

HW #1: Linear and Energy Stability Theory

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1 Thin Liquid Films on a Deformable Substrate:

Governing equations in dimensionless form:

$$\partial_t h^+ + \frac{1}{3} \partial_x [(h^+)^3 (\partial_x^3 h^+ + \partial_x \kappa)] = 0, \quad (1)$$

$$\partial_t h^- + \frac{1}{3} \partial_x [(h^-)^3 (\partial_x^3 h^- - \partial_x \kappa)] = 0, \quad (2)$$

$$B \partial_x^2 \kappa + [A(1 - \Lambda) - 2] \kappa = \partial_x^2 h^+ - \partial_x^2 h^-. \quad (3)$$

Here, h^+, h^- are the thicknesses of the ‘upper’ and ‘lower’ liquid films, respectively, while κ is the curvature of the substrate, which in the small curvature limit is related to the transverse deflection of the substrate $\eta(x, t)$ via the relation $\partial_x^2 \eta = \kappa$ (i.e. η is a displacement measured in the direction normal to the ‘horizontal’ x axis). A , B , and Λ are positive external (dimensionless) control parameters related to the axial stiffness, the flexural (bending) stiffness, and the imposed compression of the axial substrate, respectively. The base state is given by $\eta_B = \kappa_B = 0$ and $h_B^+ = h_B^- = 1$. Perturbing the base state fields and substituting $\phi = \phi_B + \phi$, linearizing, we obtain:

$$\partial_t h^+ + \frac{1}{3} \partial_x [(\partial_x^3 h^+ + \partial_x \kappa)] = 0, \quad (4)$$

$$\partial_t h^- + \frac{1}{3} \partial_x [(\partial_x^3 h^- - \partial_x \kappa)] = 0, \quad (5)$$

$$B \partial_x^2 \kappa + [A(1 - \Lambda) - 2] \kappa = \partial_x^2 h^+ - \partial_x^2 h^-. \quad (6)$$

Substituting the normal mode ansatz $\phi = \phi_0 e^{\sigma t} e^{i\alpha x} + c.c.$, we get:

$$\begin{aligned} \sigma h_0^+ &= -\frac{1}{3} (i\alpha)^4 h_0^+ - \frac{1}{3} (i\alpha)^2 \kappa_0, \\ \sigma h_0^- &= -\frac{1}{3} (i\alpha)^4 h_0^- + \frac{1}{3} (i\alpha)^2 \kappa_0, \\ B(i\alpha)^2 \kappa_0 + [A(1 - \Lambda) - 2] \kappa_0 &= (i\alpha)^2 (h_0^+ - h_0^-). \end{aligned} \quad (7)$$

Simplifying,

$$\begin{aligned}\sigma h_0^+ &= -\frac{1}{3}\alpha^4 h_0^+ + \frac{1}{3}\alpha^2 \kappa_0, \\ \sigma h_0^- &= -\frac{1}{3}\alpha^4 h_0^- - \frac{1}{3}\alpha^2 \kappa_0, \\ -B\alpha^2 \kappa_0 + [A(1-\Lambda) - 2]\kappa_0 &= -\alpha^2(h_0^+ - h_0^-).\end{aligned}\tag{8}$$

Subtracting the first two equations,

$$\sigma(h_0^+ - h_0^-) = \frac{1}{3}\{2\alpha^2 \kappa_0 - \alpha^4(h_0^+ - h_0^-)\}\tag{9}$$

This gives

$$(\sigma + \frac{\alpha^4}{3})(h_0^+ - h_0^-) = \frac{2}{3}\alpha^2 \kappa_0.\tag{10}$$

Substituting in the third equation,

$$-B\alpha^2 \kappa_0 + [A(1-\Lambda) - 2]\kappa_0 = -\alpha^2 \frac{(2/3)\alpha^2 \kappa_0}{(\sigma + \frac{\alpha^4}{3})}\tag{11}$$

This gives

$$\sigma(\alpha) = -\frac{\alpha^4}{3} \left[1 + \frac{2}{([A(1-\Lambda) - 2] - B\alpha^2)} \right].\tag{12}$$

If $[A(1-\Lambda) - 2] \ll B\alpha^2$, it is the Type-IIs instability.

