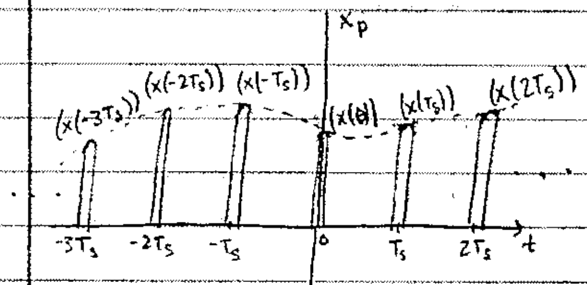
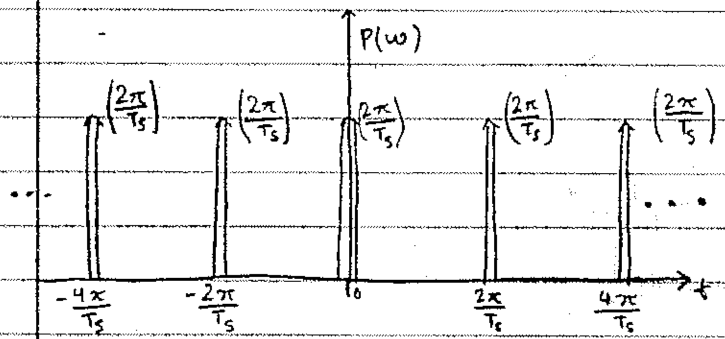
Pratool Gadtaula Problem Set 8

1. a) $x_p(t) = x(t)\,p(t)$

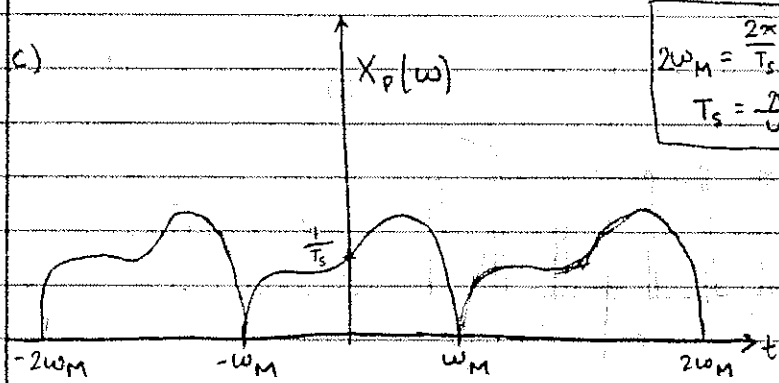


b) $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$

$P(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T_s}k\right)$



c)



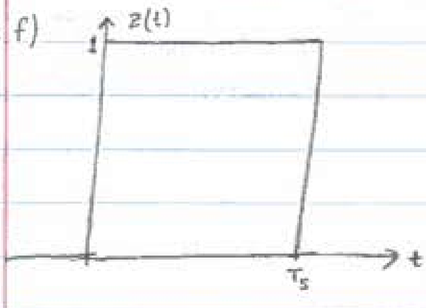
$$\boxed{\begin{aligned} 2\omega_M &= \frac{2\pi}{T_s} \\ T_s &= \frac{\pi}{\omega_M} \end{aligned}}$$

d.) $\omega_M \geq \frac{2\pi}{T_s}$

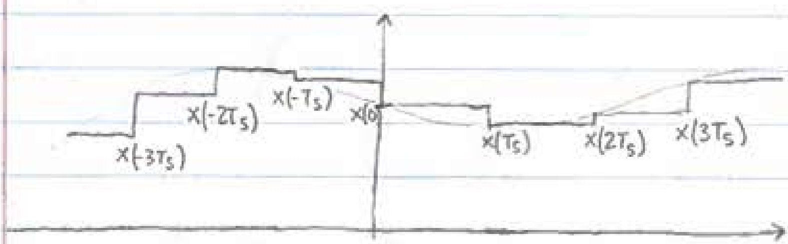
e) Low-pass filter $X_p(\omega)$ with a cut-off frequency of ω_M.

