HW #2: Stokes Flow (Flow at Low Re)

Pratik Aghor

March 28, 2021

1 Q 1: A squirming sheet at zero Re:

We consider an infinitely-long extensible sheet at $y = 0$ in a viscous fluid. (x_s, y_s denote the co-ordinates of any particle on the sheet.

$$x_s = x_0 + a\sin(kx_0 - \omega t), y_s = 0, \tag{1}$$

with x_0 being the time averaged position of any given particle on the sheet and $ak = \epsilon \ll 1$.

In order to find the induced flow, we resort to the stream-function formulation of the Stokes equations. In addition, we impose no-normal flow and no-slip at the surface of the sheet and demand that u, v, the $x-$ and $y-$ induced velocities remain bounded. The dimensional governing equation and boundary conditions (BCs) then become:

$$\begin{aligned}
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]^2 \psi &= 0, \\
\left.\frac{\partial \psi}{\partial x}\right|_{x_s,0} &= 0, \\
\left.\frac{\partial \psi}{\partial y}\right|_{x_s,0} &= \frac{dx_s}{dt} = -a\omega\cos(kx_0 - \omega t), \\
&\text{finally, } \psi \text{ cannot grow more than linear (in x, y) away from the sheet.}
\end{aligned} \tag{2}$$

The condition at infinity ensures that u, v remain bounded at infinity, since $u = \psi_y, v = -\psi_x$.

We non-dimensionalize the problem by choosing $1/k$ to be the length scale. We choose $a\omega$ to be the velocity scale, inspired by the boundary conditions.

$$\tilde{x} = kx, \tilde{y} = ky, \tilde{t} = \omega t, \tilde{u} = u/(a\omega), \tilde{v} = v/(a\omega), \tilde{\psi} = \frac{a\omega}{k}\psi. \tag{3}$$

With these scalings, we write the non-dimensional form of Eqns.(1) as follows: (dropping tildes):

$$
\begin{aligned}
x_s = x_0 + \epsilon \sin(x_0 - t), &y_s = 0,\\
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]^2 \psi &= 0,\\
\left.\frac{\partial \psi}{\partial x}\right|_{x_s,0} &= 0,\\
\left.\frac{\partial \psi}{\partial y}\right|_{x_s,0} &= -\cos(x_0 - t),\\
&\text{finally, } \psi \text{ cannot grow more than linear (in x, y) away from the sheet.}
\end{aligned} \tag{4}
$$

Taylor expanding BCs, around $(x_0, 0)$, we obtain:

$$
\begin{aligned}
\left.\frac{\partial \psi}{\partial x}\right|_{x_s,0} &= \left.\frac{\partial \psi}{\partial x}\right|_{x_0,0} + [\epsilon \sin(x_0 - t)] \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{x_0,0} + O(\epsilon^2) = 0.\\
\left.\frac{\partial \psi}{\partial y}\right|_{x_s,0} &= \left.\frac{\partial \psi}{\partial y}\right|_{x_0,0} + [\epsilon \sin(x_0 - t)] \left.\frac{\partial^2 \psi}{\partial x \partial y}\right|_{x_0,0} + O(\epsilon^2) = -\cos(x_0 - t).
\end{aligned} \tag{5}
$$

Now, posing a regular perturbation ansatz for ψ

$$\psi \sim \psi_0 + \epsilon \psi_1 + ..., \tag{6}$$

and substituting into the dimensionless governing equations and BCs, collecting terms at different orders of ϵ, we get, at $O(1)$:

$$
\begin{aligned}
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]^2 \psi_0 &= 0,\\
\left.\frac{\partial \psi_0}{\partial x}\right|_{x_0,0} &= 0\\
\left.\frac{\partial \psi_0}{\partial y}\right|_{x_0,0} &= -\cos(x_0 - t).
\end{aligned} \tag{7}
$$

