

# HW #3: Thin-film Flows and Inertia-less Convection

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## 1 Q 1: Adhesive force in a ‘squeeze film’:

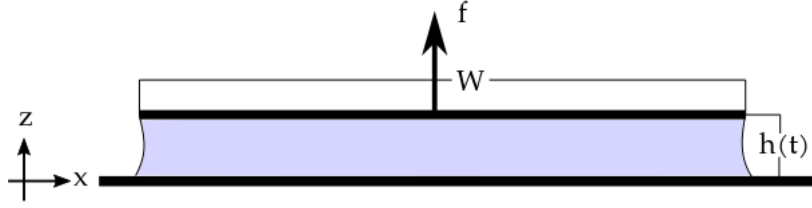


Figure 1: Thin film beneath a knife

Thin-film equations are valid here.

$$\begin{aligned}\frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial z^2}, \\ \frac{\partial p}{\partial z} &= \mu \frac{\partial^2 w}{\partial z^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.\end{aligned}\tag{1}$$

Let  $x \sim W, z \sim h_0, u \sim U, p \sim P$ . The continuity equation demands  $\frac{\partial u}{\partial x}$  and  $\frac{\partial w}{\partial z}$  to balance each other, hence  $U/L \sim \tilde{W}/h$ , giving a scale for  $w$  in terms of  $U$ , i.e.,  $\tilde{W} \sim U h_0/L = U\epsilon$ .

The  $x$ -momentum equation gives  $P$  in terms of  $U$ .  $\frac{P}{W} \sim \frac{\mu U}{h_0^2}$ , giving  $P \sim \frac{\mu U W}{h_0^2} = \mu U/(\epsilon^2 W)$ .

The dimensionless equations then become:

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2}, \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0,\end{aligned}\tag{2}$$

where all the terms are now dimensionless. The boundary conditions (BCs) are:

$$\begin{aligned}u = w = 0 &\quad \text{at } z = 0, \\ u = 0 &\quad \text{at } z = h, \\ w = \partial_t h + u \partial_x h &\quad \text{at } z = h \\ w = \partial_t h \quad \text{using } u = 0 &\quad \text{at } z = h, \\ p = p_0 &\quad \text{at } x = 0, 1.\end{aligned}\tag{3}$$

The leading order  $z$ -momentum equation ( $\partial_z p = 0$ ) tells us that  $p$  is not a function of  $z$ , i.e.,  $p \equiv p(x, t)$ . Integrating the  $x$ -momentum equation wrt  $z$ , obtain

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial p}{\partial x} \int_0^z dz \\ \frac{\partial u}{\partial z} &= \frac{\partial p}{\partial x} z + c_1(x, t) \\ \Rightarrow u &= \frac{\partial p}{\partial x} \frac{z^2}{2} + c_1(x, t)z + c_2(x, t) \\ u = 0 &\quad \text{at } z = 0, h, \\ c_2 &= 0 \\ c_1 &= -\frac{\partial p}{\partial x} \frac{h}{2} \\ \boxed{u} &= \frac{1}{2} \frac{\partial p}{\partial x} [z^2 - hz].\end{aligned}\tag{4}$$

Integrating the continuity equation across the domain wrt  $z$ :

$$\begin{aligned}
& \int_{z=0}^{h(x)} [\partial_x u + \partial_z w_z = 0] dz, \\
& w|_0^h + \int_{z=0}^{h(x)} \partial_x u dz = 0, \\
& \partial_t h + u|_h \partial_x h - 0 + \int_{z=0}^{h(x)} (\partial_x u) dz = 0 \quad \dots \text{using BCs for } w, \\
& \partial_t h + \cancel{u|_h \partial_x h} + \partial_x \int_{z=0}^{h(x)} u dz - \cancel{u|_h \partial_x h} = 0 \quad \dots \text{Leibniz rule,} \\
& \partial_t h + \partial_x \left[ \int_{z=0}^{h(x)} u dz \right] = 0 \\
& \frac{dh}{dt} + \partial_x \left[ \frac{1}{2} \frac{\partial p}{\partial x} [z^3/3 - h z^2/2]_0^h \right] = 0 \\
& \frac{dh}{dt} - \partial_x \left[ \frac{1}{2} \frac{\partial p}{\partial x} \frac{h^3}{6} \right] = 0 \\
& \frac{dh}{dt} - \frac{h^3}{12} \frac{\partial^2 p}{\partial x^2} = 0 \quad \dots h \equiv h(t) \text{ only.}
\end{aligned} \tag{5}$$

Integrating twice wrt  $x$ , we get  $p$ .

$$\begin{aligned}
p &= \frac{12}{h^3} \frac{dh}{dt} \frac{x^2}{2} + c_1 x + c_2 \\
p &= p_0 \quad \text{at } x = 0, 1, \\
c_2 &= p_0 \\
c_1 &= -\frac{6}{h^3} \frac{dh}{dt} \\
p - p_0 &= \frac{6}{h^3} \frac{dh}{dt} (x^2 - x)
\end{aligned} \tag{6}$$

