Predictor-corrector for Bratu's problem :-

$$u''(x) + \lambda e^{u(x)} = 0 \qquad u(0) = u(1) = 0$$

$\hookrightarrow x \in [0, 1]$

18

$x = 0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1$

$0 \quad 1 \quad 2 \quad 3 \quad 4$ ... $N = 5 \rightarrow 0$ to 4.

$$u(0) = 0 \qquad u(4) = u(N-1) = 0.$$

$$u''(x) \simeq \frac{u^{j+1} - 2u^j + u^{j-1}}{h^2}$$

$j=1 \rightarrow \quad \dfrac{u^2 - 2u^1 + \cancel{u^0}^{\,0\ \because\ bc.}}{h^2} + \lambda e^{u(1)} = 0$

$j=2 \rightarrow \quad \dfrac{u^3 - 2u^2 + u^1}{h^2} + \lambda e^{u(2)} = 0$

$j=3 \rightarrow \quad \dfrac{\cancel{u^4}^{\,0\ bc} - 2u^3 + u^2}{h^2} + \lambda e^{u^3} = 0$

$F(u, \lambda)$

$(N-2, 1)$

$$\underbrace{F_u(u, \lambda)}_{(N-2 \times N-2)} = \begin{bmatrix} -\frac{2}{h^2} + \lambda e^{u^1} & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u^2} & \frac{1}{h^2} \\ & \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u^3} \end{bmatrix}$$

given (u_0, λ_0)

step #1 :- find $\dot{u}_0 = \left.\frac{du}{d\lambda}\right|_{u_0}$ by solving

$$F_u(u_0, \lambda_0)\, \dot{u}_0 = - F_\lambda (u_0, \lambda_0)$$

step #(2a) :- $\lambda_1 = \lambda_0 + \Delta\lambda$

$U_1^{(0)} = U_0 + \Delta\lambda\, \dot{U}_0$ -- Initial guess for Newton iterations

step #(2b) :- Do the Newton iterations

$$\underbrace{F_u(U_1^{(\nu)}, \lambda_1)}_{(N-2)\times(N-2)}\, \underbrace{\Delta U_1^{(\nu)}}_{(N-2)\times 1} = \underbrace{-F(U_1^{(\nu)}, \lambda_1)}_{(N-2)\times 1}$$

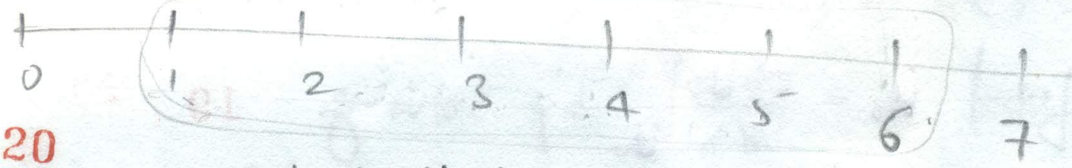
$$\Delta U_1^{(0)} = \left[F_u(U_1^{(0)}, \lambda_1)\right]^{-1} \left(-F(U_1^{(0)}, \lambda_1)\right)$$

$$U_1^{(1)} = U_1^{(0)} + \Delta U_1^{(0)}$$

Repeat until $\| U_1^{(\nu+1)} - U_1^{\nu} \| <$ tolerance.

$$\boxed{U_1 = U_0 + \Delta U_0} \quad \to \Delta U_0$$

go to step #1. & do the same thing



N =

$u'' + \lambda e^{u(x)} = 0$ $\qquad u(0) = u(\underset{(N-1)}{7}) = 0$

$\cancel{u_0 \qquad = 0 \qquad F(0)}$

$\dfrac{\cancel{u_0}^{\,0} - 2u_1 + u_2}{h^2} + \lambda e^{u_1} = 0 \longrightarrow F(0)$