Guess $\psi_0 = F(y) \cos(x - t)$. Substituting in Eqn.(7),

$$F'''' - 2F'' + F = 0 \tag{8}$$

where primes denote differentiation wrt argument, here, y. Linear, constant coefficient, homogeneous ODE, $\Rightarrow F = ce^{\lambda y}$, giving $\lambda^4 - 2\lambda^2 + 1 = 0$. This yields $\lambda = \pm 1, \pm 1$. We discard the positive roots owing to the boundedness of velocity fields at infinity, therefore

$$\psi_0 = (A + By)e^{-y} \cos(x - t) \tag{9}$$

Applying boundary conditions, we get

$$
\begin{aligned}
&\left.\frac{\partial \psi_0}{\partial x}\right|_{x_0,0} = 0 \\
&\Rightarrow \left.(A+By)e^{-y}\sin(x-t)\right|_{(x_0,0)} = 0 \\
&\Rightarrow \boxed{A=0}. \\
&\left.\frac{\partial \psi_0}{\partial y}\right|_{x_0,0} = -\cos(x_0-t) \\
&\left.Be^{-y}\cos(x-t) - Bye^{-y}\cos(x-t)\right|_{x_0,0} = -\cos(x_0-t) \\
&\Rightarrow \boxed{B=-1}. \\
&\boxed{\psi_0 = -ye^{-y}\cos(x-t)}
\end{aligned}
\tag{10}
$$

Going to $O(\epsilon)$, we obtain:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]^2 \psi_1 = 0. \tag{11}$$

The boundary conditions become:

$$
\begin{aligned}
&\left.\frac{\partial \psi_1}{\partial x}\right|_{x_0,0} + \sin(x_0-t)\left.\frac{\partial^2 \psi_0}{\partial x^2}\right|_{x_0,0} = 0 \\
&\left.\frac{\partial \psi_1}{\partial x}\right|_{x_0,0} + \sin(x_0-t)\Big[\cos(x_0-t)(-\cancelto{0}{y}e^{-y})\Big] = 0 \\
&\boxed{\left.\frac{\partial \psi_1}{\partial x}\right|_{x_0,0} = 0}.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\left.\frac{\partial \psi_1}{\partial y}\right|_{x_0,0} + \sin(x_0-t)\left.\frac{\partial^2 \psi_0}{\partial x \partial y}\right|_{x_0,0} = 0 \\
&\left.\frac{\partial \psi_1}{\partial y}\right|_{x_0,0} + \sin(x_0-t)\Big[(\cancelto{0}{y}-1)\cancelto{1}{e^{-y}}(-\sin(x_0-t))\Big] = 0 \\
&\boxed{\left.\frac{\partial \psi_1}{\partial y}\right|_{x_0,0} = -\sin^2(x_0-t) = \frac{-1+\cos 2(x_0-t)}{2}}.
\end{aligned}
\tag{13}
$$

There is a $-1/2$ factor and $\cos 2(x_0-t)$ factor in the boundary condition, so we guess
$\boxed{\psi_1 = f(y)\cos 2(x_0-t) + g(y)}$. Substituting in the Eqn. (11),

$$
\begin{aligned}
&16f - 8f'' + f'''' = 0 \\
&g'''' = 0
\end{aligned}
\tag{14}
$$

Integrating the g equation first, we get $g(y) = A_0 + A_1 y + A_2 y^2 + A_3 y^3$. We put $A_2 = A_3 = 0$, owing to boundedness of velocity fields at infinity. Hence, $\boxed{g(y) = A_0 + A_1 y}$.

Now, solving the f equation, $f = ce^{\lambda y}$ gives, $\lambda^4 - 8\lambda^2 + 16 = 0$, yielding $\lambda = \pm 2, \pm 2$. Again, we reject positive roots owing to boundedness of velocity fields at infinity. $\boxed{f = (C + Dy)e^{-2y}}$. Combining $\boxed{\psi_1 = A_0 + A_1 y + (C + Dy)e^{-2y}\cos 2(x_0 - t)}$. Applying boundary conditions,

$$
\left.\frac{\partial \psi_1}{\partial y}\right|_{x_0,0} = A_1 - 2Ce^{-2y}\cos 2(x_0 - t) + e^{-2y}\cos 2(x_0 - t) + D\cancelto{0}{y}\ldots|_{x_0,0} = \frac{-1 + \cos 2(x_0 - t)}{2}
$$

