Digital Signal Processing

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Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software installation

1.1 Run the following commands.

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1
    python3−scipy python3−numpy python3−
    matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

2.2 You will find a spectogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in problem 2.1 in the spectrogram and play. Observe the spectogram. What do you find?
Solution: There are a lot of yellow lines betweeen 440Hz to 5.1KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band nosie and execute the code.
Solution: Download and run the following code.

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

run the above code using the command

```
python3 2.3_noise.py
```

2.4 The output of the python scripy in problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectogram in problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also the signal is blank for frequencies above 5.1KHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch x(n).
Solution: Download and run the following code.Below code plots fig(3.1)

```
https://github.com/PratyakshRaj/signal−
    proccessing
```

run the above code using the command

```
python3 3.1.py
```

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), y(n) = 0, n < 0 \tag{3.2}$$

Sketech y(n).
Solution: Download and run the following code.Below code plots fig(3.2)

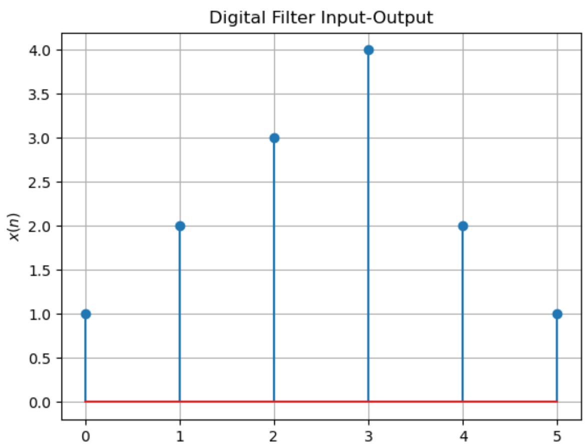

Fig. 3.1: Sketch of $x(n)$

```
wget https://github.com/PratyakshRaj/signal–
    proccessing
```

run the above code using the command

```
python3 3.2.py
```

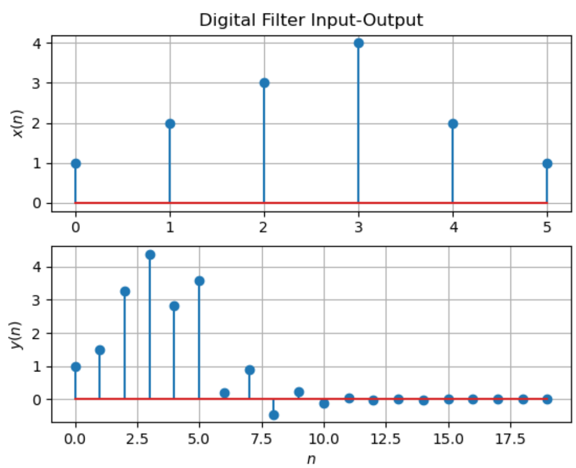

Fig. 3.2: Sketch of $x(n)$ and $y(n)$

3.3 Repeat the above exercise using C code.
Solution: Download and run the following code.Below code plots fig(3.3)

```
wget https://github.com/PratyakshRaj/signal–
    proccessing
```

```
3.3_plot.py
```

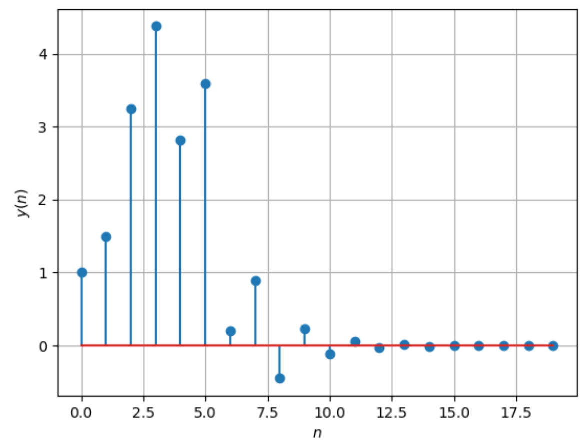

Fig. 3.3: Sketch of $y(n)$

4 $\mathcal{Z}$-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1)

