ST 516: Foundations of Data Analytics

Wilcoxon Signed-Rank Test

Wilcoxon Signed-Rank Test: Introduction

The Wilcoxon signed-rank test is used to assess whether a particular value is the 'center' of a distribution of values (one-sample setting) or differences (paired two-sample setting). We will try to clarify what we mean by 'center' later, but it turns out to be rather challenging to precisely define what the signed-rank test actually tests.

Be careful not to confuse the *signed-rank* test (today) with the *rank-sum* test (different lecture). The signed-rank test is used on single-sample or paired-sample data; the rank-sum test is used on two independent samples.

- Setting: One sample of independent observations
  - Sample of size n from population of interest: $X_1, X_2, \ldots, X_n$

  OR a sample of independent *differences* between paired observations

  - n differences between paired observations in population 1 and population 2: $D_1, D_2, \ldots, D_n$

Wilcoxon Signed-Rank Test: Procedure

To test that C_0 is the population 'center', we do the following:

1. Calculate the distance of each observation/difference from C_0.
2. Rank the observations by their distance (absolute value) from C_0. The observation closest to C_0 gets rank 1, the next closest gets rank 2, and so on.
3. Let the test statistic S be the sum of the ranks that correspond to observations *larger* than C_0.

Wilcoxon Signed-Rank Test: Example

Suppose we have observed the following data, and we wish to test the hypothesis that the ‘center’ of the differences is $C_0 = 3$ vs. the alternative that the ‘center’ is not $C_0 = 3$.

Sample A	8.6	12.3	14.0	18.1	6.3	7.5	10.7	6.2	7.0
Sample B	7.4	5.4	4.9	4.0	3.8	4.4	6.9	11.8	11.6
Differences	1.2	6.9	9.1	14.1	2.5	3.1	3.8	-5.6	-4.6

Wilcoxon Signed-Rank Test: Example

Suppose we have observed the following data, and we wish to test the hypothesis that the 'center' of the differences is $C_0 = 3$ vs. the alternative that the 'center' is not $C_0 = 3$.

1. Compute the distance of each *difference* from C_0.

Sample A	8.6	12.3	14.0	18.1	6.3	7.5	10.7	6.2	7.0
Sample B	7.4	5.4	4.9	4.0	3.8	4.4	6.9	11.8	11.6
Differences	1.2	6.9	9.1	14.1	2.5	3.1	3.8	-5.6	-4.6
Distance from $C_0 = 3$	1.8	3.9	6.1	11.1	0.5	0.1	0.8	8.6	7.6

Wilcoxon Signed-Rank Test: Example

Suppose we have observed the following data, and we wish to test the hypothesis that the 'center' of the differences is $C_0 = 3$ vs. the alternative that the 'center' is not $C_0 = 3$.

2. Rank the distances from smallest to largest.

Sample A	8.6	12.3	14.0	18.1	6.3	7.5	10.7	6.2	7.0
Sample B	7.4	5.4	4.9	4.0	3.8	4.4	6.9	11.8	11.6
Differences	1.2	6.9	9.1	14.1	2.5	3.1	3.8	-5.6	-4.6
Distance from $C_0 = 3$	1.8	3.9	6.1	11.1	0.5	0.1	0.8	8.6	7.6
Distance Ranks	4	5	6	9	2	1	3	8	7

Wilcoxon Signed-Rank Test: Example

Suppose we have observed the following data, and we wish to test the hypothesis that the 'center' of the differences is $C_0 = 3$ vs. the alternative that the 'center' is not $C_0 = 3$.

3. Add up ranks corresponding to differences *greater* than C_0.

Sample A	8.6	12.3	14.0	18.1	6.3	7.5	10.7	6.2	7.0
Sample B	7.4	5.4	4.9	4.0	3.8	4.4	6.9	11.8	11.6
Differences	1.2	6.9	9.1	14.1	2.5	3.1	3.8	-5.6	-4.6
Distance from $C_0 = 3$	1.8	3.9	6.1	11.1	0.5	0.1	0.8	8.6	7.6
Distance Ranks	4	5	6	9	2	1	3	8	7

S = 5 + 6 + 9 + 1 + 3 = 24

Reference Null Distribution

How do we decide whether a resulting signed-rank test statistic is 'significant'? How do we compute p-values?

We use a *permutation-type* approach to decide how unusual an observed test statistic is if each rank is equally likely to correspond to an observation larger than C_0 or smaller than C_0.

- If the population distribution were symmetric about C_0, each rank $1, \ldots, n$ has probability $\frac{1}{2}$ of being assigned to an observation above C_0.
- We can consider all possible ways of assigning the ranks $1, \ldots, n$ above and below C_0 to work out the exact reference distribution for the test statistic S.

Reference Null Distribution: Normal Approximation

For large sample sizes, it is time-consuming to work out the exact reference distribution. Instead, we can use a Normal approximation to the reference null distribution of S:

$$S \mathrel{\dot\sim} \text{Normal}\left(\frac{n(n+1)}{4}, \frac{n(n+1)(2n+1)}{24}\right)$$

when the population distribution is symmetric around C_0.

We can therefore construct a z-statistic and compare it to a standard normal reference distribution:

$$Z = \frac{S - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \mathrel{\dot\sim} \text{Normal}(0,1)$$

Reference Null Distribution: Example

Sample A	8.6	12.3	14.0	18.1	6.3	7.5	10.7	6.2	7.0
Sample B	7.4	5.4	4.9	4.0	3.8	4.4	6.9	11.8	11.6
Differences	1.2	6.9	9.1	14.1	2.5	3.1	3.8	-5.6	-4.6
Distance from $C_0 = 3$	1.8	3.9	6.1	11.1	0.5	0.1	0.8	8.6	7.6
Distance Ranks	4	5	6	9	2	1	3	8	7

S = 5 + 6 + 9 + 1 + 3 = 24

Using our example from before, we compute the z-statistic:

$$Z = \frac{S - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} = \frac{24 - \frac{9(10)}{4}}{\sqrt{\frac{9(10)(19)}{24}}} = 0.177$$

and find the p-value:

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2*(1 - pnorm(abs(0.1777)))
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##   0.859
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Signed-Rank Test Assumptions and Summary

Assumptions required to perform the signed-rank test:

- Individual observations in each sample are independent of each other (as usual)... and that's about it!

What have we learned after performing the Wilcoxon Signed-Rank test?

- It is very difficult to say: we compute the null distribution assuming the population is symmetric around its center—but the signed-rank test does not answer the question 'Is the population symmetric around C_0?'.
- The signed-rank test also ***does not*** answer questions about means or medians (...unless we assume that the population distribution *is* symmetric—a pretty big and unverifiable assumptions!).

It is important that you know how the signed-rank test works and what it does so that you can understand what people are doing if they use it—but keep in mind the above cautions: it is difficult to precisely say what you learn from a signed-rank test.