DSP: Part II

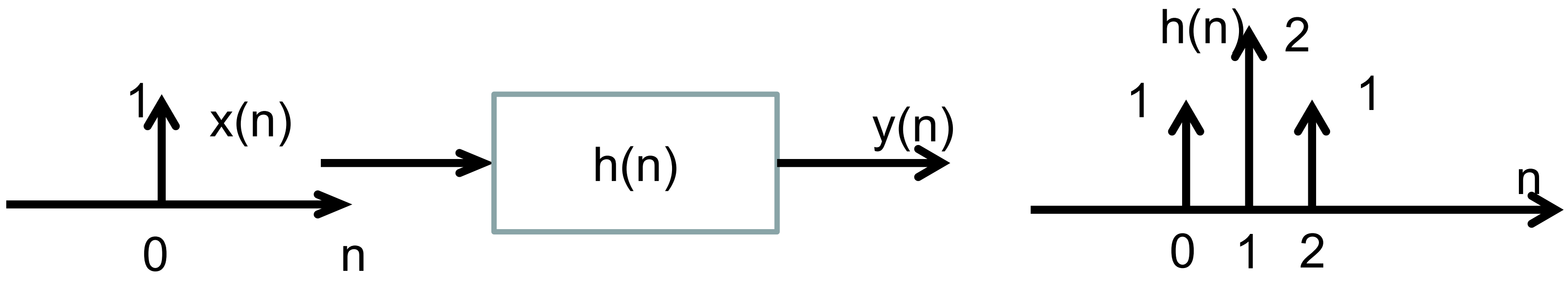
Lecture.2.2

Finite Impulse Response (FIR) Filter

Methodology/Logic for FIR

1) Digital FIR filters
2) Linear Phase FIR Filters
3) Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω) (Reviewing Z-transform and Fourier Transform)
4) Design of FIR Filters using MATLAB.
5) Design of FIR Filters – using the windowing technique

1) What is a DIGITAL FIR filter



- A digital FIR filter: an impulse response h(n) that is finite in length.
- In the part of this course that follows, we restrict our attention to causal filters, i.e, impulse response comes after the arrival of impulse.
- We only study Linear Time Invariant (LTI) filter; hence the output signal = input signal convolved with impulse response:

$$y(n) = \mathrm{x(n)} * \mathrm{h(n)} = \sum_{k=-\infty}^{\infty} \mathrm{x(k)h}(n-k) = \sum_{k=-\infty}^{\infty} h(\mathrm{k})x(n-k)$$

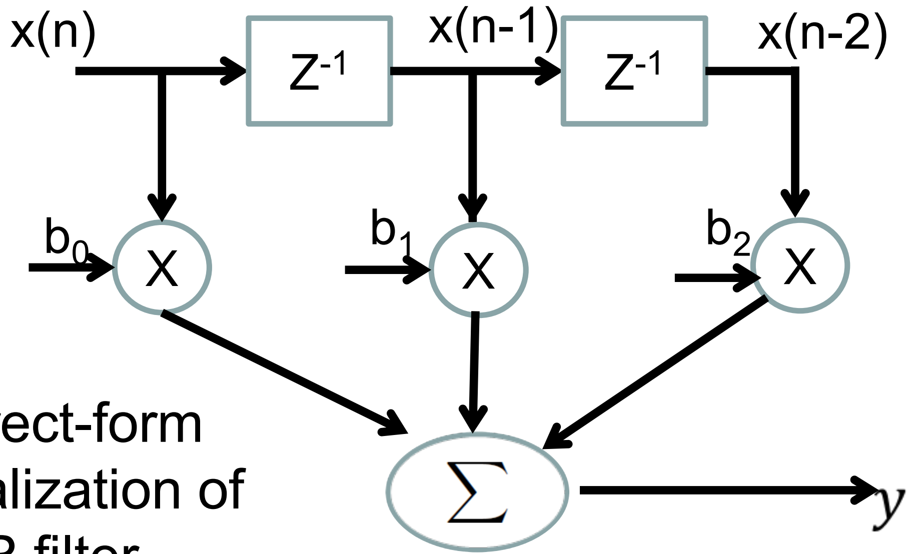
- In the example above (3-tap FIR filter), the output and input can be describe by:

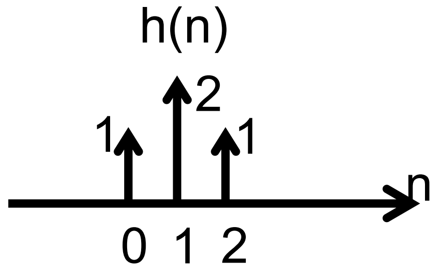
$$y(n) = \mathrm{b}_0\, \mathrm{x(n)} + \mathrm{b}_1\, \mathrm{x(n - 1)} + \mathrm{b}_2\, \mathrm{x(n - 2)}$$

where b_0, b_1 , b_2 are the coefficients of the FIR filter and are identical to the coefficients h_0, h_1 , h_2 of the impulse response.

