VECTOR SPACE

$(V,+)$ & $(F,+,\cdot)$

$(V,+)$ → Commutative Group

$(F,+,\cdot)$ → set of Scalars (Field)

Operat^n's across above 2 sets → Scalar Multiplicat^n

(1) Definition Let $(F,+,\cdot)$ be a given field whose elements are c/a scalars. The set V is a non-void set whose elements are c/a vectors.

The set $(V,\oplus)$ as a vector space over the field F, if the following ~~actions~~ axioms are specified:

V_1: For any 2 vectors, $\bar\alpha,\bar\beta \in V$, $\bar\alpha \oplus \bar\beta \in V$.

V_2: —"— , $\bar\alpha \oplus \bar\beta = \bar\beta \oplus \bar\alpha$

V_3: For any 3 vectors, $\bar\alpha,\bar\beta,\bar\gamma \in V$, $\bar\alpha \oplus (\bar\beta \oplus \bar\gamma) = (\bar\alpha \oplus \bar\beta) \oplus \bar\gamma$

V_4: There exists → $\exists$ unique vector $\bar\phi \in V$, s.t. $\bar\alpha \oplus \bar\phi = \bar\alpha = \bar\phi + \bar\alpha$ $\forall\, \bar\alpha \in V$.

V_5: For any vector $\bar\alpha \in V$, $\exists$ a vector $-\bar\alpha \in V$ s.t. $\bar\alpha \oplus (-\bar\alpha) = (-\bar\alpha) \oplus \bar\alpha = \bar\phi$

$(F,+,\cdot)$ (10) → (5) $(F,+)$ → Commutative group
→ (5) $(F-\{0\},\cdot)$ → —"—

$\bar\alpha + (-\bar\alpha) = \bar 0$

$\bar\phi + \bar\alpha = \bar\alpha$ $\quad (a+0)\bar\alpha = a\bar\alpha$

(V6) For any element $a \in F$ and any vector $\bar{\alpha} \in V$. $a\bar{\alpha} \in V$.

(V7) For any element $a \in F$ and any vector $\bar{\alpha}, \bar{\beta} \in V$, $a(\bar{\alpha} \oplus \bar{\beta}) = a\bar{\alpha} \oplus a\bar{\beta}$.

(V8) For any 2 scalars $a, b \in F$, and any vector $\bar{\alpha} \in V$, $(a+b)\bar{\alpha} = a\bar{\alpha} \oplus b\bar{\alpha}$.

(V9) For any 2 scalars $a, b \in F$ and any vector $\bar{\alpha} \in V$ $(a \cdot b)\bar{\alpha} = a(b\bar{\alpha})$

(V10) For the unit scalar $1 \in F$ and any vector $\bar{\alpha} \in V$, $1\bar{\alpha} = \bar{\alpha}$.

$V_1 - V_5 \rightarrow$ Internal Operations
$V_6 - V_{10} \rightarrow$ External —"—

$V(F)$ → $\mathbb{R}$, real vector space.
$V(F)$ → $\mathbb{C}$; complex vector space

Q. If $V(F)$ and $\bar{\alpha}, \bar{\beta}, \bar{\phi} \in \bar{V}$ and $0, a, b \in F$ then show that

(i) $a\bar{\phi} = \bar{\phi}$ (ii) $0\bar{\alpha} = \bar{\phi}$ (iii) $a(-\bar{\alpha}) = -(a\bar{\alpha})$

(iv) $(-a)\bar{\alpha} = -(a\bar{\alpha})$ (v) $a(\bar{\alpha} \oplus (-\bar{\beta})) = a\bar{\alpha} \oplus (-a\bar{\beta})$

(vi) $a\bar{\alpha} = \bar{\phi} \Rightarrow a = 0$ or $\bar{\alpha} = \bar{\phi}$

(vii) $a\bar{\alpha} = b\bar{\alpha} \Rightarrow a = b$

(viii) $a\bar{\alpha} = a\bar{\beta} \Rightarrow \bar{\alpha} = \bar{\beta}$

(i) $a\bar{\phi} = \bar{\phi}$

$$a\bar{\phi} = a(\bar{\phi} \oplus \bar{\phi})$$

$$\cancel{a(\bar{\alpha}+\bar{\alpha})} = a\bar{\phi} \oplus a\bar{\phi} \quad \text{(Prop. V7)}$$

$$a\bar{\phi} = a\bar{\phi} \oplus \bar{\phi} = a\bar{\phi} \oplus a\bar{\phi}$$

Identity

Cancellation holds bcoz V is a group.

$$\Rightarrow \boxed{\bar{\phi} = a\bar{\phi}}$$

(ii) $0\bar{\alpha} = \bar{\phi}$

$$0\bar{\alpha} = (0+0)\bar{\alpha} = 0\bar{\alpha} \oplus 0\bar{\alpha}$$

(Prop. V8)

$$\bar{\phi} + 0\bar{\alpha} = 0\bar{\alpha} \oplus 0\bar{\alpha} = 0\bar{\alpha}$$

By Cancellation prop.

$$\Rightarrow \boxed{0\bar{\alpha} = \bar{\phi}}$$

(iii) $a(-\bar{\alpha}) = -(a\bar{\alpha})$

$$a(\bar{\alpha} \oplus (-\bar{\alpha})) = a\bar{\alpha} \oplus a(-\bar{\alpha})$$

$$\Rightarrow a\bar{\phi} = a\bar{\alpha} \oplus a(-\bar{\alpha})$$

$$\Rightarrow \bar{\phi} = a\bar{\alpha} \oplus a(-\bar{\alpha})$$

$\Rightarrow a(-\bar{\alpha})$ is inverse of $a\bar{\alpha}$.

& inverse of $a\bar{\alpha} = -(a\bar{\alpha})$. Thus,

$$-(a\bar{\alpha}) = a(-\bar{\alpha})$$

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(iv) $(a+(-a))\,\bar{\alpha} = a\bar{\alpha} \oplus (-a)\bar{\alpha}$

$-(1\bar{\alpha}) = (-1)\bar{\alpha} = a(-\bar{\alpha}))$

$\Rightarrow 0\,\bar{\alpha} = a\bar{\alpha} \oplus (-a)\,\bar{\alpha} = \bar{\phi}$

$\Rightarrow (-a\bar{\alpha})$ is inverse of $a\bar{\alpha}$ $\Rightarrow \boxed{-(a\bar{\alpha}) = (-a)\bar{\alpha}}$

(v) $a(\bar{\alpha} \oplus (-\bar{\beta})) = a\bar{\alpha} \oplus (-a\bar{\beta})$

$\Rightarrow a\bar{\alpha} \oplus a(-\bar{\beta}) = a\bar{\alpha} \oplus (-a\bar{\beta})$ $\boxed{\text{From (iv)}}$

$\boxed{-a\bar{\beta} = a(-\bar{\beta})}$

(vi) $\because a\bar{\alpha} = \bar{\phi}$

$\Rightarrow a\bar{\alpha} - a\bar{\phi} = 0 \Rightarrow a(\bar{\alpha} - \bar{\phi}) = 0$

$\Rightarrow \boxed{a = 0 \text{ or } \bar{\alpha} = \bar{\phi}}$

(vii) $a\bar{\alpha} = b\bar{\alpha} \Rightarrow (a-b)\,\bar{\alpha} = 0$

$\Rightarrow \boxed{a = b}$

(viii) $a\bar{\alpha} = a\bar{\beta} \Rightarrow \boxed{\bar{\alpha} = \bar{\beta}}$ (By Cancellat^{n} Prop.)

① $\bar{\alpha} \oplus \bar{\beta} \in S \quad \forall\ \bar{\alpha}, \bar{\beta} \in S$
② $c\bar{\alpha} \in S \quad \forall\ \bar{\alpha} \in S \ \&\ c \in F$

} Sufficient for subspace.

S1 gives $\boxed{(S2, S3)}$
Commutative, Associative.

If $c\bar{\alpha} \in S$ & $c = -1 \Rightarrow$

$c\bar{\alpha} = (-1)\bar{\alpha} = -(1\bar{\alpha}) = -\bar{\alpha} \in S$

$\Rightarrow \bar{\alpha} \in S$ & $-\bar{\alpha} \in S \Rightarrow \bar{\alpha} \oplus (-\bar{\alpha}) \in S$

$\Rightarrow \boxed{\bar{\phi} \in S}$

$\checkmark \boxed{S4, S5}$

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Def^n S is a subspace of V iff $a\bar{\alpha} \oplus b\bar{\beta} \in S$ $\forall$ $\bar{\alpha}, \bar{\beta} \in S$ and $\forall$ $a, b \in F$.

