Birla Institute of Technology and Science

K K Birla Goa Campus

Dept. of Physics

Dynamical Systems in Cosmology

Submitted for fulfilling the requirements of the course PHY F376

Authors:
K Pratyush Kumar
K Karthik

Supervisor:
Dr.Kinjal Banerjee

Aug 2018 - Dec 2018

Acknowledgements

We are profoundly grateful to Dr. Kinjal Banerjee for his expert guidance and continuous encouragement throughout to see that this project is true to its target since its commencement to its completion.

At last we must express our sincere heartfelt gratitude to our friends, family members and teachers who helped us directly or indirectly through out the project.

- Pratyush and Karthik

Contents

Abstract

The project involved a review of analysis of the ΛCDM Model of Cosmology from a Dynamical Systems theory point of view, linear Stability Analysis was primarily used for this. The next phase of the project dealt with the existence of Chaos in the BKL-Mixmaster Cosmology. This report assumes that the reader is familiar with basic nonlinear dynamics.

1 Introduction

1.1 Motivation

The equations of physical cosmology are derived from theories of Gravity, primarily the Einstein Field Equations of General Theory of Relativity. The EFE are a set of 10 coupled, nonlinear partial differential equations. Certain assumptions are made in cosmology to simplify the EFEs, but it can be done only to an extent. As is evident it is not always possible to solve the equations analytically while numerical solvers require us to know the values of the parameters in the equations which are not always available. Looking at these equations from a dynamical systems perspective gives us a broad picture of the behaviour of the universe even if the exact parameters are not known.

1.2 ΛCDM Model of Cosmology

The standard model of cosmology is the ΛCDM. Λ is for Dark Energy while CDM stands for Cold Dark Matter. The crux of the model is as follows : The model is based on the Friedmann-Lemaître-Robertson-Walker metric. The universe is homogenous and isotropic and is expanding at an accelerated rate, started by a singularity called the Big Bang. This expansion can be verified by redshift of the galaxies. However what causes this expansion is not clear yet, this hypothetical entity which is causing the expansion is termed Dark Energy.

1.2.1 What ΛCDM doesn't explain

Since the Universe is expanding, and at every instant of time the light from our planet explores new regions of the universe that were never seen before. Infact these new regions are causally disconnected from us, so there is no reason for the universe to be homogenous and isotropic, this is called the Horizon problem. Several theories have been proposed to explain this, one of them being inflation.

1.3 Chaos in Bianchi Cosmology - BKL and Mixmaster

Charles Misner hoped to solve the Horizon problem using a metric from Bianchi IX cosmology. Belinskii-Khalatnikov-Lifshitz also did a similar work to show that this model is Chaotic. Since, analysis of Chaos is an integral part of Nonlinear Dynamics this model attracted our attention.

2 Standard Model - Dynamical Systems analysis

2.1 Brief Introduction to Cosmological Equations

The Einstein Field Equations yield the following differential equations.

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} - \Lambda = \kappa\rho$$
$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \lambda = \kappa p$$

a is the scale factor, it is a parameter for the size of the universe. k decides the curvature of the space, it can take values $-1, 0, 1$. ρ and p are energy density and pressure of some form of matter or radiation respectively. κ is a constant from General Relativity relating curvature and matter. The equation of energy with time is :

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

We assume a linear equation of state for the matter i.e $p = \omega\rho$. The Hubble parameter is defined as $H = \frac{\dot{a}}{a}$.

$$\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2 \tag{1}$$

$$\tag{2}$$

The scale factor equations can be rewritten as

$$3H^2 + 3k/a^2 - \Lambda = \kappa\rho \tag{3}$$

$$-2\dot{H} - 3H^2 - k/a^2 + \Lambda = \kappa p \tag{4}$$

$$\frac{\kappa\rho}{3H^2} + \frac{\Lambda}{3H^2} - \frac{k}{a^2H^2} = 1 \tag{5}$$

$$\Omega \equiv \frac{\kappa\rho}{3H^2} \tag{6}$$

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2} \tag{7}$$

2.2 A brief history of the universe

This section will explain the history of the universe based on the standard model and experiments. Any model that is proposed should follow at least some of these observations.

