Rumour Models on Complex Networks

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1 Introduction

Rumours are ubiquitous in human societies, rumour spreading either by word of mouth or through the internet can heavily affect behaviour of individuals in a community. We study rumour models on complex networks - Erdos Renyi, Small World, and Scale Free networks. We also present some new results on the Maki-Thompson model.

Pioneering work on rumour models was done by Daley and Kendall, they provided a mean field or homogeneous mixing model with three classes of individuals : Spreaders (S), Ignorants (I) and Stiflers (R) and two parameters λ - spreading probability and α - stifling probability. When a spreader contacts an ignorant, the latter converts to a spreader with probability λ. When a spreader contacts another spreader or stifler, both are converted to a stifler with rate α. Stiflers are individuals who know the rumour but are not interested in spreading the rumour. The equilibrium state has zero spreaders. Maki and Thompson gave a modified version, where the contacts are directed. Only the spreader who makes a move to spread the rumour to another spreader/stifler gets stifled.

The Maki-Thompson processes can be described through pair wise reactions :

$$S + I \to 2S \tag{1}$$
$$S + S \to R + S \tag{2}$$
$$S + R \to 2S \tag{3}$$

The differential equations describing the mean field behaviour are given below, along with the final stifler density [1]. If implemented on a homogeneous network, $\langle k \rangle$ is the average degree of the network. Notice that the final stifler density (r_∞) is independent of the degree.

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \lambda\langle k\rangle s(t)i(t) - \alpha\langle k\rangle s(t)[s(t) + r(t)] \tag{4}$$
$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} = -\lambda\langle k\rangle s(t)i(t) \tag{5}$$
$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = \alpha\langle k\rangle s(t)[s(t) + r(t)] \tag{6}$$
$$r_\infty = 1 - \exp(-\beta r_\infty); \beta = 1 + \lambda/\alpha \tag{7}$$

Both the models have been extensively studied in homogeneous mixing conditions, Erdos-Renyi network, Watts-Strogatz network and scale free networks. The topics covered in this work are :

1. Review of the above mentioned results, and report some previously unexplored aspects in the Erdos-Renyi network.
2. Study the Maki-Thompson model on the Watts-Strogatz network and report a new result related to the critical transition.
3. Effect of noise on r_∞ in the model parameters of the model.

4. Introduce an adaptive network modification rule to the Erdos-Renyi network.

In our Maki Thompson implementation, at each time step - all the spreaders are selected in a random order (without replacement), and the spreaders make an attempt on a randomly selected neighbouring node.

2 Dynamics on networks

2.1 Erdos-Renyi Network

Erdos-Renyi network is the most basic random network model. Two parameters are needed to generate the network - the network size (N) and connection density (ρ). A pair of nodes is selected and with a probability ρ, it is decided if an edge exists between the pair. For large N the average degree is ρN. A phase transition (percolation transition) is observed in the network - when $\rho < 1/N$ the network has disconnected components, when $\rho > 1/N$ the network contains a large connected component whose size is approximately the same as the network.

The average degree is immaterial to the analytical result [**MK˙analytic**] except when the network has disconnected components. However, in our Monte Carlo simulations we observe that even if the giant component is as large as the network, the analytical result is not obtained. We see a transition in Fig. 1 in the final stifler density with increasing average degree. In Fig. 2 we see that the giant component is large as the network from $\langle k \rangle = 6$ and beyond, however, the stifler density does not saturate to the analytical value. The average degree needs to be much higher, and depends on the model parameters λ and α. This is in direct contrast to the result reported in [2] (Table I : r_∞ for $N = 10000$, $\langle k \rangle = 6$ is reported to be very close to the analytical result). Fig. 3 suggests that the average minimum degree required increases somewhat linearly with α. We use a relative tolerance $= (r_{\infty,MC} - r_{\infty,MF})/r_{\infty,MF}$ to find out the minimum degree.