$\dfrac{u_1 - 2u_2 + u_3}{h^2} + \lambda e^{u_2} = 0 \longrightarrow F(1)$

$\dfrac{u_2 - 2u_3 + u_4}{h^2} + \lambda e^{u_3} = 0$

$\dfrac{u_3 - 2u_4 + u_5}{h^2} + \lambda e^{u_4} = 0$

$\dfrac{u_4 - 2u_5 + u_6}{h^2} + \lambda e^{u_5} = 0 \longrightarrow F(N-3)$

$\dfrac{u_5 - 2u_6 + \cancel{u_7}^{\,0}}{h^2} + \lambda e^{u_6} = 0 \longrightarrow F(N-2)$

$\cancel{u_7 \qquad = 0 \longrightarrow F(N-1)}$

size of $F(U, \lambda) = (N-2) \times 1$

$u_1 = u\_cropped(0)$

$u_2 = \text{—"—}(1)$

$\boxed{u_i = \text{—"—}(i-1)}$

$F_U(U, \lambda) = \dfrac{\partial F_i}{\partial U_j}$ $\qquad F_U \to (N-2) \times (N-2)$ matrix

$i \to 0$ to $N-3$

~~$j \to 0$ to $N-3$~~

$$F_u = \begin{bmatrix} -\frac{2}{h^2} + \lambda e^{u_1} & \frac{1}{h^2} & & \\ \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u_2} & \frac{1}{h^2} & \\ 0 & \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u_3} & \frac{1}{h^2} \\ & & & \ddots \\ & & \frac{1}{h^2} & -\frac{2}{h^2} + \lambda e^{u_6} \end{bmatrix}$$

$F_u(5, 4)$

$F_u(5, 5)$

$$\underline{u_1} = \begin{bmatrix} 0 \\ * \\ * \\ * \\ * \\ * \\ * \\ 0 \end{bmatrix} \begin{array}{l} \to 0 \\ \to 1 \\ \to 2 \\ \vdots \\ \\ \\ n-2 \\ n-1 \end{array} \qquad u_{1\text{-cropped}} = \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix}$$

$$F_\lambda = \begin{bmatrix} e^{\lambda u\_cropped(0)} \\ e^{\lambda u\_cropped(1)} \\ \vdots \\ e^{\lambda u\_cropped(n+1)} \end{bmatrix}$$

22

* Arclength Continuatⁿ :-

(1) $\underline{F}(\underline{u}, \lambda) = 0$ — (1) (u_0, λ_0) known

(2) add an extra eqⁿ

$$(u_1 - u_0)\frac{\partial u_0}{\partial s} + (\lambda_1 - \lambda_0)\frac{\partial \lambda_0}{\partial s} - \Delta s = 0$$

$$(u_1 - u_0)\dot{u}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s = 0 \quad \text{— (2)}$$

(3) The extended system of eqⁿs now becomes →

$$\begin{bmatrix} \underline{F}(\underline{u}, \lambda) \\ (u_1 - u_0)\dot{u}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s \end{bmatrix} = 0 \quad \text{— (3)}$$

↳ $(n+1) \times 1$ eqⁿs

$(n+1)$ unknowns → $(\underline{u}_1, \lambda_1)$

(4) Find the tangent vector → $\dot{u}_1, \dot{\lambda}_1$ → how?

solve → (1) ⇒ $\frac{\partial F}{\partial u}\dot{u}_1 + \frac{\partial F}{\partial \lambda}\dot{\lambda}_1 = 0$ — (4a)

Normalizatⁿ conditⁿ →

$$\sum_{i=1}^{n}(du_{0_i})^2 + (d\lambda)^2 = (ds)^2$$

$$\Rightarrow \quad \|\dot{u}_0\|^2 + \dot{\lambda}^2 = 1 \quad \text{— (4b)}$$

(4b) ≈ $\dot{u}_1\dot{u}_0 + \dot{\lambda}_1\dot{\lambda}_0 \simeq 1$ — (4c)