1. Universe is expanding at an accelerated rate i.e. $\Lambda > 0$.
2. The curvature $k = 0$.
3. The general past features of our universe are as follows : Inflation, Radiation dominated, Matter dominated, Cosmological Term.

In the coming sections we will see different models and if they conform to the above observations.

2.3 ΛCDM Model

In this model we will consider a universe which is spatially flat k=0,and its matter content is radiation ρ_r with $w = 1/3$ and a perfect fluid density ρ_m with $w=0$.[2] From Einstein's equations as well as continuity equations we get the following four equations.

$$3H^2-\Lambda=\kappa(\rho_m+\rho_r) \tag{8}$$

$$-2\dot{H}-3H^2+\Lambda=\kappa\frac{1}{3}\rho_r \tag{9}$$

$$\dot{\rho_r}+4H\rho_r=0 \tag{10}$$

$$\dot{\rho_m}+3H\rho_m=0 \tag{11}$$

$$\tag{12}$$

We have already defined the quantities $\Omega_m,\Omega_r,\Omega_\Lambda$ substituting these values in eq.(1) we get the following constraint equations.

$$1=\Omega_m+\Omega_r+\Omega_\Lambda \tag{13}$$

Since we have a constant value of Λ we have a constant value of Ω_Λ therefore we only have two independent quantities,since these quantities are representations of energy densities we expect them to be positive,these constraint leads to the following relations,

$$0\leq\Omega_m\leq 1$$

$$0\leq\Omega_r\leq 1$$

These equations bound the values of Ω_m Ω_r to within a right triangle between the points (0,0) (0,1) (1,0).This is the part of the phase space of (Ω_m,Ω_r) which is allowed.

Now we need to use the previous to derive equations for the evolution of the quantities Ω_m and Ω_r.

$$\frac{d\Omega_m}{dt}=\frac{d}{dt}(\frac{\kappa\rho_m}{3H^2})=\frac{\kappa}{3H}(\frac{\dot{\rho}_m}{H}-2\rho_m\frac{\dot{H}}{H^2}) \tag{14}$$

From the conservation equation for ρ_m we get the following equations,

$$\frac{1}{H}\frac{d\Omega_m}{dt}=-3\Omega_m+3\Omega_m(1-\Omega_\Lambda+\frac{\Omega_r}{3}) \tag{15}$$

using the constraint equations for Ω_m,Ω_r,Ω_Λ we get

$$\frac{1}{H}\frac{d\Omega_m}{dt}=\Omega_m(3\Omega_m+4\Omega_r-3) \tag{16}$$

Instead of using the coordinate time, we use a different time :

$$N = ln(a)$$
$$\frac{dN}{dt} = \frac{1}{a}\frac{da}{dt} = \frac{\dot{a}}{a} = H$$
$$dN = Hdt$$

this gives the final equations for the evolution of Ω_m,

$$\Omega_m^{'} = \Omega_m(3\Omega_m + 4\Omega_r - 3)$$

Similar analysis for Ω_r gives us the following equations for the evolution of Ω_r

$$\Omega_r^{'} = \Omega_r(3\Omega_m + 4\Omega_r - 4)$$

2.3.1 Linear stability analysis on ΛCDM model.

$$\Omega_m^{'} = \Omega_m(3\Omega_m + 4\Omega_r - 3)$$
$$\Omega_r^{'} = \Omega_r(3\Omega_m + 4\Omega_r - 4)$$
$$1 = \Omega_m + \Omega_r + \Omega_\Lambda$$

To perform linear stability analysis on this system we first need to find the fixed points of the two differential equations which corresponds to $\Omega_m^{'} = 0$ and $\Omega_r^{'} =$ this gives us the set of fixed points in the phase space the points are $O(0,0)$ $R(0,1)$ and $M(1,0)$ to check the stability of the points we find the Jacobian matrix of the system around these three points.

