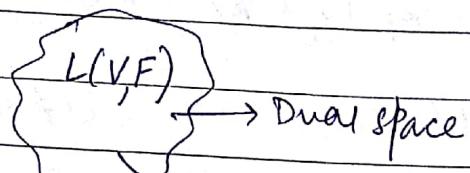


Dual Space (Also known as conjugate space).

• linear operator: L^T from V to V .

• functional: L^T from V to F .

Dual Space involves ~~all these sets of operators~~ the set of all LFs.



Forms a vector space.

Let $V(F)$ be a vector space over F . A linear transformation from $V(F)$ to F is called linear functional. That is $\phi: V \rightarrow F$, satisfying that

$$\phi(ax+by) = a\phi(x) + b\phi(y) \text{ for every } x, y \in V, a, b \in F.$$

The vector space $L(V, F)(F)$ consisting of linear functionals is called a dual space of $V(F)$ and is denoted by $V^*(F)$.

$$\therefore \phi_i \in V^*(F).$$

g) $V(F) \rightarrow n$ square matrix.

$$\phi: V \rightarrow F.$$

$$\phi(A) = \text{tr}[A]. \text{ Show that } \phi \text{ is a linear functional.}$$

$$\begin{aligned}
 \phi(A+B) &= \text{tr}[A+B] \\
 &= \sum_{i=1}^n C_{ii} \quad \text{where } C = A+B \\
 &= \sum_{i=1}^n (A+B)_{ii} \\
 &= \sum_{i=1}^n (A_{ii} + B_{ii}) \\
 &= \sum_{i=1}^n A_{ii} + \sum_{i=1}^n B_{ii} \\
 &= \phi(A) + \phi(B).
 \end{aligned}$$

$$\phi(cA) = \text{tr}[cA]$$

$$\begin{aligned}
 &= \sum_{i=1}^n ca_{ii} \\
 &= c \sum_{i=1}^n a_{ii} \\
 &= c \text{tr}[A].
 \end{aligned}$$

$$= c \phi(A).$$

q) Dim. of dual space?

$$\begin{aligned}
 m \times n &= m \times 1 \quad [n=1] \\
 &\approx m.
 \end{aligned}$$

Q) Let $\{x_1, x_2, \dots, x_n\}$ be a basis of $V(F)$ over F .

Let $\phi_1, \dots, \phi_n \in V^*$ be linear functionals defined by

$$\begin{aligned}\phi_i(x_j) &= \delta_{ij} = 1 \text{ if } i=j \\ &= 0 \text{ if } i \neq j\end{aligned}$$

Then show that $\{\phi_1, \dots, \phi_n\}$ is a basis of $V^*(F)$.

Ans)

Proof of
check if L.I.

$$(a_1\phi_1 + a_2\phi_2 + \dots + a_n\phi_n)(x_i) = a_i(x_i)$$

$$a_i = 0$$

⇒ Similarly, for all values of i , $a_i = 0$

∴ $\{\phi_1, \dots, \phi_n\}$ is L.I.

Proof of
check if spans $V^*(F)$

~~Let $f \in V^*(F)$, $f: V \rightarrow F$.~~

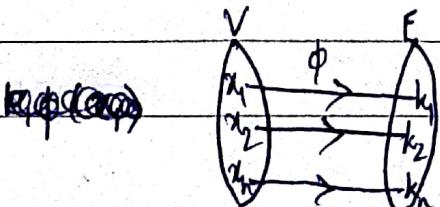
~~$$f(x) = f(c_1x_1 + \dots + c_nx_n)$$~~

~~$$\text{or, } f(x) = c_1f(x_1) + \dots + c_nf(x_n)$$~~

~~$$\therefore \text{for } x = x_i, f(x) = c_i$$~~

Let ϕ be any arbitrary element in V^* and let

$$\phi(x_1) = k_1, \dots, \phi(x_n) = k_n$$



$$k_1\phi_1 + k_2\phi_2 + \dots + k_n\phi_n = \sigma \in V^*$$

$$\begin{aligned}\therefore \sigma(x_i) &= (k_1\phi_1 + \dots + k_n\phi_n)(x_i) \\ &= k_1\phi_1(x_i) + \dots + k_i\phi_i(x_i) + \dots + k_n\phi_n(x_i) \\ &= k_i.\end{aligned}$$

$$k_i = \phi(x_i) = \sigma(x_i) + x_i.$$

$$\Rightarrow \phi = \sigma.$$

$$\therefore \phi = k_1\phi_1 + k_2\phi_2 + \dots + k_n\phi_n.$$

$\therefore \{\phi_1, \dots, \phi_n\}$ spans $V^*(F)$.

$\therefore \{\phi_1, \dots, \phi_n\}$ is a basis of $V^*(F)$.