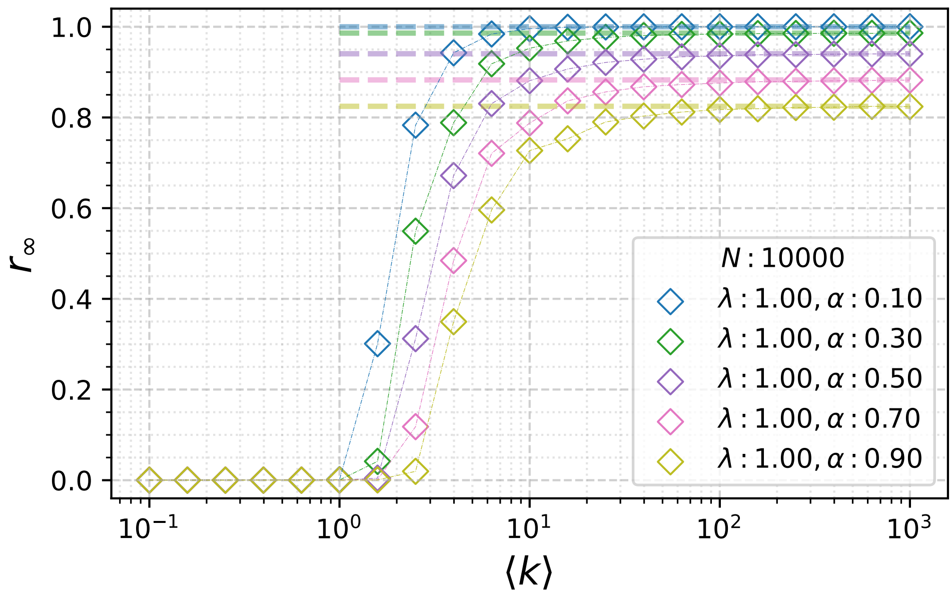


Figure 1: Comparison of Monte Carlo simulation with analytical mean field result. The horizontal lines indicate the analytical solution. Averaged over 50 instances.

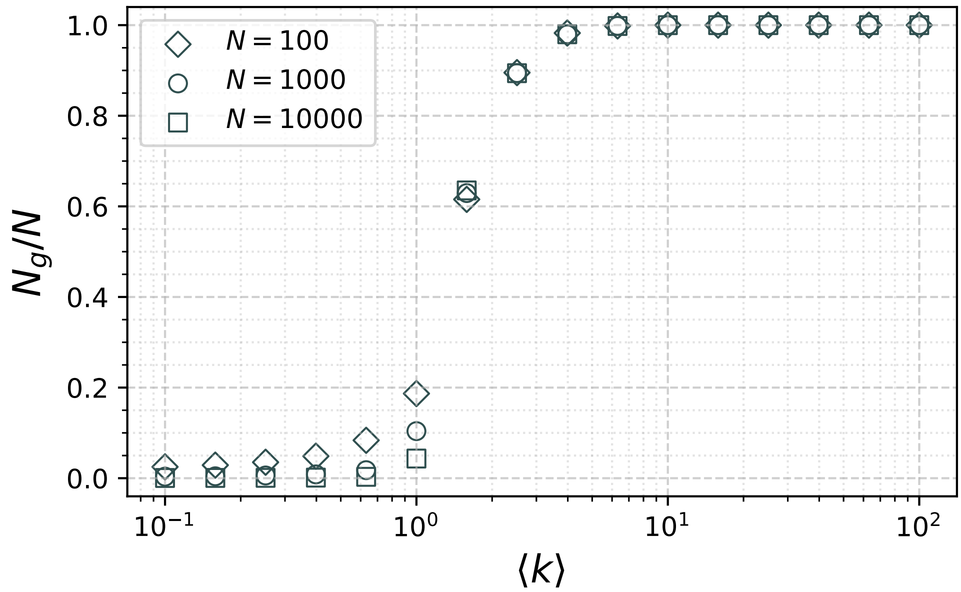


Figure 2: Giant component in ER network. Averaged over 50 instances.