Pratool Gadtaula Signals and Systems Problem Set 8

1. f)



g) $x_z(t) = x_p * z(t)$



h) $z(t)$ is a boxcar window

$z(t+\frac{T_s}{2})$ looks like a low-pass filter. $FT\{z(t+\frac{T_s}{2})\} = Z(\omega)e^{-j\omega\frac{T_s}{2}}$

$$FT\{z(t)\} = Z(\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,dt = \int_0^{T_s} z(t)e^{-j\omega t}\,dt = \int_0^{T_s}(1)e^{-j\omega t}\,dt$$

$$Z(\omega) = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^{T_s} = -\frac{1}{j\omega}\left[e^{-j\omega T_s} - 1\right]$$

$$Z(\omega)e^{-j\omega\frac{T_s}{2}} = -\frac{1}{j\omega}\left[e^{-j\omega T_s} - 1\right]e^{-j\omega\frac{T_s}{2}} = -\frac{1}{j\omega}\left[e^{-\frac{3}{2}j\omega T_s} - e^{-j\omega\frac{T_s}{2}}\right]$$

$$Z(\omega)e^{-j\omega\frac{T_s}{2}} = -\frac{1}{j\omega}e^{-\frac{3}{2}j\omega T_s} + \frac{1}{j\omega}e^{-j\omega\frac{T_s}{2}}$$

Let's take the Fourier Transform of $z(t+\frac{T_s}{2})$

$$FT\{z(t+\frac{T_s}{2})\} = \int_{-\infty}^{\infty} z(t+\frac{T_s}{2})e^{-j\omega t}\,dt = \int_{-T_s/2}^{T_s/2}(1)e^{-j\omega t}\,dt = \frac{1}{-j\omega}\left[e^{-j\omega\frac{T_s}{2}} - e^{-j\omega(-\frac{T_s}{2})}\right]$$

$$= \frac{2}{2j\omega}\left[e^{j\omega\frac{T_s}{2}} - e^{-j\omega\frac{T_s}{2}}\right] = \frac{2}{\omega}\sin\left(\frac{T_s}{2}\omega\right)$$

Now shift the time back in the frequency domain

$$Z(\omega) = e^{-j\omega(-\frac{T_s}{2})}\cdot\frac{2}{\omega}\sin\left(\frac{T_s}{2}\omega\right)$$

$$Z(\omega) = \frac{2}{\omega}\sin\left(\frac{T_s}{2}\omega\right)\cdot e^{j\omega\frac{T_s}{2}}$$

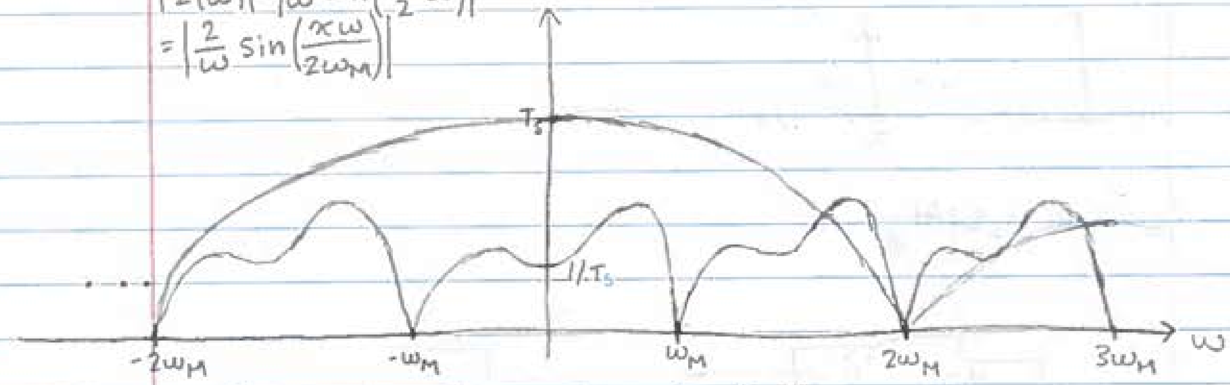
h) cont'd $Z(\omega) = \frac{2}{\omega}\sin\left(\frac{T_s}{2}\omega\right) e^{j\omega\frac{T_s}{2}}$ → this is complex, so we just want the magnitude

↓ this complex number has magnitude of 1

$X_Z(\omega) = X_p(\omega)\, Z(\omega)$

$|Z(\omega)| = \left|\frac{2}{\omega}\sin\left(\frac{T_s}{2}\omega\right)\right|$

