Digital Signal Processing:

Part II

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DSP: Part II

Lecture.1.1

Sampling and Reconstruction

Major Concepts in Part II

- Sampling and Reconstruction
- Digital Filters—FIR and IIR

Methodology/Logic for Sampling and Reconstruction

A. Overview of sampling
B. Sampling theorem & aliasing
C. A mathematical model of sampling in frequency domain
D. Reconstruction
E. Discrete time processing of continuous time signals
F. Up and down sampling
G. Quantization

A. Why Sampling?

- In order to process analog signals digitally, three steps are involved:

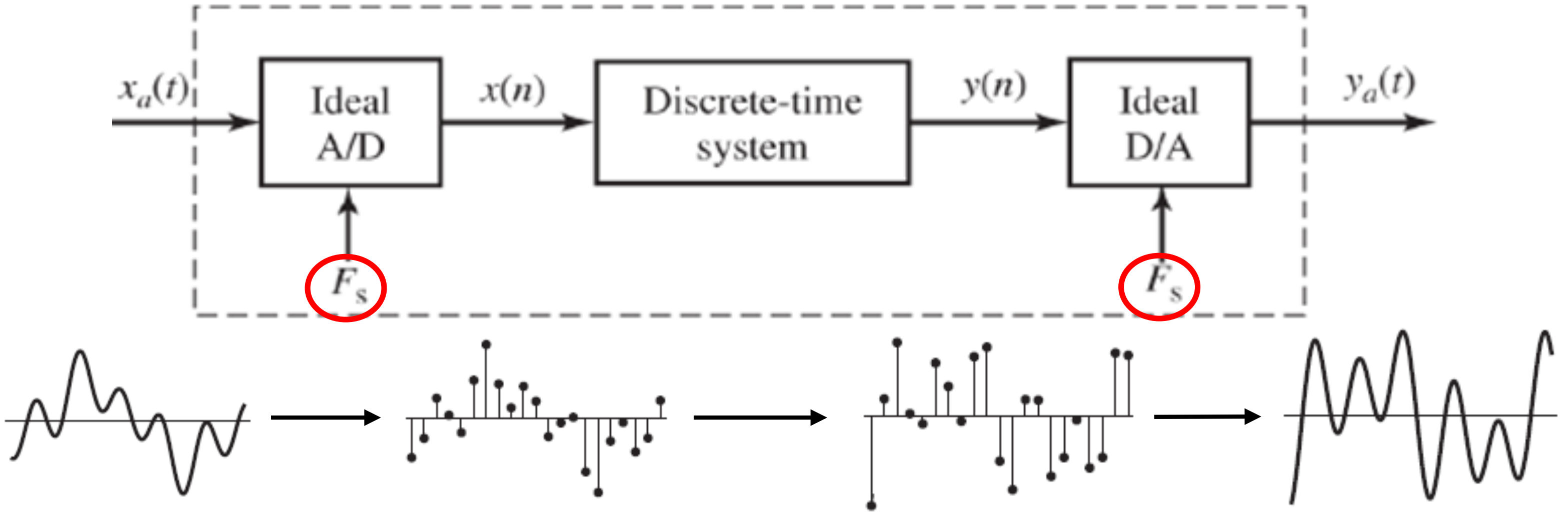
  - **Digitization**: **sampling** (digitization of time axis) and **quantization** (digitization of amplitude axis); also known as *Analog-to-Digital Conversion (A/D)*.

  - **Processing**: Digital samples are processed by a digital signal processor.

  - **Reconstruction**: Resultant digital signal is converted back into analog form by an analog reconstructor also known as *Digital-to-Analog Conversion* (D/A)

- Illustration in the next slide

A. Block diagram of a DSP system
(3 steps)



- A/D (Analog/Digital) converts analog signal to discrete sequence using sampling rate F_s .
- A discrete time system processes the signal in the digital domain.
- D/A (Digital/Analog) converts the digital signal y[n] back to analog signal y(t) using F_s as a parameter.

B. Sampling Theorem & Aliasing

- Nyquist Theorem

We can digitally represent only (analog) frequencies up to half the sampling rate

The sampling frequency should be at least twice the highest frequency contained in the signal

  - Example:
    - CD recording with F_s = 44,100Hz
      Maximum captured frequency = $F_s/2$ = 22,050 Hz
    - Telephone recording with F_s = 8000 Hz
      Maximum captured frequency = $F_s/2$ = 4000 Hz

B. Terminologies

- The maximum (analog) frequency that can be reconstructed correctly (i.e., without aliasing) by a certain sampling rate is called Nyquist frequency

$$\boxed{\text{Nyquist frequency} = \tfrac{1}{2}\ \text{Sampling rate}}$$

- The Nyquist rate is the minimum sampling rate in order to represent digitally, an analog signal with maximum frequency F_{max}

$$\boxed{\text{Nyquist rate } (F_s) > 2 \times F_{max}}$$

- A signal is
  - *under-sampled* if sampled below the Nyquist rate
  - *critically sampled* if sampled at Nyquist rate!
  - *over-sampled* if sampled higher than the Nyquist rate

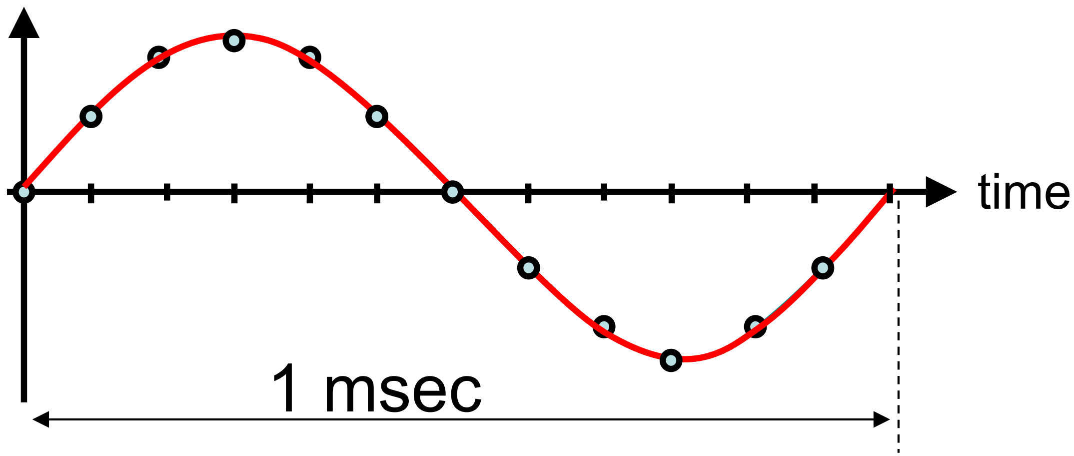
Sampling of Signal - *over-sampled*

Frequency of signal f_{SIG} = 1 KHz

- 1 cycle in 1 msec

Sampling frequency f_S = 12 KHz

- 12 samples in 1 msec



Observed frequency = 1 KHz (i.e. correct)

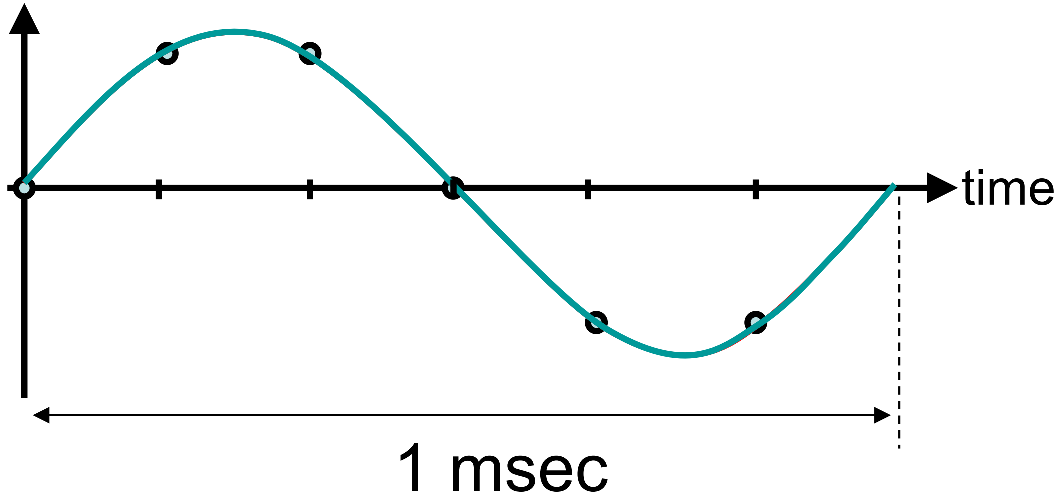
Ratio: f_S/f_{SIG} = 12/1 = 12

Sampling of Signal - *over-sampled*

(fewer samples per cycle)

Frequency of signal f_{SIG} = 1 KHz

Sampling frequency f_S = 6 KHz (6 samples in 1 msec)



Observed frequency = 1 KHz (correct)

