VECTOR SPACES

9.1. DEFINITION AND EXAMPLES OF VECTOR SPACES

So far we have studied the mathematical systems with one binary operation and with two binary operations known as groups and rings, integral domains, and fields. In the present chapter we shall discuss the matter of combining two different mathematical systems (algebraic systems) into a single system known as a vector space.

To combine two algebraic systems we have to define two binary operations which are called internal binary operation and external binary operation respectively. That is, if $(V, +)$ and $(F, +, .)$ be two systems, the system $(F, +, .)$ is a field. Then the binary operation $f: V X\, V \rightarrow V$ is called internal binary operation (addition) and the binary operation $g: F\ X\ V \rightarrow V$ is called external binary operation (scalar multiplication). Now we define the required mathematical systems.

Definition 9.1 (Vector Space) - A system $((V, \oplus), (F, +, .), .)$ is called a vector space (linear space) over the field $(F, +, .)$ if and only if

(a) $(F, +, .)$ is a field (scalars) whose identity elements with respect to addition and multiplication are denoted by 0 and 1 respectively.

(b) $(V, \oplus)$ is a commutative group, whose elements are called vectors, with respect to the operation of addition of two vectors, (called internal operation). That is,

(1) for each pair of vectors $\alpha, \beta \in V$ a vector $\alpha \oplus \beta \in V$,

(2) $\oplus$ addition is commutative, $\alpha \oplus \beta = \beta \oplus \alpha$.

(3) $\oplus$ addition is associative, $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$,

(4) there is a unique vector O is V, called the zero vector, such that $\alpha \oplus O = O \oplus \alpha = \alpha$, for all $\alpha \in V$,

(5) for each vector $\alpha \in V$ there is unique vector $-\alpha \in V$ such that $\alpha + (-\alpha) = O$.

(c) A rule $\odot$ or (external operation), called scalar multiplication, which associates with each scalar $a \in F$ and a vector $\alpha \in V$, a vector $a \odot \alpha \in V$, called the product of a and α, in such way that for all $a, b \in F$ and all $\alpha, \beta \in V$:

(1) $(a+b) \odot \alpha = (a \odot \alpha) \oplus (b \odot \alpha)$.

(2) $a \odot (\alpha \oplus \beta) = (a \odot \alpha) \oplus (a \odot \beta)$.

(3) $(a \,.\, b) \odot \alpha = a \odot (b \odot \alpha)$,

(4) $1 \odot \alpha = \alpha$., where 1 is the field identity.

Fortunately, the notation used above can be simplified. From the type of letters involved it is always clear whether we are adding two scalars or two vectors, or multiplying two scalars or a scalar and a vector. Therefore, we can use + to indicate both types of addition and juxtaposition to indicate both type of multiplications.

we shall denote a vector space over the field $(F, +, .)$ by $V(F)$.

We observe that the hypothesis $1\, x = x$ is quite essential; without it, every field and commutative group would yield a vector space under the trivial scalar multiplication $c\,\alpha = 0$ for all $c \in F$, $\alpha \in V$.

Example 9.2: Let $(F, +, .)$ be any field, and let V be the set of all n-tuples $\alpha = (x_1, x_2, \ldots, x_n)$ of scalars $x_i \in F$. If $\beta = (y_1, y_2, \ldots, y_n)$ of scalars $y_i \in F$, the sum of α and β is defined by

$$(\alpha + \beta) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

The product of scalar c and vector α is defined by $c\,\alpha = (cx_1, cx_2, \ldots cx_n)$.

Then $V(F)$ is a vector space over the field $(F, +, .)$.

The fact that this vector addition and scalar multiplication satisfy conditions given in the definition 9.1 of vectors space is easy to verify, using the similar properties of addition and multiplication of elements of F the verification of vector space axioms are left to the reader.

This space is called n-tuple space, denoted by $V_n(F)$ or F^n.

Example 9.3: Let $(F, +, .)$ be any field and let m and n be positive integers. Let V be the set of all $m \times n$ matrices over the field F.

$A = [a_{ij}]$ of scalars $a_{ij} \in F$,

where $i = 1, 2, \ldots m$, and $j = 1, 2, \ldots, n$.

If $B = [b_{ij}]$ of scalars $b_{ij} \in F$ and $i = 1, 2 \ldots m$. $J = 1, 2, \ldots n$,

the sum of A and B is defined by

$$A + B = [a_{ij} + b_{ij}].$$

The product of scalar $k \in F$ and vector A is defined by $kA = [ka_{ij}]$, called scalar multiplication.

Then we observe that the system $(V, +)$ is an abelian group under addition.

Now, to prove $V(F)$ is a vector space, we verify other conditions of vector space.

Let $A, B \in V$, and $k_1, k_2 \in F$, then

(1) $(k_1 + k_2) A = (k_1 + k_2)[a_{ij}] = [(k_1 + k_2) a_{ij}]$

$= [k_1 a_{ij} + k_2 a_{ij}]$

$= [k_1 a_{ij}] + [k_2 a_{ij}] = k_1 A + k_2 A.$

(2) $k_1 (A + B) = k_1 [a_{ij} + b_{ij}]$

$= [k_1 (a_{ij} + b_{ij})] = [k_1 a_{ij} + k_1 b_{ij}]$
$= [k_1 a_{ij}] + [k_1 b_{ij}] = k_1 A + k_1 B.$

(3) $(k_1 k_2) A = k_1 k_2 [a_{ij}] = k_1 [k_2 a_{ij}]$
$= k_1 (k_2 A).$

(4) $1 . A = 1 \{a_{ij}] = [1 . a_{ij}] = [a_{ij}] = A,$

where 1 is the identity element of the field.

Thus $V(F)$ is a vector space.

Example 9.4 Let $(R, +, .)$ be the field of real numbers and $(C, +, .)$ be the field of complex numbers then the field $(C +, .)$ of complex numbers forms a vector space over the field $(R, +, .)$ of real numbers.

Example 9.5 Let $(F, +, .)$ be a field and $(F [x], +, .)$ be the ring of polynomials in the indeterminant x with coefficients from F. Then the operations necessary to give $F [x]$ a vector space structure are the followings:

if

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$ in $F [x]$

and $g (x) = b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_m x^m$ in $F [x]$

where $m \leq n$, then

$f(x) + g (x) = (a_0 + b_0) + (a_1 + b_1) x + \ldots\ldots (a_m + b_m) x^m + \ldots + a_n x^n$ and for $c \in F$,

$cf (x) = (c . a_0) + (c . a_1) x + (c . a_2) x^2 + \ldots + (c . a_n) x^n$ belong to $F [x]$.

Let us retain symbol $F [x]$ for this vector space, in preference to the correct but awkward $F [x] (F)$.

It is referred to as the space of polynomial functions over the field $(F, +, .)$.

Example 9.6 Let $(F, +, .)$ be any field and let S be any nonempty set. Let V be the set of all functions from the set S into F.

The sum of two vectors $f, g \in V$ is the vector $f + g$, *i.e.*, the function from S into F, defined by

$$(f + g) (s) = f (s) + g (s).$$

The product of the scalar c and the function f is the function cf defined by

$$(cf) (s) = cf (s).$$

Now we shall verify that the operations we have defined satisfy vector space conditions.

(*a*) Since addition in F is commutative,

$$f (s) + g (s) = g (s) + f (s) \Rightarrow (f + g) (s) = (g + f) (s)$$

so $f + g = g + f$.

(*b*) Since addition in F is associative,

$$f (s) + [g (s) + h (s)] = [f (s) + g (s)] + h (s)$$
$$f (s) + [g (s) + h (s)] = (f + g) + h$$

so for each $s \in S$, $f + (g + h) = (f + g) + h$

(c) The unique zero vector is the zero function which assigns to each element $s \in S$ the scalar $f (s) = 0 \in F$.

(d) For each f in V, $(-f)$ is the function which is given by

$$(-f)(s) = -f(s).$$

Thus, $(V, +)$ is a commutative group.

Now, we see that the scalar multiplication satisfies the properties: For all $f, g \in F$, we have

(i) $((c_1 + c_2) f)(s) = (c_1 + c_2) f(s)$
$= c_1 f(s) + c_2 f(s) = (c_1 f + c_2 f)(s)$
for each $s \in S$, so $(c_1 + c_2) f = c_1 f + c_2 f$,

(ii) $(c_1 c_2) f(s) = c_1 (c_2 f(s))$
for each $s \in S$, so $(c_1 c_2 f) = c_1 (c_2 f)$

(iii) $(c_1 (f + g))(s) = c_1 ((f + g)(s))$
$= c_1 (f(s) + g(s))$
$= c_1 f(s) + c_1 g(s)$
$= (c_1 f + c_1 g)(s)$
for each $s \in S$,
so $c_1 (f + g) = c_1 f + c_1 g$,

and

(iv) $(I . f)(s) = 1 . f(s) = f(s)$
for each $s \in S$, so $1 . f = f$.

Which shows that the set of functions from S into F is a vector space over the field $(F, +, .)$

Theorem 9.7. If $V(F)$ is a Vector Space and $x \in V$, $y \in V$, $c \in F$, then

(1) $0x = \bar{0}$

(2) $c\,\bar{0} = \bar{0}$,

(3) $-(c\,x) = (-c)\,x = c\,(-x)$,

(4) $a(x - y) = ax - ay$,

(5) $ax = \bar{0}$, $a \neq 0 \Rightarrow x = \bar{0}$,

(6) $ax = ay$, $a \neq 0 \Rightarrow x = y$.

Proof : (1) To establish (1), we see the field result

$0 + 1 = 1$. Then

$$0x + x = 0x + 1x = (0 + 1)\,x = 1x = x = \bar{0} + x.$$

Since $(V, +)$ is a commutative group, the cancellation law gives

$$0\,x = \bar{0}.$$

(2) The proof of this follows from the group result

$\bar{0} + x = x$. We have

$$c\,\bar{0} + c\,x = c\,(\bar{0} + x) = cx = \bar{0} + cx.$$

Again the cancellation law gives

$$c\,\bar{0} = \bar{0}$$

(3) To obtain this, we observe that

$$\bar{0} = 0\,x = (c + (-c))\,x = c\,x + (-c)\,x.$$

This means that $(-c)\,x = -(c\,x)$. Similarly

$\bar{0} = c\,\bar{0} = c\,[x + (-x)] = c\,(x) + c\,(-x)$ which implies

$c\,[-x] = -[cx)$

(4) We have

$a\,(x - y) = a\,[x + (-y)] = a\,x + a\,(-y)$

$\qquad\qquad = a\,x - (a\,y)$

since by (3) we have $a\,(-y) = -(a\,y)$.

(5) We have

$ax = \bar{0},\ a \neq 0 \Rightarrow a^{-1}$ exists such that

$a \,.\, a^{-1} = a^{-1} \,.\, a = 1.$

By premultiplying $a\,x = \bar{0}$ by a^{-1}, we obtain

$a^{-1}\,(a \,.\, x) = a^{-1} \,.\, \bar{0}$

$\Rightarrow (a^{-1} \,.\, a) \,.\, x = \bar{0}$

$\Rightarrow 1\,x = \bar{0}$

$\Rightarrow x = \bar{0}$

(6) We have

$a\,x = a\,y,\ a \neq 0 \Rightarrow a^{-1}$ exists such that

$\qquad a^{-1} \,.\, a = a \,.\, a^{-1} = 1.$

Premultiplying $ax = ay$ by a^{-1}, we have

$\qquad a^{-1} .\, (a\,x) = a^{-1}\,(a\,y)$

$\Rightarrow (a^{-1} \,.\, a) \,.\, x = (a^{-1} \,.\, a) \,.\, y$

$\Rightarrow 1 \,.\, x = 1 \,.\, y$

$\Rightarrow x = y.$

9.2. SUBSPACES

Whenever a mathematical system has been considered, the question of subsystems arose. In the case of vector spaces, the sub-vector spaces are referred to as subspaces.

Definition 9.8 : Let $V\,(F)$ be a vector space over the field $(F, +, .)$ and $W \subseteq V$, $W \neq \phi$. Then $W\,(F)$ is a subspace of $V\,(F)$ provided that $W\,(F)$ satisfies the vector space axioms when equipped with the same operations defined on $V(F)$.

Since $W \subseteq V$, much of the algebraic structure of $W\,(F)$ is inherited from $V(F)$. The minimum conditions that $W\,(F)$ must satisfy to be space are :

(1) $(W, +)$ is a subgroup of $(V, +)$

(2) W is closed under scalar multiplication.

From the definition of subspace it is clear that usual criterion for deciding whether $(W, +)$ is subgroup of $(V, +)$ is to see if W is closed under differences. Thus, if $\alpha, \beta \in W$, then $\alpha - \beta \in W$. The second of the above conditions implies that $-\alpha = (-1)\,\alpha \in W$, whenever $\alpha \in W$. Because $\alpha - \beta = \alpha + (-\beta)$, and condition (2), together with the closure of W under addition is sufficient to guarantee that W be closed under differences.

Example 9.9: Every vector space $V(F)$ has two trivial subspaces, namely $V(F)$ itself and the zero subspace $\{0\}(F)$. Subspaces distinct from $V(F)$ are said to be proper subspaces.

Example 9.10: Consider the set W of vectors in $V_3(F)$ whose components add up to zero:

$W = \{(a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0\}$.

If $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W$, then their difference

$$(a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

And

$$\begin{aligned}(a_1 - b_1 + a_2 - b_2 + a_3 - b_3)\\ = (a_1 + a_2 + a_3) - (b_1 + b_2 + b_3) = 0 - 0 = 0\end{aligned}$$

which shows W is a subgroup under addition.

And for any $c \in F$,

$$c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$$

And

$$\begin{aligned} & ca_1 + ca_2 + ca_3\\ &= c(a_1 + a_2 + a_3)\\ &= c\,0 = 0.\end{aligned}$$

Thus, W is closed under scalar multiplication. Hence $W(F)$ is a subspace of $V_3(F)$.

Example 9.11: Let W denote the collection of all elements from the space $M_2(F)$ of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Let $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \in W$, then

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -(b-d) & a-c \end{bmatrix} \in W.$$

$$k\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} k.a & k.b \\ -(k.b) & k.c \end{bmatrix} \in W.$$

Hence $W(F)$ is subspace of the vector space $M_2(F)$.

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Example 9.12: Let W be the set of all symmetric matrices of order n taken from the set V of all $n \times n$ matrices over the field $(F, +, .)$ of the form

$[a_{ij}] = [a_{ji}]$, $i, j = 1, 2, 3, \ldots, n$ and $a_{ij} \in F$.

Then $W(F)$ is a subspace of the vector space $V(F)$.

Theorem 9.13: *A nonvoid subset W of V is a subspace $W(F)$ of the vector space $V(F)$ over the field $(F, +, .)$ if and only if for every $\alpha, \beta \in W$ and every $c \in F$,*

(1) $\alpha + \beta \in W$,

(2) $c\,\alpha \in W$.

Proof: The proof of the theorem consists of two parts. In the first part if $W(F)$ is a subspace of the space $V(F)$, then $W(F)$ is closed under addition of vectors and scalar multiplication of vectors by scalars. That is, if $\alpha, \beta \in W$, $c \in F$, then $\alpha + \beta \in W$ and $c\,\alpha \in W$.

Conversely, assume that W is closed under the operations. Then $(-1)\alpha = -\alpha \in W$ for every $\alpha \in W$, and $\alpha + (-\alpha) = \bar{0} \in W$. Vectors addition in W is the same as in V, so it is associative and commutative, and W forms a commutative group. The other axioms are satisfied in W because those properties are inherited from V.

Theorem 9.14: *$W(F)$ is a subspace of the vector space $V(F)$ if and only if $\phi \neq W \subseteq V$ and $c\,\alpha + d\,\beta \in W$, whenever $\alpha, \beta \in W$, $c, d \in F$.*

Proof: If $W(F)$ is a subspace, then by definition, W is nonempty and contains $c\alpha + d\,\beta$ for all $\alpha, \beta \in W$, $c, d \in F$.

Conversely, if this condition holds, that $c\,\alpha + d\,\beta \in W$ for all $\alpha, \beta \in W$, $c, d \in F$, W must contain $1\,\alpha + 1\beta = \alpha + \beta$ and $c\,\alpha + 0\,\beta = c\,\alpha$ for every α, $\beta \in W$, $c \in F$. Accordingly, W is closed with respect to the vector space operations.

Corollary 1: *$W(F)$ is a subspace of the vector space $V(F)$ if and only if $\phi \neq W \subseteq V$ and $c\,\alpha + \beta \in W$ whenever $\alpha, \beta \in W$, $c \in F$.*

Proof: Let $W \neq \phi$ and $c\,\alpha + \beta \in W$, whenever $\alpha, \beta \in W$, $c \in F$. Since W is nonempty, there is a vector $\alpha \in W$ and hence $(-1)\,\alpha + \alpha = \bar{0} \in W$. Then if α is any vector in W and $c \in F$, the vector

$$c\,\alpha = c\,\alpha + \bar{0} \in W.$$

In particular, $(-1)\,\alpha = -\alpha \in W$, finally if $\alpha\ \beta \in W$, then $\alpha + 1\,\beta = 1\,\alpha + \beta \in W$. Thus $W(F)$ is a subspace. Conversely, if W is a subspace of V, $\alpha, \beta \in W$, $c \in F$, certainly $c\,\alpha + \beta \in W$.

Theorem 9.15: If $W_1(F)$ and $W_2(F)$ are subspaces of the vector space $V(F)$, then so is $(W_1 \cap W_2)(F)$.

Proof: The set $W_1 \cap W_2$ is not empty, for $\bar{0} \in W_1 \cap W_2$.

Let $\alpha, \beta \in W_1 \cap W_2$ and $c \in F$, then since both $W_1(F)$ and $W_2(F)$ are subspaces,

$$\alpha + \beta \in W_1,\ c\,\alpha \in W_1.$$

and $\alpha + \beta \in W_2$, $c\,\alpha \in W_2$,

that is, $\alpha + \beta \in W_1 \cap W_2$

and $c\,\alpha \in W_1 \cap W_2$.

Thus $(W_1 \cap W_2)(F)$ is a subspace of $V(F)$, since $W_1 \cap W_2$ is closed under its operations.

Corollary: *If $W_i(F)$ is an indexed collection of subspaces of the vector space $V(F)$, then $(\cap W_i)(F)$ is a subspace of $V(F)$.*

***Theorem 9.16:** If $W_1(F)$ and $W_2(F)$ are subspaces of the vector space $V(F)$, then $(W_1 \cup W_2)(F)$ is not necessarily a vector space.*

Proof: $(W_1 \cup W_2, +)$ should be a subgroup of the group $(V, +)$ to be a subspace. But we have seen in the chapter of group theory that if $(W_1, +)$ and $(W_2, +)$ are subgroups of the group $(V, +)$, then $(W_1 \cup W_2, +)$ is not necessarily a subgroup of $(V, +)$. By the definition of a subspace, if $(W_1 \cup W_2, +)$ is not a subgroup of $(V, +)$, then $(W_1 \cup W_2)(F)$ is not a subspace.

Example 9.17 : Let $M_2(F)$ be the vector space of all 2-rowed square matrices over the field $(F, +, .)$. Take U to be the set of all scalar matrices

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} a \in F.$$

and W to be the set of all matrices of the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} b \in F.$$

$U(F)$ and $W(F)$ are both subspaces of $M_2(F)$,

For, if $$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \in U, c_1, d_1 \in F,$$

then $$c_1 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + d_1 \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} c_1 a & 0 \\ 0 & c_1.a \end{bmatrix} + \begin{bmatrix} d_1.c & 0 \\ 0 & d_1.c \end{bmatrix}$$

$$= \begin{bmatrix} c_1 a + d_1 c & 0 \\ 0 & c_1 a + d_1 c \end{bmatrix} \in U$$

and if $$\begin{bmatrix} 0 & b_1 \\ -b_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \in W, c, d \in F,$$

then $$c \begin{bmatrix} 0 & b_1 \\ -b_1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & cb_1 \\ -cb_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & db \\ -db & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & cb_1 + bd \\ -(cb_1 + db) & 0 \end{bmatrix} \in W.$$

But we note that while the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

belong to $U \cup W$, and their sum

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \notin U \cup W.$$

Thus, $(U \cup W)$ (F) is not a subspace of M_2 (F).

Example 9.18: Let V_3 (F) be a vector space. Let

$U = \{(a, 0, 0) \mid a \in F\}$ and $W = \{(0, a, 0) \mid a \in F\}$.

We can easily verify that U (F) and W (F) are both subspaces. Further, we note that if

$(a, 0, 0)$ and $(0, b, 0)$ belong to $U \cup W$, then their sum $(a, 0, 0) + (0, b, 0) = (a, b, 0)$

belongs to neither U nor W.

Thus, $(U \cup W)$ (F) is not a subspace of the vector space V_3 (F).

Definition 9.19: If U and W are non-empty subsets of the vector space V (F), their (linear) sum is defined to be the set

$U + W = \{u + w \mid u \in U, w \in W\}$.

Theorem 9.20: If $U(F)$ and $W(F)$ are subspaces of the vector space $V(F)$, then $(U + W)$ (F) is also a subspace.

Proof : The sum $U + W \neq \phi$, since $\bar{0}, \in U$, $\bar{0} \in W$, hence

$\bar{0} = \bar{0} + \bar{0} \in U + W$.

Let $\alpha, \beta \in U + W$, then

$\alpha = u_1 + w_1$ and $\beta = u_2 + w_2$, where $u_1, u_2 \in U$, $w_1, w_2 \in W$. For scalars c, $d \in F$,

$$c\,\alpha + d\,\beta = c\,(u_1 + w_1) + d\,(u_2 + w_2)$$
$$= (cu_1 + du_2) + (cw_1 + dw_2).$$

Since U (F) and W (F) are subspaces of the space V (F), it follows $c\,u_1 + du_2 \in U$ and $cw_1 + dw_2 \in W$. Thus

$$c\,\alpha + d\,\beta \in U + W.$$

Hence $(U + W)$ (F) is a subspace.

Theorem 9.21: If U (F) and W (F) are subspaces of the vector space V (F), then $(U + W)$ (F) is the smallest subspace containing both U and W, that is, $(U + W)$ (F) is the subspace generated by $U \cup W$, $(U + W)$ $(F) = [U \cup W]$ (F).

Proof: We have $U + W = \{u + w \mid u \in U, w \in W\}$. So for any scalars a, $b \in F$, $a\,u \in U$ and $b\,w \in W$ and $a\,u + b\,w \in U + W$. Therefore every element of $U + W$ can be written as the linear combination of elements of $U \cup W$. But the set $[U \cup W]$ of all linear combinations of elements of $U \cup W$ forms a subspace $[U \cup W]$ (F) of V (F). This implies $U + W \subseteq [U \cup W]$.

since $U \subset U + W$, $W \subset U + W$, then

$U \cup W \subseteq U + W$. That is, $(U + W)$ (F) is the subspace of the vectorspace V (F) which contains $U \cup W$. As the $[U \cup W]$ (F) is the smallest subspace containing $U \cup W$, we have $[U \cup W] \subseteq U + W \Rightarrow U \cup W = [U \cup W]$ since $U + W \subseteq [U \cup W]$.

***Theorem 9.22:** Let $U(F)$ and $W(F)$ be two subspaces of the vector space $V(F)$. Then following conditions are equivalent :*

(1) $U \cap W = \{\bar{0}\}$

(2) Every vector $\alpha \in U \cup W$ is uniquely represented in form $\alpha = u + w$ where $u \in U$, $w \in W$.

Proof: Let $U \cap W = \{\bar{0}\}$. Let the vector $\alpha \in U \cup W$ be respresented in two forms

$$\alpha = u_1 + w_1 = u_2 + w_2,\ u_1, u_2 \in U,\ w_1, w_2 \in W.$$

Then $u_1 - u_2 = w_2 - w_1$, but $u_1 - u_2 \in U$ and $w_2 - w_1 \in W$, since $U(F)$ and $W(F)$ are subspaces.