$$
A_1 + (-2C + D)\cos 2(x_0 - t) = \frac{-1 + \cos 2(x_0 - t)}{2}.
$$

$$
\boxed{A_1 = \frac{1}{2}}, \boxed{D - 2C = \frac{1}{2}}.
\tag{15}
$$

$$
\begin{aligned}
&\left.\frac{\partial \psi_1}{\partial x}\right|_{x_0,0} = 0 \\
&C \sin{(x_0 - t)} = 0 \\
&\boxed{C = 0}
\end{aligned}
\tag{16}
$$

Implying $\boxed{D = 1/2}$. Without loss of generality we substitute $A_0 = 0$ (as we are interested in the gradients of ψ, not ψ itself).

$$
\boxed{\psi_1 = \frac{y}{2}e^{-2y}\cos 2(x - t) - \frac{y}{2}}.
$$

$$
u = \frac{\partial \psi}{\partial y} = (y-1)e^{-y}\cos{(x - t)} + \epsilon\left[\frac{1}{2}e^{-2y}\cos 2(x - t) - ye^{-2y}\cos 2(x - t)\right] - \frac{\epsilon}{2}.
$$

At infinity, away from the sheet, we obtain a steady streaming flow: $\boxed{U = -\epsilon/2}$.

In dimensional terms, $\boxed{U = -2\pi^2\left(\frac{a}{\lambda}\right)^2 c}$, where $\lambda = \frac{2\pi}{k}$ is the wavelength and $c = \frac{\omega}{k}$ is the phase speed. This induced flow is in the opposite direction as that obtained from an undulating sheet ([Acheson, 1991]).

2 Q 2: Drag on a sphere in Stokes flow:

We derived the Stokes streamfunction for a Stokes flow past a sphere in class:

$$\psi(r,\theta) = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta \tag{17}$$

Remember the standard $z-$ direction is x-direction in our case, in the sense that the polar angle θ is measured from the x- axis.

The radial and polar velocities are then given by

$$\begin{aligned} u_r &= \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\cos\theta \\ u_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\left(1 - \frac{1}{4r^3} - \frac{3}{4r}\right)\sin\theta \end{aligned} \tag{18}$$

Integrating the r-momentum equation, we find the pressure distribution in the domain:

$$\begin{aligned} \nabla p &= \nabla^2\boldsymbol{u} \\ &= \nabla^2\boldsymbol{u} - \nabla(\nabla.\boldsymbol{u}) \;...\because \nabla.\boldsymbol{u} = 0 \\ &= -\nabla\times(\nabla\times\boldsymbol{u}) \\ \frac{\partial p}{\partial r} &= -[\nabla\times(\nabla\times\boldsymbol{u})]_r \\ &= -\left[\nabla\times\left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\right]_r \\ &= -\left[\nabla\times\left(\frac{-3\sin\theta}{2r^2}\hat{e}_\phi\right)\right]_r \\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\cancel{r}\sin\theta\cdot\frac{3\sin\theta}{2r^{\cancel{2}}}\right)\right] \\ &= \frac{3\cos\theta}{r^3}. \end{aligned} \tag{19}$$

Integrating from r to ∞, we obtain

$$\begin{aligned} \int_r^\infty \frac{\partial p}{\partial r}dr &= \int_r^\infty \frac{3\cos\theta}{r^3}dr \\ p_\infty - p &= \frac{3}{2}\frac{\cos\theta}{r^2} \\ &\boxed{p = p_\infty - \frac{3}{2}\frac{\cos\theta}{r^2}}. \end{aligned} \tag{20}$$

We first obtain the stress vector τ in terms of the stress tensor $\mathcal{T}$. We know, $\tau_i = \mathcal{T}_{ij}n_j$. For evaluation at the surface of the sphere $r = 1$, $n_j = \hat{e}_r = [1,0,0]^T$

in spherical polar co-ordinates. We evaluate τ_i's at the surface of the sphere, because we are interested in the drag on the sphere.