1) Example of a digital FIR filter
(cont'd)

$$y(n) = b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2)$$





Direct-form realization of FIR filter

$$y(n) = \sum_{l=0}^{L-1} b_l x(n-l)$$

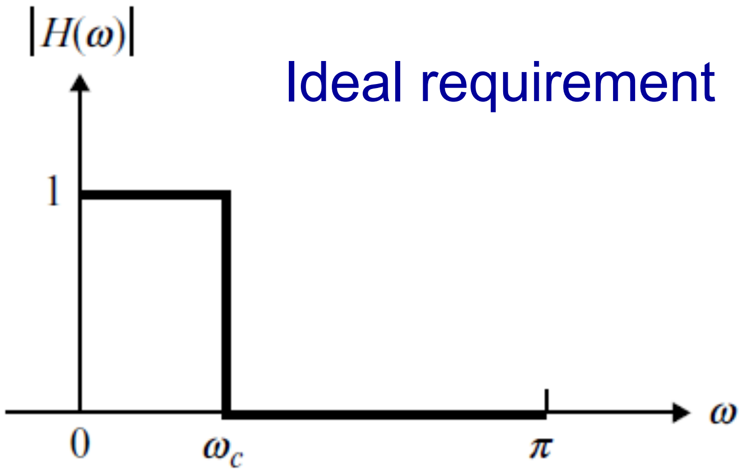
Number of computation for performing convolution:

- For each sample, L multiplications and L-1 additions are needed, where L = FIR filter length.
- Hence for a block of N samples of input x(n), computation to generate y(n): N*L multiplications and N*(L-1) additions.

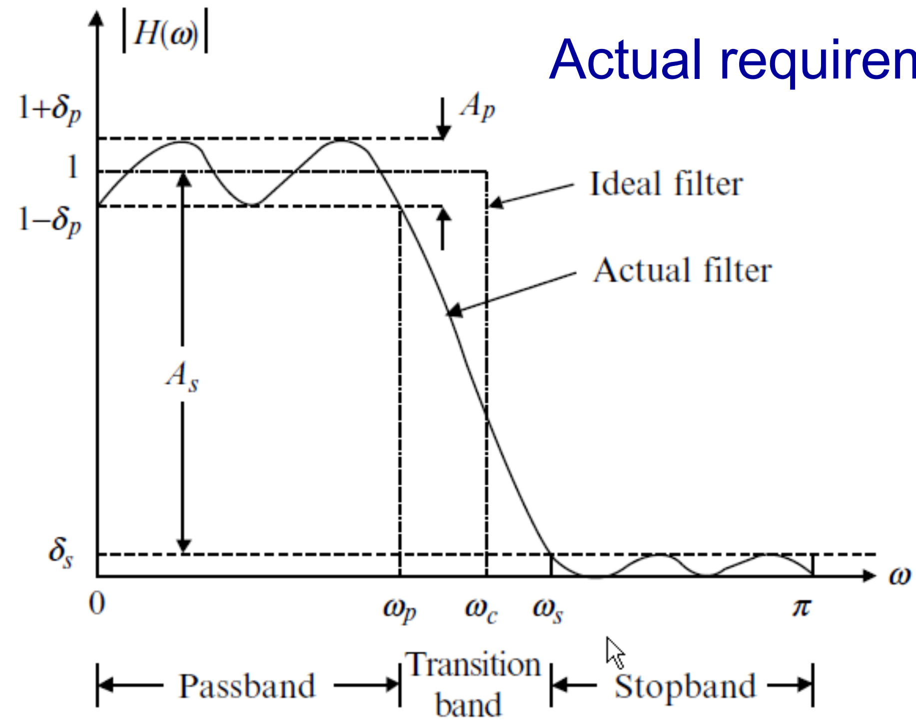
1) So how do we spec a filter? (magnitude response)

1) Given a specification of a filter transfer function:

Ideal requirement



Actual requirement



For FIR filter design, our task is to find:

- The length (L) of the filter and
- The coefficients b_l of the FIR filter, i.e the impulse response values

Practise with Q1(a), Tut 5.

Methodology/Logic for FIR

1) Digital FIR filters
2) Linear Phase FIR Filters
3) Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω) (Reviewing Z-transform and Fourier Transform)
4) Design of FIR Filters using MATLAB.
5) Design of FIR Filters – using the windowing technique

What is phase?

- We can express a complex Fourier transform function with its amplitude and phase separately:

$$X(\omega) = |X(\omega)|\, e^{j\theta(\omega)}$$

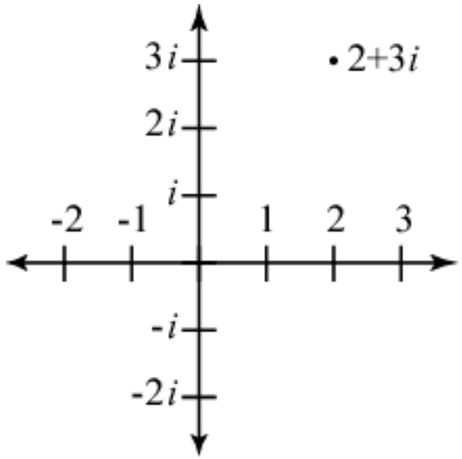
$|X(\omega)|$ - amplitude

$\theta(\omega)$ - phase

For any complex number:

$$x = a + jb \rightarrow x = r(\cos\theta + j\sin\theta)$$

$$r = \sqrt{x x^*} = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$



Euler's formula: $\quad e^{j\theta} = \cos\theta + j\sin\theta$

$$x = a + jb \rightarrow x = r(\cos\theta + j\sin\theta) \rightarrow re^{j\theta}$$