The equation $u_1 - u_2 = w_2 - w_1$ means that

$$u_1 - u_2 \in U \cap W \text{ and } w_2 - w_1 \in U \cap W,$$

therefore it follows that

$$u_1 - u_2 = 0,\ w_2 - w_1 = 0$$

or $u_1 = u_2$, $w_1 = w_2$.

In other words, α is uniquely represented in the form $u + w$. Conversely, assume the statement (2) holds and the vector $\beta \in U \cap W$. We may express β in two different ways as the sum of a vector in U and a vector in W, namely $\beta = \beta + \bar{0}$, where $\beta \in U$, $\bar{0} \in W$ and $\beta = \bar{0} + \beta$, where $\bar{0} \in U$, $\beta \in W$. The uniqueness gives $\beta = \bar{0}$, that is, $U \cap W = \{\bar{0}\}$.

Definition 9.23: The two subspaces $U(F)$ and $W(F)$ of the vector space $V(F)$ are complementary if $U \cap W = \{\bar{0}\}$ and $U + W = V$. The sum $U + W$ is called the direct sum and denoted by $U \oplus W$, if $U + W = V$ and $U \cap W = \{\bar{0}\}$.

Definition 9.24: Two subspaces $U(F)$ and $W(F)$ are said to be disjoint if $U \cap W = \{\bar{0}\}$.

Theorem 9.25: Let $U(F)$ and $W(F)$ be two subspaces of the vector space $V(F)$. Every vector $x \in V$ is uniquely expressible in the form $x = u + w$, where $u \in U$, $w \in W$, if and only if $V = U \oplus W$.

Proof: Let $V = U \oplus W$. Then $U \cap W = \{\bar{0}\}$ and $U + W = V$.

Let $v \in V$ and Let $v = x_1 + x_2 = y_1 + y_2$, $\quad x_1, y_1 \in U$
$\quad x_2, y_2 \in W$

So

$$x_1 + x_2 = y_1 + y_2 \Rightarrow x_1 - y_1 = y_2 - x_2$$

Since $x_1 - y_1 \quad U$, $y_2 - x_2 \in W$, then

$$x_1 - y_1 = y_2 - x_2 \Rightarrow x_1 - y_1 \in W \text{ and } y_2 - x_2 \in U$$
$$\Rightarrow x_1 - y_1, y_2 - x_2 \in U \cap W = \{\bar{0}\}$$
$$\Rightarrow x_1 - y_1 = \bar{0} \text{ and } y_2 - x_2 = \bar{0}$$
$$\Rightarrow x_1 = y_1 \text{ and } x_2 = y_2.$$

Hence every element of V is uniquely expressed as a sum of an element of U and of an element of W.

Conversely, if $x = u + w \in U + W$, where $x \in V$, then

$U + W = V$.

$x \in U \cap W \Rightarrow x \in U$ and $x \in W$

$\Rightarrow x = u + \bar{0}$ and $x = \bar{0} + w$

$\Rightarrow u + \bar{0} = \bar{0} + w$

$\Rightarrow u = w = \bar{0}$. Since the representation of x is unique. Hence

$x = \bar{0} \Rightarrow U \cap W = \{\bar{0}\}$.

Hence the theorem.

Remark: For given a subspace $U(F)$ of $V(F)$ there exist several complementary subspaces. In case of $V_2(F)$, if

$U = \{(a, 0) \mid a \in F\}$, $W = \{(0, a) \mid a \in F\}$, $W' = \{(a, a) \mid a \in F\}$

Then $W(F)$ and $W'(F)$ are complementary subspaces of the subspace $U(F)$.

Theorem 9.26: *Let $W(F)$ be a subspace of the vector space $V(F)$ and the cosets $\alpha + W$, $\beta + W \in V/W$. If vector addition and scalar multiplication are given by $(\alpha + W) + (\beta + W) = (\alpha + \beta) + W$,*

$c(\alpha + W) = c\alpha + W$, $c \in F$,

then $(V/W)(F)$ is itself a vector space, known as the quotient space of V by W.

Proof: Since $(V, +)$ is a commutative group, the subgroup $(W, +)$ of the group $(V, +)$ is also commutative. Hence $(W, +)$ is a normal subgroup of $(V, +)$. Thus, $(V/W, +)$ is a quotient group of V by W. The quotient group $(V/W, +)$ is a commutative group since $(V, +)$ is a commutative group.

For other vector space axioms, we note that if $\alpha + W$, $\beta + W \in V/W$, $c_1, c_2 \in F$, then

(i) $(c_1 + c_2)(\alpha + W)$

$$= (c_1 + c_2)\alpha + W$$
$$= (c_1\alpha + c_2\alpha) + W$$
$$= (c_1\alpha + W) + (c_2\alpha + W)$$
$$= c_1(\alpha + W) + c_2(\alpha + W).$$

(ii) $(c_1 . c_2)(\alpha + W)$

$$= (c_1 . c_2)\alpha + W$$
$$= c_1 . ((c_2\alpha) + W)$$
$$= c_1(c_2(\alpha + W))$$

(iii) $c[(\alpha + W) + (\beta + W)] = c[(\alpha + \beta) + W]$

$$= (c(\alpha + \beta)) + W$$
$$= c\alpha + c\beta) + W$$
$$= (c\alpha + W) + (c\beta + W)$$
$$= c(\alpha + W) + c(\beta + W).$$

(iv) $1 . (\alpha + W) = \alpha + W$.

this proves the theorem.

Definition 9.27: Let A be a set of vectors in a vector space $V(F)$. The subspace spanned by A is defined to be the intersection W of all subspaces of V which contain A. When A is a finite set of vectors, $A = \{\alpha_1, \alpha_2, ...\alpha_n\}$, we shall simply call W the subspace spanned by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition 9.28: Suppose $V(F)$ is a vector space and $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$, any finite sum of the form

$$c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n,$$

where each $c_i \in F$, is said to be a linear combination (over F) of the vectors $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$. A linear combination is called trivial if all its coefficients $c_i = 0$, and nontrivial if at least one coefficient is different from zero.

If the set $A = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$, the set [A] of all linear combinations of vectors of A is the collection of all finite sums of the form

$$c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n.$$

where $c_i \in F$, $\alpha_i \in A$, and $i = 1, 2, \ldots\ldots n$.

Theorem 9.29: The set [A] of all linear combinations of any non-void subset A of vectors of V is a subspace [A] (F) of the vector space V (F).

Proof: It is clear that the set $[A]$ of all linear combinations contains A. So the set $[A]$ is non-empty.

Let $\alpha, \beta \in [A]$, $c, d \in F$.

Since $\alpha, \beta \in [A]$, α and β are linear combinations of $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n \in A$. Thus

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n, \quad (a_i \in F),$$

and $$\beta = b_1\alpha_1 + b_2\alpha_2 + \ldots\ldots + b_n\alpha_n. \quad (b_i \in F)$$

we note that

$$\begin{aligned} c\alpha + d\beta &= c(\textstyle\sum a_i\alpha_i) + d(\sum b_i\alpha_i) \\ &= \textstyle\sum (ca_i)\alpha_i + \sum (db_i)\alpha_i \\ &= \textstyle\sum (ca_i + db_i)\alpha_i \in [A], \end{aligned}$$

since $ca_i + db_i \in F$,

which shows $[A](F)$ is a subspace of the vector space $V(F)$.

Example 9.30: Let $(F, +, .)$ be a subfield of the field $(C, +, .)$ of complex numbers. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$

Suppose

$$\alpha_1 = (1, 2, 0, 3, 0),$$
$$\alpha_2 = (0, 0, 1, 4, 0),$$
$$\alpha_3 = (0, 0, 0, 0, 1).$$

A vector α is in the subspace $[A](F)$ of F^5 if and only if there exists $c_1, c_2, c_3, \in F$ such that

$$\begin{aligned} \alpha &= c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \\ &= c_1(1, 2, 0, 3, 0) + c_2(0, 0, 1, 4, 0) + c_3(0, 0, 0, 0, 1) \\ &= (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3). \end{aligned}$$

Thus, $[A]$ consists of the vectors of the form

$$\alpha = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3),$$

Thus for example $(-3, -6, -1, -5, 2) \in [A]$, whereas

$$(2, 4, 6, 7, 8) \notin [A].$$

Example 9.31: Let A be the subset of $V_n(F)$ (or F^n) whose elements are the n vectors $e_1 = (1, 0, 0, \ldots\ldots 0)$, $e_2 = (0, 1, \ldots\ldots 0), \ldots\ldots e_n = (0, 0, \ldots\ldots 1)$; in general, e_k is the vector with 1 in the k^{th} component position and 0 elesewhere :

$e_k = (\delta_{k1}, \delta_{k2}, \ldots\ldots \delta_{kn})$, $k = 1, 2, \ldots\ldots n$, $\delta_{ij} = 1, i = j$ $\delta_{ij} = 0, i \neq j$

Here $[A](F)$, the subspace spanned by these vectors, is all of $V_n(F)$. Indeed, for any n-tube $(a_1, a_2, \ldots\ldots a_n)$ over F, we have $(a_1, a_2, \ldots\ldots a_n) = a_1(1, 0, 0, \ldots\ldots 0) + a_2(0, 1, 0, \ldots\ldots 0) + \ldots\ldots + a_n(0, 0, \ldots\ldots 1)$

$= a_1 e_1 + a_2 e_2 + \ldots\ldots + a_n e_n \in [A]$.

Example 9.32: Let $V(F)$ be the space of all polynomial functions over F.

Let S be the subset of V consisting of the polynomial functions $f_1, f_2, \ldots\ldots$ defined by

$f_n(x) = x^n$, $n = 0, 1, 2, \ldots\ldots$

Then $V(F)$ is the subspace spanned by the set S.

PROBLEMS

In the exercises below, the symbol F stands for an arbitrary field.

1. If $V(F)$ is a vector space, prove that

   (*a*) $n(cx) = (nc)x = c(nx)$

   for all $x \in V$, $c \in F$ and $n \in Z$.

   (*b*) If $x \in V$ with $x \neq 0$, then $c_1 x = c_2 x \Rightarrow c_1 = c_2$.

   (*c*) verify that

   $(x_1 + x_2) + (x_3 + x_4) = [x_2 + (x_3 + x_1)] + x_4$

   for all x_1, x_2, x_3, and x_4 in V.

2. Let V be the set of all pairs (x, y) of real numbers and $(F, +, .)$ be the field of real numbers. Define $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$, $c(x, y) = (cx, y)$. Is V, with these operations, a vector space over the field of real numbers ?

3. Let V be the set of all ordered pairs (x, y) of real numbers and let $(F, +, .)$ be a field of real numbers.

   Define

   $(x, y) + (x_1, y_1) = (3y + 3y_1, -x - x_1)$,

   $c(x, y) = (3cy, -cx)$

   Verify that V, with these operations, is not a vector space over the field of real numbers.

4. Consider the set of all the triples of real numbers (a_1, a_2, a_3), subject to the conditions stated below. In each case determine whether the system forms a vector space relative to addition of triples and multiplication of a criple by a real number.

   (1) $a_1 = 0$; a_2 and a_3 arbitrary.

   (2) $a_1 = -a_2$; a_3 arbitrary.

   (3) a_1, a_2, arbitrary, $a_3 = 1 + a_1 - a_2$.

(4) a_1, a_2 arbitrary, $a_3 = 3a_1 - 4a_2$

(5) $a_1\ a_2 \geq 0$, a_3 arbitrary.

5. Consider the set S of all solutions of a homogeneous differential equation of order n:

$y^n + a_1(x)\, y^{n-1} + ... + a_{n-2}(x)\, y'' + a_{n-1}(x)\, y' + a_n(x)\, y = 0.$

Prove that S forms a vector space relative to add' ion and scalar multiples of functions.

N2 6. If V is the space of all functions from R into R which are continuous on $-1 \leq x \leq 1$, which of the following subsets are subspaces of V ?

(1) All differentiable functions.

(2) All polynomials of degree two.

(3) All polynomials of degrees less than five.

(4) All odd functions $f(-x) = -f(x)$.

(5) All even functions $f(-x) = f(x)$.

(6) All functions for which $f(0) = 0$.

(7) All negative functions.

(8) All constant functions.

7. For each of the following sets W, determine whether $W(R)$ is a subspace of the vector space $V_n(R)$ over the field of real numbers.

(1) $W = \{(a_1, a_2, ..., a_n) \mid a_1 + a_2 + ... + a_n \neq 0\}$.

(2) $W = \{(a_1, a_2, ..., a_n) \mid a_1 = a_2 = ... = a_n\}$.

(3) $W = \{(a_1, a_2, ..., a_n) \mid a_1\ a_2 = 0\}$.

(4) $W = \{(a_1, a_2, ..., a_n) \mid a_k \in Z, \forall k\}$.

8. Consider the set of all triples of real numbers, (a_1, a_2, a_3) subject to conditions stated below. In each case determine the set forms a subspace of $V_3(R)$.

(*i*) $a_1 = 5a_2$

(*ii*) $a_1 + a_2 = a_3^{-1}$.

(*iii*) $a_1 \geq 0$.

(*iv*) $a_1 = 1$.

(*v*) a_1 is a rational number.

9. Let $U(F)$ and $W(F)$ be subspace of the vector space $V(F)$. Prove that $(U \cup W)(F)$ forms a subspace of $V(F)$ if and only if $U \supseteq W$ or $W \supseteq U$.

10. Find all of the subspace of $V_2(Z_2)$ and $V_3(Z_2)$.

11. In the vector space $V_3(F)$, define the subsets U and W by $U = \{(a, b, a + b) \mid a, b \in F\}$, $W = \{(c, c, c) \mid c \in F\}$ verify $V_3 = U \oplus W$.

12. If S and T are subsets of the vector space $V(F)$ over the field F, then

(*a*) $S \subseteq T \Rightarrow [S] \subseteq [T]$.

(*b*) $[S\ U\ T] \Rightarrow [S] + [T]$.

(*c*) $[[S]] \Rightarrow [S]$

(*d*) $S \subseteq [T] \Rightarrow [S] \subseteq [T]$.

13. Let $V(R)$ be the vector space of all functions form the set R of real numbers into itself, and Let

$U = \{f \in V \mid f(-x) = -f(x)\}$, and

$W = \{f \in V \mid f(-x) = f(x)\}$, Then prove that
$V = U \oplus W$.

14. Determine whether $W(F)$ is a subspace of the indicated vector space:.
 (a) $V_3(F)$; for fixed scalars $a_1, a_2, a_3 \in F$,
 $W = \{(x_1, x_2, x_3) \mid a_1 x_1 + a_2 x_2 + a_3 x_3 = 0\}$.
 (b) $V(F)$; $W = \{(a_1, a_2, ... a_n, 0, 0, ...) \mid a_k \in F, n \in N\}$.
 (c) $\underset{\infty}{F}[x]$; W consists of all polynomials of degree greater then 4.
 (d) $V_n(F)$; for fixed $n \times n$ matrix $[a_{ij}]$,
 $W = \{x \in V_n(F) \mid [a_{ij}]\, x = 0\}$.
 (e) $M_3(F)$, $W = \begin{bmatrix} a & b & 0 \\ 0 & a+b & 0 \\ 0 & 0 & b \end{bmatrix} \in M_3(F) \{a, b \in F\}$

15. Suppose $U(F)$, $W(F)$, are subspaces of the vector space $V(F)$. Prove that
 (a) If $U \subseteq V$, $W \subseteq V$, then $U + W \subseteq V$,
 (b) $(U \cap V) + (W \cap V) \subseteq (U + W) \cap V$,
 (c) If $U \subseteq V$, then $U + (W \cap V) = (U + W) \cap V$.

9.3. LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

We now come to one of the most useful concepts in the theory of vector spaces, that of linear dependence and linear independence. We give below some useful definitions.

Definition 9.33: Let $S = \{\alpha_1, \alpha_2, ... \alpha_n)$ be a finite subset of the vector space $V(F)$. (*i*) S is said to be linearly dependent (*L.D.*) if there exist n scalars.

$c_1, c_2, ..., c_n \in F$, not all zero, Such that

$c_1 \alpha_1 + c_2 \alpha_2 + ... + c_n \alpha_n = \bar{0}$

(*ii*) On the other hand, S is said to be linearly independent (L, I) if and only if every equation of the form

$c_1 \alpha_1 + c_2 \alpha_2 + ... + c_n \alpha_n = 0$

implies that $c_1 = c_2 = c_3 = ... = c_n = 0$

(*iii*) A vector β which can be expressed in the form

$\beta = c_1 \alpha_1 + c_2 \alpha_2 + ... + c_n \alpha_n$

is said to be a linear combination of vectors $\alpha_1, \alpha_2, ... \alpha_n$ where $c_i \in F$.

Example 9.34: Let $(F, +, .)$ be a subfield of the complex numbers. In $V_3(F)$ the vectors

$\alpha_1 = (3, 0, -3)$,
$\alpha_2 = (-1, 1, 2)$,
$\alpha_3 = (4, 2, -2)$,
$\alpha_4 = (2, 1, 1)$,

are linearly dependent, since

$c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3 + c_4 \alpha_4 = c_1 (3, 0, -3) + c_2 (-1, 1, 2) + c_3 (4, 2, -2) + c_4 (2, 1, 1)$

$= (3 c_1 - c_2 + 4 c_3 + 2 c_4, c_2 + 2 c_3 + c_4, - 3 c_1 + 2 c_2 - 2 c_3 + c_4).$

$= (0, 0, 0)$

implies $3 c_1 - c_2 + 4 c_3 + 2 c_4 = 0,$

$c_2 + 2 c_3 + c_4 = 0$

$-3 c_1 + 2 c_2 - 2 c_3 - 1 c_4 = 0,$

which give $c_1 = 2, c_2 = 2, c_3 = - 1, c_4 = 0,$

The vectors $e_1 = (1, 0, 0)$

$e_2 = (0, 1, 0)$

$e_3 = (0, 0, 1)$

are linarly independent, since

$c_1 e_1 + c_2 e_2 + c_3 e_3 = c_1 (1, 0, 0) + c_2 (0, 1, 0) + c_3 (0, 0, 1)$

$= (c_1, c_2, c_3) = (0, 0, 0)$

implies $c_1 = c_2 = c_3 = 0.$

Examples 9.35: In $R[x]$, the vector space of polynomials in x over the field of real numbers $(R, +, .)$, consider the three polynomials $1 + x + 2x^2$, $2 - x + x^2$, $- 4 + 5x + x^2$ of degree two, Since

$2 (1 + x + 2x^2) + (-3)(2 - x + x^2) + (-1)(-4 + 5x + x^2) = 0$, then the given polynomials are linarly dependent,

Example 9.36: Show that the vectors $(1, 2, 3), (3, -2, 0)$ form a linearly independent set.

Let $k_1 (1, 2, 3) + k_2 (3, -2, 0) = (0, 0, 0)$

which means $k_1 + 3 k_2 = 0,$

$2 k_1 - 2 k_2 = 0$

$3 k_1 + 0 k_2 = 0.$

Therefore, on solving these equations we have

$k_1 = 0 = k_2.$

Hence the given vectors are linearly independent.

Here we shall consider the vactors taken from the given vectorspace $V(F)$ over the field $(F, +, .)$

Theorem 9.37: If β is linear combination of the set of vectors $\alpha_1, \alpha_2 ... \alpha_n$ then the set $\{\beta, \alpha_1, \alpha_2 ... \alpha_n\}$ is linearly dependent.

Proof: We have

$\beta = a_1 \alpha_1 + a_2 \alpha_2 + ... + a_n \alpha_n, a_i \in F$

or $\beta - (a_1 \alpha_1 + a_2 \alpha_2 + ... + a_n \alpha_n) = \bar{0}$

It is clear that the coefficient of β is not zero, which is sufficient to show that the set $\{\beta, \alpha_1, ... \alpha_n\}$ is a linearly dependent.

Theorem 9.38: If $\{\alpha, \alpha, ..., \alpha_n\}$ is a linearly independent set and $\{\beta, \alpha_1,$

$\alpha_2, \ldots \alpha_r$) is linearly dependent set, β is a lineary combination of $\alpha_1, \alpha_2, \ldots, \alpha_r$.

Proof we have

$a\beta + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = \bar{0}$, $a, a_i \in F$

where the coefficients are not all zero.

Now, if $a \neq 0$,

$\beta = a^{-1}(-1)(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n)$

So that β is a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_n$.

If $a = 0$,

$$a_1\alpha_1 + a_2\alpha_2 + \ldots a_n\alpha_n = \bar{0}$$

implies $a_1 = a_2 = a_3 = \ldots = a_n = 0$, since $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are linearly independent and we have a contradiction.

Theorem 9.39: Every super set of a linearly dependent set is linearly dependent.

Proof: If $S_1 \subseteq S_2$, then S_2 is called the super set of S_1.

Let $S_2 = \{\alpha_1, \alpha_2, \ldots \alpha_r, \alpha_{r+1}, \ldots \alpha_n\}$,

and $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_r\}$ be linearly dependent set. Thus, S_2 is a super set of the set S_1.

Since S_1 is linearly dependent, there exists a relation of the form

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_r\alpha_r = \bar{0}$ $a_i \in F$ .... (1)

where $a_1, a_2, \ldots a_r$ are not all zero.

We have relation of the form

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_r\alpha_r + a_{r+1}\alpha_{r+1} \ldots + a_n\alpha_n = , \bar{0}$ .... (2)

where $a_{r+1} = a_{r+2} = \ldots = a_n = 0$,

which shows the set $\{\alpha_1, \alpha_2, \ldots \alpha_{r-1}, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n\}$ is linearly dependent since all a_i's are not zero in the relation

Theorem 9.40: A set

$\{\alpha_1, \alpha_2, \ldots \alpha_r\}$

of non-zero vectors is linearly dependent if, and only if, some

$\alpha_p, 2 \leq p \leq r$

is a linear combination of the preceding vectors.

Proof: Suppose that the set is L.D., being non-zero set consisting only of a_1, is L.I. Now there must exist a number $p \geq 2$ such that

$\{\alpha_1, \alpha_2, \ldots \alpha_{p-1}\}$

is linearly independent (L.I.) and

$\{\alpha_1, \alpha_2, \ldots \alpha_{p-1}, \alpha_p\}$

is a linearly dependent set for, at most $p = r$ since $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}\ \alpha_p$ are L.D. and $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$ are L.I. then α_p is a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_{p-1}$ by theorem 9.38

Conversely, if α_p is a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_{p-1}$, then the set $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}, \alpha_p$ is linearly dependent. And any super set $\{\alpha_1, \alpha_2, \ldots, \alpha_p, \ldots, \alpha_r\}$ of a linearly dependent set $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ is linearly dependent by the theorem 9.39.

Theorem 9.41: If the set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a linearly independent set, then any vector $\alpha \in [A]$ is uniquely expressible as a linear combination of the vectors $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$.

Proof: Let α have two representations,

$\alpha = a_1\,\alpha_1 + a_2\,\alpha_2 + \ldots + a_n\,\alpha_n = b_1\,\alpha_1 + b_2\,\alpha_2 + \ldots + b_n\,\alpha_n$.

This leads to the relation

$(a_1 - b_1)\,\alpha_1 + (a_2 - b_2)\,\alpha_2 + \ldots + (a_n - b_n)\,\alpha_n = 0.$

Since the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent,

$a_k - b_k = 0$, for all $k = 1, 2, \ldots n$

or $a_k = b_k$.

Exercise 9.43: If α, β, γ are L.I. vectors in $V(F)$,

Prove that $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also L.I.