Ratio: f_S/f_{SIG} = 6/1 = 6

Sampling of Signal *- over-sampled*

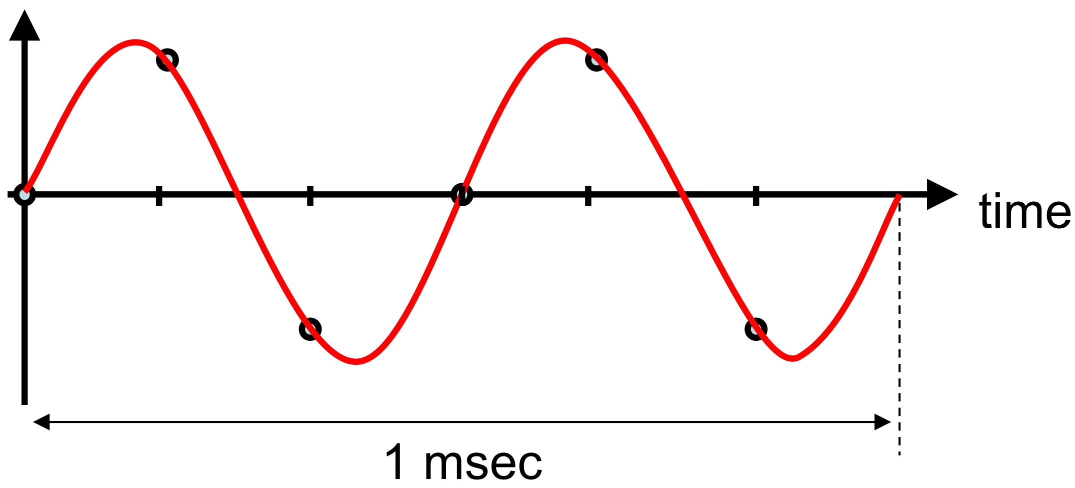
(further reduced in samples per cycle)

Frequency of signal f_{SIG} = 2 KHz

• 2 cycles in 1 msec

Sampling frequency f_S = 6 KHz

• 6 samples in 1 msec



Observed frequency = 2 KHz (correct)

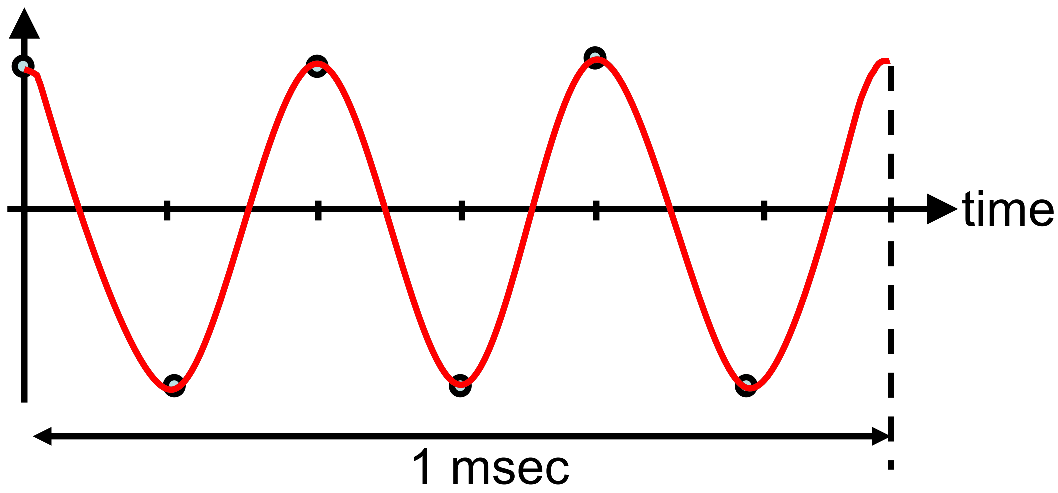
Ratio: f_S/f_{SIG} = 6/2 = 3

Sampling of Signal *- critically sampled*

Frequency of signal f_{SIG} = 3 KHz

- (3 cycles in 1 msec)

Sampling frequency f_S = 6 KHz

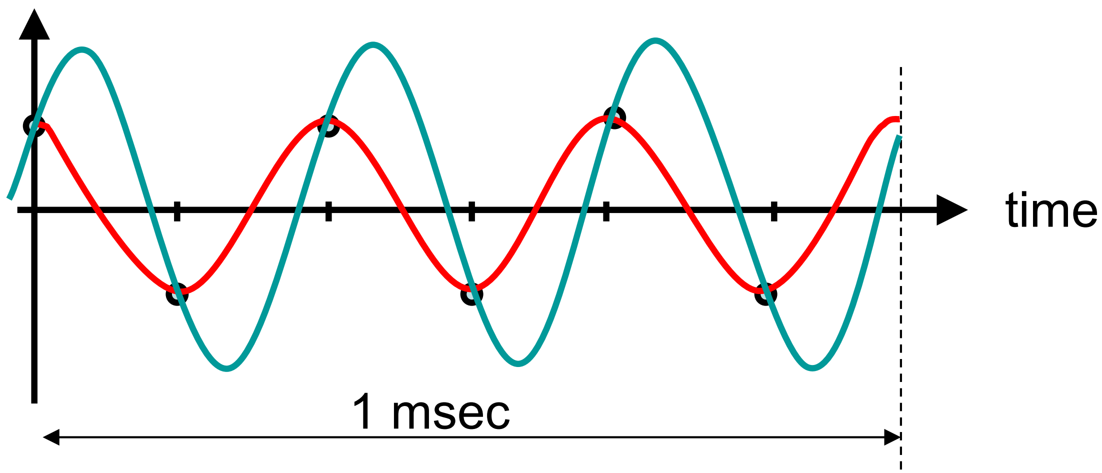


Observed frequency = 3KHz (correct)

Ratio: f_S/f_{SIG} = 6/3 = 2

Minimum Sampling Frequency - *critically sampled*

Frequency of signal f_{SIG} = 3 KHz (3 cycles in 1 msec)
Sampling frequency f_S = 6 KHz (6 samples in 1 msec)

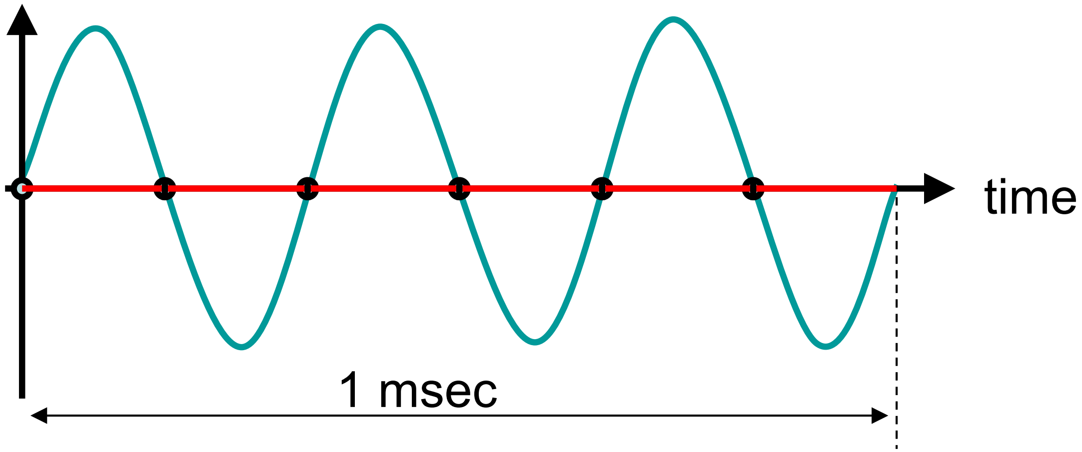


Observed frequency = 3KHz
i.e. correct frequency but wrong amplitude & phase.
Ratio: f_S/f_{SIG} = 6/3 = 2

Minimum Sampling Frequency - *critically sampled*

Frequency of signal f_{SIG} = 3 KHz

Sampling frequency f_S = 6 KHz



Observed frequency = 0Hz!