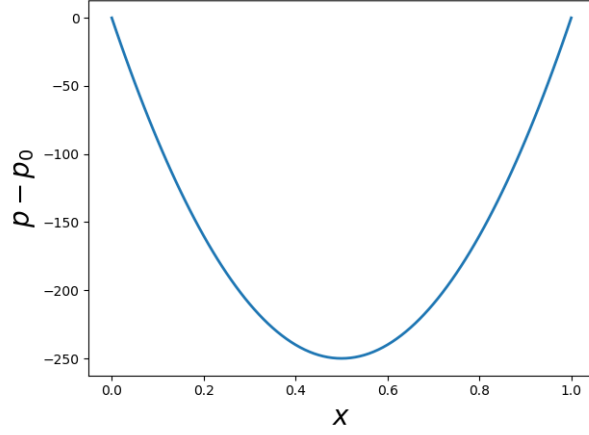


Figure 2: Gauge pressure  $p - p_0$  in the film

Force per unit length (into the paper) exerted by the fluid on the knife is

$$\begin{aligned}
 f_1 &= \int_0^1 (p - p_0) dx, \\
 f_1 &= \frac{6}{h^3} \frac{dh}{dt} \int_0^1 (x^2 - x) dx \\
 f_1 &= \frac{6}{h^3} \frac{dh}{dt} \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 \\
 f_1 &= -\frac{1}{h^3} \frac{dh}{dt}
 \end{aligned} \tag{7}$$

Therefore, the force (per unit length into the plane of paper) needed to pull the knife upward is  $f = -f_1 = \frac{1}{h^3} \frac{dh}{dt}$ , which is huge if dimensionless  $h \sim \epsilon$  is small.

**2 Q 2: Static shape of a pendant droplet with uniform surface tension and gravity:**

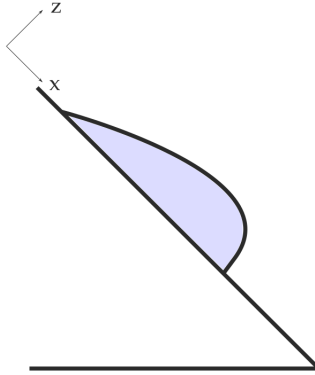


Figure 3: Pendant droplet on an incline

The dimensional governing equations can be written as:

$$\begin{aligned}\frac{\partial p}{\partial x} &= \mu \frac{\partial^2 u}{\partial z^2} + \rho g \sin \alpha \\ \frac{\partial p}{\partial z} &= \mu \frac{\partial^2 w}{\partial z^2} - \rho g \cos \alpha \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.\end{aligned}\tag{8}$$

and the dimensional boundary conditions (BCs) are:

$$u = w = 0 \quad \text{at} \quad z = 0,$$

$$\text{kinematic BC: } \frac{D(z-h)}{Dt} = 0 \Rightarrow w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{at } z = h(x), \quad \text{0: steady}$$

Dynamic BC (tangential):  $t_i \sigma_{ij} n_j - \underbrace{t_i \sigma_{\alpha ij} n_j}_{\substack{\nearrow 0 \\ \nearrow \frac{\partial \gamma}{\partial s}}} = \frac{\partial \gamma}{\partial s} \Rightarrow (\partial_z u + \partial_x w)(1 - (\partial_x h)^2) - 4\partial_x h \partial_x u = 0$   
 at  $z = h(x)$ ,

$$\text{Dynamic BC (normal): } n_i \sigma_{ij} n_j - \cancel{n_i \sigma_{aij} n_j} \stackrel{p_0}{=} \gamma K \quad \text{at } z = h(x), \quad (9)$$

Let us non-dimensionalize the governing equations and BCs. The scalings used are as follows:

$$x \sim L, z \sim h, u \sim U, w \sim W, p \sim P \quad (10)$$

From the continuity equation  $O(\partial_x u) \sim O(\partial_z w)$  for balancing each other.

This immediately yields the scaling for  $W$  in terms of  $U$ , i.e.

$$\begin{aligned} \frac{U}{L} &\sim \frac{W}{h_0}, \\ \Rightarrow W &\sim \frac{U h_0}{L} = \epsilon U. \end{aligned} \quad (11)$$

where  $\epsilon = h_0/L \ll 1$  is the thin-film approximation. We now turn to the  $x$ -momentum equation to obtain the scale for pressure in terms of  $U$ . Balancing the pressure gradient and the viscous terms, we obtain

$$\begin{aligned} \frac{P}{L} &\sim \frac{\mu U}{h^2}, \\ P &\sim \frac{\mu U L}{h^2} = \frac{\mu U}{\epsilon^2 L}. \end{aligned} \quad (12)$$