$= \left|\frac{2}{\omega}\sin\left(\frac{\pi\omega}{2\omega_M}\right)\right|$



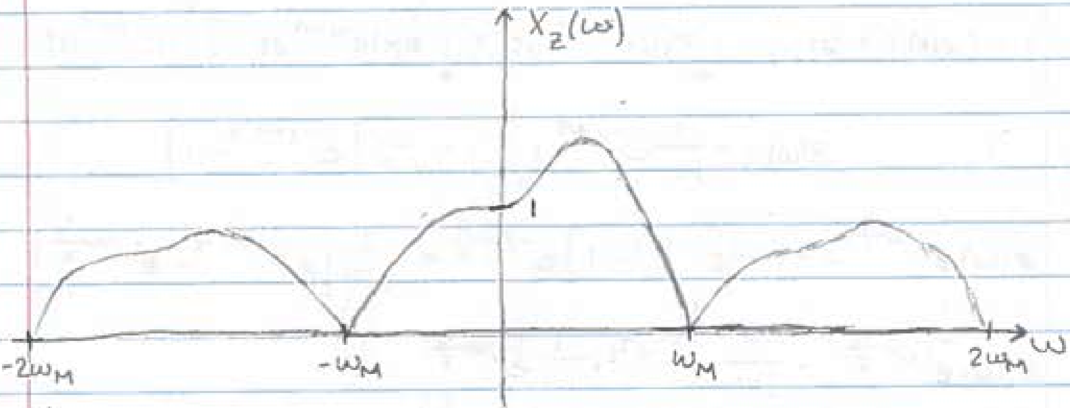
$\lim_{\omega\to 0}\left[\frac{2}{\omega}\sin\left(\frac{T_s}{2}\omega\right)\right] = \lim_{\omega\to 0}\frac{2\cos\left(\frac{T_s}{2}\omega\right)\cdot\frac{T_s}{2}}{1} = T_s$

For the sake of graphing, assume $T_s > 1$ and $T_s = \frac{\pi}{\omega_M}$

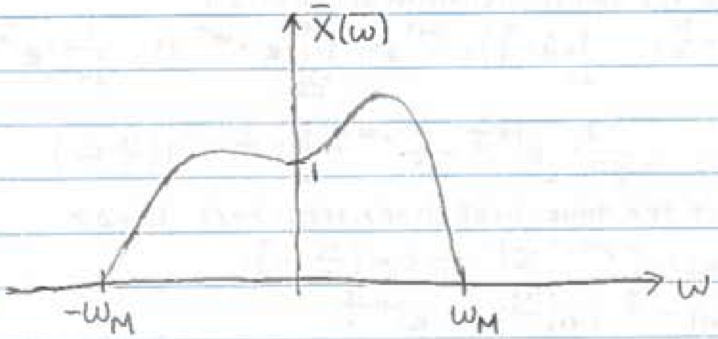
$\left|\frac{2}{\omega}\sin\left(\frac{\pi\omega}{2\omega_M}\right)\right| = 0$ when $\frac{\pi\omega}{2\omega_M} = \pi$

$\omega = 2\omega_M$

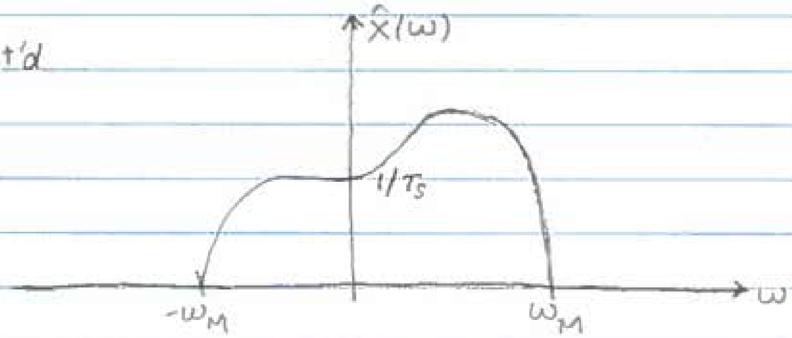
$\forall k,\ k\in\mathbb{Z},\ k\neq 0,\ |Z(2k\omega_M)| = 0$



i)