Solution: To show that the elements $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are linearly independent, there are scalars x, y, z such that

$$x(\alpha + \beta) + y(\beta + \gamma) + z(\gamma + \alpha) = \bar{0}$$

or $$(x + z)\,\alpha + (x + y)\,\beta + (y + z)\,\gamma = \bar{0} \qquad \ldots (1)$$

Since α, β, Y are inearly independent, then we ontain from the relation (1), that

$x + y = 0,$ ... (2)

$x + z = 0,$ ... (3)

$x + x = 0,$ ... (4)

On adding (2), (3) and (4) we get

$x + y + z = 0$ ... (5)

Subtracting (2), (3), and (4) from (5) respectively we get

$z = 0, x = 0, y = 0.$

Which shows, $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are linearly independent.

(V) N7 ***Exercise 9.44:*** Show that the set $\{\alpha_1\, \alpha_2 \alpha_3\}$ is linearly dependent if the set $\{\alpha_1 + a\,\alpha_2 + b\,\alpha_3, \alpha_2, \alpha_3\}$ is linearly dependent, where $\alpha_1, \alpha_2, \alpha_3 \in V(F)$, $a, b \in F$.

Solution: Since the set $\{\alpha_1 + a\,\alpha_2 + b\,\alpha_3, \alpha_2, \alpha_3\}$ is linearly dependent, there exist scalars $l, m, n, \in F$ not all zero, such that

$$l(\alpha_1 + a\,\alpha_2 + b\,\alpha_3) + m(\alpha_2) + n(\alpha_3) = \bar{0}$$

or $$l\,\alpha_1 + (a\,l + m)\,\alpha_2 + (b\,l + n)\,\alpha_3 = \bar{0} \qquad \ldots (1)$$

If $l \neq 0$, then $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent.

If $l = 0$, the relation (1) reduces to the form

$m\,\alpha_2 + n\,\alpha_3 = \bar{0}$

Since l, m, n are not all zero, and $l = 0$, at least one of m, n is non zero. Which implies α_2, α_3 are linearly dependent and $\{\alpha_2, \alpha_3\} \subset \{\alpha_1, \alpha_2, \alpha_3\}$ which means $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent.

9.4. BASES AND DIMENSION

Theorem 9.45: Let n vectors $v_1, v_2, \ldots, v_n$ span a vector space containing m linearly independent vectors $u_1, u_2, \ldots u_m$, then $n \geq m$.

Proof: Since $v_1, v_2, \ldots v_n$ span the space, u_1 can be expressed linear combination of the v_i. Therefore this equation can be solved for some one of the v_i, say v_1 in terms of u_1 and the rest of the v_i, consequently the set consisting of u_1 and the rest of the v_i spans the vector space, since any linear combination of the v_i becomes a linear combination of u_1 and all the v_i except V_1 when the expression for v_1 interms of u_1, and other v_i is used to eliminate v_1. Then u_2 can be expressed as a linear combination of u_1 and all the v_i except v_1. Since u_i are are linearly independent, some v_i must have a non-zero coefficient, and therefore this v_i can be expressed in terms of u_1, u_2 and the remaining $(n-2)$ v_i, and these n vectors span the space. The process can be continued untill all m of the v_i vectors are used, and since at each stage one u_i vector is replaced, the number of vector v_i must have been at least as great as the number of vectors u_i. That is, $n \geq m$.

Theorem 9.46: If two sets of linearly independent vectors span the same space, there are the same number of vectors in each set.

Proof: If there are m vectors in one set and n in the other, then by theorem. 9.45 $m, \geq n$ and $n \geq m$, and thus $m = n$.

Definition 9.47: A basis of vector space is a linearly independent subset which generates (spans) the whole space. A vector space is finite-dimensional if and only if it has a finite basis.

Example 9.48: Let $(F, +, .)$ be a field and in vector space $V_n(F)$ (or F^n) let S be the subset consisting of the vectors

$e_1, e_2, \ldots\ldots, e_n$ defined by

$e_1 = (1, 0, 0, \ldots 0)$.

$e_2 = (0, 1, 0, \ldots 0)$.

$e_3 = (0, 0, 1, \ldots 0)$,

... ...

... ...

$e_n = (0, 0, 0, \ldots 1)$.

Let $a_1, a_2, \ldots a_n$ be scalars in F and put

$\alpha = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n$

Then $\alpha = (a_1, a_2, \ldots a_n)$.

This shows $e_1, e_2, \ldots, e_n$ span $V_n(F)$ (or F^n). Since $\alpha = \bar{0}$ if and only if $a_1 = a_2 = \ldots = a_n = 0$, the vectors $e_1, e_2, \ldots, e_n$ are linearly independent. Thus the set $S = \{e_1, e_2, \ldots, e_n\}$ is a basis for $V_n(F)$. This particular basis is called standard basis of $V_n(F)$.

Example 9.49: Let $M_n(F)$ be the vector space of $n \times n$ matrices over a field $(F, +, .)$. This space has a basis consisting of the n^2 matrices E_{ij}, where E_{ij} is the square matrix of order n having as its $(i, j)^{th}$ entry one and zeros elsewhere, any matrix $(a_{ij}) \in M_n(F)$ can be written as

$(a_{ij}) = a_{11} E_{11} + a_{12} E_{12} + \ldots + a_{nn} E_{nn}$. More over $[a_{ij}] = \bar{0}$ if and only if $a_{ij} = \ldots a_{nn} = 0$ Hence $E_{11}, E_{12}, \ldots E_{nn}$, are linearly independent over F.

Example 9.50: Let $F[x]$, the vector space of polynomials in x with coefficients from F. A basis for $F[x]$ is formed by the set

$S = \{1, x, x^2, \ldots, x^n, \ldots\}$

By definition, each polynomial $p(x) = a_0 1 + a_1 x + \ldots + a_n x^n \in F[x]$ is linear combination of the elements of S. The set S is linearly independent. For any finite subset

$\{x^{n_1}, x^{n_2}, \ldots, x^{n_k}\}, (0 \le n_1 < n_2 < \ldots < n_k)$, the relation

$c_1 x^{n_1} + c_2 x^{n_2} + \ldots + c_k x^{n_k} = \bar{0}$

holds if and only if $c_1 = c_2 = \ldots = c_k = 0$.

Corollary: If $V(F)$ is a finite dimensional vector space, then the two bases have the same (finite) number of elements.

Proof: Since $V(F)$ is a finite dimensional it has a finite basis

$\beta_1, \beta_2, \ldots \beta_m$

Then, by the Theorem 9.45 every basis of $V(F)$ is finite and contains no more than m elements Thus if $\alpha_1, \alpha_2, \ldots \alpha_n$ is a basis, $n \le m$. By the same argumant $m \le n$. Hence $m = n$.

Definition 9.51: The number of elements in a basis of the finite dimensional vector space $V(F)$ is known as the dimension of $V(F)$ denoted by dim $V(F)$.

Example 9.52: If $(F, +, .)$ be a field, the dimension of $V_n(F)$ is n. For the standard basis of $V_n(F)$ contains n vectors.

Corollary: Let $V(F)$ be n-dimensional vector space, then

(*a*) any set of vectors in $V(F)$ which contains more than n vectors is linearly dependent.

(*b*) no set which contains less than n vectors can span $V(F)$.

Theorem 9.53: Let S be a L.I. subset of a (finite dimensional) vector space $V(F)$. Suppose $\beta \in V$ which is not in the subspace $[S](F)$, then the set obtained by adjoining β to S is L.I.

Proof: Let S have distinct vectors $\alpha_1, \alpha_2, \ldots \alpha_m$, and that $c_1 \alpha_1 + c_2 \alpha_2 + \ldots + c_m \alpha_m + b\beta = \bar{0}$, $c_i \in F$ then $b = 0$, for otherwise,

$$\beta = \frac{(-c_1)}{b} \alpha_1 + \ldots + \left(\frac{-c_m}{b}\right) \alpha_m$$

and $\beta \in [S](F)$. Thus $c_1 \alpha_1 + c_2 \alpha_2 + \ldots + c_m \alpha_m = \bar{0}$ and since S is a L.I. set, each $c_i = 0$,

which completes the proof.

Theorem 9.54: If $W(F)$ is a subspace of a finite-dimensional vector space $V(F)$, every linearly independent subset of W is finite and is a part of a (finite) basis for $W(F)$.

Proof: Let S_0 be a linearly independent (L.I.) subset of W. If S is L.I. subset of W containing S_0, then S is also a L.I. subset of V. Since $V(F)$ is a finite dimensional, S contains no more than dim V elements. Thus there is a maximal L.I. subset S of W which contains S_0. Since S is a maximal L.I. subset of W containing S_0, the preceeding lemma shows that W is the subspace $[S]$ spanned by S. Hence S is a basis for $W(F)$ and S_0 is the part of a basis of $W(F)$.

Corollary 2: In a finite dimensional vector space $V(F)$ every non-empty L.I. set of vectors is a part of a basis.

Corollary 2: If $W(F)$ is a proper subspace of a finite dimensional vector space $V(F)$, then W is finite dimensional and dim $W <$ dim V.

Theorem 9.55: Any linearly independent set of vectors in n-dimensional space $V(F)$ can be extended to a basis.

Proof: Let $B_1 = \{\alpha_1, \alpha_2, \ldots\ldots \alpha_k\}$ be linearly independent and $A = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a basis of $V(F)$. If $\beta_i \in B_1$ for $i = 1, 2, \ldots n$, then $[B_1] = V$. Otherwise, for some $\beta_i \notin B_1$, $\{B_1, \beta_i\}$ is linearly independent thus the original set has been extended to a larger independent set and theorem follows by repeating the argument until the enlarged set spans $V(F)$.

Example 9.56: Show that the set

$(1, 2, 1), (2, 1, 0), (1, -1, 2)$ forms a basis for $V_3(F)$.

Solution: Directly let (x, y, z) be any vector in $V_3(F)$. We want to know if there exist a_1, a_2, a_3 such that

$a_1(1, 2, 1) + a_2(2, 1, 0) + a_3(1, -1, 2) = (x, y, z)$.

For this we get the equations.

$$a_1 + 2a_2 + a_3 = x \qquad \ldots(1)$$

$$2a_1 + a_2 - a_3 = y \qquad \ldots(2)$$

$$a_1 + 0a_2 + 2a_3 = z \qquad \ldots(3)$$

From (1) & (2) $a_1 + a_2 = \frac{1}{3}(x + y)$.

From (1) & (3) $a_1 + 2a_3 = 2x - z$. ...(4)

$$a_2 = \frac{5x}{9} - \frac{y}{9} - \frac{z}{3} \qquad \ldots(5)$$

$$a_3 = \frac{-2x}{9} + \frac{4}{9}y + \frac{z}{3} \qquad \ldots(6)$$

Putting the value of a_2 in (3) $a_1 = \frac{x}{9} - \frac{2y}{9} + \frac{z}{3}$

Hence $S = \{(1, 2, 1), (2, -1, 0)\ (1, -1, 2)\}$ spans $V_3(F)$. As the dimension of $V_3(F)$ is 3 and S has only 3 vectors, these must be all independent.

N **Theorem 9.57:** If $W(F)$ is a subspace of a finite dimensional vector space $V(F)$, then

(a) $W(F)$ is finite dimensional with $\dim W \leq \dim V$,

(b) $\dim W = \dim V$ if and only if $W = V$.

Proof: (a) Let $V(F)$ be a finite dimensional vector space with $\dim V = n$. Then there is a linear independent set $\{x_1, x_2, ..., x_n\}$ of n vectors which spans $V(F)$. So the L.I. set $\{x_1, x_2, ..., x_n\}$ is a basis of $V(F)$. Since $W(F)$ is a subspace of $V(F)$, it must have basis, that is, there are L.I. vectors $y_1, y_2, ...y_m$ which span $W(F)$. that is, $\dim W = m$. The set of L.I. vectors $y_1, y_2, ... y_m$ can be extended to the set of L.I independent vectors $y_1, y_2, ... y_m, y_{m+1}, ..., y_n$ which spans $V(F)$. That is, the basis of $W(F)$ is a subset of the basis of $V(F)$ which implies $\dim W \leq \dim V$.

(b) Let $\dim W = \dim V = n$. Then there exist a basis $\{y_1, y_2, ..., y_n\}$ for $W(F)$ and a basis $\{x_1, x_2, ..., x_n\}$ for $V(F)$. Two basis of a vector space have the same number of L.I. vectors by corrollary of Theorem 9.45. This implies both basis $\{y_1, y_2, ..., y_n\}$ and $\{x_1, x_2, ..., x_n\}$ *span* $V(F)$. Hence $W = V$.

Conversely, suppose that $W = V$. We have by (a) $\dim W \leq \dim V$. if the finite number of vectors $x_1, x_2, ..., x_n$ span $V = W$ then any set of indenpendent vectors in $W(F)$ contains no more then m elements, that is, $n \leq m$ or $\dim V \leq \dim W$. Thus $\dim W = \dim V$.

N **Theorem 9.58:** If $W(F)$ is a subspace of a finite dimensional vector space $V(F)$, then the quotient space $(V/W)(F)$ is also a finite dimensional and $\dim V = \dim W + \dim V/W$.

Proof: Since $W(F)$ is a subspace of a finite dmensional vectorspace $V(F)$, $W(F)$ is finite dimensional with $\dim W \leq \dim V$. Let $\{x_1, x_2, ..., x_n\}$ be a basis for $W(F)$. But this basis $\{x_1, x_2, ..., x_n\}$ can be extended to the basis of $V(F)$ by adding vectors $y_1, y_2, ..., y_m$. Thus we have the basis $\{x_1, x_2, ..., x_n, y_1, y_2, ..., y_m\}$ for $V(F)$.

Any vector $x \in V$ can be written as

$x = a_1 x_1 + a_2 x_2 + ... + a_n x_n + b_1 y_1 + ... + b_m y_m$,

where a'^s, and b'^s are in F.

Since $W(F)$ is a subspace, $a_1 x_1 + a_2 x_2 + .. + a_n x_n \in W$,

and

$$a_1 x_1 + a_2 x_2 + .. + a_n x_n = x - (b_1 y_1 + ... + b_m y_m) \in W.$$

$$\begin{aligned} \Rightarrow \quad x + W &= b_1 y_1 + b_2 y_2 + ... + b_m y_m + W \\ &= (b_1 y_1 + W) + (b_2 y_2 + W) + ... + (b_m y_m + W) \\ &= b_1 (y_1 + W) + b_2 (y_2 + W) + ... + b_m (y_m + W). \end{aligned}$$

Thus the coset $x + W \in V/W$ is expressed as the linear combination of the cosets $y_1 + W, y_2 + W, ..., y_m + W$, that is, they span $V(F)$.

Now we have to show that $y_1 + W, y_2 + W, ..., y_m + W$. form a basis for $(V/W)(F)$. For this, we assume

$$c_1 (y_1 + W) + c_2 (y_2 + W) + ... + c_m (y_m + W) = \bar{0} + W = W.$$

or $\quad c_1 y_1 + W + c_2 y_2 + W + ... + c_m y_m + W = W$

or $\quad (c_1 y_1 + c_2 y_2 + ... + c_m y_m) + W = W$ which implies

$c_1 y_1 + c_2 y_2 + ... + c_m y_m \in W$ which must be a linear combination of the vectors $x_1, x_2, ..., x_n$ of W. So

$c_1 y_1 + c_2 y_2 + ... + c_m y_m = d_1 x_1 + d_2 x_2 + ... + d_n x_n$,

for some $d_1, d_2, ..., d_n$ of F.

This meane that

$c_1 y_1 + c_2 y_2 + ... + c_m y_m - d_1 x_1 - d_2 x_2 ..... - d_n x_n = \bar{0}$

Since $x_1, x_2, ..., x_n, y_1, ..., y_m$ are linearly independent, $c_1 = c_2 = ... = c_m = d_1 = d_2 = ... = d_n = 0$.

Thus $y_1 + W, y_2 + W_2, ..., y_m + W$ form a basis of $(V/W)(F)$ which has m elements.

Hence $\dim (V/W) = m = (m + n) - n$

or $\dim V = \dim W + \dim \dfrac{V}{W} \quad = \dim V - \dim W$.

Example 9.59: We consider the vector space $V_3 (F)$ and subspace $W(F)$,

Where $W = \{(a, 0, 0) \mid a \in F\}$.

Hence the quotient space $(V_3/W)(F)$ consists of the consets of the form $(o, b, c) + W$. $W(F)$ has the basis $\{(1, 0, 0)\}$ which can be extended to the basis of V (F) by adjoining the vector $(0, 1, 0)$ and $(0, 0, 1)$.

The cosets $(0, 1, 0) + W$ and $(0, 0, 1) + W$ will form the basis of $(V_3/W)(F)$.

If $(a, b, c) \in V_3$, then

$(a, b, c) + W = b\,[(0, 1, 0) + W] + c\,[(0, 0, 1) + W]$

Thus

$\dim V/W = 2 = \dim V - \dim W$.

Theorem 9.60: If $U(F)$ and $W(F)$ are subspaces of a finite dimensional vectorspace $V(F)$, then

$$\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$$

If $\quad V = U \oplus W$,

$$\dim (U \oplus W) = \dim U + \dim W.$$

Proof: Since $U(F)$ and $W(F)$ are subspaces of $V(F)$, then $(U \cap W)(F)$ and $(U + W)(F)$ are also subspaces of $V(F)$. Since $V(F)$ is finite dimensional, all its subspaces are finite dimensional.

Let $\{x_1, x_2, ..., x_n\}$ be a basis for $(U \cap W)(F)$

Since $U \cap W \subset U$ and $U \cap W \subset W$, the basis $\{x_1, ... x_n\}$ of $(U \cap W)(F)$ can be extended to a basis $\{x_1, .. x_n, u_1, ..., u_m\}$ of $U(F)$ by adjoining vectors $u_1, u_2, ..., u_m$ and to a basis $\{x_1, x_2, ... x_n, w_1, w_2, ... w_r\}$ of $W(F)$ by adjoining vectors $w_1, w_2, ..., w_r$, respectively. Thus the basis of $U(F)$ has $n + m$ elements and of $W(F)$ $n + r$ elements.

Since $(U + W)(F)$ is the smallest subspace generated by $U \cup W$, every element of $U + W$ can be represented as a linear combination of the vectors of the set

$\{u_1, u_2, \ldots, u_m, x_1, x_2, \ldots x_n, w_1, w_2, \ldots, w_r\}$. Therefore this set spans $(U + W)(F)$. If this is linearly independent, then we can say $(U + W)(F)$ has diminsion $m + n + r$. For this we assume that

$a_1 u_1 + a_2 u_2 + \ldots + a_m u_m + b_1 x_1 + \ldots + b_n x_n + c_1 w_1 + \ldots + c_r w_r = \bar{0}$ where all a's, b's, and c's are in F.

Let $z = c_1 w_1 + c_2 w_2 + c_3 w_3 + \ldots + c_r w_r$, $z \in W$ as W (F) is a subspace. Then we have

$z = -[a_1 u_1 + \ldots + a_m u_m + b_1 x_1 + \ldots + b_n x_n] \in U$ as

$U(F)$ is a subspace with the bass $\{u_1, u_2, \ldots u_m, x_1, \ldots, x_n\}$.

This means that $z \in U \cap W$. So z can be written as a linear combination of the basis elements $\{x_1, x_2, \ldots, x_n\}$ of $(U \cap W)(F)$. Therefore for some $d_1, d_2, \ldots d_n$ of F, we have

$z = d_1 x_1 + d_2 x_2 + \ldots + d_n x_n$.

Hence

$z = c_1 w_1 + c_2 w_2 + \ldots + c_r w_r = d_1 x_1 + d_2 x_2 + \ldots + d_n x_n$

$\Rightarrow \quad c_1 w_1 + c_2 w_2 + \ldots + c_r w_r - d_1 x_1 - d_2 x_2 \ldots\ldots - d_n x_n = \bar{0}$

$\Rightarrow \quad c_1 = c_2 = \ldots\ldots c_r = d_1 = d_2 = \ldots = d_n = 0$ since $w_1, w_2, \ldots w_r, x_1, x_2, \ldots, x_n$ is a basis for $W(F)$.

Since $\{x_1, x_2, \ldots x_n, u_1 u_2, \ldots u_m\}$ is a basis of $U(F)$, then $a_1 u_1 + a_2 u_2 + \ldots + a_m u_m + b_1 x_1 + b_2 x_2 + \ldots + b_n x_n = \bar{0}$

$a_1 = a_2 = \ldots = a_m = b_1 = \ldots = b_n = 0$.

Therefore this proves.

$a_1 u_1 + \ldots + a_m u_m + b_1 x_1 + \ldots b_n x_n + c_1 w_1 + c_2 w_2 + \ldots + c_r w_r = \bar{0} \Rightarrow a_1 = a_2 = \ldots = a_m = b_1 = \ldots = b_n = c_1 = \ldots = c_r = 0$.

Hence the set $\{u_1, u_2, \ldots u_m, x_1, x_2, \ldots x_n, w_1, w_2, \ldots w_r\}$ is L.I. and forms a basis of $(U + W)(F)$ with dim $(U + W) = m + n + r$.

Hence

$$\begin{aligned} \dim(U + W) &= m + n + r = (m + n) + (n + r) - n \\ &= \dim U + \dim W - \dim(U \cap W). \end{aligned}$$

In particular, If $U \oplus W = V$, then $U \cap W = \{\bar{0}\}$, $U + W = V$.

$$\begin{aligned} \text{So } \dim(U \oplus W) &= \dim U + \dim W - \{\bar{0}\} \\ &= \dim U + \dim W, \text{ since } \dim\{\bar{0}\} = 0. \end{aligned}$$

Hence the theorem.

PROBLEMS

1. For each of the following vector spaces, determine whether the sets listed are linearly dependent or independent:
   (1) $V_4(F) : \{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}$,
   (2) $F[x] : \{x^2 + x - x, x^2 - x - 2, x^2 + x + 1\}$,
   (3) $V_3(Z_5) : \{(4, 1, 3), (2, 3, 1), (4, 1, 0)\}$,
   (4) $V_3(C) : \{(1, 2 + i, 3), (2 - i, i, 1), (i, 2 + 3i, 2)\}$.