If $a=1, b=1$ If $a=c$ & $b=0$.

Q. $V = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, x_i \in R \right\}$

$(V, \oplus) \to$ over matrix addition

Show that $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_4 \end{bmatrix}, x_i \in R \right\}$ is a subspace of V.

Sol. $\bar{\alpha} = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_4 \end{bmatrix}, \bar{\beta} = \begin{bmatrix} x_1' & x_2' \\ x_2' & x_4' \end{bmatrix}$

$$\begin{bmatrix} ax_1 + bx_1' & ax_2 + bx_2' \\ ax_2 + bx_2' & ax_4 + bx_4' \end{bmatrix} \in \begin{bmatrix} \cancel{X_1} & \cancel{X_2} \\ X_3 & X_4 \end{bmatrix}.$$

Q. Let $R^3 = \{(x_1, x_2, x_3) ; x_1, x_2, x_3 \in R\}$

$(R^3, \oplus)$ be a vector space over $(R, +, \cdot)$

$\oplus \to (x_1, x_2, x_3) \oplus (y_1, y_2, y_3)$
$= (x_1+y_1, x_2+y_2, x_3+y_3)$

Show that $S = \{(x_1, x_2, 0), x_i \in R\}$ is a subspace of R^3.

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$\cancel{f(g(a))}$

$h = f \circ g$ $\quad \forall a, b$

$h(a*b) = f(g(a*b)) = f(g(a) * g(b))$

$= \cancel{f(g(a)) * f(g(b))}$

$= h(a) * h(b)$

$[(f \circ g) \circ h](a) = [f \circ (g \circ h)](a) \quad \forall a$

$I(a) = a$

$(f \circ I)(a) = f(a)$

$f \circ I = f = I \circ f$

$(f+g) \circ h = T.$

$T(a) = (f+g)(h(a))$

$= f(h(a)) * g(h(a))$

$= (f \circ h)(a) * (g \circ h)(a)$

$= (f \circ h + g \circ h)(a)$

$\underbrace{f \circ (g+h)}_{T} = \underbrace{(f \circ g)}_{U} + \underbrace{(f \circ h)}_{V}$

$\cancel{f \circ (g+h)}$

$T(a) = f((g+h)(a)) = f(g(a) * h(a))$

$= f(g(a)) * f(h(a))$

$\cancel{((f \circ g) + (f \circ h))\, a}$

$= U(a) * V(a)$

$= (U+V)(a) \quad \forall a$

Theorem: The intersection of any 2 subspace of a vector space $V(F)$ is also sub space.

Proof: let $W_1(F)$ & $W_2(F)$ are 2 subspaces of $V(F)$.

$$\phi \in W_2, \ \phi \in W_1 \Rightarrow \phi \in \underbrace{W_1 \cap W_2}_{\text{Non empty set.}}$$

Case 1: $W_1 \cap W_2 = \{\phi\}$ ← Sub space.

Case 2: $W_1 \cap W_2 \neq \{\phi\}$

$\bar{\alpha}, \bar{\beta}$ belongs to W_1 & W_2.

$\Rightarrow \bar{\alpha}, \bar{\beta} \in W_1$ & $\bar{\alpha}, \bar{\beta} \in W_2$

$\Rightarrow a\bar{\alpha} + b\bar{\beta} \in W_1$ & $a\bar{\alpha} + b\bar{\beta} \in W_2$. (For any $a, b \in F$)

(as W_1 & W_2 are subspaces)

$\Rightarrow \boxed{a\bar{\alpha} + b\bar{\beta} \in W_1 \cap W_2}$ So, $W_1 \cap W_2$ is a subspace.

Theorem: If union of 2 sub spaces is a sub space iff one is contained in the other.

Proof: Let $W_1(F)$ & $W_2(F)$ be subspaces of $V(F)$.

$\Rightarrow W_1 \cap W_2 = \{\phi\}$

$\Rightarrow W_1 \cup W_2 = \{\phi\}$

If $W_1 \subset W_2$, $W_1 \cup W_2 = W_2$
If $W_2 \subset W_1$, $W_1 \cup W_2 = W_1$
} $\Rightarrow W_1 \cup W_2$ is a subspace as W_1 & W_2 is a subspace.

Let $W_1 \cup W_2$ is a subspace. ($W_1 \not\subset W_2$, $W_2 \not\subset W_1$)
let $\bar{\alpha} \in W_1$ & $\bar{\alpha} \notin W_2$ | $\bar{\beta} \notin W_1$ & $\bar{\beta} \in W_2$

$\because W_1 \cup W_2$ contains $a\bar{\alpha} + b\bar{\beta}$. $a, b \in F$ as $W_1 \cup W_2$ is a subspace.
So, $a\bar{\alpha} + b\bar{\beta} \in W_1$ or $a\bar{\alpha} + b\bar{\beta} \in W_2$.

$-a\bar{\alpha} + a\bar{\alpha} + b\bar{\beta} \in W_1$ (as $-a\bar{\alpha} \in W_1$) subspace.
$a(\bar{\alpha} + (-\bar{\alpha})) + b\bar{\beta}$
$a\,\bar{\phi} + b\bar{\beta}$
$\Rightarrow \cancel{\bar{\phi}} + b\bar{\beta} \in W_1$ But $\bar{\beta} \notin W_1$ contradiction.
$b\bar{\beta} \in W_1$

LINEAR COMBINATION OF VECTORS

If $V(F)$ is a vector space and $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_k \in V$, and $c_1, c_2, \dots, c_k \in F$ then

$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \dots + c_k\bar{\alpha}_k$ is called linear combinatn of vectors.

$$\Rightarrow \bar{\alpha} = \sum_{i=1}^{k} c_i \bar{\alpha}_i \in V$$

Q_2 Show that $\bar{\alpha} = (8, 17, 36)$ is a linear combinatn of $\bar{\alpha}_1 = (1, 0, 5)$, $\bar{\alpha}_2 = (0, 3, 4)$, $\bar{\alpha}_3 = (1, 1, 1)$

Sol.

$$c_1(1,0,5) + c_2(0,3,4) + c_3(1,1,1) = (8, 17, 36)$$

$c_1 + c_3 = 8$, $3c_2 + c_3 = 17$, $5c_1 + 4c_2 + c_3 = 36$

$4c_1 + 4c_2 = 28$, $5c_1 + c_2 = 19$
$c_1 + c_2 = 7$, $5c_1 + c_2 = 19$ $\Rightarrow 4c_1 = 12$
$\Rightarrow \boxed{c_1 = 3}$

$\Rightarrow \boxed{c_2 = 4}$ & $\boxed{c_3 = 5}$ } Ans

Linear Span

The linear span of a non empty subset of V(F) is the set of all linear combination of any finite number of elements of S and is denoted by L(S)

$$L(S) = \left\{ \sum_{i=1}^{k} c_i \bar{\alpha}_i \;, \alpha_i \in S \text{ and } c_i \in F \right\}$$

$L(S) \subset V$, S is subset.

- $V \subset L(S)$ only if every vector in V can be represented as an element of L(S). Then $V = L(S)$ and we can say 'S' spans 'V'.

Q: Show that the linear span L(S) of any subset 'S' of a vector space V(F) is a subspace of V.

Sol. To Show : $a\bar{\alpha} + b\bar{\beta} \in L(S)$ $(a, b \in F)$

Let $\bar{\alpha} = \sum_{i=1}^{k} a_i \bar{\alpha}_i$ $(a_i \in F$ & $\bar{\alpha}_i \in S)$

$\bar{\beta} = \sum_{j=1}^{m} b_j \bar{\beta}_j$ $(b_j \in F, \bar{\beta}_j \in S)$

$$\Rightarrow a\bar{\alpha} + b\bar{\beta} = a \sum_{i=1}^{k} a_i \bar{\alpha}_i + b \sum_{j=1}^{m} b_j \bar{\beta}_j$$

$$= \sum_{i=1}^{k} (a \cdot a_i) \bar{\alpha}_i + \sum_{j=1}^{m} (b b_j) \bar{\beta}_j$$

$$= \sum_{i=1}^{k} (c_i \bar{\alpha}_i) + \sum_{j=1}^{m} (d_j \bar{\beta}_j)$$

$(c_i = a \cdot a_i$ & $d_j = b b_j)$

$\Rightarrow a\bar{\alpha} + b\bar{\beta} \Rightarrow \{\underbrace{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k, \bar{\beta}_1, \cdots, \bar{\beta}_m}_{S}\}$.