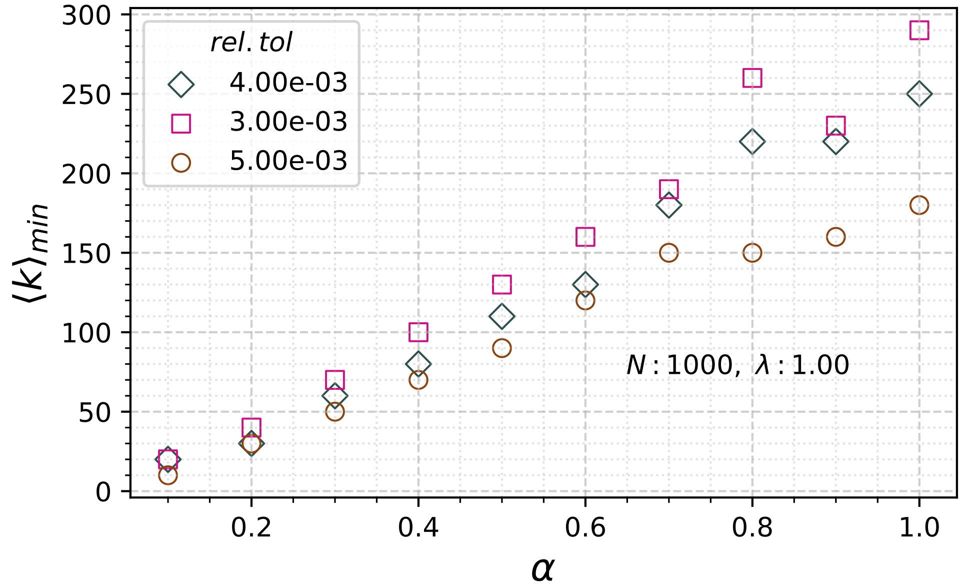


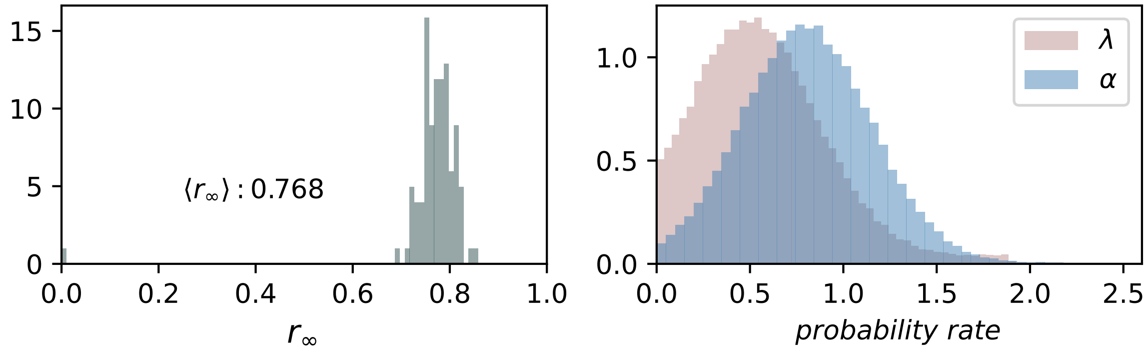
Figure 3: Minimum average degree required to replicate the mean field result.

2.2 Heterogeneity in the spreading and stifling rates

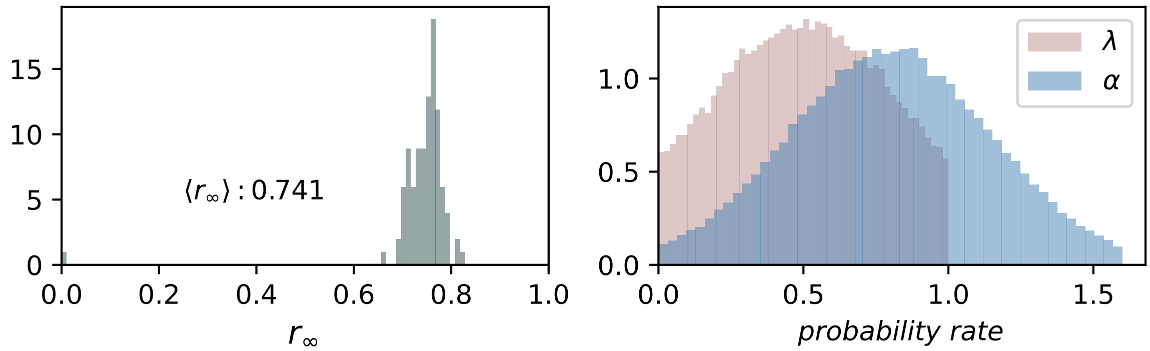
The spreading and stifling rates are usually assumed to be the same for all nodes. However, this assumption is not true for human beings, some people can spread the rumour to more people and some may not spread the rumour at all. Network topology may not be adequate in factoring in these differences, even if a person is highly connected they may be not be inclined to spread a rumour because of a variety of factors like - personal interests, talkativeness or veracity of the rumour. Therefore, instead of using fixed values, we use a distribution.