$$\begin{aligned}
\tau_r &= \mathcal{T}_{rr} = [-p + 2e_{rr}]|_{r=1} \\
&= -p_\infty + \frac{3}{2}\cos\theta \;...\; \because e_{rr}|_{r=1} = \left.\frac{\partial u_r}{\partial r}\right|_{r=1} = 0 \\
\tau_\theta &= \mathcal{T}_{\theta r} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\
&= -\frac{3}{2}\sin\theta \\
\tau_\theta &= \mathcal{T}_{\phi r} = 0.
\end{aligned} \tag{21}$$

To calculate the $x-$component of the stress vector, with a bit of geometry, we can get

$$\begin{aligned}
\tau_x &= \tau_r\cos\theta - \tau_\theta\sin\theta \\
&= -p_\infty\cos\theta + \frac{3}{2}.
\end{aligned} \tag{22}$$

The drag on the surface of the sphere ($r = 1$) then is given by

$$\begin{aligned}
D &= \int_0^{2\pi}\int_0^{\pi} \tau_x \sin\theta d\theta d\phi \\
&= \cancelto{0}{\left[\int_0^{2\pi}\int_0^{\pi} -p_\infty \cos\theta\sin\theta d\theta d\phi\right]} + \frac{3}{2}(4\pi) \\
&= 6\pi.
\end{aligned} \tag{23}$$

In dimensional terms, we obtain the famous Stokes drag formula $\boxed{D = 6\pi\mu U a}$, where a is the radius of the sphere, moving with velocity U and μ is the dynamic viscosity. The drag coefficient C_D can then be written as

$$C_D = \frac{D}{\rho U^2 a^2} = \frac{6\pi\mu U a}{\rho U^2 a^2} = \frac{6\pi}{\frac{\rho U(a)}{\mu}} = \frac{6\pi}{Re} \tag{24}$$

where $Re = \frac{\rho U(a)}{\mu}$ is the Reynolds number based on the radius of the sphere.

Oseen's Improvement

The dimensionless Oseen's equation in the frame of reference of the moving sphere (so the sphere is at rest in this frame) are given by:

$$\begin{aligned}
Re\frac{\partial \boldsymbol{u}}{\partial x} &= -\nabla p + \nabla^2\boldsymbol{u}, \\
\nabla\cdot\boldsymbol{u} &= 0.
\end{aligned} \tag{25}$$

Taking the curl of Eqn.(25), and noting that $\nabla \times \nabla p = 0$ identically, we obtain the vorticity form of Oseen's equations.

$$
\begin{aligned}
Re\frac{\partial \boldsymbol{\omega}}{\partial x} &= \nabla^2 \boldsymbol{\omega} \\
Re\left(\frac{1}{\cos\theta}\frac{\partial \boldsymbol{\omega}}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{\omega}}{\partial \theta}\right) &= \nabla^2 \boldsymbol{\omega} \;... \because x = r\cos\theta
\end{aligned} \tag{26}
$$

In the axisymmetric case, substituting u_r, u_θ from Eqn.(18),

$$
\begin{aligned}
\boldsymbol{\omega} &= \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{e}_\phi \\
&= \left(\frac{1}{r}\frac{\partial}{\partial r}\left[\cancel{r}\left(-\frac{1}{\cancel{r}\sin\theta}\frac{\partial \psi}{\partial r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}\right]\right)\hat{e}_\phi \\
&= \left(-\frac{1}{r\sin\theta}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^3}\left[-\frac{\cot\theta}{\sin\theta}\frac{\partial \psi}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \theta^2}\right]\right)\hat{e}_\phi \\
&= -\frac{1}{r\sin\theta}D^2\psi
\end{aligned} \tag{27}
$$

where D^2 is given by

$$
D^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\right] \;...(\text{NOTE:}D^2 \neq \nabla^2). \tag{28}
$$

Substituing Eqn. (27) in Eqn. (26), we get

$$
\left(\frac{1}{Re}D^2 - \cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)D^2\psi = 0. \tag{29}
$$