$\Rightarrow a\bar{\alpha} + b\bar{\beta} \in L(S) \Rightarrow \boxed{L(S) \text{ is a subspace}}$.

Q. Show that subset $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ spans entire space $\mathbb{R}^3$ i.e. $L(S) = \mathbb{R}^3$.

Sol. Proof: $L(S) \subset \mathbb{R}^3$ from definition

To Prove: $\mathbb{R}^3 \subset L(S)$

$\cancel{(a,b,c) \in L(S)}$

$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$\Rightarrow (a,b,c) \in L(S)$

$\Rightarrow \mathbb{R}^3 \in L(S)$ Hence, $\mathbb{R}^3 = L(S)$. ✓

LINEAR DEPENDENCE AND INDEPENDENCE

A finite set of $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k\}$ of k vectors of $V(F)$ is said to L.D. if there exists a set $\{c_1, c_2, \cdots, c_k\}$ of scalars (not all zeros), such that

$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_k\bar{\alpha}_k = \phi$.

If $c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_k\bar{\alpha}_k = \phi$ but each $c_i = 0$, then linearly independent.

Q. If 2 vectors are LD then show that one is scalar multiple of other.

Sol.

$a\bar{\alpha} + b\bar{\beta} = \phi$ $(a, b \in F)$ & $a, b \neq 0$

$\Rightarrow a\bar{\alpha} = -b\bar{\beta}$ $\Rightarrow \boxed{\bar{\alpha} = (-ba^{-1})\bar{\beta}}$ Hence proved.

Q. Show that any super set of L.D. set of vector is also L.D.

Sol. Let $V(F)$ be a vector space over the field F. Let $\alpha_i \in V(F)$, $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k\}$ are L.D.

$\exists$ $c_1, c_2, \ldots, c_k \in F$ s.t.

$$\sum_{i=1}^{k} c_i \bar{\alpha}_i = \phi$$ where not all c_i's are zero.

Let us consider a super set

$\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k, \bar{\alpha}_{k+1}, \ldots, \bar{\alpha}_m\}$, $m > k$.

$$\underbrace{c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_k\bar{\alpha}_k}_{\phi} + 0\cdot\bar{\alpha}_{k+1} + \ldots + 0\bar{\alpha}_m = \phi$$

$\Rightarrow$ Not all c_i's are zero.

$$\sum_{i=1}^{m} c_i \bar{\alpha}_i = \phi$$ where not all c_i's are 0.

Theorem

Q. The set of non-zero vectors $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k\}$ is L.D., iff one of them say $\bar{\alpha}_r$, $2 \leq r \leq k$ is the linear combination of the preceding vector.

Sol.

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_k\bar{\alpha}_k = \phi$$

$$\Rightarrow c_2\bar{\alpha}_2 + \ldots + c_k\bar{\alpha}_k = -(c_1)\bar{\alpha}_1$$

$$\Rightarrow \boxed{\left(\frac{-c_2}{c_1}\right)\bar{\alpha}_2 + \ldots + \left(\frac{-c_k}{c_1}\right)\bar{\alpha}_k = \bar{\alpha}_1}$$

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Proof: (i) Case I Let the set of vectors be L.D.

$\exists$ scalars $c_i \in F$, Not all zero s.t.

$$\sum_{i=1}^{k} c_i \bar{\alpha}_i = \bar{\phi}$$

Let c_r be the last non zero coeff.

If $r=1$, $c_1\alpha_1 = \bar{\phi} \Rightarrow \boxed{\bar{\alpha}_1 = \bar{\phi}}$ (But set of vectors are non 0.)

Hence, $r > 1$.

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + \underset{\text{Non zero}}{c_r}\bar{\alpha}_r = \phi$$

$\because c_r \neq 0$, c_r^{-1} exists.

$$\Rightarrow c_r\bar{\alpha}_r = -c_1\bar{\alpha}_1 - c_2\bar{\alpha}_2 - c_3\bar{\alpha}_3 - \ldots - c_{r-1}\bar{\alpha}_{r-1}$$

$$\Rightarrow \bar{\alpha}_r = (-c_1 c_r^{-1})\bar{\alpha}_1 + (-c_2 c_r^{-1})\bar{\alpha}_2 - \ldots + (-c_{k-1} c_r^{-1})\bar{\alpha}_{r-1}$$

$$\{(-c_1 c_r^{-1}), (c_2 c_r^{-1}), \ldots, c_{r-1} c_r^{-1}\} \in F \checkmark.$$

Case II. If $\bar{\alpha}_r = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_{r-1}\bar{\alpha}_{r-1}$ $(2 \leq r \leq k)$

$$\Rightarrow c_1\bar{\alpha}_1 + \ldots + c_{r-1}\bar{\alpha}_{r-1} + \underset{\text{not } 0}{(-1)}\bar{\alpha}_r = \bar{\phi}$$

$\Rightarrow (\bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_r) \to$ LD as all c_i's not 0.

$\Rightarrow \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_k\}$ super set of $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_r\}$.

$\Rightarrow$ L.D. $\checkmark$

Theorem: If a vector space $V(F)$ is spanned by L.D set $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k\}$ then V can be generated by a proper subset of $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}$

Proof: Let $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k\}$ be L.D.

Therefore, one of the vectors is linear combinatn of others.

Let $\bar{\alpha}_r = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_{r-1}\bar{\alpha}_{r-1} + c_{r+1}\bar{\alpha}_{r+1} + \cdots + c_k\bar{\alpha}_k$.

where $c_1, \cdots c_k \in F$.

Let $\bar{\beta} \in V$, $V = L\{\bar{\alpha}_1, \cdots, \bar{\alpha}_k\}$.

$$\bar{\beta} = b_1\bar{\alpha}_1 + b_2\bar{\alpha}_2 + \cdots + b_r\bar{\alpha}_r + \cdots + b_k\bar{\alpha}_k$$

Replace $\bar{\alpha}_r$,

$$\bar{\beta} = (b_1 + c_1 b_r)\bar{\alpha}_1 + (b_2 + c_2 b_r)\bar{\alpha}_2 + \cdots + (b_{r-1} + b_r c_{r-1})\bar{\alpha}_{r-1} + \cdots + (b_k + b_r c_k)\bar{\alpha}_k$$

$\Rightarrow \bar{\beta}$ represented as $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{r-1}, \bar{\alpha}_{r+1}, \cdots, \bar{\alpha}_k\}$

$\Rightarrow \bar{\beta} \in L\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{r-1}, \bar{\alpha}_{r+1}, \cdots, \bar{\alpha}_k\}$

$\Rightarrow V \subset L\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{r-1}, \bar{\alpha}_{r+1}, \cdots, \bar{\alpha}_k\}$

& $L\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_{r-1}, \bar{\alpha}_{r+1}, \cdots, \bar{\alpha}_k\} \subset V$

$\boxed{V_{m^2}(F)}$.

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$\Rightarrow \{\bar{\alpha}_1, \bar{\alpha}_2, ----, \bar{\alpha}_{r-1}, \bar{\alpha}_{r+1}, ----, \bar{\alpha}_k\}$ spans V.

BASIS

A subset S of a vector space V(F) is said to be basis if

(i) S is linearly independent.
(ii) any vector in V other than that of S is linear combination of the vectors of S.

Dimension

The number of elements in a basis of the vector space V(F) is called the dimension of V and in denoted by dim (V).

Q. (1, 0, 0), (0, 1, 0), (0, 0, 1) forms a basis of $\mathbb{R}^3$.
$(\mathbb{R}^3, \oplus) \rightarrow (\mathbb{R}, +, \cdot)$

Sol. (i) L.ID :-

$$c_1(1,0,0) \oplus c_2(0,1,0) \oplus c_3(0,0,1) = \phi$$

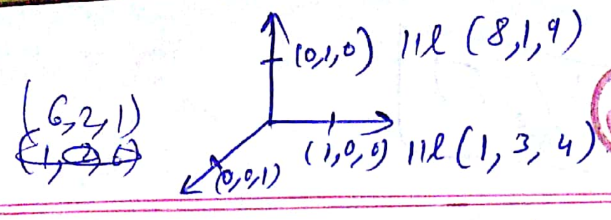
$\Rightarrow (c_1, c_2, c_3) = \underset{(0,0,0)}{\phi}$ only when $\boxed{c_1 = c_2 = c_3 = 0}$

$\Rightarrow$ Linearly independent.