1. i) cont'd



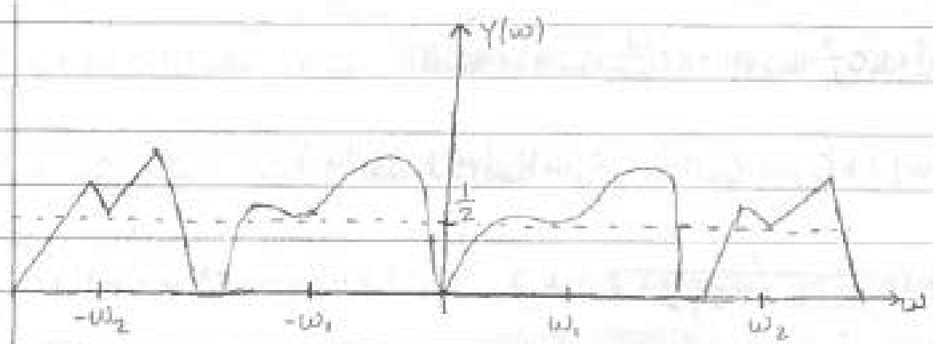
j) $\bar{X}(\omega)$ is a more squished down form of 1 period of $X_p(\omega)$ because the sinc of $Z(\omega)$ causes that to happen. $\hat{X}(\omega)$ is exactly 1 period of $X_p(\omega)$.

k. $$\frac{\bar{X}(\omega_M)}{\hat{X}(\omega_M)} = \frac{X_z(\omega_M)}{X_p(\omega_M)} = \frac{X_p(\omega_M)\, Z(\omega_M)}{X_p(\omega_M)} = Z(\omega_M)$$

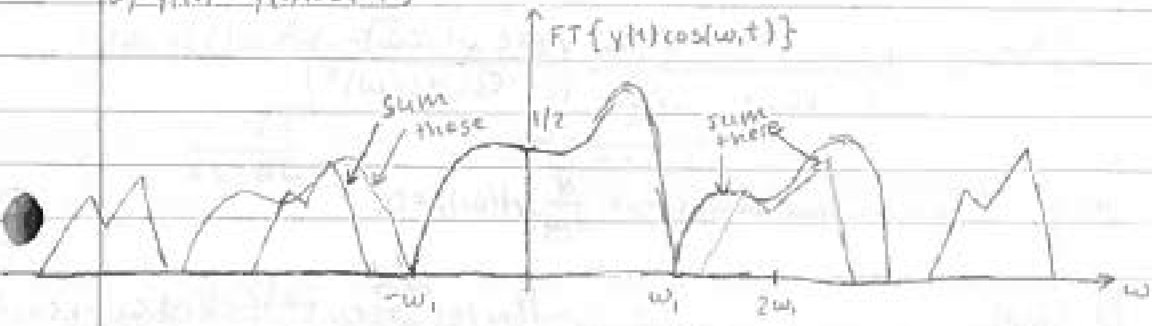
$$Z(\omega_M) = \frac{2}{\omega_M} \sin\left(\frac{\pi \omega_M}{2\omega_M}\right) = \boxed{\frac{2}{\omega_M}}$$

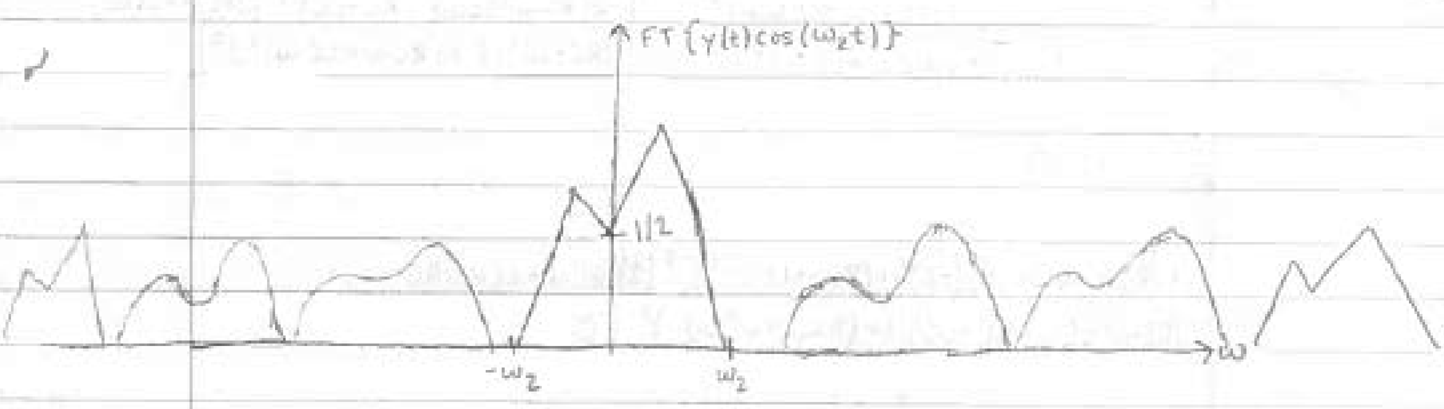
2. a) $y(t) = x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t)$