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

$\mathcal{Z}$- transform of $x(n)$ is

$$\mathcal{Z}\{x(n)\} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.7}$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k}(1 + 2z^{-1} + 3z^{-2} + \quad (4.8)$$
$$4z^{-3} + 2z^{-4} + z^{-5}) \quad (4.9)$$
$$\mathcal{Z}\{x(n-k)\} = z^{-k} + 2z^{-(k+1)} + 3z^{-(k+2)} \quad (4.10)$$
$$+ 4z^{-(k+3)} + 2z^{-(k+4)} + z^{-(k+5)} \quad (4.11)$$

4.2 Obtain $X(z)$ for $x(n)$ in problem (3.1)
Solution:

$$\mathcal{Z}\{x(n)\} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \quad (4.12)$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k}(1 + 2z^{-1} + 3z^{-2} + \quad (4.13)$$
$$4z^{-3} + 2z^{-4} + z^{-5}) \quad (4.14)$$
$$\mathcal{Z}\{x(n-k)\} = z^{-k} + 2z^{-(k+1)} + 3z^{-(k+2)} \quad (4.15)$$
$$+ 4z^{-(k+3)} + 2z^{-(k+4)} + z^{-(k+5)} \quad (4.16)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.17)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.18)$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.19)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.20)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.21)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.22)$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \quad (4.23)$$

and from (4.21),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.24)$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.25)$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.26)$$

Solution:

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \sum_{n=0}^{\infty} (az^{-1})^n \quad (4.27)$$
$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.28)$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \quad (4.29)$$

Plot $|H(e^{\jmath\omega})|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution: Download and run the following code. The following code plots Fig. 4.6.

```
wget https://github.com/PratyakshRaj/signal-
    proccessing
```

```
4.5.py
```

We observe that $|H(e^{\jmath\omega})|$ is periodic with fundamental period 2π.

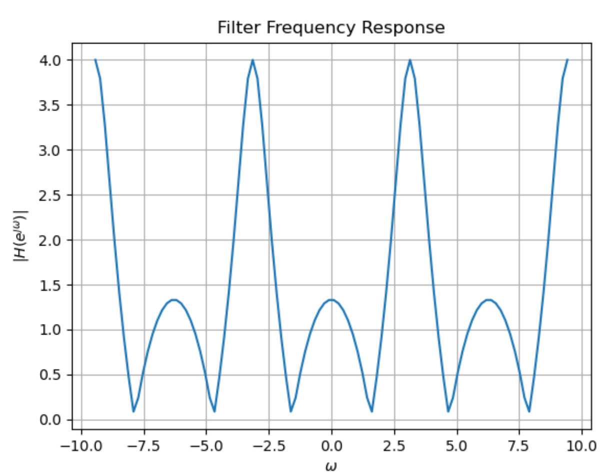

Fig. 4.6: $|H(e^{\jmath\omega})|$

$$H\left(e^{\jmath\omega}\right) = \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} \tag{4.30}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1+\cos 2\omega - \jmath\sin 2\omega\right|}{\left|1+\frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.31}$$

$$= \sqrt{\frac{(1+\cos 2\omega)^2+(\sin 2\omega)^2}{(1+\frac{1}{2}\cos\omega)^2+(\frac{1}{2}\sin\omega)^2}} \tag{4.32}$$

$$= \sqrt{\frac{2+2\cos 2\omega}{\frac{5}{4}+\cos\omega}} \tag{4.33}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5+4\cos\omega}} \tag{4.34}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5+4\cos\omega}} \tag{4.35}$$

period of $|cos\omega|$ is π and period of $\sqrt{5+4\cos\omega}$ is 2π.
Now period of $\left|H\left(e^{\jmath\omega}\right)\right|$ is $\frac{LCM(\pi,2\pi)}{HCF(\pi,2\pi)} = \frac{2\pi}{1} = 2\pi$