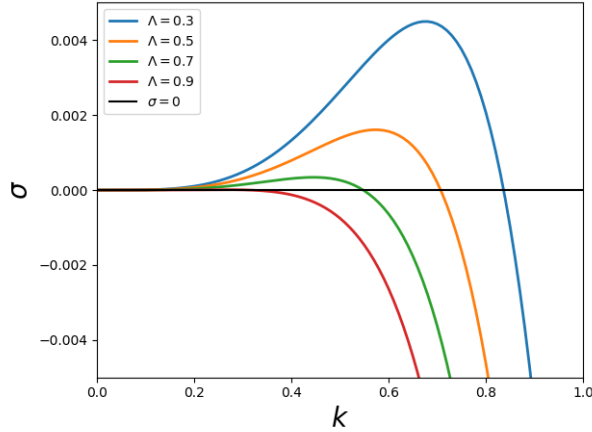


Figure 1: As Λ decreases, we see a Type-IIs type instability. Here, $A = B = 0.5$.

For marginal stability curve, we substitute $\sigma = 0$ in Eqn.(12). This yields:

$$\frac{[A(1-\Lambda)]}{B} = \alpha^2\tag{13}$$

Assuming Λ to be our control parameter,

$$\Lambda = 1 - B\alpha^2/A \quad (14)$$

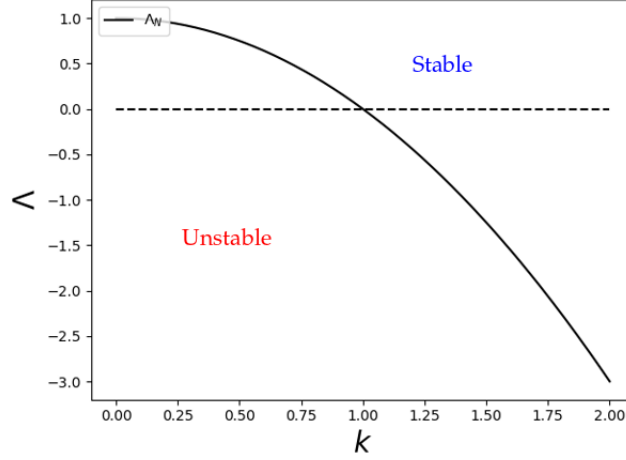


Figure 2: Marginal stability curve. Here, $A = B = 0.5$.

Fastest growing mode can be found by finding where $\frac{\partial \sigma}{\partial \alpha} = 0$. However, since $\sigma \equiv \sigma(\alpha^2)$, we can equivalently find $\frac{\partial \sigma}{\partial (\alpha^2)} = 0$ to find the fastest growing mode. We represent $[A(1 - \Lambda) - 2] \equiv \#$ for simplicity.

$$\begin{aligned} \frac{\partial \sigma}{\partial (\alpha^2)} &= \frac{-2\alpha^2}{3} - \frac{2}{3} \frac{[(\# - B\alpha^2)(2\alpha^2) - \alpha^4(-B)]}{(\# - B\alpha^2)^2} \\ 0 &= -\alpha^2 - \frac{-B\alpha^4 + 2\#\alpha^2}{(\# - B\alpha^2)^2} \\ 0 &= -1 + \frac{B\alpha^2 - 2\#}{(\# - B\alpha^2)^2} \\ (\# - B\alpha^2)^2 &= B\alpha^2 - 2\# \\ \#^2 + B^2\alpha^4 - 2\#B\alpha^2 &= B\alpha^2 - 2\# \\ 0 &= B^2\alpha^4 - (2\# + 1)B\alpha^2 + (\#^2 + 2\#) \\ \alpha^2 &= \frac{(2\# + 1) \pm \sqrt{(1 - 4\#)}}{2B} \end{aligned} \quad (15)$$

Therefore, the fastest growing modes will be $\alpha_f = \pm \sqrt{\frac{(2\# + 1) \pm \sqrt{(1 - 4\#)}}{2B}}$.

(Have to check which of the inner \pm modes will be fastest. Can be done by substituting into the dispersion relation and varying parameters.)

2 2D pattern forming system (one confined direction): Two-component porous medium convection:

2D convection in a fluid-saturated porous layer confined between plane parallel rigid boundaries coincident with $z = 0$ and $z = 1$ but now with two (dimensionless) scalar fields, namely the temperature $T(x, z, t)$ and the salt concentration $S(x, z, t)$, that determine the density of the fluid.