2. If F is the field of real, numbers, show the vectors $(1, 1, 0, 0)$, $(0, 1, -1, 0)$, $(0, 0, 0, 3)$ in $V_4(F)$ are L.I. over F.
3. Show that the system of three vectors $(1, 3, 2)$, $(1, -7, -8)$, $(2, 1, -1)$ of $V_3(R)$ is L.D.
4. In $V(R)$, where R is the field of real numbers, determine whether each of the following set of vectors is L.I. or L.D.
   (1) $(1, 1, 1)$, $(1, 0, 1)$, $(0, 1, 0,)$,
   (2) $(8, 4, 8)$, $(2, 1, 2)$, $(0, 0, 1)$,
   (3) $(1, -1, 1)$ $(2, 1, -2)$, $(8, 1, -4)$.
5. Show that the set
   $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$
   spans $V_3(R)$ but does not form a basis.
6. Show that the set of all 2×2 matrices over real numbers forms a vector space under the usual addition and scalar multiplication and the

   $$\text{set} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

   forms the basis for $M_2(F)$.
7. Given
   $u_1 = (1, -1, 3)$, $u_2 = (2, 3, 5)$, $u_3 = (-1, 4, -2)$ and $u_4 = (4, 1, -2)$, find c_1, c_2, c_3, c_4 not all zero, such that $c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 = \bar{0}$
8. Prove that a set consisting of a single nonzero vector is L.I.
9. Prove that Every sub-set of L.I. set is L.I.
10. Prove that any $n + 1$ vectors from n-dimensional vector space are L.D.
11. prove that if $x \in [x_1, x_2, \ldots, x_n]$ is uniquely expressible in the form $x = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n$ for some b's of F, the vectors $x_1, x_2, \ldots, x_n$ are linearly independent.
12. (a) Find a basis for the vector space $C(R)$ and all bases for the space $V_3(Z_2)$.
    (b) For what values of a do the vectors $(1 + a, 1, 1)$, $(1, 1 + a, 1)$, and $(1, 1, 1 + a)$ form a basis of $V_3(R)$.
13. Assume $\{x_1, x_2, x_3\}$, is a basis for the vector space $V_3(R)$. Verify that the sets $\{x_1 + x_2, x_2 + x_3, x_3 + x_1\}$ and $\{x_1, x_1 + x_2, x_1 + x_2 + x_3\}$ also serve as bases of $V_3(R)$. is this situation true in the space $V_3(Z_2)$?
14. If diag Mn is the set of all diagonal matrices of order n, show that diag Mn (F) is a subspace of Mn (F) and determine its dimension.
15. Prove that if $W(F)$ is a proper subspace of the finite dimensional vector space $V(F)$, then dim $W <$ dim V.
16. Let $V(F)$ be a finte dimensional vector space with bases $\{x_1, x_2, \ldots, x_n\}$. If $W_k(F)$ is the subspace generated by x_k $(k = 1, 2, \ldots n)$, verify that $V = W_1 \oplus W_2 \oplus \ldots \oplus Wn$
17. Let $U(F)$ and $W(F)$ be subspaces of $V_n(F)$ such that dim $U > n/2$ and dim $W > n/2$. Show that $U \cap W \neq \{0\}$.
18. Determine the dimension of the quotient space $(V_3/W)(F)$, where the set $W = \{(a, b, a + b \mid a, b \in F\}$.

19. Prove that every subspace of finite dimensional vector space has a complement.

Hint. Let $U(F)$ be subspace with dim $U = m$ of the finite dimensional vector space $V(F)$ with dim $V = n$.

Let $\{x_1, x_2, \ldots, x_m\}$ be a basis of $U(F)$. Then this set $\{x_1, x_2, \ldots, x_m\}$ can be extended to the basis $\{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n\}$ of $V(F)$ by adjoining $n-m$ vectors $x_{m+1}, x_{m+2}, \ldots, x_n$. If we suppose that $W(F)$ is a subspace spanned by $x_{m+1}, x_{m+2}, \ldots, x_n$ then $\{x_{m+1}, \ldots, x_n\}$ is a basis for $W(F)$.

For any $x \in V$, we have

$$x = a_1 x_1 + a_2 x_2 + \ldots + a_m x_m + a_{m+1} x_{m+1} + \ldots + a_n x_n$$

So

$$x \in U + W \text{ since } a_1 x_1 + a_2 x_2 + \ldots + a_m x_m \in U$$

$$\text{and } a_{m+1} x_{m+1} + \ldots + a_n x_n \in W$$

This means $V = U + W$.

Now

$$y \in U \cap W \Rightarrow y \in U \text{ and } y \in W$$

$$\Rightarrow y = b_1 x_1 + b_2 x_2 + \ldots + b_m x_m = c_{m+1} x_{m+1} + \ldots + c_n x_n \text{ since } \{x_1, \ldots, x_m\}$$

is a basis of $U(F)$ and $\{x_{m+1}, \ldots, x_n\}$ is basis of $W(F)$

$$\Rightarrow b_1 x_1 + b_2 x_2 + \ldots + b_m x_m - c_{m+1} x_{m+1} - \ldots - c_n x_n = 0$$

$$\Rightarrow b_1 = b_2 = \ldots = b_m = c_{m+1} = \ldots = c_n = 0,$$

since $\{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n\}$ is L.I. being a basis of $V(F)$. Hence $y = \Rightarrow U \cap W = \{\bar{0}\}$

This shows that $U \cap W = \{\bar{0}\}$ and $V = U + W$ that is $V = U \oplus W$. Hence $U(F)$ has its complement $W(F)$.

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10
LINEAR TRANSFORMATION

10.1 DEFINITION AND EXAMPLES OF LINEAR TRANSFORMATIONS

We have studied homomorphisms from one algabraic system to another algebraic system, namely, group homomorphism, ring homomorphism. On parallel lines we shall study vector space homomorphism, Since the vactor space $U(F)$ is comprised of two algebraic system, group $(V, +)$ and *a* field $(F, +,.)$, there may be some confusion as to what operations are to be preserved by such functions. Generally vector space homomorphism are called *Linear mappings* or linear transformation.

Definition 10.1: Let $U(F)$ and $V(F)$ be two vector spaces over the same field $(F, +,.)$. A function f from U to V, $f: U \rightarrow V$, is said to be *linear transformation* from $U(F)$ into $V(F)$ if

(1) $f(x + y) = f(x) + f(y)$,

(2) $f(c\,x) = cf(x)$,

$\forall\ x, y \in U$ and $c \in F$.

Thus $f: U \rightarrow V$ is linear if it preserves the vector addition and scalar multiplication. Since $U(F)$ and $V(F)$ are vector spaces ove rthe same field, F, then some times we call the linear transformation f an F – linear transformation. The set of all linear mappings from $U(F)$ into $V(F)$ will be denoted by $L(U, \not{}\,)$.

From (2), if $c = 0$, we get $f(0\,.\,x) = f(0) = 0f(x) = 0$, that is, svery linear transformation maps zero vector of U onto zero vector of V.

Theorem 10.2: *Let $U(F)$ and $V(F)$ be two vector spaces over the same field F. Then $f: U \rightarrow V$ is a linear mapping if and only if $f(ax + by) = af(x) + bf(y), \forall\ x, y \in U$ and $a, b \in F$.*

Proof: Let $f: U \rightarrow V$ be a linear mapping. Then

$\forall\ x, y \in U$, $a, b \in F$, $a\,x + by \in U$, since $U(F)$ is a vector space over F.

Then $f(ax + by) = f(a\,x) + f(by)$ by 1 of Def.

$= af(x) + bf(y)$ by 2 of Def.

Conversely, let $f(ax + by) = a f(x) + b f(y)$.

Taking $a = 1 = b \in F$, the identity of F, then

$f(x + y) = f(1.x + 1.y) = 1.f(x) + 1.f(y)$

$= f(x) + f(y)$.

Again taking $a \neq 0, b = 0$, then

$f(a x) = f(a x + o.y) = a f(x) + 0 f(y) = a f(x)$.

Hence f is a linear transformation.

Example 10.3: Let $M_{m \times n}(F)$ be a vector space over the field F, and let $[a_{ij}]$ be $m \times m$ a fixed matrix over F.

Define a function $f: M_{mm} \to M_{mm}$ by

$f[b_{ij}] = [a_{ij}].[b_{ij}], \quad \forall [b_{ij}] \in M_{mm}$,

we see that for $[b_{ij}], [c_{ij}] \in M_{mm}$ and for $r, s \in F$,

$$\begin{aligned} f(r[b_{ij}] + s[c_{ij}]) &= [a_{ij}].[r[b_{ij}] + s[c_{ij}]] \\ &= r[a_{ij}].[b_{ij}] + s[a_{ij}][c_{ij}] \\ &= r f[b_{ij}] + s f[c_{ij}]. \end{aligned}$$

Hence f is a linear transformation from $M_{mm}(F)$ into $M_{mm}(F)$.

Example 10.4: Let $F[x]$ be a vector space of polynomials in the indeterminant x with coefficients from F. A Function $f: F[x] \to F[x]$ is defined by the differentiation, that is, if $p(x) = a_0 + a_1 + + \ldots + a_n x^n \in F[x]$, then $f(p(x)) = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} \in F[x]$ i.e., $f(p(x)) = p'(x)$. We see that $\forall\ p(x), g(x) \in F(x)$

$$\begin{aligned} f(a p(x) + b g(x)) &= [a(p(x) + b g(x)]' \\ &= (a(p(x))' + (b g(x))' \\ &= a p'(x) + b g'(x) \\ &= a f(p(x)) + bf(g(x)). \end{aligned}$$

which shows f, that is, the differention is a L.T. (Linear transformation).

Example 10.5: Let F^3 and F^2 be two vector spaces over the same field F. Define a mepping $f: F^3 \to F^2$

by $(a, b, c) = (a, b), \forall\ (a, b, c) \in F^3$

For any $(a, b, c), (a', b', c') \in F^3$,

$$\begin{aligned} f[(a, b, c) + (a', b', c')] &= f(a + a', b + b', + c + c') \\ &= (a + a', b + b') \\ &= (a, b) + (a', b') \\ &= f(a, b, c) + f(a', b', c'). \end{aligned}$$

And

$$\begin{aligned} f(r(a, b, c) &= f(ra . rb, rc) = (ra, rb) \\ &= r(a.b) = r f(a, b, c). \end{aligned}$$

Hence f is *a* L.T.

Example 10.6: Let $W(F)$ be a subspace of the vector space $V(F)$. Then the natural mapping $f: V \to V/W$ defined by $f(x) = x + W, \quad x \in V$, is a linear transformation. We can check that

$f(a x + by) = \quad a x + by + W$

$= (ax+W)+(by+W)$
$= a(x+W)+b(y+W)$
$= af(x)+bf(y).$

Example 10.7: Let $f: R^2 \to R^2$ be the translation mapping defined by $f(x,y)=(x+1, y+2)$, $\forall (x,y) \in R^2$. We observes that $f(0,0)=(1,2) \neq (0,0)$, that is, f does not map $(0,0) \in R^2$ onto $(0,0) \in R^2$. This shows that f is not a linear transformation.

Example 10.8: Let $f: U \to V$ be the mapping which assings $0 \in V$ to every $u \in U$, i.e., $f(u)=0$, $\forall u \in U$. Than for any $u, v \in U$ and for any $k \in F$, we have

$f(u+v)=0=0+0=f(u)+f(v)$ and
$f(ku)=0=k\,0=kf(u)$

Hence f is L.T. which is called the *zero mapping* usually denoted by 0.

Example 10.9: We consider the identity mapping $I: V \to V$ defined by $f(v)=v$, $v \in V$. Then for any $u, v \in V$, $I(u+v)=u+v=I(u)+I(v)$ and $I(ku)=k\,I(u)$. Hence I is L.T.

Example 10.10: In R^2 Let f_1, f_2, and f_3 be defined by

$$f_1(x,y) = (x,0),$$
$$f_2(x,y) = (0,y),$$
$$f_3(x,y) = (y,x).$$

Hence f_1, f_2, and f_3 are all L.T. f_1 is a projection of each point of the plane onto the x-axis, f_2 is a projection of each point of a plane onto y-axis, and f_3 is reflextion across the line $y=x$.

We observe that $f_1(x,y) = (x,0)$ and $f_2(x,0)$
$=(0,0)$, that is, $f_2 \circ f_1(x,y) = f_2(f_1(x,y))$
$= f_2(x,0)=(0,0).$

So $f_2 \circ f_1 = 0$ but $f_1 \neq 0$ and $f_2 \neq 0$. Hence the product of non-zero linear transformation can be the zero transformation. Also $f_3 \circ f_2((x,y)) = f_3((0,y)) = (y,0)$: however $f_2 \circ f_3(x,y) = f_2(y,x) = (0,x)$. Hence $f_3 \circ f_2 \neq f_2 \circ f_3$, so the multiplication of transformation is not commutative. Finally, we observe that $f_1(f_1(x,y)) = f_1(x,0) = (x,0) = f_1(x,y)$. So $f_1^2 = f_1$. Thus there exists *idempotent transformation* other than I and 0.

Example 10.11: In the vector space $F[x]$ of polynomials $p(x)$ of degree not exceeding n, Let

$D(p(x)) = \frac{d}{dx}(p(x)) = p'(x)$. we have seen that D is linear. We also note that $D^{n+1}(P(x)) = 0$ for every $p(x) \in F[x]$, so $D^{n+1}=0$. Thus there exists a non-zero transformation f such that a finite power of f is 0. A transformation f is called *nilpotent* of index n if $f^n = 0$ but $f^{n-1} \neq 0$.

Definition 10.12: Two linear transformations f_1 and f_2 from $U(F)$ to $V(F)$ are said to be equal if and only if $f_1(x) = f_2(x)$, $\forall x \in U$.

Definition 10.13: A linear transormation $f: U \to V$ is called an isomorphism if f is one-to-one. The vector spaces $U(F)$ and $V(F)$ are said to be isomorphic if there is an isomorphism of U onto V. Symmbolically, we write $U(F) \cong V(F)$.

Example 10.14: Let $V(F)$ be a vector space over the field F of dimension n and let $\{e_1, e_2, \ldots, a_n\}$ be the basis of V we define $f: V \to F^n$ by $f(v) = f(a_1 e_1 + a_{-2} e_{-2} + \ldots a_n e_{-n}) = (a_{-1}, a_{-2}, \ldots, a_n)$, since $v \in V$ can be written as $v = a_1 . e_1 + a_2 e_2 + \ldots + a_n e_n$. It can be easily seen that f is one to one mapping. Hence it is isomorphism from V onto F^n, So $V(F) \cong F^n$.

Theorem 10.15: *Let $f: U \to V$ be a linear transformation from $U(F)$ into $V(F)$. If $\{x_1, x_2, \ldots, x_n\}$ is L.D. in U, then $f(x_1), f(x_2), \ldots, f(x_n)$ are also L.D. in V.*

Proof: Since $x_1, x_2, \ldots, x_n$ are L.D., then there exists $a_1, a_2, \ldots, a_n \in F$ not all zero such that

$$a_1 x + a_2 x_2 + \ldots + a_n x_n = 0 \in U.$$

$$\text{Therefore } 0 = f(0) = f(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n)$$
$$= a_1 f(x_1) + a_2 f(x_2) + \ldots + a_n f(x_n),$$

which shows $f(x_1), f(x_2), \ldots, f(x_n)$ are L.D. in V.

Theorem 10.16: Let $f: U \to V$ be linear, and suppose $x_1, x_2, \ldots, x_n \in U$ have the property that their images $f(x_1), f(x_2), f(x_3), \ldots f(x_n)$ are linearly independent (L.I.). Then the vectors $x_1, x_2, \ldots x_n$ are also L.I.

Proof: Since $f(x_1), f(x_2), \ldots, f(x_n)$ are L.I., than these exists scalars $a_1, a_2, \ldots, a_n \in F$ with $a_1 = a_2 = \ldots = a_n = 0$ such that

$$a_1 f(x_1) + a_2 f(x_2) + \ldots + a_n f(x_n) = 0 \in V$$
$$\Rightarrow \quad f(a_1 x + a_2 x_2 + \ldots + a_n x_n) = 0 = f(0)$$
$$\Rightarrow \quad a_1 x_2 + a_2 x_2 + \ldots + a_n x_n = 0 \in U, \text{ where}$$

$a_1 = a_2 = \ldots = a_n = 0$. This shows that $x_1, x_2, \ldots, x_n$ are linearly independent.

But the converse is not true, that is, if $x_1, x_2, \ldots, x_n$ are L.I, then their images $f(x_1), f(x_2), \ldots, f(x_n)$ may not be linearly independent. This can be shown by an example. In R^2 let f be a linear transformation defined by

$$f(x, y) = (0, y).$$

We know $\{(1, 1), (0, 1)\}$ is a basis of R^2, that is, $(1, 1)$ and $(0, 1)$ are linearly independent. Their images $f(1, 1) = (0, 1)$ and $f(0, 1) = (0, 1)$ are not L.I. For this, let $k_1, k_2 \in F$, then

$$k_1 f(1, 1) + k_2 f(0, 1) = (0, 0)$$
$$\Rightarrow k_1 (0, 1) + k_2 (0, 1) = (0, 0) \quad \Rightarrow (0, k_1) + (0, k_2) = (0, 0)$$
$$\Rightarrow (0 + 0, k_1 + k_2) = (0, 0)$$
$$\Rightarrow k_1 + k_2 = 0$$
$$\Rightarrow k_1 = -k_2.$$

Thus, $f(1, 1)$ and $f(0, 1)$ are not L.I., since k_1 and k_2 are not zero.

Theorem 10.17: Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for the finite dimensional vector space $V(F)$ and $\{y_1, y_2, \ldots, y_n\}$ an arbitrary set of n vectors from $W(F)$. Then there exists a unique linear mapping $f: V \to W$ such that

$f(x_1) = y_1, f(x_2) = y_2, \ldots\ldots, f(x_n) = y_n$.

Proof: Let $x \in V$, since $\{x_1, x_2, \ldots\ldots, x_n\}$ is a basis for V, then there exist scalars $a_1, a_2, \ldots\ldots, a_n \in F$ for which

$x = a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n$

Let us define $f : V \to W$ at the vector x by taking

$f(x) = a_1 y_1 + a_2 y_2 + \ldots\ldots, a_n y_n$.

Since $a_1, a_2, \ldots\ldots, a_n$ are unique, the mapping f is well defined. For $i = 1, 2, \ldots\ldots,$

$$x_i = 0\, x_1 + 0\, x_2 + \ldots\ldots + 1\, x_i + \ldots\ldots + 0\, x_n.$$

$$\Rightarrow f(x_i) = 0\, y_1 + 0\, y_2 + \ldots\ldots + 1\, y_i + \ldots\ldots + 0\, y_n.$$

Now we see that f is linear mapping. Let

$x, y \in V$, where $x = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$ and

$y = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n$.

$$\begin{aligned}
\text{Then } f(x + y) &= f\{(a_1 + b_1) x_1 + (a_2 + b_2) x_2 + \ldots + (a_n + b_n) x_n\} \\
&= (a_1 + b_1) y_1 + (a_2 + b_2) y_2 + \ldots + (a_n + b_n) y_n \\
&= (a_1 y_1 + a_2 y_2 + \ldots + a_n y_n) + (b_1 y_1 + b_2 y_2 + \ldots + b_n y_n) \\
&= f(x) + f(y).
\end{aligned}$$

$$\begin{aligned}
\text{Further} \quad cx &= c\, a_1 x_1 + c\, a_2 x_2 + \ldots\ldots + ca_n x_n, \\
f(c\,x) &= c\, a_1 y_1 + c\, a_2 y_2 + \ldots\ldots + ca_n y_n. \\
&= c\,(a_1 y_1 + a_2 y_2 + \ldots\ldots + a_n y_n) \\
&= c\, f(x).
\end{aligned}$$

Thus f is linear.

Finally we shall show that f is unique linear mapping. Now suppose $g : V \to W$ is linear with the property that $g(x_i) = y_i$, $i = 1, 2, \ldots., n$.

If $x = a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n$, then

$$\begin{aligned}
g(x) &= a_1 g(x_1) + a_2 g(x_2) + \ldots\ldots + a_n g(x_n) \\
&= a_1 y_1 + a_2 y_2 + \ldots\ldots + a_n y_n = f(x).
\end{aligned}$$

Since $g(x) = f(x)$, $\forall\ x \in V$, $g = f$. Thus f is unique. Hence the theorem is proved.

Example 10.18: Let $f : R^2 \to R$ be the linear mapping for which $f(1, 1) = 3$ and $f(0, 1) = -2$. Since $\{(1, 1), (0, 1)\}$ is a basis for R^2, such mapping exists and is unique by above theorem Find $f(a, b)$.

Solution: Since $\{(1, 1), (0, 1)\}$ is a basis of R^2 then $(a, b) \in R^2$ can be expressed uniquely as

$$\begin{aligned}
(a, b) &= x(1, 1) + y(0, 1), \text{ where } x, y \text{ are scalars.} \\
&= (x, x) + (0, y) \\
&= (x, x + y) \Rightarrow x = a \text{ and } x + y = b
\end{aligned}$$

Thus $x = a$, $y = b - a$.

$$\begin{aligned}
\text{Now } f(a, b) &= f(x(1, 1) + y(0, 1)) = x f(1, 1) + y f(0, 1) \\
&= x(3) + y(-2) \\
&= a \,.\, 3 + (b - a)(-2) \\
&= 3a - 2b + 2a = 5a - 3b
\end{aligned}$$

10.2. KERNEL AND IMAGE OF LINEAR MAPPING

Definition 10.19: Let $f : U \to W$ be a linear mapping from a vector space $U(F)$ into a vector space $W(F)$. Then ker (f) is the set of all $x \in U$ which are mapped on 0, the additive identity of W, by f. That is,

Ker $(f) = \{x \in U \mid f(x) = 0 \in W\}$.

Definition 10.20: Let $f: V \to W$ be a linear mapping. Then the se of all images of all element $x \in U$ is called the Range of f, written as R_f or the image of f, written as $f(V)$. Thus

$R_f = f(V) = \{y \in W \mid y = f(x)$ for some $x \in U\}$.

***Theorem 10.21:** Let $f: V \to W$ be a Linear mapping from a vector space $V(F)$ into $W(F)$ over the same field F.*

Then

(1) Ker $(f)(F)$ is a subspace of $U(F)$;

(2) $f(V)$ is a subspace of $W(F)$;

(3) dim V = dim ker (f) + dim $f(V)$ (Sylvesters law)

Proof: (1) Let $x, y \in$ ker (f), then $x, y \in V$ and

$f(x) = 0$ and $f(y) = 0$.

Now for $a, b \in F$, we have

$f(ax + by) = af(x) + bf(y) = a.0 + b.0 = 0 + 0 = 0$

so $ax + by \in$ ker (f).

Hence ker $(f)(F)$ is a subspace of $V(F)$

Definition: (1) Ker $(f)(F)$ is called a null space and the nullity $\nu(f)$ of a linear transformations f is the dimension of its null space $\nu(f)$ = dim Ker (f).

(2) Let $y_1, y_2 \in f(V)$. Then there exist $x_1, x_2 \in V$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. Now for $a, b \in F$, we have

$$\begin{aligned} f(a_1x_1 + a_2x_2) &= a_1f(x_1) + a_2f(x_2) \\ &= a_1y_1 + a_2y_2 \in f(V). \end{aligned}$$

Thus $f(V)(F)$ is a subspace of $W(F)$.

Definition: $f(V)(F)$ is called range space and dim $f(V)$ is called the rank $r(f)$ of a linear transformation f.

(3) Let dim $V = n$. Suppose ker $(f) \neq \{0\}$, so that ker (f) is a subspace of $V(F)$ of finite dimension, say, dim ker $(f) = r \leq n$.

Let $\{x_1, x_2, \ldots, x_r\}$ be a bases of ker (f). Then we can extend $\{x_1, x_2, \ldots, x_r\}$ to a basis

$\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r}\}$ of V

Now for any $y \in f(V)$, there exists $x \in V$ such that $y = f(x)$. In terms of basis for $V(F)$, the vector x can be written as

$x = a_1x_1 + a_2x_2 + \ldots + a_rx_r + b_1y_1 + b_2y_2 + \ldots + b_{n-r}y_{n-r}$

$$\begin{aligned} \text{But } f(a_1x_1 + a_2x_2 + \ldots + a_rx_r) &= a_1f(x_1) + \ldots + a_rf(x_r) \\ &= a_1 0 + \ldots + a_r 0 = 0. \end{aligned}$$

which implies

$$y = f(x) = a_1 f(x_1) + a_2 f(x_2) + \dots + a_r f(x_r) + b_1 f(y_1) + \dots + b_{n-r} f(y_{n-r})$$
$$= b_1 f(y_1) + b_2 f(y_2) + \dots + b_{n-r} f(y_{n-r}).$$

Thus from this, we get that every element $y \in f(V)$ can be expressed as a Linear combination of vectors of $f(y_1), f(y_2), \dots, f(y_{n-r})$.

