let $B \to -B$, and $\gamma_1 = \Lambda/h$ ($\Rightarrow \partial_x \gamma_1 = \frac{-\Lambda}{h^2}\partial_x h$), we get:

$$\partial_T h + \partial_x \left[-\frac{C\Lambda}{2}\partial_x h + \left(B\partial_x h + \partial_x^3 h\right)\frac{h^3}{3} \right] = 0 \tag{37}$$

The base state is $h_b = 1$. Introduce a perturbation of the form $h = 1 + \eta$. Substituting in Eqn.(37), obtain:

$$\begin{aligned} &\partial_T \eta + \partial_x \left[-\frac{C\Lambda}{2}\partial_x \eta + \left(B\partial_x \eta + \partial_x^3 \eta\right)\frac{1}{3} \right] = 0, \\ &\partial_T \eta + \frac{1}{3}\partial_x^4 \eta + \frac{B}{3}\partial_x^2 \eta - \frac{C\Lambda}{2}\partial_x^2 \eta = 0. \end{aligned} \tag{38}$$

Now, we start by "modal analysis", i.e., seek solutions of the form $\eta = Ae^{\sigma t}e^{ikx} +$ c.c. where k is the (known) real wavenumber of the perturbation, σ is the possibly complex growth rate and c.c. denotes the complex conjugate. Substituting into Eqn.(38):

$$\sigma + \frac{k^4}{3} - \left(\frac{B}{3} - \frac{C\Lambda}{2}\right)k^2 = 0 \tag{39}$$

Hence, we get the dispersion relation $\sigma \equiv \sigma(k)$.

$$\sigma = \left(\frac{B}{3} - \frac{C\Lambda}{2}\right)k^2 - \frac{k^4}{3} \tag{40}$$

Unstable when $Re(\sigma) > 0$.

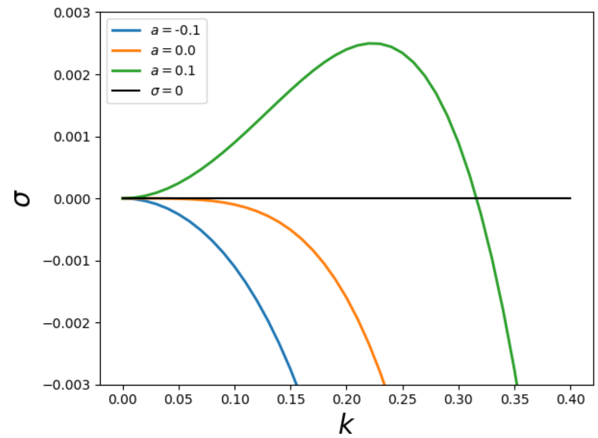


Figure 6: Dispersion relation ($\sigma \equiv \sigma(k)$) for the inear stability of a uniformly thick film lining the underside of a rigid flat horizontal substrate with $a = 2B - 3C\Lambda$

From Fig.(6), it is clear that when $a > 0$, a band of modes become unstable. Hence, the condition for instability is $2B - 3C\Lambda \geq 0$ or $\boxed{B > 3C\Lambda/2}$.

4 Q 4: Marangoni convection in the inertia-less limit:

The aim of the analysis is to investigate the possibility that, even in the absence of buoyancy, convection may be possible provided that the temperature-dependence of the surface tension coefficient γ is accounted for.

The dimensional governing equations are the incompressible Stokes equations w/o gravity:

$$\begin{aligned}
&\partial_x p = \mu(\partial_x^2 u + \partial_z^2 u), \\
&\partial_z p = \mu(\partial_x^2 w + \partial_z^2 w), \\
&0 = \partial_x u + \partial_z w, \\
&\partial_t T + u\partial_x T + w\partial_z T = \kappa(\partial_x^2 T + \partial_z^2 T),
\end{aligned} \tag{41}$$

The BCs are:

$$\begin{aligned}
&u = w = 0 \quad \text{at } z = 0, \\
&\mu \partial_z u = \partial_x \gamma \quad \text{at } z = H, \\
\text{where} \quad &\gamma = \gamma_0 - \Lambda(T - T_0), \\
&w = 0 \quad \text{at } z = H, \\
&T = T_0 \quad \text{at } z = 0, \\
&\partial_z T = -Q_0 \quad \text{at } z = H.
\end{aligned} \tag{42}$$

The surface height H remains constant throughout this analysis.