Ratio: f_S/f_{SIG} = 6/3 = 2

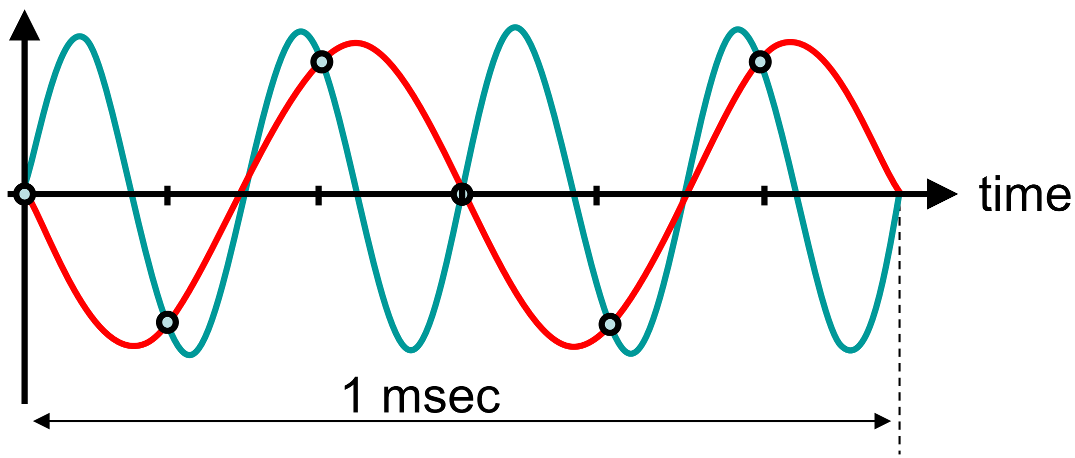
conclusion

- Nyquist sampling rate: the necessary (but not the sufficient) condition for signal reconstruction
  - It is desirable to use a sampling rate F_s **>** $2*F_{max}$ (instead of F_s **=** $2*F_{max}$).

Sampling of Signal – *under-sampled*

Frequency of signal f_{SIG} = 4 KHz (4 cycles in 1 msec)

Sampling frequency f_S = 6 KHz (6 samples in 1 msec)



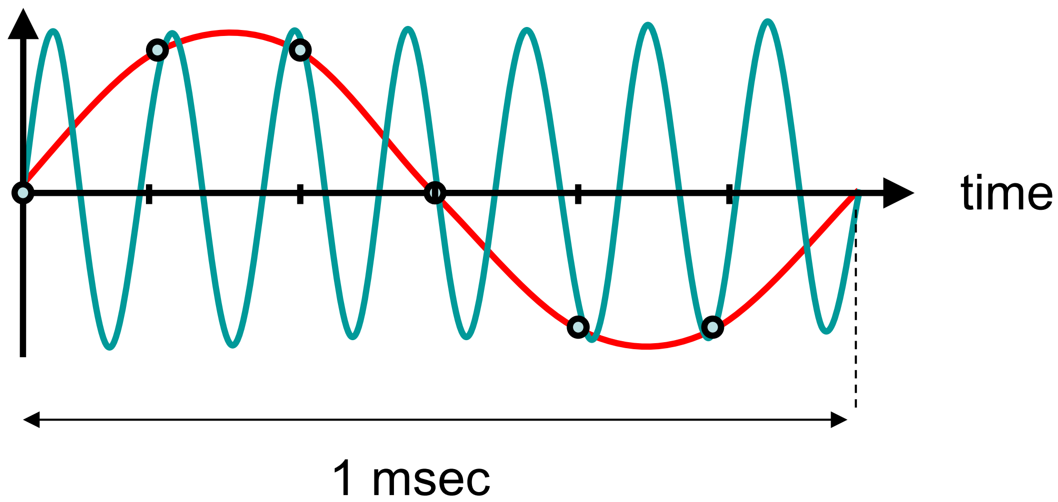
Observed frequency = 2 KHz! (wrong!)

Ratio: f_S/f_{SIG} = 6/4 = 1.5

Sampling of Signal – *under-sampled*

Frequency of signal f_{SIG} = 7 KHz (7 cycles in 1 msec)

Sampling frequency f_S = 6 KHz (6 samples in 1 msec)



Observed frequency = 1KHz (wrong)

Ratio: f_S/f_{SIG} = 6/7 = 0.86

B. Aliasing

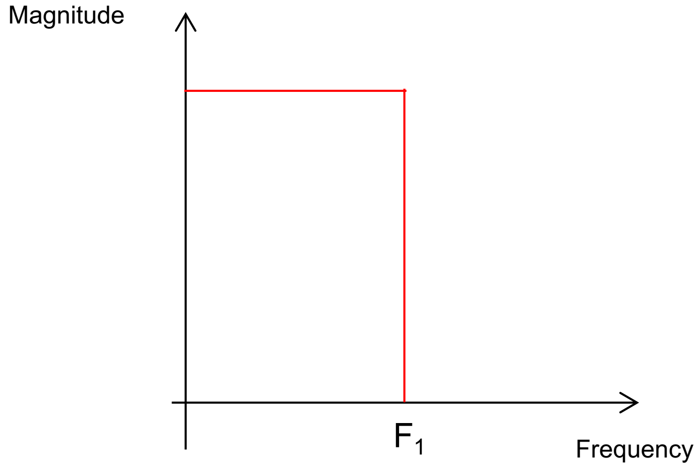
- What happens when sampling rate is too slow?
  - The frequency of reconstructed signal will be not the same as frequency of original signal → ALIASING phenomenon

B. Effect of Aliasing

- An aliased signal provides a poor representation of the analog signal
- Aliasing is an effect that causes different signals to become indistinguishable (or *aliases* of one another) when sampled.
- Aliasing causes false frequency component to appear in the reconstructed signal (as *examples above and to be further explored next)*

B. Avoid Aliasing

- Approach 1: Increasing sampling rate at least twice the highest frequency component in the signal regarding to Nyquist theorem
- Approach 2: Use an anti-aliasing analog lowpass filter before the A/D converter to remove frequencies higher than the Nyquist frequency

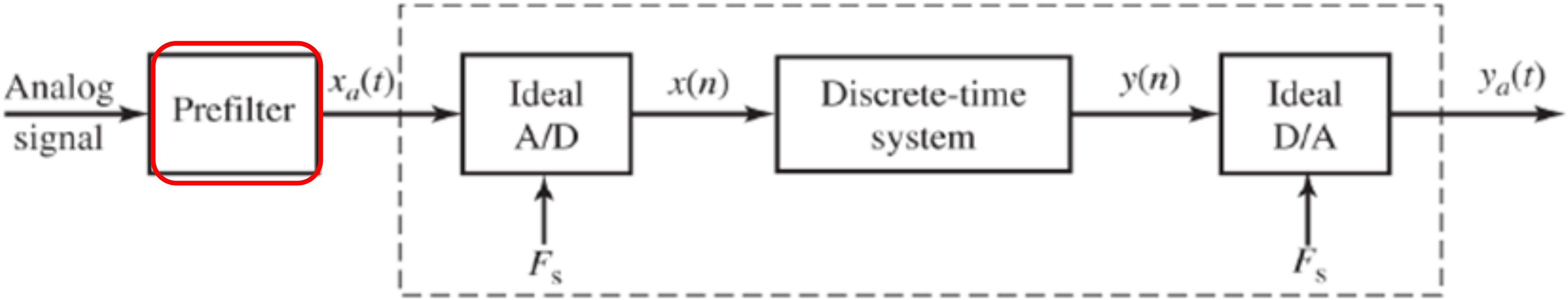


Ideal Anti-alias Filter:

- F_1 is maximum input frequency
- Frequencies < F_1 are desired frequencies
- Frequencies > F_1 are undesired frequencies

B. Avoid Aliasing

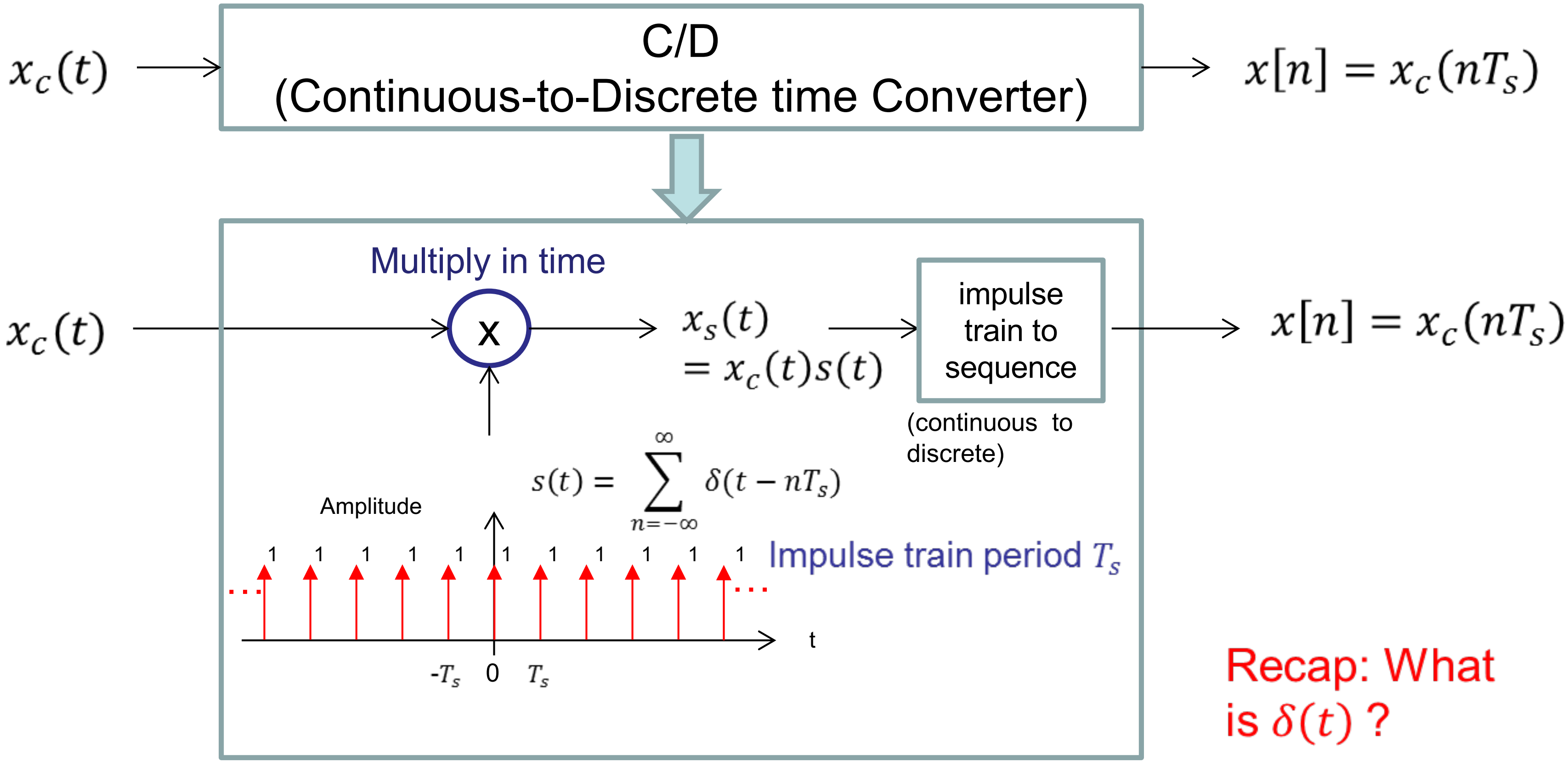
- The block diagram of a DSP system with a prefilter to avoid aliasing



