ST 516: Foundations of Data Analytics

The z-test, p-values, and CI/Hypothesis Test Duality

Hypothesis Test for Population Mean: The z-test

Recall from the previous lecture that a hypothesis test has the following components:

- Parameter of interest
- Sample
- Hypotheses: Null hypothesis and Alternative hypothesis
- Test statistic
- Rejection region

We want to choose a rejection region that has the desired *significance level* α.

Hypothesis Test for Population Mean: The z-test

When our parameter of interest is a population mean μ, we perform what is called a **z-test** (or t-test, which we will discuss in later modules).

Hypotheses: $H_0 : \mu = \mu_0$ vs. $H_A : \mu \neq \mu_0$.

Remember the z-statistic from last lecture:

$$Z = \frac{\overline{X} - \mu_0}{\sqrt{\sigma^2 / n}}$$

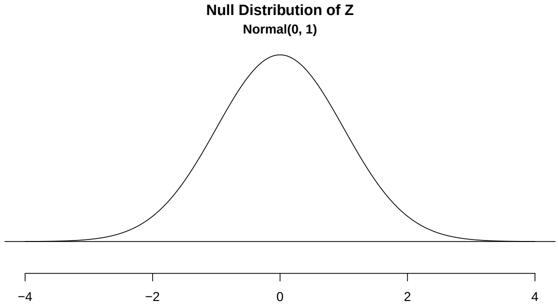
Note, however, that we almost never know the population variance σ^2, so we use the sample variance s^2 in place of σ^2, giving the following more commonly used z-statistic:

$$\boxed{Z = \frac{\overline{X} - \mu_0}{\sqrt{s^2 / n}}}$$

Hypothesis Test for Population Mean: The z-test

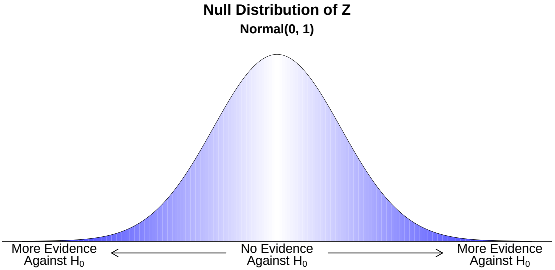
When the null hypothesis is true, we have $Z \dot{\sim} N(0,1)$. That is, the z-statistic has an approximately standard normal distribution when the null hypothesis is true.

We call the distribution of a test statistic when the null hypothesis is true the **null distribution** of the statistic.



Hypothesis Test for Population Mean: The z-test

Remember that very large (positive) or very small (negative) values of Z (*large absolute values of Z*) mean that the observed sample mean $\overline{X}$ is *far* from the hypothesized population mean μ_0. This should lead us to *reject* the null hypothesis $H_0 : \mu = \mu_0$.



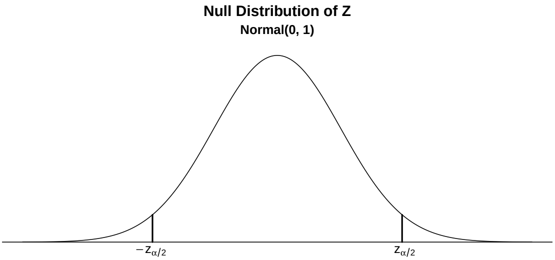
The question is: how far is far enough to reject?

z-test Critical Value

We want a test procedure with a specified significance level α.

Definition

The **critical value** $z_{\alpha/2}$ is the value such that the probability of a Normal(0, 1) random variable being larger than $z_{\alpha/2}$ is $\alpha/2$. By symmetry, this also means the probability of a Normal(0, 1) random variable being less than $-z_{\alpha/2}$ is also $\alpha/2$.



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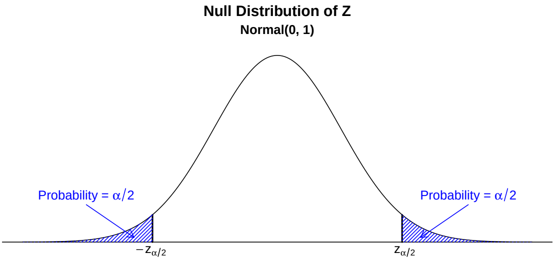


z-test Critical Value

You can find the critical value $z_{\alpha/2}$ in R by first setting the level α that you would like your test to have. Here we want a level $\alpha = 0.05$ test, so we use

```
alpha <- 0.05
```

Then you have R calculate the critical value

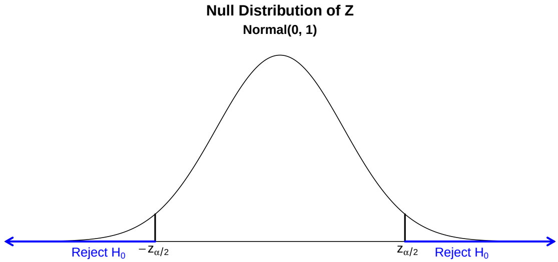
```
qnorm(1 - alpha/2)
```

```
## [1] 1.959964
```

This gives us the appropriate value of $z_{\alpha/2}$, which when $\alpha = 0.05$ is $z_{0.025} \approx 1.96$.

Hypothesis Test for Population Mean: The z-test

We reject the null hypothesis $H_0 : \mu = \mu_0$ when the z-statistic is less than $-z_{\alpha/2}$ or greater than $z_{\alpha/2}$, so we have probability $\alpha/2 + \alpha/2 = \alpha$ of rejecting the null hypothesis *when the null hypothesis is true*. That means this gives a level α test of H_0.