Let $B = \{f(y_1), f(y_2), \dots, f(y_{n-r})\}$

If we show that the set B is L.I., then theorem is proved. for this,

Let
$$a_1 f(y_1) + a_2 f(y_2) + \dots + a_{n-r} f(y_{n-r}) = 0, \text{ then}$$
$$f(a_1 y_1 + a_2 y_2 + \dots + a_{n-r} y_{n-r}) = 0$$
$$\Rightarrow \quad a_1 y_1 + a_2 y_2 + \dots + a_{n-r} y_{n-r} \in \ker(f).$$

Since $\{x_1, x_2, \dots, x_r\}$ is basis for ker (f), then there must exist scalars $d_1, d_2, \dots, d_r$ such that
$$d_1 x_1 + d_2 x_2 + \dots d_r d_r = a_1 y_1 + a_2 y_2 + \dots + a_{n-r} y_{n-r}$$
or $d_1 x_1 + d_2 x_2 + \dots + d_r x_r - a_1 y_1 - a_2 y_2 \dots - a_{n-r} y_{n-r} = 0.$

Since $\{x_1, x_2, \dots, x_r, y_1, y_2, \dots y_{n-r}\}$ is a basis for $V(F)$, it is L.I., consequently $d_1 = d_2 = \dots = d_r = a_1 = a_2 = \dots = a_{n-r} = 0.$

Hence $\{f(y_1), f(y_2), \dots f(y_{n-r})\}$ is a basis of $f(V)(F)$.

Thus
$$\dim V = r + (n - r) = \dim \ker(f) + \dim f(V).$$

Hence the theorem is proved.

If ker $(f) = 0$, then f maps any basis of $V(F)$ onto a basis of $f(V)(F)$. Thus $\dim V = \dim f(V)$.

Corollary: *Let $V(F)$ and $W(F)$ be finite-dimensional vactor spaces with $\dim V = \dim W$ and let $f : V \to W$ be a L.T. Then f is one-to-one mapping if and only if f maps V onto W.*

Proof: First of all suppose f is one-to-one, so that ker $(f) = \{0\}$ which means dim ker $(f) = 0$. By Sylvester's law we have $\dim V = \dim f(V)$. We know that if $\dim V = \dim f(V)$ and f is one-to-one, then f is an onto mapping.

Conversely, if f maps V onto W, $f(V) = W$.

Now from $f(V) = W$ and $\dim V = \dim W = \dim f(V)$, the equation $\dim V = \dim \ker(f) + \dim f(V)$ yields to dim ker $(f) = 0$ which implies ker $(f) = \{0\}$. Hence f is one-to-one mapping.

Theorem 10.22: *If $V(F)$ is a finite dimensional vector space of dimension n, then $V(F) \cong V_n(F)$ or F^n.*

Proof: Let $\dim V = n$, then there exists a basis of n elements $x_1, x_2, \dots, x_n$ in V. So every $x \in V$ can be expressed as $x = a_1 x_1 + a_2 x_2 + \dots + a_n x_n, a_i \in F$, $1 \le i \le n$.

Now we define a mapping $f : V \to V_n$ by
$$f(x) = f(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = (a_1, a_2, \dots, a_n).$$

Now we prove that f is linear. For any $x, y \in V, a_i \in F, b_i \in F, 1 \le i \le n$, we have

$x = a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n,$
$y = b_1 x_1 + b_2 x_2 + \ldots\ldots + b_n x_n.$

For $k_1, k_2 \in F$, we have

$$
\begin{aligned}
f(k_1 x + k_2 y) &= (k_1 a_1 + k_2 b_1, k_1 a_2 + k_2 b_2, \ldots\ldots, k_1 a_n + k_2 b_n) \\
&= (k_1 a_1, k_1 a_2, \ldots\ldots, k_1 a_n) + (k_2 b_1, k_2 b_2, \ldots, k_2 b_n) \\
&= k_1 (a_1, a_2, \ldots\ldots, a_n) + k_2 (b_1, b_2, \ldots, b_n) \\
&= k_1 f(x) + k_2 f(y).
\end{aligned}
$$

Again we show that f is one-to-one mapping.

For any $x, y \in V$, assume

$$
\begin{aligned}
f(x) = f(y) &\Rightarrow (a_1, a_2, \ldots a_n) = (b_1, b_2, \ldots\ldots, b_n) \\
&\Rightarrow a_1 = b_1, a_2 = b_2, \ldots\ldots, a_n = b_n. \\
&\Rightarrow x = y.
\end{aligned}
$$

Further for $(a_1, a_2, \ldots\ldots, a_n) \in V_n$, $x \in V$ such that $f(x) = (a_1, a_2, \ldots\ldots, a_n)$. Which shows f is an onto mapping. Hence f is an isomorphism. Hence $V(F) \cong V_n(F)$

Corollary 1: Two finite-dimensional vector spaces $V(F)$ and $W(F)$ are isomorphic if and only if dim V = dim W.

Proof: Let dim V = dim $W = n$, then we shall prove $V(F) \cong W(F)$. Since dim V = dim $W = n$, then by the above theorem, we have

$V(F) \cong V_n(F)$ and $W(F) \cong V_n(F)$ which implies

$V(F) \cong W(F)$.

Conversely, if $V(F) \cong W(F)$, then there exists one-to-one and onto mapping $f: V \rightarrow W$, so that ker $(f) = \{0\}$ By sylvester's Law, we have dim V = dim ker (f) + dim $f(V)$ = dim $\{0\}$ + dim W = dim W.

Corrollary 2: Let $U(F)$ and $W(F)$ be complementary subspace relative to finite-dimensional vector space $V(F)$, that is $V = U \oplus W$.

$(V/W)(F) \cong U(F)$.

Proof: Let dim $V = n$. We have by theorem 9.58 dim V/W = dim V – dim W = dim U, since dim (V) = dim $(U \oplus W)$ = dim U + dim W. Since dim V/W = dim U, $(V/W)(F) \cong U(F)$.

Theorem 10.23: *Let $f \in L(V, W)$. A mapping $\bar{f}$ is defined from $(V/\ker(f))(F)$ into $W(F)$ as follows:*

$\bar{f}(x + \ker(f)) = f(x)$.

Then the mapping $\bar{f}$ is well defined, linear, one-to-one, and onto R_f. In short $\bar{f}$ is an isomorphism of $\left(\dfrac{V}{\ker f}\right)(F)$ onto $R_f(F)$.

(1) To prove $\bar{f}$ is well defined we have to show that if $x + \ker(f) = y + \ker(f)$, then $\bar{f}(x + \ker(f) = \bar{f}(y + \ker ff)$ That is, $f(x) = f(y)$.

$$
\begin{aligned}
\text{Now } x + \ker(f) = y + \ker(f) &\Rightarrow x - y \in \ker(f) \\
&\Rightarrow f(x - y) = 0 \in W \text{ since } f: V \rightarrow W \\
&\Rightarrow f(x) - f(y) = 0 \text{ Since } f \text{ is linear}
\end{aligned}
$$

$$\Rightarrow f(x) = f(y)$$
$$\Rightarrow \bar{f}(x + \ker(f)) = \bar{f}(y + \ker(f)),$$ by definition of $\bar{f}$.

Hence $\bar{f}$ is well defined.

(2) For any $x + \ker(f), y + \ker(f) \in V/\ker(f)$ $a, b \in F$,

$$\begin{aligned}\bar{f}(a(x + \ker(f)) + b(y + \ker(f))) &= \bar{f}(ax + \ker(f) + (by + \ker(f)) \\ &= \bar{f}(ax + by + \ker(f)) \\ &= f(ax + by) \\ &= af(x) + bf(y), \text{ Since } f \text{ is linear} \\ &= a\bar{f}(x + \ker(f)) + b\bar{f}(y + \ker(f)).\end{aligned}$$

Hence $\bar{f}$ is a linear transformation from $(V/\ker(f))(F)$ into $W(F)$.

(3) To show that $\bar{f}$ is one-to-one. We have, for $x + \ker(f), y + \ker(f) \in V/\ker(f)$,

$$\begin{aligned}\bar{f}(x + \ker(f)) = \bar{f}(y + \ker(f)) &\Rightarrow f(x) = f(y) \\ &\Rightarrow f(x) - f(y) = 0 \in W \\ &\Rightarrow f(x - y) = 0 \text{ Since } f \text{ is linear} \\ &\Rightarrow x - y \in \ker(f) \\ &\Rightarrow x + \ker(f) = y + \ker(f).\end{aligned}$$

This shows that $\bar{f}$ is one-to-one linear function.

(4) For any $w \in R_f$, there exists some $x \in V$ for which $w = f(x)$. $w = f(x) = \bar{f}(x + \ker(f))$ which implies that every element $w \in R_f$ is an image element of some $x + \ker(f)$ under $\bar{f}$. Hence $\bar{f}$ is an onto linear mapping.

It follows

$(V/\ker(f))(F) \cong R_f(F)$.

If $R_f = W$, then $(V/\ker(f))(F) \cong W(F)$.

N ***Theorem 10.24:*** *If $W_1(F)$ and $W_2(F)$ are subspaces of a vector space $V(F)$, then $(W_1 + W_2)/W_2)(F) \cong \left(\dfrac{W_1}{W_1 \cap W_2}\right)(F)$.*

Proof: We know that if $W_1(F)$ and $W_2(F)$ are sub-spaces of $V(F)$, then $(W_1 + W_2)(F)$ is a subspace of $V(F)$ and $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$. Thus $W_2 \subseteq W_1 + W_2 \subseteq V$ which means $W_2(F)$ is a subspace of the space $(W_1 + W_2)(F)$.

So we have the quotient space $(W_1 + W_2)/W_2(F)$.

Now we define a mapping $f ! W_1 \rightarrow \dfrac{W_1 + W_2}{W_2}$ by

$f(x) = x + W_2, \forall \; x \in W_1$.

Now we observe the following things about f.

(1) f is well defined, since for $x, y \in W_1$,

$x = y \Rightarrow x + W_2 = y + W_2 \Rightarrow f(x) = f(y)$.

(2) For any $x, y \in W_1$ and $a, b \in F$, we have

$$\begin{aligned} f(ax + by) &= ax + by + W_2 \\ &= (ax + W_2) + (by + W_2) \\ &= a(x + W_2) + b(y + W_2) \\ &= af(x) + bf(y). \end{aligned}$$

Thus f is linear.

(3) For any $x + W_2 \in \dfrac{W_1 + W_2}{W_2}$, $x \in W_1 + W_2 \Rightarrow x = x_1 + x_2$ for $x_1 \in W_1$, $x_2 \in W_2$

$$\begin{aligned} &\Rightarrow W_2 + x = W_2 + (x_1 + x_2) \\ &\Rightarrow W_2 + x = W_2 + x_1 \\ &\Rightarrow x - x_1 + W_2 = f(x'), x - x_1 = x' \in W_2. \end{aligned}$$

which proves f is onto linear function. Hence f is a homomorphism from $W_1(F)$ onto $\dfrac{W_1 + W_2}{W_2}(F)$, Since f is not one-to-one as more than one $x \in W_1$ are mapped on the same $x + W_2$ by f.

4. Since f is not one-to-one, ker (f) exists.

$$\begin{aligned} \text{So ker } (f) &= \{x \in W_1 \mid f(x) = x + W_2 = W_2\}. \\ &= \{x \mid x \in W_1, x + W_2 = W_2 \Rightarrow x \in W_2\} \\ &= \{x \mid x \in W_1 \text{ and } x \in W_2\} \\ &= W_1 \cap W_2. \end{aligned}$$

5. Now $f : W_1 \to \dfrac{W_1 + W_2}{W_2}$ is linear and

ker (f) = $W_1 \cap W_2$. Then by theorem 10.23

$$\left(\frac{W_1}{W_1 \cap W_2}\right)(F) \cong \left(\frac{W_1 + W_2}{W_2}\right)(F).$$

This completes the theorem.

N

Corollary: If $V = W_1 \oplus W_2$, then $\left(\dfrac{V}{W_2}\right)(F) \cong W_1(F)$.

Proof: Since $V = W_1 \oplus W_2$, then $W_1 \cap W_2 = \{0\}$.

So ker (f) = $W_1 \cap W_2 = \{0\}$ of the mapping

$f : W_1 \to \dfrac{W_1 \oplus W_2}{W_2}$ Thus by the above theorem

$$W_1(F) \cong \frac{W_1 + W_2}{W_2}(F).$$

10.3. NON-SINGULAR TRANSFORMATIONS

Definition 10.25: A linear transformation f from $V(F)$ into $W(F)$ is said to be non-singular if and only if there exists mapping f^* from $R_f(F)$ onto $V(F)$ such that

$f^* o f = I$, where I is identity mapping on $V(F)$.

Now we have that f is linear and $f^* o f = I$ and we prove that with these conditions f^* is also linear. If $x, y \in R_f$, $v_1, v_2 \in V$ such that $f(v_1) = x$ and $f(v_2) = y$.

Then $(f^* o f)(v_1) = f^*(f(v_1)) = f^*(x) = v_1$ and $f^*(y) = v_2$. For any $a, b \in F$,

$$\begin{aligned} f^*(ax + by) &= f^*(a f(v_1) + b f(v_2)) \\ &= f^*(f(av_1) + f(bv_2)) \\ &= f^*(f(av_1 + bv_2)) \\ &= (f^* o f)(av_1 + bv_2) \\ &= av_1 + bv_2 = af^*(x) + bf^*(y). \end{aligned}$$

Hence f^* is linear when f is linear and $f^* of = I$.

Theorem 10.26: Let $f \in L(V, W)$ be linear mapping; the following statements are equivalent.

(a) f is non-singular.

(b) For all $x, y \in V$, if $f(x) = f(y)$, then $x = y$.

(c) $\ker(f) = \{0\}$.

(d) $\gamma(f) = 0$

(e) $r(f) = \dim V$.

(f) f maps any basis of $V(F)$ onto a basis of $W(F)$.

Proof: (a) implies (b). Assume f is non-singular and also assume $f(x) = f(y)$. Then

$$\begin{aligned} f^*(f(x)) = f^*(f(y)) &\Rightarrow (f^* of)(x) = (f^* of)(y) \\ &\Rightarrow I(x) = I(y) \\ &\Rightarrow x = y. \end{aligned}$$

(b) implies (c). If $x \in \ker(f)$, then

$$\begin{aligned} f(x) = 0 = f(0) &\Rightarrow f^*(f(x)) = f^*(f(0)) \\ &\Rightarrow I(x) = I(0) \\ &\Rightarrow x = 0. \end{aligned}$$

(c) implies (d). $\nu(f) = \dim \ker(f) = \dim \{0\} = 0$.

(d) implies (e). By Sylvester's law

$$\dim V = \nu(f) + r(f) = r(f) \text{ as } \nu(f) = 0$$

(e) implies (f). By (e) $\dim V = \dim f$

$$\Rightarrow V(F) \cong R_f(F).$$

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of V, then there exist $a_1, a_2, \ldots a_n \in F$ such that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0. \text{ where } a_1 = a_2 = \ldots = a_n = 0.$$

$\Rightarrow f(a_1 x_1 + a_2 x_2 + ... + a_n x_n) = f(0) = 0$

$\Rightarrow a_1 . f(x_1) + a_2 f(x_2) + ... a_n f(x_n) = 0$

$\Rightarrow f(x_1), f(x_2), ... f(x_n)$ are L.I since $a_1 = a_2 = ... a_n = 0$

(f) implies (a). Let $\{x_1, x_2, ... x_n\}$ be a basis of V and $f(x_1), f(x_2), ..., f(x_n)$ base of R_f. Hence each $y \in R_g$ has unique expression of the form $y = bf(x_1) + b_2 f(x_2) + .. + b_n f(x_n)$, $b_k \in F$, $1 \leq k \leq n$. Let f^* be a mapping from R_f onto V defined by $f^*(y) = b_1 x_1 + b_2 x_2 + ... + b_n x_n = \sum_{k=1}^{n} b_k x_k$. We must show that $f^* of = I$ on $V(F)$. For each $x \in V$, $x = \sum_{k=1}^{n} a_k x_k$,

$$f(x) = f\left(\sum_{k=1}^{n} a_k x_k\right) = \sum_{k=1}^{n} a_k f(x_k) \in R_f,$$

$$f^*(f(x)) = \sum_{k=1}^{n} a_k x_k = x.$$ Hence $f^* of = I$.

N **Example 10.27:** Prove that a linear mapping $f: V \longrightarrow W$ is one-to-one if and only if the image of an independent set is independent.

Proof: Let f be one-to-one, and let $\{x_1, x_2, ... x_n\}$ be an independent subset of V.

Now For, $a_1, a_2, ..., a_n \in F$, we assume

$a_1 f(x_1) + a_2 f(x_2) + ... + a_n f(x_n) = 0$

$\Rightarrow f(a_1 x_1 + a_2 x_2 + ... + a_n x_n) = 0$, since f is linear

$\Rightarrow a_1 x_1 + a_2 x_2 + ... + a_n x_n \in \text{Ker} f$.

Since f is one-to-one, Ker $f = \{0\}$.

So $a_1 x_1 + a_2 x_2 + ... + a_n x_n = 0$. Since $x_1, x_2, ... x_n$ are linearly independent, $a_1 = a_2 = ... = a_n = 0$ which implies $a_1 f(x_1) + a_2 (x_2) + ... + a_n f(x_n) = 0$ where all a's are zero.

Hence $f(x_1), f(x_2), ..., f(x_n)$ are L.I.

Conversely, Let the image of any independent set be independent. If $v \in V$ is non-zero, $\{v\}$ is L.I. By assumption $\{f(v)\}$ is also L.I. which means $f(v) \neq 0$. Thus non-zero elements of V have non-zero images which implies that ker (f) does not contain non-zero elements of V. So ker (f) contains only $0 \in V$, i.e., Ker (f) = $\{0\}$ which implies f is one-to-one.

Remark: If f is one-to-one, then f is non-singular. Therefore if $f: V \longrightarrow W$ is a linear mapping, then f is non-singular if and only if the image of L.I. set is L.I. set.

N **Theorem 10.28:** A linear mapping $f: V \longrightarrow W$ is an isomorphism of a vector space $V(F)$ to a vectors space $W(F)$ if and only if f is non-singular.

Proof: Let $f ! V \longrightarrow W$ be linear mapping from $V(F)$ onto $U(F)$. Let f be non-singular. Assume $f(x) = f(y) \Rightarrow f(x) - f(y) = 0 \Rightarrow f(x - y) = 0 \Rightarrow (x - y) \in \ker f \Rightarrow x - y = 0 \Rightarrow x = y$, Since f is non-singular, ker $(f) = \{0\}$ which implies f is one-to-one. Hence f is an isomorphism.

Conversely, if f is an isomorphism, then $f: V \to W$ is one-to-one. So ker $(f) = \{0\} \Rightarrow f$ is non-singular.

Theorem 10.29: If f is a linear mapping from $V(F)$ to $W(F)$ and if f^* is a mapping from R_f to $V(F)$ such that $f^* \circ f = I$, then $f \circ f^* = I$ on R_f

Proof: By hypothesis, f is non-singular, so any $y \in R_f$ can be written as $y = f(x)$ for some $x \in V$.

Then

$$(f \circ f^*)(y) = (f \circ f^*)(f(x)) = f(f^*(f(x))$$
$$= f(x) = y$$

Hence $f \circ f^* = I$ on R_f

This shows that f is invertible and its inverse is f^* which is written $f^{-1} = f^*$

10.4. ALGEBRA OF LINEAR TRANSFORMATIONS

Let $V(F)$ and $W(F)$ be two vector spaces over the same field F. The set of all Linear transformation is denoted by $L(V, W)$. Now we have to see that $L(V, W)$ under defined operations over the field F is a vector space. Some authors write $L(V, W)$ as $\text{Hom}_F(V, W)$.

Definition 10.30: The sum $f + g$ of two Linear mappings $f, g \in L(V, W)$ is defined by the rule

$$(f + g)(x) = f(x) + g(x), \quad x \in V, \text{ and}$$

the scalar multiplication is given by

$$(cf)(x) = c\,f(x), \text{ where } c \in F, x \in V.$$

Lemma 10.31: If $f, g \in L(V, W)$, then sum

$$f + g \in L(V, W)$$

Proof: Let $x, y \in V$, $a, b \in F$. Since f and g are L.T., then $f(ax + by) = af(x) + bf(y)$, and $g(ax + by) = a\,g(x) + bg(y)$

$$\begin{aligned} \text{Now } (f+g)(ax+by) &= f(ax+by) + g(ax+by) \\ &= a f(x) + b f(y) + a\,g(x) + b\,g(y) \\ &= a(f(x) + g(x)) + b(f(y) + g(y)) \\ &= a(f+g)(x) + b(f+g)(y) \end{aligned}$$

Which shows $f + g$ is linear.

Lemma 10.32: For any $f \in L(V, W)$, $c \in F$, then $cf \in L(V, W)$.

Proof: Let $x, y, \in V$, $a, b \in F$. Since f is linear, $f(ax + by) = af(x) + bf(y)$.

$$\begin{aligned} \text{Now } (cf)(ax+by) &= c\,f(ax+by) \\ &= c(af(x) + bf(y)) \\ &= (c.a)f(x) + (cb)f(y) \\ &= (ac)f(x) + (bc)f(y) \\ &= a(cf(x)) + b(cf(y)). \end{aligned}$$

Hence $cf \in L(V, W)$.

Theorem 10.33: Let $V(F)$ and $W(F)$ be vector spaces over the field F. With addition and scalar multiplication defined as in Definition 10.30 for mappings, $L(V, W)(F)$ is itself a vector space.

Proof: To prove the theorem we shall first prove that $(L(V, W), +)$ is a commutative group under the addition defined by $(f+g)(x) = f(x) + g(x)$, $\forall\ f, g \in L(V, W)$ where $x \in V$.

Now we verify group axioms.

1. ***Closure property:*** For any $f, g \in L(V, W)$,

$f + g \in L(V, W)$ by Lemma 10.31

which shows $L(V, W)$ is closed under the operation.

2. ***Associativity:*** For any $f, g, h \in L(V, W)$, and $x \in V$,

$$[(f+g)+h](x) = [(f+g)(x)] + h(x)$$
$$= [f(x) + g(x)] + h(x)$$
$$= f(x) + [g(x) + h(x)] \text{ since } f(x),$$

$g(x), h(x) \in W$ and addition in W is associative.

$$= [f + (g+h)](x).$$

Thus $(f+g)+h = f+(g+h)$. Hence addition in $L(V, W)$ is associative.

3. ***Existence of identity:*** We define a zero function $Z! \ V \longrightarrow W$ which maps every element of V on 0 of W. that is $Z(x) = 0$, $\forall\ x \in V$. Now we see that Z is linear. For this, if $x_1, x_2 \in V$, $a, b \in F$, then

$Z(a x_1 + b x_2) = 0 = 0 + 0 = a Z(x_1) + bZ(x_2)$.

which show $Z \in L(V, W)$.

Now for any $f \in L(V, W)$, $x \in V$, we have

$(f+Z)(x) = f(x) + Z(x) = f(x) + 0 = f(x)$.

$(f+Z) = f \Rightarrow Z$ is the additive identity in $L(V, W)$.

4. ***Existence of inverse:*** For any $f \in L(V, W)$. Let $-f$ be defined by $(-f)(x) = -f(x)$. Now for any $x, y \in V$, $a, b \in F$, $(-f)(ax + by) = -f(ax + by)$

$$= -[af(x) + bf(y)]$$
$$= -af(x) - bf(y)$$
$$= a(-f(x) + b(-f(y)).$$

which shows $-f \in L(V, W)$.

$$\text{Now } ((f + (-f))(x) = f(x) + (-f)(x)$$
$$= f(x) - f(x) = 0 \text{ additive identity of } W$$
$$= Z(x).$$

Thus $f + (-f) = Z$. Which shows $-f$ is the additive inverse of f. Hence every $f \in L(V, W)$ has additive inverse $-f \in L(V, W)$.

