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DEPT. OF PHYSICS

Blackholes and Gravitational Collapse

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Sept 2018 - Nov 2018

Acknowledgements

I am profoundly grateful to Prof. Kinjal Banerjee for his expert guidance and continuous encouragement throughout to see that this project is true to its target since its commencement to its completion.

I would like to express my deepest appreciation towards Mr. Surendra Padamata for his help in finding appropriate resources.

At last I must express my sincere heartfelt gratitude to my friends, family members and teachers who helped me directly or indirectly through out the project.

- Pratyush

Abstract

The project involved a review of Schwarzschild Blackholes, The Chandrasekhar Limit and Pressure Free Gravitational Collapse.

1 Introduction

1.1 Motivation

Singularities are an essential features of General Relativity as proved by Penrose and Hawking and many interesting phenomena can be observed around singularities. One of the widely studied phenomena are blackholes, the aim of the project was to review features of Schwarzchild Blackholes, conditions for formation of Blackholes (Chandrasekhar Limit) and process of gravitational collapse from a star to Blackhole.

1.2 Schwarzschild Metric

Karl Schwarzschild proposed a solution of Einstein Field Equations in 1916. The solution described an asymptotically flat space-time around a spherically symmetric, non rotating and chargeless but static massive body.[2]

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 + r^2 d\Omega$$

Since the body is static, there exists a time symmetry. Outside the body the $R_{\mu\nu} = 0$. To derive this metric we start from a metric of a very general form with the following features : spherical symmetry and time symmetry.

$$\begin{aligned}
ds^2 &= -e^{-2\alpha(t)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega \\
\bar{r} &= e^{\gamma(r)}r \\
d\bar{r} &= e^{\gamma}\Big(1 + r\frac{d\gamma}{dr}\Big) \\
ds^2 &= e^{=2\alpha}dt^2 + e^{2\beta}e^{-2r}\Big(1 + r\frac{d\gamma}{dr}\Big)^{-2}d\bar{r}^2 + r^2\Omega \\
\bar{r} &\to r \qquad e^{2\beta}e^{-2r}\Big(1 + r\frac{d\gamma}{dr}\Big)^{-2} \to e^{2\beta} \\
ds^2 &= e^{-2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2
\end{aligned}$$

Since outsde the star $R_{\mu\nu}$ is zero. Making each component of the tensor zero yeilds us the following :

$$\begin{aligned}
R_{tt} = R_{rr} = 0 &\Rightarrow \exp 2(\beta - \alpha)R_{tt} + R_{rr} = 0 \Rightarrow \\
\alpha &= -\beta \\
R_{\theta\theta} = 0 &\Rightarrow \exp 2\alpha = 1 - R_s/r
\end{aligned}$$

R_s is called the Schwarzschild radius. Metric equation becomes :

$$ds^2 = -\Big(1 - \frac{R_s}{r}\Big)dt^2 + \Big(1 - \frac{R_s}{r}\Big)^{-1}dr^2 + r^2 d\Omega$$

In the weak field limit, the tt component of metric around a point mass gives $R = 2GM$. This parameter M can interpreted as the Newtonian mass of the star. The mass here is what can be measured by observing orbits of bodies around the star.

1.2.1 Singularities

We notice that the tt component of the metric blows up at r=0 and rr component blows up at R_s. However this doesn't necessarily mean that the metric or the space-time has blown up there, they could be coordinate singularities. Curvature is a reliable measure of whether sapce-time has been blown up , so if the Ricci scalar is finite it means that there is no singularity there. The Ricci scalar is $\frac{48G^2M^2}{r^6}$. $r = 0$ is a singularity but R_s is not. Nonetheless R_s

has very interesting features associated with it.

A singularity is the situation when the observables derived from the equations of the theory at hand take arbitrarily large values and hence can not be measured. The singularity at the center of the star is one such singularity, they also have the characteristics that the geodesics through it are incomplete i.e. they have a terminate at the point. One of the contemporary objectives of physics is to remove singularities from General Relativity because it breaks down at singularities and a better description of nature is required for such cases.