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$
Solution: $h(n)$ is given by the inverse DTFT of $H(e^{j\omega})$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.36}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.37}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.38}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} h(k)2\pi\delta[n-k] \tag{4.39}$$

$$= h(n) \tag{4.40}$$

Since

$$\int_{-\pi}^{\pi} e^{j\omega n}d\omega = 2\pi\delta[n] \tag{4.41}$$

$$and \tag{4.42}$$

$$\delta[n-k] = \begin{cases} 1 & n=k \\ 0 & \text{otherwise} \end{cases} \tag{4.43}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n<5 \tag{5.1}$$

for $H(z)$ in (4.19)
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{rl} & \quad\ \ 2x-4 \\ \frac{1}{2}x+1 \!\!\!& \overline{\big)\ \ x^2 \qquad +1} \\ & \underline{-x^2-2x} \\ & \qquad -2x+1 \\ & \qquad\ \ \underline{2x+4} \\ & \qquad\qquad\ 5 \end{array}$$

$$\implies 1+z^{-2} = \left(1+\frac{1}{2}z^{-1}\right)\left(-4+2z^{-1}\right)+5 \tag{5.3}$$

$$\implies H(z) = -4+2z^{-1}+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

$$\frac{5}{1+\frac{1}{2}z^{-1}} = 5\left(1+\frac{1}{2}z^{-1}\right)^{-1} \tag{5.5}$$

$$= 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^n \tag{5.6}$$

$$H(z) = -4+2z^{-1}+5-\frac{5}{2}z^{-1}+\frac{5}{4}z^{-2} - \frac{5}{8}z^{-3}+\frac{5}{16}z^{-4}-\frac{5}{32}z^{-5}+\cdots \tag{5.7}$$

$$H(z) = -4+2z^{-1}+5\left(-\frac{1}{2}\right)^{n-2} \text{ for } n \geq 2 \tag{5.8}$$

Therefore, by comparing coefficients

$$h(n) = \begin{cases} 1 & n=0 \\ -\frac{1}{2} & n=1 \\ \frac{5}{4} & n=2 \\ -\frac{5}{8} & n=3 \\ \frac{5}{16} & n=4 \end{cases} \tag{5.9}$$

on applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.10}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.11}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.12}$$

$$\frac{5}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^{n} u(n) \tag{5.13}$$

$$\tag{5.14}$$

Therefore,

$$h(n) = -4\delta(n)+2\delta(n-1)+5\left(-\frac{1}{2}\right)^{n} u(n) \tag{5.15}$$

following Python code that plots Fig. 5.1.

```
wget https://github.com/PratyakshRaj/signal-
    proccessing
```

```
        5.1.py
```

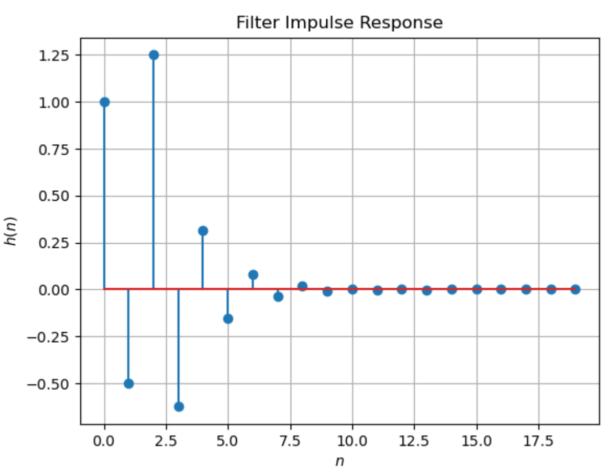

Fig. 5.1: Plot of $h(n)$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.16}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.19),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.17}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.18}$$

using (4.26) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution: following code plots Fig. 5.3.

```
wget https://github.com/PratyakshRaj/signal-
    proccessing

            python3 5.2.py
```