5. ***Commutativity:*** For any $f, g \in L(V, W)$, $x \in V$,

We have $(f+g)(x) = f(x) + g(x)$

$= g(x) + f(x)$ since $f(x)$ and $g(x) \in W$ and addition in W is commutative.

$$= (g+f)(x).$$

Therefore $f + g = g + f$. Hence addition in $L(V, W)$ is commutative.

Hence $(L(V, W), +)$ is an abelian group under the addition defined above.

For showing $(L(V, W))(F)$ is a vector space we have to verify the following properties with respect to scalar multiplication. By Lemma 10.32 for any $f \in L(V, W)$ and $c \in F$, $(cf) \in L(V, W)$. Hence $L(V, W)$ is closed under scalar multiplication. For any $f, g \in L(V, W)$, $a, b \in F$, we have

6. (a):
$$\begin{aligned} c(f+g)(x) &= c[(f+g)(x)] \\ &= c[f(x)+g(x)] = cf(x) + cg(x) \\ &= (cf)(x) + (cg)(x) \\ &= (cf+cg)(x). \end{aligned}$$

Hence $c(f+g) = cf + cg$.

(b)
$$\begin{aligned} ((a+b)f)(x) = (a+b)f(x) &= af(x) + bf(x) \\ &= (af)(x) + (bf)(x) \\ &= (af+bf)(x). \end{aligned}$$

Therefore $(a+b)f = af + bf$.

(c)
$$\begin{aligned} ((ab)f)(x) = (ab)f(x) &= a(bf(x)) \\ &= a(bf)(x) \end{aligned}$$

Hence $(ab)f = a(bf)$.

(d) $(If)(x) = I(f(x)) = f(x)$ which implies

$If = f$.

Hence $L(V, W)(F)$ is a vector space over the field F.

***Theorem 10.34:** If $V(F)$ and $W(F)$ are vector spaces of demension m and n, respectively, over F, then the space $L(V, W)(F)$ is of dimension $m.n$ over F.*

Proof: To prove this theorem we shall show that there exists a basis of $L(V, W)(F)$ of $m.n$ elements. Let $(x_1, x_2, \dots, x_m)$ be a basis of $V(F)$ and let $\{y_1, y_2, \dots, y_n\}$ be a basis of $W(F)$. By theorem. 10.17 there exists a linear mapping in $L(V, W)$ which maps elements of the basis of $V(F)$ onto arbitrary elements of $W(F)$, so we define f_{ij} $V \rightarrow W$ by, $1 \le i \le m$, $1 \le j \le n$,

$$\begin{aligned} f_{ij}(x_k) &= 0 \text{ if } i \neq k, \\ &= y_j \text{ if } i = k. \end{aligned}$$

That is, f_{ij} maps x_i onto y_j and other x's onto 0.

We observe that the set $\{f_{i,j}\}$ contains $m.n$ elements. Hence the theorem is proved if the set $\{f_{i,j}\}$ forms a basis of $L(V, W)(F)$.

First we shall see that $\{f_{i,j}\}$ generates $L(V, W)$. Let $f \in L(V, W)$. For each index i $(1_1 \le i \le m)$, $f(x_i) \in W$. So $f(x_i)$ can be expressed as a linear combination of vectors $y_1, y_2, \dots, y_n$ of basis vectors of W.

Thus

$$\begin{aligned} f(x_i) &= a_{i1}y_1 + a_{i2}y_2 + \dots + a_{in}y_n, \quad a_{ij} \in F.\ 1 \le i \le m \\ &= \sum_{j=1}^{n} a_{ij}\, y_j \end{aligned}$$

Since $f_{ij}(x_i) = y_j$ and $f_{ij}(x_k) = 0$ if $i \neq k$.,

$$f(x_i) = \sum_{j=1}^{n} a_{ij}\, y_j$$

$$f(x_k) = \sum_{j=1}^{n} a_{kj}\, f_{kj}(x_k) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\, f_{ij}(x_k)\right)$$

$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\, f_{ij}(x_k)\right).$$

which implies $f = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\, f_{ij}\right)$ Thus every $f \in L(V, W)$ can be written as the Linear combination of the elements of the set $\{f_{ij}\}$. Thus we conclude the set $\{f_{ij}\}$ spans $L(V, W)(F)$.

In order to prove that $\{f_{ij}\}$ forms a basis of $L(V, W)$. There remains to show that $\{f_{ij}\}$ is linearly independent. Suppose that there is some linear combination

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, f_{ij} = 0, \text{ where } c_{ij} \in F.$$

If we evaluate this expression at the vector x_k,

$$0 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, f_{ij}(x_k) = \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij}\, f_{ij}(x_k))$$

$$= \sum_{j=1}^{n} c_{kj}\, y_j \text{ Since } f_{ij}(x_i) = y_j$$

if $i = k$ and $f_{ij}(x_i) = 0$, if $i \neq k$. $1 \le j \le n$.

Thus $\sum_{j=1}^{n} c_{kj}\, y_j = 0$. Since $\{y_1, y_2, \ldots, y_n\}$ is a basis for $W(F)$, then $\sum_{j=1}^{n} c_{kj}\, y_j = 0$ gives $c_{k1} = c_{k2} = \ldots = c_{kn} = 0$.

By varying k, we conclude that all the coefficient $c_{ij} = 0$.

Hence the theorem is proved.

Corollary 1: If dim $V = n$, then dim $L(V, V) = n^2$.

Corollary 2: If dim $V = n$, then dim $L(V, F) = n$.

Proof: Since the field F is a vactor space over itself and the field F is generated by its multiplicative identity, then dim $F = 1$. So by the theorem dim $(V, F) =$ dim V. dim $F = n.1 = n$.

Note: Sometimes we denote the set $L(V, V)$ of all linear transformation from the vector space $V(F)$ into itself by $A(V)$ or by Hom (V, V).

Theorem 10.35: *Let $V(F)$, $U(F)$, and $W(F)$ be vector spaces over F. Let $f: V \to U$ be a linear mapping from $V(F)$ into $U(F)$ and let $g\,!\,U \to W$ be a linear mapping from $U(F)$ into $W(F)$. Then $g \circ f$ is a linear mapping from $V(F)$ into $W(F)$.*

Proof: Since $f: V \to U$ be a linear mapping, then For $x, y \in V$, $a, b \in F$ we have, $f(ax+by) = a f(x) + b f(y) \in U$.

Again, since $g\,!\,U \to W$ be a linear mapping,

Then for $f(x), f(y) \in U$, we have

$$(g \circ f)(a x + by) = g(f(a x + by))$$
$$= g(a f(x)) + b f(y)) = a g(f(x)) + b g(f(y))$$
$$= a (g \circ f)(x) + b (g \circ f)(y).$$

This shows $g \circ f$ is a Linear mapping from $V(F)$ into $W(F)$.

Corollary :1: $k(g \circ f) = (kg) \circ f = g \circ (kf)$.

Corollary 2: Let $V(F)$ be a vector space. Then $L(V, V)(F)$ is also a vector space over F. If $f, g \in L(V, V)$, then gof and $fog \in L(V, V)$

Theorem 10.36: **For each vector space $V(F)$,** *the triple $(L(V, V), +, .)$* is ring with identity under addition defined in $L(V, V)$ and composition of functions.

Proof: The proof is left as an exercise for the students. But from this discussion we have a very important another fact.

Corollary: If $GL(V)$ denotes the set of all invertible mappings in $L(V, V)$, then $(GL(V), o)$ forms a group, called the general linear group.

Proof: Let $f, g \in GL(V)$. That is, f and g are invertible mappings from V into V. f is invertible, if and only if it is an one-to-one and onto mapping. Since f and g are invertible, that is, they are one-to-one mappings, their compositions f og and *gof* are also one-to-one. Thus *fog* and *gof* both are invertible. Since f and g are linear, *fog* and *gof* are also linear by the theorem 10.35.

Hence *fog*, $gof \in GL(V)$. Hence $GL(V)$ is closed under the composition 'o'.

Since the composition o of functions in $L(V, V)$ is associative, it is also associative in $GL(V)$ as $GL(V) \subset L(V, V)$.

Identity mapping e_v from V onto V is always one-to-one, so it is invertible. Since $e_v(a x + b y) = a x + b y = a e_v(x) + be_v(y)$, and $e_v \in L(V, W)$.

Since f is invertible, f^{-1} exists and f^{-1} is one to one. So $(f^{-1})^{-1} = f \in GL(V)$ $\Rightarrow f^{-1}$ is invertible. Thus every $f \in V$ has its inverse $f^{-1} \in GL(V)$. To see that f^{-1} is linear, Let $x_1, x_2 \in V$. Since f is one-to-one and onto, there exists $y_1, y_2 \in V$ such that $f(x_1) = y_1, f(x_2) = y_2$.

Since f is linear $f(x_1 + x_2) = f(x_1) + f(x_2) = y_1 + y_2$ and $f(a x_1) = a f(x_1) = a y_1$

By definition of f^{-1}, we have $f^{-1}(y_1) = x_1, f^{-1}(y_2) = x_2$ and

$$f^{-1}(y_1 + y_2) = x_1 + x_2 = f^{-1}(y_1) + f^{-1}(y_2),$$
$$f^{-1}(ay_1) = ax_1 = a f^{-1}(y_1).$$

Hence f^{-1} is Linear.

The structure of $L'(V, V)$ can be approached from one more direction. For this we define what is meant by algebra over a field.

Definition 10.37: A vector space $V(F)$ is said to be an algebra over the field F if its elements can be multiplied in such a way that $V(F)$ becomes a ring in which scalar multiplication is related to ring multiplication (denoted by.,) by the following mixed associative law:

$c(x \,.\, y) = (c\,x) \,.\, y = x.(c\,y), (\forall\, x, y \in V, c \in F).$

Theorem 10.38: The triple $(L(V, V), +, o)$ is an algebra over the field F.

Proof: We observe the following things about the triple $(L(V, V), +, o)$.

(*i*) $L(V, V)$ is a vector space over F by theorem 10.33.

(*ii*) $L(V, V)$ is a vector space of demension n^2 over F by corollary I of theorem 10.34.

(*iii*) $(L(V, V), +, 0)$ is ring with identity under the addition of functions and composition of functions by theorem 10.36.

To show that $(L(V, V), +, o)$ is an algebra we have to prove that mixed associative law holds. To get this, Let $c \in F$, $g, f \in L(V, V)$, and $x \in V$, Then

$$(fo\,(cg))\,(x) = f((cg)\,(x)) = f(c\,g\,(x)) = c\,f(g\,(x)$$
$$= c\,(fog)\,(x).$$

Since this equality holds for $x \in V$, we conclude that

$f\,o\,(cg) = c\,(f\,o\,g).$

Similarly,

$$((cf)\,og\,(x) = (cf)\,(g\,(x)) = c\,(f\,g\,(x)$$
$$= c\,(fog)\,(x).$$

Hence $(c\,f)\,og = c\,(fog)$.

Thus $c\,(fog) = (cf)\,og = f\,o\,(cg)$.

Therefore the triple $(L(V, V), +, o)$ is an algebra over F of dimension n^2. Hence the theorem

Example 10.39: We consider the set $M_n(F)$ of all square matrices of order n over the field F. We know that $(M_n(F), +,.)$ is a ring and vector space over F. The mixed associative Law also holds in $M_n(F)$. For this Let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n} \in M_n(F)$

Then $A.B = [c_{ij}]_{n \times n}$

For $k \in F$,

$(k\,A) \,.\, B = k\,(A).\,B = k\,(A.B)$

$A.(k\,B) = k\,(A \,.\, B)$. So

$k\,(A \,.\, B) = (k\,A).\,B = A\,(k\,B)$

(2) The Rings $(R, +,.)$ and $(C, +,.)$ of real and complex numbers are also algebras over R and C respectively.

Theorem 10.40: Let $f : V \to U$ and $g : U \to W$ be linear. Hence (gof) : $V \to W$ is Linear. Then

(i) rank $(gof) \leq$ rank g,
(ii) rank $(gof) \leq$ rank f.

Proof: (i) Since $f(V) \subseteq U$, $g(f(V)) \subseteq g(U)$ and so dim $g(f(V)) \leq$ dim $g(U)$. Then

rank $(g \circ f) =$ dim $(g \circ f)(V) =$ dim $g(f(V)) \leq$ dim $g(U) =$ rank g. Hence rank $g \circ f \leq$ rank g.

(ii) Since $f(V) \subseteq U$ and $g : f(V) \to W$, then by Sylvester's law dim $f(V) =$ dim ker (g) + dim $g(f(V))$

which implies dim $g(f(V)) \leq$ dim $f(V)$

Now rank $(g \circ f) =$ dim $(g \circ f)(V) =$ dim $g(f(V)) \leq$ dim $f(V) =$ rank f

Hence rank $(g \circ f) \leq$ rank f.

Theorem 10.41: For any Linear transformation $f: V \to W$, $r(f) \leq \min(\dim V, \dim W)$.

Proof: Let dim $V = n$ and dim $W = m$. Since $f(V) \subseteq W$, then dim $f(V) \leq$ dim W. That is, $r(f) \leq$ dim W.

Again, since dim $V = n$, then there exists $n + 1$ vectors $x_1, x_2, \ldots, x_n, x_{n+1}$ which are L.D. Therefore for some $a_1, a_2, \ldots, a_n, a_{n+1} \in F$, not all zero, we have $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + a_{n+1} x_{n+1} = 0$. Since f is linear $f(0) = 0$. Thus

$0 = f(0) = f(a_1 x_1 + a_2 x_2 + \ldots + a_n + a_{n+1} x_{n+1})$

$= a_1 f(x_1) + a_2 f(x_2) + \ldots + a_n f(x_n) + a_{n+1} f(x_{n+1})$. which shows $f(x_1), f(x_2), \ldots f(x_n), f(x_n), f(x_{n+1})$ are L.D., since all $a's$ are not zero. Hence $f(V)$ can not contain $(n + 1)$ L.I. vectors. which implies dim $f(V) \leq n =$ dim V. Thus $r(f)$ is less then dim V and dim W which means $r(f) \leq \min(\dim V, \dim W)$.

Theorem 1042: Let $f: V \to W$ be a linear transformation. For any subspace $H(F)$ of $V(F)$, dim $f(H) \geq$ dim $H - \nu(f)$.

Proof: Let f_1 be a restriction of f to H, that is, $f(H) = f_1(H)$. Since f is linear, f_1 is linear of H into W. For this, if $x, y \in H$, $a, b \in F$, we have

$f_1(ax + by) = f(ax + by) = af(x) + bf(y) = af_1(x) + bf_1(y)$. By Sylvester's law dim $H =$ dim ker (f_1) + dim $f_1(H)$

or dim $f_1(H) =$ dim $H - \nu(f_1)$ .... (1)

Now since $f(H) = f_1(H)$, dim $f(H) =$ dim $f_1(H)$ .... (2)

$$\begin{aligned} \ker f_1 &= \{x \in H \mid f_1(x) = 0\} \\ &= \{x \in H \mid f(x) = 0\} \subseteq \{x \in V \mid f(x) = 0\} \\ &\Rightarrow \operatorname{Ker}(f_1) \subseteq \ker(f) \Rightarrow \nu(f_1) \leq \nu(f) \quad \ldots\ldots\ldots (3) \end{aligned}$$

From (1), (2) and (3) we have

$$\begin{aligned} \dim f(H) = \dim f_1(H) &= \dim H - \nu(f_1) \\ &\geq \dim H - \nu(f). \end{aligned}$$

Hence the result is proved.

Theorem 10.43: If $f, g \in L(V, W)$, then

(i) $r(af) = r(f), \forall\ a \neq 0 \in F$,

(ii) $|r(f) - r(g)| \leq r(f + g) \leq r(f) + r(g)$

(i) Since $f(V)(F)$ is a subspace of $W(F)$, then $(af)(V) = af(V) \subseteq f(V)$ and $a^{-1} f(V) \subseteq f(V)$.

Therefore $a\,[a^{-1} f(V)] \subseteq a\, f(V)$.

or $(a\,a^{-1}) f(V) \subseteq a\, f(V) \Rightarrow f(V) \subseteq a\, f(V)$.

Hence $f(V) = a\, f(V) \Rightarrow \dim f(V) = \dim a\, f(V)$

Hence $r(f) = r(a\,f)$.

(*ii*) We know, for any $x \in V$,

$(f+g)(x) = f(x) + g(x)$.

Therefore $(f+g)(V) \subseteq f(V) + g(V)$

which implies $\dim (f+g)(V) \leq \dim (f(V) + g(V))$

$\leq \dim f(V) + \dim g(V)$

$\Rightarrow r(f+g) \leq r(f) + r(g)$ ...... (1)

Now $f = f + g - g$. By (1) we have

$r(f) = r[(f+g) + (-g)] \leq r(f+g) + r(-g)$

By (*i*) we have $r(a\,g) = r(g)$, if $a = -1$. then

$r(-g) = r(g)$.

So $r(f) \leq r(f+g) + r(-g) = r(f+g) + r(g)$

or $r(f) - r(g) \leq r(f+g) \leq r(f) + r(g)$ ...... (2)

Similary $r(g) - r(f) \leq r(f+g) \leq r(f) + r(g)$ ...... (3)

From (2) and (3) we have

$|r(f) - r(g)| \leq r(f) + r(g)$.

N

Theorem 10.44: Let $U(F)$, $V(F)$ and $W(F)$ be vector spaces over F, Let $f: U \to V$ and $g: V \to W$ be linear mappings.

then

(*i*) $r(g\,o\,f) \leq \min\{r(f), r(g)\}$,

(*ii*) $r(f) + r(g) - n \leq r(g\,of)$, where $n = \dim V$.

Proof 1: By theorem 10.41,

$r(g\,of) \leq \min(\dim U, \dim W)$

$r(g\,o\,f) = \dim (g\,o\,f)(U) = \dim (g\,(f(U))$

$\leq \min(\dim f(U), \dim W)$

Since, by applying theorem 10.41 to,$g: f(U) \to W$, we have $r(g) = \dim g\,(fU)) \leq \min(\dim f(U), \dim W,)$

$\leq \dim f(U) = r(f)$.

Thus $r(g\,o\,f) \leq r(f)$ ...... (1)

Since $(g\,o\,f)(x) = g\,(f(x))$, it follows that range of $(g\,o\,f)$ is contained in range of g, so that $(g\,o\,f)(U)$

$\subseteq g(V) \Rightarrow r(g\,o\,f) \leq r(g)$ ...... (2).

From (1) and (2), we have

$r(g\text{ of}) \leq \min(r(f), r(g))$. ...... (3)

(*ii*) Since $\dim V = n$, then by sylvester's law, $n = \dim V = \dim \ker(g) + \dim(g(V))$.

or $n = \nu(g) + r(g)$

or $\nu(g) = n - r(g)$ ...... (4)

Since $f(U) \subseteq V$, then we can choose g_1 to be the restriction of g to $f(U)$.

Then

$\ker g_1 = \{x \in f(U) \mid g_1(x) = 0\} = \{x \in f(U) \mid g(x) = 0\}$

$\subseteq \{x \in V \mid g(x) = 0\} = \ker g.$

Thus $\ker g_1 \subseteq \ker g \Rightarrow v(g_1) \leq v(g).$ ...... (5)

From (4) and (5) we get

$$v(g_1) \leq v(g) = n - r(g)$$

or

$$v(g_1) \leq n - r(g)$$ ...... (6)

Now $g_1(f(U)) = g(f(U)) = (g \circ f)(U) \Rightarrow r(g_1) = r(g \circ f).$

But by sylvester's law

$$\dim f(U) = v(g_1) + r(g_1)$$
$$= v(g_1) + r(g \circ f).$$

Thus we get

$$r(f) = r(g_1) + r(g \circ f)$$
$$\leq r(g \circ f) + n - r(g) \text{ by (6)}$$

i.e.

$$r(f) + r(g) \leq r(g \circ f) + n$$

or

$$r(f) + r(g) - n \leq r(gof).$$

Thus the theorem is proved.

Corollary: If one of f and g is an isomorphism, the $r(g \circ f) = \min(r(f), r(g))$.

Let g be an isomorphism, then, $f(U) \subseteq V$

and $(g \circ f)(U) = g(f(U) \subseteq W$, say $W_1 = g(f(U))$.

Since g is an isomorphism, $f(U)(F) \cong W_1(F)$.

or $\dim f(U) = \dim W_1 = \dim g(f(U)$

$$r(f) = r(g \circ f)$$ ...... (1)

Again, If f is an isomorphism, then

$$f(U) = V \Rightarrow r(f) = \dim V.$$

and

$r(g \circ f) = \dim (g \circ f)(U) = \dim g(f(U)) = \dim g(V) \Rightarrow r(g \circ f) = r(g).$

From (1), and (2)

We have $r(g \circ f) = \min(r(f), r(g)).$

Example 10.45: Let $V(F)$ be a vector space over F of dimension 5. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be a basis of $V(F)$. Let $f: V \to V$, and $g: V \to V$ be defined by

$$f(x) = a_1 x_1 + a_2 x_2 + a_3 x_3,$$

and $g(x) = a_2 x_2 + a_3 x_3 + a_4 a_4$ For any $x = a_1x_1 + a_2x_2 + v_3x_3 + a_4x_4 + a_5x_5 \in V, a_i \in F, 1 \leq i \leq 5.$

It is clear that $f(V)$ has a basis $\{x_1, x_2, x_3\}$ and $g(V)$ has a bases $\{x_2, x_3, x_4\}$. ...... (1)

Thus $r(f) = r(g) = 3.$

Now $(f \circ g)(x) = f(g(x)) = f(a_2 x_2 + a_3 x_3 + a_4 x_4)$

$$= a_2 x_2 + a_3 x_3.$$

This gives $(f \circ g)(V)$ has a bases $\{x_2, x_3\}$ and so $r(f \circ g) = 2.$

Thus from (1) and (2) it is clear that

$$r(f) + r(g) - 5 < r(f \circ g) < \min(r(f), r(g))$$

$$\Rightarrow \quad 3 + 3 - 5 < 2 < \min\{3, 3\}. = 3.$$

PROBLEMS

1. If we regard the complex numbers as a vector space over the real field, is the conjugate mapping, $f(a + ib) = a - ib$, a linear transformation?
2. Determine which of the following functions are linear mappings of $V_3(R)$ over the real field R into itself:
   (a) $f(a_1, a_2, a_3) = (a_1 + 1, a_2 + 1, 0)$,
   (b) $f(a_1, a_2, a_3) = (a_2, a_1, a_3)$,
   (c) $f(a_1, a_2, a_3) = (a_1, a_2, I)$,
   (d) $f(a_1, a_2, a_3) = (a_1, -a_2 - a_3)$,
   (e) $f(a_1, a_2, a_3) = (a_1, a_1 + a_2, a_1 + a_2 + a_3)$,
   (f) $f(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, -a_1 + a_2, 2a_1 + a_2)$.
3. Let $f \in L(V, W)$. then
   (1) Show that any subspace $V_1(F)$ of $V(F)$ is mapped by f into a subspace $W_1(F)$ of $W(F)$.
   (2) Conversely, show that if $W_1(F)$ is a subspace of $W(F)$, the set of all vectors of $V(F)$ which are mapped into $W_1(F)$ is a subspace of $V(F)$.
4. Show that a linear transformation f from $V_n(F)$ to $W(F)$ is determined by the effect of f on any basis of $V_n(F)$.
5. Let $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \in R[x]$, the vector space of polynomials in x over the real field R. Let the mapping D and M be defined by

$$D(p(x)) = \frac{d}{dx}(p(x)),$$

$$M p(x) = x p(x).$$

   (I) Show that both D and M are linear transformations.
   (II) Is D nilpotent on this space?
   (III) Prove that $MD - DM = I$
   (IV) Deduce that $(DM)^2 = D^2 M^2 + DM$.
6. Let the mapping $f \in L(V, V)$ and S denote the set of vectors of V which are left fixed by f:

$$S = \{x \in V \mid f(x) = x\}.$$

   Verify that $S(F)$ forms a subspace of the vector space $V(F)$
7. Show that by example that the conclusion of the corollary to theorem 10.2) is false if $V(F)$ is infinite dimensional.
8. Suppose the mapping $f \in L(V, W)$ with $\dim V > \dim W$. Show that there exists a non zero vector $x_0 \in V$ for which $f(x_0) = 0$.
9. Let $V(F)$ be finite-dimensional with basis $\{x_1, x_2, ..., x_n\}$, and let $\{y_1, y_2, ..., y_n\}$ be any n elements of V. If the function $f: V \to V$ is defined by taking $f(a_1 x_1 + ... + a_n x_n) = a_1 y_1 + ... + a_n y_n$, $a_i \in F$,

Prove that f is Linear; determine when f will be an isomorphism.