Writing $\cos\theta = c$ we get

$$
\begin{aligned}
\left(\frac{1}{Re}D^2 - \cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)D^2\psi &= 0. \\
\left(\frac{1}{Re}D^2 - c\frac{\partial}{\partial r} + \frac{(1-c^2)^{1/2}}{r}\frac{\partial}{\partial \theta}\right)D^2\psi &= 0. \\
\left(\frac{1}{Re}D^2 - c\frac{\partial}{\partial r} + \frac{(1-c^2)^{1/2}}{r}\frac{\overset{-(1-c^2)^{1/2}}{\cancel{dc}}}{\cancel{d\theta}}\frac{\partial}{\partial c}\right)D^2\psi &= 0. \\
\left(\frac{1}{Re}D^2 - c\frac{\partial}{\partial r} - \frac{(1-c^2)}{r}\frac{\partial}{\partial c}\right)D^2\psi &= 0.
\end{aligned} \tag{30}
$$

We are asked to verify by direct substitution that $\psi(r,\theta;Re) = (1+c)[1 - e^{-\frac{1}{2}Re r(1-c)}]$ satisfies Eqn. (30).

First, we convert the $D^2(r,\theta)$ into $D^2(r,c)$.

Using chain rule, we have $\frac{\partial}{\partial\theta} = \frac{\partial c}{\partial\theta}\frac{\partial}{\partial c} = -(1-c^2)^{1/2}\frac{\partial}{\partial c}$. Similarly, applying $\frac{\partial}{\partial\theta}$ one more time, we obtain:

$$
\begin{aligned}
\frac{\partial^2}{\partial\theta^2} &= \overset{-(1-c^2)^{1/2}}{\cancel{\frac{\partial c}{\partial\theta}}}\frac{\partial}{\partial c}\left(-(1-c^2)^{1/2}\frac{\partial}{\partial c}\right) \\
&= (1-c^2)^{1/2}\left((1-c^2)^{1/2}\frac{\partial^2}{\partial c^2} - \frac{c}{(1-c^2)^{1/2}}\frac{\partial}{\partial c}\right) \\
&= (1-c^2)\frac{\partial^2}{\partial c^2} - c\frac{\partial}{\partial c}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
D^2 &= \left[\frac{\partial^2}{\partial r^2} + \frac{1-c^2}{r^2}\frac{\partial^2}{\partial c^2} - \cancel{\frac{c}{r^2}\frac{\partial}{\partial c}} + \cancel{\frac{c}{r^2}\frac{\partial}{\partial c}}\right] \\
&\boxed{D^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{1-c^2}{r^2}\frac{\partial^2}{\partial c^2}\right]}
\end{aligned} \tag{32}
$$

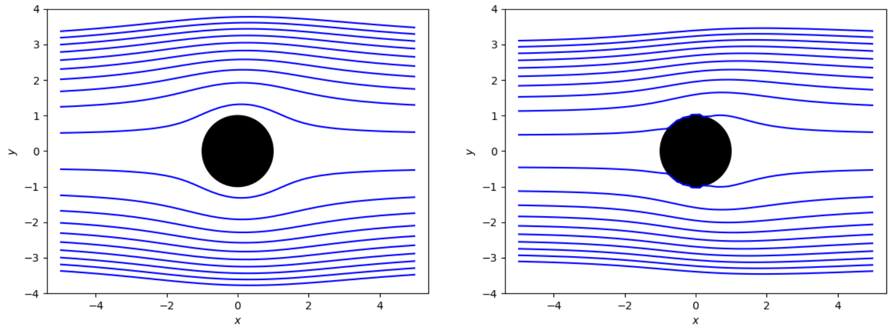
Let us find $D^2\psi$ now.