1.3 Geodesics of Schwarzschild Metric

The Python package 'gravipy' can been used to obtain Christoffel connections, Riemann and Ricci tensors.[1] The geodesics of the Schwarzschild Metric are :

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^2}\Big(\frac{dt}{d\lambda}\Big)^2 - \frac{GM}{r(r-2GM)}\Big(\frac{dr}{d\lambda}\Big)^2 - (r-2GM)\Big[\Big(\frac{d\theta}{d\lambda}\Big)^2 + sin^2(\theta)\Big(\frac{d\phi}{d\lambda}\Big)^2\Big] = 0$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\Big(\frac{d\phi}{d\lambda}\Big)^2 = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\cot(\theta)\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

There could be 4 Killing Vectors; one for time symmetry and 3 for spherical symmetry. We know that $K_\mu T^\mu =$ const. This implies $K_\mu \frac{dx^\mu}{d\lambda} = c$. Since there exists a time symmetry, it implies $K^\mu = (\partial_t)^\mu = (1,0,0,0)$.

$$K_\mu = g_{\mu\nu}K^\nu = \Big(2GM/r - 1, 0, 0, 0\Big)$$

2 Schwarzschild Blackholes

The calculations done above were valid for $r > 2GM$ which is an apparent singularity. Now, let us consider radial null curves i.e.

$\theta = c_1, \phi = c_2$ and $ds^2 = 0$. c_1 and c_2 are constants.

$$ds^2 = 0 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 \Rightarrow \frac{dt}{dr} = \pm\Big(1 - \frac{2GM}{r}\Big)^{-1}$$

$$r \to 2GM, \frac{dt}{dr} \to \pm\infty$$

$$r \to 0, \frac{dt}{dr} \to \pm 1/\infty$$

However this is a result of inappropriate coordinates. We make a coordinate transformation :

$$\frac{dt}{dr} = \pm\frac{1}{1 - 2GM/r} \Rightarrow t = \pm(r + a\ln(r - 2GM)) + C$$

$$t = \pm r^* + C$$

$$r^* = r + 2GM\ln(r/2GM - 1)$$

At $r = 2GM$, r^* goes to $-\infty$. That is the coordinate singularity has been pushed to infinity. This new coordinate is called the tortoise coordinate. By differentiating the above equation, we get

$$dr^* = dr\Big(\frac{r/2GM}{r/2GM - 1}\Big) \tag{1}$$

and the metric can be written in terms of the tortoise coordinate as :

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)dr^{*2} + r^2 d\Omega \tag{2}$$

We observe that the metric is non singular at $r = 2GM$. Since, we were looking at radial null geodesics $\frac{dt}{dr^*} = \pm 1$. Integrate this and we get $t \pm r = C$, C is a constant. Now, we define the two quantities $t \pm r$ as our new coordinates.

$$v = t + r^*$$

$$u = t - r^*$$

$$dv = dt + dr^*$$

And the metric (1) can be written as :

$$ds^2 = \Big(1 - \frac{2GM}{r}\Big)(-dt^2 + dr^{*2}) + r^2 d\Omega$$
$$dt^2 = (dv - dr^*)^2 = dr^{*2} + dv^2 - 2dr^* dv$$
$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dv^2 + (drdv + dvdr) + r^2 d\Omega$$

These are the Eddington-Finkelstein coordinates. Even in these coordinates $r = 2GM$ is not a singularity however r= 0 is. For null radial curves

$$dv^2\Big(1 - \frac{2GM}{r}\Big) = drdv + dvdr$$
$$\Big(\frac{dv}{dr}\Big)^2 \alpha = 2\frac{dv}{dr}$$
$$\alpha = 1 - 2GM/r$$

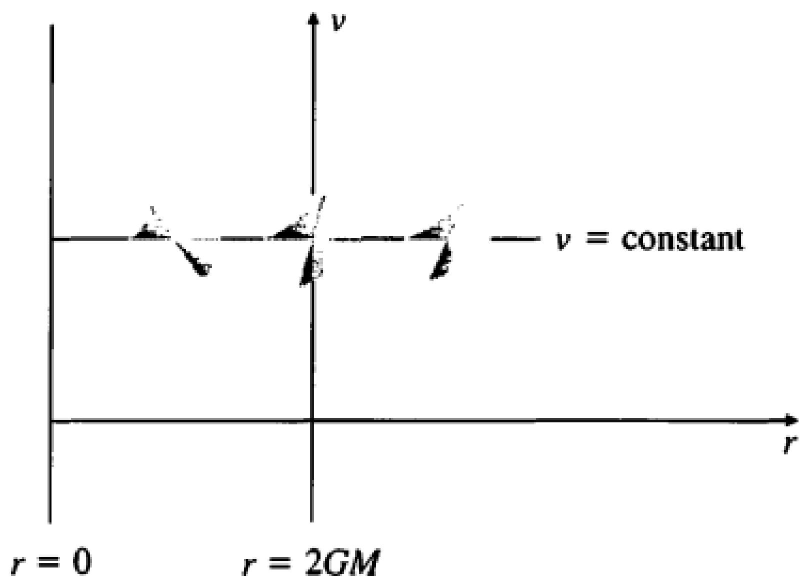
Using the above equation we have two solutions for $\frac{dv}{dr}$. We can construct space-time diagrams from them to find their physical meaning.