First, we cast the governing equations in terms of the streamfunction ψ, such that

$$u = \partial_z \psi, \quad w = -\partial_x w. \tag{43}$$

The incompressibility condition is then automatically satisfied. Eliminating pressure by taking the curl of the momentum equations:

$$\begin{aligned}
&\mu(\partial_x^2 \partial_z u + \partial_z^3 u - \partial_x^3 w - \partial_z^2 \partial_x w) = 0, \\
\Rightarrow \quad &(\partial_x^4 + 2\partial_x^2 \partial_z^2 + \partial_z^4)\psi = 0, \\
&\boxed{\nabla^4 \psi = 0}.
\end{aligned} \tag{44}$$

The dimensional equations and BCs, in terms of the streamfunction ψ can be written as:

$$\begin{aligned}
&\nabla^4 \psi = 0, \\
&\partial_t T + [u \cdot \nabla] T = \kappa \nabla^2 T.
\end{aligned} \tag{45}$$

The BCs become:

$$\begin{aligned}
&\partial_z \psi = \partial_x \psi = 0 \quad \text{at } z = 0, \\
&\mu \partial_{zz} \psi = \partial_x \gamma \quad \text{at } z = H, \\
\text{where} \quad &\gamma = \gamma_0 - \Lambda(T - T_0), \\
&\partial_x \psi = 0 \quad \text{at } z = H, \\
&T = T_0 \quad \text{at } z = 0, \\
&\partial_z T = -Q_0 \quad \text{at } z = H.
\end{aligned} \tag{46}$$

Scaling $x \sim H, y \sim H, u \sim \kappa/H, T \sim Q_0 H$, we obtain scalings for time and streamfunction. The scaling for time is obtained from the energy equation, where $\partial_t T$ must balance $\kappa \nabla^2 T$, yielding $t \sim H^2/\kappa$. From the definition of the streamfunction, we get $\psi \sim \kappa$. Using these scales, we obtain the dimensionless equations:

$$\begin{aligned}
&\nabla^4 \psi = 0, \\
&\partial_t T + [u \cdot \nabla] T = \nabla^2 T.
\end{aligned} \tag{47}$$

The BCs become:

$$
\begin{aligned}
&\partial_z\psi = \partial_x\psi = 0 \quad \text{at } z=0,\\
&\partial_{zz}\psi = -\tilde{\Lambda}\partial_x T \quad \text{at } z=1, \quad \text{where } \tilde{\Lambda} = \frac{\Lambda Q_0 H^2}{\kappa\mu},\\
&\partial_x\psi = 0 \quad \text{at } z=1,\\
&T = T_0/(Q_0 H) \quad \text{at } z=0,\\
&\partial_z T = -1 \quad \text{at } z=1.
\end{aligned}
\tag{48}
$$

All the quantities in the above BCs are now dimensionless. If $\psi =$ const, $\nabla^4\psi$ is definitely zero and $\boldsymbol{u}_b = \mathbf{0}$ is the base state velocity. Without loss of generality, we take $\boxed{\psi_b = 0}$. We assume a steady conduction base state for the temperature with no x-variation. $\partial_{zz}T_b = 0$, giving $T_b = Az + B$. With $T_b = T_0/(Q_0H)$ at $z = 0$, we get $B = T_0/(Q_0H)$ and $\partial_z T = -1$ at $z = 1$ yields $A = -1$. Therefore, the steady state base temperature profile is $\boxed{T_b = T_0/(Q_0H) - z}$. Perturbing about the base state and substituting $\psi \equiv \psi_b + \psi$ and $T = T_b + \theta$ (noting that $T_{b_z} = -1$), into the governing equations and BCs,

$$
\begin{aligned}
&\boxed{\nabla^4\psi = 0},\\
&\partial_t\theta + \psi_z\theta_x - \psi_x(-1+\theta_z) = \nabla^2\theta.\\
&\text{Neglecting nonlinear terms}\\
&\boxed{\partial_t\theta + \psi_x = \nabla^2\theta}.
\end{aligned}
\tag{49}
$$

The BCs become:

$$
\begin{aligned}
&\boxed{\psi_z = \psi_x = \theta = 0} \quad \text{at } z=0,\\
&\boxed{\psi_x = \theta_z = 0, \quad \psi_{zz} = -\tilde{\Lambda}\partial_x\theta} \quad \text{at } z=1.
\end{aligned}
\tag{50}
$$

Substituting

$$
\begin{bmatrix}\theta\\ \psi\end{bmatrix} = \begin{bmatrix}\hat{\theta}\\ \hat{\psi}\end{bmatrix} e^{ikx}e^{\sigma t} + \text{c.c.},
\tag{51}
$$

we obtain a linear eigenvalue problem in z.

$$
\begin{aligned}
&[k^4 - 2k^2D^2 + D^4]\hat{\psi} = 0\\
&\sigma\hat{\theta} + ik\hat{\psi} = [-k^2 + D^2]\hat{\theta}\\
&\text{combining the above, we obtain,}\\
&\boxed{\hat{\psi} = \frac{1}{ik}[-k^2 + D^2 - \sigma]\hat{\theta}} \quad \text{and}\\
&\boxed{[D^4 - 2k^2D^2 + k^4][D^2 - k^2]\hat{\theta} = \sigma[D^4 - 2k^2D^2 + k^4]\hat{\theta}},
\end{aligned}
\tag{52}
$$

where $D \equiv d_z$.