$$
\begin{aligned}
D^2\psi &= \left[\frac{\partial^2}{\partial r^2}(1+c)[1-e^{-\frac{1}{2}Rer(1-c)}] + \frac{1-c^2}{r^2}\frac{\partial^2}{\partial c^2}(1+c)[1-e^{-\frac{1}{2}Rer(1-c)}]\right] \\
&= \frac{Re^2}{4}(1-c^2)^2 e^{-\frac{1}{2}Rer(1-c)} - \frac{1-c^2}{r^2}e^{-\frac{1}{2}Rer(1-c)}\left[\frac{1}{4}Re^2r^2 + \frac{cRer}{2} + 1\right]
\end{aligned} \tag{33}
$$

Substituing Eqn.(33)into Eqn. (30), I verified that $\psi(r,\theta;Re) = (1+c)[1-e^{-\frac{1}{2}Rer(1-c)}]$ satisfies Eqn. (30). I did not have enough time to verify or type out that the following stream function also satisfies Eqn. (30) and the boundary conditions $\psi = \frac{\partial\psi}{\partial r} = 0$ on $r=1$ and $\psi \sim \frac{r^2}{2}\sin\theta$ as $r\to\infty$.

$$
\psi(r,\theta;Re) \sim \left(\frac{r^2}{2} + \frac{1}{4r}\right)\sin^2\theta - \frac{3}{2Re}(1+c)[1-e^{-\frac{1}{2}Rer(1-c)}]. \tag{34}
$$

I did plot it though. As can be seen fro Fig.(1), the flow at $Re = 0.1$ is fore-aft symmetric, while the flow at $Re = 1$ breaks the fore-aft symmetry.



(a) $Re = 0.1$ (b) $Re = 1$

Figure 1: Flow at $Re = 1$ breaks the fore-aft symmetry.

3 Q 3: Superposition of singular solutions of Stokes flow:

Velocity field due to a single Stokeslet:

We derived in class the strema-function ψ due to a Stokeslet of strength F situated at the origin.

$$\psi \sim -\frac{3}{4} r \sin^2 \theta \tag{35}$$

This gives the induced velocity field to be

$$u(r, \theta, \phi) = \frac{F}{8\pi\mu} \left(\frac{2\cos\theta}{r}, -\frac{\sin\theta}{r} \right) \tag{36}$$

We can write this in terms of Cartesian co-ordinates, with

$$u(x, y) = \frac{F}{8\pi\mu} \left(\frac{2x}{(x^2+y^2)}, -\frac{y}{(x^2+y^2)} \right) \tag{37}$$

The unit vectors are still $\hat{e}_r$, $\hat{e}_\theta$. In order to get to Cartesian co-ordinates, $\hat{e}_x = \cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta$, $\hat{e}_y = \sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta$. Moreover, we have $\tan\theta = y/x$, giving $\cos\theta = x/(x^2+r^2)^{1/2}$ and $\sin\theta = y/(x^2+r^2)^{1/2}$, yielding

$$\boldsymbol{u}(x, y) = \frac{F}{8\pi\mu} \left(\frac{2x^2+y^2}{(x^2+y^2)^{3/2}} \hat{e}_x + \frac{xy}{(x^2+y^2)^{3/2}} \hat{e}_y \right) \tag{38}$$

Similarly, the stremfunction $\psi(x, y) = -\frac{3}{4} \frac{y^2}{(x^2+y^2)^{1/2}}$. I plotted this using python:

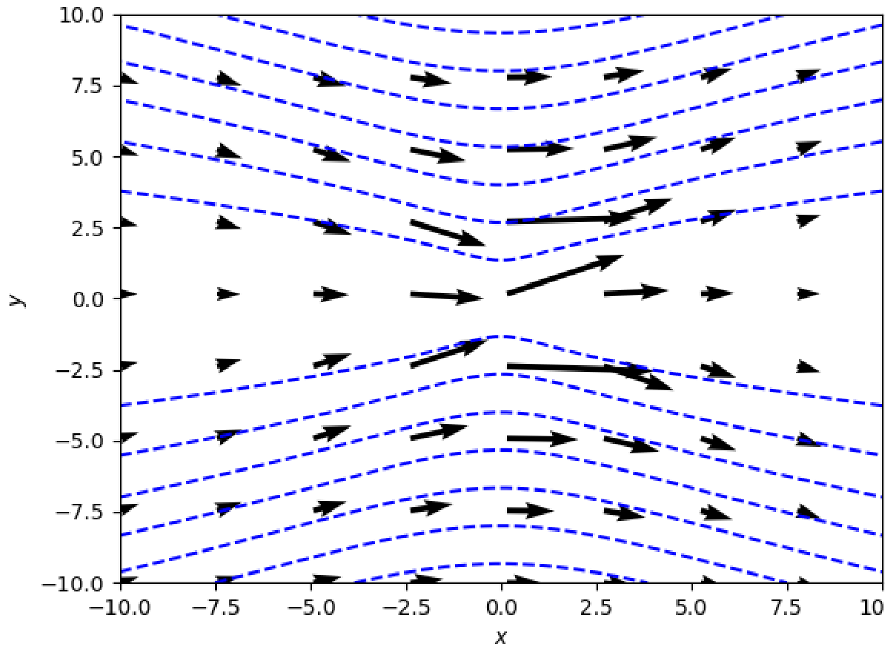


Figure 2: Flow induced due to a single Stokeslet. Near (0,0) we get singular behavior. The dashed blue lines represent contours of the stream function ψ.

Finite slender cylinder:

Consider a finite slender cylinder ($x = -b$ to $x = c$), with radius a ($b, c \gg a$), moving in a direction parallel to its axis. We want to derive an expression for the induced velocity field due to a line of Stokeslets of strength $(f dx, 0, 0)$, distributed along its centerline.

Without loss of generality, we consider a Stokeslet sitting at $X = 0$. The distance r of any point in the domain from the surface of the cylinder is then approximately given by $r^2 = (x - X)^2 + y^2 + z^2$. On the surface of the cylinder $y^2 + z^2 = a^2$, hence, $\boxed{r^2 = x^2 + a^2}$.

$$\boldsymbol{du}_{Stokeslet}\Big|_{X=0, y^2+z^2=a^2} = \frac{f}{8\pi\mu}\left(\frac{x^2 + r^2}{r^3}, \frac{xy}{r^3}, \frac{xz}{r^3}\right). \tag{39}$$

In order to calculate the induced field due to a continuum of Stokeslets, we must integrate

$$\boldsymbol{u}_{Stokeslet} = \frac{f}{8\pi\mu}\int_{x=-b}^{c}\left(\frac{x^2 + r^2}{r^3}, \frac{xy}{r^3}, \frac{xz}{r^3}\right) = I_1\hat{e}_x + I_2\hat{e}_y + I_3\hat{e}_z. \tag{40}$$

Let us focus on the I_2 and I_3 integrals first. The integrals will be of the form $\int_{-b}^{c} \frac{x}{r^3} dx$. Changing variables to $\chi = x/a$

$$I_{2,3} \sim \int_{-b/a\to-\infty}^{c/a\to\infty} \frac{x}{r^3} dx \tag{41}$$

But $\frac{x}{(x^2+a^2)^{3/2}}$ is an odd function of x, hence on a symmetric interval, I_2, I_3 will vanish. In this case, they will vanish owing to the 'slender-ness' of the cylinder.

Now, let's focus on the I_1 integral.

$$\begin{aligned}
I_1 &\sim \int_{x=-b}^{c} \frac{x^2+r^2}{r^3} dx \\
&\sim \int_{x=-b}^{c} \frac{x^2+x^2+a^2}{r^3} dx \\
&\sim \int_{x=-b}^{c} \frac{2(x^2+a^2)-a^2}{r^3} dx \\
&\sim 2\int_{x=-b}^{c} \frac{dx}{(x^2+a^2)^{1/2}} - a^2 \int_{x=-b}^{c} \frac{dx}{(x^2+a^2)^{3/2}} \\
&\text{chaning variables to } \chi = x/a, \text{ and noting } b/a, c/a \to \infty \\
&\sim 2\int_{-b/a}^{c/a} \frac{d\chi}{(1+\chi^2)^{1/2}} - \int_{-b/a}^{c/a} \frac{d\chi}{(1+\chi^2)^{3/2}} \\
&\text{substituting } \chi = \sinh(\phi) \\
&\sim 2\left[\sinh^{-1}\left(\frac{c}{a}\right) + \sinh^{-1}\left(\frac{b}{a}\right)\right] - 2 \\
&\sim 2\ln\frac{4cb}{a^2} - 2
\end{aligned} \tag{42}$$