The Jacobian matrix of a system

$$\dot{x} = f(x,y)$$

$$\dot{y} = gx,y)$$

is

$$\begin{bmatrix} f_{,x} & f_{,y} \\ g_{,x} & g,x \end{bmatrix}$$

where $f_{,x}$ and $f_{,y}$ represents partial derivative of f with respect to x and y respectively.
Similarly $g_{,x}$ and $g_{,y}$ represents partial derivative of g with respect to x and y respectively.

the sign of eigenvalues of the Jacobian matrix at the fixed points of the system gives us a indication of the stability of the fixed point. if λ_1 and λ_2 corresponds to the eigenvalues of the Jacobian matrix at the fixed point then the conditions for stability of the fixed points are:

Eigenvalues	Description
$\lambda_1 < 0, \lambda_2 < 0$	The fixed point is stable
$\lambda_1 > 0, \lambda_2 > 0$	The fixed point is unstable
$\lambda_1 < 0, \lambda_2 > 0$	The fixed point is a saddle node.
$\lambda_1 = 0, \lambda_2 < 0$	The fixed point is a stable node.
$\lambda_1 = 0, \lambda_2 > 0$	The fixed point is an unstable node.
$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$	The fixed point is a stable center if α is negative
$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$	The fixed point is a unstable center if α is positive

The Jacobian matrix at the three fixed points of the ΛCDM model are:

$$J(O) = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$$

$$J(R) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$J(S) = \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}$$

and the corresponding eigenvalues are :
for O $\lambda_1 = -3, \lambda_2 = -4$
for R $\lambda_1 = 1, \lambda_2 = 4$
for S $\lambda_1 = -1, \lambda_2 = 3$

Therefore we can clearly infer that the point O behaves like a stable center,whereas the point R behaves like an unstable node,the point S is a saddle node.

2.3.2 Cosmological Implications of Linear Stability Analysis on the ΛCDM Model.

We see that O is the only attractor of the system therefore all trajectories must eventually reach O,the point R is an unstable node so it is the only point which can at as a "past time" attractor therefore all trajectories began at the point R,some trajectories move towards the point saddle point S but eventually move towards the point O.

Cosmologically,this corresponds to an early universe which is dominated by radiation which corresponds to the point R as the system evolves it moves towards a more matter dominated universe, eventually moving towards the point O which corresponds to a universe dominated by the cosmological constant.

2.4 Scalar Field Model

[2] Inflation (Repellor $\lambda > 0$) $\rightarrow$ Radiation (Saddle $\lambda_i > 0$ $\lambda_j < 0$) $\rightarrow$ Matter (Saddle $\lambda_i > 0$ $\lambda_j < 0$) $\rightarrow$ Cosmological term (Attractor $\lambda_i < 0$)

Assume $\lambda_i \neq 0$. In this model we replace the cosmological constant with a scalar field which is changing with time $\psi(t)$ and its potential $V(\psi) = V_0 \exp(-\lambda k \psi)$; $V_0 > 0; \lambda$ is just a parameter. Using a different parameter for equation of state $P = \omega\rho = (\gamma - 1)\rho$. For radiation $\gamma = \frac{4}{3}$ and for matter $\gamma = 1$. Now we take a look at the field equations :

$$H^2 = \frac{\kappa^2}{3}(\rho + \frac{1}{2}\dot{\psi}^2 + V) \tag{17}$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho + P + \dot{\psi}^2) \tag{18}$$

Define (19)

$$\rho_\psi = \frac{1}{2}\dot{\psi}^2 + V \tag{20}$$

$$P_\psi = \frac{1}{2}\dot{\psi}^2 - V \tag{21}$$

The conservation equations give : (22)

$$\dot{\rho} = -3h(\rho + P) \tag{23}$$

$$\ddot{\psi} = -3H\dot{\psi} - \frac{dV}{d\psi} = -3H\dot{\psi} + \lambda\kappa V \tag{24}$$

Divide eqn. (1) by H^2 (25)

$$1 = \frac{\kappa^2\rho}{3H^2} + \frac{\kappa^2\dot{\psi}^2}{6H^2} + \frac{\kappa^2 V}{3H^2} \tag{26}$$