A realistic approach would be to consider a normal distribution, and see the effect of increasing standard deviation. We sample the rates from normal distributions $N(\langle\lambda\rangle, \sigma^2)$ and $N(\langle\alpha\rangle, \sigma^2)$, they are symmetric about the mean and its support is $(-\infty, \infty)$, the rates however can not be less than zero. In order to overcome this issue, we try two approaches. The first approach is to impose a lower and upper bound in the random numbers generated - any number less than zero or greater than twice the mean is replaced with a uniform random number between zero and twice mean. This preserves the symmetry about the mean and ensures that the parameters are positive. We observe that r_∞ increases as σ is increased. The problem with the above described approach is that for higher σ, the distribution ends up approaching a uniform distribution and is no longer realistic for modeling a population. Instead, we now clip only one tail of the distribution. If any parameter is less than zero, it is replaced by a uniformly distributed number from $U(0, \langle\lambda\rangle + 4\sigma)$ or $U(0, \langle\alpha\rangle + 4\sigma)$. The symmetry is lost but the distribution seems to be more realistic. We use the mode of the distribution to compare different cases since the mean will change due to clipping of the left tail. Note that σ represents the standard deviation of the original distribution, and not of the clipped distribution. In Fig 5 and Fig. 6, the results are summarised for the case when σ_λ **and σ_α are equal - heterogeneity increases the spread**.

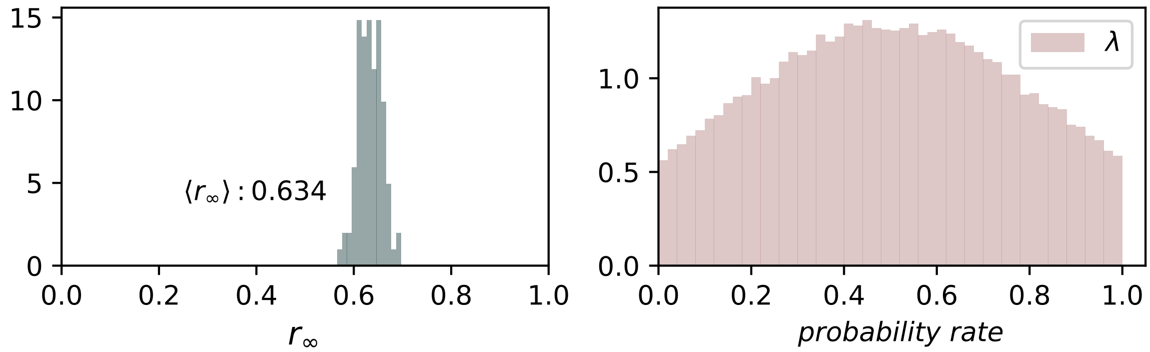
We also find (Fig. 4c and 4d) that if only one parameter is heterogeneous, the effect can be quite different. To explore this further, we vary σ for the two parameters independently (for different peak values of λ and α) and represent the $(\sigma_\lambda, \sigma_\alpha)$ space through heat maps in Fig. 7, where the system shows a variety of non trivial behaviours. Note that here only the left tail of the distribution is clipped and redistributed with $U(0, \langle\lambda\rangle + 4\sigma)$ or $U(0, \langle\alpha\rangle + 4\sigma)$. In figures 7e and 7f we see that increasing noise in stifling parameter can cause an non monotonic change in r_∞.



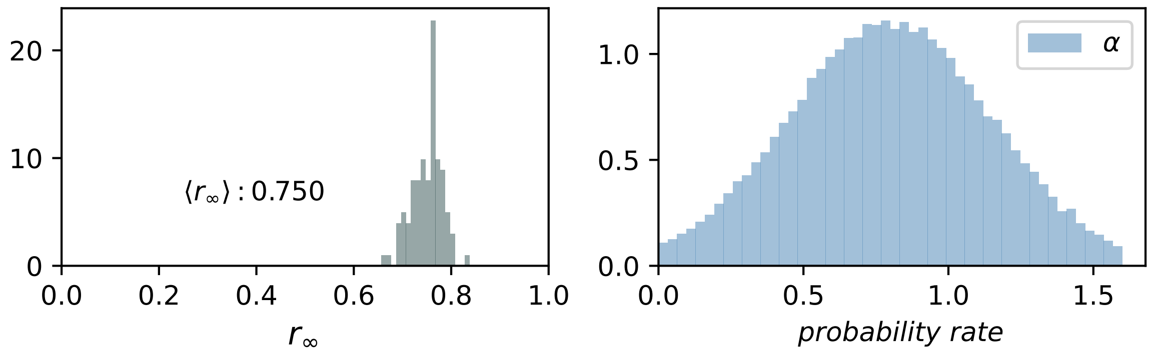
(a) $\lambda_{peak} = 0.5$, $\alpha_{peak} = 0.8$, $\sigma = 0.35$. Left tail clipped off.