phase

amplitude

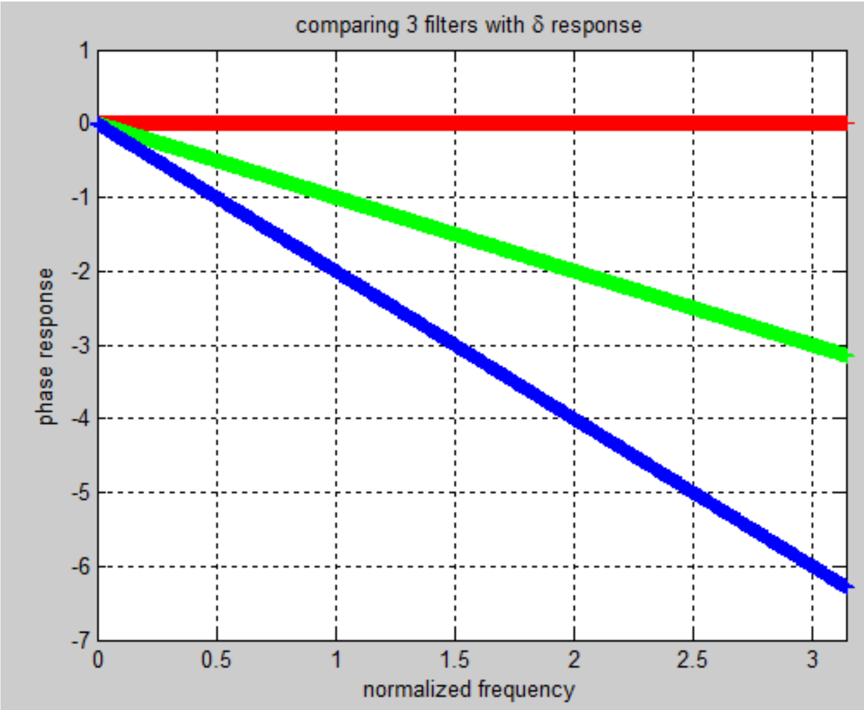
2) Why is Linear Phase important in filters?

- Look at an example first: if the magnitude gain of a filter =1, and the phase is linear; what is the impulse response of the filter?

Consider 3 filters: $h_0[n] = [1]$ vs $h_1[n] =[0\ 1]$ vs $h_2[n] =[0\ 0\ 1]$
  - The magnitude response of the 3 filters are the same: gain = 1 for all frequency (**can be verified via DTFT(h[n])**).
  - However, the phase response are different:

Red: $h_0[n]= [1\ 0\ 0]$;
Green: $h_1[n]= h_0[n-1]= [0\ 1\ 0]$;
Blue: $h_2[n]= h_0[n-2]= [0\ 0\ 1]$;

$$\mathrm{DTFT}(h_0) = H_0(\omega)$$
$$\mathrm{DTFT}(h_0[n-d]) = e^{-j\omega d} H_0(\omega)$$



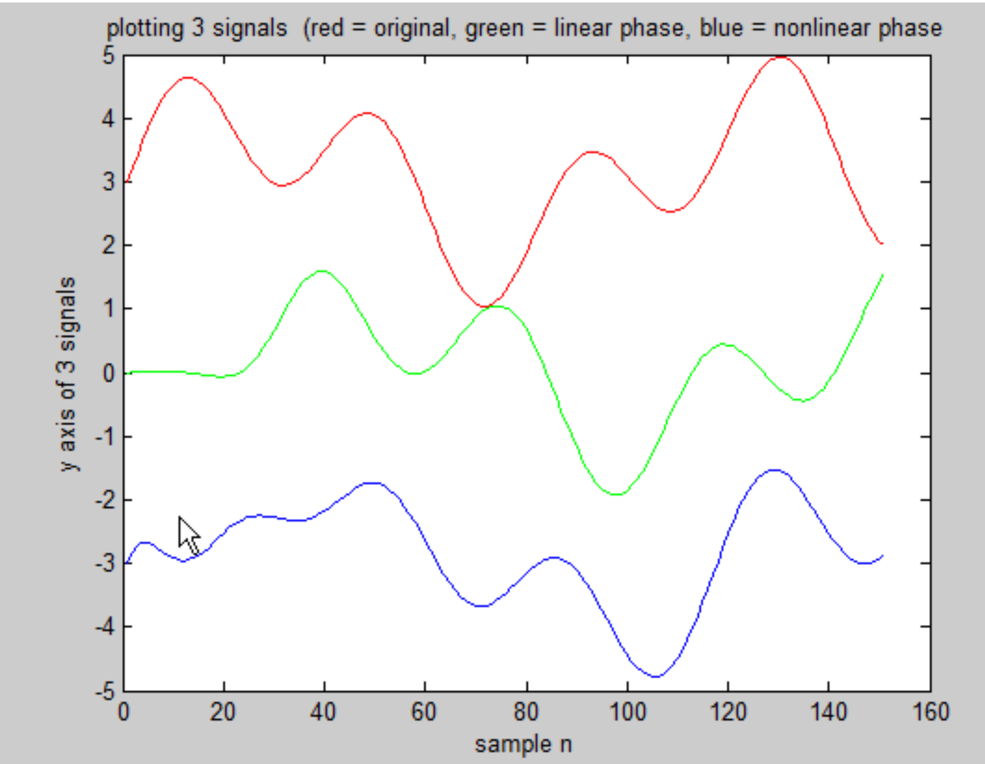
Matlab: plot_3LinearPhaseResponse.m

Find the transfer functions corresponding to the following impulse responses:

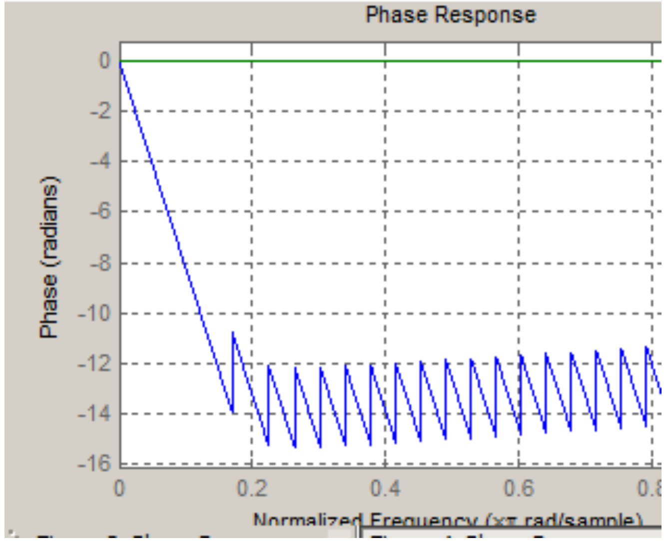
$h_0[n] = [1]$, $h_1[n] =[\ 0\ 1]$ and $h_2[n] =[\ 0\ 0\ 1]$.

2) Why is Linear Phase important?

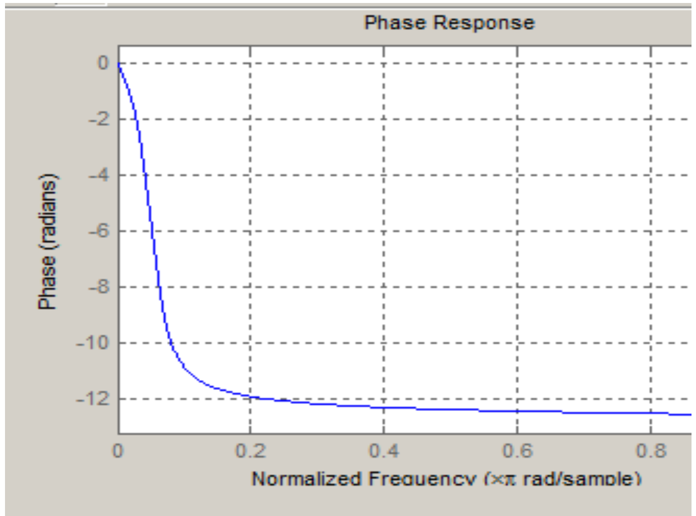
In the following examples, we pass a signal through 2 filters with same magnitude gain =1. The second filter produces nonlinear phase, observe phase distortion!



Linear Phase



Non-Linear Phase



Signal red = original,

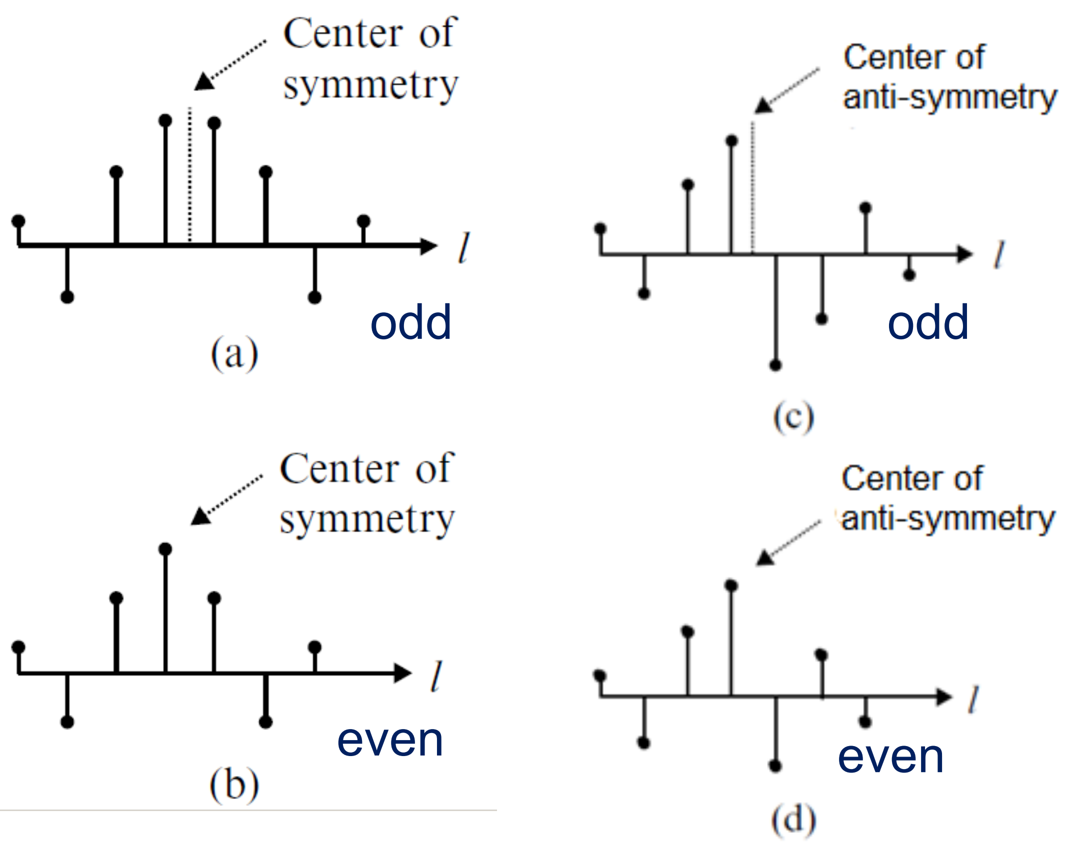
Green = passed through linear phase, gain =1