Rewrite it in terms of new variables (27)

$$1 = s^2 + x^2 + y^2 \tag{28}$$

$$1 \geq 1 - x^2 - y^2 = s^2 \geq 0 \tag{29}$$

$$\Omega_\psi = \frac{\kappa^2\rho_\psi}{3H^2} = \frac{\kappa^2}{3H^2}(\frac{1}{2}\dot{\psi}^2 + V) = x^2 + y^2 \tag{30}$$

$$\gamma_\psi = 1 + \omega_\psi \tag{31}$$

$$= 1 + \frac{P_\psi}{\rho_\psi} = \frac{2x^2}{x^2 + y^2} \tag{32}$$

Instead of using the coordinate time, we use a different time :

$$N = ln(a)$$
$$\frac{dN}{dt} = \frac{1}{a}\frac{da}{dt} = \frac{\dot{a}}{a} = H$$
$$dN = Hdt$$

Now we setup our dynamical system by finding the differential equations for x and y. Then we will use Linear stability analysis to find properties of the system.

$$x = \frac{\kappa}{\sqrt{6}}\frac{\dot{\psi}}{H},\;\; y = \frac{\kappa\sqrt{V}}{\sqrt{3}H},\;\; s = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$$
$$\frac{dx}{dt} = \frac{\kappa}{\sqrt{6}}\Big(-3\dot{\psi} - \frac{1}{H}\frac{dV}{d\psi} - \dot{\psi}\frac{\dot{H}}{H}\Big)$$

Eqn. (15) has been used for double derivative of ψ

Then use Einstein's equation for $\dot{H}$

$$\frac{dx}{dt} = \frac{\kappa}{\sqrt{6}}\Big[-3\dot{\psi} + \frac{\lambda\kappa V}{H} + \frac{\kappa^2\dot{\psi}\gamma\rho}{2H^2} + \frac{\kappa^2\dot{\psi}^3}{2H^2}\Big]$$
$$= H\Big(-3x + \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2}x\big((1-x^2-y^2)\gamma + 2x^2\big)\Big)$$

use the new time variable and denote differentiation by $'$

$$x' = -3x + \sqrt{\frac{3}{2}}\lambda y^2 + \frac{3}{2}x\big((1-x^2-y^2)\gamma + 2x^2\big) = f(x,y)$$

now, for y

$$y = \sqrt{\frac{V}{3}}\frac{\kappa}{H}$$
$$y' = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y\big(\gamma(1-x^2-y^2) + 2x^2\big) = g(x,y)$$

We observe that the equations are invariant under $y \rightarrow -y$, so we just focus on $y > 0$. Next, we compute the zeros of the differential equations and the jacobian.

$$\frac{\partial f}{\partial x} = -3 + 3x^2 + \frac{3}{2}\gamma(1 - x^2 - y^2) + \frac{3}{2}x^2(4 - 2\gamma)$$
$$\frac{\partial f}{\partial y} = \sqrt{6}\lambda y - 3\gamma xy$$
$$\frac{\partial g}{\partial x} = -\lambda\sqrt{\frac{3}{2}}y + 3xy(2 - \gamma)$$
$$\frac{\partial g}{\partial y} = -\lambda\sqrt{\frac{3}{2}}x + \frac{3}{2}\big(\gamma(1 - x^2 - y^2) + 2x^2\big) - 3\gamma y^2$$

We list the fixed points of the equations, and the cosmological parameters.

1. $O\ (0,0)\quad \Omega_\psi = 0\quad \gamma_\psi = \infty$ Saddle point.