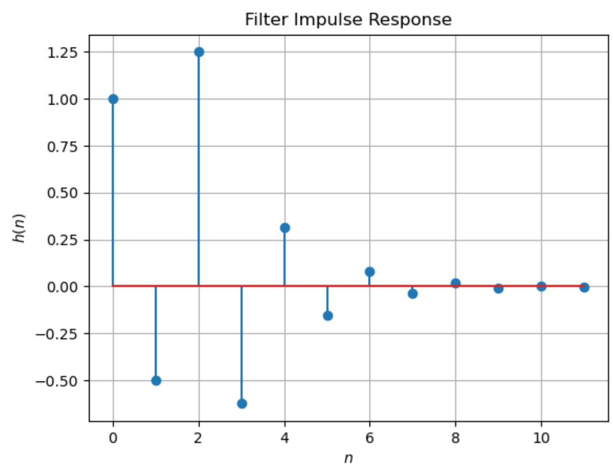

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

From the plot, it is clear that $h(n)$ is bounded.

$$|u(n)| \le 1 \tag{5.19}$$

$$\left|\left(-\frac{1}{2}\right)^{n}\right| \le 1 \tag{5.20}$$

$$\implies \left|\left(-\frac{1}{2}\right)^{n} u(n)\right| \le 1 \tag{5.21}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \le 1 \tag{5.22}$$

$$\implies h(n) \le 2 \tag{5.23}$$

Therefore $h(n)$ is bounded.

5.4 Convergent? Justify using the ratio test.
Solution:

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \quad (5.24)$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \quad (5.25)$$

$$= \frac{1}{2} < 1 \quad (5.26)$$

Therefore, $h(n)$ is convergent

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.27)$$

Is the system defined by (3.2) stable for the impulse response in (5.16)?
Solution: Note that

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.28)$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) = \frac{4}{3} \quad (5.29)$$

Thus, the given system is stable.

5.6 Verify the above result using a python code.
Solution: following code yields the result.

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
python 5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.30)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
python3 5.4.py
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.31)$$

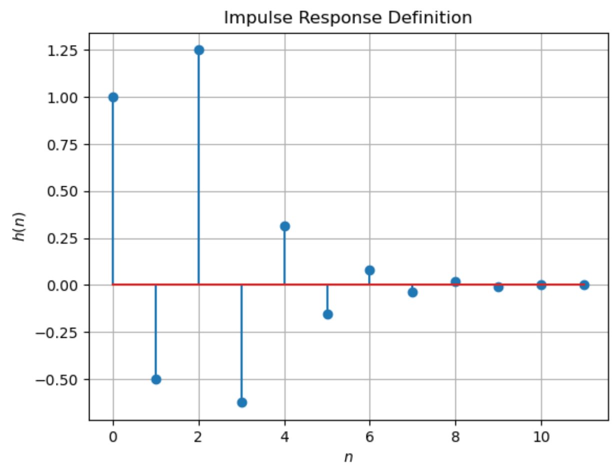

Fig. 5.7: $h(n)$ from the definition

Comment. The operation in (5.31) is known as *convolution*.
Solution: The following code plots Fig. 5.8. it is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
python3 5.5.py
```

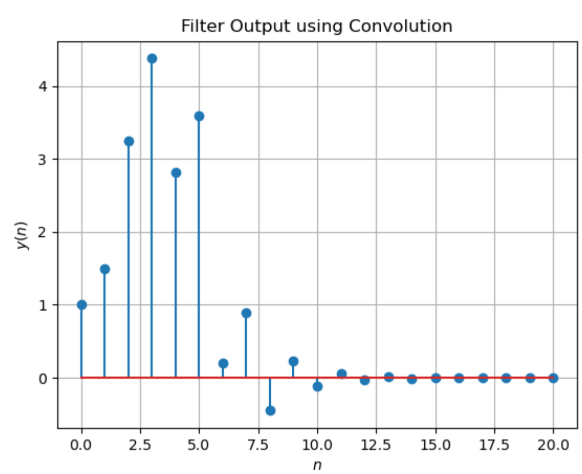

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \tag{5.32}$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.33}$$

and $\mathbf{x}$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.34}$$

following code plot the convolution.