$$\frac{dv}{dr}\Big(\alpha\frac{dv}{dr} - 2\Big) = 0$$
$$\frac{dv}{dr} = 0$$
or
$$\frac{dv}{dr} = \frac{2}{1 - 2GM/r}$$

Since we started of with a radial null curve, our two solutions correspond to an infalling and outgoing path. The outgoing path is the second solution

$$\frac{dv}{dr} = \begin{cases} > 0 & \text{for } r > 2GM \\ \infty & \text{for } r = 2GM \\ < 0 & \text{for } r < 2GM \end{cases} \tag{3}$$

The first solution $\frac{dv}{dr} = 0$ always holds while the second one changes with radial coordinate, these are the equations describing the slopes of the light cone. Outside $r = 2GM$ Going inside as well as going outside is possible. At $r = 2GM$ only going inside is possible since the other slope is infinity or $\frac{dr}{dv} = 0$. Inside $r = 2GM$ the angle of the light cone becomes even smaller and the only possible direction of motion is towards $r = 0$.



captionTilting of light cone

2.1 A Note on Event Horizon

$R = 2GM$ is called the event horizon. Black Holes contrary to the popular perception do not suck everything in, neither the statement that their escape velocity is c entirely accurate. Outside the event horizon the gravitational pull (in Newtonian terms) is just like a star or any other massive body. However once the event horizon has been crossed, it is impossible to get out on timelike or null like paths. For an ordinary star, planet or massive body an object can escape to infinity even without attaining escape velocity i.e by a consistent acceleration however inside the event horizon even this is not possible since $\dfrac{dv}{dr} = 0$. This is what makes escaping from a blackhole different from escaping from other bodies like stars or planets.

2.2 Maximally Extended Schwarzschild Solution

In the Eddington-Finkelstein coordinates we saw that only future directed paths are allowed inside $R = 2GM$. Now, we take a look at the other coordinate $u = t - r^*$.

$$\begin{aligned}
ds^2 &= \Big(1 - \frac{2GM}{r}\Big)(-dt^2 + dr^{*2}) \\
u &= t - r^* \\
du &= dt - dr^* \\
dt^2 &= du^2 + dr^{*2} + dudr^* + dr^* du \\
ds^2 &= \Big(1 - \frac{2GM}{r}\Big)(-du^2 - dudr^* dr^* du)
\end{aligned}$$

Using eqn(1.4.1)

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)du^2 - (dudr + drdu)$$

Previously we solved for $\dfrac{dv}{dr}$ here we solve for for $\dfrac{du}{dr}$ by a similar approach and get two solutions.

$$\begin{aligned}
\frac{du}{dr} &= 0 \\
\frac{du}{dr} &= \frac{-2}{1 - 2GM/r}
\end{aligned}$$

Here, the light cones have the opposite behaviour.

$$\frac{du}{dr} = \begin{cases} < 0 & \texttt{for } r > 2GM \\ -\infty & \texttt{for } r = 2GM \\ > 0 & \texttt{for } r < 2GM \end{cases} \tag{4}$$

Outside the Event Horizon there is no restriction, however to cross the event horizon we need to go along the past directed curves or the past directed light cone allows crossing the horizon.

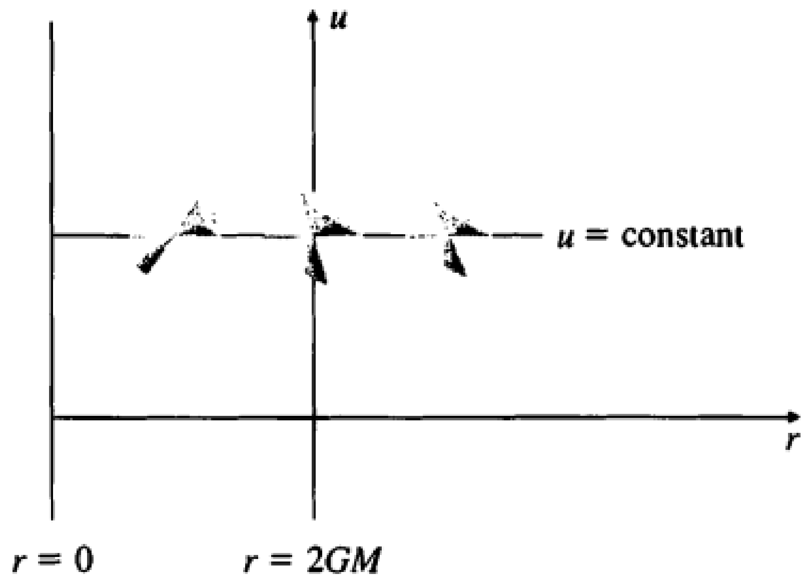


Figure 1: Tilting lightcones

We have extended the spacetime in two directions $t \to \infty$ and $t \to -\infty$. We now look at space like curves to explore other regions of the spacetime. Redefining coordinates as :