(b) $\langle\lambda\rangle = 0.5$, $\langle\alpha\rangle = 0.8$, $\sigma = 0.35$. Both tails clipped off.



(c) $\langle\lambda\rangle = 0.5$, $\langle\alpha\rangle = 0.8$, $\sigma_\lambda = 0.35$. Only λ is heterogeneous.



(d) $\langle\lambda\rangle = 0.5$, $\langle\alpha\rangle = 0.8$, $\sigma_\alpha = 0.35$. Only α is heterogeneous.

Figure 4: Distributions for certain model parameters for various types of heterogeneity

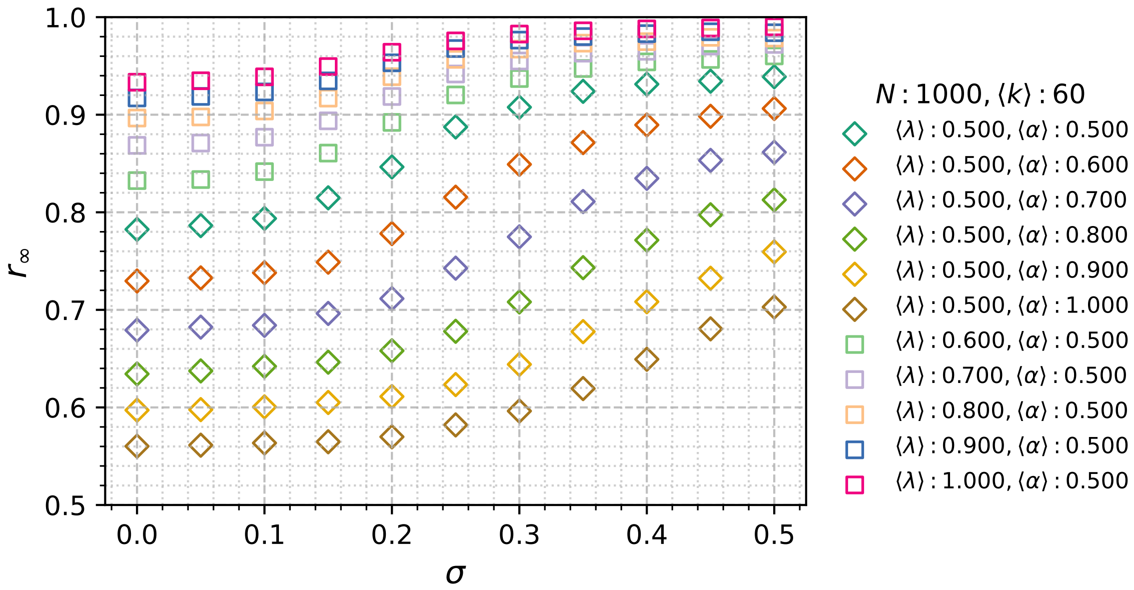


Figure 5: Stifler density as a function of standard deviation. Both tails clipped.