10. Prove that if the mapping $f \in L(V, V)$ is such that $\ker(f) = \ker(f^2)$, then $V = \operatorname{Ker}(f) \oplus f(V)$.

11. For a fixed element $a \in F$, define the Scalar multiplication $f_a : V \to V$ by $f_a(x) = ax, x \in V$.
Given $R = \{f_a \mid a \in F\}$ and $R' = R - \{0\}$, show that
(a) the triple $(R, +, o)$ forms a subring of $(L(V, V), +, o)$ isomorphic to $(F, +, .)$.
(b) The pair $(R', 0)$ is a normal subgroup of the Linear group $(GL(V), o)$.

12. A Linear mapping $f \in L(V, V)$ is said to be nilpotent if $f^n = 0$ for some $n \in N$. If f is nilpotent and if $f^{n-1}(x_o) \neq 0$,
Prove that
$\{x_o, f(x_o), f^2(x_o), \ldots, f^{n-1}(x_o)\}$
is a linearly independent set of vectors.

13. If the linear mappings $f\ g \in L(V, W)$
(a) $\operatorname{Ker}(g\ o\ f) \supseteq \operatorname{Ker}(f)$ and $\nu(g\ o\ f) \geq \nu(f)$
(b) if $f, g \in L(V_n, V_n)$, then
$\nu(f + g) \geq \nu(f) + \nu(g) - n$.
and $\nu(f) + \nu(g) \geq \nu(g\ o\ f) \geq max\{\nu(f), \nu(g)\}$

14. Let $V(F)$, $U(F)$, and $W(F)$ be vector spaces over F. Let f, f' be linear mappings from V into U and Let g, g' be linear mappings from U into W; then prove that
(i) $go(f + f') = g\ o\ f + g\ o\ f'$,
(ii) $(g + g')$ of $= g\ o\ f + g'\ of$.

10.5. DUAL SPACES

We have studied in the last section the structure of $L(V, W)$. Now if we take vector space $V(F)$ in place of $W(F)$, then we get the set of linear transformations $L(V, V)$ which we have also done in brief. Now we take up a special case when $W(F) = F$. The field F is considered a vector space over itself, so our aim in this section is to confine our attention to the set $L(V, F)$ which is called the set of linear functionals. So scalar valued functions are Linear functionals which are linear mappings from a vector space $V(F)$ to F.

Defintion 10.46: Let $V(F)$ be a vector space over F. A Linear Transformation from $V(F)$ into F is called a Linear functional. That is, if $\phi : V \to F$ is a mapping satisfying that $\phi(a\,x + b\,y) = a\,\phi(x) + b\,\phi(y)$ for every $x, y \in V$ and $a, b \in F$ is a linear functional. The set of all linear functionals is denoted by V^n.

Example 10.47: Let $V_n(F)$ is a vector space over F. Then $\phi_i : v_n(F) \to F$, defined by
$\phi_i(a_1, a_2, \ldots, a_n) = a_i$, is a linear functional on $V_n(F)$.

Example 10.48: Let $V(F)$ be a vector space of all polynomials in x over F. Let $I : V \to R$ be the integral operator defined by $I(p(x)) = \int_a^b p(x)\,dx$, where $a \le x \le b$. Now for $k, k \in F$, $p(x), q(x) \in V(F)$,

$$\begin{aligned} I(k_1 p(x) + k_2 q(x)) &= \int_a^b (k_1 p(x) + k_2 q(x))\, dx \\ &= \int_a^b k_1 p(x)\, dx + \int_a^b k_2 q(x)\, dx \\ &= k_1 \int_a^b (x)\, dx + k_2 \int_a^b q(x)\, dx \\ &= k_1 I(p(x)) + k_2 I(q(x)). \end{aligned}$$

Hence I is linear, and hence it is a Linear functional on $V_a(F)$.

Example 10.49: Let $V(F)$ be the vector space of n-square matrices over F. Let $\phi : V \to F$ be the trace mapping defined by $\phi(A) = a_{11} + a_{22} + \ldots + a_{nn}$, where $A = [a_{ij}]$.

That is, ϕ assigns to the matrix A the sum of the diagonal elements. To see that ϕ is Linear, let

$A = [a_{ij}]$, $B = [b_{ij}] \in V(F)$ and $k_1, k_2 \in F$, then

$$\begin{aligned} \phi(k_1 A + k_2 B) &= \phi(k_1 [a_{ij}] + k_2 [b_{ij}]) \\ &= \phi([k_1 a_{ij}] + [k_2 b_{ij}]) \\ &= \phi([k_1 a_{ij} + k_2 b_{ij}]) \\ &= k_1 a_{11} + k_1 a_{22} + \ldots + k_1 a_{nn} + \\ &\quad k_2 b_{11} + k_2 b_{22} + \ldots + k_2 b_{nn} \\ &= k_1 f(A) + k_2 f(B). \end{aligned}$$

Hence ϕ is linear and so it is linear functional on $V(F)$.

In the theorem 10.33 we have seen that $L(V, W)$ is a Vector space. So the set V^* of all Linear functionals on $V(F)$ also forms a vector space under addition and scalar multiplication defined by

$(\phi + \sigma)(x) = \phi(x) + \sigma(x)$, and $(k\phi)(x) = k\phi(x)$. where $\phi, \sigma \in V^*$ and $k \in F$.

Definition 10.50: The vector space $L(V, F)(F)$, consisting of all linear functionals is called *dual space* of $V(F)$ and is denoted by $V^*(F)$. A dual space is also called a *Conjugate space.*

Since F is vector space over itself of dimension one, then For any finite-dimensional vector space $V(F)$, dim V^* = dim $(L(V, F))$ = din V. dim F = dim V. by theorem 10.34. Now we establish an important relation between a vector space and its dual.

Theorem 10.51: Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of $V(F)$ over F. Let $\phi_1, \ldots \phi_n \in V^*$ be the linear functionals defined by

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ \theta & \text{if } i \ne j \end{cases}$$

Then $\{\phi_1, \phi_2, \ldots \phi_n\}$ is a bases of $V^*(F)$, where δ_{ij} is Kronecker delta.

Proof: Kronecker delta δ_{ij} is a short form of writing

$\phi_1(x_1) = 1, \phi_1(x_2) = 0, \ldots, \phi_1(x_n) = 0$

$\phi_2(x_1) = 0, \phi_2(x_2) = 1, \ldots, \phi_2(x_n) = 0$

....

$\phi_n(x_1) = 0, \phi_n(x_2) = 0, \ldots, \phi_n(x_n) = 1.$

We first show that $\{\phi_1, \phi_2, \ldots, \phi_n\}$ spans V^*. Let ϕ be an arbitrary element of V^* and let

$\phi(x_1) = k_1, \phi(x_2) = k_2, \ldots, \phi(x_n) = k_n$

now $k_1\phi_1 + k_2\phi_2 + \ldots + k_n\phi_n = \sigma \in V^*$.

So, for $1 \le i \le n$, we have

$$\sigma(x_i) = (k_1\phi_1 + k_2\phi_2 + \ldots + k_i\phi_i + \ldots + k_n\phi_n)(x_i)$$
$$= k_1\phi_1(x_i) + \ldots + k_i\phi_i(x_i) + \ldots + k_n\phi_n(x_i)$$
$$= k_1 0 + .. + k_i . I \ldots + k_n 0 = k_i$$

Thus $\phi(x_i) = \sigma(x_i) = k_i$ for $i = 1, 2, \ldots n$. Since ϕ and σ agree on the basis vectors, $\phi = \sigma = k_1\phi_1 + k_2\phi_2 + \ldots + k_n\phi_n$. This shows $\{\phi_1, \phi_2 \ldots \phi_n\}$ spans V^*.

To prove the theorem there remains to show that $\phi_1\phi_2, \ldots, \phi_n$ is $L.I.$ Assume that $a_1\phi_1 + a_2\phi_2 + \ldots + a_n\phi_n = 0 \in V^*, a_i \in F$.

So $(a_1\phi_1 + a_2\phi_2 + \ldots + a_n\phi_n)x_i = 0(x_i) = 0 \in F$

$a_1\phi_1(x_1) + a_2\phi_2(x_i) + a_i\phi_i(x_i) + \ldots + a_n\phi_n(x_i) = 0$

$a_1 . 0 + a_2 0 + .. + a_i + \ldots a_n 0 = 0 \Rightarrow a_i = 0$.

That is, $a_1 = a_2 \ldots = a_n = 0$. Hence $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is $L.I.$ and so it is a basis of V^*.

The set $\{\phi_1, \phi_2, \ldots \phi_n\}$ is termed dual basis. It also follows that dim $V = n$ = dim V^*.

Example 10.52: consider the following basis of R^2: $\{x_1 = (2, 1), x_2 = (3, 1)\}$. Find the dual bases $\{\phi_1, \phi_2\}$.

Solution: We seek Linear functionals

$\phi_1(x, y) = ax + by$ and $\phi_2(x, y) = cx + dy$ such that

$\phi_1(x_1) = I, \phi_1(x_2) = 0, \phi_2(x_1) = 0, \phi_2(x_2) = 1$.

Thus $\phi_1(x_1) = \phi_1(2, 1) = a . 2 + b . 1 = I$

$\phi_1(x_2) = \phi_1(3, 1) = a . 3 + b . I = 0$

both expressions give $a = -1, b = 3$.

$\phi_2(x_1) = \phi_2(2, 1) = c . 2 + d . 1 = 0$

$\phi_2(x_2) = \phi_2(3, 1) = c . 3 + d . 1 = 1$

these both expressions give $c = 1$ or $d = -2$.

Hence the dual basis is $\{\phi_1(x; y) = -x + 3y, \phi_2(x, y) = x - 2y\}$.

Theorem 10.53: Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of V and let $(\phi_1, \phi_2, \ldots \phi_n\}$ be the dual basis of V^* then any vector $x \in V$,

$$x = \phi_1(x)x_1 + \phi_2(x)x_2 + \ldots + \phi_n(x)x_n. \quad \ldots (1)$$

and, for any linear functional $\sigma \in V^*$,

$$\sigma = \sigma(x_1)\phi_1 + \sigma(x_2)\phi_2 + \ldots + \sigma(x_n)\phi_n \quad \ldots (2)$$

Proof: Since $\{x_1, x_2, \ldots, x_n\}$ is a basis, then any $x \in V$ can be written as $x = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ for some $a_i \in F$.

Then $\phi_i(x) = \phi_i(a_1x_1 + . + a_ix_i + \ldots + a_nx_n)$

$= a_1\phi_i(x_1) + \ldots a_i\phi_i(x_i) + \ldots + a_n\phi_i(x_n)$

$= a_1 0 + \ldots + a_i I + \ldots a_n 0 = a_i$

that is, $\phi_1(x) = a_1, \phi_2(x) = a_2, \ldots \phi_n(x) = a_n$.

Hence $x = \phi_1(x)x_1 + \phi_2(x)x_2 + \ldots + \phi_n(x)x_n$.

Again let $\sigma \in V^*$ then

$$\begin{aligned}\sigma(x) &= \sigma(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n)\\ &= a_1 \sigma(x_1) + a_2 \sigma(x_2) + \ldots + a_n \sigma(x_n)\\ &= \phi_1(x)\sigma(x_1) + \phi_2(x)\sigma(x_2) + \ldots + \phi_n(x)\sigma(x_n).\\ &= \sigma(x_1)\phi_1(x) + \sigma(x_2)\phi_2(x) + \ldots + \sigma(x_n)\phi_n(x).\end{aligned}$$

Since $\sigma(x_i) \in F$ and $\phi_i(x) \in F$, and multiplication in F is commutative.

$$= (\sigma(x_1)\phi_1 + \sigma(x_2)\phi_2 + \ldots + \sigma(x_n)\phi_n)(x)$$

which implies

$$\sigma = (\sigma(x_1))\phi_1 + (\sigma(x_2))\phi_2 + \ldots + (\sigma(x_n))\phi_n.$$

This completes the proof of the theorem.

Theorem 10.54: Let $V(F)$ be a finite-dimensional vector space over F. Then there exists $\phi \in V^*$ such that $\phi(x) \neq 0$, for any $x \in V$.

Proof: Since $x \neq 0$, $\{x\}$ is $L.I$ subset of V. So it can be extended to a basis $\{x = x_1, x_2, \ldots, x_n\}$ of V. Let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be a basis of V^* dual to the basis $\{x_1, x_2, \ldots, x_n\}$. Then by definition $\phi_1(x_1) = 1$. Thus taking $\phi = \phi_1$, we get $\phi(x) \neq 0$,

Since

$$\begin{aligned}\phi(x) = \phi(a_1 x_1 + a_2 x_2 + \ldots + a_n x_n) &= \phi_1(a_1 x_1 + a_2 x_2 + \ldots\ldots + a_n x_n)\\ &= a_1\phi_1(x_1) + a_2\phi_1(x_2) \ldots\ldots + a_n\phi_1(x_n)\\ &= a_1 + a_2\, 0 + \ldots + a_n.0\\ &= a_1 \neq 0\end{aligned}$$

We have seen that every space $V(F)$ has its dual space $V^*(F)$. So cosequently the dual space will have its dual space We shall denote the dual space of $V^*(F)$ by $V^{**}(F)$ and refer to it as the *second dual* of $V(F)$ or the bidual space of $V(F)$. That is, V^{**} is the set of linear functionas on $V^*(F)$ which map $f \in V^*$ onto some $k \in F$. Thus any mapping $f \in V^*$ is a linear functional on V and at the same time f is a vector for finding a linear functional on V^*, More over we know that dim V = dim V^*. Similarly, dim V^* = dim V^{**}. Which results that dim V = dim V^{**} and by the corollary of the Theorem 10.22 $V(F) \cong V^{**}(F)$.

For any vector $x \in V$ and $f \in V^*$, $f(x)$ is scaler (here it is not a function of x). In this case x ranges over V and f is fixed. Now we keep x fixed and allow f to range over V^*.

Now we specially define the function

$T_x : V^* \to F$ by

$T_x(f) = f(x), f \in V^*$.

To see T_x is linear, we have, for $f, g \in V^*$, $a, b \in F$

$$\begin{aligned}T_x(af + bg) &= (af + bg)(x)\\ &= af(x) + bg(x)\\ &= aT_x(f) + bT_x(g).\end{aligned}$$

which means T_x is linear functional on V^* and so $T_x \in V^{**}(F)$. T_x, defined in this way, is called the evalution functional induced by the vector x.

Theorem 10.56: If $V(F)$ is a finite-dimensional vector space, then $V(F) \cong V^{**}(F)$ by the mapping

$\phi : V \rightarrow V^{**}$ defined by $\phi(x) = T_x$.

Proof: First we show that ϕ is Linear. For this, let $x, y \in V$ and $a, b \in F$, then $\phi(ax + by) = T_{ax+by}$.

Now for any $f \in V^*$, we have

$$\begin{aligned} T_{(ax+by)}(f) &= f(ax + by) \\ &= af(x) + bf(y) \\ &= aT_x(f) + bT_y(f) \\ &= (aT_x + bT_y)(f) \end{aligned}$$

which implies $T_{(ax+by)} = aT_x + bT_y$.

Thus $\phi(ax + by) = T_{(ax+by)} = aT_x + bT_y = a\phi(x) + b\phi(y)$.

This shows that ϕ is linear.

Now we shall see that ϕ is one-to-one function. For this,

$x \in Ker\,\phi \Rightarrow T_x(f) = 0,\ f \in V^*$

$\Rightarrow f(x) = 0,\ f \in V^*$.

But by theorem 10.54 there exists $f \in V^*$ for which $f(x) \neq 0$ when $x \neq 0$, which implies that $x \in \ker(\phi) \Rightarrow f(x) = 0,\ f \in V^* \Rightarrow x = 0 \Rightarrow \ker\phi = \{0\}$.

Hence ϕ is one-to-one linear function. Since $\dim V = \dim V^{**}$, ϕ is also onto function which establishes that ϕ is an isomorphism from $V(F)$ onto $V^{**}(F)$. ϕ is called canonioal isomorphism.

Hence the theorem

Corallary: If $\dim V$ = finite, then each linear functional in V^{**} is of the form T_x for some unique $x \in V$.

Theorem 10.57: If $V(F)$ is a finite-dimensional vector space. If $\{\phi_i\}$ is the basis of $V^*(F)$ dual to a basis $\{x_i\}$ of $V(F)$, then $\{x_i\}$ is the basis of $V(F) = V^{**}(F)$ which is dual to $\{\phi_i\}$.

Proof: Let $\{\phi_1, \phi_2, \ldots\ldots \phi_n\}$ be a basis of $V^*(F)$. Then we can find a basis $\{T_1, T_2, \ldots, T_n\}$ which is dual to the basis $\{\phi_1, \phi_2 \ldots \phi_n\}$. This implies

$$T_i(\phi_j) = \delta_{ij} = \begin{array}{l} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array}$$

But according to the preceeding corollary, there exist vectors $x_1, x_2, \ldots\ldots x_n$ for which $Tx_1, Tx_2, \ldots\ldots, Tx_n \in V^{**}$

So we can have

$T_i(\phi) = T_{x_i}(\phi) = \phi(x_i)$, $\phi \in V^*$. In particular $\phi = \phi_j \in \{\phi_1, \phi_2, \ldots\ldots, \phi_n\} \subset V^*$, we have

$T_i(\phi_j) = \delta_{ij} = Tx_i(\phi_j) = \phi_j(x_i)$

Hence we have proved that

$\phi_i (x_j) = \delta_{ij} \ (i, j = 1, 2, \ldots\ldots n)$

where $\{x_1, x_2, \ldots\ldots x_n\}$ is a basis for $V(F)$ having $\{\phi_1, \phi_2, \ldots\ldots \phi_n\}$ as its dual.

Definition 10.58: Let $V(F)$ be a vectorspace over F. and $V^*(F)$ dual space of $V(F)$. For any subset W of V, if there exists $f \in V^*$ such that $f(x) = 0$ for every $x \in W$, i.e., $f(W) = 0$, then f is called an annihilator of W, the set $W^{\perp} = \{f \in V^* \mid f(x) = 0$, for all $x \in W\}$ is called the annihilator or orthogonal of W in V^*. For any set $T \subset V^*$, the set $T^{\perp} = \{x \in V \mid f(x) = 0, \forall f \in T\}$ is called the orthogonal or annihilator of T in V.

Lemma 10.59: Let $V^*(F)$ be a dual space of vector space $V(F)$ over F. For any subsets W and T of V and V^* respectively $W^{\perp}(F)$ and $T^{\perp}(F)$ are subspace of $V^*(F)$ and $V(F)$ respectively.

Proof: Since $V^*(F)$ is a vectorspace, there is zero function Z which maps every element of V onto 0, zero of F. i.e.,

$Z(x) = 0, \quad x \in V$. So $Z \in W^{\perp}$ and $W^{\perp} \neq \phi$.

Now for $f, g \in W^{\perp}$ and $a, b \in F$, we have

$(af + bg)(x) = af(x) + bg(x) = a.0 + b.0 = 0$. Since $f(x) = g(x) = 0, \forall x \in W$. Thus $af + bg \in W^{\perp}$ Hence $W^{\perp}(F)$ is a subspace of $V^*(F)$.

Again $f(0) = 0, \quad f \in T \Rightarrow 0 \in T^{\perp} \Rightarrow T^{\perp} \neq \phi$.

Let $x, y \in T^{\perp}$, $a, b \in F$, and $f \in T$. Then $f(x) = f(y) = 0$

by definition of $T^{\perp}$ and $f(ax + by) = af(x) + bf(y)$

$= a.0 + b.0 = 0 \Rightarrow ax + by \in T^{\perp}$.

This shows that $T^{\perp}(F)$ is a subspace of a vector space $V(F)$. This completes the proof.

Lemma 10.60: Let $V^*(F)$ be the dual space of a vectorspace $V(F)$ over F. Then

(*i*) For any subsets $W_1, W_2 \subseteq V$, $W_1 \subseteq W_2 \Rightarrow W_2^{\perp} \subseteq W_1^{\perp}$

(*ii*) For any subsets $T_1, T_2 \subseteq V^*$, $T_1 \subseteq T_2 \Rightarrow T_2^{\perp} \subset T_1^{\perp}$

(*iii*) For any subset $W \subseteq V$, $W \subseteq W^{\perp\perp}$,

(*iv*) For any subset $T \subseteq V^*$, $T \subseteq T^{\perp\perp}$

we leave the proof of (*II*) and *IV* as exercise. However, we shall prove (1) and (*III*).

Proof: (*i*) Let $\phi \in W_2^{\perp}$ Then $\phi(x) = 0, \ x \in W_2$. But $W_1 \subseteq W_2$ so for $x \in W_1, \Rightarrow \phi(x) = 0, \forall \ x \in W_1$. Hence $\phi \in {}_1^{\perp}$. Therefore ${}_2^{\perp} \subseteq W_1^{\perp}$.

(*ii*) Let $x \in W$. Then for every linear functional $f \in W^{\perp}$, $f(x) = 0 \Rightarrow T_x(f) = f(x) = 0, \forall f \in W^{\perp} \Rightarrow T_x \in W^{\perp\perp}$. Therefore, under the identification of $V(F)$ and $V^{**}(F)$, $x \in W^{\perp\perp}$ which implies $W \subseteq W^{\perp\perp}$.

Theorem 10.61: If $W(F)$ is a subspace of the finite dimensional vector space $V(F)$, the annihilator of W is the set $W^{\perp}$. Then

(*i*) $\dim W + \dim W^{\perp} = \dim V$,

(*ii*) $W^{\perp\perp} = W$.

Proof: (i) Let dim $V = n$ and dim $W = r$. Then there are r independent vectors $x_1, x_2, \ldots, x_r$ which form the basis of $W(F)$. This basis $\{x_1, x_2, \ldots, x_r\}$ of W can be extended to a basis $\{x_1, x_2, \ldots, x_r, y_1, \ldots, y_{n-r}\}$ of $V(F)$.

Let $\{\phi_1, \phi_2, \ldots \phi_r, f_1, f_2, \ldots f_{n-r}\}$ be a dual basis of $\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots y_{n-r}\}$, the bases of $V(F)$. Then $\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j, \end{cases}$

and $f_r(x_s) = \delta_{rs} \;\; = 1$ if $r = s$

$\qquad\qquad\qquad\quad 0$ if $r \neq s$

Thus it is clear that each $f_r(x_i) = 0, \quad x_i \in \{x_1, x_2, \ldots, x_r\}$. Hence $f_1, f_2, \ldots, f_n \in W^{\perp}$. Now claim that $\{f_j\}$ is a basis of $W^{\perp}(F)$. It is obvious that $\{f_j\}$ is a part of the basis of $V^*(F)$ and so it is $L.I.$ To show that $\{f_j\}$ is a basis of $W^{\perp}(F)$, there remains to show that $\{f_j\}$ spans $W^{\perp}(F)$. For this, Let $f \in W^{\perp}$. By theorem 10.53, we have,

$f = f(x_1)\,\phi_1 + f(x_2)\,\phi_2 + \ldots + f(x_r)\,\phi_r$

$+ f(y_1) f_1 + f(y_2) f_2 + \ldots + f(y_{n-r}) f_{n-r}$

$= 0\,\phi_1 + 0\,\phi_2 + \ldots + 0.\phi_r + f(y_1) f_1 + f(y_2) f_2$

$+ \ldots\ldots + f(y_{n-r}) f_{n-r}$

$= f(y_1) f_1 + f(y_2) f_2 + \ldots + f(y_{n-r}) f_{n-r}$

Thus $\{f_1, f_2, \ldots, f_{n-r}\}$ spans $W^{\perp}$, which follows that dim $W^{\perp} = n - r =$ dim V - din W or dim W+ dim $W^{\perp}$ = dim V.