$$v' = \exp\frac{v}{4GM}$$
$$u' = -\exp\frac{u}{4GM}$$

In terms of r and t :

$$v' = \sqrt{\frac{r}{2GM} - 1}\exp{(r+t)/4GM}$$
$$u' = -\sqrt{\frac{r}{2GM} - 1}\exp{(r-t)/4GM}$$
$$dv' = \frac{v'}{4GM}dv$$
$$du' = \frac{u'}{4GM}du$$

We had the metric in terms of u and v, substitute the above equations to get

$$ds^2 = -\frac{1}{2}\Big(1 - \frac{2GM}{r}\Big)(dvdu + dudv) + r^2 d\Omega \Rightarrow$$
$$ds^2 = \frac{1}{2}\Big(1 - \frac{2GM}{r}\Big)\frac{16G^2M^2}{u'v'}(dv'du' + du'dv')$$
$$u'v' = -\Big(\frac{r}{2GM} - 1\Big)\exp{(r/2GM)} \Rightarrow$$
$$ds^2 = -16\frac{G^3M^3}{r}(du'dv' + dv'du')\exp{(-r/2GM)} + r^2 d\Omega$$

The coordinate singularity has been totally removed at 2GM and we have the extended Schwarzschild solution as stated in previous section.

2.2.1 Kruskal-Szekeres Coordinates

Since, our common coordinate system is one timelike and three spacelike coordinates, we will make a further transformation to the above coordinates even though the coordinate singularity has been entirely removed.

$$T \equiv \frac{1}{2}(v' + u') \tag{5}$$

$$R \equiv \frac{1}{2}(v' - u') \tag{6}$$

$$T = \sqrt{\left(\frac{r}{2GM} - 1\right)} e^{r/4GM} \sinh \frac{t}{4GM} \tag{7}$$

$$R = \sqrt{\left(\frac{r}{2GM} - 1\right)} e^{r/4GM} \cosh \frac{t}{4GM} \tag{8}$$

$$T + R = \exp\left(v/4GM\right) = v' \tag{9}$$

$$T - R = \exp\left(-u/4GM\right) = u' \tag{10}$$

Let us see the event horizon in these new coordinates

$r \to 2GM \quad \Rightarrow r^* \to -\infty \quad \Rightarrow v \to -\infty \quad \Rightarrow v' \to 0$

Similarly, $u' \to 0 \quad \Rightarrow$

$T = \pm R$

$T^2 - R^2 = 0$

For a constant radius surface we get

$T^2 - R^2 = v'u' = \exp\left(\frac{t + r^*}{4GM}\right) \exp\left(\frac{-t + r^*}{4GM}\right) = \exp\left(r^*/2GM\right) = const.$

$\frac{T}{R} = \tanh \frac{t}{4M}$

For $t \to \pm\infty \Rightarrow \frac{T}{R} = \pm 1$

Since the only singularity present now is $r = 0$, T and R should take all values except when they correspond to $r = 0$. Let R vary from $(-\infty, \infty)$. From eqn.(9) and (10) we get $T^2 - R^2$.

$$T^2 - R^2 = (1 - r/2GM)\exp(r/2GM) \tag{11}$$

This function is always less than or equal to unity and at $r = 0$ the function becomes unity therefore (T, R) can take the following values :

$$-\infty \leq R \leq \infty \tag{12}$$

$$T^2 - R^2 < 1 \tag{13}$$

If we construct a diagram of (T, R) coordinates, we will have certain allowed regions due to eqn.(12) and (13) and points in this region will correspond to a given sphere or surface of constant r due to eqn.(11).

