Q1

$U = \{ f \in V \mid f(x) = -f(-x) \}$

$W = \{ f \in V \mid f(-x) = f(x) \}$

we first prove that U and V are both subspaces.

Consider $U \to$ Let $f_1 \in U$ and $f_2 \in U$.

then $g(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$

$g(-x) = \alpha_1 f_1(-x) + \alpha_2 f_2(-x) = -\alpha_1 f_1(x) - \alpha_2 f_2(x)$

$= -g(x)$

Similarly we can prove that W is a subspace

~~Now, if $x \in U \cap V$ then~~

Now any function f can be expressed as

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= g_1(x) + g_2(x)$$

Notice that $g_1(-x) = g_1(x) \Rightarrow g_1 \in W$

Also $g_2(-x) = -g_2(x) \Rightarrow g_2 \in U$

Thus any vector $v \in V$ can be expressed as $g_1 + g_2$ where $g_1 \in U$ and $g_2 \in W$ . . (i)

Also if $f \in$ ~~U~~ $U \cap W$ then

$f \in U \Rightarrow f(-x) = -f(x)$ $f \in W$, and $f(-x) = f(x)$

$\Rightarrow f(x) = 0 \ \forall x \in \mathbb{R}$.

Thus $f(x) = 0$ is our identity element

since for any function $g + f = f + g = g$

$\Rightarrow U \cap W = \{0\}$ . . (ii)

Thus using (i) and (ii) $\to$ we have $V = U \oplus W$.

→ Q2) $f : V \to W$

Let $B = (v_1, v_2 \ldots v_k)$ be a basis of V

Since it is a basis, every vector $v \in V$ can be expressed as a linear combination of these vectors

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_k v_k \quad \alpha_k \in F$$

$$v_1, v_2 \ldots v_k \in B.$$

Now consider that we need to find $f(v)$ for some $v \in V$.

$$f(v) = f(\alpha_1 v_1 + \alpha_2 v_2 \ldots \alpha_k v_k)$$

$$= f(\alpha_1 v_1) + f(\alpha_2 v_2) + f(\alpha_3 v_3) + \ldots f(\alpha_k v_k)$$

[Since it is a linear transformation $f(x+y) = f(x) + f(y)$]

$$= \alpha_1 f(v_1) + \alpha_2 f(v_2) + \alpha_3 f(v_3) \ldots \alpha_k f(v_k)$$

[linear $f(c\alpha) = c f(\alpha)$]

Hence if we know the values for $f(v_1) \ldots f(v_k)$ we can find the mapping for any vector v.

This also means that if we have the effect of f on any basis of $V(F)$, then the range of complete linear transformation can be evaluated.

(3) For the given mapping to be a vector space, it should satisfy all properties.

$(x, y) + (x_1, y_1)$
$= (3y + 3y_1, -x - x_1)$
$c(x, y) = (3cy, -cx)$

Now to consider associative property.

$((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$
$= (3y_1 + 3y_2, -x_1 - x_2) + (x_3, y_3)$
$= (-3x_1 - 3x_2 + 3y_3, -3y_1 - 3y_2 - x_3)$ (i)

$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$
$= (x_1, y_1) + (3y_2 + 3y_3, -x_2 - x_3)$
$= (3y_1 - 3x_2 - 3x_3, -x_1 - 3y_2 - 3y_3)$ (ii)

Clearly (i) and (ii) are not same.

But to satisfy this associativity property they must be equal

$\Rightarrow$ Associativity does not hold

$\Rightarrow$ Not a vector space

(proved)

(Q4) $V \rightarrow$ vector space, X and $W \rightarrow$ subspace.

To prove $X \cup W$ is a subspace iff $W \subseteq X$ or $X \subseteq W$.

(i) If $W \subseteq X$ ~~or $X \subseteq W$~~

$X \cap W = X$ which is a subspace $\Rightarrow X \cap W \rightarrow$ subspace

Similarly for $X \subseteq W$.

(ii) $X \cup W$ is a subspace.

Let it hold that $W \not\subseteq X$ and $X \not\subseteq W$.

Then $\exists$ elements $x_1 \in W - X$ and $x_2 \in X - W$

Now $x_1, x_2 \in$ ~~$X \cap W$~~ ~~hence~~ $X \cup W$

Since it is a subspace $x_3 = x_1 + x_2 \in X \cap W$.