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u} &= -\nabla P + (Ra_T T - Ra_S S)\hat{\mathbf{e}}_z, \\ \partial_t T + \mathbf{u} \cdot \nabla T &= \nabla^2 T, \\ \partial_t S + \mathbf{u} \cdot \nabla S &= \tau \nabla^2 S.\end{aligned}\tag{16}$$

where Ra_T and Ra_S are thermal and solutal Rayleigh numbers and $0 < \tau < 1$ is the diffusivity ratio. We define a streamfunction ψ such that $u = \partial_z \psi$ and $w = -\partial_x \psi$. Eliminating pressure by taking the curl of the Darcy's law (momentum equation) and taking the y component, we obtain:

$$\begin{aligned}\cancel{\left[\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right]} &= \cancel{\frac{\partial}{\partial x}} [Ra_T T - Ra_S S] \\ -\psi_{xx} - \psi_{zz} &= Ra_T T_x - Ra_S S_x, \\ \Rightarrow \nabla^2 \psi &= Ra_S S_x - Ra_T T_x.\end{aligned}\tag{17}$$

2.1 Linear Stability Analysis:

Base state $T_B(z) = C_B(z) = 1 - z$ and $\psi_B = 0$. Substituting $T = T_B + \theta, S = C_B + c, \psi = \psi_B + \psi$ and linearizing, obtain:

$$\begin{aligned}\psi_{xx} + \psi_{zz} &= Ra_S S_x - Ra_T \theta_x \\ \partial_t \theta - \psi_x(-1) &= \theta_{xx} + \theta_{zz} \\ \partial_t c - \psi_x(-1) &= c_{xx} + c_{zz}\end{aligned}\tag{18}$$

$$\begin{aligned}\psi_{xx} + \psi_{zz} + (Ra_T \theta_x - Ra_S c_x) &= 0 \\ -\psi_x + \theta_{xx} + \theta_{zz} &= \partial_t \theta \\ -\psi_x + c_{xx} + c_{zz} &= \partial_t c\end{aligned}\tag{19}$$

Substituting the normal mode ansatz $\phi = \hat{\phi}(z)e^{ikx} + \text{c.c.}$, where c.c. is the complex conjugate, we obtain the following linear eigenvalue problem:

$$\begin{bmatrix} D^2 - k^2 & (ik)Ra_T & -(ik)Ra_S \\ -(ik) & D^2 - k^2 & 0 \\ -(ik) & 0 & \tau(D^2 - k^2) \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\theta} \\ \hat{c} \end{bmatrix} = \sigma \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{\theta} \\ \hat{c} \end{bmatrix}\tag{20}$$

However, for this particular problem, we can do better than this. We have Dirichlet BC's on θ , c and w , i.e., $\theta = c = \partial_x \psi = 0$ at $z = 0, 1$. The last condition, however, can be interpreted as follows: At the boundaries ($z = 0, 1$), ψ is constant in x . Since ψ is independent of x , we can set the value of $\psi = 0$ at the boundaries, without loss of generality. Anyway, we are interested in the velocity fields and those can be obtained from the derivatives of ψ , not ψ itself. Hence, the boundary conditions can be taken to be $\theta = c = \psi = 0$ at $z = 0, 1$. With this information, we can guess the z -dependence of the perturbation fields to be $\sim \sin(m\pi z)$. Writing $\phi = \phi_0 \sin(m\pi z)$, we further simplify the above system as:

$$\begin{bmatrix} -(k^2 + m^2\pi^2) & (ik)Ra_T & -(ik)Ra_S \\ -(ik) & -(k^2 + m^2\pi^2) & 0 \\ -(ik) & 0 & -\tau(k^2 + m^2\pi^2) \end{bmatrix} \begin{bmatrix} \psi_0 \\ \theta_0 \\ c_0 \end{bmatrix} = \sigma \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \theta_0 \\ c_0 \end{bmatrix} \quad (21)$$

We can solve this system analytically for each k , yielding a quadratic in σ , giving two roots for each k . However, I found the expression to be too cumbersome and resorted to numerics at this point. For $Ra_S = \tau = 0$, I was able to recover the dispersion relation for the one-component porous medium convection that we derived in class, with $k_c = \pi$ and $Ra_{Tc} = 4\pi^2$.

