PA Q4

1) Linear system : $x_{t+1} = Ax_t + Bu_t$

Cost Function: $J(X,U) = x_T^T Q_f x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$

Different linear system : $x_{t+1} = Ax_t + Bu_t + C$

$$\bar{A} = \begin{pmatrix} A & C - BR^{-1}r \\ 0 & 1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

$$\bar{Q_f} = \begin{pmatrix} Q_f & q_f \\ q_f^T & \eta \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} Q & q \\ q^T & \eta \end{pmatrix}$$

where η is an arbitrary constant.

$$\bar{x_t} = \begin{pmatrix} x_t \\ 1 \end{pmatrix} \quad \& \quad \bar{u_t} = u_t + R^{-1}r$$

$$\bar{x}_{t+1} = \bar{A}\bar{x_t} + \bar{B}\bar{u_t} = \begin{pmatrix} A & C - BR^{-1}r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_t \\ 1 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix}\begin{pmatrix} u_t + R^{-1}r \end{pmatrix}$$

$$= Ax_t + C - \cancel{BR^{-1}r} + Bu_t + \cancel{BR^{-1}r}$$

The above equation resembles our original linear equation. Therefore, it solves the affine problem.

2) $$J(\bar{X},\bar{U}) = \bar{x_t}^T \bar{Q_f}\, \bar{x_t} + \sum_{t=1}^{T-1} \bar{x_t}^T \bar{Q}\, \bar{x_t} + \bar{u_t}^T R \bar{u_t}$$

$$= \begin{pmatrix} x_t \\ 1 \end{pmatrix}^T \begin{pmatrix} Q_f & q_f \\ q_f^T & \eta \end{pmatrix}\begin{pmatrix} x_t \\ 1 \end{pmatrix} + (u_t + R^{-1}r)^T R\,(u_t + R^{-1}r)$$

$$+ \sum_{t=1}^{T-1} \begin{pmatrix} x_t \\ 1 \end{pmatrix}^T \begin{pmatrix} Q & q \\ q^T & \eta \end{pmatrix}\begin{pmatrix} x_t \\ 1 \end{pmatrix}.$$

$$= \left(x_t Q_f + q_f^T \quad x_t q_f + \eta\right)\begin{pmatrix} x_t \\ 1 \end{pmatrix} + u_t^T R u_t + u_T^T r$$

$$+ (R^{-1} r) R u_t + (R^{-1} r)^T r + \sum_{t=1}^{T-1} x_t Q x_t + q^T x_t +$$

$$x_t q + \eta \quad \text{----} (1)$$

The above equation ~~is~~ remains quadratic w.r.t the original cost function. After cost function derivation cost function is almost identical ; has one additional term which depends upon the variance in the noise (η)

3) Autonomous driving on an inclined plane.
If a car is moving on a flat surface it is a linear system. However if it is ~~to~~ going up an inclined plane, it has some additional forces acting on it (gravity) & its coordinate axis is slightly rotated.

4) a) $\bar{A} = \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}$

$$\bar{x}_{t+1} = \begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_t \\ 1 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \bar{u}_t$$

$$= Ax_t + C + B\bar{u}_t$$

For system dynamics to be $x_{t+1} = Ax_t + Bu_t + C$
$\bar{u}_t$ should be equal to u_t.

b) Cost Function:
Everything in Eq (1) above will remain same except the last term.

$\bar{u}_t^T R \bar{u}_t$ will be $u_t R u_t$ (as in the original equation)