We now turn to the normal stress boundary condition at  $z = h(x)$ . As shown in class, the normal stress boundary condition at the leading order reduces to the so called Young-Laplace equation  $p - p_0 = -\epsilon^3 \bar{c}^{-1} \partial_x^2 h$ , where all the variables are dimensionless and  $\bar{c} = \frac{\mu U}{\gamma}$  is the capillary number. In order to retain the effects of surface tension at the leading order, we demand  $\epsilon^3 \bar{c}^{-1} = O(1)$ . Specifically, re-scaling  $\epsilon^{-3} \bar{c} = C$ , where  $C = O(1)$  is the new capillary number. This yields the scaling for  $u$ ,

$$\begin{aligned} C &= \bar{c}/\epsilon^3, \\ &= \mu U / \gamma \epsilon^3, \\ U &\sim \gamma \epsilon^3 / \mu. \end{aligned} \quad (13)$$

Using these scales, the dimensionless governing equations and boundary conditions become:

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2} + (\rho g L^2 \sin \alpha / \gamma \epsilon) \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2} - (\rho g L^2 \cos \alpha / \gamma) \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \quad (14)$$

Defining  $G = \rho g L^2 / \gamma$  to be the “gravity” number, we obtain

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2} + \frac{G \sin \alpha}{\epsilon} \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2} - G \cos \alpha \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \quad (15)$$

And the BCs become:

$$\begin{aligned}
u = w = 0 & \quad \text{at} \quad z = 0, \\
w = uh_x & \quad \text{at} \quad z = h(x), \\
u_z = 0 & \quad \text{at} \quad z = h(x), \\
p - p_0 = -C^{-1}h_{xx} & \quad \text{at} \quad z = h(x), \\
h = 0, & \quad \text{at} \quad x = 0, 1.
\end{aligned} \tag{16}$$

At the leading order, the  $z$ -momentum equation becomes  $p_z = -G \cos \alpha$ , which is just hydrostatic balance. Integrating wrt  $z$ , we get:

$$p = -[G \cos \alpha]z + \tilde{p}(x), \tag{17}$$

where  $\tilde{p}(x)$  is a constant of integration. Applying the normal-stress BC (the Young-Laplace condition), we obtain,  $p|_h = p_0 - C^{-1}h_{xx}$ .

$$\begin{aligned}
p_0 - h_{xx} &= -[G \cos \alpha]h + \tilde{p}(x), \\
\Rightarrow \tilde{p}(x) &= p_0 + [G \cos \alpha]h - C^{-1}h_{xx} \\
\Rightarrow p &= p_0 + [G \cos \alpha](h - z) - C^{-1}h_{xx}.
\end{aligned} \tag{18}$$

Therefore,  $\boxed{p = p_0 + [G \cos \alpha](h - z) - C^{-1}h_{xx}}$ .

Substituting in the  $x$ -momentum equation and integrating wrt  $z$  twice, obtain  $u$ :

$$\begin{aligned}
& \int_{z=0}^z \left[ (G \cos \alpha)h_x - C^{-1}h_{xxx} = u_{zz} + \frac{G \sin \alpha}{\epsilon} \right] dz, \\
u_z &= \left[ (G \cos \alpha)h_x - C^{-1}h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] z + \tilde{u}_z(x), \\
\because u_z &= 0 \quad \text{at} \quad z = h(x), \\
\tilde{u}_z(x) &= - \left[ (G \cos \alpha)h_x - C^{-1}h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] h, \\
& \boxed{u_z = \left[ (G \cos \alpha)h_x - C^{-1}h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] (z - h)}, \\
\Rightarrow u &= \left[ (G \cos \alpha)h_x - C^{-1}h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] \left( \frac{z^2}{2} - hz \right) + \tilde{u}(x), \\
\because u &= 0 \quad \text{at} \quad z = 0, \\
\tilde{u} &= 0, \\
\Rightarrow & \boxed{u = \left[ (G \cos \alpha)h_x - C^{-1}h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] \left( \frac{z^2}{2} - hz \right)}.
\end{aligned} \tag{19}$$

Now, integrating continuity equation across the domain wrt  $z$ , we obtain:

$$\begin{aligned}
& \int_{z=0}^{h(x)} [\partial_x u + \partial_z w_z = 0] dz, \\
& w|_0^h + \int_{z=0}^{h(x)} \partial_x u dz = 0, \\
& \cancel{\partial_t h} + u|_h \partial_x h - 0 + \int_{z=0}^{h(x)} (\partial_x u) dz = 0 \quad \dots \text{using BCs for } w, \quad (20) \\
& \cancel{u|_h \partial_x h} + \partial_x \int_{z=0}^{h(x)} u dz - \cancel{u|_h \partial_x h} = 0 \quad \dots \text{Leibniz rule,} \\
& \int_{z=0}^{h(x)} u dz = c
\end{aligned}$$

However,  $Q = \int_{z=0}^{h(x)} u dz$  corresponds to the volume flux and there is no volume flux here. So  $c = 0$ . We then get the equation for  $h$ .