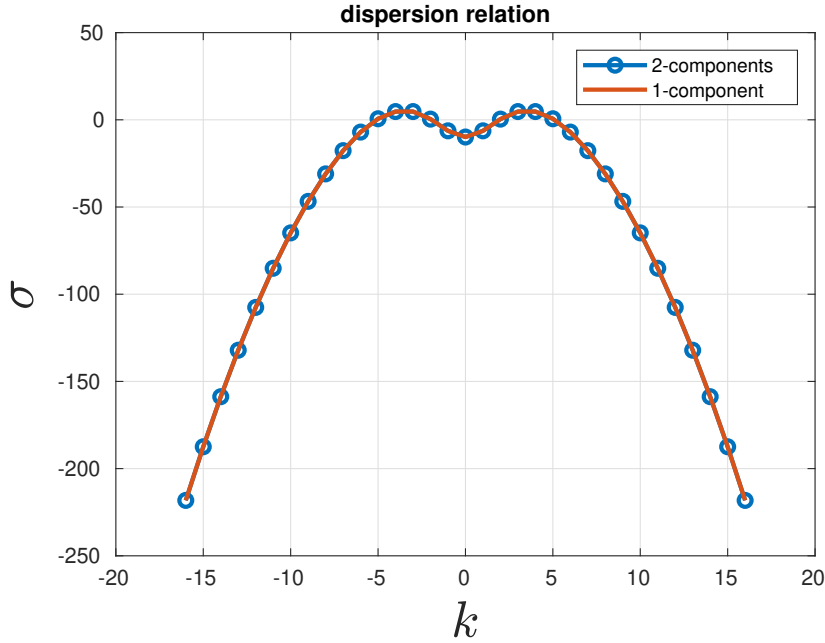


Figure 3: Comparison between dispersion relations obtained for $Ra_S = \tau = 0$ (2-components) and for the one-component porous medium convection. $Ra_T = 4\pi^2 + 10$.

For marginal stability we substitute $\sigma = 0$ and the condition for marginal stability then becomes $\det(A) = 0$, where A is the LHS matrix in Eqn.(21). It is easy to calculate the determinant using third row and expansion in minors.

$$k^2(\tau Ra_T - Ra_S) = \tau(k^2 + m^2\pi^2)^2. \quad (22)$$

2.2 Energy Stability Analysis:

To perform energy stability analysis, we first need to create an energy-like quantity. We do not linearize equations at any point and deal with full nonlinear evolution equations in this approach. Multiplying θ -equation by θ , c -equation by c and adding them, we create a quadratic, positive definite energy-like quantity. Let us first focus on the θ -equation.

$$\begin{aligned}
& \int_0^1 \int_0^{L_x} \theta [\partial_t \theta + \mathbf{u} \cdot \nabla \theta = w + \nabla^2 \theta] dx dz \\
& \int_0^1 \int_0^{L_x} \theta \left[\partial_t \theta + \nabla \cdot \left(\mathbf{u} \frac{\theta^2}{2} \right) - \frac{\theta^2}{2} \nabla \cdot \mathbf{u} = w + \nabla^2 \theta \right] dx dz \\
& \text{NOTE: } \nabla \cdot \left(\mathbf{u} \frac{\theta^2}{2} \right) = \int_0^1 \left[\int_0^{L_x} \partial_x (u \theta^2 / 2) dx \right] dz + \int_0^1 \left[\int_0^{L_x} \partial_z (w \theta^2 / 2) dz \right] dx \\
& \partial_t \left[\int_0^1 \int_0^{L_x} \frac{\theta^2}{2} dx dz \right] = \int_0^1 \int_0^{L_x} [w \theta + \theta \nabla^2 \theta] dx dz \\
& \text{using integration by parts (IBP) and BCs...} \\
& \partial_t \left[\int_0^1 \int_0^{L_x} \frac{\theta^2}{2} dx dz \right] = \int_0^1 \int_0^{L_x} \left[w \theta + \nabla \cdot \theta \nabla \theta - \frac{1}{2} \frac{\partial \theta^2}{\partial t} \right] dx dz \Big|_{z=0,1} \\
& \partial_t \left[\int_0^1 \int_0^{L_x} \frac{\theta^2}{2} dx dz \right] = \int_0^1 \int_0^{L_x} [w \theta - |\nabla \theta|^2] dx dz
\end{aligned} \tag{23}$$

Similarly, we can obtain another equation by multiplying c to the c -equation and integrating over the domain.

$$\partial_t \left[\int_0^1 \int_0^{L_x} \frac{c^2}{2} dx dz \right] = \int_0^1 \int_0^{L_x} [wc - \tau |\nabla c|^2] dx dz \quad (24)$$

Adding these two equations and defining

$$E(t) = \int_0^1 \int_0^{L_x} \left[\frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2} \right] dx dz, \text{ we}$$

obtain:

$$\frac{dE}{dt} = \int_0^1 \int_0^{L_x} \left[\tilde{w}(\tilde{\theta} + \tilde{c}) - |\nabla \tilde{\theta}|^2 - \tau |\nabla \tilde{c}|^2 \right] dx dz \quad (25)$$