$$Y(\omega) = \frac{1}{2\pi} X_1(\omega) * [\pi\delta(\omega - \omega_1) + \pi\delta(\omega + \omega_1)] + X_2(\omega) * [\pi\delta(\omega - \omega_2) + \pi\delta(\omega + \omega_1)]$$



b) $y_1(t) = y(t)\cos(\omega_1 t)$





c) You can recover $x_1(t)$ from $y(t)$ by multiplying $y(t)$ with a cosine wave with the frequency $x_1(t)$ was originally multiplied with (ω_1). Then, because the original $x_1(t)$ signal was band limited from $-\omega_M$ to ω_M, you would low-pass filter $x_1(t)$ with a cut-off frequency of ω_M. To recover $x_2(t)$ from $y(t)$, you would would do the exact same procedure as above, except that the multiplied cosine's frequency is ω_2. Then you would have to scale both of these signals by 2 for the original amplitude.

3. a) $i(t) = C\frac{d}{dt} v_{out}(t)$ $\quad v_{in}(t) = v_R(t) + v_L(t) + v_{out}(t)$

$v_L(t) = L\frac{d}{dt} i(t)$ $\quad v_{in}(t) = i(t)R + L\frac{d}{dt} i(t) + v_{out}(t)$

$$v_{in}(t) = RC\frac{d}{dt} v_{out}(t) + LC\frac{d}{dt}\left(\frac{d}{dt} v_{out}(t)\right) + v_{out}(t) = RC\dot{v}_{out}(t) + LC\ddot{v}_{out}(t) + v_{out}(t)$$

$$\boxed{v_{in}(t) = RC\dot{v}_{out}(t) + LC\ddot{v}_{out}(t) + v_{out}(t)}$$

b) $V_{in}(\omega) = RCj\omega V_{out}(\omega) + LC(j\omega)^2 V_{out}(\omega) + V_{out}(\omega)$

$$\frac{V_{in}(\omega)}{V_{out}(\omega)} = RCj\omega - LC\omega^2 + 1$$

$$\boxed{H(\omega) = \frac{1}{RCj\omega - LC\omega^2 + 1}}$$

c) $|H(\omega)| = \left|\frac{1}{RCj\omega - LC\omega^2 + 1}\right|$

$$\boxed{|H(\omega)| = \frac{1}{\sqrt{(1-LC\omega^2)^2 + (RCj\omega)^2}} = \left[(1-LC\omega^2)^2 + (RCj\omega)^2\right]^{-1/2}}$$

d) $\frac{d}{d\omega}|H(\omega)| = -\frac{1}{2}\left[(1-LC\omega^2)^2 + (RCj\omega)^2\right]^{-3/2} \cdot \left[2(1-LC\omega^2)\cdot(-2LC\omega) + 2(RCj)^2\omega\right]$

$$\frac{d}{d\omega}|H(\omega)| = 0$$

$$\frac{\frac{1}{2}\left[2(1-LC\omega^2)(-2LC\omega) + 2(RCj)^2\omega\right]}{\left[(1-LC\omega^2)^2 + (RCj\omega)^2\right]^{3/2}} = 0$$