Now if $x_3 \in W$ then since $x_1 \in W$

then $x_2 = x_3 - x_1 \in W$

$[(-x_1) \in W \rightarrow$ inverse$]$

But we took $x_2 \in X - W \Rightarrow x_2 \notin W$

$\Rightarrow$ Contradiction

~~Hence~~ Using the same argument for x_1 we have $x_1 \notin X$

$\Rightarrow$ Contradiction

Thus our assumption was false and indeed one subspace has to be a subset of the other.

(proved)

(5) $V \to \dim(V) = n$

U and W subspace of $V \to$ finitedimensional.

Now, for two finite dimensional subspaces U and W, ~~dim~~

$$\dim(V) = \dim(U) + \dim(W) - \dim(U \cap W)$$

This is because,

$$U+W/W \approx U/(U \cap W) \quad \text{and} \quad \dim(A/B) = \dim(A) - \dim(B)$$

Now,

Given that $\dim(U) > n/2$, $\dim(W) > n/2$ $\Rightarrow \dim(U) + \dim(W) \geq n+2$.

~~Thus~~ But $\dim(V) = n$.

$$n + \dim(U \cap W) = \dim(U) + \dim(W) \geq n+2$$

$$\Rightarrow \dim(U \cap W) \geq 2$$

Now since $\dim(U \cap W) \geq 2 \Rightarrow \exists$, 2 vectors that are linearly independent.

If ~~AH~~ $U \cap W = \{0\}$ $\dim(U \cap W) = 0$ $\Rightarrow$ contradiction

$\Rightarrow U \cap W \neq \{0\}$

(6) $f \in L(V, V)$

$f^n = 0$ for some $n \Rightarrow$ nilpotent.

To prove: $\{x_0, f(x_0), \ldots, f^{n-1}(x_0)\}$ is linearly independent classmate

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Assume that they are linearly dependent.

Then,

$\alpha_0 x_0 + \alpha_1 f(x_0) + \alpha_2 f^2(x_0) + \dots \alpha_{n-1} f^{n-1}(x_0) = 0$

$\Rightarrow \alpha_i \neq 0$ for some i

Let p be the smallest index such that $\alpha_p \neq 0$.

Thus we have

$\alpha_p f^p(x_0) + \alpha_{p+1} f^{p+1}(x_0) + \dots \alpha_{n-1} f^{n-1}(x_0) = 0$

Now $f^{n-p-1}\left(\alpha_p f^p(x_0) + \alpha_{p+1} f^{p+1}(x_0) + \dots \alpha_{n-1} f^{n-1}(x_0)\right) = f(0)$

$= 0$

$= \alpha_p f^{n-1}(x_0) + \alpha_{p+1} f^n(x_0) + \dots \alpha_{n-1} f^{2n-p-1}(x_0) = 0$

[using property of linear transformation]

Now since $f^n(x) = 0$

we have $f^{n+i}(x) = 0 \quad \forall x, i \geq 0$

Hence

all terms $\underbrace{\alpha_{p+1} f^n(x_0)}_{0} + \dots \underbrace{\alpha_{n-1} f^{2n-p-1}(x_0)}_{0} = 0$

Thus we have $\boxed{\alpha_p f^{n-1}(x_0) = \vec{0}}$

But $f^{n-1}(x_0) \neq \vec{0} \Rightarrow \alpha_p = 0$ OEF.

But we assumed $\alpha_p \neq 0$

Contradiction

Hence there is no $\alpha_p \neq 0$ Thus

The vectors are linearly independent.

(7) $g \circ (f+f_1)(x)$

$= g \circ (f(x)+f_1(x))$ ← Using linearity property of Transformation

$= g(f(x)) + g(f_1(x))$

$= g \circ f + g \circ f_1$

Since $g \circ f(x)$

$= g(f(x)) =$

Thus we get a mapping from $V \to W$

$(g+g_1) \circ f(x)$

$= (g+g_1) \circ f(x)$

$= g(f(x)) + g_1(f(x))$

$= g \circ f + g_1 \circ f$

This is because $g(f(x)) = (g \circ f)(x)$

Thus proved.