- Signal is pre-filtered to limit highest frequency –
band-limiting the signal so that A/D conversion would not have aliasing. Pre-filter is a typically a Low Pass filter with cutoff freq uency = ½ Fs

C. A mathematical model of sampling in frequency domain

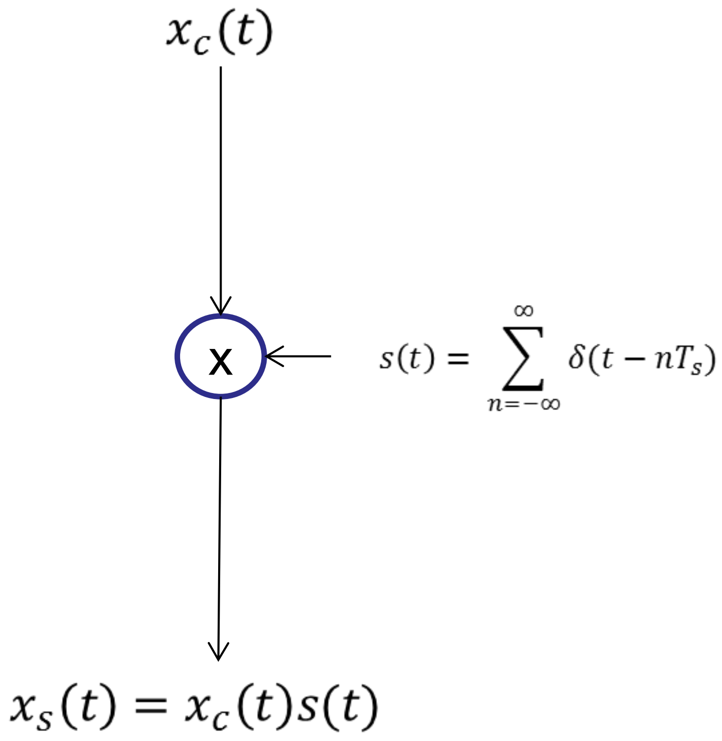
- A sampling model: input is continuous signal and output is a sequence of discrete time samples

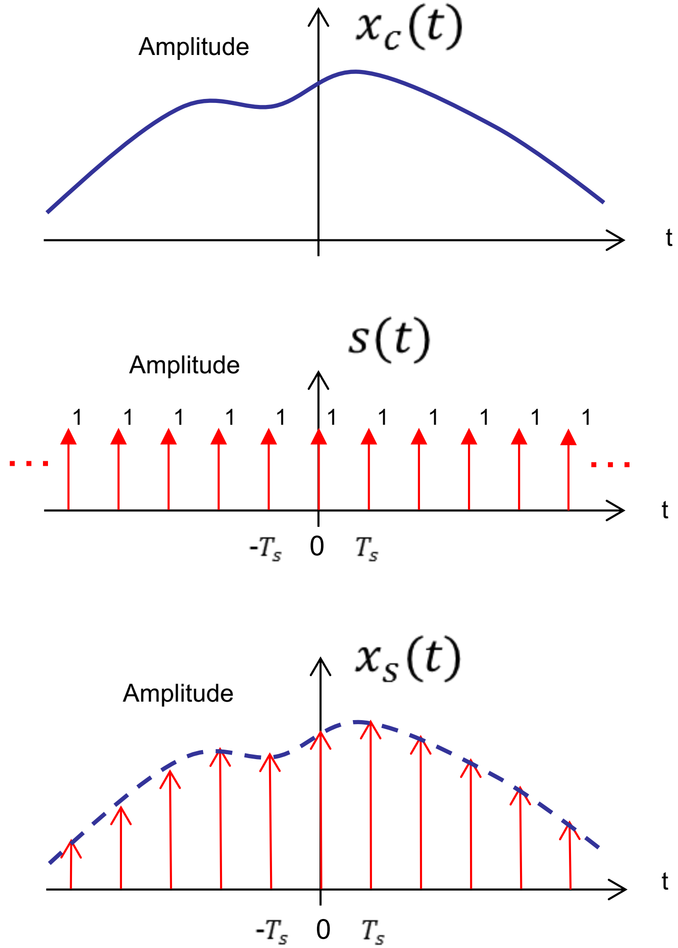


Recap: What is $\delta(t)$?

C. Sampling in frequency domain

- Multiply in time between input and impulse train period T_s

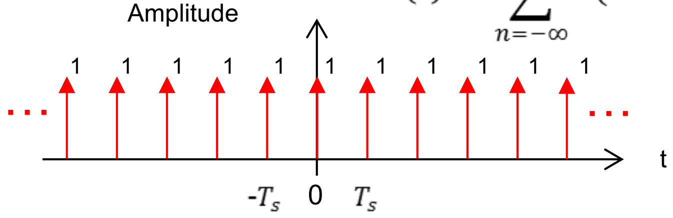




C. Sampling impulse train

- Periodic Impulse Signal

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



- Recap:
  - Fourier transform $F(\Omega)$ of real function $f(t)$:

$$F(\Omega) = \int_{-\infty}^{\infty} f(x) e^{-j\Omega t} dt$$

(or $F(j\Omega)$)

  - Fourier series of a periodic signal

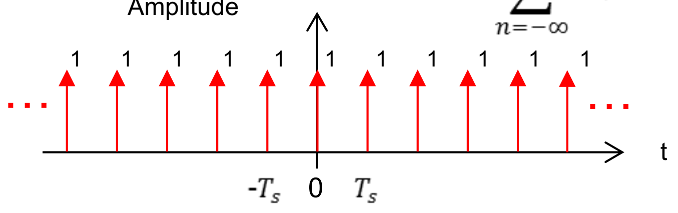
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_s t}$$

which $c_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} f(t) e^{-jk\Omega_s t}$ and $\Omega_s = 2\pi / T_s$

C. Sampling pulse train

- Fourier Transform of a periodic impulse train

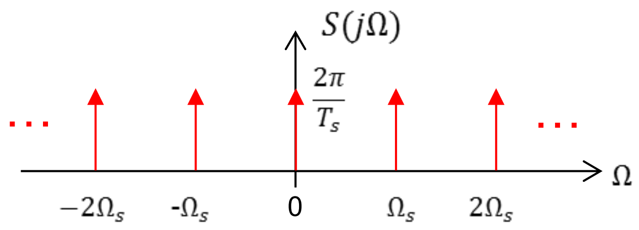
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



Fourier Transform of a periodic impulse train is a periodic impulse train

$$S(j\Omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{2\pi}{T_s})$$

(can be proved)