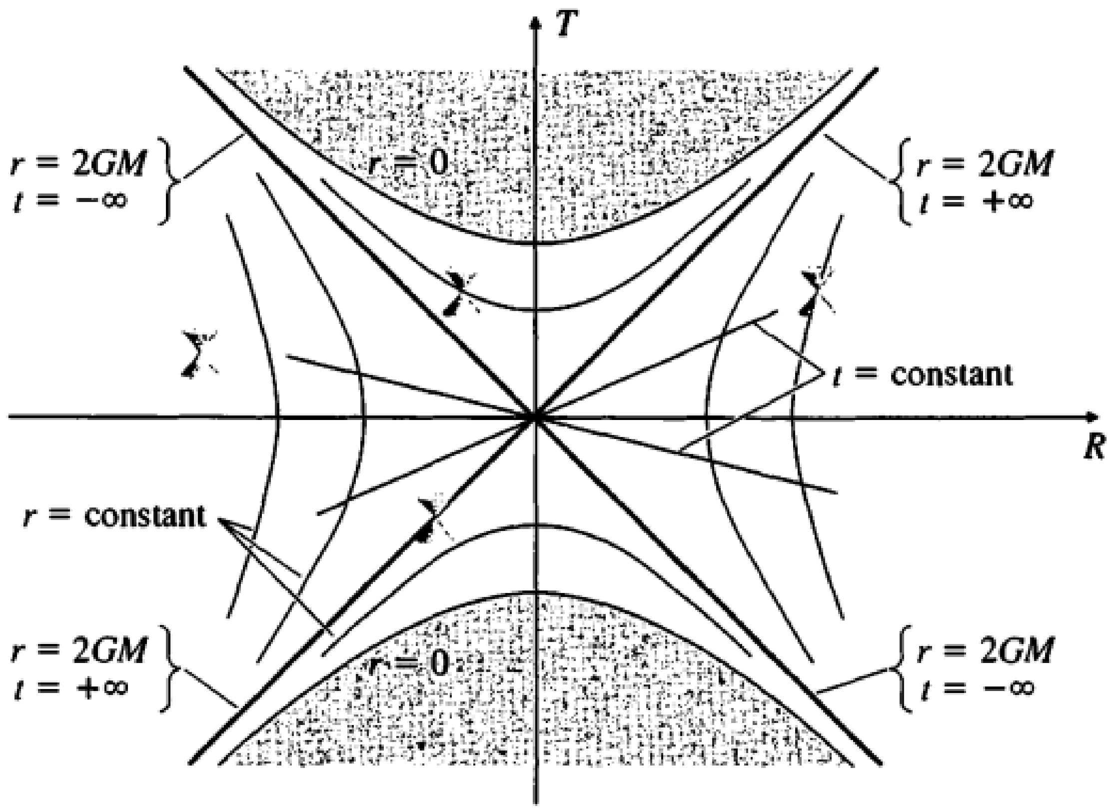


Figure 2: Kruskal Diagram

The above image is caled the Kruskal Diagram with angular coordinates not taken into account. The Hyperbolae are curves of constant r and straight lines of constant t. It is divided into 4 regions. Region 1 is for $R > 2GM$, region 2 can be explored by following a future directed path from 1 to 2. Region 3 can be accessed only by past directed curves and region 4 is accesible only by spacelike curves which is a physical impossibility. Region 2 represents the blackholes, once entered one will evenetually hit $r = 0$. While

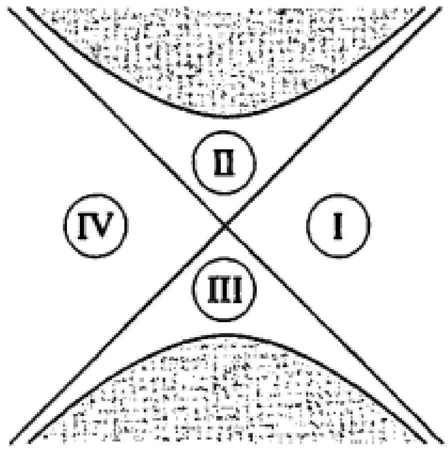


Figure 3: Kruskal regions

region 3 is its opposite, things can only escape from 3 to 1, but it can not be entered from region 1. This is called a Whitehole.