2. $A_+\ (1,0)\quad \Omega_\psi = 1\quad \gamma_\psi = 2$

$\lambda < \sqrt{6}$: Unstable Node
$\lambda > \sqrt{6}$: Saddle

3. $A_-\ (-1,0)\quad \Omega_\psi = 1\quad \gamma_\psi = 2$

$\lambda > \sqrt{6}$: Unstable Node
$\lambda < \sqrt{6}$: Saddle

4. $B\ (\frac{\lambda}{\sqrt{6}}, \sqrt{1 - \lambda^2/6})\quad \Omega_\psi = 1\quad \gamma_\psi = \frac{\lambda^2}{3}$

$\lambda^2 > 3\gamma$: Saddle
$\lambda^2 < 3\gamma_\psi$: Stable

5. $C\ \big(\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}, \sqrt{\frac{3\gamma(2-\gamma)}{2\lambda^2}}\big)$ We observe that for some values this point won't fall in our phase space.

$\lambda^2 > \frac{24\gamma^2}{9\gamma - 2}$: Stable Node.
The eigenvalues of this fixed point are very complicated therefore a general condition on its nature was not calculated. The eigenvalues have been found using MATLAB codes in the appendix.

Now, that we have seen the fixed points and their characteristics we will look at the trajectories in the phase space for some parameter values.

2.4.1 Converting x and y to Cosmological parameters

We use the following equations to get measurable quantities from dynamical phase variables.

$$\Omega_\psi = x^2 + y^2$$
$$\rho_\psi = \frac{1}{2}\dot{\psi}^2 + V$$
$$\gamma_\psi = \frac{2x^2}{x^2+y^2} \Rightarrow P_\psi = (\gamma_\psi - 1)\rho_\psi = \frac{1}{2}\dot{\psi}^2 - V$$
$$s = 1 - x^2 - y^2 = \frac{\kappa^2 \rho}{3H^2} \quad P = (\gamma - 1)\rho$$

2.4.2 Phase Space for $\gamma = 1$ and $\lambda = 1$

The point O is saddle and $P = 0$, A_+ and A_- are unstable nodes with $\rho = 0$ and B is stable with $\rho = 0$; $P_\psi = -2\rho_\psi/3$ while C is not present in the region of interest. We note that all the points lie on the boundary of phase space.

2.4.3 Phase Space for $\gamma = 1$ and $\lambda = 2$

The point O is saddle with matter domination, A_+ and A_- are unstable nodes with zero matter content and $P_\psi = \rho_\psi$ and B is saddle with the equation of state $P_\psi = \left(\frac{\lambda^2}{3} - 1\right)\rho_\psi$ and matter content zero. C is an attracting spiral with $\rho = 3H^2/4\kappa^2$, $P = 0$, $\rho_\psi = 9H^2/4\kappa^2$, $P_\psi = 0$. We mention the eigenvalues of C since it is a special case :

$$\mu_1 = -\frac{3}{4} - \frac{3\iota}{4} \quad \mu_2 = -\frac{3}{4} + \frac{3\iota}{4}$$

C lies inside the boundary, while the rest of the points are on the boundary.

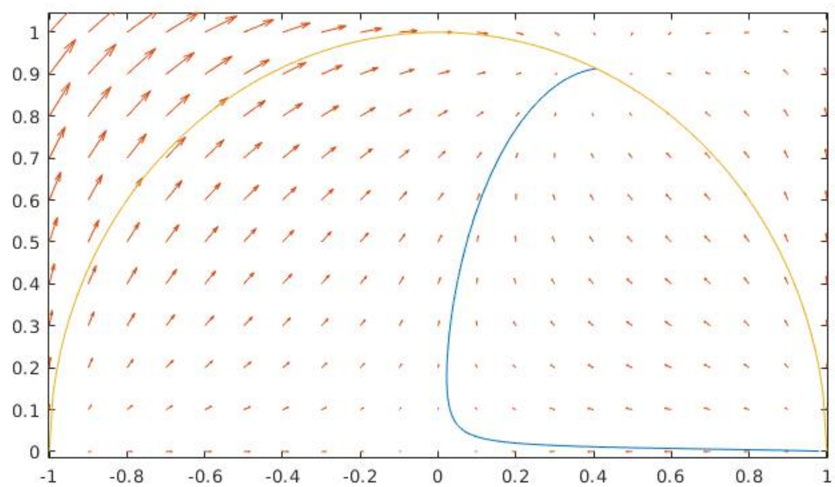

Figure 1: Phase Space for $\gamma = 1, \lambda = 1$. Blue line is a trajectory while orange curve indicates the boundary of phase space