Blue = passed through nonlinear phase, gain =1

Matlab: plot_3nonLinearPhaseResponse

2) How to recognize a linear phase filter?

- Hence, if a filter has linear phase, then no-phase distortion!
  - In addition, if magnitude == 1 and the phase is linear, then the whole signal is simply delayed.
- How to get a linear phase filter?
  - Usually, linear phase is achieved using an FIR filter.
  - It can be shown that if the filter response is symmetrical or anti-symmetrical, then the phase is linear.



symmetrical or anti-symmetrical impulse response h[n]→ linear phase.

order of an FIR filter can be even or odd.

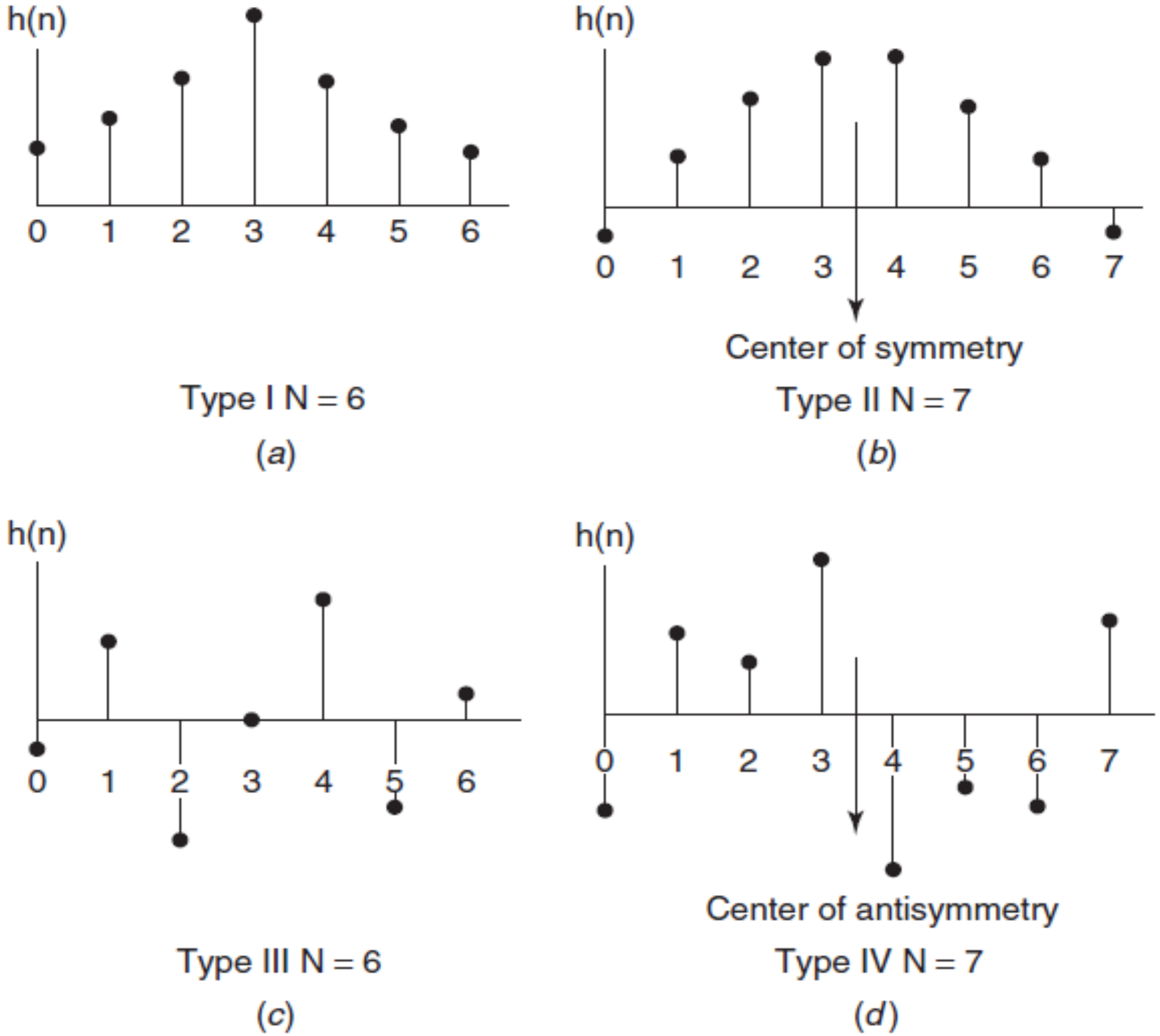
Hence,
4 different possibilities.