8) $R_B^2 = \{x_1 = (2,1), x_2 = (3,1)\} \leftarrow \text{basis in } R^2.$

Find the dual basis $\{\phi_1, \phi_2\}$.

$$\phi_1(x_1) = 1, \quad \phi_1(x_2) = 0$$

$$\phi_2(x_1) = 0, \quad \phi_2(x_2) = 1.$$

~~$$\begin{aligned}\phi_1 &= \begin{cases} 1 & \text{if } x_i = (2,1) \\ 0 & \text{otherwise} \end{cases} \\ \phi_2 &= \begin{cases} 1 & \text{if } x_i = (3,1) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$~~

Let $\phi_1((x,y)) = ax + by$, $\phi_2((x,y)) = cx + dy$ [linear functional]

$$\phi_1((2,1)) = 1, \quad \phi_1((3,1)) = 0.$$

$$\Rightarrow 2a + b = 1$$

$$3a + b = 0$$

$$a = -1, \quad b = 3.$$

$$\phi_2((2,1)) = 0, \quad \phi_2((3,1)) = 1.$$

$$2c + d = 0$$

$$3c + d = 1$$

$$\Rightarrow c = 1, \quad d = -2.$$

$$\therefore \phi_1((x,y)) = -x + 3y$$

$$\phi_2((x,y)) = x - 2y.$$

9)

Let $\{x_1, x_2, \dots, x_n\}$ be a basis of V and let $\{\phi_1, \phi_2, \dots, \phi_n\}$ be a dual basis of V^* , then show that any vector $x \in V$ is written as

$$x = \phi_1(x)x_1 + \phi_2(x)x_2 + \dots + \phi_n(x)x_n \quad (1)$$

and any linear functional $\sigma \in V^*$ is written as

$$\sigma = \sigma(x_1)\phi_1 + \sigma(x_2)\phi_2 + \dots + \sigma(x_n)\phi_n \quad (2)$$

Ans). Since $\{x_1, \dots, x_n\}$ is a basis in V .

$x \in V$, $x = a_1 x_1 + \dots + a_n x_n$ ($a_i \in F$).

$$\boxed{\phi_i(x) = a_i}$$

since $\{\phi_1, \dots, \phi_n\}$ is a basis in V^* .

$\sigma \in V^*$, $\sigma = c_1 \phi_1 + \dots + c_n \phi_n$. ($c_i \in F$).

Could have used the above result as well.

$$\sigma(x_i) = c_i \phi_i(x_i)$$

[since $\phi_j(x_i) = 0$]
 $\phi_i(x_i) = 1$.

$$\therefore \boxed{\sigma(x_i) = c_i}$$

Alt.

$$\sigma(x) = \sigma(a_1 x_1 + \dots + a_n x_n)$$

$$= a_1 \sigma(x_1) + \dots + a_n \sigma(x_n)$$

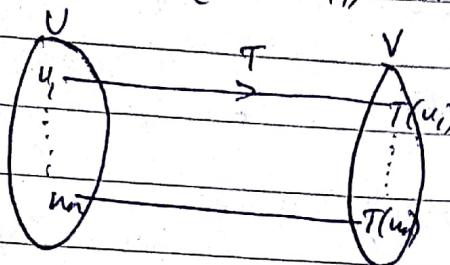
$$= \phi_1(x_1) \sigma(x_1) + \dots + \phi_n(x_n) \sigma(x_n)$$

$$= \sigma(x_1) \phi_1(x_1) + \dots + \sigma(x_n) \phi_n(x_n)$$
 [F is commutative]

$$= (\sigma(x_1) \phi_1 + \dots + \sigma(x_n) \phi_n)(x)$$

Matrices and linear transformations

Let $U(F)$ and $V(F)$ be 2 vector spaces over the field $(F, +, \cdot)$ and let $T: U \rightarrow V$ be a linear transformation. Again, let $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ be the bases of $U(F)$ and $V(F)$ respectively. Since T maps U into V , $\{T(u_1), \dots, T(u_n)\}$ will belong to V .



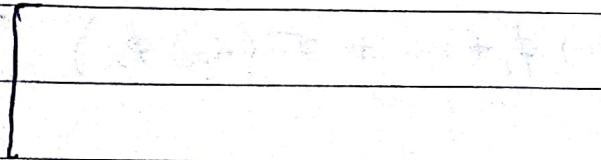
$$T(u_1) = a_{11}v_1 + a_{12}v_2 + \dots + a_{1m}v_m$$

$$T(u_2) = a_{21}v_1 + a_{22}v_2 + \dots + a_{2m}v_m.$$

⋮

$$T(u_n) = a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nm}v_m.$$

$[T(u_1) \quad T(u_2) \quad \dots \quad T(u_n)]$, where $T(u_i)$ is a $m \times 1$ column vector.



Continued after 'System of Linear Equations'

23/10/18.

System of linear Equations

$$ax + by = c$$

$$ax + by + cz + d = 0$$

For an eqⁿ to be linear, the no. of variables don't matter, just their degree should be 1.

A linear equation in the n -variables x_1, x_2, \dots, x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \quad (\text{where } a_1, \dots, a_n \text{ are coeff.})$$

(i) i) $3x - 4y = 1$ linear

ii) $\frac{x}{2} - \frac{y}{3} - \frac{z}{5} = 9$ linear.

(iii) $x_1 + 5x_2 = 3 - x_3 + 2x_4$ linear.

(iv) $\sqrt{2}x + \frac{\pi}{4}y - (\sin \frac{\pi}{5})z = 1$ ~~linear~~

(v) $xy + 2z = 1$ Non linear.

(vi) $\frac{x}{y} + z = 2$ Non linear.

(vii) $x_1^2 - x_2^3 = 3$ Non linear.

Solution of a linear eqⁿ.