```
https://github.com/PratyakshRaj/signal−
    proccessing
```

```
5.9.py
```

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.35}$$

Solution: from 5.31, we substitute $k : n - k$ to

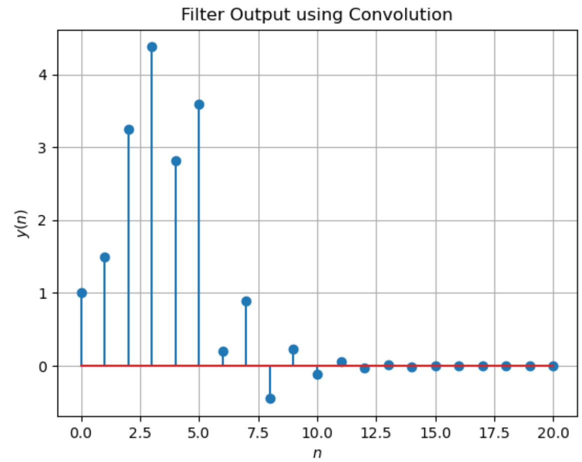

Fig. 5.9: Plot of the convolution of $x(n)$ and $h(n)$

get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.36}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.37}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.38}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: The following code plots Fig. 6.1.

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
python3 6.1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution:

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
6.2.py
```