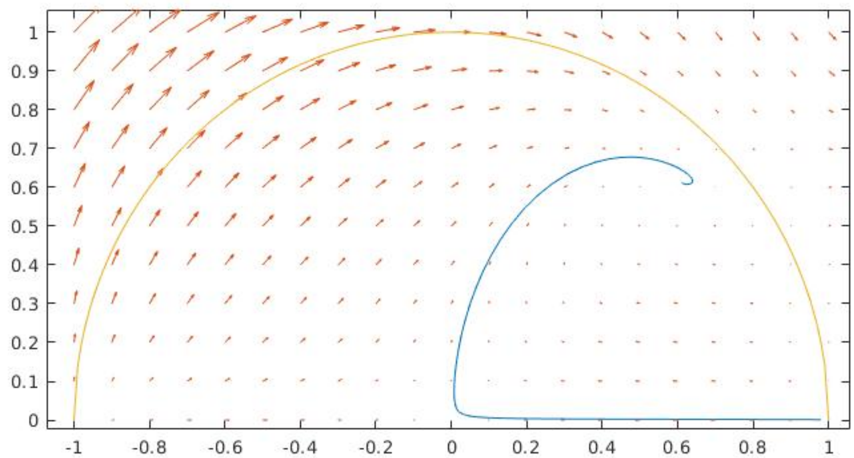

Figure 2: Phase Space for $\gamma = 2, \lambda = 1$. Blue line is a trajectory while orange curve indicates the boundary of phase space

3 Alternate Cosmology - BKL approximation to Bianchi IX Metric

We take a look at an alternate metric for cosmology called the Bianchi IX metric. This is not a standard model, however this model has been shown to exhibit chaos which is why considered studying this model. There is also some literature which has shown that this chaos is dependent on coordinates and is not an intrinsic feature of this metric.

The Bianchi IX Metric was explored by Charles Misner in 1969, his model was an anisotropic metric with three scale factors, and unlike the FLRW metric this universe could contract in one direction while expand in other directions and vice versa, Misner called this a 'Mixmaster Universe'. In 1970, three Russian physicists Belinsky, Khalatnikov and Lifshitz showed that the dynamics of this Universe can be approximated by a discrete one dimensional map which has been shown to be chaotic.[1] The metric used is called the Kasner Metric which is a diagonal metric with the spatial coefficients of the form t^p.

3.1 The metric and some of its characteristics

Consider the spatial part of the metric as :

$$dl^2 = t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{33}$$

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \tag{34}$$

$$p_1 < p_2 < p_3 \tag{35}$$

$$p_1 \in [-1/3, 0] \tag{36}$$

$$p_2 \in [0, 2/3] \tag{37}$$

$$p_3 \in [2/3, 1] \tag{38}$$

Later we show that (39)

$$p_1(u) = \frac{-u}{1+u+u^2} \;\; p_2(u) = \frac{1+u}{1+u+u^2} \;\; p_3(u) = \frac{u(1+u)}{1+u+u^2} \tag{40}$$

$$p_i(1/u) = p_i(u) \text{ where } u \in [1, \infty) \tag{41}$$

3.2 New Variables and Definitions

A synchronous frame is a transformation of our primary $x,\ y,\ z$ axes such that the time coordinate doesn't change, so the spatial metric in a general synchronous frame will look like :

$$dl^2 = (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)dx^\alpha dx^\beta$$

$$a = t_l^p; \;\; b = t_m^p; \;\; c = t_n^p$$

$l,\ ,m,\ n$ are vectors. The three $p's$ are functions of spatial coordinates and only constraint eqn.(34) holds for them. Now, we write the components of the Ricci Tensor in vacuum

$$-R_l^l = \frac{(a'bc)'}{abc} + \frac{1}{2a^2b^2c^2}[\lambda^2 a^4 - (\mu b^2 - \nu c^2)] = 0$$