$$\begin{aligned}
& \int_0^h \left[ (G \cos \alpha) h_x - C^{-1} h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] \left( \frac{z^2}{2} - hz \right) dz = 0 \\
& \left[ (G \cos \alpha) h_x - C^{-1} h_{xxx} - \frac{G \sin \alpha}{\epsilon} \right] \left( \frac{z^3}{6} - h \frac{z^2}{2} \right) \Big|_0^h = 0 \quad (21) \\
& (G \cos \alpha) h_x - C^{-1} h_{xxx} - \frac{G \sin \alpha}{\epsilon} = 0.
\end{aligned}$$

For gravity to do anything, it must have an  $O(1)$  effect in the  $x$ -direction. Redefining  $\tilde{G} = G/\epsilon$  and demanding  $\tilde{G} \sim O(1)$ , we get

$$(\epsilon \tilde{G} \cos \alpha) h_x - C^{-1} h_{xxx} - \tilde{G} \sin \alpha = 0. \quad (22)$$

Neglecting the  $O(\epsilon)$  term at the leading order, we obtain

$$C^{-1} h_{xxx} + \tilde{G} \sin \alpha = 0 \quad (23)$$

Finally defining the Bond number to be  $B = \tilde{G}C = \rho g L^2 C / \epsilon \gamma$ , and integrating thrice in  $x$ , we get:

$$h_{xxx} = (-B \sin \alpha) \quad (24)$$

$$h_{xx} = (-B \sin \alpha)x + c_1 \quad (25)$$

$$h_x = (-B \sin \alpha) \frac{x^2}{2} + c_1 x + c_2 \quad (26)$$

$$h = (-B \sin \alpha) \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3 \quad (27)$$

Since  $h = 0$  at  $x = 0, 1$ ,  $c_3 = 0$  and  $c_1/2 + c_2 = B \sin \alpha/6$ . Hence,  $h = (-B \sin \alpha) \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x$ .



Also, the volume  $V_0$  is preserved. In dimensionless terms  $V_0 = \int_0^1 h dx$ . Therefore, we get,

$$V_0 = -B \sin \alpha / 24 + c_1 / 6 + c_2 / 2 \quad (28)$$

Solving  $c_1 / 2 + c_2 = B \sin \alpha / 6$  and Eqn.(28) simultaneously, we obtain:  $c_1 = \frac{-24V_0 + B \sin \alpha}{2}$  and  $c_2 = \frac{72V_0 - B \sin \alpha}{12}$ .

### 3 Q 3: Linear stability of a liquid film with non-uniform surface tension and destabilizing gravity.

#### Governing Equations and BCs:

The governing equations, as before, can be written as

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial p}{\partial z} &= \epsilon^2 \frac{\partial^2 w}{\partial z^2} - G \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \quad (29)$$

These can be easily retrieved from Eqns.(15) by setting  $\alpha = 0$ . Here,  $G = \rho g L^2 / \gamma_0$  as defined in the previous problem.

The tangential stress boundary condition can be written as follows:

$$\begin{aligned} t_i \sigma_{ij} n_j - t_i \sigma_{ij}^0 n_j &= \partial_s \gamma \quad \text{is the arclength along the surface,} \\ \hat{t} &= \frac{\hat{e}_x + \partial_x h \hat{e}^z}{\sqrt{1 + (\partial_x h)^2}}, \quad \hat{n} = \frac{-\partial_x h \hat{e}_x + \hat{e}^z}{\sqrt{1 + (\partial_x h)^2}} \\ \therefore t_1 \sigma_{11} n_1 + t_1 \sigma_{12} n_2 + t_1 \sigma_{12} n_2 + t_2 \sigma_{22} n_2 &= \partial_s \gamma \\ \mu(\partial_z u + \partial_x w)(1 - (\partial_x h)^2) - 4\mu \partial_x h (\partial_x u - \partial_z w) &= \partial_s \gamma. \end{aligned} \quad (30)$$

But  $ds \approx \sqrt{dx^2 + dy^2} = dx(\sqrt{1 + (\partial_x h)^2})$  at  $y = h$ . This gives  $\partial_s = \frac{\partial_x}{\sqrt{1 + (\partial_x h)^2}}$ . In dimensionless terms, using the scaling for  $u$  from Eqn. (13),  $U \sim \gamma_0 \epsilon^3 / \mu$  we

get:

$$\begin{aligned}
& \frac{1}{1 + \epsilon^2(\partial_x h)^2} [\mu(\partial_z u + \partial_x w)(1 - \epsilon^2(\partial_x h)^2)U/(L\epsilon) - 4\mu\partial_x h(\partial_x u - \partial_z w)\epsilon U/L] \\
&= (\gamma_0/L) \frac{\partial_x \gamma}{\sqrt{1 + (\partial_x h)^2}}, \\
& \frac{1}{\sqrt{1 + \epsilon^2(\partial_x h)^2}} [\mu(\partial_z u + \partial_x w)(1 - \epsilon^2(\partial_x h)^2)(\gamma_0\epsilon^3/\mu)/(\epsilon) - 4\mu\partial_x h(\partial_x u - \partial_z w)\epsilon(\gamma_0\epsilon^3/\mu)] \\
&= \gamma_0\partial_x \gamma, \\
& \boxed{\epsilon^2\partial_z u = \partial_x \gamma} \quad \text{at the leading order.}
\end{aligned} \tag{31}$$