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

The vector $[s_1, s_2, \dots, s_n]$ which satisfies the eqⁿ is a solution.

$$3x - 4y = -1$$

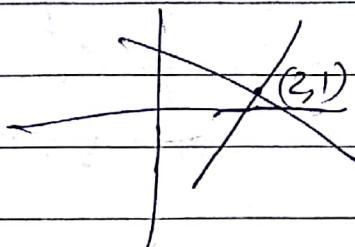
$$(1, 1) \quad (0, \frac{1}{4})$$

Basically, represents a straight line.

\therefore Infinite solⁿ.

$$\begin{cases} 2x - y = 3 \\ x + 3y = 5 \end{cases} \rightarrow (2, 1)$$

Unique solⁿ.



Examples

$$\begin{cases} x - y = 1 \\ x + y = 3 \end{cases}$$

$$\begin{cases} x - y = 2 \\ 2x - 2y = 4 \end{cases}$$

$$\begin{cases} x - y = 1 \\ x - y = 3 \end{cases}$$

Unique solⁿ.

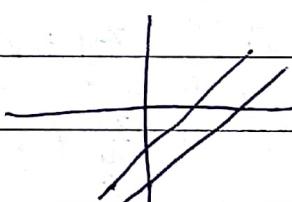
Geometrically,
intersecting.

Infinite solⁿ.

Both represent the
same line.

Parallel lines \Rightarrow 0 solⁿ.

There is solution.



Called consistent system of
equations.

Inconsistent system
of equations.

If the constant part of the eqⁿs are 0, we can
be assured that there's atleast 1 solⁿ \rightarrow origin.

\therefore consistent sys. of eqⁿs.

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{array} \right\} \text{CONSISTENT.}$$

Case 1.

$$x - y = 1$$

$$x + y = 3$$

Case 2

$$x - y = 1$$

$$y = 1$$

Substitute $y=1$ in ① called backward substitution.

Same solⁿ. o. called Equivalent systems.
(2,1)

In case 2, we can find the solⁿ by merely looking.

If we have a mechanism by which case 1 can be converted to case 2 easily, ~~without~~ solving of the system of eqⁿs would be easy.

$$x - y - 2 = 2$$

$$y + 3z = 5 \rightarrow (3, -1, 2)$$

$$5z = 10$$

$$\text{However, } \left. \begin{array}{l} x - y - 2 = 2 \\ 3x - 3y + 2z = 16 \\ 2x - y + 2 = 9 \end{array} \right\} \rightarrow (3, -1, 2).$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array} \right]$$

Coefficient
Matrix

Augmented Matrix

We need to show how a sys. of eqⁿs is equivalent to another sys. of eqⁿs.

Let's convert this to a form suitable for backward substitution.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array} \right] \quad \begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned}$$

Step 1. Subtract $3 \times \textcircled{1}$ from $\textcircled{2}$, i.e. $R_2' \leftarrow R_2 - 3R_1$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array} \right] \quad \begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ 2x - y + z &= 9 \end{aligned}$$

Step 2. Subtract $2 \times \textcircled{1}$ from $\textcircled{3}$, i.e. $R_3' \leftarrow R_3 - 2R_1$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array} \right] \quad \begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ y + 3z &= 5 \end{aligned}$$

Step 3. Interchange 2nd & 3rd eqns. i.e. $R_2 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array} \right] \quad \begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned}$$

Defⁿ,

A matrix is in row echelon form if it satisfies the following properties :-

- 1) Any row(s) consisting entirely of zeros is (are) at bottom.
- 2) In each nonzero row, the first nonzero entry (called the **leading entry**) is in a column to the left of any leading entries below it.

Q)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

Is this a Row-Echelon matrix?

YES

Row Operations

- 1) Interchange of 2 rows.
- 2) Multiply a row by a non-zero constant.
- 3) Add a multiple of a row to another row.

Using these row operations,

$$[\quad] \rightarrow \text{Row Echelon matrix.}$$

called Gaussian Elimination, (Direct Method).

Steps of Gaussian Elimination :-

- 1) Write the augmented matrix.
- 2) Use row operations to reduce the augmented matrix to row echelon form.
- 3) Solve the sys. of eqn's., using back substitution.

9)

Solve :-

0	2	3	8
2	3	1	5
1	-1	-2	-5

 $R_1 \leftrightarrow R_2$

2	3	1	5
0	2	3	8
1	-1	-2	-5

$$R_3' \leftarrow R_3 - \frac{1}{2} R_1$$

2	3	1	5
0	2	3	8
0	-2.5	-2.5	-7.5

$$R_3' \leftarrow 2R_3$$

2	3	1	5
0	2	3	8
0	-5	-5	-15

$$R_3' \leftarrow R_3 + \frac{5}{2} R_2$$

2	3	1	5
0	2	3	8
0	0	2.5	5

i.e. $2x + 3y + z = 5$

$$2y + 3z = 8 \rightarrow (0, 1, 2)$$

$$2.5z = 5$$

Q)