$$-R_m^m = \frac{(ab'c)'}{abc} + \frac{1}{2a^2b^2c^2}[\mu^2 a^4 - (\lambda b^2 - \nu c^2)] = 0$$

$$-R_n^n = \frac{(abc')'}{abc} + \frac{1}{2a^2b^2c^2}[\nu^2 a^4 - (\lambda b^2 - \mu c^2)] = 0$$

$$-R_0^0 = \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} = 0$$

$$a = t_l^p = e^\alpha, \;\; b = t_m^p = e^\beta, \;\; t_n^p = e^\gamma$$

$$dt = abcd\tau$$

The full derivation can be found in the original paper.[1] We have defined a new time variable τ. Now we express derivatives of new variables $\alpha,\ \beta,\ \gamma$ in terms of derivatives of old variables.

$$\alpha_\tau = \frac{d\alpha}{dt}\frac{dt}{d\tau} = a'bc$$
$$\alpha_{\tau\tau} = (abc)'abc$$

This transforms our old equations from the Ricci scalar to the following form.

$$2\alpha_{\tau\tau} = (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4 \tag{42}$$
$$2\beta_{\tau\tau} = (\lambda a^2 - \nu c^2)^2 - \mu^2 b^4 \tag{43}$$
$$2\gamma_{\tau\tau} = (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4 \tag{44}$$
$$(\alpha+\beta+\gamma)_{\tau\tau} = \alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau \tag{45}$$

Add eqns. (42, 43 and 44) and use eqn. 45 to get

$$\alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau = \frac{1}{4}(\lambda^2 a^4 + \mu^2 b^4 + \nu^2 c^4 - 2\lambda\mu a^2 b^2 - 2\lambda\nu a^2 c^2 - 2\mu\nu b^2 c^2) \tag{46}$$

In a Kasner regime the right hand side of eqns. 42 to 44 are zero. However this is not possible when $t = 0$. Since $a,\ b,\ c$ depend on a power of t, two of them will go to zero for $t \to 0$ because one of the exponents $p's$ will be negative and others positive. Using this we get :

$$\alpha_{\tau\tau} = -\frac{1}{2}\lambda^2 e^{4\alpha} \tag{47}$$
$$\beta_{\tau\tau} = \gamma_{\tau\tau} = \frac{1}{2}\lambda^2 e^{4\alpha} \tag{48}$$

$\lambda^2 = 1$ for Bianchi IX metric. Now, let $p_l = p_1,\ p_m = p_2,\ p_n = p_3$.

$$a \sim t^{p_1}$$
$$b \sim t^{p_2} \quad c \sim t^{p_3}$$
$$abc \sim t^{p_1+p_2+p_3} \sim t$$
$$abc = \Lambda t$$
$$dt = \Lambda t d\tau \Rightarrow \tau = \frac{1}{\Lambda}\ln t + C$$

From the relation between τ and t we see that

$$t \to \infty \Rightarrow \tau \to \infty$$

$$t \to 0 \Rightarrow \tau \to -\infty$$

$$\alpha_\tau = a'bc = \frac{abc}{a}a' = \Lambda p_1$$
$$\beta_\tau = \Lambda p_2 \;\; \gamma_\tau = \Lambda p_3$$

The solution to these differential equations has been provided in the paper by BKL. In the limit $\tau \to -\infty$ i.e $t = 0$ are given by :

$$a \sim e^{-\Lambda p_1 \tau}$$
$$b \sim e^{\Lambda(p_2+2p_1)\tau}$$
$$c \sim e^{\Lambda(p_3+2p_1)\tau}$$
$$t \sim e^{\Lambda(1+2p_1)\tau}$$

Switching from τ to t we get the following relations.

$$\begin{aligned}
&a \sim t^{p'_l} \\
&b \sim t^{p'_m} \;\; c \sim t^{p'_n} \\
&p'_l = \frac{|p_1|}{1-2|p_1|} \;\; p'_m = -\frac{2|p_1| - p_2}{1-2|p_1|} \;\; p'_n = \frac{p_3 - 2|p_1|}{1-2|p_1|} \\
&abc = \Lambda' t, \;\; \Lambda = (1-2|p_1|)\Lambda.
\end{aligned}$$