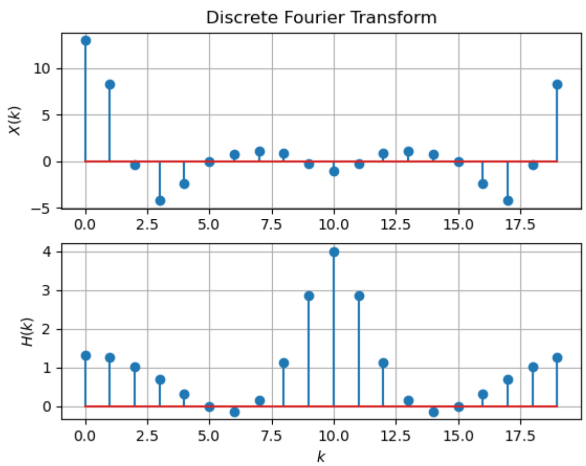

Fig. 6.1: Plots of the real parts of the DFT of $x(n)$ and $h(n)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
6.3.py
```

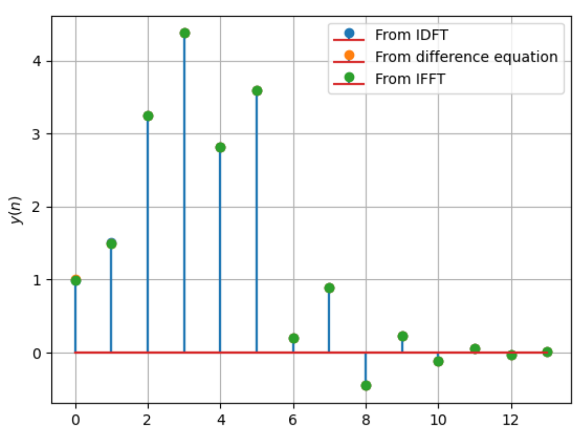

Fig. 6.3: $y(n)$ from the DFT

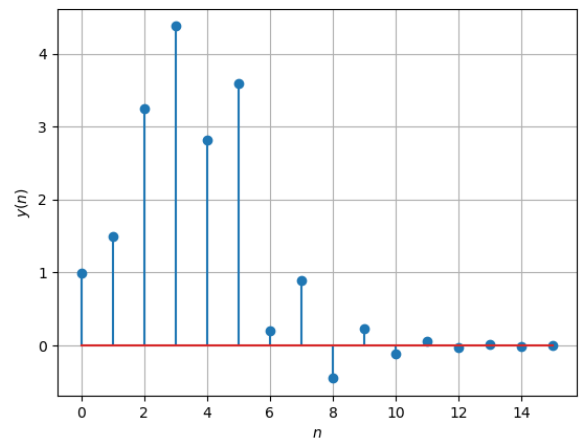

Fig. 6.4: $y(n)$ using FFT and IFFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: code:

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
6.4.py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

6.5 Wherever possible, express all the above equations as matrix equations.
Solution: We use the DFT Matrix, where $\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \ldots & \omega^0 \\ \omega^0 & \omega^1 & \ldots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \ldots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.4}$$

i.e. $W_{jk} = \omega^{jk}$, $0 \leq j, k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \tag{6.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{6.6}$$

Using (6.3), the inverse Fourier Transform is

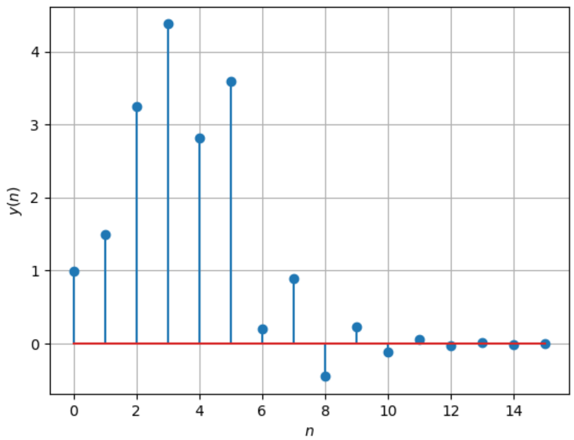

Fig. 6.6: $y(n)$ using DFT matrix

given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \tag{6.7}$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \tag{6.8}$$

where H denotes hermitian operator. We can rewrite (6.2) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{W}\mathbf{h}) \cdot (\mathbf{W}\mathbf{x}) \tag{6.9}$$

6.6 Verify the above equation by generating the DFT matrix in python.
Solution: code:

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
6.6.py
```

The above code plots (6.6)

6.7 Compute the 8-point FFT in C.
Solution: code at https://github.com/PratyakshRaj/signal-proccessing

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N = e^{-\jmath 2\pi/N} \tag{7.8}$$

$$\implies W_{N/2} = e^{-\jmath 4\pi/N} \tag{7.9}$$

And

$$W_N^2 = e^{-\jmath 4\pi/N} \tag{7.10}$$

$$\boxed{W_N^2 = W_{N/2}} \tag{7.11}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$

Solution:

$$\mathbf{F}_2 = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.13}$$

$$\mathbf{D}_{4/2} = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \tag{7.14}$$

$$\text{R.H.S} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_{4/2}\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_{4/2}\mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.15}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{bmatrix} \mathbf{P}_4 \tag{7.16}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{7.17}$$

$$\text{LHS} = \mathbf{F}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.18}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{7.19}$$

Hence, LHS = RHS

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.20}$$

Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the i^{th} column of $\mathbf{F}_N$, from 7.12 and 7.13,

$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.21}$$

and

$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \tag{7.22}$$

Thus,

$$\begin{pmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{pmatrix} \begin{pmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.23}$$

and so,

$$\begin{pmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{pmatrix} \begin{pmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{pmatrix} \mathbf{P}_N$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} = \mathbf{F}_N \tag{7.24}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.25}$$

Solution:

$$\mathbf{P_4} = \begin{pmatrix} \mathbf{e_4}^1 & \mathbf{e_4}^3 & \mathbf{e_4}^2 & \mathbf{e_4}^4 \end{pmatrix} \tag{7.26}$$

