The two variable Newton method for solving equations

Let $a(x,y)$ and $b(x,y)$ be two differentiable functions of x and y. In calculus we sometimes need to solve the equations

(*) $a(x,y) = 0$ and $b(x,y) = 0$

simultaneously. Here are two examples.

Example 1 Find the critical points of a function f(x,y).

We need to find simultaneous solutions to the equations $\frac{\partial f}{\partial x}(x,y) = 0$ and $\frac{\partial f}{\partial y}(x,y) = 0$. This is (*), where $a(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $b(x,y) = \frac{\partial f}{\partial y}(x,y)$.

Example 2 Find the maximum and minimum values of a function f(x,y) on a curve of the form $g(x,y) = c$. The method of Lagrange multipliers says that we must solve the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, which becomes the two equations

(1) $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ and (2) $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$

Multiplying (1) by $\frac{\partial g}{\partial y}$ and multiplying (2) by $\frac{\partial g}{\partial x}$ we obtain $\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$. Hence we must solve the equations (*), where $a(x,y) = g(x,y) - c$ and $b(x,y) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

Newton's method of solution

In general there is no method for getting solutions to (*) that you can write down. However the Newton method gives a numerical procedure that solves the equation (*) to any desired degree of accuracy. The idea behind the Newton method is very simple. We describe it first and then illustrate the method with two examples.

Step 1 By some method we find an approximate solution (x_o,y_o) to (*).

One way to find an approximate solution to (*) would be to graph the curves $a(x,y) = 0$ and $b(x,y) = 0$, and then look to see where these two curves intersect. A computer program with a zoom finder can be useful here.

Step 2 We replace the linear equations (*) by the approximate equations

$(*)_o$ $a_o(x,y) = 0$ and $b_o(x,y) = 0$

where $a_o(x,y)$ and $b_o(x,y)$ are the linear approximations of $a(x,y)$ and $b(x,y)$ at (x_o,y_o). The equation $(*)_o$ is a linear equation in two unknowns x and y, and it can be easily solved by hand. The solution (x_1,y_1) to $(*)_o$ is not the true solution to the original equation (*), but in general it is a better approximation than (x_o,y_o).

Recall that the linear approximation of a function f(x,y) at a point (x_o,y_o) is given by

$$f_o(x,y) = A + B(x - x_o) + C(y - y_o)$$

where $A = f(x_o,y_o)$, $B = \frac{\partial f}{\partial x}(x_o,y_o)$ and $C = \frac{\partial f}{\partial y}(x_o,y_o)$, all real numbers.

Step 3 We replace the linear equations (*) by the approximate equations

$(*)_1$ $a_1(x,y) = 0$ and $b_1(x,y) = 0$

where $a_1(x,y)$ and $b_1(x,y)$ are the linear approximations of a(x,y) and b(x,y) at (x_1,y_1). Let (x_2,y_2) be the solution to $(*)_1$.

Step 4 Continue as described above. In general we get a sequence of approximate solutions (x_1,y_1), (x_2,y_2), ... , (x_k,y_k) that converge rapidly to the true solution if the beginning approximate solution (x_o,y_o) is reasonably accurate. To obtain the next approximate solution $(x_{k+1}y_{k+1})$ we solve the equations

$(*)_k$ $a_k(x,y) = 0$ and $b_k(x,y) = 0$

where $a_k(x,y)$ and $b_k(x,y)$ are the linear approximations of a(x,y) and b(x,y) at (x_k,y_k).

A matrix formula for the approximate solutions (x_k,y_k)

In order to solve the approximate equations $(*)_k$ above rapidly with a computer it is useful to have a formula that one can program into the computer. First we define the Jacobian matrix $J(x,y) = \begin{bmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{bmatrix}$. If we write (x_k,y_k) and (x_{k+1},y_{k+1}) as column vectors $\begin{bmatrix} x_k \\ y_k \end{bmatrix}$ and $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix}$ then the equation $(*)_k$ can be written in matrix form as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a(x_k) \\ b(y_k) \end{bmatrix} + J(x_k,y_k)\begin{bmatrix} x-x_k \\ y-y_k \end{bmatrix} = \begin{bmatrix} a(x_k) \\ b(y_k) \end{bmatrix} + J(x_k,y_k)\begin{bmatrix} x \\ y \end{bmatrix} - J(x_k,y_k)\begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

The solution $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix}$ to $(*)_k$ in matrix form now becomes

$$(*)_k \quad \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J(x_k,y_k)^{-1}\begin{bmatrix} a(x_k) \\ b(y_k,) \end{bmatrix}$$

where $J(x_k,y_k)^{-1}$ denotes the inverse of the matrix $J(x_k,y_k)$.

Example 1 Find an intersection point of the circle $x^2 + y^2 = 1$ and the parabola $y = x^2$.

We need to solve the equations

(*) $a(x,y) = 0$ where $a(x,y) = y - x^2$

$b(x,y) = 0$ where $b(x,y) = x^2 + y^2 - 1$

We pick $(x_o,y_o) = (1,1)$ as an approximate solution. If $a_o(x,y)$ and $b_o(x,y)$ denote the linear approximations at (x_o,y_o), then we obtain $a_o(x,y) = -2x + y + 1$ and $b_o(x,y) = 2x + 2y - 3$. Hence the equations $a_o(x,y) = 0$ and $b_o(x,y) = 0$ have the solution $\mathbf{(x_1, y_1) = (5 / 6, 2 / 3)}$.

If $a_1(x,y)$ and $b_1(x,y)$ denote the linear approximations at (x_1,y_1), then we obtain $a_1(x,y) = -(5/3)x + y + (25/36)$ and $b_1(x,y) = (5/3)x + (4/3)y - (77/36)$. Hence the equations $a_1(x,y) = 0$ and $b_1(x,y) = 0$ have the solution $\mathbf{(x_2, y_2) = (331 / 420, 13 / 21)}$.

If we continue this process we obtain the following approximate solutions

$\mathbf{(x_1, y_1) = (5 / 6, 2 / 3)} \approx (.8333, .6667)$

$\mathbf{(x_2, y_2) = (331 / 420, 13 / 21)} \approx (.7881, .6190)$

$\mathbf{(x_3, y_3)} \approx (.7861, .6180)$. In two more steps we obtain 15 digit accuracy

$\mathbf{(x_5, y_5) \approx (.786151377757423, .618033988749894)}$

Example 2 Find a critical point of the function $f(x,y) = 2x^2y^2 + x^2y - 2x - y^2$

We calculate $a(x,y) = \frac{\partial f}{\partial x}(x,y) = 4xy^2 + 2xy - 2$

$b(x,y) = \frac{\partial f}{\partial y}(x,y) = 4x^2y + x^2 - 2y$

The critical points of $f(x,y)$ are the solutions to the equations $a(x,y) = 0$ and $b(x,y) = 0$.

We start with the approximate solution $(x_o,y_o) = (1,-1)$ and we obtain the equations

$0 = a_o(x,y) = 0 + 2(x-1) - 6(y+1) = 2x - 6y - 8$

$0 = b_o(x,y) = -1 - 6(x-1) + 2(y+1) = -6x + 2y + 7$

The solution to these equations is the next approximate solution

$\mathbf{(x_1, y_1) = (13/16 , -17/16) = (.8125, -1.0625)}$ Continuing we obtain

$\mathbf{(x_2, y_2) \approx (.8069, -1.0759)}$ In two more steps we obtain 15 digit accuracy

$\mathbf{(x_4, y_4) \approx (.807068419897086, -1.075848752008690)}$