Substitute

$$\begin{bmatrix} \hat{\theta} \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \psi_0 \end{bmatrix} e^{imz}. \tag{53}$$

We first solve the $\hat{\psi}$ equation, since it is independent. Get $(k^2+m^2)^2 = 0$, degenerate solutions. Therefore, basis would be formed by $\{e^{kx}, xe^{kx}, e^{-kx}, xe^{-kx}\}$. General solution: $\hat{\psi} = c_1 e^{kx} + c_2 x e^{kx} + c_3 e^{-kx} + c_4 x e^{-kx}$.

BCs on $\hat{\psi}$:

$$\begin{aligned} \frac{d\hat{\psi}}{dz} = \hat{\psi} &= 0 \quad \text{at } z = 0, \\ \hat{\psi} &= 0 \quad \text{at } z = 1, \\ \frac{d^2\hat{\psi}}{dz^2} &= -\tilde{\Lambda} i k \hat{\theta} \quad \text{at } z = 1. \end{aligned} \tag{54}$$

Can find $\hat{\psi}$ in terms of $\hat{\theta}$ from here and substitute into the $\hat{\theta}$ equation to obtain the dispersion relation $\sigma \equiv \sigma(k)$.

5 Q 5: A lubrication approximation for Darcy flow in semi-saturated porous media:

Consider a $2d$ shallow-water flow over a porous medium of length L. The lubrication approximation here would be $\epsilon \equiv h/L \ll 1$. We assume incompressibility and use Darcy's law as the momentum equations. The dimensional governing equations become:

$$\begin{aligned} \boldsymbol{u} &= -\frac{\kappa}{\mu} \nabla (p + \rho g z), \\ \nabla . \boldsymbol{u} &= 0. \end{aligned} \tag{55}$$

In component form:

$$\begin{aligned} u &= -\frac{\kappa}{\mu} \frac{\partial p}{\partial x}, \\ w &= -\frac{\kappa}{\mu} \frac{\partial p}{\partial z} - \frac{\kappa \rho g}{\mu}, \\ \frac{\partial u}{\partial x} &+ \frac{\partial w}{\partial z} = 0. \end{aligned} \tag{56}$$

The boundary conditions are:

$$\begin{aligned} & w(x, z = 0, t) = 0, \\ & w(x, z = h(x,t), t) = \partial_t h + u \partial_x h, \\ & p(x, z = h(x,t), t) = p_0, \end{aligned} \tag{57}$$

where p_0 is the constant atmospheric pressure impressed on the top of the groundwater layer (and capillary effects are being neglected). Scaling $x \sim L, z \sim h, u \sim U, p \sim P$. The continuity equation implies $\frac{U}{L} \sim \frac{W}{h}$ or $W \sim Uh/L = \epsilon U$. The x-momentum equation implies $U \sim \kappa P/\mu L \Rightarrow P \sim \mu UL/\kappa$. In the z-momentum equation, the relative size of w and $\frac{\kappa}{\mu}\frac{\partial p}{\partial z}$ term can be found to be:

$$
\begin{aligned}
|w| \bigg/ \left|\frac{\kappa}{\mu}\frac{\partial p}{\partial z}\right| &\sim \frac{\epsilon U}{UL/h}, \\
|w| \bigg/ \left|\frac{\kappa}{\mu}\frac{\partial p}{\partial z}\right| &\sim \epsilon^2.
\end{aligned}
\tag{58}
$$

Hence, we neglect w at the leading order in the z-momentum equation. At the leading order, the dimensional z-momentum equation reads:

$$
\frac{\partial p}{\partial z} = -\rho g. \tag{59}
$$

Integrating, we obtain $p = -\rho g z + c(x)$. Using the boundary condition at the top surface $z = h$, obtain $\boxed{p - p_0 = \rho g(h - z)}$.

Substituting in the x-momentum equation, obtain: $\boxed{u = -\frac{\kappa\rho g}{\mu}\partial_x h}$.

Now, using the depth-averaged version of the continuity equation (see Eqns. (5) and (20))

$$
\begin{aligned}
&\partial_t h + \partial_x \left[\int_0^h u dz\right] = 0, \\
&\partial_t h - \partial_x \left[\frac{\kappa\rho g}{\mu}\partial_x h[z]_0^h\right] = 0, \\
&\partial_t h - \partial_x \left[\frac{\kappa\rho g}{\mu} h\partial_x h\right] = 0, \\
&M\partial_t h = \partial_x \left[h\partial_x h\right],
\end{aligned}
\tag{60}
$$

where $M = \mu/(\kappa\rho g)$. This is nonlinear diffusion equation for $h(x,t)$. Notice that pressure and h are linearly related in this problem. If there is a Gaussian pressure anomaly localized at $x = 0$ at $t = 0$, it will diffuse as time goes on. The time-scale for this would be governed by the above nonlinear diffusion equation. Namely, $M/t \sim h/L^2$ or $t \sim ML^2/h = ML/\epsilon$. This pressure diffusion is typical of porous media flows.