Writing  $\gamma = 1 + \epsilon^2\gamma$  and equating terms of the same order in  $\epsilon$ , the tangential BC at the leading order reduces to  $\boxed{\partial_z u = \partial_x \gamma_1}$ . Also, the dynamic boundary condition (in the normal direction), in dimensional terms is the Young-Laplace equation  $p - p_0 = -\gamma\partial_x^2$ . In dimensionless terms, we remember that  $\gamma$  is no longer a constant. The dimensionless version will read  $p - p_0 = C^{-1}\gamma\partial_x^2 h$ , where all the quantities are now dimensionless.

Hence the BCs become:

$$\begin{aligned}
u &= w = 0 \quad \text{at} \quad z = 0, \\
w &= \partial_t h + u\partial_x x \quad \text{at} \quad z = h(x), \\
u_z &= (\partial_x \gamma_1) \quad \text{at} \quad z = h(x), \\
p - p_0 &= -C^{-1}(1 + \epsilon^2\gamma_1)\partial_x^2 h \quad \text{at} \quad z = h(x), \\
h &= 0, \quad \text{at} \quad x = 0, 1.
\end{aligned} \tag{32}$$

As before, integrating the  $z$ -momentum equation at the leading order is just the hydrostatic balance. Integrating the  $z$ -momentum equation in  $z$ , we obtain the pressure distribution. This is similar to the previous question and we directly write  $p$ , by setting  $\alpha = 0$  in Eqn.(18).

$$p = p_0 + G(h - z) - C^{-1}(1 + \epsilon^2\gamma_1)h_{xx}. \tag{33}$$

Let us define  $\pi = p - p_0$  to be the gauge pressure. Therefore,  $\pi = G(h - z) - C^{-1}h_{xx}$ .

Substituting in the  $x$ -momentum equation and integrating twice wrt  $z$ , we get:

$$\begin{aligned}
& \partial_z^2 u = \partial_x \pi \quad \dots \cdot \partial_x p = \partial_x \pi \\
\Rightarrow & \partial_z u = (\partial_x \pi)z + c_1(x, t) \\
& \partial_z u = (\partial_x \gamma_1) \quad \text{at} \quad z = h(x), \\
\Rightarrow & u_z = (\partial_x \pi)(z - h) + (\partial_x \gamma_1) \\
\Rightarrow & u = (\partial_x \pi) \left( \frac{z^2}{2} - hz \right) + (\partial_x \gamma_1)z + c_2(x, t) \\
& u = 0 \quad \text{at} \quad z = 0 \Rightarrow c_2(x, t) = 0. \\
\Rightarrow & \boxed{u = (\partial_x \pi) \left( \frac{z^2}{2} - hz \right) + (\partial_x \gamma_1)z}.
\end{aligned} \tag{34}$$

Substituting into conservation of mass Eqn.(20),

$$\begin{aligned}
& \partial_t h + \partial_x \int_0^{h(x)} \partial_x \pi \left( \frac{z^2}{2} - hz \right) + (\partial_x \gamma_1)z dz = 0 \\
& \partial_t h + \partial_x \left[ \partial_x \pi \left( \frac{z^3}{6} - \frac{hz^2}{2} \right) + (\partial_x \gamma_1) \frac{z^2}{2} \right]_0^h = 0 \\
& \partial_t h + \partial_x \left[ (\partial_x \gamma_1) \frac{h^2}{2} - (\partial_x \pi) \frac{h^3}{3} \right] = 0 \\
& \partial_t h + \partial_x \left[ (\partial_x \gamma_1) \frac{h^2}{2} - (G \partial_x h - C^{-1}(1 + \epsilon^2 \gamma_1) \partial_x^3 h - \epsilon^2 C^{-1} (\partial_x \gamma_1) (\partial_x^2 h)) \frac{h^3}{3} \right] = 0.
\end{aligned} \tag{35}$$

Rescaling time to  $T = t/C$ , old time scale  $U_0/L$  changes to  $T = t' / (\mu L / \epsilon^3 \gamma_0)$ , where  $t'$  is the dimensional time. Also, defining  $CG = C \rho g L^2 / \gamma_0$  to be the “Bond number”, we obtain:

$$\partial_T h + \partial_x \left[ C (\partial_x \gamma_1) \frac{h^2}{2} - (B \partial_x h - (1 + \epsilon^2 \gamma_1) \partial_x^3 h - \epsilon^2 (\partial_x \gamma_1) (\partial_x^2 h)) \frac{h^3}{3} \right] = 0.$$

at the leading order  $O(1)$

$$\boxed{\partial_T h + \partial_x \left[ C (\partial_x \gamma_1) \frac{h^2}{2} - (B \partial_x h - \partial_x^3 h) \frac{h^3}{3} \right] = 0}. \tag{36}$$