Therefore

$$\boldsymbol{u}_{Stokeslet} = \frac{f}{8\pi\mu}\left(2\ln\frac{4cb}{a^2} - 2, 0, 0\right) \tag{43}$$

This already gives the uniform flow at the surface of a cylinder, so we expect that the doublet contribution would be zero or a constant. Let us calculate the induced velocity due to a source-doublet distribution of to-be-determined strength $(g dx, 0, 0)$. As for the Stokeslet, to obtain the total induced velocity we integrate the velocity field induced by doublets along the axis of the cylinder.

$$\boldsymbol{u}\Big|_{doublet} = \frac{g}{4\pi}\int_{x=-b}^{c} \left(\frac{1}{r^3} - \frac{3x^2}{r^5}, \frac{-3xy}{r^5}, \frac{3xz}{r^5}\right) dx = I_{1d}\hat{e}_x + I_{2d}\hat{e}_y + I_{3d}\hat{e}_z. \tag{44}$$

The subscript 'd' denotes the contribution due to doublets. As before, the I_{2d}, I_{3d} vanish owing to the 'odd-ness' of x/r^5 in x, in the limit $b/a, c/a \to \infty$.

Consider

$$
\begin{aligned}
&I_{1d} \sim \int_{-b}^{c} \frac{dx}{(x^2+a^2)^{3/2}} - \int_{-b}^{c} \frac{3x^2}{(x^2+a^2)^{5/2}} dx \\
&\quad \text{the first part, as above } = \frac{2}{a^2} \\
&\quad \sim \frac{2}{a^2} - 3 \int_{-b}^{c} \frac{x^2+a^2-a^2}{(x^2+a^2)^{5/2}} dx \\
&\quad \text{changing variable to } \chi = x/a \\
&\quad \sim \frac{2}{a^2} - \frac{3}{a^2} \left(\cancelto{2}{\int_{-\infty}^{\infty} \frac{d\chi}{(1+\chi^2)^{3/2}}} - \int_{-\infty}^{\infty} \frac{d\chi}{(1+\chi^2)^{5/2}} \right) \\
&\quad \text{substituting } \chi = \sinh\phi \\
&\quad \sim \frac{2}{a^2} - \frac{3}{a^2} \left(2 - \int_{-\infty}^{\infty} \operatorname{sech}^4 \phi d\phi \right) \\
&\quad \sim \frac{2}{a^2} - \frac{3}{a^2} \left(2 - \frac{1}{3} \left[\left(\frac{\cancel{e^{2\phi}+e^{-2\phi}}}{2} + 2 \right) \tanh\phi \frac{4}{\cancel{(e^{\phi}+e^{-\phi})^2}} \right]_{-\infty}^{\infty} \right) \\
&\quad \text{The cancelled terms above cancel each other in the limit } \pm\infty \\
&\quad \sim \frac{2}{a^2} - \frac{3}{a^2} \left(2 - \frac{1}{3} \left[\frac{1}{2} \right] \cdot (1-(-1)) \cdot 4 \right) \\
&\quad \sim \frac{2}{a^2} - \frac{3}{a^2} \left(2 - \frac{4}{3} \right) \\
&\quad = 0
\end{aligned}
\tag{45}
$$

As expected, the doublet contribution comes out to be exactly **0**. Hence any finite g would do the job, since the contribution from the doubltes is nil. The total induced velocity field is then

$$
\boldsymbol{u} = \boldsymbol{u}_{Stokeslet} = \frac{f}{8\pi\mu} \left(2 \ln \frac{4cb}{a^2} - 2, 0, 0 \right) \tag{46}
$$

References

[Acheson, 1991] Acheson, D. J. (1991). Elementary fluid dynamics.