$$w - x - y + 2z = 1$$

$$2w - 2x - y + 3z = 3$$

$$-w + x - y = -3.$$

~~$$\textcircled{2} - \textcircled{1} \Rightarrow z = -w + x + 2$$~~

~~$$\text{Now, } w - x - y + 2(-w + x + 2) = 1 \Rightarrow -w + x - y = -3$$~~

~~$$2w - 2x - y + 3(-w + x + 2) = 3 \Rightarrow -w + x - y = -3$$~~

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array} \right]$$

$$\downarrow R_2' \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -3 \end{array} \right] \xrightarrow{R_3' \leftarrow R_3 + R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array} \right]$$

$$\downarrow R_3' \leftarrow R_3 + 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\therefore w - x - y + 2z = 1$$

$$y - 2 = 1 \Rightarrow y = z + 1$$

$$\text{Let } z = t, y = t + 1.$$

$$\Rightarrow w - x - t - 1 + 2t = 1 \Rightarrow w - x + t = 2$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+s-t \\ s \\ 1+t \\ t \end{bmatrix}$$

$$\Rightarrow w = x - t + 2$$

$$\text{Let } x = s, w = 2 + s - t.$$

~~Row~~ Reduced Echelon Form

- 1) It is in Row Echelon form.
- 2) The leading entry in each nonzero row is 1.
- 3) Each column containing a leading 1 has 0 everywhere else.

Hence, once converted to ~~Row~~ reduced ^{row} Echelon form, the backward substitution step isn't reqd. You directly get $x_i = b_i$, etc + i.

This is called Gauss-Jordan Elimination (Direct Method)
Steps:-

- 1) Write augmented matrix.
- 2) Use elementary row operations to reduce the augmented matrix to reduced row echelon form.

26/10/18

Q) Solve the system of eqn's by Gauss Jordan ^{elimination} Method.

$$\sqrt{2}x + y + 2z = 1$$

$$\sqrt{2}y - 3z = -\sqrt{2}$$

$$-y + \sqrt{2}z = 1$$

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array} \right] \xrightarrow{R_1' \rightarrow \frac{1}{\sqrt{2}} R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{\sqrt{2}} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array} \right]$$

$R_2' \rightarrow \frac{R_2}{\sqrt{2}}$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{\sqrt{2}} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & -1 & \sqrt{2} & 1 \end{array} \right] \xrightarrow{R_1' \rightarrow R_1 + \frac{1}{\sqrt{2}} R_3} \left[\begin{array}{ccc|c} 1 & 0 & (\sqrt{2}+1) & \sqrt{2} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & -1 & \sqrt{2} & 1 \end{array} \right]$$

$R_3' \rightarrow R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & (\sqrt{2}+1) & \sqrt{2} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array} \right] \xrightarrow{R_1' \rightarrow R_1 + (2+5)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array} \right]$$

$R_2' \rightarrow R_2 - 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array} \right] \xrightarrow{R_3' \rightarrow -\sqrt{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Ans : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 0 \end{bmatrix}$.

$$g) \quad a + b + c + d = 4.$$

$$a+2b+3c+4d = 10$$

$$at \ 3bt + 6c + 10d = 20$$

$$at + 4bt + 10c + 20d = 35.$$

Solve by Gaussian Elimination.

1	1	1	1	4
1	2	3	4	10
1	3	6	10	20
1	4	10	20	35

$$R_2' \rightarrow R_2 - R_1$$

1	1	1	1	4
0	1	2	3	6
1	3	6	10	20
1	4	10	20	35

$$R_3' \rightarrow R_3 - R_1 - 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$R_1' \rightarrow \frac{R_1 - R_1 - 3R_2}{-3R_3}$$

1	1	1	1	4
0	1	2	3	6
0	0	1	3	4
1	4	10	20	35

$$d=1 \Rightarrow c=1 \Rightarrow b=1 \Rightarrow a=1.$$

$$\therefore \text{Ans} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

CONT'D after Compilers (last section).

Indirect / Intuitive Methods

$$7x_1 - 3x_2 = 5$$

$$3x_1 - 5x_2 = -7$$

quite easy to guess: (1, 2)

Jacobi's Method

$$x_1 = \frac{5 + x_2}{7}$$

$$x_2 = \frac{7 + 3x_1}{5}$$

Initial Guess

$$x_1^{(0)} = 0$$

$$x_2^{(0)} = 0$$

$$x_1^{(1)} = \frac{5 + x_2^{(0)}}{7} = \frac{5}{7}$$

$$x_1^{(2)} = \frac{5 + \frac{7}{5}}{7} = \frac{32}{35}$$

$$x_2^{(1)} = \frac{7 + 3x_1^{(0)}}{5} = \frac{7}{5}$$

$$x_2^{(2)} = \frac{7 + 3 \cdot \frac{5}{7}}{5} = \frac{64}{35}$$

	(0)	(1)	(2)	(3)	(4)	(5)	(6) \leftarrow
x_1	0	$5/7$	$32/35$	$239/245$	$1216/1225$	0.998	0.999
x_2	0	$7/5$	$64/35$	$341/175$	$2432/1225$	1.996	1.999

$$x_1^{(3)} = \frac{5 + \frac{64}{35}}{7} = \frac{239}{245} \quad x_1^{(4)} = \frac{5 + \frac{341}{175}}{7} = \frac{1216}{1225}$$

$$x_2^{(3)} = \frac{7 + 3 \cdot \frac{32}{35}}{5} = \frac{341}{175} \quad x_2^{(4)} = \frac{7 + 3 \cdot \frac{239}{245}}{5} = \frac{2432}{1225}$$