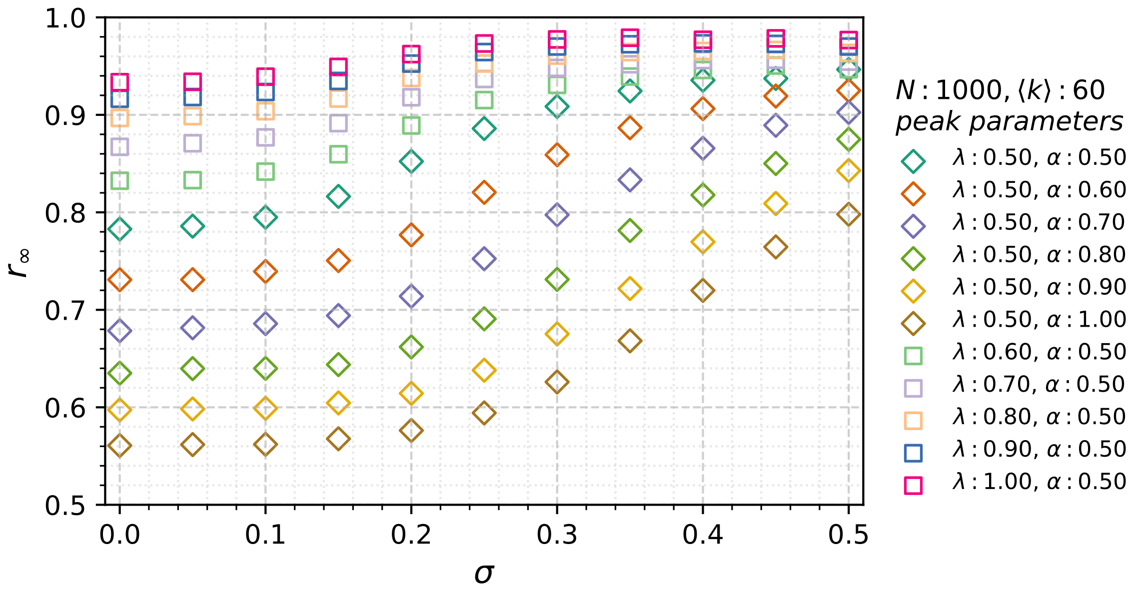
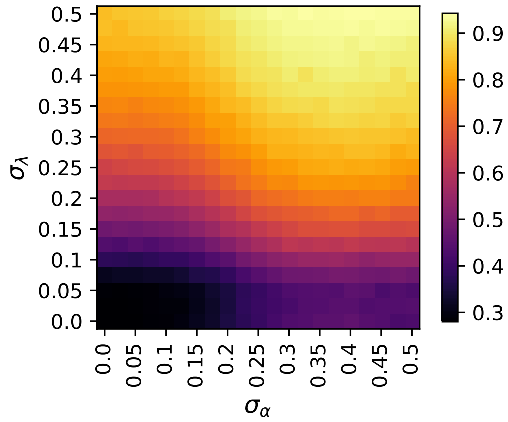
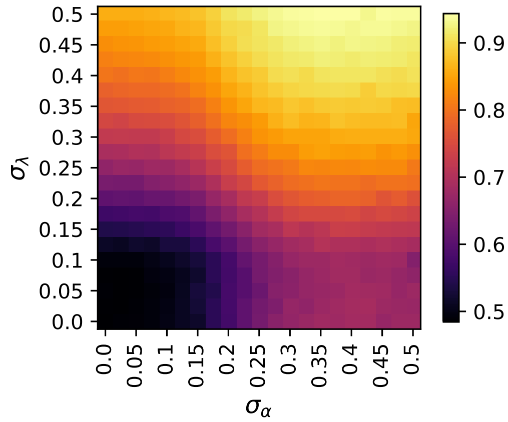


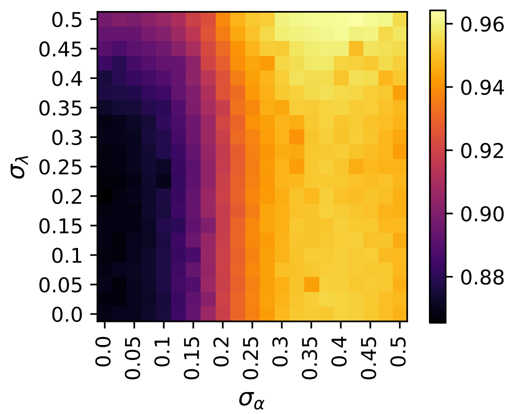
Figure 6: Stifler density as a function of standard deviation. Left tail clipped.



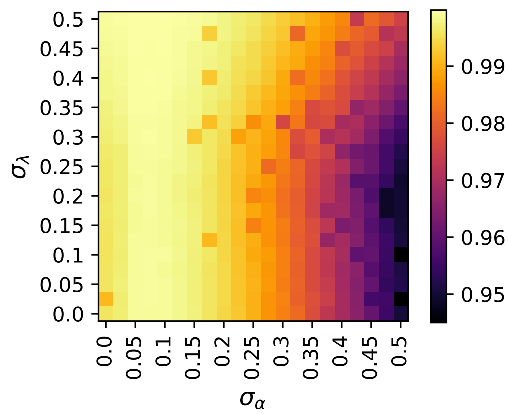
(a) $\lambda_{peak} = 0.1$, $\alpha_{peak} = 0.5$