3 The Chandrasekhar Limit

S. Chandrasekhar derived an expression in 1930 for the maximum mass of a star that could become a white dwarf by balancing electron degeneracy pressure with gravitational attraction. Since electrons are Fermions, they follow Fermi-Dirac distribution and therefore their degeneracy is limited. So if a sufficient number of electrons are forced into a small volume of space we will experience a pressure because we are essentially forcing all electrons to a single state. This pressure is called LEctron-Degenracy pressure. Once a collapsing star overcomes this, the next stage could be a blackhole or Neutron star depending on the Neutron degeneracy pressure. [3]

$$\frac{< N_r >}{g_r} = \text{Probability of a state being occupied.} \tag{14}$$

$$\frac{< N_r >}{g_r} = f(E_r) = \frac{1}{1 + \exp \frac{E_r - \mu}{k_b T}} \tag{15}$$

The energy is given by special relativity :

$$E_r = \sqrt{p^2 c^2 + m^2 c^4} \tag{16}$$

The number density of states : $\dfrac{d\Pi}{d^3 x d^3 p}$

Number of states in a cell is density times volume : $f(p) = \dfrac{d\Pi}{d^3 x d^3 p} \times \frac{h^3}{g}$. Notice that eqn. (16) is constraint relating energy and momentum.

Density in coordinate space is :

$$n = \int \frac{d\Pi}{d^3 x d^3 p} d^3 p = \int \frac{g \times f(p)}{h^3} d^3 p$$

Energy density in coordinate space is :

$$\int E \times P(E) d^3 p = \int E \frac{d\Pi}{3x d^3 p} d^3 p = \int E \frac{g f(p)}{h^3} d^3 p$$

Since, pressure is associated with momentum flux

$$P = \frac{1}{3} n < vp >= \frac{1}{3} \int pv \frac{d\Pi}{d^3 x d^3 p} d^3 p = \frac{1}{3} \int pv g f(p)/h^3 d^3 p$$

We assume that at temperature zero we have ideal Fermions and $g = 2$, the equations can be integrated with the upper limit Fermi momentum given by $E_f^2 = p_f^2 c^2 + m^2 c^4$.

$$n_e = \frac{g}{h^3} \int_0^{p_f} d^3 p = \frac{g}{h^3} \int_0^{p_f} p^2 dp (4\pi) = \frac{4\pi g p_f^3}{3h^3} = \frac{8\pi p_f^3}{3h^3}$$

Since a star is not entirely composed of electrons, thye can be considered as embedded in Baryonic matter. We assume that each Baryon is associated with an average number of electrons.
Baryon rest mass : m_B.
Baryon volume : V_B
Mean number of electrons per Baryon : Y_e.
Electron number density : n_e.

$$\rho_0 = \frac{m_B}{V_B}$$
$$V_B = Y_e/n_e$$
$$\rho_0 = \frac{n_e m_B}{Y_e} \text{ Rest mass density of Baryon.}$$
$$n_e = \frac{8\pi p_f^3}{3h^3} = \frac{Y_e\rho_0}{m_B} \Rightarrow \rho_0 = \frac{8\pi m_B p_f^3}{3h^3 Y_e}$$

Pressure :

$$P = \frac{2}{3h^3}\int pvf(p)d^3p = \frac{2}{3h^3}\int_0^{p_f} pvd^3p = \frac{2}{3h^3}\int_0^{p_f} 4\pi p^3 v d^3p$$

Integral limits from Inverse Transform Sampling | $p_f = f^{-1}(f(p))$

The integral will have different values for non relativistic and ultra relativistic case.

1.
$$p = m_e v \Rightarrow P = \frac{8\pi}{15h^3 m_e}p_f^5$$

2.
$$v \sim c \Rightarrow P = \frac{2\pi c}{3h^3}p_f^4$$

$$\text{We had } : \rho_0 = \frac{8\pi m_B p_f^3}{3h^3 Y_e} \tag{17}$$
$$\Rightarrow P = K\rho_0^\gamma \tag{18}$$
$$\text{Non relativistic } K = \frac{h^2 Y_e^{5/3}}{m_B^{5/3}}\left(\frac{3}{8\pi}\right)^{2/3}\frac{1}{5m_e} \qquad \gamma = \frac{5}{3} \tag{19}$$
$$\text{Relativistic } K = \frac{h Y_e^{4/3}}{m_B^{4/3}}\left(\frac{3}{8\pi}\right)^{1/3}\frac{c}{4} \qquad \gamma = \frac{4}{3} \tag{20}$$

Now, we take a look at the different forces in action inside a star, the gravitational pull and force due to degeneracy pressure.

$$dF_g = \frac{Gm(r)}{r^2} 4\pi r^2 \rho_0 dr \tag{21}$$

$$dF_P = -4\pi r^2 dP \tag{22}$$

In equilbrium they will be equal $\frac{dP}{dr} = -m(r)G\rho_0/r^2$ (23)

$$m(r) = \int_0^r \rho_0 4\pi r^2 dr \Rightarrow \frac{dm}{dr} = 4\pi r^2 \rho_0 \tag{24}$$

$$\frac{dm}{dr} = -\frac{d}{dr}\Big(\frac{dP}{dr} r^2/(\rho_0 G)\Big) = 4\pi r^2 \rho_0 \tag{25}$$