## Linear stability of a uniformly thick film lining the underside of a rigid flat horizontal substrate

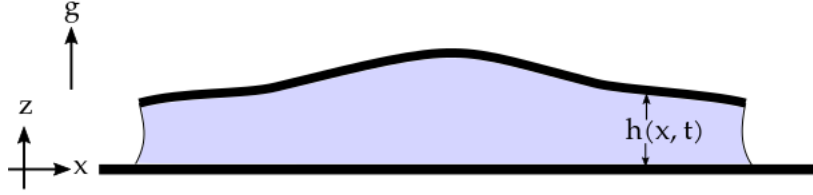


Figure 4: A film lining the underside of a rigid flat horizontal substrate.

We say that a film lining underside of a rigid horizontal substrate is equivalent to the case of regular film on a horizontal substrate, with gravity pointing upwards. So we let  $B \rightarrow -B$ , and  $\gamma_1 = \Lambda/h$  ( $\Rightarrow \partial_x \gamma_1 = \frac{-\Lambda}{h^2} \partial_x h$ ), we get:

$$\partial_T h + \partial_x \left[ -\frac{C\Lambda}{2} \partial_x h + (B \partial_x h + \partial_x^3 h) \frac{h^3}{3} \right] = 0 \quad (37)$$

The base state is  $h_b = 1$ . Introduce a perturbation of the form  $h = 1 + \eta$ . Substituting in Eqn.(37), obtain:

$$\begin{aligned} \partial_T \eta + \partial_x \left[ -\frac{C\Lambda}{2} \partial_x \eta + (B \partial_x \eta + \partial_x^3 \eta) \frac{1}{3} \right] &= 0, \\ \partial_T \eta + \frac{1}{3} \partial_x^4 \eta + \frac{B}{3} \partial_x^2 \eta - \frac{C\Lambda}{2} \partial_x^2 \eta &= 0. \end{aligned} \quad (38)$$

Now, we start by “modal analysis”, i.e., seek solutions of the form  $\eta = Ae^{\sigma t} e^{ikx} + \text{c.c.}$  where  $k$  is the (known) real wavenumber of the perturbation,  $\sigma$  is the possibly complex growth rate and c.c. denotes the complex conjugate. Substituting into Eqn.(38):

$$\sigma + \frac{k^4}{3} - \left( \frac{B}{3} - \frac{C\Lambda}{2} \right) k^2 = 0 \quad (39)$$

Hence, we get the dispersion relation  $\sigma \equiv \sigma(k)$ .

$$\sigma = \left( \frac{B}{3} - \frac{C\Lambda}{2} \right) k^2 - \frac{k^4}{3} \quad (40)$$

Unstable when  $Re(\sigma) > 0$ .

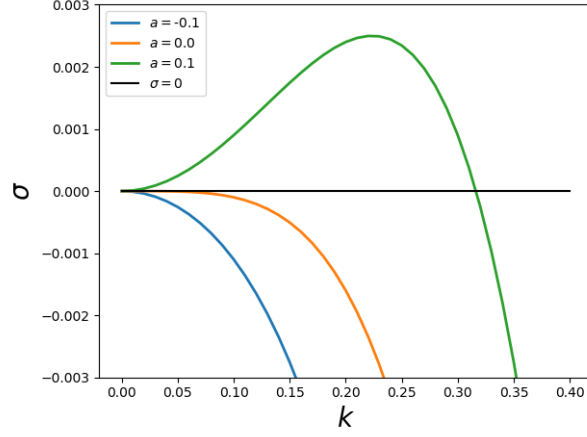


Figure 5: Dispersion relation ( $\sigma \equiv \sigma(k)$ ) for the linear stability of a uniformly thick film lining the underside of a rigid flat horizontal substrate with  $a = 2B - 3C\Lambda$

From Fig.(5), it is clear that when  $a > 0$ , a band of modes become unstable. Hence, the condition for instability is  $2B - 3C\Lambda \geq 0$  or  $B > 3C\Lambda/2$ .

#### 4 Q 4: Marangoni convection in the inertia-less limit:

The aim of the analysis is to investigate the possibility that, even in the absence of buoyancy, convection may be possible provided that the temperature-dependence of the surface tension coefficient  $\gamma$  is accounted for.