From (**??**)

$$\mathbf{x} = \{1, 2, 3, 4\} \tag{7.27}$$

$$\mathbf{P}_4\mathbf{x} = (1, 3, 2, 4) \tag{7.28}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.29}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution:

$$(\mathbf{F}_N\mathbf{x})_k = \sum_{m=0}^{N-1} W_N^{mk} x(m) \tag{7.30}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N} = X(k) = \mathbf{X}_k \tag{7.31}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.32}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.33}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.34}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.37}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.38}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.39}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.40}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.44}$$

Solution: We write out the values of performing an 8-point FFT on $\mathbf{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{j2kn\pi}{8}} \tag{7.45}$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-\frac{j2kn\pi}{4}} + e^{-\frac{j2k\pi}{8}} x(2n+1) e^{-\frac{j2kn\pi}{4}} \right) \tag{7.46}$$

$$= X_1(k) + e^{-\frac{j2k\pi}{4}} X_2(k) \tag{7.47}$$

where $\mathbf{X}_1$ is the 4-point FFT of the even-numbered terms and $\mathbf{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.48}$$

$$e^{-\frac{j2k\pi}{8}} = -e^{-\frac{j2(k-4)\pi}{8}} \tag{7.49}$$

we can now write out $X(k)$ in matrix form as in 7.32 and 7.33. We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{j2kn\pi}{8}} \tag{7.50}$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{j2kn\pi}{4}} + e^{-\frac{j2k\pi}{8}} x_2(2n+1) e^{-\frac{j2kn\pi}{4}} \right) \tag{7.51}$$

$$= X_3(k) + e^{-\frac{j2k\pi}{4}} X_4(k) \tag{7.52}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.54}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.55}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.56}$$

But observe that from 7.25,

$$\mathbf{P}_8 \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{7.57}$$

$$\mathbf{P}_4 \mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \tag{7.58}$$

$$\mathbf{P}_4 \mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \end{pmatrix} \tag{7.59}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.60}$$

compte the DFT using (7.29)

Solution:

$$\mathbf{X} = \mathbf{F}_6 \mathbf{x} \tag{7.61}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right) & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^5 \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^9 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{16} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^5 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} & \left(e^{\frac{-j2\pi}{6}}\right)^{25} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.62}$$

$$= \begin{pmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{pmatrix} \tag{7.63}$$

12. Repeat the above exercise using the FFT after zero padding **x**.
Solution:
The code:

```
wget https://github.com/PratyakshRaj/signal−
    proccessing/blob/main/7.12.ipynb
```

The result:

$$\begin{bmatrix} 13 \\ -3.1213 - 6.5355j \\ j \\ 1.1213 - 0.5355j \\ -1 \\ 1.1213 + 0.5355j \\ -j \\ -3.1213 + 6.5355j \end{bmatrix} \tag{7.64}$$

13. Write a C program to compute the 8-point FFT.
Solution:
The code:

```
wget https://github.com/PratyakshRaj/signal−
    proccessing/blob/main/7.13.c
```

The output:

$$\begin{bmatrix} 13 \\ -3.1327 - j6.5545 \\ j \\ 1.1327 - j0.5545 \\ -1 \\ 1.1327 + j0.5545 \\ -j \\ -3.1327 + j6.5545 \end{bmatrix} \tag{7.65}$$

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: code:

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
7.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.
Solution: For the given values, the difference equation is

$$\begin{aligned} &y(n) - (4.44)\, y(n-1) + (8.78)\, y(n-2) \\ &- (9.93)\, y(n-3) + (6.90)\, y(n-4) \\ &- (2.93)\, y(n-5) + (0.70)\, y(n-6) \\ &- (0.07)\, y(n-7) = \left(5.02 \times 10^{-5}\right) x(n) \\ &+ \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2) \\ &+ \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4) \\ &+ \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6) \\ &+ \left(5.02 \times 10^{-5}\right) x(n-7) \end{aligned} \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.26),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned}
h(n) = {} & [(2.76)(0.55)^n \\
& + (-1.05 - 1.84\jmath)(0.57 + 0.16\jmath)^n \\
& + (-1.05 + 1.84\jmath)(0.57 - 0.16\jmath)^n \\
& + (-0.53 + 0.08\jmath)(0.63 + 0.32\jmath)^n \\
& + (-0.53 - 0.08\jmath)(0.63 - 0.32\jmath)^n \\
& + (0.20 + 0.004\jmath)(0.75 + 0.47\jmath)^n \\
& + (0.20 - 0.004\jmath)(0.75 - 0.47\jmath)^n]u(n) \\
& + \left(-6.81 \times 10^{-4}\right)\delta(n)
\end{aligned} \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