We define a quantity $\lambda = \min\{\int_0^1 \int_0^{L_x} [|\nabla\tilde{\theta}|^2 + \tau|\nabla\tilde{c}|^2 - \tilde{w}(\tilde{\theta} + \tilde{c})] dx dz\}$, where \tilde{w} is slaved to $\tilde{\theta}, \tilde{c}$ in the same way as w is slaved to θ and c , i.e. $\nabla^2 w = -Ra_s c_{xx} + Ra_T \theta_{xx}$ or $w = \mathcal{L}(\theta, c)$. If we can prove $\lambda > 0$, this would imply $dE/dt < 0$ and that the system is energy stable.

2.2.1 Variational Formulation:

Let

$$J[\tilde{\theta}, \tilde{c}] = \int_0^1 \int_0^{L_x} [|\nabla\tilde{\theta}|^2 + \tau|\nabla\tilde{c}|^2 - \tilde{w}(\tilde{\theta} + \tilde{c})] dx dz \quad (26)$$

, such that $\tilde{w} = \mathcal{L}(\tilde{\theta}, \tilde{c})$. Also, we impose a normalization constraint on the energy-like quantity to keep book-keeping simple (The idea of normalizing is similar to matrix norms $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq 0, \|x\|=1} \|Ax\|$). We demand, $\int_0^1 \int_0^{L_x} [\frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2}] dx dz = 1$. We now have a variational problem at hand, with the ‘cost functional’ $J(\tilde{\theta}, \tilde{c})$ (to be minimized) subject to constraints $\tilde{w} = \mathcal{L}(\tilde{\theta}, \tilde{c})$ and $\int_0^1 \int_0^{L_x} [\frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2}] dx dz = 1$. This leads us to define the Lagrangian/ Lagrange functional L :

$$\begin{aligned} L(\tilde{\theta}, \tilde{c}, \tilde{w}, v, \Lambda) \equiv & \int_0^1 \int_0^{L_x} [|\nabla\tilde{\theta}|^2 + \tau|\nabla\tilde{c}|^2 - \tilde{w}(\tilde{\theta} + \tilde{c})] dx dz \\ & - \int_0^1 \int_0^{L_x} v(\nabla^2 \tilde{w} + Ra_s c_{xx} - Ra_T \theta_{xx}) dx dz \\ & - \Lambda \left[\int_0^1 \int_0^{L_x} \frac{\tilde{\theta}^2}{2} + \frac{\tilde{c}^2}{2} dx dz - 1 \right], \end{aligned} \quad (27)$$

where v is a Lagrange multiplier field and Λ is a scalar Lagrange multiplier. Now, we must set the first variations of L with respect to each of its arguments. We start with $\frac{\delta L}{\delta \tilde{\theta}} = 0$. From variational calculus, we know this is equivalent to $\frac{d}{d\epsilon} L(\tilde{\theta} + \epsilon\eta, \tilde{c}, \tilde{w}, v, \Lambda) \Big|_{\epsilon=0} = 0$, with η satisfying $\eta = 0$ at $z = 0, 1$.

$$\begin{aligned}
L(\tilde{\theta} + \epsilon\eta, \tilde{c}, \tilde{w}, v, \Lambda) &= \int_0^1 \int_0^{L_x} \left[|\nabla \tilde{\theta} + \epsilon\eta|^2 + \tau |\nabla \tilde{c}|^2 - \tilde{w}(\tilde{\theta} + \epsilon\eta + \tilde{c}) \right] dx dz \\
&\quad - \int_0^1 \int_0^{L_x} v(\nabla^2 \tilde{w} + Ra_s c_{xx} - Ra_T \theta_{xx} - \epsilon Ra_T \eta_{xx}) dx dz \\
&\quad - \Lambda \left[\int_0^1 \int_0^{L_x} \left(\frac{(\tilde{\theta} + \epsilon\eta)^2}{2} + \frac{\tilde{c}^2}{2} \right) dx dz - 1 \right], \\
&= \int_0^1 \int_0^{L_x} \left[|\nabla \tilde{\theta}|^2 + 2\epsilon \nabla \tilde{\theta} \cdot \nabla \eta + \epsilon^2 |\nabla \eta|^2 + \tau |\nabla \tilde{c}|^2 - \tilde{w}(\tilde{\theta} + \epsilon\eta + \tilde{c}) \right] dx dz \\
&\quad - \int_0^1 \int_0^{L_x} v(\nabla^2 \tilde{w} + Ra_s c_{xx} - Ra_T \theta_{xx} - \epsilon Ra_T \eta_{xx}) dx dz \\
&\quad - \Lambda \left[\int_0^1 \int_0^{L_x} \left(\frac{\tilde{\theta}^2 + 2\epsilon \tilde{\theta} \eta + \epsilon^2 \eta^2}{2} + \frac{\tilde{c}^2}{2} \right) dx dz - 1 \right]
\end{aligned} \tag{28}$$