Let $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in V$ or S.

$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = \alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1)$

$\Rightarrow$ can be expressed as L.C. of vector in S.



Find out another basis for $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$c_1(8,1,9) + c_2 \overset{(6,2,1)}{\cancel{(1,2,6)}} + c_3(1,3,4)$$

$$\cancel{(8c_1 + c_2 + c_3),\ (c_1 + 2c_2 + 3c_3),\ 9c_1 + 6c_2 + 4c_3}$$

$$\left(\underset{0}{8c_1 + 6c_2 + c_3},\ \underset{0}{c_1 + 2c_2 + 3c_3},\ \underset{0}{9c_1 + c_2 + 4c_3}\right)$$

Q. $(1,1,1,1)$, $(0,1,1,1)$, $(0,0,1,1)$, $(0,0,0,1)$ do they form basis?

Sol. Yes, $\alpha_1(1,1,1,1) + \alpha_2(0,1,1,1) + \alpha_3(0,0,1,1) + \alpha_4(0,0,0,1) = \phi$

$$\Rightarrow \left(\underset{0}{\alpha_1},\ \underset{0}{\alpha_1+\alpha_2},\ \underset{0}{\alpha_1+\alpha_2+\alpha_3},\ \underset{0}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right)$$

$$\Rightarrow \boxed{\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0} \quad \text{LI.}$$

$$(a,b,c,d) = a_1(1,1,1,1) + b_1(0,1,1,1) + c_1(0,0,1,1) + d_1(0,0,0,1)$$

$$\Rightarrow a = a_1 \qquad a_1 + b_1 = b \Rightarrow \boxed{b_1 = b - a}$$

$$c_1 + \overset{b}{\cancel{a_1 + b_1}} = c \Rightarrow \boxed{c_1 = c - b}$$

$$d_1 + (\overset{a_1+b_1+c_1}{c}) = d \Rightarrow \boxed{d_1 = d - c}$$

Hence, form Basis.

Steinitz Replacement Theorem

Theorem: In a vector space $V_n(F)$ with the basis set $B = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m\}$ every vector $\bar{\alpha} \in V$ is uniquely expressed as the linear combinatn. of vector in V

Proof: $\bar{\alpha} = \sum_{i=1}^{m} c_i \bar{\alpha}_i$ —①

$\bar{\alpha} = \sum_{i=1}^{m} d_i \bar{\alpha}_i$ —②

①-② $\Rightarrow \phi = \sum_{i=1}^{m} (c_i - d_i) \bar{\alpha}_i$

$\because \{\bar{\alpha}_i\} \rightarrow$ L.I. $\Rightarrow c_i - d_i = 0$

$\Rightarrow \boxed{c_i = d_i \ \forall i}$

Theorem: If $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m\}$ is a basis of $V_n(F)$ and $\bar{\beta}$ is a non-zero vector belonging to V, then $\bar{\beta} = \sum_{i=1}^{m} c_i \bar{\alpha}_i$ (where $c_i \in F$ and all c_i's are not 0) If $c_j \neq 0$, then show that $\rightarrow$

$\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{j-1}, \bar{\beta}, \bar{\alpha}_{j+1}, \ldots, \bar{\alpha}_m\}$ is a basis of $V_n(F)$

Proof: $\bar{\beta} = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_{j-1}\bar{\alpha}_{j-1} + c_j\bar{\alpha}_j + c_{j+1}\bar{\alpha}_{j+1} + \ldots + c_m\bar{\alpha}_m$

$\Rightarrow 0 = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \ldots + c_{j-1}\bar{\alpha}_{j-1} + (c_j\bar{\alpha}_j - \bar{\beta}) + \ldots + c_m\bar{\alpha}_m$

$\bar{\beta} = \sum_{i=1}^{m} c_i\bar{\alpha}_i, \quad c_j \neq 0$

$\Rightarrow \bar{\beta} = c_j\bar{\alpha}_j + \sum_{\substack{i=1 \\ i\neq j}}^{m} c_i\bar{\alpha}_i$

$\Rightarrow \boxed{\bar{\alpha}_j = c_j^{-1}\beta - c_j^{-1}\sum_{\substack{i=1 \\ i\neq j}}^{m} c_i\bar{\alpha}_i}$ —①

(c_j^{-1} exists $\because c_j \neq 0$)

To Prove: $S = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{j-1}, \beta, \bar{\alpha}_{j+1}, \ldots, \bar{\alpha}_n\}$ is basis of $V_n(F)$.

(i) S is L.ID.

(ii) S spans in vector space $V_n(F)$.

(i) Let $\sum_{\substack{i=1 \\ i \neq j}}^{n} \delta_i \bar{\alpha}_i + \delta_j \bar{\beta} = \bar{\phi}$ $(\delta_i, \delta_j \in F)$

(we need to show $\forall_{i,j}\ \delta_i, \delta_j = 0$)

$\Rightarrow \sum_{\substack{i=1 \\ i \neq j}}^{n} \delta_i \bar{\alpha}_i + \delta_j \left(\sum_{i=1}^{n} c_i \bar{\alpha}_i \right) = \bar{\phi}$

$\Rightarrow \sum_{\substack{i=1 \\ i \neq j}}^{n} (\delta_i + c_i \delta_j) \bar{\alpha}_i + \delta_j c_j \bar{\alpha}_j = \bar{\phi}$

all α_i's covered.

$\because$ $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n$ are L.D.

$\Rightarrow \delta_i + c_i \delta_j = 0, \quad \delta_j c_j = 0 \quad (\because c_j \neq 0)$

$\Rightarrow \boxed{\delta_j = 0}$

$\Rightarrow \boxed{\delta_i = 0}$

$\Rightarrow \forall i, j \quad \delta_i, \delta_j = 0 \Rightarrow \boxed{\text{L.ID}}$

(ii) $\bar{\gamma} = d_1 \bar{\alpha}_1 + d_2 \bar{\alpha}_2 + \ldots + d_j \bar{\beta} + \ldots + d_n \bar{\alpha}_n$.

$\sum_{\substack{i=1 \\ i \neq j}}^{n} d_i \bar{\alpha}_i + d_j \bar{\beta} = \bar{\gamma}$

Let $\bar{\gamma} \in V$ $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ is a basis of $V_n(F)$

$\Rightarrow \bar{\gamma} = \sum_{i=1}^{n} \mu_i \bar{\alpha}_i$

$$\Rightarrow \bar{\gamma} = \sum_{\substack{i=1 \\ i\neq j}}^{m} \mu_i \bar{\alpha}_i + \mu_j \bar{\alpha}_j$$

Replace $\bar{\alpha}_j$ from ①:

$$= \sum_{\substack{i=1 \\ i\neq j}}^{m} \mu_i \bar{\alpha}_i + \mu_j \left(c_j^{-1} \bar{\beta} - c_j^{-1} \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i \bar{\alpha}_i \right)$$

$$= \sum_{\substack{i=1 \\ i\neq j}}^{m} (\mu_i - \mu_j c_j^{-1} c_i) \bar{\alpha}_i + \mu_j c_j^{-1} \bar{\beta}$$

$\Rightarrow \bar{\gamma}$ is expressed as Linear combinatⁿ of $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\beta}, \ldots \bar{\alpha}_m\}$

Thus, S spans in vector space $V_m(F)$. $\Rightarrow \bar{\gamma} \in \{\ \text{—}\ \}$

Hence proved.

Q₂ Show that set $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3\}$ is linearly dependent if the set $\{\bar{\alpha}_1 + a\bar{\alpha}_2 + b\bar{\alpha}_3, \bar{\alpha}_2, \bar{\alpha}_3\}$ is L.D., where $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \in V(F)$, $a, b \in F$.

