Gödel's Incompleteness Theorems Summary

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1 Summary

Gödel's incompleteness theorems are set of two theorem of mathematical logic that deals with logic and philosophy of mathematics. The incompleteness theorem is applied to **formal system** which is defined as an abstract structure used for inferring theorems from axioms according to a set of rules. The application of this theorems are in formal systems that are of sufficient complexity to express the basic arithmetic of the natural numbers and which are **consistent, and effectively axiomatized**. The incompleteness theorems shows that system that contain sufficient amounnt of arithmetic can not posses all the three properties. The incompleteness theorems are as follow :

1. **First Incompleteness Theorem** : Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete,i.e., there are statements of the language of F which can neither be proved nor disproved in F.

2. **Second Incompleteness Theorem** : Assume F is a consistent formalized system which contains elementary arithmetic. $F \forall Cons(F)$.The standard proof of the second incompleteness theorem assumes that the provability predicate $Prov_A(P)$ satisfies the **Hilbert–Bernays provability** conditions

The second theorem is stronger than the first incompleteness theorem because the statement constructed in the first incompleteness theorem does not directly express the consistency of the system. The proof of the second incompleteness theorem is obtained by formalizing the proof of the first incompleteness theorem within the system F itself.In Godel's second theorem the formula $Cons(F)$ is a particular expression of consistency.
Employing a diagonal argument, Gödel's incompleteness theorems were the first of several closely related theorems on the limitations of formal systems. They were followed by **Tarski's undefinability** theorem on the formal undefinability of truth, **Church's proof** that **Hilbert's Entscheidungs** problem is unsolvable, and **Turing's theorem** that there is no algorithm to solve the halting problem.
The theorems are widely, but not universally, interpreted as showing that Hilbert's program to find a complete and consistent set of axioms for all mathematics is impossible. Rosser proved in 1936 that the hypothesis of $\omega - consistency$, which was an integral part of Gödel's original proof, could be replaced by simple consistency, if the Gödel sentence was changed in an appropriate way. These developments left the incompleteness theorems in essentially their modern form.The theorem received criticism from some known pioneer's such as **Finsler, Zermelo, Wittgenstein** but still we apply this theorem as per our need.