Taking the first derivative of the above expression wrt ϵ and evaluating at $\epsilon = 0$, we get:

$$\begin{aligned}
\frac{\delta L}{\delta \tilde{\theta}} &= \int_0^1 \int_0^{L_x} \left[2\nabla \tilde{\theta} \cdot \nabla \eta - \tilde{w} \eta \right] dx dz \\
&\quad - \int_0^1 \int_0^{L_x} v(-Ra_T \eta_{xx}) dx dz \\
&\quad - \Lambda \left[\int_0^1 \int_0^{L_x} \tilde{\theta} \eta dx dz \right], \\
0 &= \int_0^1 \int_0^{L_x} \left[2\nabla \tilde{\theta} \cdot \nabla \eta - \tilde{w} \eta + Ra_T v \eta_{xx} - \Lambda \tilde{\theta} \eta \right]
\end{aligned} \tag{29}$$

We want everything to be proportional to η , so using IBP to factor out η from each term - For example,

$$\begin{aligned}
\int_0^1 \int_0^{L_x} 2\nabla \tilde{\theta} \cdot \nabla \eta dx dz &= \int_0^1 \int_0^{L_x} 2\nabla \cdot \eta \nabla \tilde{\theta} - 2\eta \nabla^2 \tilde{\theta} dx dz \\
&= \int_0^1 \left(\int_0^{L_x} \cancel{\partial_x [\eta \partial_x \tilde{\theta}]} dx \right) dz + \int_0^1 \left(\int_0^{L_x} \cancel{\partial_z [\eta \partial_z \tilde{\theta}]} dz \right) dx \\
&\quad - \int_0^1 \int_0^{L_x} 2\eta \nabla^2 \tilde{\theta} dx dz,
\end{aligned} \tag{30}$$

Both the boundary terms vanish since $\eta(0) = \eta(1) = 0$ in z and periodic BCs in x . Similarly for other terms, we obtain

$$0 = \int_0^1 \int_0^{L_x} \eta \left[-2\nabla^2 \tilde{\theta} - \tilde{w} + Ra_T \partial_{xx} v - \Lambda \tilde{\theta} \right] dx dz. \tag{31}$$

Since this must be true for all η (satisfying appropriate BCs), we must have

$$\boxed{-2\nabla^2\tilde{\theta} - \tilde{w} + Ra_T\partial_{xx}v = \Lambda\tilde{\theta}}. \quad (32)$$

Similarly, setting variation of L wrt \tilde{c} to be zero ($\frac{\delta L}{\delta \tilde{c}} = 0$), we get

$$\boxed{-2\tau\nabla^2\tilde{c} - \tilde{w} - Ra_S\partial_{xx}\tilde{v} = \Lambda\tilde{c}}. \quad (33)$$

$\frac{\delta L}{\delta \tilde{w}} = 0$ gives

$$\boxed{\nabla^2\tilde{v} + \tilde{\theta} + \tilde{c} = 0}. \quad (34)$$

$\frac{\delta L}{\delta \tilde{v}} = 0$ gives

$$\boxed{\nabla^2\tilde{w} + (Ra_S\tilde{c}_{xx} - Ra_T\tilde{\theta}_{xx}) = 0}. \quad (35)$$

And finally, $\frac{\delta L}{\delta \tilde{\Lambda}} = 0$ gives back the normalization condition. Substituting $\tilde{\phi} = \phi_E \sin(m\pi z)e^{ikx}$, we get:

$$\begin{aligned} 2(m^2\pi^2 + k^2)\theta_E - w_E - k^2Ra_Tv_E &= \Lambda\theta_E, \\ 2\tau(m^2\pi^2 + k^2)c_E - w_E + k^2Ra_Sv_E &= \Lambda c_E, \\ -(m^2\pi^2 + k^2)v_E + \theta_E + c_E &= 0, \\ -(m^2\pi^2 + k^2)w_E + k^2(Ra_T\theta_E - Ra_Sc_E) &= 0. \end{aligned} \quad (36)$$