(5)
24

$$\begin{bmatrix} \frac{\partial F}{\partial u}\Big|_{(u_0,\lambda_0)} & \frac{\partial F}{\partial \lambda}\Big|_{(u_0,\lambda_0)} \\ \dot{u}_0 & \dot{\lambda}_0 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{\lambda}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{——(5)}$$

use LU to solve (5) → get $(\dot{u}_1, \dot{\lambda}_1)$ ✓

(6) Guess for Newton →

$$\left.\begin{aligned} u_1^{(0)} &= u_0 + \dot{u}_0 \Delta s \\ \lambda_1^{(0)} &= \lambda_0 + \dot{\lambda}_0 \Delta s \end{aligned}\right\} \text{——(6)}$$

(7) Newton iterat^ns →

suppose $g(u,\lambda) = 0$

$u_1^{(0)}$ & $\lambda_1^{(0)}$ are guesses

$$\underbrace{g(u_1^{(0)} + \Delta u, \lambda_1^{(0)} + \Delta\lambda)}_{0} = g(u_1^{(0)}, \lambda_1^{(0)}) + \left[\frac{\partial g}{\partial u}\Big|_{u_1^{(0)},\lambda_1^{(0)}}\right](\Delta u) + \left[\frac{\partial g}{\partial \lambda}\Big|_{u_1^{(0)},\lambda_1^{(0)}}\right]\Delta\lambda$$

solve $\left[\frac{\partial g}{\partial u}\Big|_{u_1^{(0)},\lambda_1^{(0)}}\right]\Delta u + \left[\frac{\partial g}{\partial \lambda}\Big|_{u_1^{(0)},\lambda_1^{(0)}}\right]\Delta\lambda$

$= -g(u_1^{(0)}, \lambda_1^{(0)})$ ——(7a) do it for 1 & 2.

$$\begin{bmatrix} \dfrac{\partial F}{\partial u}\Big|_{(u_1^{(0)},\lambda_1^{(0)})} & \dfrac{\partial F}{\partial \lambda}\Big|_{(u_1^{(0)},\lambda_1^{(0)})} \\ \cancel{\dfrac{\dot u^{(0)}}{\phi}}\;\dot u_0 & \cancel{\dfrac{\dot \lambda^{(0)}}{\phi}}\;\dot \lambda_0 \end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\lambda\end{bmatrix} = -\begin{bmatrix} F\left(u_1^{(0)},\lambda_1^{(0)}\right) \\ \left(u_1^{(0)}-u_0\right)\dot u_0 + \left(\lambda_1^{(0)}-\lambda_0\right)\dot\lambda_0 - \Delta s\end{bmatrix} \quad \text{— } 8$$

Solve 8 with Newton-Raphson to get $(\Delta u, \Delta \lambda)$

$$\left.\begin{aligned} u_1 &= u_0 + \Delta u \\ \lambda_1 &= \lambda_0 + \Delta \lambda \end{aligned}\right\} \quad \text{— } 9$$

Repeat!

How to find $\dot{u}_0$ & $\dot{\lambda}_0$?

We know $\rightarrow \|\dot{u}_0\|^2 + \dot{\lambda}_0^2 = 1$ — (4b)

also, from (4a)

$$\Rightarrow \left[\frac{\partial F}{\partial u}\Big|_{u_0,\lambda_0}\right]\dot{u}_0 + \left[\frac{\partial F}{\partial \lambda}\Big|_{u_0,\lambda_0}\right]\dot{\lambda}_0 = 0 -$$

(4a) $\Rightarrow$

$$\dot{u}_0 = -\left[\frac{\partial F}{\partial u}\Big|_{u^0,\lambda^0}\right]^{-1}\left[F_\lambda(u_0,\lambda_0)\right]\dot{\lambda}$$

(4d)

(4d) in (4b) $\Rightarrow$

$$\left\{\left(\left[F_u(u_0,\lambda_0)\right]^{-1}\left[F_\lambda(u_0,\lambda_0)\right]\right)^2 + 1\right\}\dot{\lambda}_0^2 = 1$$

$$\Rightarrow \boxed{\dot{\lambda}_0 = \frac{1}{\sqrt{\left\{\left(\left[F_u(u_0,\lambda_0)\right]^{-1}\left[F_\lambda(u_0,\lambda_0)\right]\right)^2 + 1\right\}}}}$$