(9) $f: V \to W$

$r(f) = \dim(\mathrm{Im}(f))$

We need to prove $r(f) \leq$ ~~dim~~ $\min(\dim(V), \dim(W))$

(i) $r(f) \leq \dim(W)$

~~Proo~~ $\Rightarrow \dim(\mathrm{Im}(f)) \leq \dim(W)$

$\mathrm{Im}(V)$ is a subspace of W

and dimension of subspace $\leq$ dimension of vector parent space

[It is subspace because

~~α_1 Im~~ $\alpha_1 f(x_1) + \alpha_2 f(x_2) = f(\underbrace{\alpha x_1 + \alpha_2 x_2}_{\in V})$

$\Rightarrow$ ~~$\in$~~ $f(\alpha x_1 + \alpha x_2) \in \mathrm{Im}(f)$]

If $\dim(V) > \dim(W)$ then we are done.

(ii) $r(f) \leq \dim(V)$ and $\dim(V) \leq \dim(W)$

If $v_1, v_2, v_3 \ldots v_n$ for a basis for V.

Then $f(v_1) \ldots f(v_n)$ form a basis of $\mathrm{Im}(f)$.

Also if $v_1, v_2 \ldots v_n$ are linearly independent

Then $f(v_1), f(v_k)$ are also linearly independent

$\forall v_i \ni f(v_i) \neq 0$.

~~As~~ Since $\dim(V) \to$ vectors linearly independent and forming basis $\to (v_1, v_2 \ldots v_n)$

$\Rightarrow (f(v_1), f(v_2) \ldots f(v_n))$ linearly independent and basis of $\mathrm{Im}(f)$

$\Rightarrow \dim(\mathrm{Im}(f)) \leq \dim(V)$

(proved)

Also using rank-nullity theorem,

$r(f) +$ ~~dim~~ $\dim(N(f)) = \dim(V)$

$\Rightarrow r(f) \leq \dim(V) \quad \because \dim \geq 0$

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(10) $f \in L(V,W)$ with $\dim(V) > \dim(W)$

Let $\dim(V) = n \Rightarrow$

$(v_1, v_2 \ldots v_n) \rightarrow$ linearly independent and generate V.

Also $v_i \neq 0 \ \forall i$ trivially.

Now consider
$f(v_1), f(v_2) \ldots f(v_n)$

~~The~~ Now if $f(v_i) = 0$ for any v_i then we are done. If not then we have $f(v_1) \ldots f(v_n)$ spanning the entire space and also linearly independent. If they are linearly dependent, then

$$f(v_i) = \alpha_1 f(v_1) + \ldots \alpha_{i-1} f(v_{i-1}) + \alpha_{i+1} f(v_{i+1}) \ldots + \ldots \alpha_n f(v_n)$$

$$\Rightarrow f(v_i) = f(\alpha_1 v_1 + \ldots \alpha_{i-1} v_{i-1} + \ldots \alpha_n v_n)$$

$$\Rightarrow f(v_i - \alpha_1 v_1 \ldots - \alpha_n v_n) = 0$$

$\in \text{Ker}(f)$

$$\Rightarrow \sum_{j=k+1}^{n} c_j w_j = \sum_{j=1}^{k} a_j w_j$$

$j = 1$ to K $\quad v_1$ to $v_K \in \text{Ker}(f)$

$$\Rightarrow a_1 = a_2 \ldots a_k = c_k \ldots c_n = 0.$$

~~Thus~~ But if they are linearly independent then $\dim(W) < \dim(V)$ fails because any number of vectors greater than dimension are linearly dependent.

Thus at least one v_i was such that $f(v_i) = 0$ (proved)

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(11) $ST(x,y)$
$= S(T(x,y)) = S(0,x) = (x,0)$

$TS(x,y)$
$= T(S(x,y)) = T(y,x) = (0,y)$

$S^2(x,y) = S(S(x,y)) = S(y,x) = (x,y)$

$T^2(x,y) = T(T(x,y)) = T(0,x) = (0,0)$

(12) $F = R$ $\quad f : C \to C$

$f(a+ib) = a-ib$

Now for any $x_1, x_2 \in C$

$f(x_1+x_2) = f(a+ib+c+id)$
$= f((a+c)+i(b+d))$
$= (a+c)-i(b+d)$
$= (a-ib)+(c-id)$
$= f(x_1)+f(x_2)$

Also,

$cf(x) = f(c(a+ib))$
$c \in F$
$= f(ca+icb)$
$= ca-icb = c(a-ib)$
$= cf(x)$

Thus proved that
the conjugate mapping can be
defined as a linear transformation