Multiplying the first two equations by $(m^2\pi^2 + k^2)$ and substituting the third and fourth equations into them, we get

$$\begin{aligned} 2(m^2\pi^2 + k^2)^2\theta_E - k^2(Ra_T\theta_E - Ra_Sc_E) - k^2Ra_T(\theta_E + c_E) &= \Lambda(m^2\pi^2 + k^2)\theta_E, \\ 2\tau(m^2\pi^2 + k^2)^2c_E - k^2(Ra_T\theta_E - Ra_Sc_E) + k^2Ra_S(\theta_E + c_E) &= \Lambda(m^2\pi^2 + k^2)c_E. \end{aligned} \quad (37)$$

Separating coefficients of θ_E, c_E , get:

$$\begin{aligned} [2(m^2\pi^2 + k^2)^2 - 2k^2Ra_T]\theta_E + k^2(Ra_S - Ra_T)c_E &= \Lambda(m^2\pi^2 + k^2)\theta_E, \\ k^2(Ra_S - Ra_T)\theta_E + [2\tau(m^2\pi^2 + k^2)^2 + 2k^2Ra_S]c_E &= \Lambda(m^2\pi^2 + k^2)c_E. \end{aligned} \quad (38)$$

Writing into a matrix-form

$$\begin{bmatrix} (2(m^2\pi^2 + k^2)^2 - 2k^2Ra_T) & k^2(Ra_S - Ra_T) \\ k^2(Ra_S - Ra_T) & (2\tau(m^2\pi^2 + k^2)^2 + 2k^2Ra_S) \end{bmatrix} \begin{bmatrix} \theta_E \\ c_E \end{bmatrix} = \Lambda(m^2\pi^2 + k^2) \begin{bmatrix} \theta_E \\ c_E \end{bmatrix}. \quad (39)$$

For energy stability, want $\Lambda > 0$, Minimum $\Lambda = 0$. To get the marginal energy-stability curve, we solve for $\Lambda = 0$. The determinant of the LHS matrix must

vanish for $Ax = 0$ to have a nontrivial solution. Hence, the condition becomes:

$$\begin{aligned}
& 4\tau(m^2\pi^2 + k^2)^4 + 4k^2Ra_S(m^2\pi^2 + k^2)^2 - 4\tau k^2Ra_T(m^2\pi^2 + k^2)^2 - 4k^4Ra_TRa_S \\
& - k^4(Ra_S^2 + Ra_T^2 - 2Ra_TRa_S) = 0, \\
\Rightarrow & 4\tau(m^2\pi^2 + k^2)^4 + 4k^2Ra_S(m^2\pi^2 + k^2)^2 - 4\tau k^2Ra_T(m^2\pi^2 + k^2)^2 \\
& - 2k^4Ra_TRa_S - k^4(Ra_S^2 + Ra_T^2) = 0, \\
\Rightarrow & \boxed{4\tau(m^2\pi^2 + k^2)^4 + 4k^2(m^2\pi^2 + k^2)^2(Ra_S - \tau Ra_T) - k^4(Ra_S + Ra_T)^2 = 0}.
\end{aligned} \tag{40}$$