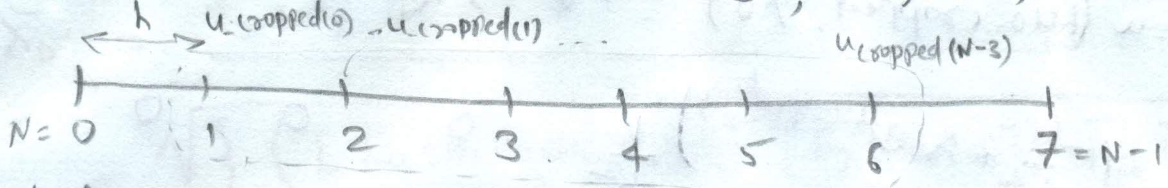
(4e)

put (4e) in (4d) $\Rightarrow$ $\dot{u}_0$

* Arclength continuat^n for Bratu's problem

$u'' + \lambda e^{u(x)} = 0$ for $x \in [0,1]$, $u(0) = u(1) = 0$.

N = 8



F(u0_cropped, λ_0) →

F(i = 0) ⇒ $$\frac{u_2 - 2u_1 + \cancel{u_0}^{\,0}}{h^2} + \lambda e^{u_1} = 0 \Rightarrow$$

F(i = 1) ⇒ $$\frac{u_3 - 2u_2 + u_1}{h^2} + \lambda e^{u_2} = 0$$

F(i = 0) ⇒ $$\frac{u\_cropped(1) - 2\,u\_cropped(0)}{h^2} + \lambda e^{u\_cropped(0)} = 0$$

F(i = 1) ⇒ $$\frac{u\_cropped(2) - 2u\_cropped(1) + u\_cropped(0)}{h^2} + \lambda e^{u\_cropped(1)} = 0$$

F(i = i) ⇒ $$\frac{u\_cropped(i+1) - 2u\_cropped(i) + u\_cropped(i-1)}{h^2} + \lambda e^{u\_cropped(i)} = 0$$

$$F(\ldots) \to \frac{u_3 - 2u_2 + u_1}{h^2} + \lambda e^{u_2} = 0$$

$$F(i=0) \to \frac{u\_cropped(1) - 2u\_cropped(0)}{h^2} + \lambda e^{u\_cropped(0)} = 0$$

$$F(i=1) \to \frac{u\_cropped(2) - 2u\_cropped(1) + u\_cropped(0)}{h^2} + \lambda e^{u\_cropped(1)} = 0$$

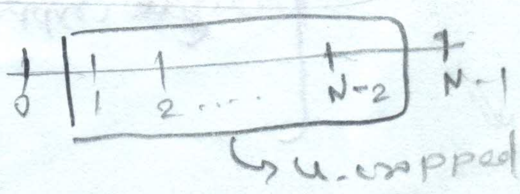
$$F(i=i) \to \frac{u\_cropped(i+1) - 2u\_cropped(i) + u\_cropped(i-1)}{h^2} + \lambda e^{u\_cropped(i)} = 0$$

$\vdots$

$$F(i=N-3) \to \frac{-2u\_cropped(N-3) + u\_cropped(N-4)}{h^2} + \lambda e^{u\_cropped(N-3)} = 0$$

$F(u_0\_cropped, \lambda_0) \to (N-2) \times 1$ vector.