$$(1-LC\omega^2)(-2LC\omega) + (RCj)^2\omega = 0$$

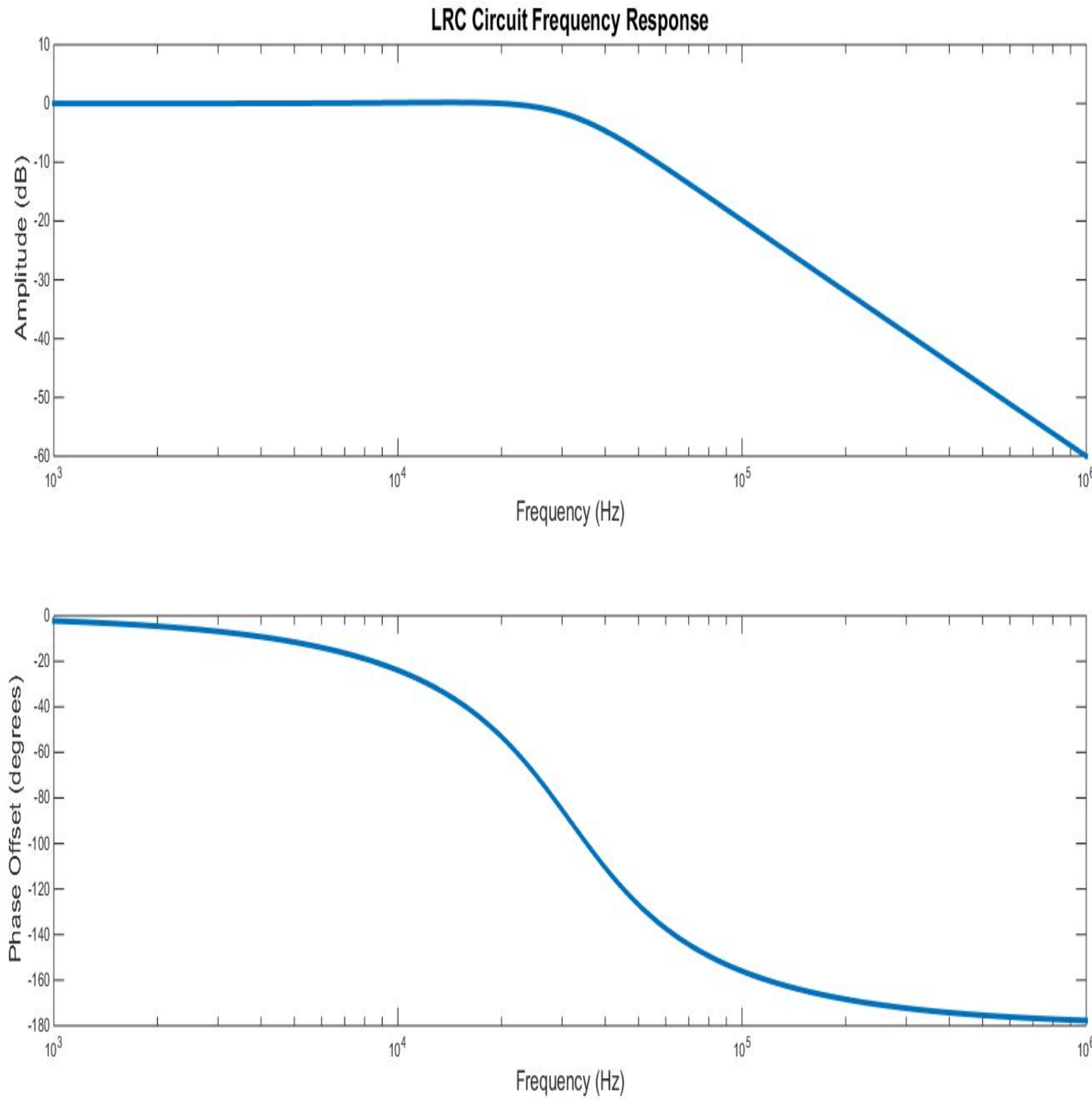
$$-2LC\omega + 2(LC)^2\omega^3 + (RC)^2(-1)\omega = 0$$