$$= \frac{2432}{1225}$$

Gauss-Seidal Method

$$x_1 = \frac{5 + x_2}{7}$$

$$x_1^{(0)} = 0$$

$$x_2^{(0)} = 0$$

$$x_2 = \frac{7 + 3x_1}{5}$$

$$x_1^{(1)} = \frac{5 + x_2^{(0)}}{7} = \frac{5}{7}$$

$$x_1^{(2)} = \frac{5 + x_2^{(1)}}{7} = \frac{5 + \frac{64}{35}}{7} = \frac{239}{245}$$

$$x_2^{(1)} = \frac{7 + 3x_1^{(1)}}{5} = \frac{64}{35}$$

$$x_2^{(2)} = \frac{7 + 3x_1^{(2)}}{5} = \frac{7 + 3 \cdot \frac{239}{245}}{5} = \frac{2432}{1225}$$

	(0)	(1)	(2)	(3)	(4)	(5) ~
x_1	0	$\frac{5}{7}$	$\frac{239}{245}$	$\frac{8557}{8575}$		
x_2	0	$\frac{64}{35}$	$\frac{2432}{1225}$	$\frac{85696}{42875}$		

$$x_1^{(3)} = \frac{5 + \frac{2432}{1225}}{7} = \frac{8557}{8575}$$

$$x_2^{(3)} = \frac{7 + 3 \cdot \frac{8557}{8575}}{5} = \frac{85696}{42875}$$

g) $3x_1 - x_2 = 1$

$$-x_1 + 3x_2 - x_3 = 0$$

Solve by

$$-x_2 + 3x_3 - x_1 = 1$$

(a) Jacobi

$$-x_3 + 3x_1 = 1$$

(b) Gauss-Seidal.

Yenice opt. compiler path l inp. program

Matrices (continued)

$$V^m \rightarrow W^n$$

$$\begin{bmatrix} & \\ & \end{bmatrix}$$

$n \times m$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Diagonal Entries

Off diagonal entries

• Trace of a matrix,

$$\text{tr}[A] = \sum_i a_{ii}.$$

• If $m=n$ and $A = [a_{ii}]$

it is called a diagonal matrix.

• $A = B$ if their orders are equal and
 $[a_{ij}] = [b_{ij}]$ entries.

• $A+B = [a_{ij}] + [b_{ij}] = [a_{ij}+b_{ij}]$.
(have to be of same order).

• A^T $a_{ij} \rightarrow a_{ji}^T$

• Symmetric Matrix : $A = A^T$, i.e. $a_{ij} = a_{ji} \neq b_{ij}$.

• Skew-Symmetric Matrix : $A = -A^T$ i.e. $a_{ij} = -a_{ji} \neq b_{ij}$.
 $\therefore a_{ii} = 0 \forall i$. [$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$].

$AB \neq BA$

$A_{m \times n} \quad B_{n \times p}$ necessary condition for mult. to
be possible,

$$\begin{bmatrix} : \\ a_{11} \ a_{12} \ \dots \ a_{1n} \end{bmatrix}_{m \times n} \times \begin{bmatrix} b_{11j} & b_{12j} & \dots & b_{1pj} \\ b_{21j} & b_{22j} & \dots & b_{2pj} \\ \vdots & \vdots & \ddots & \vdots \\ b_{nj} & b_{(n-1)j} & \dots & b_{pj} \end{bmatrix}_{n \times p} \rightarrow [c_{ij}]$$

$$c_{ij} = \sum_{k=1}^n a_{ik}^o b_{kj}.$$

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Inverse of a Matrix

If A is a matrix of size $(n \times n)$, then the inverse A' ($n \times n$) is a matrix such that:

$$AA' = A'A = I_n$$

$$A^{-1} = \frac{\text{adj}(A)}{\det A}.$$

Adjoint = (Cofactor matrix)^T.

If $\det A \neq 0$, the matrix is ~~not~~ invertible.

If A is invertible, then show that the inverse is unique.

let A' and A'' be 2 inverses of A .

$$\text{then, } AA' = A'A = I \quad \text{--- (1)}$$

$$AA'' = A''A = I \quad \text{--- (2)}$$

$$A' = A' I$$

$$= A' (A A'')$$

$$= (A' A) A''$$

$$= I A''$$

$$= A''$$

[Proved].

If A is invertible, $n \times n$ matrix, then the system of linear equations $Ax = b$ has the unique soln $x = A^{-1}b$ for any b in \mathbb{R}^n .

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2.$$

:

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n.$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \equiv Ax = b.$$

If $x = A^{-1}b$, $A(A^{-1}b) = (AA^{-1})b = Ib = b$.

Let's show that it is unique.

