<Ans 8 and Ans 8.1>

For a single instance, univariate normal distribution probabilistic function is :-

$$p(x ; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

For N samples, the function would be :-

$$p(X ; \mu, \sigma) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

Assumptions :

- Each data point is sampled independently from the same distribution
- Independent and Identically Distributed

Log likelihood function :

$$\ln p(X : \mu, \sigma^2) = \ln \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \sum_{i=1}^{N} \ln \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} \right)$$

$$= \sum_{i=1}^{N} \left(\ln e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} - \ln 1/\sqrt{2\pi}\,\sigma \right)$$

$$= \sum_{i=1}^{N} \frac{-1}{2\sigma^2}(x_i-\mu)^2 + \sum_{i=1}^{N} \ln \sqrt{2\pi} + \sum_{i=1}^{N} \ln \sigma$$

$$= \sum_{i=1}^{N} -\frac{1}{2\sigma^2}(x_i-\mu)^2 + n \cdot \ln \sqrt{2\pi}$$

$$+ n \cdot \ln \sigma \quad \text{--- (1)}$$

Maximum likelihood estimation for mean of the distribution μ_{ML}.

$$\Rightarrow \max_{\mu_{ML}} \ln\left(p(X : \mu, \sigma)\right)$$

$$= \frac{\delta}{\delta \mu_{ML}} J = 0 .$$

$$\Rightarrow \frac{\delta}{\delta \mu} \sum_{i=1}^{N} \cdot \frac{-1}{2\sigma^2} (x_i - \mu)^2 + n \cdot \ln\sqrt{2\pi} + n \ln \sigma$$

$$\Rightarrow -\frac{\delta}{\delta \mu} \sum_{i=1}^{N} (x_i - \mu)^2 \cdot \frac{1}{2\sigma^2} = 0$$

$$\Rightarrow -\frac{1}{2\sigma^2} \sum_{i=1}^{N} 2(x_i - \mu) = 0$$

$$\Rightarrow \sum_{i=1}^{N} (x_i - \mu) = 0$$

$$\Rightarrow \sum_{i=1}^{N} x_i - n\mu = 0$$

$$\Rightarrow \sum_{i=1}^{N} x_i = n \mu_{ML}$$

$$\Rightarrow \boxed{\hat{\mu}_{ML} = \frac{1}{N} \sum_{i=1}^{N} x_i}$$

M

Maximum likelihood estimation for standard deviation σ_{ML}.

$$\Rightarrow \max_{\sigma_{ML}} \; p(X;\mu,\sigma)$$

$$\Rightarrow \frac{\delta}{\delta\sigma}\left(\sum_{i=1}^{N} \frac{-1}{2\sigma^2}(x_i-\mu)^2 + n\cdot\ln\sigma + n\ln\sqrt{2\pi}\right)$$

$$\Rightarrow \frac{\delta}{\delta\sigma} n\ln\sigma - \frac{\delta}{\delta\sigma}\sum_{i=1}^{N}(x_i-\mu)^2\cdot\frac{1}{2\sigma^2} = 0$$

$$\Rightarrow \frac{n}{\sigma} - \sum_{i=1}^{N}(x_i-\mu)^2\frac{2}{2\sigma^3} = 0$$

$$\Rightarrow \frac{n}{\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\Rightarrow \boxed{\sigma^2 = \frac{1}{n}\sum_{i=1}^{N}(x_i-\mu)^2}$$