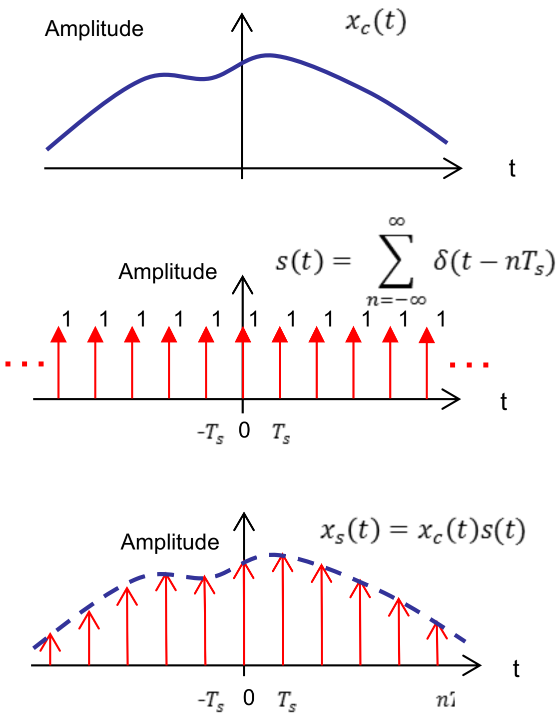


$$\Omega_s(rad/\sec) = 2\pi/T_s$$

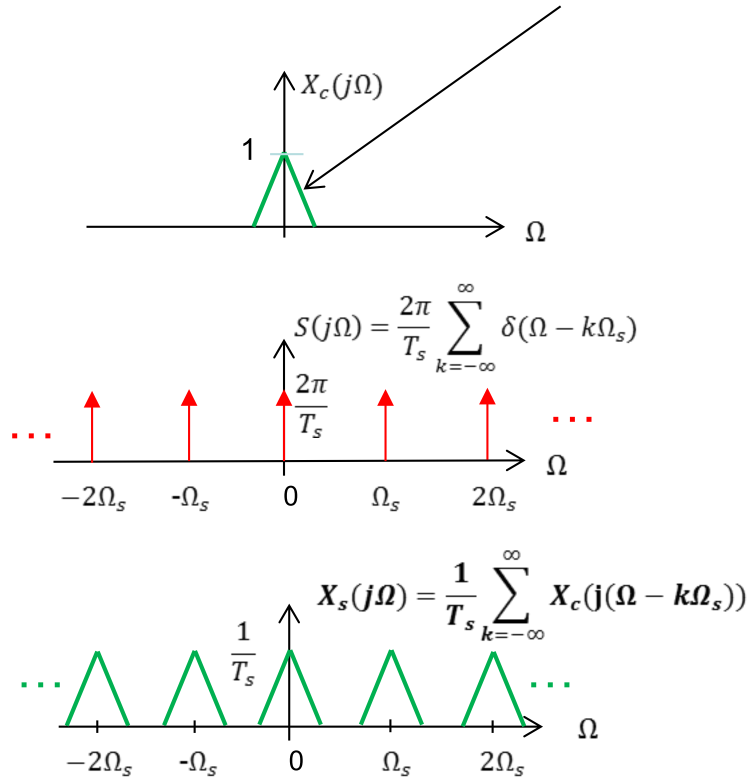
C. Sampled signal in Frequency domain

- Fourier Transform of sampled signal

In a case where we do not know the spectrum, we can specify as this in general.



Multiply in time



Convolution in frequency

Using $\boxed{f_1(t)f_2(t) \overset{Fourier}{\Longleftrightarrow} \frac{1}{2\pi} F_1(\omega) * F_2(\omega)}$

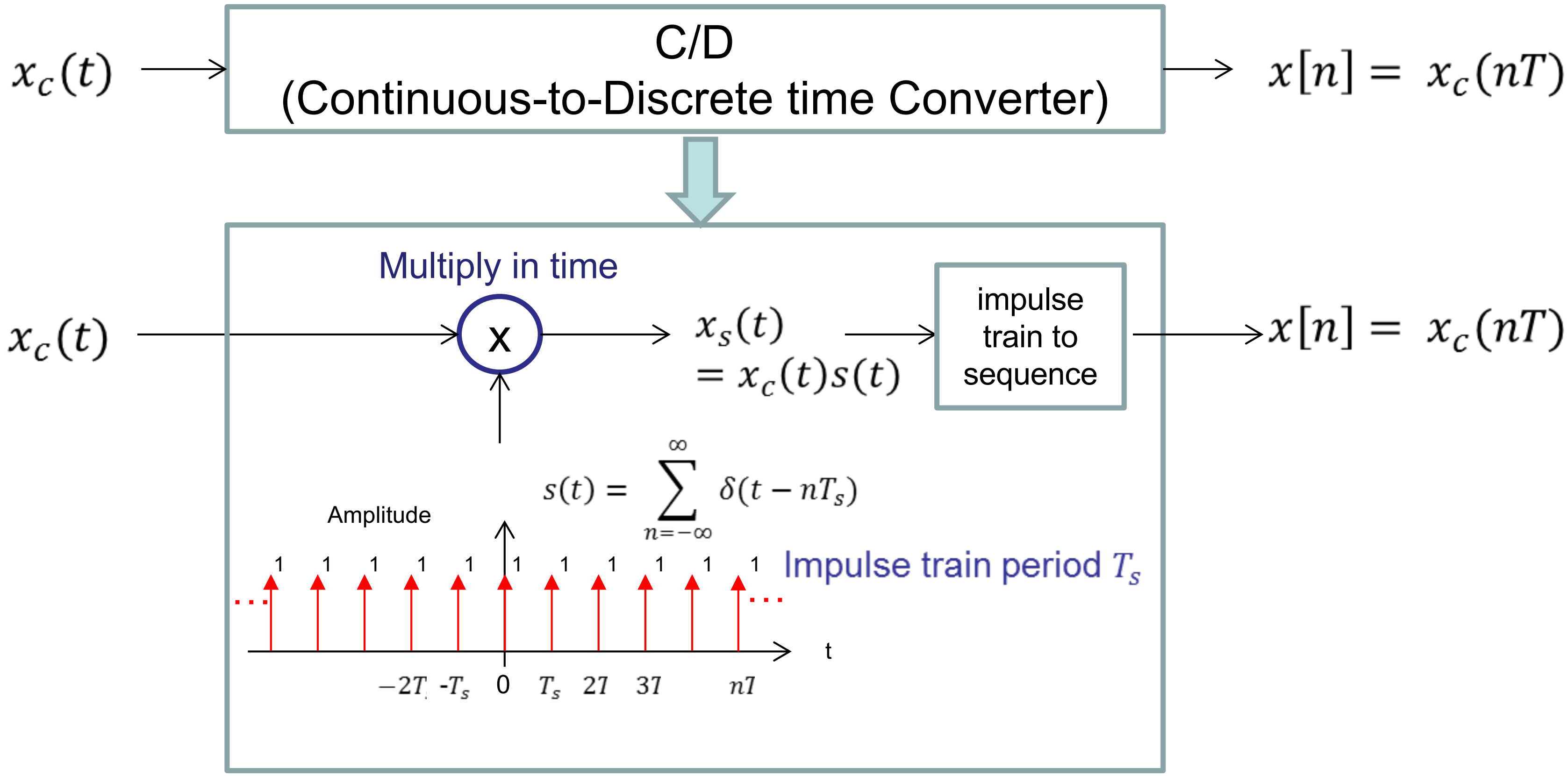
Show that $X_s(j\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$ in the previous slide.

TABLE 5.1 Fourier Transform Properties

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

C. Sampled impulse train to discrete-time sequence

- Recap: C/D Block



C. Sampled impulse train to discrete-time sequence

- Example:

  Given the continuous time signal $x_a(t)$ is sampled at $F_s = 1/T_s$

$$x_a(t) = A\sin(2\pi F\, t) = A\sin(\Omega t)$$

Sampling sequence:

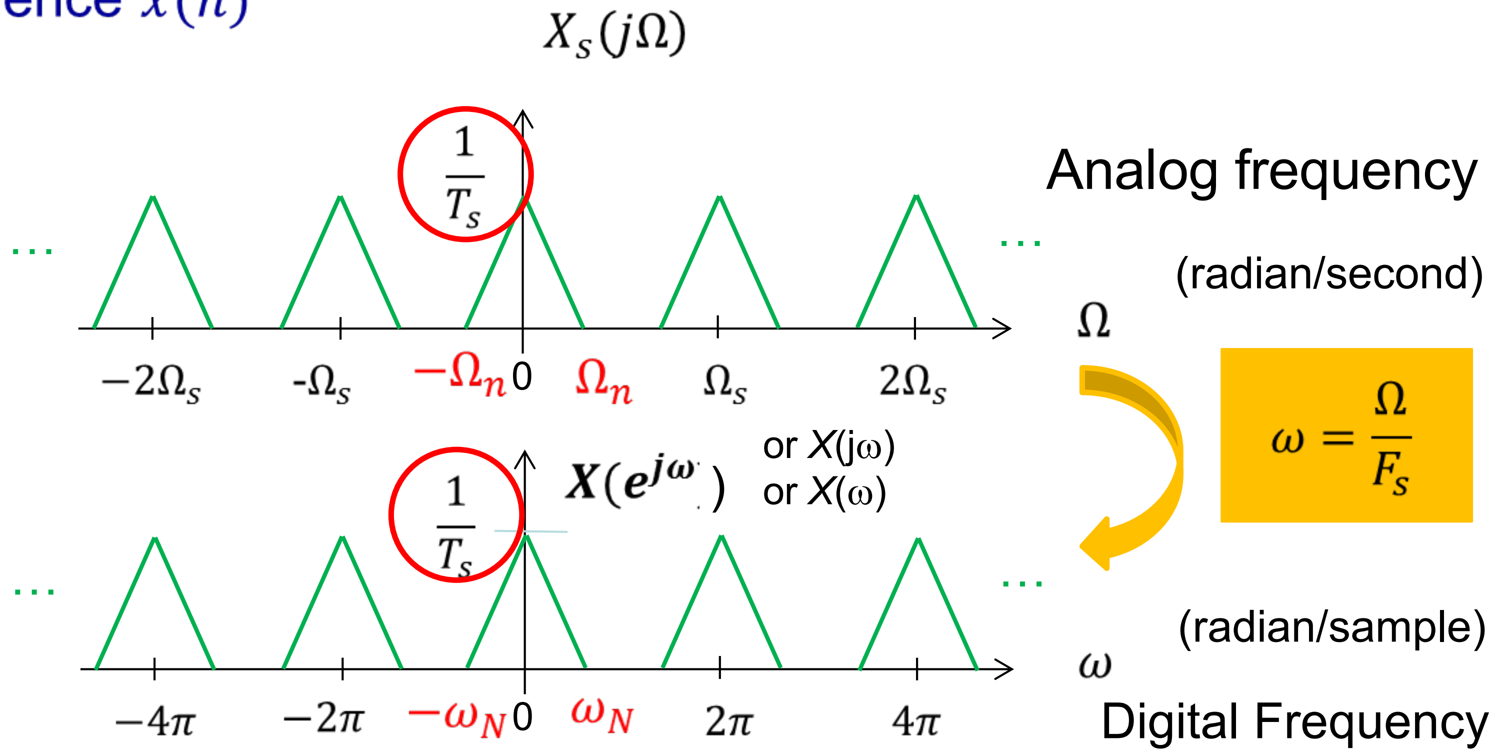
$$\begin{aligned} x[n] &= A\sin(2\pi\, F\, nT_s) \\ &= A\sin\left(2\pi \frac{F}{F_s}\, n\right) \\ &= A\sin\left(\frac{\Omega}{F_s}\, n\right) \\ &= A\sin(\omega\, n) \end{aligned}$$

What is the difference between F, Ω and ω?

Description	Notation	Unit
Continuous signal	$x_a(t)$	
Sampled signal	$x_a(nT)$	
Discrete-time signal	***x[n] or x(n)***	
Analog frequency	F	Hz
Analog frequency	$\Omega = 2\pi F$	rad/sec
Digital Frequency	$\omega = {}^{2\pi F}/_{F_s}$	rad/sample
FT of $x_a(t)$	$X_a(\Omega)$ or $X_a(j\Omega)$	
DTFT of $x(n)$	$X(e^{j\omega})$ or $X(j\omega)$ or $X(\omega)$	

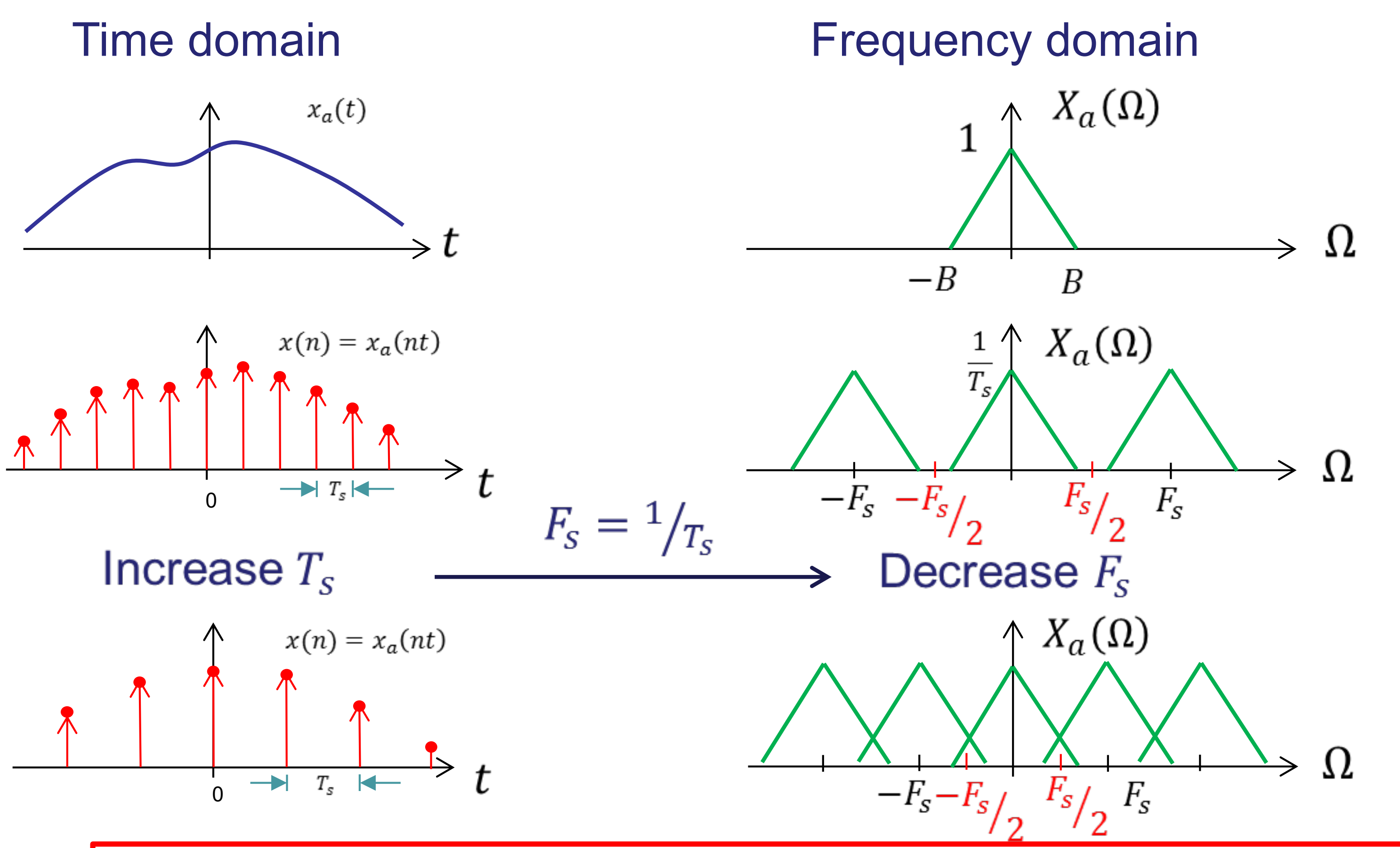
C. Sampled impulse train to discrete-time sequence

- Spectrum of sampled impulse train $x_s(t)$ and discrete-time sequence $x(n)$

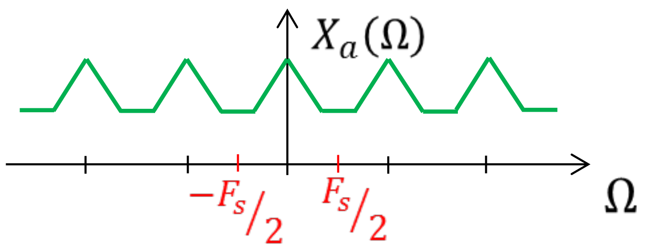


Note: the change in x-axis from Ω to ω can be considered as a change signal x(t) from x(t) to x(at) where a == Fs. The y-axis scale remains the SAME.