2) Four Types of Linear Phase Filters

N: order of an FIR filter
Length (tap): N+1

Type	impulse response	
I	symmetric	length is odd
II	symmetric	length is even
III	anti-symmetric	length is odd
IV	anti-symmetric	length is even

The longer the length (L) of the filter, i.e., more coefficients, the more complex the filter is, hence more computation; but it will allow more flexibility and generate sharper transition. This is the classic tradeoff of performance vs complexity.



impulse responses of the four types of linear phase FIR filters.

2) Example A: Type I FIR Filter

- In this example, h(0) = h(6), h(1) = h(5), h(2) = h(4), and h(3) is at the center of the symmetry

$$\begin{aligned}
\mathrm{H(z)} &= \textstyle\sum_{n=-\infty}^{\infty} \mathrm{h(n)z^{-n}} \\
&= h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} \\
&= h(0)(1+z^{-6}) + h(1)(z^{-1}+z^{-5}) + h(2)(z^{-2}+z^{-4}) + h(3)z^{-3} \\
&= z^{-3}\{h(0)[z^{3}+z^{-3}] + h(1)[z^{2}+z^{-2}] + h(2)[z^{1}+z^{-1}] + h(3)\}
\end{aligned}$$

$$H(e^{j\omega}) = \mathrm{H(z)}|_{z=e^{j\omega}}$$

$$\begin{aligned}
&= e^{-j3\omega}\{h(0)[e^{j3\omega}+e^{-j3\omega}] + h(1)[e^{j2\omega}+e^{-j2\omega}] + h(2)[e^{j\omega}+e^{-j\omega}] + h(3)\} \\
&= e^{-j3\omega}\{2h(0)\cos(3\omega) + 2h(1)\cos(2\omega) + 2h(2)\cos(\omega) + h(3)\} \\
&= e^{-j3\omega}\, H_r(e^{j\omega})
\end{aligned}$$

Has no contribution to phase, except for possibility of 180°.

where $H_r(e^{j\omega})$ is real valued and can be positive or negative. If $H_r(e^{j\omega})$ is negative, it introduces a phase shift of π radian (is 180°). The term in front $e^{-j3\omega}$ affects the phase. This term shows that the phase delay introduced is linearly dependent on frequency, specifically is equal to 3ω.

Practise with Q3, Q4(a), Tut 5.

2) Delay relating to Linear Phase Filter

$$H(e^{j\omega}) = e^{-j3\omega} H_r(e^{j\omega})$$

In this example, the linear phase delay is 3ω, where the 3 represents 3 samples delay. Hence if we know the sampling rate, e.g, F_s =1/T, then 3 samples delay is = 3T seconds. For a linear phase filter, this also means that all frequencies are equally delayed by 3T seconds (group delay).

2)The 4 Types of Frequency Responses

The frequency responses of the four types of FIR filters are summarized below:

$$H(e^{j\omega}) = e^{-j[(N/2)\omega]}\left\{h\left(\frac{N}{2}\right)+2\sum_{n=1}^{N/2}h\left[\frac{N}{2}-n\right]\cos(n\omega)\right\}$$

We have shown the case with N=6 above.

phase delay: -(N/2)*ω
group delay in time: ~N/2

for type I

$$H(e^{-j\omega}) = e^{-j[(N/2)\omega]}\left\{2\sum_{n=1}^{(N+1)/2}h\left[\frac{N+1}{2}-n\right]\cos\left(\left(n-\frac{1}{2}\right)\omega\right)\right\}$$

for type II

$$H(e^{-j\omega}) = e^{-j[(N\omega-\pi)/2]}\left\{2\sum_{n=1}^{N/2}h\left[\frac{N}{2}-n\right]\sin(n\omega)\right\}$$

for type III

$$H(e^{-j\omega}) = e^{-j[(N\omega-\pi)/2]}\left\{2\sum_{n=1}^{(N+1)/2}h\left[\frac{N+1}{2}-n\right]\sin\left(\left(n-\frac{1}{2}\right)\omega\right)\right\}$$