We had expressed our $p's$ in terms of u, if we do the same for the new p' variables we see that

$$p'_l = p_2(u-1) \quad p'_m = p_1(u-1) \quad p'_n = p_3(u-1)$$

So essentially at each oscillation u is decreased by one, and the the exponents exchange places. The map can defined as follows : [3]

$$u_{i+1} = \begin{cases} u_i - 1 \;\;, & \texttt{for } u_i \in [2, \infty) \\ \frac{1}{u_i - 1} \;\;, & \texttt{for } u_i \in [1, 2) \end{cases}$$

Since u is not bounded in this map, we define a new variable

$$\tilde{u} = \frac{1}{u}$$

$$\tilde{u}_{i+1} = \begin{cases} \tilde{u} - 1 \ , & \texttt{for } \frac{\tilde{u}_i}{1-\tilde{u}_i} \in [0, 0.5) \\ \frac{1}{\tilde{u}_i - 1} \ , & \texttt{for } \frac{1-\tilde{u}_i}{\tilde{u}_i} \in [0.5, 1] \end{cases}$$

The above transformation in the map can be verified very easily by using the transformation in the variable u. To prove that the map is chaotic we invoke the following theorem.
If a discrete map has a fixed point of period-3, then the map is chaotic. [3]
We now look for a period 3 point of the map. We start with $\tilde{u}_1 < 0.5$ and reach $\tilde{u}_2 > 0.5$, then $\tilde{u}_3 < 0.5$ and the final application of the map gives $\tilde{u}_4$. It is equated to $\tilde{u}_1$ and is solved for, we get $\tilde{u}_1 = \frac{1}{6}(\sqrt{13}-1)$. This proves that the map is chaotic. As it has been stated before that there exist papers which claim that this chaos is coordinate dependent, however we will not be reviewing this literature in the report.

A MATLAB Code - Scalar Field Model

```
clear all;
h=0.1;
t=0:h:20;
a = 2; %lambda
b = 1; %gamma
f=@(x,y) -3*x + (3/2)^0.5 * a * y.^2 + 1.5 * x .* (2*x.^2 + b*(1 - x
g=@(x,y) -1.5^0.5 *a*x.*y + 1.5*y.*(b*(1 - x.^2 -y.^2) + 2*x.^2);
X_0 = 0.98; Y_0 = 0.001;
X(1) = X_0;
Y(1) = Y_0;

for i = 1:length(t)-1
    X(i+1) = X(i) + f(X(i),Y(i))*h;
    Y(i+1) = Y(i) + g(X(i),Y(i))*h;
end
```

```
[x,y] = meshgrid(-1:0.1:1,0:0.1:1);
u = f(x,y);
v = g(x,y);
figure

plot(X,Y)
hold on
scale = 6;
quiver(x,y,u,v)

hold on
p = -1:0.01:1;
q = sqrt(1 - p.^2);
plot(p,q);
```

References

[1] V.A. Belinskii, I.M. Khalatnikov, and E.M. Lifshitz. "Oscillatory approach to a singular point in the relativistic cosmology". In: *Advances in Physics* 19.80 (1970), pp. 525-573. DOI: 10.1080/00018737000101171. eprint: https://doi.org/10.1080/00018737000101171. URL: https://doi.org/10.1080/00018737000101171.

[2] C. G. Boehmer and N. Chan. "Dynamical systems in cosmology". In: *ArXiv e-prints* (Sept. 2014). arXiv: 1409.5585 [gr-qc].

[3] D. W. Hobill. "Deterministic chaos and Bianchi cosmologies". In: *Dynamical Systems in Cosmology*. Ed. by J. Wainwright and G. F. R.Editors Ellis. Cambridge University Press, 1997, pp. 229-246. DOI: 10.1017/CBO9780511524660.013.