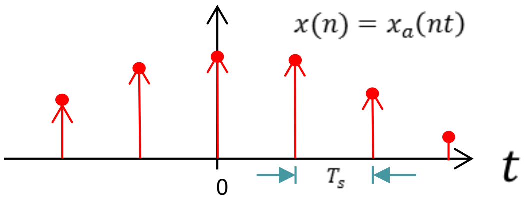
C. Effect of sampling rate on discrete-time signal

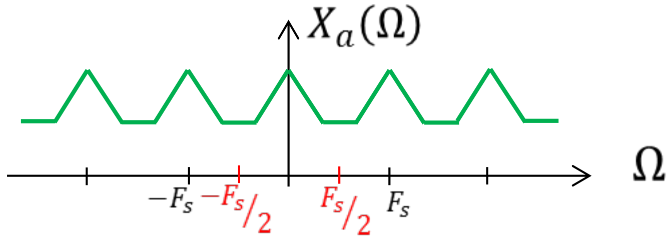


If $F_s/2$ < B, the replicated spectrum overlaps and alias occur

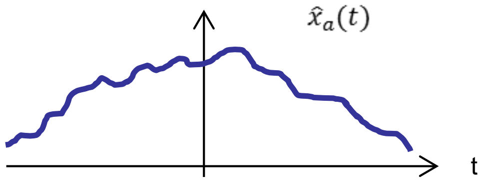


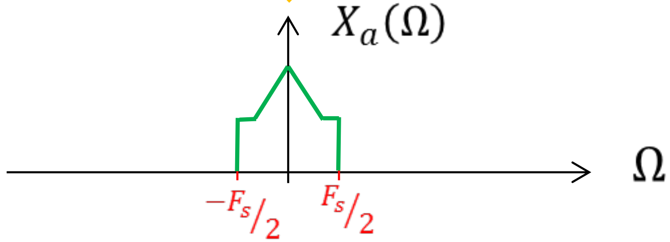
C. Sampling Frequency Effect on Reconstruction



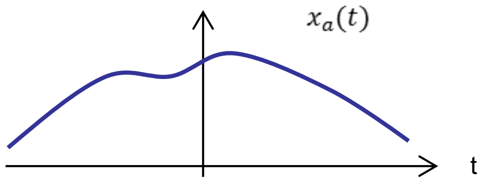


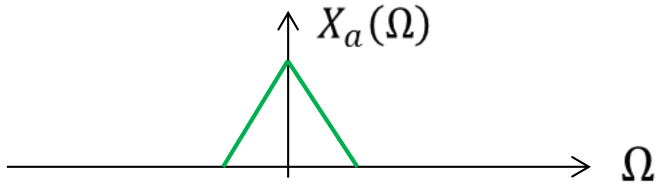
Filter away replicated frequencies





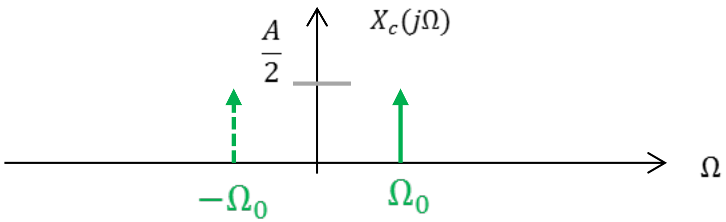
Compare with original signal (in time & frequency domain)





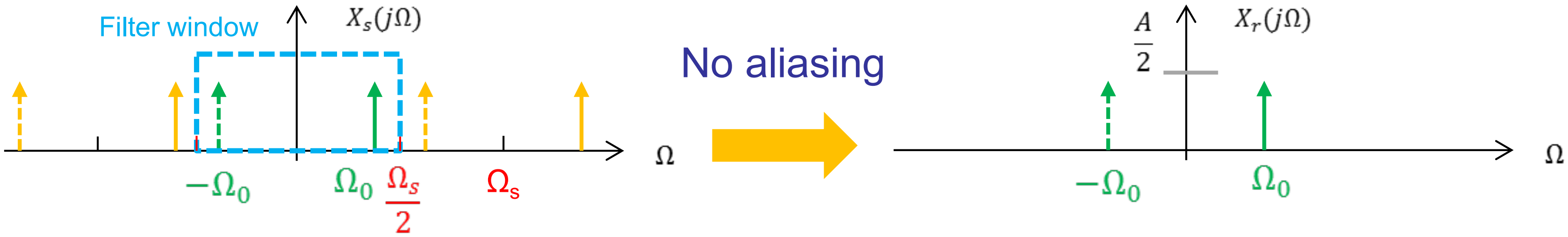
Observation: Sampling at an insufficient rate precludes perfect reconstruction and introduces aliasing

C. Example: A single sine wave example of without & with aliasing

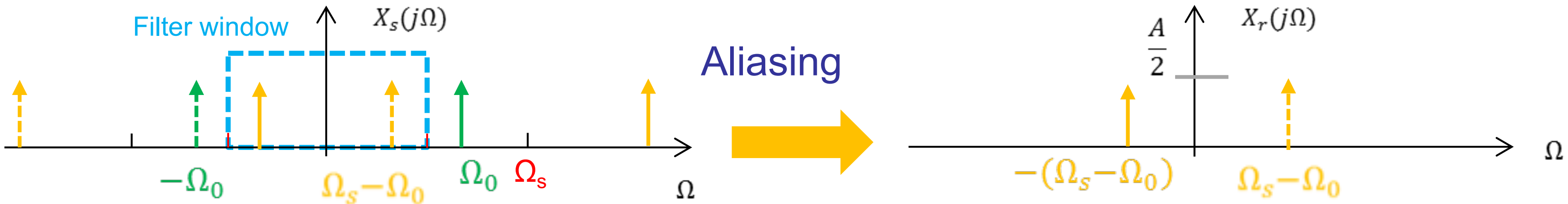


Given a sine wave with amplitude A at frequency Ω_0, Its frequency representation $X_c(j\Omega)$ is given on the left.