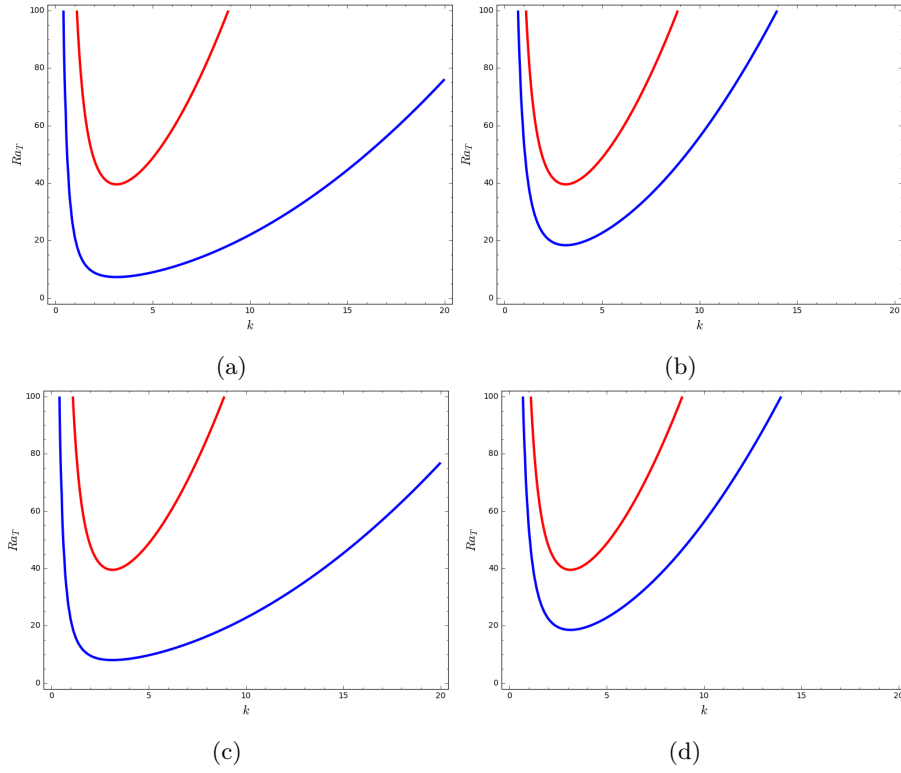


Figure 4: (a) $Ra_S = 10^{-2}, \tau = 10^{-2}$, (b) $Ra_S = 10^{-2}, \tau = 10^{-1}$, (c) $Ra_S = 10^{-1}, \tau = 10^{-2}$, (d) $Ra_S = 10^{-1}, \tau = 10^{-1}$. Here, the red curve is the marginal energy stability expression we derived in class for one-component porous medium convection: $Ra_T k^2 = (m^2\pi^2 + k^2)^2$. The blue curve is for the two-component case with different Ra_S and τ -values. Here, $m = 1$ in all the calculations. We can see that the marginal stability curves are more sensitive to τ as $\tau, Ra_S \rightarrow 0$.

Plots in Figs.(4) were generated in sagemath. For now, we fix $\tau = 0.1$, vary $Ra_S = [100, 150, 200, 250, 300]$ and obtain the marginal energy-stability curves.

When the curves go below $Ra_T = 0$, we see that even though $Ra_T < 0$, i.e., in a stably-stratified regime, we can have energy instability.

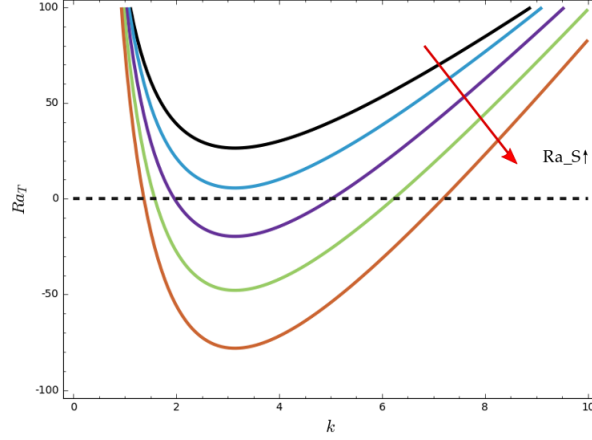


Figure 5: Marginal energy-stability curve for $\tau = 0.1$, increasing $Ra_S = [100, 150, 200, 250, 300]$.

Finally we compare the energy vs linear stability thresholds. Comparing Eqns. (22) and (40), we see that there is a gap between the linear and energy stability thresholds.

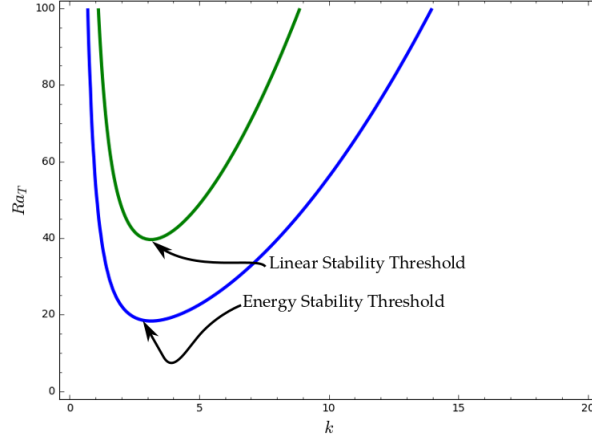


Figure 6: Comparing linear- vs energy-marginal stability curves. $m = 1$, $Ra_S = 10^{-2}$, $\tau = 10^{-1}$. We see a clear gap between the thresholds.