Since we assumed a polytropic equation of state $P = K\rho_0^\gamma$ (26)

$$\frac{dP}{dr} = K\gamma\rho_0^{\gamma-1}\frac{rho_0}{dr} \tag{27}$$

Using this in (25) we get $\frac{1}{r^2}\frac{d}{dr}\Big(r^2 K\gamma\rho_0^{\gamma-1}\frac{d\rho_0}{dr}\Big) = -4\pi G\rho_0$ (28)

We have an equation in density and radius, (29)

so we choose appropriate boundary conditions (30)

$\rho_0 = \rho_c \ for \ r = 0 \ and \ \rho_0 = 0 \ at \ r = R.$ (31)

We apply the following transformations in sequence : (32)

$$\gamma - 1 = 1/n \tag{33}$$

$$\rho_0 = \rho_c\theta^n \tag{34}$$

$$a = \sqrt{(n+1)K\rho_c^{\frac{1-n}{n}}/(4\pi G)} \tag{35}$$

$$r = a\xi \tag{36}$$

$$\Rightarrow \frac{d}{d\xi}\Big(\xi^2\frac{\theta}{d\xi}\Big) = -\theta^n \tag{37}$$

This is called Lame Emden Equation of index n. For our values of n it can only be solved numerically. The roots of θ as a function of ξ are given below. We are looking at the function's roots because there ρ_0 will be zero i.e surface of the star.

$$\gamma = 5/3 \quad n = 3/2 \quad \xi_1 = 3.65375 \quad \xi_1^2|\theta'(\xi_1)| = 2.71406$$
$$\gamma = 4/3 \quad n = 3 \quad \xi_1 = 6.89685 \quad \xi_1^2|\theta'(\xi_1)| = 2.01824$$

To find the mass of the star,

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho_0 dr = 4\pi\rho_c \int_0^{\xi_1} a^3 \xi^2 \theta^n d\xi \\ &= -4\pi\rho_c a^3 \left(\xi^2 \frac{d\theta}{d\xi}\Big]_{\xi_1} - \xi^2 \frac{d\theta}{d\xi}\Big]_0 \right) = -4\pi\rho_c a^3 \xi_1^2 \theta'(\xi_1) \\ &= 4\pi\rho_c a^3 \xi_1^2 |\theta'(\xi_1)| \end{aligned}$$

It can be seen from numerical solutions that the derivative is negative, hence its modulus is used in the last step. Now for the Non relativistic Case :

$$\gamma = 5/3 \quad n = 3/2$$
$$a = \sqrt{\frac{(n+1)K}{4\pi G}} \sqrt{\rho_c^{\frac{1-3/2}{3/2}}}$$
$$= \omega_1 \rho_c^{-1/6}$$

Since $R = a\xi \Rightarrow R = \omega_2 \rho_c^{-1/6}$

$$M \sim a^3 \rho_c \sim \sqrt{\rho_c} \sim \sqrt{\frac{1}{R^6}} = R^{-3}$$

As we keep adding mass to the star its radius will decrease. This means that the space available for electrons to move about is decreasing. From uncertainity principle it follows that

$$\Delta x \sim \frac{1}{\Delta p}$$

The uncertainity in momentum will increase, and this will cause more and more electrons to achieve near light speed. So as the star becomes more massive we need to switch to the Relativistic case.

$$\gamma = 4/3 \quad n = 3$$
$$a = \sqrt{\frac{(n+1)K}{4\pi G}}\sqrt{a^{-2/3}}$$
$$a = \sqrt{\frac{K}{\pi G}\rho_c^{-1/3}} \Rightarrow$$
$$a \sim \rho_c^{-1/3} \Rightarrow R \sim \rho_c^{-\frac{1}{3}}$$
$$M \sim a^3\rho_c \sim (\rho_c^{\frac{-1}{3}})^3\rho_c = \texttt{const.}$$

As electrons approach light speed M tends to become constant w.r.t R. M is not very sensitive to changes in R or R is highly sensitive to changes in M. It implies that by adding more and more mass a critical mass is reached where R shrinks to zero. This is critical mass is the Chandrasekhar limit. We had an expression for M from the Lame-Emden equation for Relativistic case.

$$M_{ch} = 4\pi\rho_c a^3\xi_1^2|\theta'(\xi_1)| \Rightarrow$$
$$M_{ch} = 4\pi 2.01824\left(\frac{K}{\pi G}\right)^{\frac{3}{2}}$$