Sol.

$$c_1 \bar{\alpha}_1 + c_1 a \bar{\alpha}_2 + c_1 b \bar{\alpha}_3 + c_2 \bar{\alpha}_2 + c_3 \bar{\alpha}_3 = \phi \quad (\text{all } c_i \neq 0)$$

$$\Rightarrow \underbrace{c_1}\bar{\alpha}_1 + \underbrace{(c_1 a + c_2)}\bar{\alpha}_2 + \underbrace{(c_3 + c_1 b)}\bar{\alpha}_3 = \phi.$$

all c_i's $\neq 0 \Rightarrow$ If $c_1 = 0$ but $c_2, c_3 \neq 0$ then also L.D.

If $c_2 = 0$ but $c_1, c_3 \neq 0$ — " —

— " — $c_3 = 0$ but $c_2, c_1 \neq 0$ — " —

$\Rightarrow \boxed{\text{L.D.}}$ when all c_i's $\neq 0$.

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$\therefore \{\bar{\alpha}_1 + a\bar{\alpha}_2 + b\bar{\alpha}_3, \bar{\alpha}_2, \bar{\alpha}_3\} \rightarrow$ LD.

l, m, n not all zeros,

s.t.,

$$l(\bar{\alpha}_1 + a\bar{\alpha}_2 + b\bar{\alpha}_3) + m(\bar{\alpha}_2) + n\bar{\alpha}_3 = 0$$

$$\Rightarrow l\bar{\alpha}_1 + (la + m)\bar{\alpha}_2 + (lb + n)\bar{\alpha}_3 = 0$$

If $l \neq 0$, then L.D.

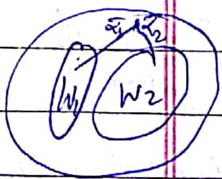
If $l = 0 \Rightarrow m\bar{\alpha}_2 + n\bar{\alpha}_3 = 0$

$m/n \rightarrow$ atleast one of them has to be non zero.

$\{\alpha_1, \alpha_2, \alpha_3\} \Rightarrow \boxed{L.D.}$

Linear Sum Of Two Subspaces

If W_1 and W_2 are 2 subspaces of $V(F)$, then the linear sum of 2 subspaces is denoted by $W_1 + W_2$ in the set of sums $\bar{\alpha}_1 + \bar{\alpha}_2$ such that $\bar{\alpha}_1 \in W_1$, $\bar{\alpha}_2 \in W_2$



$$W_1 + W_2 = \{\underbrace{\bar{\alpha}_1 + \bar{\alpha}_2}_{\text{vector addition}}, \bar{\alpha}_1 \in W_1, \bar{\alpha}_2 \in W_2\}$$

Theorem: If W_1 & W_2 are subspaces of $V(F)$, then show that $W_1 + W_2$ is subspace of V.

Proof: To show: $a\bar{\alpha} + b\bar{\beta} \in W_1 + W_2$.

$\bar{\alpha} \in W_1 + W_2 \Rightarrow \bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2$ (where $\bar{\alpha}_1 \in W_1, \bar{\alpha}_2 \in W_2$)

$\bar{\beta} \in W_1 + W_2 \Rightarrow \bar{\beta} = \bar{\beta}_1 + \bar{\beta}_2$ (—"—, $\bar{\beta}_1 \in W_1, \bar{\beta}_2 \in W_2$)

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$\because W_1$ is subspace $\Rightarrow a\alpha_1 + b\beta_1 \in W_1$
$\alpha_1, \beta_1 \in W_1$

$$a\bar{\alpha} + b\bar{\beta} = a(\bar{\alpha}_1 + \bar{\alpha}_2) + b(\bar{\beta}_1 + \bar{\beta}_2)$$
$$= \underbrace{(a\bar{\alpha}_1 + b\bar{\beta}_1)}_{\bar{\gamma}_1 \in W_1} + \underbrace{(a\bar{\alpha}_2 + b\bar{\beta}_2)}_{\text{||ly, } \bar{\gamma}_2 \in W_2} \quad \text{(Associative \& Commutative)}$$

$$= \bar{\gamma}_1 + \bar{\gamma}_2$$

$$\Rightarrow \boxed{a\bar{\alpha} + b\bar{\beta} \in W_1 + W_2}$$

Definition (Direct Sum)

Let W_1 & W_2 be 2 subspaces of vector space $V(F)$ s.t. $V = W_1 + W_2$. This linear sum is called direct sum denoted by $V = W_1 \oplus W_2$ if every vector $\bar{\xi} \in V$ can be written in one & only one way as $\bar{\xi} = \bar{\alpha} + \bar{\beta}$ when $\bar{\alpha} \in W_1$, $\bar{\beta} \in W_2$.

Theorem: The necessary & sufficient condition that a vector space $V(F)$ is a direct sum of 2 sub spaces W_1 & W_2 are $V = W_1 + W_2$ and $W_1 \cap W_2 = \bar{\phi}$.
additive identity

Proof: (1) If V is a direct sum $\Rightarrow V$ is linear sum $W_1 + W_2$.

let $W_1 \cap W_2 \neq \bar{\phi}$, $W_1 \cap W_2 = \bar{\alpha}$, $\bar{\alpha} \neq \bar{\phi}$
$\Rightarrow \bar{\alpha} \in W_1, W_2, V$.

Proof : (i) Given (Let $W_1 \cap W_2 = \{\emptyset\}$, $V = W_1 + W_2$) $\longrightarrow$ Prove ($V = W_1 \oplus W_2$)

Let us assume, V is not a direct sum.

$\bar{\alpha} \in V$, $\bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2 = \bar{\beta}_1 + \bar{\beta}_2$ ($\bar{\alpha}_1, \bar{\alpha}_2 \in W_1$, $\bar{\beta}_1, \bar{\beta}_2 \in W_2$) check it.

$\bar{\alpha}_1 + \bar{\alpha}_2 = \bar{\beta}_1 + \bar{\beta}_2$

$\Rightarrow \bar{\alpha}_1 - \bar{\beta}_1 = \bar{\beta}_2 - \bar{\alpha}_2 = \bar{x}$ $\left| \begin{array}{l} \bar{\alpha}_1 - \bar{\beta}_1 \in W_1 \\ \bar{\beta}_2 - \bar{\alpha}_2 \in W_2 \end{array} \right.$

$\Rightarrow \bar{x} \in W_1$ & $\bar{x} \in W_2$.

$\Rightarrow \bar{x} \neq \bar{\emptyset} \in W_1 \cap W_2$. Hence, contradiction to what was given.

Hence Proved.

(ii) Given ($V = W_1 \oplus W_2$) $\Rightarrow$ Prove ($W_1 \cap W_2 = \{\emptyset\}$, $V = W_1 + W_2$)

Let $\bar{\alpha} \in V \Rightarrow \bar{\alpha} = \bar{\alpha}_1 \oplus \bar{\alpha}_2$ ($\bar{\alpha}_1, \bar{\alpha}_2 \in W_1$)

Trivial as vector sum $\Rightarrow$ linear sum.

Let $W_1 \cap W_2 \neq \{\emptyset\} \Rightarrow W_1 \cap W_2 = \bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2$.
$= \{\emptyset, \bar{\alpha}\}$

For some element $\bar{w} \in V$

$\Rightarrow \bar{w} = \underset{\in W_1}{\bar{\emptyset}} + \underset{\in W_2}{\bar{\alpha}} = \underset{\in W_2}{\bar{\alpha}} + \underset{\in W_1}{\bar{\emptyset}}$

$\Rightarrow$ 2 representations for $\bar{w} \in V$.

Hence, contradiction as can't be represented as uniquely.

Hence Proved.

$\boxed{L: V \to V}$ → Linear Operator

$\boxed{L: V \to F}$ Linear Function (Dot Product)
Vector → Scalar

LINEAR TRANSFORMATION

$(V, +) \longrightarrow (W, +)$

$\vec{x} \qquad \vec{y} \longrightarrow f(\vec{y})$

$\vec{x} \longrightarrow f(\bar{x})$

$f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y})$
$f(c\bar{x}) = cf(\bar{x})$

Let ϕ is the null element of V.

