Gödel's Incompleteness Theorems Summary

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1 Summary

Gödel's incompleteness theorems are set of two theorem of mathematical logic that deals with logic and philosophy of mathematics. The incompleteness theorem is applied to **formal system** which is defined as an abstract structure used for inferring theorems from axioms according to a set of rules. The application of this theorems are in formal systems that are of sufficient complexity to express the basic arithmetic of the natural numbers and which are **consistent, and effectively axiomatized**. The incompleteness theorems shows that system that contain sufficient amounnt of arithmetic can not posses all the three properties. The incompleteness theorems are as follow :

1. **First Incompleteness Theorem** : Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete,i.e., there are statements of the language of F which can neither be proved nor disproved in F. The first incompleteness theorem shows that the Gödel sentence G_F of an appropriate formal theory F is unprovable in F.
   Although the Gödel sentence refers indirectly to sentences of the system F, when read as an arithmetical statement the Gödel sentence directly refers only to natural numbers. It asserts that no natural number has a particular property, where that property is given by a primitive recursive relation. Because, when interpreted as a statement about arithmetic, this unprovability is exactly what the sentence (indirectly) asserts, the Gödel sentence is, in fact, true. For this reason, the sentence G_F is often said to be "true but unprovable."

2. **Second Incompleteness Theorem** : Assume F is a consistent formalized system which contains elementary arithmetic. $F\forall Cons(F)$.The standard proof of the second incompleteness theorem assumes that the provability predicate $Prov_A(P)$ satisfies the **Hilbert–Bernays provability** conditions.
   The second theorem is stronger than the first incompleteness theorem because the statement constructed in the first incompleteness theorem does not directly express the consistency of the system. The proof of the second incompleteness theorem is obtained by formalizing the proof of the first incompleteness theorem within the system F itself.In Godel's second theorem the formula $Cons(F)$ is a particular expression of consistency.

Employing a diagonal argument, Gödel's incompleteness theorems were the first of several closely related theorems on the limitations of formal systems. They were followed by **Tarski's undefinability** theorem on the formal undefinability of truth, **Church's proof** that **Hilbert's Entscheidungs** problem is unsolvable, and **Turing's theorem** that there is no algorithm to solve the halting problem.