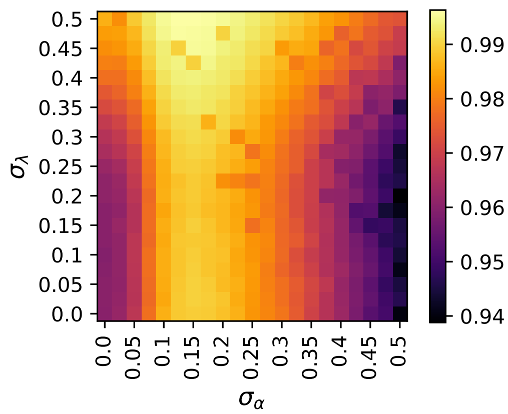
(b) $\lambda_{peak} = 0.2$, $\alpha_{peak} = 0.5$



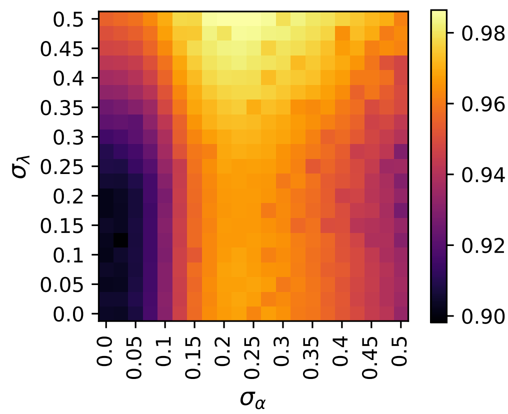
(c) $\lambda_{peak} = 0.7$, $\alpha_{peak} = 0.5$



(d) $\lambda_{peak} = 0.5$, $\alpha_{peak} = 0.1$



(e) $\lambda_{peak} = 0.5$, $\alpha_{peak} = 0.1$



(f) $\lambda_{peak} = 0.5$, $\alpha_{peak} = 0.2$

Figure 7: r_∞ as a function of $\sigma_\lambda, \sigma_\alpha$ for various λ_{peak} and α_{peak}. Note that the color scales are different in all the sub figures. (a) and (b) σ_λ causes a larger change than σ_α. (c) σ_λ does not have any appreciable effect on r_∞, (d) Unlike other cases, increasing σ_α leads to a decrease in r_∞. (e) and (f) Increasing σ_α causes a non monotonic change in r_∞

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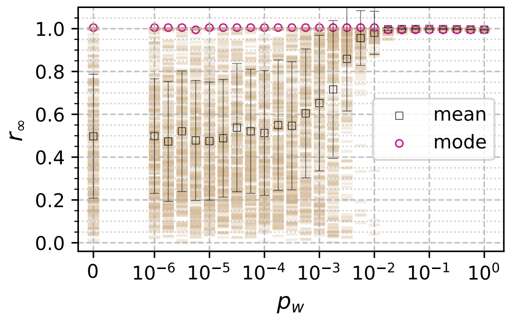
2.3 Small world networks

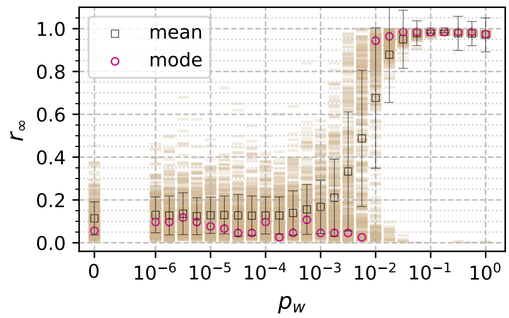
The Erdos-Renyi network has a very low clustering coefficient and average path length, unlike real world networks where clustering coefficient is very high. The Watts-Strogatz model provides a method to generate networks which have high clustering and low path length. Three parameters are required to generate the network - N, $\langle k \rangle$, p_w. Initially, a regular ring network is generated with the given size and degree, then each edge in the network is rewired with the probability p_w.

The Maki-Thompson model has been studied on small world networks [3], and it was established that a critical transition occurs as the rewiring probability (p_w) is increased. It was shown that for $\lambda = 1, \alpha = 1$, there exists a critical p_w beyond which the rumour spreads to a significant fraction of the population. We study how this critical transition changes as the model parameters are varied. Our simulation uses a network of 1000 nodes, with $\langle k \rangle = 6$. We get a lower and upper bound on the transition point using two consecutive p_w values where the mode of r_∞ distribution changes by a large amount. It is seen that the critical p_w decreases as α is decreased. The distributions are shown in Fig. 8 (a), (b), (c). A peculiarity is seen in (a), the regular graph ($p_w = 0$) shows a bimodality in the distribution. The results are summarised in Fig. 8d.