Let y be another solⁿ to $Ax=b$.

$$Ay = b \Rightarrow A^{-1}(Ay) = A^{-1}b$$

$$\Rightarrow Iy = A^{-1}b$$

$$\Rightarrow y = A^{-1}b$$

$\therefore y = x$. \Rightarrow Unique.
Proved

Q) Use the inverse of the matrix to solve :-

$$x + 2y = 3$$

$$3x + 4y = -2$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\text{adj}(A) \stackrel{\text{def}}{=} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}^T = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}b = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -16 \\ 11 \end{bmatrix}.$$

Properties of inverse of a matrix.

$$1) (A^{-1})^{-1} = A.$$

$$2) (cA)^{-1} = \frac{1}{c} A^{-1}.$$

$$3) (AB)^{-1} = B^{-1}A^{-1}.$$

$$4) (AT)^{-1} = (A^{-1})^T$$

$$5) (A^n)^{-1} = (A^{-1})^n.$$

If A is an invertible matrix, and n is a positive integer, then A^{-n} is defined by \star

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}.$$

Elementary Matrices

An Elementary matrix is any matrix that is obtained by performing an elementary row operation ~~on~~ on an identity matrix.

$$g) E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$4 \times 4 \qquad 4 \times 4 \qquad 4 \times 4$

What operations have been applied to get E_i 's?

$$\underline{E_1 : R'_2 \leftarrow 3R_2}$$

$$\underline{E_2 : R'_1 \leftarrow R_3}$$

$$\underline{E_3 : R'_4 \leftarrow -2R_2 + R_4}$$

$$g) A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \quad \text{Find } E_1 A, E_2 A, E_3 A.$$

$$E_1 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \quad E_2 A = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

$$E_3 A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} - 2a_{21} & a_{42} - 2a_{22} & a_{43} - 2a_{23} \end{pmatrix}$$

\therefore Equivalent to applying the row operations on the matrix A itself.

Let E be an elementary matrix obtained by performing row operation on I_n . If the same elementary row operation is performed on $n \times n$ matrix A , the result is same EA .

$$I \xrightarrow{3R_2} E_1 \quad I \xrightarrow{R_{13}} E_2$$

$$I \xrightarrow{R_4 - 2R_2} E_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I.$$

\downarrow \downarrow

E_1 E_1^{-1}

(Also an Elementary Matrix)

$$E_2^{-1} = E_2$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The fundamental theorem of Invertible Matrices

Let A be $n \times n$ matrix. Then the following statements are equivalent :-

- 1) A is invertible
- 2) $Ax = b$ has a unique sol n for every $b \in \mathbb{R}^n$.
- 3) $Ax = 0$ has only the trivial sol n .
- 4) The reduced row echelon form of A is I_n .
- 5) A is a product of elementary matrix.

Assume (1) to be true.

(1) \Rightarrow (2) (already proved).

Now, for $Ax = 0$, $x = A^{-1}0 = 0$ [since $X0 = 0$]
 $\nexists x$.

$\therefore (2) \Rightarrow (3)$.

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = 0$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = 0$$

↓

$$a_n x_1 + a_{n2} x_2 + \dots + a_{nn} x_n = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_1 + x_2 + \dots + x_n = 0$$

On applying Gaussian Elimination on this, we'll get In.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 \end{array} \right] \quad \therefore (4).$$

Now since (4) is true,

$$A \xrightarrow{R_1} \xrightarrow{R_2} \dots \xrightarrow{R_k} I_n$$

$$(E_k - E_1 E_1) A = I_n$$

$$A = (E_k - E_1)^{-1} = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

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Q) If possible, express $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

$$(E_k \dots E_1) A = I \Rightarrow A = (E_k \dots E_1)^{-1}.$$

~~$(E_{R2} \dots E_{R1}) \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = (E_k \dots E_1)$~~

~~$\therefore (E_k \dots E_1)^{-1} \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$~~

~~$\therefore (E_k \dots E_1)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix}$~~

~~$= \frac{1}{3} \begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix}$~~

~~$\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \in E_k \dots E_1$~~

$$\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_1' \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_2' \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_2' \leftarrow \frac{1}{3} R_2$$

~~$E_3 E_2 E_1 A = I$~~

I

$$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} .$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A = (E_3 E_2 E_1)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right) = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}.$$

#. Let A be a square matrix. If a sequence of elementary row operations reduces A to I , then the same sequence will transform I to A^{-1} .

$$A \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} I \quad (E_k E_{k-1} \cdots E_1) A = I.$$

$$I \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} A^{-1} \quad \underbrace{(E_k E_{k-1} \cdots E_1)}_B$$

$$\therefore BA = I.$$

$$\therefore (E_k E_{k-1} \cdots E_1) I = A^{-1}. \quad \Rightarrow B = A^{-1}.$$

$$\Rightarrow BI = A^{-1}.$$

f) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

$$[A | I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array} \right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array} \right] \quad [R'_2 \leftarrow R_2 - 2R_1]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array} \right] \quad [R_3' \leftarrow R_3 - R_1 + \frac{1}{2}R_2].$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array} \right] \quad [R_2' \leftarrow -\frac{1}{2}R_2].$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & -3 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array} \right] \quad [R_1' \leftarrow R_1 - 2R_2 - 5R_3]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 0 \end{array} \right] \quad [R_2' \leftarrow R_2 + 3R_3]$$

$$\stackrel{0}{\text{P.O.}} \quad A^{-1} = \left[\begin{array}{ccc} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 0 \end{array} \right].$$

(i) Find the inverse of $\begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & 2 \end{bmatrix}$ using

Gauss Jordan method.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ -4 & -1 & 6 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array} \right]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 10 & 0 & 1 \end{array} \right] \quad \begin{aligned} R_2' &\leftarrow R_2 + 2R_1 \\ R_3' &\leftarrow R_3 + R_1 \end{aligned}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 6 & -2 & 0 & 1 \end{array} \right] \quad R_3' \leftarrow R_3 - 3R_1$$

$$= \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 6 & -2 & 0 & 1 \end{array} \right] \quad R_1' \leftarrow R_1 -$$

Upper Triangular Matrix

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ \vdots & & & \vdots \\ 0 & 0 & 0 & * \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} * & & & \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{bmatrix}$$

Let A be a square matrix. A factorization of A as $A = LU$ where L is the unit lower triangular and U is the upper triangular, is called LU factorization.

g) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$ Express A as LU .