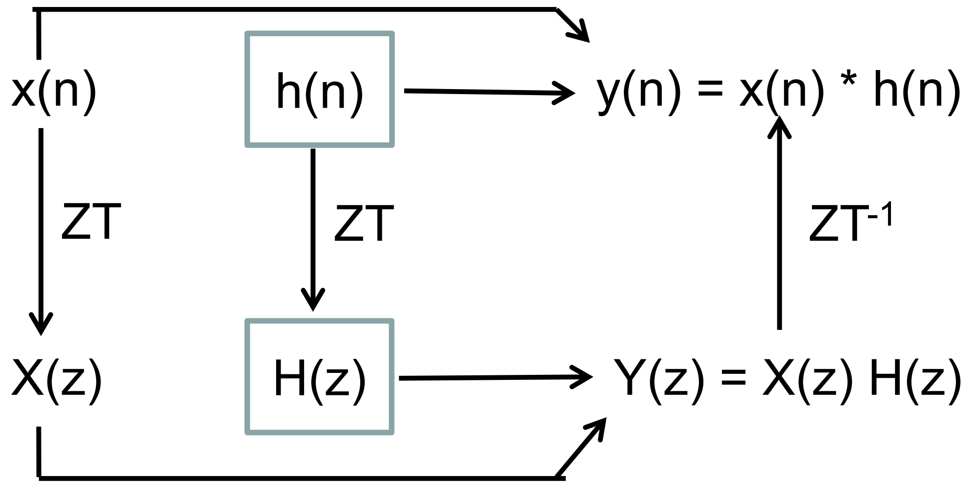
for type IV

Methodology/Logic for FIR

1) Digital FIR filters
2) Linear Phase FIR Filters
3) Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω) (Reviewing Z-transform and Fourier Transform)—good for both FIR and IIR, although we only concentrate on FIR in Part P2.2.
4) Design of FIR Filters using MATLAB.
5) Design of FIR Filters – using the windowing technique

3) What is H(Z) –a review

- H(z) is the transfer function of the filter: Z-transform of h(n)



$$H(z) = \frac{Y(Z)}{X(Z)} = ZT[h(n)] = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

3)Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω)

We need both for a complete analysis.

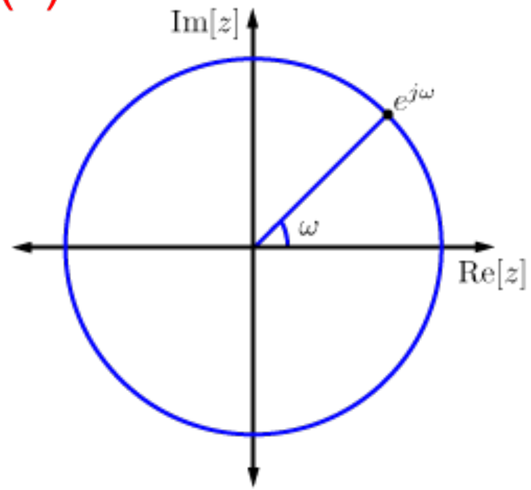
- What is 'z' in H(z)?

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

- 'z' is a complex number, and H(z) is a polynomial function of 'z' in the complex plane and uses h(n) as its coefficients.

H(z) at 'z' on the unit circle, i.e.,

$z = e^{j\omega}$ for $\omega = 0..\pi$, is $H(\omega)$.



$$H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) z^{-n} |_{z=e^{j\omega}}$$
$$= \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = H(\omega)$$

Remember on the unit circle (because $|z|=1$):
$z = \cos(\omega) + j \sin(\omega) = e^{j\omega}$ (Euler formula)

3) Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω)

- H(z) is a polynomial, and it can be factored. E.g,

  $H(z) = h_0 + h_1 z + h_2 z^2 = (z-a)(z-b)$

  - The values when z = *a* or *b* are values that will make H(z) to be 0. These are known as the roots of the polynomial.

3) Recap: MATLAB codes for roots and converting roots to polynomials

Let consider: $H(z) = \underline{z^2 + 3z + 2} = (z+1)(z+2)$

To find the factors of H(z), use roots([1 3 2]); result = [-2 -1];

To get polynomial from roots, use poly([-1 -2]), result = [1 3 2]

Syntax

```
r = roots(c)
```

Description

`r = roots(c)` returns a column vector whose elements are the roots of the polynomial `c`.

Row vector `c` contains the coefficients of a polynomial, ordered in descending powers. If `c` has `n+1` components, the polynomial it represents is $c_1 s^n + \ldots + c_n s + c_{n+1}$.

Syntax

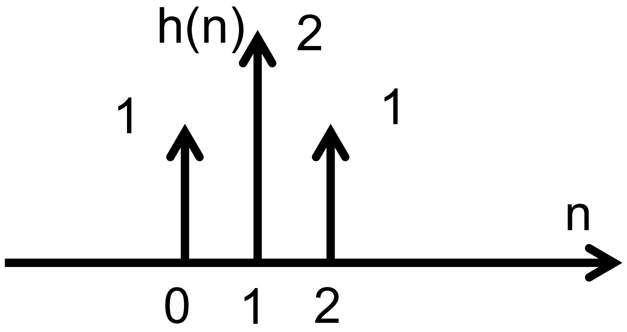
```
p = poly(r)
```

Description

`p = poly(r)` where `r` is a vector returns a row vector whose elements are the coefficients of the polynomial whose roots are the elements of `r`.

3) Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω)

Given h(n) = {1,2,1}, we have H(z) = 1 + 2z^{-1} + 1z^{-2}



Therefore $H(z) = \frac{1+2z^{\{-1\}}+1z^{\{-2\}}}{1} = \frac{z^{\{2\}}+2z^{\{1\}}+1}{z^{\{2\}}} = \frac{(z+1)(z+1)}{(z)(z)}$