$u_0\_cropped(i) = u_0(i+1)$



$F\_u\,(h, u_{0}\text{-cropped}, \lambda_0) \longrightarrow (N-2)\times(N-2)$ matrix

$$F\_u = \begin{bmatrix} \left(-\frac{2}{h^2} + \lambda_0 e^{u\_cropped(0)}\right) & \frac{1}{h^2} & 0 & 0 & 0 & 0 \\ \frac{1}{h^2} & \left(-\frac{2}{h^2} + \lambda_0 e^{u\_cropped(1)}\right) & \frac{1}{h^2} & 0 & 0 & 0 \\ 0 & \frac{1}{h^2} & (\quad) & \frac{1}{h^2} & 0 & 0 \\ 0 & 0 & \frac{\phi}{h^2} & (\quad) & \frac{1}{h^2} & 0 \\ 0 & 0 & 0 & \frac{1}{h^2} & (\,) & \frac{1}{h^2} \\ 0 & 0 & 0 & 0 & \frac{1}{h^2} & (\end{bmatrix}$$

$$(\quad) \equiv -\frac{2}{h^2} + \lambda_0 e^{u\_cropped(i)}$$

ith row

$F\_\lambda(u\_cropped, \lambda_0) = (N-2)\times 1$ vector

$$F\_\lambda = \begin{bmatrix} e^{u\_cropped(0)} \\ e^{u\_cropped(1)} \\ \vdots \\ e^{u\_cropped(i)} \\ \vdots \\ e^{u\_cropped(N-3)} \end{bmatrix}$$

once we have F_u (h, u0-cropped, λ0) & F_λ(u0-cropped, λ0), we can find $\dot{u}_0$ & $\dot{\lambda}_0$ by → 4d & 4e 29

$$\dot{\lambda}_0 = \frac{\pm 1}{\sqrt{\left\{ \left([F_u(h, u_0\text{-cropped}, \lambda_0)]^{-1} [F_\lambda(u_0, \lambda_0)] \right)^2 + 1 \right\}}} \quad \text{— } 4e$$

&

$$\dot{u}_0 = - \left[F_u (h, u_0\text{-cropped}, \lambda_0) \right]^{-1} \left[F_\lambda(u_0\text{-cropped}, \lambda_0) \right] \dot{\lambda}_0 \quad \text{— } 4d$$

---

get u1_guess_cropped & λ1_guess ?

↳ $u_1^{(0)}$ $\lambda_1^{(0)}$

$$\left.\begin{aligned} u_1^{(0)} &= u_0 + \dot{u}_0 \Delta s \\ \lambda_1^{(0)} &= \lambda_1 + \dot{\lambda}_0 \Delta s \end{aligned}\right\}$$

Feed this to Newton iterat^ns →

$$\underbrace{\begin{bmatrix} (F_u)_1^{(0)} & (F_\lambda)_1^{(0)} \\ \dot{u}_0 & \dot{\lambda}_0 \end{bmatrix}}_{\text{Newton extended\_matrix}} \begin{bmatrix} \Delta u \\ \Delta \lambda \end{bmatrix} = - \underbrace{\begin{bmatrix} F(u_1^{(0)}, \lambda_1^{(0)}) \\ (u_1^{(0)} - u_0)\dot{u}_0 + (\lambda_1^{(0)} - \lambda_0)\dot{\lambda}_0 - \Delta s \end{bmatrix}}_{\text{extended\_rhs\_vector.}}$$

get u1_guess_cropped & λ1_guess ?

↳ $u_1^{(0)}$ $\lambda_1^{(0)}$?

$$\left.\begin{aligned} u_1^{(0)} &= u_0 + \dot{u}_0 \Delta s \\ \lambda_1^{(0)} &= \lambda_1 + \dot{\lambda}_0 \Delta s \end{aligned}\right\} \text{———}$$

Feed this to Newton iteratⁿs →

$$\begin{bmatrix} (F_u)_1^{(0)} & (F_\lambda)_1^{(0)} \\ \dot{u}_0 & \dot{\lambda}_0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} F(u_1^{(0)}, \lambda_1^{(0)}) \\ (u_1^{(0)} - u_0)\dot{u}_0 + (\lambda_1^{(0)} - \lambda_0)\dot{\lambda}_0 - \Delta s \end{bmatrix}$$

Newton_extended_matrix

extended_rhs_vector

get $u_1 = u_0 + \Delta u$

$\lambda_1 = \lambda_0 + \Delta u$ & repeat