This completes the first part of the theorem.

(ii) Let dim $V = n$ and dim $W = r$, then dim V = dim $V^* = n$. by (1) dim $W^{\perp} = n - r$. Thus again by (1) dim $W^{\perp\perp} = n - (n - r) = r$ which implies dim W = dim $W^{\perp\perp}$

By the Lemma 10.60: $W \subseteq W^{\perp\perp}$, which shows $W = W^{\perp\perp}$. Hence the theorem.

Theorem 10.62: Let $W_1(F)$ and $W_2(F)$ be two subspaces of a finite dimensional vector space $V(F)$, then

(a) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(b) $(W_1^{\perp} + W_2^{\perp}) = (W_1 \cap W_2)^{\perp}$

Proof: $f \in (W_1^{\perp} + W_2^{\perp}) \Rightarrow f(x) = 0 \quad x \in W_1 + W_2$

$\Rightarrow f(w_1 + w_2) = 0$ Since $x = w_1 + w_2$, $w_1 \in W, w_2 \in W_2$

$\Rightarrow f(w_1) + f(w_2) = 0$, $w_1 \in W_1$, $w_2 \in W_2$

$\Rightarrow f(w_1) = 0$ and $f(w_2) = 0$ since $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$

$\Rightarrow f \in W^{\perp}_1$ and $f \in W_2^{\perp}$

$\Rightarrow f \in \;{}_1^{\perp} \cap W_2^{\perp}$

Thus $(W_1^{\perp} + W_2^{\perp}) \subseteq W_1^{\perp} \cap W_2^{\perp}$

Conversely, $g \in W_1^{\perp} \cap W_2^{\perp} \Rightarrow g \in W_1^{\perp}$ and $g \in W_2^{\perp}$

$\Rightarrow g(w_1) = 0$ and $g(w_2) = 0$

$w_1 \in W_1$ and $w_2 \in W_2$

$\Rightarrow g(w_1) + g(w_2) = 0,\ w_1 \in W_1, w_2 \in W_2$

$\Rightarrow g(w_1 + w_2) = 0,\ \forall\ w_1 \in W_1, W_2 \in W_2$

$\Rightarrow g \in (W_1 + W_2)^{\perp}$ since $w_1 + w_2 \in W_1 + W_2$

Thus $W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$.

From (1) and (2) we have

$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

(b) We have that $W_1^{\perp}(F)$ and $W_2^{\perp}(F)$ are subspaces of $V^*(F)$. Now by (a) we obtain,

$$(W_1^{\perp} + W_2^{\perp}) = (W_1^{\perp})^{\perp} \cap (W_2^{\perp})^{\perp} = W_1 \cap W_2$$

Thus $$((W_1^{\perp} + W_2^{\perp})^{\perp})^{\perp} = (W_1 \cap W_2)^{\perp} \Rightarrow W_1^{\perp} + W_2^{\perp} = (W_1 \cap W_2)^{\perp}.$$

Hence the theorem

N

10.6 TRANSPOSE OF A LINEAR MAPPING

Definition 10.63: Let $f: V \to U$ be a Linear mapping from a vector space $V(F)$ into a vector space $U(F)$. Now for any linear functional $\phi \in U^*$, the composition $(\phi\ o f)$ is a linear mapping from $V(F)$ to F,

That is, $\phi\ o f \in V^*$. Thus the correspondence $\phi \to (\phi\ o f)$ is a mapping from $U^*(F)$ into $V^*(F)$, we denote it by f^{T} and call it transpose of f. Thus $f^{\mathrm{T}} : U^* \to V^*$ defined by

$f^{\mathrm{T}}(\phi) = \phi\ o f,\ \phi \in U^*$.

i.e., $(f^{\mathrm{T}}(\phi)(x) = \phi(f(x)),\ \forall\ x \in V$

N *Theorem 10.64:* Let $f: V \to U$ be linear mapping and let $f^{\mathrm{T}} : U^* \to V^*$ be its transpose. Then

(*i*) f^{T} is a linear mapping from $U^*(F)$, into $V^*(f)$

(*ii*) $\ker(f^{\mathrm{T}}) = f(V)^{\perp}$

Proof: (1) For any scalars $a, b \in F$ and for any linear functionals $\phi, \sigma \in U^*$,

$$\begin{aligned} f^{\mathrm{T}}(a\phi + b\sigma) &= (a\phi + b\sigma)\ o f \\ &= a(\phi\ o f) + b(\sigma\ o f) \\ &= af^{\mathrm{T}}(\phi) + bf^{\mathrm{T}}(\sigma). \end{aligned}$$

Thus f^{T} is Linear function from $U^*(F)$ into $V^*(F)$.

(*ii*) Let $\phi \in \mathrm{Ker}(f^{\mathrm{T}})$. That is, $f^{\mathrm{T}}(\phi) = \phi\ o f = 0 \in V^*$

If $u \in f(V)$, then $u = f(x)$ for some $x \in V$.

Thus we have $\phi(u) = 0$ for every $u \in f(V)$ which means $\phi \in f(V)^{\perp}$. Hence $\ker(f^{\mathrm{T}}) \subseteq f(V)^{\perp}$. Conversely, let $\sigma \in f(V)^{\perp}$. Then $\sigma(y) = 0,\ \forall\ y \in f(V)^{\perp}$ which implies $\sigma(f(V)) = \{0\}$. Then for every $x \in V$, $(f^{\mathrm{T}}(\sigma))(x) = (\sigma\ o f)(x) = \sigma(f(x)) = 0 = 0(x)$

which implies $(f'(\sigma))(x) = 0(x) \Rightarrow f'(\sigma) = 0$

$\Rightarrow \sigma \in \ker(f')$.

Therefore $f(V)^{\perp} \subseteq \ker(F')$

From (1) and (2) we conclude that

$\ker(f') = f(V)^{\perp}$.

Theorem 10.65: Let $f: V \to W$ be linear and let $V(F)$ and $W(F)$ be finite-dimensional vector spaces. Then rank (f) = rank (f').

Proof: Let dim $V = n$ and dim $W = m$ and also Let rank $(f) = \dim f(V) = r$. Now by theorem 10.62, we have

$$\dim W = \dim(f(V)) + \dim(f(V)^{\perp},$$

$$\text{or } \dim f(V)^{\perp} = \dim W - \dim f(V)$$
$$= m - r.$$

By the theorem 10.64, we have

$\ker(f') = f(V)^{\perp} \Rightarrow$ mullity $(f') = m - r$.

By sylvesters Law, we obtain

dim $W^* = \dim(\ker(f')) + \dim(f'(W^*))$

or $\dim(f'(W)^*) = \dim W^* - \dim \ker(f')$.

Since dim $W = m = \dim W^*$, then

rank $(f') = m - (m - r) = r =$ ran $k(f)$. Hence the theorem.

Theorem 10.66: If $V = U \oplus W$, then the function $f_w : V \to W$ which assigns to each vector $x \in V$ its uniquely determined component x_2 in the representation $x = x_1 + x_2$, $(x_1 \in U, x_2 \in W)$ is an idempotent Linear mapping with Ker $(f_w) = U$, $f_w(V) = W$.

Conversely, every idempotent Linear mapping $f \in L(V, V)$ defines a direct sum decomposition $V = \ker(f) \oplus f(V)$.

Proof: First we show that $f_w \in L(V, V)$. For this, suppose $x, y \in V$, then $x = u_1 + w_1$ and $y = u_2 + w_2$ where $u_1, u_2 \in U$, $w_1, w_2 \in W$. For $a, b \in F$, we have

$$\begin{aligned} ax + by &= a(u_1 + w_1) + b(u_2 + w_2) \\ &= (au_1 + bu_2) + (aw_1 + bw_2) \in V, \text{ where} \end{aligned}$$

$au_1 + bu_2 \in U$ and $aw_1 + bw_2 \in W$.

$$\begin{aligned} \text{So } f_w(ax + by) &= f_w((au_1 + bu_2) + (aw_1 + bw_2)) \\ &= aw_1 + bw_2 = af_w(x) + bf_w(y). \end{aligned}$$

which shows $f_w \in L(V, V)$.

We observe something more:

Since $u = u + 0$ and $w = 0 + w$ as every element of V is uniquely expressed as sum of elements of U and W respectively. Then we have

(i) $f_w(u) \quad = f_w(u + 0) = 0$ if and only if $u \in U$,

(ii) $f_w(w) \quad = f_w(0 + w) = w$ if and if $w \in W$.

Now suppose $x = u + w$, with $u \in U$, $w \in W$, then

$$f_w^2(x) = f_w(f_w(x)) = f_w(f_w(u + w)) = f_w(w) = w = f_w(x)$$

$f_w^2 = f_w$. Hence f_w is idempotent.

For the converse. Let $f \in L(V, V)$ with $f^2 = f$. Taking $u = x - f(x)$ and $w = f(x)$, we may write $x \in V$ as $x = u + w$. Since f is idempotent, then

$$f^2(x) = f(x) \Rightarrow f(x) - f^2(x) = 0 = f(u)$$

while $f(w) = f^2(x) = f(x) = w$.

This shows that $V = \ker(f) + W$, where the set $W = \{x \in V \mid f(x) = x\}$.

Now we have to show that $V = \ker(f) + W$ is direct sum, that is, $\ker(f) \cap W = \{0\}$.

if $x \in \ker(f)$, then $f(x) = 0$, where as if $x \in W, f(x) = x$. if $x \in \ker(f)$ and $x \in W$, $f(x) = 0$. By def. of W, it is clear that $W \subseteq f(V)$. For the reverse inclusion, we have $f^2 = f \Rightarrow f(x) \in W, \forall\ x \in V \Rightarrow f(V) \subseteq W$. Here $W = f(V)$. This completes the proof of the theorem

Example 10.67: Let $f: V_4 \to V_4$ be defined by

$f(a_1, a_2, a_3, a_4) = (a_1, a_2, 0, 0)$.

Then $\ker(f) = \{(0, 0, a_3, a_4) \mid a_3, a_4 \in F\}$

and $f(V) = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in F\}$.

Now $\ker f \cap f(V_4) = \{(0, 0, 0, 0)\}$

and any $(a_1, a_2, a_3, a_4) \in V$ can be written as $(a_1, a_2, a_3, a_4) = (0, 0, a_3, a_4) + (a_1, a_2, 0, 0)$

$\in \ker(f) + f(V_4)$.

and
$$\begin{aligned} f^2(a_1, a_2, a_3, a_4) &= f(f(a_1, a_2, a_3, a_4)) \\ &= f(a_1, a_2, 0, 0) \\ &= (a_1, a_2, 0, 0) = f(a_1, a_2, a_3, a_4) \end{aligned}$$

Hence f is idempotent.

Theorem 10.68: Let $U(F)$ and $W(F)$ be complementary subspaces of a finite dimensional space $V(F)$, i.e., $V \neq U \oplus W$. Then

(*i*) $U^*(F) \cong W^\perp(F)$, $U^\perp(F) \cong W^*(F)$.

(*ii*) $V^* = U^\perp + W^\perp$

Proof: (*i*) Since $V(F)$ is finite dimensional, and $V = U \oplus W$, then $\dim V = \dim U + \dim W$.

Let $\dim V = n$, $\dim U = m$. So
$$\begin{aligned} \dim W &= \dim V - \dim U \\ &= n - m. \end{aligned}$$

$\dim W + \dim W^\perp = \dim V \Rightarrow$
$$\begin{aligned} \dim W^\perp &= \dim V - \dim W \\ &= n - (n - m) \\ &= m = \dim U. \end{aligned}$$

Since $\dim W^\perp = \dim U$, then $W^\perp(F) \cong U(F)$

Since $\dim U = \dim U^*$, so $W^\perp(F) \cong U^*(F)$.

Again $\dim U + \dim U^\perp = \dim V \Rightarrow$
$$\begin{aligned} \dim U^\perp &= \dim V - \dim U \\ &= n - m \\ &= \dim W. \end{aligned}$$

But $\dim W = \dim W^*$, so $\dim U^\perp = \dim W^*$

which implies $U^\perp(F) \cong W^*(F)$.

(*ii*) For proving $V^* = U^\perp + W^\perp$, we shall show that $U^\perp \cap W^\perp = \{0\}$, where 0 is a zero function.

Now $f \in U^{\perp} \cap W^{\perp}$ $\Rightarrow f \in U^{\perp}$ and $f \in W^{\perp}$

$\Rightarrow f(u) = 0, \; u \in U$ and $f(w) = 0 \; w \in W$.

$\Rightarrow f(u) + f(w) = 0, \; u \in U$ and $w \in W$

$\Rightarrow f(u + w) = 0 \; u \in U$ and $w \in W$, since f is linear

$\Rightarrow f(V) = 0$ Since $v = u + w$ as $V = U \oplus W$

$\Rightarrow f$ is a zero function.

$\Rightarrow U^{\perp} \cap W^{\perp} = \{0\}$, where 0 is the zero function.

Now there remains to show that every element of $f \in V^*$ is uniquely expressible as $f = f_u + f_w$, where $f_u \in W^{\perp}$ and $f_w \in U^{\perp}$.

Since $V = U \oplus W$, any $x \in V$ can be written as

$x = u + w, \; u \in U, \; w \in W$.

Now define $f_u : V \rightarrow K$ and $f_w : V \rightarrow K$ by

$f_u(x) = f(u)$ and $f_w(x) = f(w)$

So $f_u, f_w \in V^*$

For $x, y \in V$, $x = u_1 + w_1$, $y = u_2 + w_2$ and for $a, b \in F$,

$f_u(a x + by) = f_u(au_1 + bu_2 + aw_1 + bw_2)$

where $a u_1 + bu_2 \in U$, $aw_1 + bw_2 \in W$.

$$= f(au_1 + bu_2)$$

$$= af(u_1) + b f(u_2)$$

$$= a f_u(x) + b f_u(y)$$

which implies f_u is linear. Similarly, f_w can be shown linear.

Since $u = u + 0$, $w = 0 + w$

Now for $w \in W$,

$f_u(w) = f_u(0 + w) = f(0) = 0 \Rightarrow f_u \in W^{\perp}$

and for $u \in U$

$f_w(u) = f_w(u + 0) = f(0) = 0 \Rightarrow f_w \in U^{\perp}$

And $(f_u + f_w)(x) = f_u(x) + f_w(x)$

$$= f_u(u + w) + f_w(u + w)) = f(u) + f(w)$$

$$= f(u + w) = f(x)$$

which implies $f_u + f_w = f$.

This completes the proof of the theorem.

10.7 MATRICES AND LINEAR TRANSFORMATIONS

Let $U(F)$ and $V(F)$ be two vector spaces over the field $(F, +, .)$ and let $T: U \rightarrow V$ be a linear transformation. Again let $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ be the bases of $U(F)$ and $V(F)$ respectively. We know that T is determined on any vector as soon as we know its action on a basis of U. Since T maps U into V, $T(u_1), T(u_2), \ldots, T(u_m)$ will belong to V. Since every element of V can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$ in a unique way. Thus:

$T(u_1) = a_{11} v_1 + a_{12} v_2 + \ldots + a_{1n} V_n$

$T(u_2) = a_{21} v_1 + a_{22} v_2 + \ldots + a_{2n} V_n$

$T(u_m) = e_{m1} V_1 + a_{m2} V_2 + ... + a_{m2} V_n$,
where each $a_{ij} \in F$. This system of equation can be written as

$T(u_i) = \sum_{j=1}^{n} a_{ij} V_j$, for $i = 1, 2, ..., m$.

The ordered set of $m.n.$ numbers of F completely describes T. They will serve the purpose of representing T.

Definition 10.69: Let $U(F)$ and $V(F)$ be two vector spaces of dimansion m and n over field $(F, +, .)$ respectively and Let $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ be bases of $U(F)$ and $V(F)$ respectively. If $T \in Hom(U, V)$, then the matrix of T or $\{T(u_1), T(u_2), ... T(u_m)\}$ relative to the basis $\{v_1, v_2, ..., v_n\}$, written as $m(T)$, is

$$m(T) = \begin{bmatrix} a_{11} & a_{12} & ...... & a_{1n} \\ a_{21} & a_{12} & ...... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & ...... & a_{mn} \end{bmatrix} = (a_{ij})$$

where $T(u_i) = \sum_{j=1}^{n} a_{ij} v_j$, $i = 1, 2, 3, ..., m$.

If $U = V$, $u_i = v_i$, $1 \leq i \leq n$, then the square matrix $m(T)$ of order n is called the matrix of T relative to the basis $\{v_1, v_2, ..., v_n\}$.

Theorem 10.70: Let $U(F)$ and $V(F)$ be two vector spaces of dimension m and n over the field $(F, +, .)$ respectively. Then Hom $(U, V) \cong M_{m \times n}(F)$, where $M_{m \times n}$ is a vector space of all $m \times n$ matrices over $(F, +, .)$.

Proof: Let $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ be bases of the vector spaces $U(F)$ and $V(F)$ respectively. Let $T \in$ Hom (U, V), then

$$T(u_i) = \sum_{j=1}^{n} a_{i_j} v_j, \quad for\ i = 1, 2, ..., m$$

When a vector space is defined on a field $(F, +, .)$ then there is no distinction between left and right scalar multiplication of a vector. So

$$T(u_i) = \sum_{j=1}^{n} v_j a_{i_j}, \quad for\ i = 1, 2, ..., m.$$

and $[a_{ij}]$ is an $m \times n$ matrix of T.

Let $T, T' \in$ Hom (U, V). Then

$$T(u_i) = \sum_{j=1}^{n} a_{i_j} v_j, \quad and \quad T^1(u_i) = \sum_{j=1}^{n} b_{ij} v_j$$

for $i = 1, 2, ..., m$, and a_{ij} and $b_{ij} \in F$.

Since T and T' are Linear transformation then

$$(T+T')(u_i) = T(u_i) + T'(u_i) = \sum_{j=1}^{n} a_{ij} v_j + \sum_{j=1}^{n} b_{ij} v_j$$

$$= \sum_{j=1}^{n} (a_{ij} + b_{ij}) v_j$$

Thus if $[a_{ij}]$ and $[b_{ij}]$ are $m \times n$ matrices of T and T' respectively, $[a_{ij} + b_{ij}]$ is an $m \times n$ matrix of $T + T'$.

For any $c \in F$, we have

$$(cT)(u_i) = cT(u_i) = c \sum_{j=1}^{n} a_{ij} v_j$$

$$= \sum_{j=1}^{n} (ca_{ij}) v_j$$

which implies $[ca_{ij}]$ is an $m \times n$ matrix of the linear transformation (cT).

Let $M_{m \times n}(F)$ be a vector space of all $m \times n$ matrices over the field $(F, +, .)$.

We define ϕ : Hom $(U, V) \longrightarrow M(F)$ by $\phi(T) = [a_{ij}]$. Then we see that

$\phi(T + T') = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}] = \phi(T) + \phi(T')$

and $\phi(cT) = [ca_{ij}] = c(a_{ij}] = c\,\phi(T)$.

It shows that ϕ is a Linear transformation.

Now, for any T and T', we have

$\phi(T) = \phi(T') \Rightarrow [a_{ij}] = [b_{ij}]$

$\Rightarrow a_{ij} = b_{ij}$, for all i, j

$\Rightarrow \Sigma a_{ij} v_j = \Sigma b_{ij} v_j \Rightarrow T(u_i) = T'(u_i) \Rightarrow T = T'$.

Again for any matrix $[a_{ij}] \in M_{m \times n}(F)$ there exists a linear transformation T such that

$\phi(T) = [a_{ij}]$. Hence ϕ is onto. This proves the theorem.

N **Corollary:** Hom $(V, V) \cong M_n(F)$, the algebra of all matrices over the field $(F, +, .)$.

Proof: In the above theorem we have seen that there is an isomorphism ϕ : Hom $(V, V) \longrightarrow M_n(F)$ defined by $\phi(T) = [a_{ij}]$. $[a_{ij}]$ is a square matrix of order n and such that

$$T(v_i) = \sum_{j=1}^{n} a_{ij} v_j, \text{ for any } T \in \text{Hom}(V, V).$$

Let $T' \in$ Hom (V, V) and $T'(v_i) = \sum_{j=1}^{n} b_{ij} v_j$

$$\text{Then } (TT')(V_i) = T(T'(V_i)) = T\left(\sum_{j=1}^{n} b_{ij} v_j\right)$$

$$= T\left(\sum_{j=1}^{n} v_j\, b_{ij}\right)$$

$$= \sum_{j=1}^{n} T(v_j) b_{ij}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v_i \; a_{ki} \right) b_{ij}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ki} \; b_{ij} \; v_i = \sum_{i=j}^{n} C_{kj} \; V_j$$

Where $c_{k_j} = \sum_{i=1}^{n} a_{ki} \, b_{ij}$

Hence $\phi \; (T.T') = \phi\,(T)\;\phi\,(T')$

This proves the corollary.

PROBLEMS

1. Let $V(F)$ be a finite dimensional vector space over F. Prove that
   (a) if $x_1, x_2 \in V$ with $x_1 \neq x_2$, then there exists a linear functional $f \in V^*$ for which $f(x_1) \neq f(x_2)$.
   (b) If $W(F)$ is proper non-zero subspace of $V(F)$ and the vector $x_o \notin W$, then there exists some $f \in V^*$ such that $f(x_o) = 1, f(x) = 0, \; \forall \, x \in W$.
3. If $W(F)$ is subspace of the vector space $V(F)$, and $W^{\perp}$ is the annihitator of W, then prove that
   $(V/W)^* \, (F) \equiv W^{\perp} \, (F)$ and
   $(V / W^{\perp}) \, (F) \equiv W^* \, (F)$.
4. For each linear mapping $f \in L(V, W)$ there is a transpose f^T of f. Then prove that
   $L(V, W) \equiv L(W^*, V^*)$ under the mapping that sends each function $f \in L(V, W)$ to its transpose f^T.
5. Let $\{x_1, x_2, ..., x_n\}$ be a basis for finite dimensional vector space $V(F)$, and let $\{f_1, f_2, ..., f_n\}$ be the corresponding dual basis for $V^*(F)$. Suppose that $[a_{ij}]$ is the representing matrix, relative to $\{x_1, x_2, ..., x_n\}$, of linear mapping $f \in L(V, V)$. Prove that the transpose f^T of f is represented by the matrix $[a_{ij}]^T$ relative to $\{f_1, f_2, ... f_n\}$.
6. If the functionals $f, g \in V^*$ are such that $\ker.(f) \subseteq \ker(g)$, prove that there exists a scaler a for which
   $f = ag$.
7. Suppose $f\!: U \to V$ and $g \,!\, V \to W$ are linear. Prove that $(g \, o \, f^T = f^T \, o \, g^T$.
8. Let $V(F)$ be a vector space over F. Let $\phi_1, \phi_2, \in V^*$ and suppose $f: V \to R$ sdefined by
   $f(v) = \phi_1(v).\ \phi_2(v)$ also belongs to V^*, show that either $\phi_1 = 0$ or $\phi_2 = 0.\Delta$