Step 1: Convert A to some U .

$$A \xrightarrow{R_1} \dots \xrightarrow{R_k} U$$

$$U = (E_k \dots E_1) A \Rightarrow A = (E_k \dots E_1)^{-1} U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \dots E_k^{-1}}_{\text{will be } L_u} U$$

$$A = L_u U$$

will be L_u (unit lower triangular matrix)

$$\left[\begin{array}{ccc} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{array} \right] \xrightarrow{\textcircled{1} R_2' \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{array} \right] \xrightarrow{\textcircled{2} R_3' \leftarrow R_3 + R_1} \left[\begin{array}{ccc} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{array} \right] \xrightarrow{\textcircled{3} R_3' \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{array} \right]$$

$$\therefore E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A = (E_3 E_2 E_1)^{-1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$$(E_3 E_2 E_1)^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

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Eigen values and eigen vectors

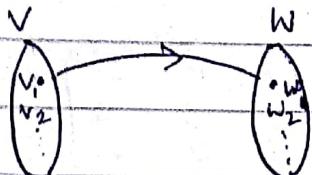
What is an eigen value of a matrix? ($N \times N$ matrix)

$$Ax = \lambda x$$

↪ Eigen value.

What is the importance of finding eigen values of a matrix?

Consider two vector spaces V, W .



An L.T. from V to W can be represented as a matrix.

$$\{v_1, v_2, \dots, v_n\} \quad \{w_1, w_2, \dots, w_m\}$$

Construct a matrix corresponding to this L.T. :-

$$\text{Let } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y-3z \end{bmatrix}$$

Let $B = \{e_1, e_2, e_3\}$ be the basis of \mathbb{R}^3 . Let $\{\bar{e}_1, \bar{e}_2\}$ be the basis of \mathbb{R}^2 . Find the matrix wrt. $B \& C$.

For B : $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

C: $e_1 = (1, 0)$, $e_2 = (0, 1)$.

Dim. of transformation matrix: 2×3 .

$$\begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

[since order of basis is changed]

If $C : (\bar{e}_1, \bar{e}_2)$ basis then

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix}$$

Similarly, if we change \bar{e}_1, \bar{e}_2 to :-

$$C : \bar{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \bar{e}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Find matrix representation of the linear transformation:-

$$T(e_1) = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(e_1)]_C = \bar{e}_2 + \bar{e}_1.$$

$$T(e_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, T(e_2) = e_2 - 2e_1.$$

$$T(e_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3e_2.$$

$$(T(e_1), T(e_2), T(e_3)) : - \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}.$$

So changing order of basis vectors makes a difference.
But something remains fundamental (eigen value remains same)

New scenario :-

$$B : \{e_1, e_2, e_3\} ; C : \{e_1, e_2, e_3\}$$

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \\ x + y \end{bmatrix}$$

Consider $C' : \{e_3, e_2, e_1\}$.

Find T_{BC} and $T_{BC'}$.

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$T(e_3) = T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Characteristic eqⁿ

$$\det(A - \lambda I_n) = 0$$

characteristic polynomial.

if you have diagonal matrix, the diagonal elements
are itself eigen values.

$$A - \lambda I_3 = \begin{bmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$$

Let $A - \lambda I = 0,$

This is an eigen basis.

There are 2 basis associated with this

- Q) Find the eigen values and eigen vectors of the following matrix.

A represents an L.T. from $\mathbb{R}^3 \rightarrow \mathbb{R}^3$.

$$A_{BC} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

wrt. B & C

if B & C were eigen basis

$\Rightarrow A_{BC}$ will be diagonal matrix.

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix}$$

we get

$$(-1)^2 (\lambda - 2) = 0 \Rightarrow \lambda = 1, 2$$

The no. of times an eigen value occurs is called the algebraic multiplicity.

Here, alg. multiplicity (1) = 2
(2) = 1.

For $\lambda = 1$, $Ax = \lambda x \Rightarrow Ax - x = 0 \Rightarrow (A - I)x = 0$

$$\text{if } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A - I & | & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving,

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

Free variable : x_3 .

Let $x_3 = t$:- $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigen space ? dim ?

↓
Corresponding to
 $\lambda = 1$ $\hookrightarrow = 1$
(geometric multiplicity).

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What are similar matrices?

Let A and B be $(n \times n)$ matrices. We say that A is similar to B if there is an invertible $(n \times n)$ matrix P such that $P^{-1}AP = B$. If A is similar to B then we write $A \sim B$.

If A is similar to B , $P^{-1}AP = B$

$$P^{-1}A = BP^{-1}$$

$$\Rightarrow A = PBP^{-1}$$

$$\Rightarrow AP = PB$$

Q) If $A \sim B$, can we say $B \sim A$?