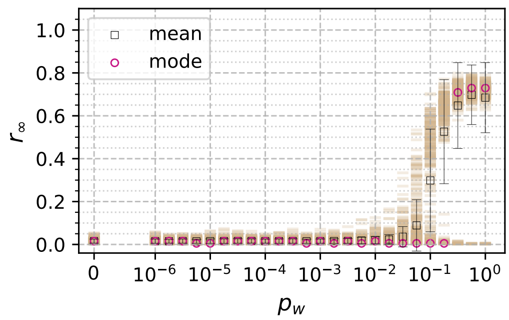
In [3] the argument presented (for the case $\lambda = 1, \alpha = 1$) to explain the transition is that in regular networks, the rumour will die locally due to high clustering. However, some shortcuts can lead to a non local spread, which causes the bimodality. The decrease in critical p_w can be explained using the same argument - lowering α causes an increase in spreading in general. Therefore, even in networks which do not have as many long range connections, one can still achieve a larger spread by decreasing α. From our simulations we also see that even in regular networks ($p_w = 0$), with $\alpha = 0.1$ the distribution of r_∞ can span across the local to non local regime. The reasons for this remain unclear to us and warrants a much deeper analysis.



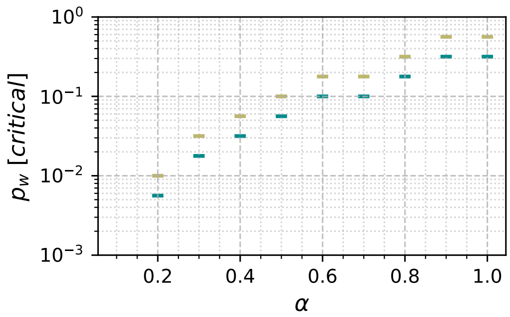
(a) $\lambda = 1.0$, $\alpha = 0.1$



(b) $\lambda = 1.0$, $\alpha = 0.2$



(c) $\lambda = 1.0$, $\alpha = 0.8$



(d) $\lambda = 1.0$

Figure 8: Watts-Strogatz model : $N = 1000, \langle k \rangle = 6$. (a), (b), (c) : Distribution, mean and mode of r_∞ as a function of p_w for various λ and α. The opacity of the color represents the probability. Note that in (a) the transition does not exist since there is a bimodality even at $p_w = 0$. (d) shows how the critical p_w changes with α. Yellow bar is the upper bound and cyan is the lower bound for critical p_w.

3 Adaptive networks

Adaptive networks are networks that change in response to the dynamics being played out on them. We devise a very simple rule for the Maki Thompson model on Erdos-Renyi network, at each time step - the spreaders make an attempt to spread the rumour with their contacts, once all the spreaders have done that, all the ignorants in the network (with a probability p_w) sever their links to a neighbour (randomly chosen) and connect to a random spreader in the network. This is done to simulate the effect that ignorants are aware that a rumour exists, but they seek out people who know the details. As expected this can lead to very high r_∞ values, the interesting part however is the structure of the final network. We end up with a network whose degree distribution is no longer a Gaussian. The process of development of such hubs with specific degree values can be understood by considering the time evolution of a particular instance. We will add further details once we have completed our analysis.

4 Discussion and Conclusion

We have shown some new results in the Maki-Thompson model :

1. A systematic study of the discrepancy in Maki-Thompson model's analytical solution and Erdos-Renyi network revealed that the minimum average degree increases almost linearly with the stifling rate.

2. Introduced random heterogeneity in the model parameters to account for the effect of talkativeness and reservedness of individuals. Adding noise in the spreading and stifling rates can lead to either increase or decrease in the spread.

3. The transition point of the model for Watts-Strogatz network decreases as α is decreased, and the transition disappears for $\alpha << 1$.

References

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[2] Yamir Moreno, Maziar Nekovee, and Amalio F. Pacheco. "Dynamics of rumor spreading in complex networks". In: *Phys. Rev. E* 69 (6 June 2004), p. 066130. DOI: 10.1103/PhysRevE.69.066130. URL: https://link.aps.org/doi/10.1103/PhysRevE.69.066130.

[3] Damián H. Zanette. "Critical behavior of propagation on small-world networks". In: *Phys. Rev. E* 64 (5 Oct. 2001), p. 050901. DOI: 10.1103/PhysRevE.64.050901. URL: https://link.aps.org/doi/10.1103/PhysRevE.64.050901.