By symmetry, this rejection region could equivalently be stated as

$$\text{Reject } H_0 \text{ when } |Z| > z_{\alpha/2}$$

z-test Example

Recall the example from last lecture: You sample $n = 20$ dogs who visit the OSU veterinary clinic in 2014, and measure their body temperature X. You would like to test whether this population of dogs has normal body temperature (101°F for healthy dogs).

Suppose the sample mean body temperature for your sample of 20 dogs is 100.1, and the sample variance is $s^2 = 4$. Would we reject the null hypothesis $H_0 : \mu = 101$ at level $\alpha = 0.05$?

First, compute the z-statistic:

$$Z = \frac{100.1 - 101}{\sqrt{4/20}} = -2.012$$

Then compare the resulting absolute value of Z to the critical value $z_{\alpha/2}$, which for $\alpha = 0.05$ we found previously to be $z_{0.025} = 1.96$:

$$|Z| = 2.012 > 1.96 \qquad \text{so we Reject } H_0 \text{ at level } \alpha = 0.05$$

p-values

Definition

A **p-value** associated with a hypothesis test of some null hypothesis H_0 vs. some alternative H_A is the probability under the null hypothesis of observing a result at least as *extreme* as the statistic you observed. 'Extreme' here means in the direction of the alternative (that is, the direction of the rejection region).

Example: if our observed sample mean dog temperature were 100.1, the p-value for the test of $H_0 : \mu = 101$ vs. $H_A : \mu \neq 101$ would be the probability of a z-statistic with a *larger* absolute value than our observed value of $|-2.012| = 2.012$ when the null hypothesis is true. We compute the p-value as

```
2*(1 - pnorm(2.012)
```

```
## 0.04421994
```

Alternative p-value Definition

If the p-value you compute is less than the significance level α that you were interested in, you would *reject* the null hypothesis H_0 at level α. This is due to the following alternative, but equivalent, definition of a p-value:

Definition

A **p-value** associated with a hypothesis test of some null hypothesis H_0 vs. some alternative H_A is the *smallest* significance level α' at which the null hypothesis H_0 would be rejected in favor of H_A.

This second definition is a bit more challenging to parse, but it means the same thing.

Recall that if we reject the null hypothesis at significance level α_1, then we would also reject at significance level α_2 for any $\alpha_2 > \alpha_1$. For instance, if we reject the null hypothesis at level $\alpha = 0.06$, then we would also reject at level $\alpha = 0.07$ or $\alpha = 0.12$.

p-value interpretations

It is important to correctly interpret a p-value:

- A p-value is the probability of observing a result at least as extreme as the data we observed, if the null hypothesis is true.
- A p-value is *NOT* the probability that the null hypothesis is true.
- CORRECT: The p-value of 0.044 tells us that when the true population mean is 101 (null hypothesis is true), the probability of observing a sample mean at least as far from 101 as 100.1 is 0.044.
- INCORRECT: The p-value of 0.044 tells us that based on this sample mean of 100.1, there is a 0.044 probability that the population mean is 101 (null hypothesis is true).

Hypothesis Test and Confidence Interval Duality

Recall that a confidence interval for a parameter θ is the set of *plausible* values for that parameter, based on the observed data.

A hypothesis test of a null hypothesis H_0 vs. an alternative hypothesis H_A *rejects* the null hypothesis if the null hypothesis is *not plausible*. Equivalently, the hypothesis test *fails to reject* the null hypothesis if the null hypothesis *is plausible*.

Combining these two ideas, we obtain the following connection between confidence intervals and hypothesis tests:

A level $(1-\alpha)100\%$ confidence interval is the set of all parameter values, θ_0, for which a null hypothesis $H_0: \theta = \theta_0$ would *not* be rejected at level α in favor of $H_A: \theta \neq \theta_0$.

Hypothesis Test and Confidence Interval Duality

Restatement of this relationship:

- If a value θ_0 *is in* a 95% confidence interval, then $H_0 : \theta = \theta_0$ would *not be rejected* at level $\alpha = 0.05$ in favor of the alternative $H_A : \theta \neq \theta_0$.
- If a hypothesis $H_0 : \theta = \theta_0$ is *not rejected* in favor of $H_A : \theta \neq \theta_0$ at level $\alpha = 0.05$, then θ_0 would be *in* the 95% confidence interval.
- This means that if you are given a $(1-\alpha)100\%$ confidence interval for a parameter, you can quickly determine whether a given hypothesis would be rejected at level α.

Hypothesis Test and Confidence Interval Duality Example

Example: If the sample mean is 100.1°F, with a sample size of $n = 20$ and a sample variance of $s^2 = 4$, the 95% confidence interval for the population mean dog temperature μ for dogs visiting the OSU Veterinary Clinic in 2014 is

$$\left(100.1 - 1.96\sqrt{\frac{4}{20}},\ 100.1 + 1.96\sqrt{\frac{4}{20}}\right) = (99.2235,\ 100.9765)$$

Would $H_0 : \mu = 101.5$ be rejected at level $\alpha = 0.05$ in favor of $H_A : \mu \neq 101.5$?

Hypothesis Test and Confidence Interval Duality Example

Example: If the sample mean is $100.1°$F, with a sample size of $n = 20$ and a sample variance of $s^2 = 4$, the 95% confidence interval for the population mean dog temperature μ for dogs visiting the OSU Veterinary Clinic in 2014 is

$$\left(100.1 - 1.96\sqrt{\frac{4}{20}}, 100.1 + 1.96\sqrt{\frac{4}{20}}\right) = (99.2235, 100.9765)$$

Would $H_0 : \mu = 101.5$ be rejected at level $\alpha = 0.05$ in favor of $H_A : \mu \neq 101.5$?

Yes. Since 101.5 is *NOT* in the 95% confidence interval, we would *reject* $H_0 : \mu = 101.5$ in favor of $H_A : \mu \neq 101.5$.