Zeros of H(z): The roots of the numerators polynomial : -1 and -1

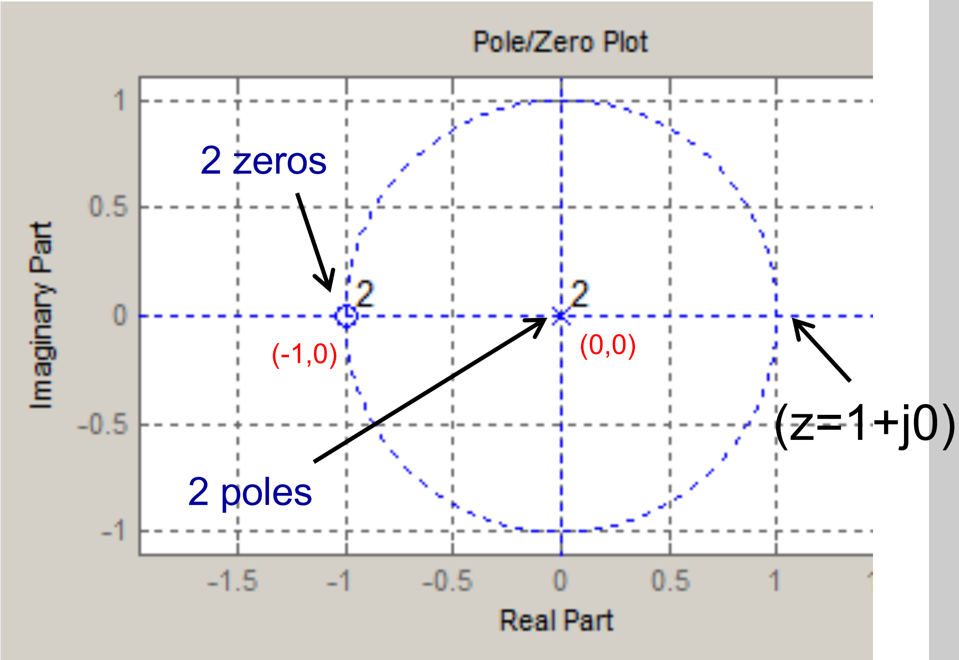
Poles of H(z): Roots of the denominators polynomial : 0 and 0

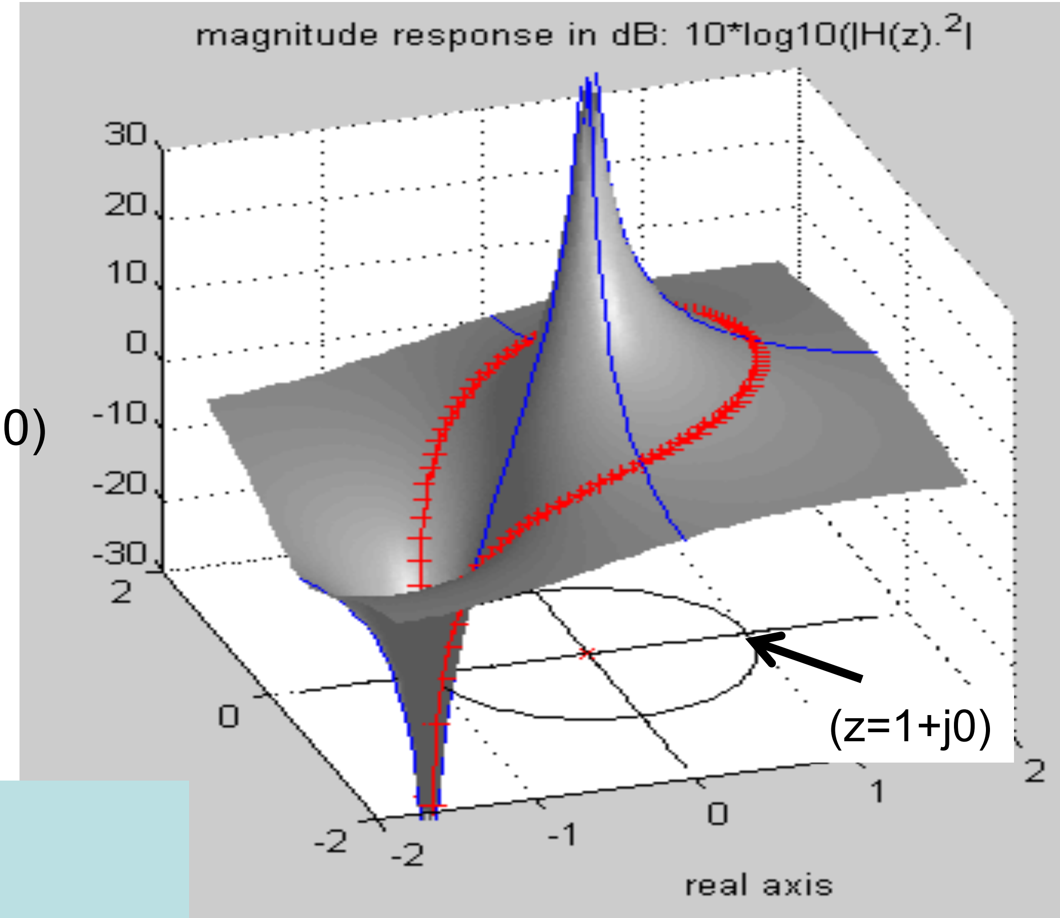
3) Analysis: From impulse response to H(z) pole/zero plot and then to frequency response H(ω)

$H(z) = 1 + 2z^{-1} + 1z^{-2}$ (cont'd)

Zeros of H(z): Two roots of the numerator polynomial are -1, corresponding to (-1,0) in the complex plane — see below.

Poles of H(z): Two roots of the denominator polynomial are 0, corresponding to (0,) in the complex plane — see below.





Matlab: use fir_3D_plot_Hz to visualize H(z) surface.