The dimensional governing equations are the incompressible Stokes equations w/o gravity:

$$\begin{aligned}
 \partial_x p &= \mu(\partial_x^2 u + \partial_z^2 u), \\
 \partial_z p &= \mu(\partial_x^2 w + \partial_z^2 w), \\
 0 &= \partial_x u + \partial_z w, \\
 \partial_t T + u\partial_x T + w\partial_z T &= \kappa(\partial_x^2 T + \partial_z^2 T),
 \end{aligned} \tag{41}$$

The BCs are:

$$\begin{aligned}
u = w = 0 & \quad \text{at } z = 0, \\
\mu \partial_z u = \partial_x \gamma & \quad \text{at } z = H, \\
\text{where } \gamma = \gamma_0 - \Lambda(T - T_0), \\
w = 0 & \quad \text{at } z = H, \\
T = T_0 & \quad \text{at } z = 0, \\
\partial_z T = -Q_0 & \quad \text{at } z = H.
\end{aligned} \tag{42}$$

The surface height  $H$  remains constant throughout this analysis.

First, we cast the governing equations in terms of the streamfunction  $\psi$ , such that

$$u = \partial_z \psi, \quad w = -\partial_x \psi. \tag{43}$$

The incompressibility condition is then automatically satisfied. Eliminating pressure by taking the curl of the momentum equations:

$$\begin{aligned}
& \mu(\partial_x^2 \partial_z u + \partial_z^3 u - \partial_x^3 w - \partial_z^2 \partial_x w) = 0, \\
\Rightarrow & (\partial_x^4 + 2\partial_x^2 \partial_z^2 + \partial_z^4) \psi = 0, \\
& \boxed{\nabla^4 \psi = 0}.
\end{aligned} \tag{44}$$

The dimensional equations and BCs, in terms of the streamfunction  $\psi$  can be written as:

$$\begin{aligned}
\nabla^4 \psi &= 0, \\
\partial_t T + [u \cdot \nabla] T &= \kappa \nabla^2 T.
\end{aligned} \tag{45}$$

The BCs become:

$$\begin{aligned}
\partial_z \psi = \partial_x \psi &= 0 \quad \text{at } z = 0, \\
\mu \partial_{zz} \psi = \partial_x \gamma & \quad \text{at } z = H, \\
\text{where } \gamma = \gamma_0 - \Lambda(T - T_0), \\
\partial_x \psi &= 0 \quad \text{at } z = H, \\
T = T_0 & \quad \text{at } z = 0, \\
\partial_z T = -Q_0 & \quad \text{at } z = H.
\end{aligned} \tag{46}$$

Scaling  $x \sim H, y \sim H, u \sim \kappa/H, T \sim Q_0 H$ , we obtain scalings for time and streamfunction. The scaling for time is obtained from the energy equation, where  $\partial_t T$  must balance  $\kappa \nabla^2 T$ , yielding  $t \sim H^2/\kappa$ . From the definition of the streamfunction, we get  $\psi \sim \kappa$ . Using these scales, we obtain the dimensionless equations:

$$\begin{aligned}
\nabla^4 \psi &= 0, \\
\partial_t T + [u \cdot \nabla] T &= \nabla^2 T.
\end{aligned} \tag{47}$$

The BCs become:

$$\begin{aligned}
\partial_z \psi &= \partial_x \psi = 0 \quad \text{at } z = 0, \\
\partial_{zz} \psi &= -\tilde{\Lambda} \partial_x T \quad \text{at } z = 1, \quad \text{where } \tilde{\Lambda} = \frac{\Lambda Q_0 H^2}{\kappa \mu}, \\
\partial_x \psi &= 0 \quad \text{at } z = 1, \\
T &= T_0/(Q_0 H) \quad \text{at } z = 0, \\
\partial_z T &= -1 \quad \text{at } z = 1.
\end{aligned} \tag{48}$$

All the quantities in the above BCs are now dimensionless. If  $\psi = \text{const}$ ,  $\nabla^4 \psi$  is definitely zero and  $\mathbf{u}_b = \mathbf{0}$  is the base state velocity. Without loss of generality, we take  $\boxed{\psi_b = 0}$ . We assume a steady conduction base state for the temperature with no  $x$ -variation.  $\partial_{zz} T_b = 0$ , giving  $T_b = Az + B$ . With  $T_b = T_0/(Q_0 H)$  at  $z = 0$ , we get  $B = T_0/(Q_0 H)$  and  $\partial_z T = -1$  at  $z = 1$  yields  $A = -1$ . Therefore, the steady state base temperature profile is  $\boxed{T_b = T_0/(Q_0 H) - z}$ .

Perturbing about the base state and substituting  $\psi \equiv \psi_b + \psi$  and  $T = T_b + \theta$  (noting that  $T_{bz} = -1$ ), into the governing equations and BCs,

$$\begin{aligned}
&\boxed{\nabla^4 \psi = 0}, \\
&\partial_t \theta + \psi_z \theta_x - \psi_x (-1 + \theta_z) = \nabla^2 \theta. \\
&\text{Neglecting nonlinear terms} \\
&\boxed{\partial_t \theta + \psi_x = \nabla^2 \theta}.
\end{aligned} \tag{49}$$

The BCs become:

$$\begin{aligned}
&\boxed{\psi_z = \psi_x = \theta = 0} \quad \text{at } z = 0, \\
&\boxed{\psi_x = \theta_z = 0, \quad \psi_{zz} = -\tilde{\Lambda} \partial_x \theta} \quad \text{at } z = 1.
\end{aligned} \tag{50}$$