$f(\phi_v + \phi_v) = f(\phi_v) + f(\phi_v)$

Definition: Let $U(F)$ & $V(F)$ be 2 vectors spaces over the same field $(F, +, \cdot)$. A function f from U to V is said to be linear ~~fun~~ transformation if (i) $f(\bar{x} + \bar{y}) = f(\bar{x}) + f(\bar{y}) \; \forall \bar{x}, \bar{y} \in V$

Linearity! ← $f(c\bar{x}) = cf(\bar{x})$

$f(a\bar{x} + b\bar{y}) = af(\bar{x}) + bf(\bar{y})$

Eg: let $M_{m \times m}(F)$ be a vector space over the field F, and let $W[a_{ij}]_{m \times m}$ be a fixed matrix over F. Define a function

$f : M_{m \times m} \longrightarrow M_{m \times m}$ by

$f[b_{ij}] = [a_{ij}] \cdot [b_{ij}] \;\; \forall \, b_{ij} \in M_{m \times m}$

Sol. Let $[b_{ij}], [c_{ij}] \in M_{m \times m}(F)$

To show: $f(r[b_{ij}] + s[c_{ij}]) = r f([b_{ij}]) + s f([c_{ij}])$

LHS : $f(r[b_{ij}] + s[c_{ij}]) = [a_{ij}]\cdot(r[b_{ij}] + s[c_{ij}])$

$$= r[a_{ij}]\cdot[b_{ij}] + s[a_{ij}]\cdot[c_{ij}]$$

$$= r f([b_{ij}]) + s f([c_{ij}])$$

Hence, linear transformation.

Q. Let $f : U \to V$ be a linear transform if $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ are L.D. in U, then show that $\{f(\bar{x}_1), f(\bar{x}_2), \ldots, f(\bar{x}_n)\}$ are L.D. in V.

Sol.
$$c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n = \bar{\phi}$$

atleast one $c_i \neq 0$ or not all c_i's $= 0$.

$$f(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) = f(\bar{\phi}) = \bar{\phi}$$

$$\Rightarrow c_1 f(\bar{x}_1) + \cdots + c_n f(\bar{x}_n) = \phi$$

for not all c_i's $= 0$.

Hence, $\{f(\bar{x}_1), \ldots, f(\bar{x}_n)\}$ are also L.D.

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for a finite dimensional space $V(F)$ and $\{y_1, y_2, \ldots, y_n\}$ be an arbitary set of 'n' vectors from $W(F)$. Then, show that there exists unique (only one) linear mapping $f : V \to W$ s.t. $f(x_i) = y_i$ for $i = 1$ to n.

linearity conditions should be satisfied.

Sol. $x = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \text{——} + \alpha_m \bar{x}_m$

Let $x \in V$

since $\{x_1, x_2, \cdots, x_m\}$ is a basis of V.

$x = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \quad (a_i \in F)$

Let us define a mapping $f : V \to V$

s.t. $f(x) = a_1 y_1 + a_2 y_2 + \cdots + a_m y_m$

when $x = x_i$

$\Rightarrow x_i = a_1 x_1 + \cdots + a_i x_i + \cdots + a_m x_m$

$\Rightarrow a_1 x_1 + \cdots + (a_i - 1) x_i + \cdots + a_m x_m = 0.$

All a_i s $= 0 \quad \Rightarrow \boxed{a_i = 1}$

$\Rightarrow f(x_i) = \overset{[\cdots 0] + a_i}{\cancel{(a_i - 1)}} y_i + [0] \cdots = a_i^{1} y_i = y_i$

Now, let $x, y \in V$

$\Rightarrow x = a_1 x_1 + \cdots + a_m x_m$

& $y = b_1 x_1 + \cdots + b_m x_m$

(i) $\Rightarrow f(x+y) = f\{(a_1 + b_1) x_1 + \cdots + (a_m + b_m) x_m\}$

$= (a_1 + b_1) y_1 + \cdots + (a_m + b_m) y_m$

$= (a_1 y_1 + a_2 y_2 + \cdots + a_m y_m) + (b_1 y_1 + b_2 y_2 + \cdots + b_m y_m)$

$= f(x) + f(y).$

Pt

(ii) $f(cx) = c f(x) \quad \forall c \in F$

$\Rightarrow f(cx) = f(a_1 c x_1 + a_2 c x_2 + \ldots + a_m c x_m)$

$= a_1 c y_1 + a_2 c y_2 + \ldots + a_m c y_m$

$= c(a_1 y_1 + \ldots + a_m y_m)$

$= \boxed{c f(x)}$

Hence, linearity is proved.

(iii) Finally, we show that f is unique.

Assume that f is not unique

Let $g : V \to W$

s.t. ~~st~~ $g(x_i) = y_i \quad i = 1, 2, \ldots, m$

g is another linear.

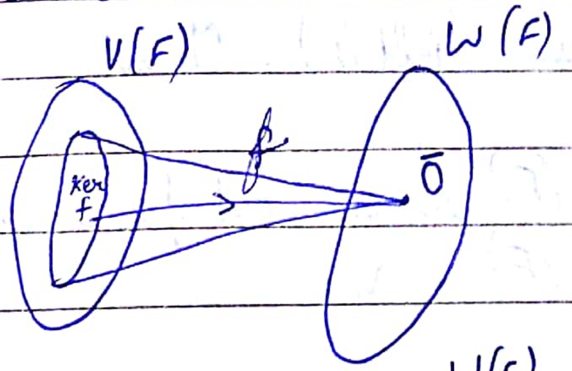
$g(x) = g(a_1 x_1 + \ldots + a_m x_m)$ $\left(\because g \text{ is linear} \right)$

$= a_1 g(x_1) + a_2 g(x_2) + \ldots + a_m g(x_m)$

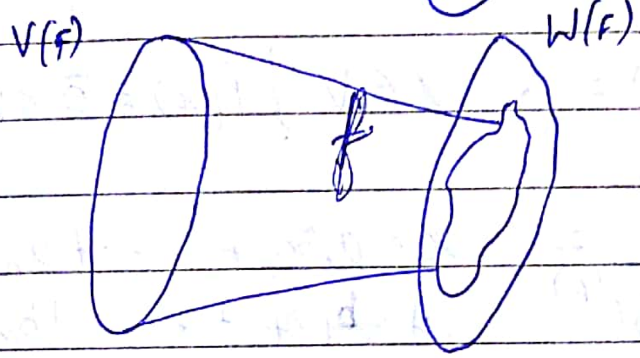
$= a_1 y_1 + \ldots + a_m y_m = f(x) \; \forall x$

Hence, ~~only~~ f is unique.

Kernel



$f(x+y) = f(x) + f(y)$
$f(cx) = cf(x)$.



- Let $f: V \to W$ be a linear mapping from a vector space $V(F)$ into a vector space $W(F)$. Then $Ker(f)$ is the set of all $x \in V$, which are mapped on $\bar{0}$, the additive identity of W by f.

$$Ker(f) = \{x \in V \mid f(x) = \bar{0} \in W\}$$

- Let $f: V \to W$ be a linear transformation (LT). Then set of all images of elements $x \in V$ is called the range set or image set.

$$R_f(V) = f(V) = \{y \in W \mid y = f(x) \text{ for some } x \in V\}.$$

Theorem: Let $f: V \to W$ be a L.T. over the same field (F). Then,

(i) $Ker(f)(F)$ is a subspace of $V(F)$

(ii) $R_f(f)(F)$ — " — $W(F)$.

Sol: (i) $Ker(f) = \{ x \in V \mid f(x) = \bar{0} \in W \}$.

$y, x \in V$ ~~for~~ $\Rightarrow$ $x = a_1 x_1 + \dots + a_m x_m \in V$

To Prove: $ax + by \in Ker(f)(F)$

$y = b_1 x_1 + \dots + b_m x_m \in V$

$f(a_1 x_1 + \dots + a_m x_m) = \bar{0} = f(y) = f(x)$

$\Rightarrow a_1 y_1 + \dots + a_m y_m = \bar{0}$

$f(ax + by) = f(a(a_1 x_1 + \dots + a_m x_m) + b(b_1 x_1 + \dots + b_m x_m))$

$= a f(x) + b f(y)$ (as f is linear)

$= a\bar{0} + b\bar{0} = \bar{0}$

$\Rightarrow \boxed{(ax + by) \in Ker(f)(F)}$

(ii) $R_f(f)(V) = \{ y \in W \mid y = f(x) \text{ for some } x \in V \}$.

Let $y_1, y_2 \in f(V)$

$ay_1 + by_2 \in f(V)$

$a, b \in F$

If $Ker f = \{\bar{0}\}$ then $\dim(Ker f) = 0$.