If the Baryonic matter is Helium then $Y_e = \frac{1}{2}$

$$K = \frac{h^2Y_e^{\frac{5}{3}}}{m_B^{5/3}}\left(\frac{3}{8\pi}\right)^{2/3}\frac{1}{5m_e}$$
$$m_B = 1.67 \times 10^{-24}$$
$$c = 3 \times 10^{10}$$
$$G = 6.67 \times 10^{-8}$$
$$h = 6.63 \times 10^{-27}$$
$$M_\circ = 1.989 \times 10^{33}$$
$$M_{ch} = 1.46 \;\; M_\circ$$

4 Spherically Symmetric Gravitational Collapse

Take $G = 1$ and $c = 1$. The Schwarzschild Metric reads

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2 d\Omega$$

The metric is valid for outside the surface of the star/body in vacuum. The metric that describes the inside of the star will be different, however at the surface of the star both the metrics should be same if we assume continuity. Take $r = R(t)$

$$ds^2 = -\Big[\Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1}(\frac{dR}{dt})^2\Big] + R(t)^2 d\Omega \tag{38}$$

On the surface there is zero pressure and spherical symmetry, so a given point on the surface will follow radial timelike curve i.e the angular component can be neglected.

$$d\Omega^2 = 0 \quad ds^2 = -d\tau^2 \tag{39}$$

$$ds^2 = -d\tau^2 = -\Big[\Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1}(\frac{dR}{dt})^2\Big] \tag{40}$$

$$1 = \Big[\Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1}(\frac{dR}{dt})^2\Big]\Big(\frac{dt}{d\tau}\Big)^2 \tag{41}$$

If K^μ is a Killing Vector (42)

$$K_\mu T^\mu = c_1 \Rightarrow K_\mu = \frac{\partial}{\partial t} = \partial_t \tag{43}$$

$$K_\mu = g_{\mu 0}K^0 \quad \Rightarrow K_0 = -\Big(1 - \frac{2M}{R}\Big) \tag{44}$$

$$\epsilon = -\Big(1 - \frac{2M}{R}\Big)\frac{dt}{d\tau} \tag{45}$$

From eqn.(41) (46)

$$1 = \Big[(1 - 2M/R) - (1 - 2M/R)^{-1}\dot{R}^2\Big]\epsilon^2(1 - 2M/R)^{-2} \tag{47}$$

$$\dot{R}^2 = \frac{1}{\epsilon^2}\Big(1 - \frac{2M}{R}\Big)^2 \tag{48}$$

$$\dot{R}^2 = \Big(\frac{2M}{R} - 1 + \epsilon^2\Big) \tag{49}$$

$$\dot{R}^2 = 0 \Rightarrow R = 2M \ \text{ and} R = 2M/(1 - \epsilon^2) \tag{50}$$

(51)

These are the points at which the Collapse stops or starts. So, it starts at the second root and ends at the first root. A plot of $\dot{R}^2$ shows that the second root is the maximum allowed radius and it goes to $R = 2M$ at a point of zero slope, i.e R goes to $2M$ as $t = \infty$. The above analysis is from the point of view of an observer at ∞. If we switch to a time coordinate that is local to the surface of the star we see a different scenario.

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{\epsilon}\Big(1 - \frac{2M}{R}\Big)\frac{d}{d\tau}$$

$$\Big(\frac{dR}{dt}\Big)^2 = \Big[\frac{1}{\epsilon}\Big(1 - \frac{2M}{R}\Big)\frac{dR}{d\tau}\Big]^2$$

Use eqn.49

$$\Big(\frac{dR}{d\tau}\Big)^2 = \Big(\frac{2M}{R} - 1 + \epsilon^2\Big) = 0$$

$$R_{max} \equiv \frac{2M}{1 - \epsilon^2}$$

$$\Big(\frac{dR}{d\tau}\Big)^2 = (1 - \epsilon^2)\Big(\frac{R_{max}}{R} - 1\Big)$$

The plot of the last equation shows that the rate of change of radius is zero only at R_{max}. $R = 2M$ is not a root of the equation, i.e. it reaches $R = 2M$ in a finite time. [4]

4.1 A Note on Naked Singularities and Cosmic Censorship

English physicist and mathematician Roger Penrose had conjectured that a singularity must necessarily be cloaked by an event horizon, therefore any information about the singularity can never be obtained. However there exist some models of gravitational collapse which lead to naked singularities. These models have not been verified yet, but also the Cosmic Censorship conjecture is yet to be proven.

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