Yes: $P^{-1}AP = B \Rightarrow A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$
∴ $B \sim A$.

Q) Is $A \sim A$? $(I^{-1})AI = A$ Yes.

Q) If $A \sim B$ and $B \sim C$, then $A \sim C$?

$$P^{-1}AP = B$$

$$\Rightarrow A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$$

~~∴~~ $A \sim C$

Q) If A, B are $(n \times n)$ matrices, such that $A \sim B$ show that $\det(A) = \det(B)$.

$$P^{-1}AP = B$$

$$\det(P^{-1}AP) = \det(B)$$

$$\det(P^{-1}) \det(A) \det(P) = \det(B) \quad \text{--- (1)}$$

$$PP^{-1} = I$$

$$\det(PP^{-1}) = \det(I) = 1$$

$$\Rightarrow \det(P) \det(P^{-1}) = 1 \Rightarrow \det(P^{-1}) = \frac{1}{\det(P)}$$

\therefore From (1),

$$\frac{1}{\det(P)} \times \det(A) \times \det(P) = \det(B)$$

$$\Rightarrow \boxed{\det(A) = \det(B)}$$

\downarrow Necessary condition but not sufficient.

Q) If A, B are $n \times n$ matrices, such that $A \sim B$ show that A and B have same characteristic polynomial.

$$P^{-1}AP = B$$

Q2) Find out whether they are similar matrices or not :-

i) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Ans 2) i) ~~Let there exist P such that~~

$$\det(A) = -3$$

$$\det(B) = 3$$

Since $\det(A) \neq \det(B) \Rightarrow A \not\sim B$.

ii) $\det(A) = -4$

$$\det(B) = -4$$

Since $\det(A) = \det(B) \Rightarrow$ they might be similar.

Check characteristic polynomial

~~Let there exist P s.t.~~

$$P^{-1}AP = B$$

$$\begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

$$\text{If } P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{i.e. } (1-\lambda)(2-\lambda) - 6 = 0$$

$$\Rightarrow 2 - 3\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$(ad-bc) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

$$(ad-bc) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a+3c & b+3d \\ 2a+2c & 2b+2d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

For B ,

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$$
$$\Rightarrow -1 + \lambda^2 - 3 = 0$$
$$\Rightarrow \lambda^2 - 4 = 0 \quad \text{--- (2)}$$

$$\textcircled{1} \neq \textcircled{2} \Rightarrow A \not\sim B$$

rotation of basis

If $B = \underbrace{U}_{\substack{\text{unitary} \\ \text{matrix}}} A \underbrace{U^{-1}}_{\substack{\text{same} \\ \text{Eigen values}}}$

Eigen values do not change by rotating basis etc.
 \downarrow
 fundamental.

Diagonalization

An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that A is similar to D .
 i.e. there exists $n \times n$ invertible matrix P such that $P^{-1}AP = D$.

Q) Let A be an $(n \times n)$ matrix then A is diagonalizable if and only if A has n linearly independent eigen vectors.

① Suppose $A \sim D$,

$$D = P^{-1}AP \quad (P \text{ is an invertible matrix})$$

$$\Rightarrow AP = PD$$

Let the columns of the matrix P be $\{P_1, P_2, \dots, P_n\}$

— / —

$$\Rightarrow A \cdot [P_1, P_2, \dots, P_n] = [P_1, P_2, \dots, P_n] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix}$$

$$\Rightarrow [AP_1, AP_2, \dots, AP_n] = [\lambda_1 P_1, \lambda_2 P_2, \dots, \lambda_n P_n].$$

$$AP_i = \lambda_i P_i.$$

∴ Entries of the diagonal matrix are the eigen values of the matrix A .

Since P is invertible, column vectors are LI.

(2)

DIY.

g) if possible, find a matrix P that diagonalizes

i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ ii) $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$.

calc. eigen vectors of A (should be L.I.).

i) $\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix}$ ~~row op~~

$$-\lambda \left[(\lambda(\lambda-4)+5) \right] + [-2] = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$(\lambda-1) [\lambda^2 - 3\lambda + 2] = 0$$

$$\Rightarrow (\lambda-1)^2(\lambda-2) = 0.$$

$$\Rightarrow \lambda = 1, 1, 2.$$

same eigen values \Rightarrow L.D. eigen vectors
 \Rightarrow Not diagonalizable.

$$Ax = \lambda x$$

$$Ax = x \quad \text{and} \quad Ax = 2x$$

$$(A - I)x = 0$$

$$(ii) \begin{bmatrix} (-1-\lambda) & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & (-1-\lambda) \end{bmatrix}$$

$$(-1-\lambda) \begin{bmatrix} -\lambda(-1-\lambda) \end{bmatrix} + \begin{bmatrix} \lambda \end{bmatrix} = 0 .$$

$$\Rightarrow -\lambda^3 - \lambda^2 - \lambda^2 - \lambda + \lambda = 0 .$$

$$\Rightarrow -\lambda^3 - 2\lambda^2 = 0 .$$

$$\Rightarrow \lambda^2 (\lambda + 2) = 0 .$$

$$\lambda = 0, 0, -2 .$$

Ans =

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$