Substituting

$$\begin{bmatrix} \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \hat{\theta} \\ \hat{\psi} \end{bmatrix} e^{ikx} e^{\sigma t} + \text{c.c.}, \tag{51}$$

we obtain a linear eigenvalue problem in  $z$ .

$$\begin{aligned}
&[k^4 - 2k^2 D^2 + D^4] \hat{\psi} = 0 \\
&\sigma \hat{\theta} + ik \hat{\psi} = [-k^2 + D^2] \hat{\theta} \\
&\text{combining the above, we obtain,} \\
&\boxed{\hat{\psi} = \frac{1}{ik} [-k^2 + D^2 - \sigma] \hat{\theta}} \quad \text{and} \\
&\boxed{[D^4 - 2k^2 D^2 + k^4] [D^2 - k^2] \hat{\theta} = \sigma [D^4 - 2k^2 D^2 + k^4] \hat{\theta}},
\end{aligned} \tag{52}$$

where  $D \equiv d_z$ .

## 5 Q 5: A lubrication approximation for Darcy flow in semi-saturated porous media:

Consider a  $2d$  shallow-water flow over a porous medium of length  $L$ . The lubrication approximation here would be  $\epsilon \equiv h/L \ll 1$ . We assume incompressibility and use Darcy's law as the momentum equations. The dimensional governing equations become:

$$\begin{aligned}\mathbf{u} &= -\frac{\kappa}{\mu} \nabla(p + \rho g z), \\ \nabla \cdot \mathbf{u} &= 0.\end{aligned}\tag{53}$$

In component form:

$$\begin{aligned}u &= -\frac{\kappa}{\mu} \frac{\partial p}{\partial x}, \\ w &= -\frac{\kappa}{\mu} \frac{\partial p}{\partial z} - \frac{\kappa \rho g}{\mu}, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.\end{aligned}\tag{54}$$

The boundary conditions are:

$$\begin{aligned}w(x, z = 0, t) &= 0, \\ w(x, z = h(x, t), t) &= \partial_t h + u \partial_x h, \\ p(x, z = h(x, t), t) &= p_0,\end{aligned}\tag{55}$$

where  $p_0$  is the constant atmospheric pressure impressed on the top of the groundwater layer (and capillary effects are being neglected). Scaling  $x \sim L, z \sim h, u \sim U, p \sim P$ . The continuity equation implies  $\frac{U}{L} \sim \frac{W}{h}$  or  $W \sim Uh/L = \epsilon U$ . The  $x$ -momentum equation implies  $U \sim \kappa P / \mu L \Rightarrow P \sim \mu UL / \kappa$ . In the  $z$ -momentum equation, the relative size of  $w$  and  $\frac{\kappa}{\mu} \frac{\partial p}{\partial z}$  term can be found to be:

$$\begin{aligned}|w| / \left| \frac{\kappa}{\mu} \frac{\partial p}{\partial z} \right| &\sim \frac{\epsilon U}{UL/h}, \\ |w| / \left| \frac{\kappa}{\mu} \frac{\partial p}{\partial z} \right| &\sim \epsilon^2.\end{aligned}\tag{56}$$

Hence, we neglect  $w$  at the leading order in the  $z$ -momentum equation. At the leading order, the dimensional  $z$ -momentum equation reads:

$$\frac{\partial p}{\partial z} = -\rho g.\tag{57}$$

Integrating, we obtain  $p = -\rho g z + c(x)$ . Using the boundary condition at the top surface  $z = h$ , obtain  $\boxed{p - p_0 = \rho g(h - z)}$ .



Substituting in the  $x$ -momentum equation, obtain:  $\boxed{u = -\frac{\kappa\rho g}{\mu}\partial_x h}$ .

Now, using the depth-averaged version of the continuity equation (see Eqns. (5) and (20))

$$\begin{aligned}\partial_t h + \partial_x \left[ \int_0^h u dz \right] &= 0, \\ \partial_t h - \partial_x \left[ \frac{\kappa\rho g}{\mu} \partial_x h [z]_0^h \right] &= 0, \\ \partial_t h - \partial_x \left[ \frac{\kappa\rho g}{\mu} h \partial_x h \right] &= 0, \\ M \partial_t h &= \partial_x [h \partial_x h],\end{aligned}\tag{58}$$

where  $M = \mu/(\kappa\rho g)$ . This is nonlinear diffusion equation for  $h(x, t)$ . Notice that pressure and  $h$  are linearly related in this problem. If there is a Gaussian pressure anomaly localized at  $x = 0$  at  $t = 0$ , it will diffuse as time goes on. The time-scale for this would be governed by the above nonlinear diffusion equation. Namely,  $M/t \sim h/L^2$  or  $t \sim ML^2/h = ML/\epsilon$ . This pressure diffusion is typical of porous media flows.