$f(V)$ x in V

$ay_1 + by_2 = a\underbrace{f(x_1)}_{x} + b\underbrace{f(x_2)}_{y}$

$= f(ax_1 + bx_2)$ [$\because f$ is linear].

$ax_1 + bx_2 \in V$. Let mapping to $ax+by$.
closure $\therefore$ $ay_1 + by_2 \in R_f(V)(F)$.
$\therefore$ subspace.

- $Ker(f)$ is known as the null space, and $\dim(Ker f)$ is called the nullity of linear transformation.

- $\dim f(v)$ is called as the rank of the linear transformation.

Sylvester law

- Rank Nullity Theorem: Let $f: V \to W$ be a linear transformation from the vector space $V(F)$ to $W(F)$ then, $\dim(V) = \dim(Ker(f)) + \dim(f(V))$
subset.

Proof: Let $\dim V = n$.

Let $Ker f \neq \{\bar{0}\}$, say $\dim(Ker f) = r \le n$.

Let $\{x_1, x_2, \dots, x_r\}$ be a basis of $Ker f$.

Check out!

Then, we can extend $\{x_1, x_2, \dots, x_r\}$ to a basis $\{x_1, x_2, \dots, x_r, y_1, \dots, y_{n-r}\}$ of V.

Now for any $y \in f(V)$, $\exists\, x \in V$ s.t. $y = f(x)$.

$$x = a_1 x_1 + a_2 x_2 + \dots + a_k x_k + b_1 y_1 + \dots + b_{n-k} y_{n-k}$$

as (they form basis)

$$y = f(x) = \underbrace{a_1 f(x_1) + \dots + a_k f(x_k)}_{} + b_1 f(y_1) + \dots + b_{n-k} f(y_{n-k})$$

(being basis of Ker f belong to) (∵ f is linear)

$$y = f(x) = b_1 f(y_1) + \dots + b_{n-k} f(y_{n-k})$$

⇒ $y \in f(V)$ can be expressed as L.C. of $\{f(y_1), f(y_2), \dots, f(y_{n-k})\}$ → spans $f(V)$ → L.I.?

Inference: Basis of $f(V)$.

$$a_1 f(y_1) + \dots + a_{n-k} f(y_{n-k}) = 0$$

$$\Rightarrow f(\underbrace{a_1 y_1 + \dots + a_{n-k} y_{n-k}}_{m \in \text{Ker} f}) = \bar{0} \quad (\because f \text{ is linear})$$

⇒ Can be expressed as basis of Ker f

$$d_1 x_1 + \dots + d_k x_k = m = a_1 y_1 + \dots + a_{n-k} y_{n-k}$$

$$\Rightarrow \boxed{d_1 x_1 + \dots + d_k x_k - a_1 y_1 - \dots - a_{n-k} y_{n-k} = 0}$$

All coeff $= 0$.

$x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{n-k}$ forms basis of V.

$\Rightarrow \{f(y_1), f(y_2), \ldots, f(y_{n-r})\}$ are L.ID.

$\Rightarrow$ Hence, it forms basis.

$\Rightarrow \dim V = n = r + (n-r)$
$= \dim(\operatorname{Ker} f) + \dim f(V)$

Q. $f: V \to W$ is L.T. If $\operatorname{Ker}(f) = \{\bar{0}\}$ then what can u say abt. $\dim[f(v)]$.

Sol. $\dim V = \dim\{\operatorname{Ker}(f)\} + \dim\{f(v)\}$
$= \dim\{f(v)\}$.

Q. Let $F(V)$ and $W(V)$ are finite dimensional vector spaces with $\dim(V) = \dim(W)$ and let $f: V \to W$ be L.T. Then f is one to one mapping iff f maps V onto W.

Sol. If f is one to one $\Rightarrow \operatorname{Ker} f = \{\bar{0}\}$

$\Rightarrow \dim(V) = \overset{0}{\dim(\operatorname{Ker} f)} + \dim(f(V))$
$= \dim(f(V))$

& $\dim(W) = \dim(f(V))$ $\quad \boxed{f \text{ is onto !}} \rightarrow$

Case II: Conversely, $W = f(V)$ (f is onto)
($\because \dim(W) = \dim(f(V))$)

$\Rightarrow \dim W = \dim V = \dim f(V)$

$\Rightarrow \dim V = \dim(\operatorname{Ker} f) + \dim(f(V))$

$\Rightarrow \dim(\operatorname{Ker} f) = 0 \Rightarrow \boxed{\operatorname{Ker} f = \bar{0}}$

$\boxed{f \text{ is one-one !}} \rightarrow$

f is one to one iff $\ker(f) = \{\bar{0}\}$.

- For f is one-one.

  $X_1 \neq X_2$, $x_1, x_2 \in V$.

  s.t. $f(X_1) = f(X_2) = a \in W$

  $\Rightarrow f(X_1) = f(X_2)$

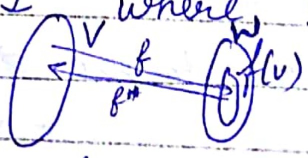
  $\Rightarrow f(X_1) - f(X_2) = f(X_1 - X_2) = \underset{\hookrightarrow \in W}{\bar{0}}$ (f is L.T.)

  $\Rightarrow (X_1 - X_2) \in \ker f$ Contradictⁿ as $\ker f = \{\bar{0}\}$

  $\Rightarrow \boxed{X_1 = X_2}$

Non ~~linear~~ Singular Transformation

A linear transformation f from $V(F)$ to $W(F)$ is said to be non singular iff there exists mapping f^* from $f(V)$ onto $V(F)$, s.t. $f^* \circ f = I$ where I is identity mapping on ~~$f(V)$~~ $V(F)$.



- We know f is L.T. Is f^* L.T.?

Sol. f^* is mapping from $f(V)$ to V.

If $x, y \in f(V)$ then $f^*(x) = v_1$, $f^*(y) = v_2$ where $v_1, v_2 \in V$.

$\boxed{ax + by \in f(V)}$ $(a, b \in F)$

So, $f(v_1) = x$ & $f(v_2) = y$.

$f: V \to W$. 32(b)

$f(V)$

$f^* \circ f(v_1) = v_1$ & $f^* \circ f(v_2) = v_2$ ($\because$

$f^*(ax+by) = f^*(a f(v_1) + b f(v_2))$

$\quad = f^* \circ (f(av_1 + bv_2))$ ($\because f$ is linear)

$\quad = a v_1 + b v_2 \in V$ ($\because f^* \circ f$ is I)

$\quad = a f^*(x) + b f^*(y) \in V$

$\Rightarrow f^*$ is L.T.

Q. Let $f: V \to W$ be a L.T., then show that following statements are equivalent:

(a) f is non singular

(b) For all $x, y \in V$ if $f(x) = f(y)$ then $x = y$.

(c) $Ker(f) = \{0\}$.

Sol. ~~(a)~~ ~~Let $x \neq y$, $f^* \circ f(x) = x$ & $f^* \circ f(y) = y$.~~

~~$\Rightarrow f^* \circ f(x) \neq f^* \circ f(y)$~~

(a) $\Rightarrow$ (b)

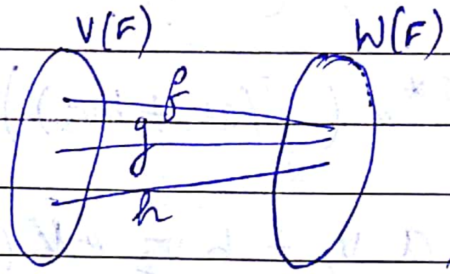
f is non singular $f^* \circ f(v) = v$

Now, $f(x) = f(y) \Rightarrow f^* \circ f(x) = f^* \circ f(y)$

$\Rightarrow \boxed{x = y}$ ($\because f$ is non singular)

$\Rightarrow f$ is one to one.

(b) $\Rightarrow$ (c) If f is one to one, then $\ker(f) = \{0\}$.



~~Set of~~ L.T. b/w V & W

$\{f, g, h, \ldots\} = L(V,W)$

$L(V, W, F)$ is also a vector space.

operations?

field

(i) If $f, g \in L(V,W)$ then $f+g \in L(V,W)$

$\Rightarrow$ $f+g$ is a L.T.

(ii) If $f \in L(V,W)$ then $cf \in L(V,W)$

$\Rightarrow$ cf is L.T.

what are ~~is~~ operation?