The filter frequency response is plotted at

```
wget https://github.com/PratyakshRaj
    /signal−proccessing
```

```
7.7.2.py
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We observe that for all i, $|p(i)| < 1$ and so, as $h(n)$ is the sum of many convergent series, we see that $h(n)$ converges and is bounded.

$$\sum_{n=0}^{\infty} h(n) = H(1) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (8.2), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2).

```
wget https://github.com/PratyakshRaj/signal−
    proccessing
```

```
7.2.py
```

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: It is low pass with order=4 and cutoff-frequency=4kHz.

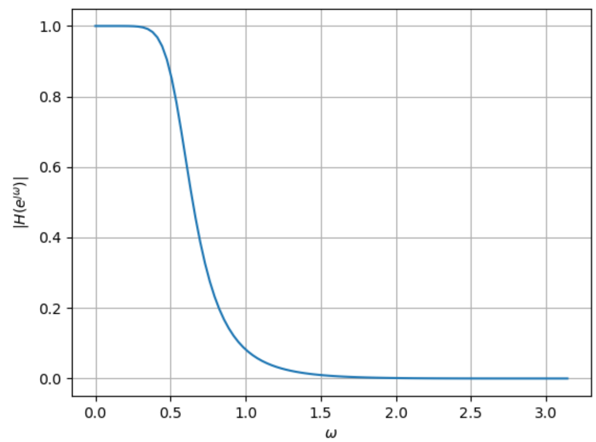

Fig. 8.2: Plot of $h(n)$

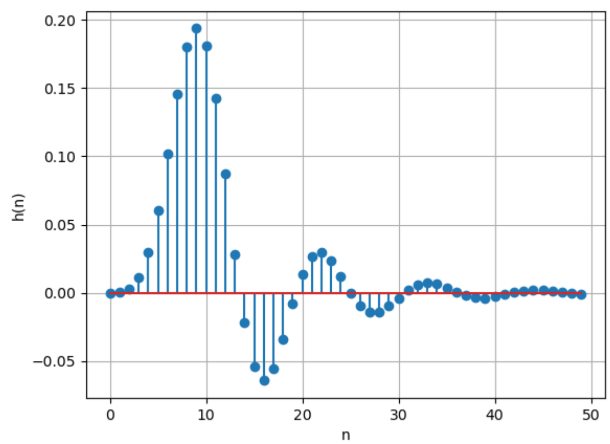

Fig. 8.2: Filter frequency response

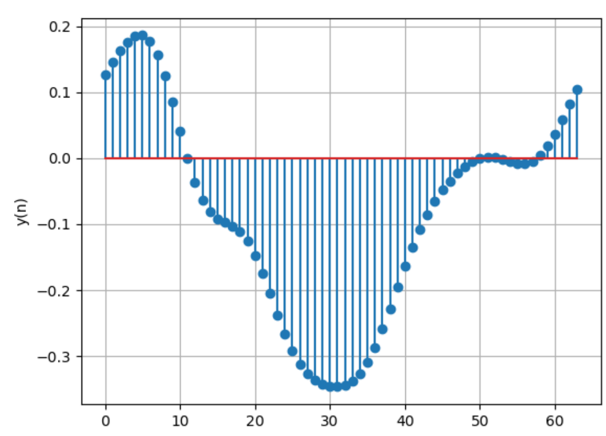

Fig. 8.2: Plot of $y(n)$

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: At the order of the filter= 7