$$2L^2C^2\omega^3 - (2LC + R^2C^2)\omega = 0$$

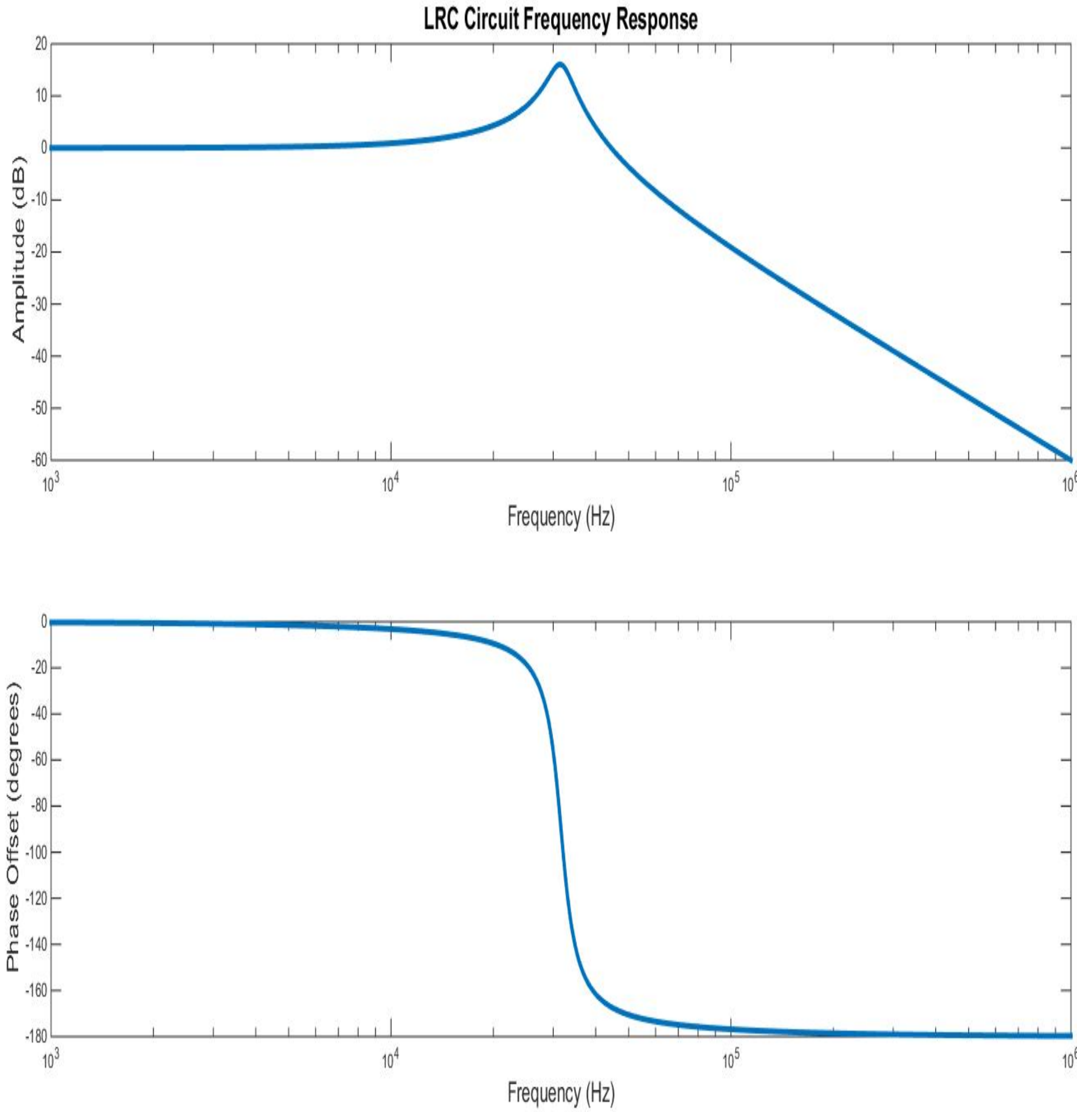
$$2L^2C^2\omega^2 - 2LC - R^2C^2 = 0$$

$$\omega = \pm\sqrt{\frac{2LC + R^2C^2}{2L^2C^2}} = \pm\sqrt{\frac{1}{LC} + \frac{R^2}{2L^2}}$$

$$\boxed{\omega = \pm\sqrt{\frac{1}{LC} + \frac{R^2}{2L^2}}}$$



C = 10^-7 F,   L = 10^-2 H,   R = 400 Ohm



C = 10^-7 F, L = 10^-2 H, R = 50 Ohm