(i) $(f+g)x = f(x)+g(x)$ $(x \in V)$

vector addition

(ii) $(cf)(x) = cf(x)$ $x \in V$

scalar multiplication

$\forall c \in F$

Let $x, y \in V$, $a, b \in F$.

$\because$ f, g are L.T.

$f(ax+by) = af(x) + bf(y)$

$g(ax+by) = ag(x) + bg(y)$.

$\Rightarrow$ $(f+g)(ax+by) = f(ax+by) + g(ax+by)$ (Defined like this)

$= af(x) + bf(y) + ag(x) + bg(y)$

$= a(f(x)+g(x)) + b(f(y)+g(y))$

$\Rightarrow$ $(f+g)$ is a L.T. $\Rightarrow$ $\boxed{(f+g) \in L(V,W)}$.

To Prove: $cf \in L(V,W)$ i.e. cf is a L.T.

$$cf(ax+by) = c[f(ax+by)]$$
$$= c[af(x) + bf(y)]$$
$$= (ca)f(x) + (cb)f(y)$$
$$= (ac)f(x) + (bc)f(y) \quad (\because ca \in F \text{ commutative } ca = a\cdot c)$$
$$= a(cf(x)) + b(cf(x))$$
$\Rightarrow$ $\boxed{c.f \in L(V,W)}$ is a L.T.

Other 8 properties to be shown:

(ii) $(f+g)+h = f+(g+h)$ $\forall f,g,h \in L(V,W)$

(iii) $f+g = g+f$ $\forall f,g \in L(V,W)$

(iv) $f+\bar{0} = \bar{0}+f = f$ $\forall f \in L(V,W)$
$\exists$ an element $\bar{0} \in L(V,W)$

(v) $f+(-f) = (-f)+f = \bar{0}$ $\forall f, f \in L(V,W)$
$\exists$ an element $-f \in L(V,W)$ s.t.

(vii) $a(f+g) = af+ag$ $\forall a \in F$ & $f,g \in L(V,W)$

(viii) $(a\cdot b)f = a(bf)$ $\forall a,b \in F$, $f \in L(V,W)$

(ix) $(a+b)f = af+bf$ $\forall a,b \in F$, $f \in L(V,W)$

(x) $1f = f$, 1 is the unit element of F, $f \in L(V,W)$.

(ii) $$((f+g)+h)(ax+by) \underset{\text{Vector Addition}}{=} (f+g)(ax+by) + h(ax+by)$$
$$= f(ax+by) + g(ax+by) + h(ax+by)$$
$$= f(ax+by) + (g+h)(ax+by)$$
$$= (f+(g+h))(ax+by).$$

(iii) $(f+g)(ax+by) = f(ax+by) + g(ax+by)$
$= g(ax+by) + f(ax+by)$
$= (g+f)(ax+by)$

(iv) $(f+\bar{0})(ax+by) = f(ax+by) + \bar{0}(ax+by)$
$= f(ax+by)$
$(\bar{0}+f)(ax+by) = \text{— " —} = (f+\bar{0})$.

$\exists$ a function $Z : V \to W$ s.t. $Z(x) = \bar{0}_W \ \forall x \in V$, $\bar{0}_W$ is identity element of W

(v) $(f + -f)(ax+by) = f(ax+by) - f(ax+by) = Z(ax) = \bar{0}_W \ \forall x \in V$

$(-f+f)(ax+by) = -f(ax+by) + f(ax+by) = \bar{0}_W = Z(x)$

(vii) $c(f+g)(ax+by) = \cancel{cf + cg}\ c\{f(ax+by) + g(ax+by)\}$
$= (cf) f(ax+by) + (cg) f(ax+by)$
$= (cf + cg)(ax+by)$ (From def. of V.P.)

(viii) $(c.d)\, f(ax+by) = (c.d)(a f(x) + b f(y))$
$= (cd)(a f(x) + b f(y))$
$= c(da f(x) + db f(y))$
$= c((d f)(ax+by))$

(ix) $(c+d)\, f(ax+by) = (c+d)\, f(a f(x) + b f(y))$
$= ca f(x) + cb f(y) + da f(x) + db f(y)$

(x) —
$= c(a f(x) + b f(y)) + d(a f(x) + b f(y))$
$= (c f(ax+by)) + (d f(ax+by))$

$\to Z(ax+by) = 0_W = 0_W + 0_W = a0_W + b0_W$
$= a Z(x) + b Z(y)$ $\{\because Z(x) = 0 \ \forall x \in V$

$\Rightarrow Z \in L(V,W)\}$ Now, $(f+Z)x = f(x) + Z(x)$
$= f(x) + 0_W$
$= \boxed{f(x)}$

$\boxed{Z \text{ is identity element in } L(V,W)}$

(v) for any $f \in L(V,W)$

Let $(-f)(\cancel{ax+by}\;x) = -f(x)$

$(-f)(ax+by) = -[f(ax+by)] = -[af(x) + bf(y)]$
as f is LT.

$= a\{-f(x)\} + b\{-f(y)\}$

Theorem: If $V(F)$ and $W(F)$ are vector spaces of dimension m & n respectively, over F, then the space $L(V,W)(F)$ is of dimension $(m \cdot n)$ over F.

Proof: let $\{x_1, \dots, x_m\}$ be a basis of V
" $\{y_1, \dots, y_n\}$ — " — W

$\exists$ a linear map $f_{ij} : V \to W$ $(1 \le i \le m, 1 \le j \le n)$
$f_{ij} \in L(V,W)$

$$f_{ij}(x_k) = \begin{cases} 0 & \text{if } i \neq k \\ y_j & \text{if } i = k \end{cases}$$

$B = \{f_{ij}\}$ has mn elements

is a basis

$f \in L(V,W)$

$(f_{ij}$ is L$)$.

$f \in L(V,W)$
$\Rightarrow f(x_i) \in W$
$f(x_i) = a_{i1}y_1 + a_{i2}y_2 + \dots + a_{in}y_n$, $1 \le i \le m$

$$\Rightarrow \boxed{f(x_i) = \sum_{j=1}^{n} a_{ij} y_j}$$

$$\left.\begin{aligned} f_{ij}(x_k) &= y_j \quad i=k \quad (1) \\ f_{ij}(x_k) &= 0 \quad i\neq k \end{aligned}\right\}.$$

$$\Rightarrow f(x_k) = \sum_{j=1}^{m} (a_{kj}\, y_j)$$

$$= \sum_{j=1}^{m} a_{kj} f_{kj}(x_k) + \overset{(i\neq k)}{0} \quad \left(\text{Substitute } y_j \text{ from (1)}\right)$$

$$\Rightarrow f(x_k) = \sum_{i=1}^{m}\left(\sum_{j=1}^{m} (c_{ij} f_{ij}(x_k))\right) \quad (i=k).$$

For arbitrary input x_k.

$$\Rightarrow f = \sum_{i=1}^{m}\left(\sum_{j=1}^{m}(c_{ij} f_{ij})\right) \quad \left(\text{as } f(x) = g(x)\ \forall x \Rightarrow f=g\right)$$

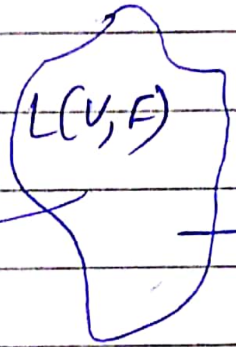
$\Rightarrow f_{ij}$ spans $L(V, W)$.

To show: Linearly Independent

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} f_{ij} = \bar{0}_W \Rightarrow c_{ij} = 0.$$

DUAL SPACE

$L(V, V) \rightarrow$ Linear operatn
$L(V, F) \longrightarrow$ Linear functnal



Dual space ← → forms a vector space called dual space.

Let $V(F)$ be a vector space over field F. A linear transformatn from $V(F)$ to F is called linear functional. Then $\phi : V \longrightarrow F$, satisfying that $\phi(ax+by) = a\phi(x) + b\phi(y)$ for every $x, y \in V$, $a, b \in F$.

The vector space $L(V,F)(F)$ consisting of linear functnals is called a dual space of $V(F)$ & is denoted by $V^*(F)$. $\boxed{\phi_i \in \{V^*(F)\}}$

Q. Let $V(F) \rightarrow$ set of all n square matrix.
$\phi : V \rightarrow F$, $\phi(A) = tr[A]$
(L. functn)