a) If $\Omega_0 < \Omega_s/2$

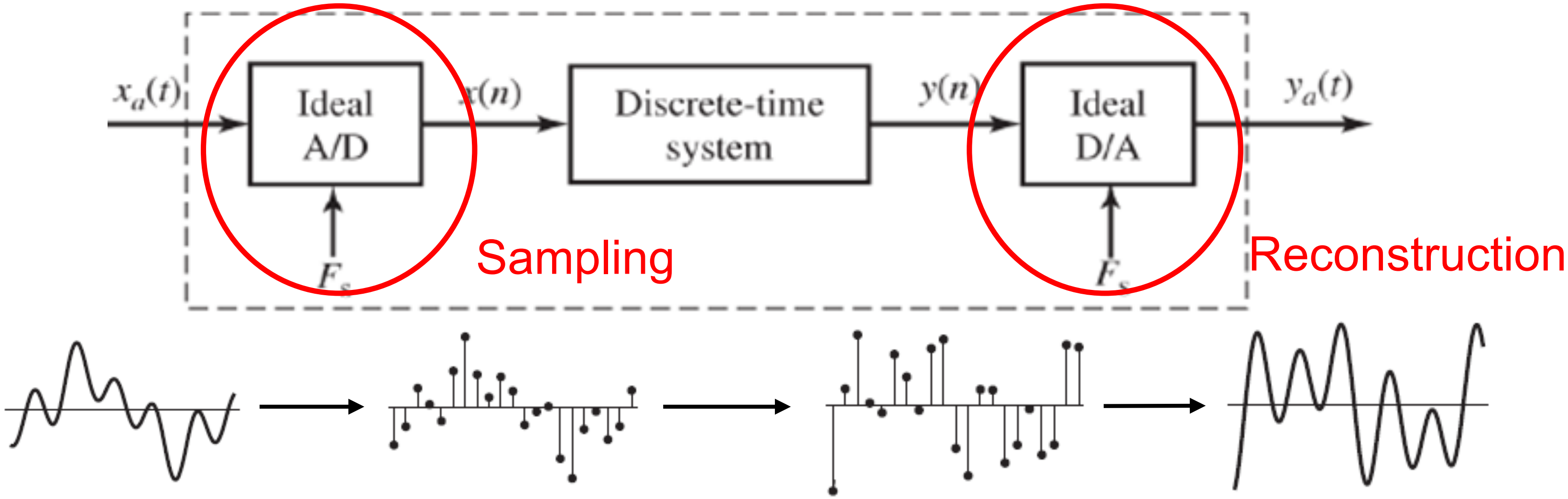


a) If $\Omega_0 > \Omega_s/2$

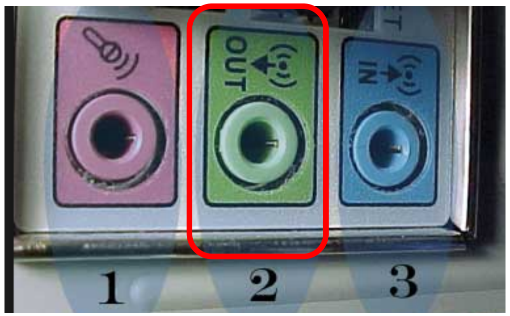


D. Reconstruction

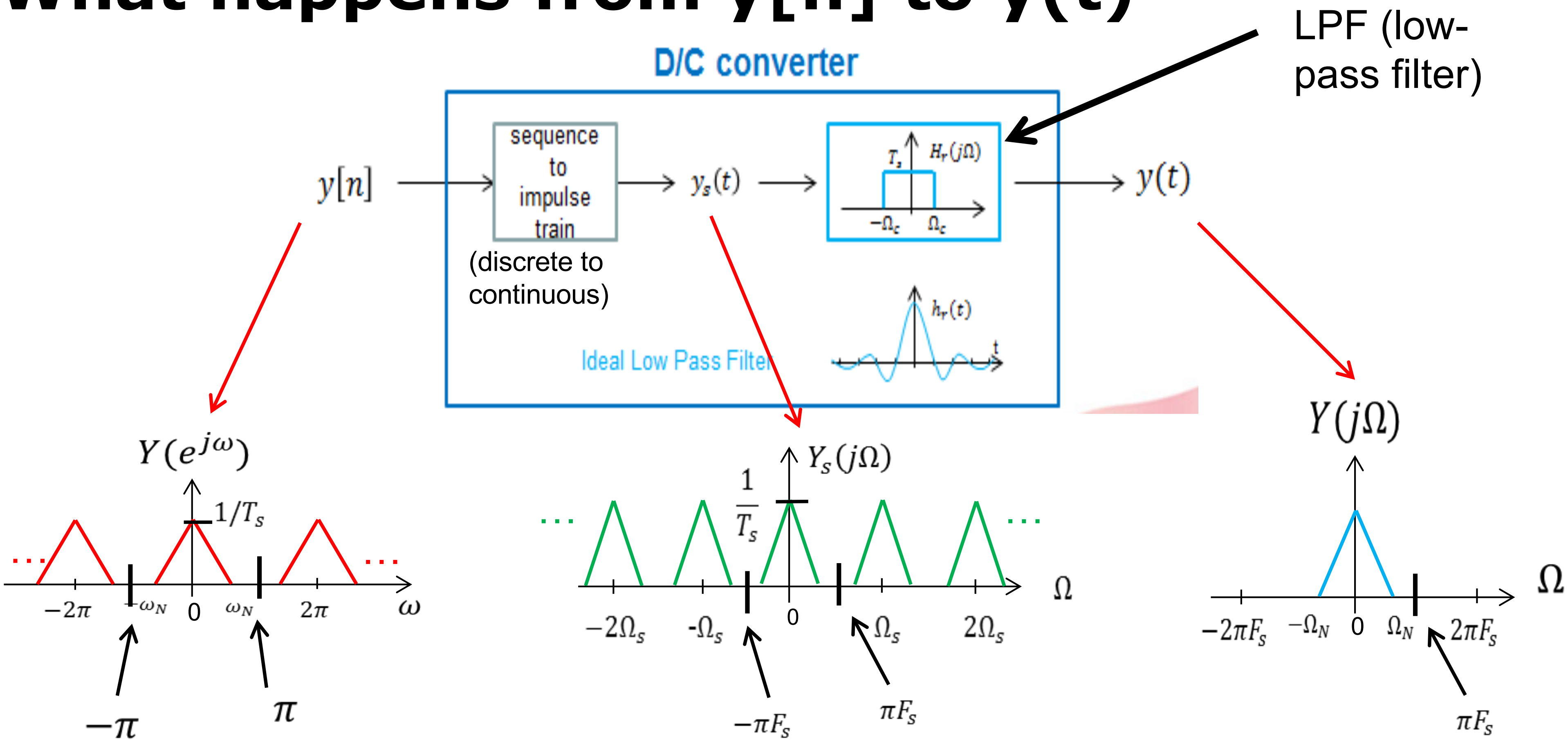
- Recap: a DSP system



- Reconstruction: given a discrete signal y[n], we wish to get back continuous time signal y(t)
  - E.g, play mp3 files into the speaker
  - One of functions of the sound card

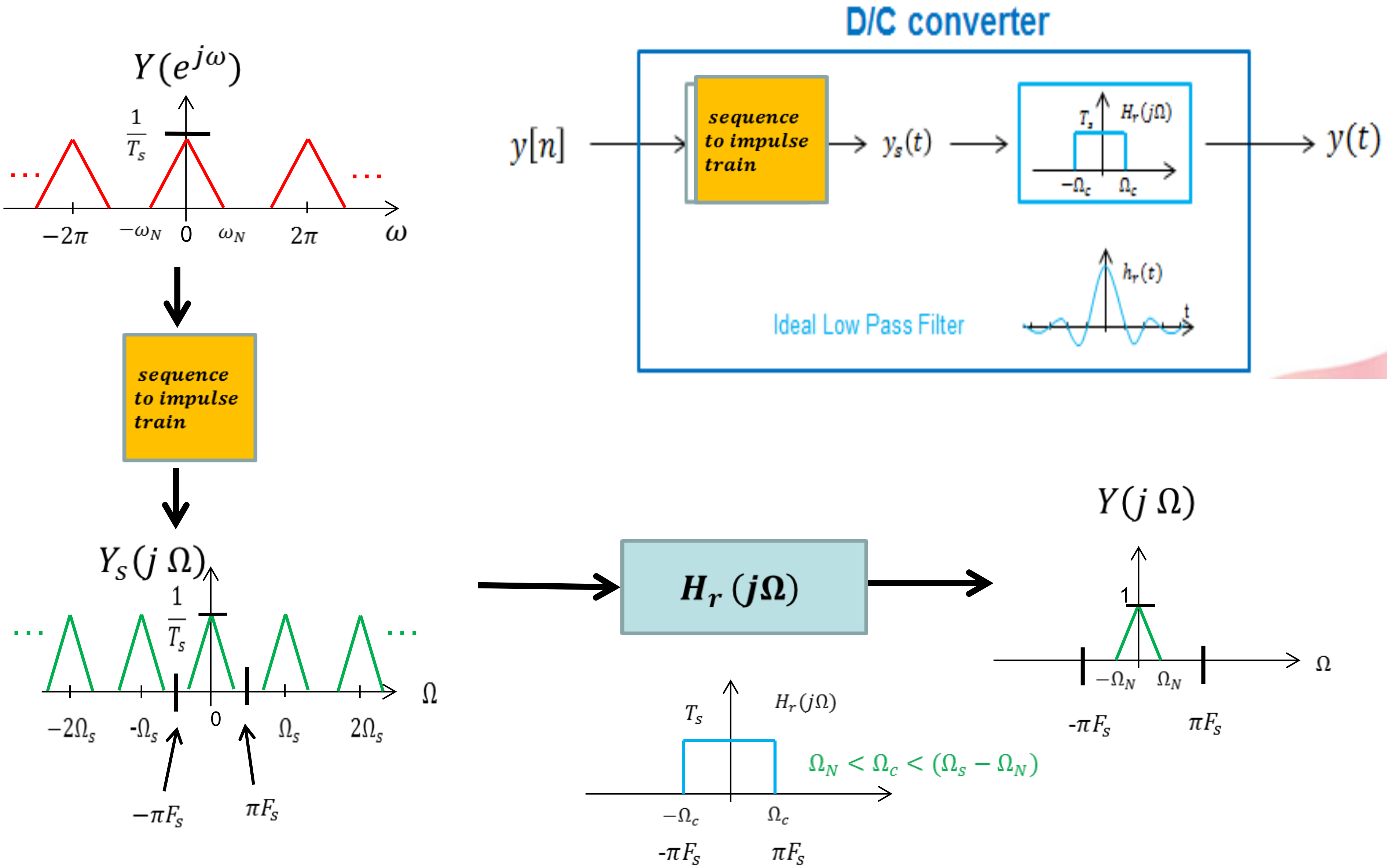


What happens from y[n] to y(t)



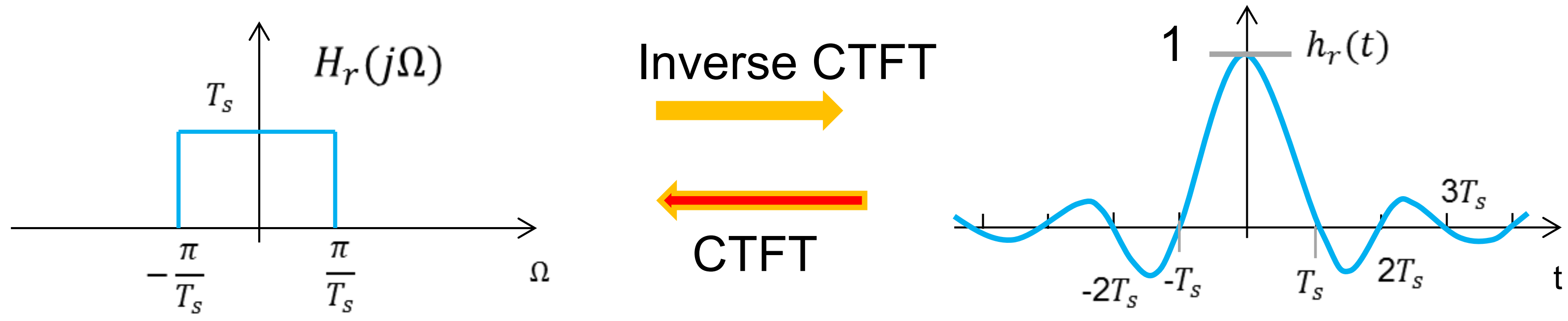
Assuming the D/C is a perfect re-constructor, i.e., it has an ideal LPF with cut-off frequency $\Omega = \pi F_s$ (1/2 sampling frequency in radian/sec), then the LPF basically removes all replicas of $Y(j\Omega)$ above $\Omega = \pi F_s$. In the time domain, the LPF interpolates to smooth the discrete sequence into a continuous sequence.

D. Ideal Reconstruction from a continuous time sampled sequence



D. Ideal reconstruction of a band-limited signal from its samples

- Ideal low pass filter



Frequency domain

Time domain

Note: the x-axis above is continuous time.

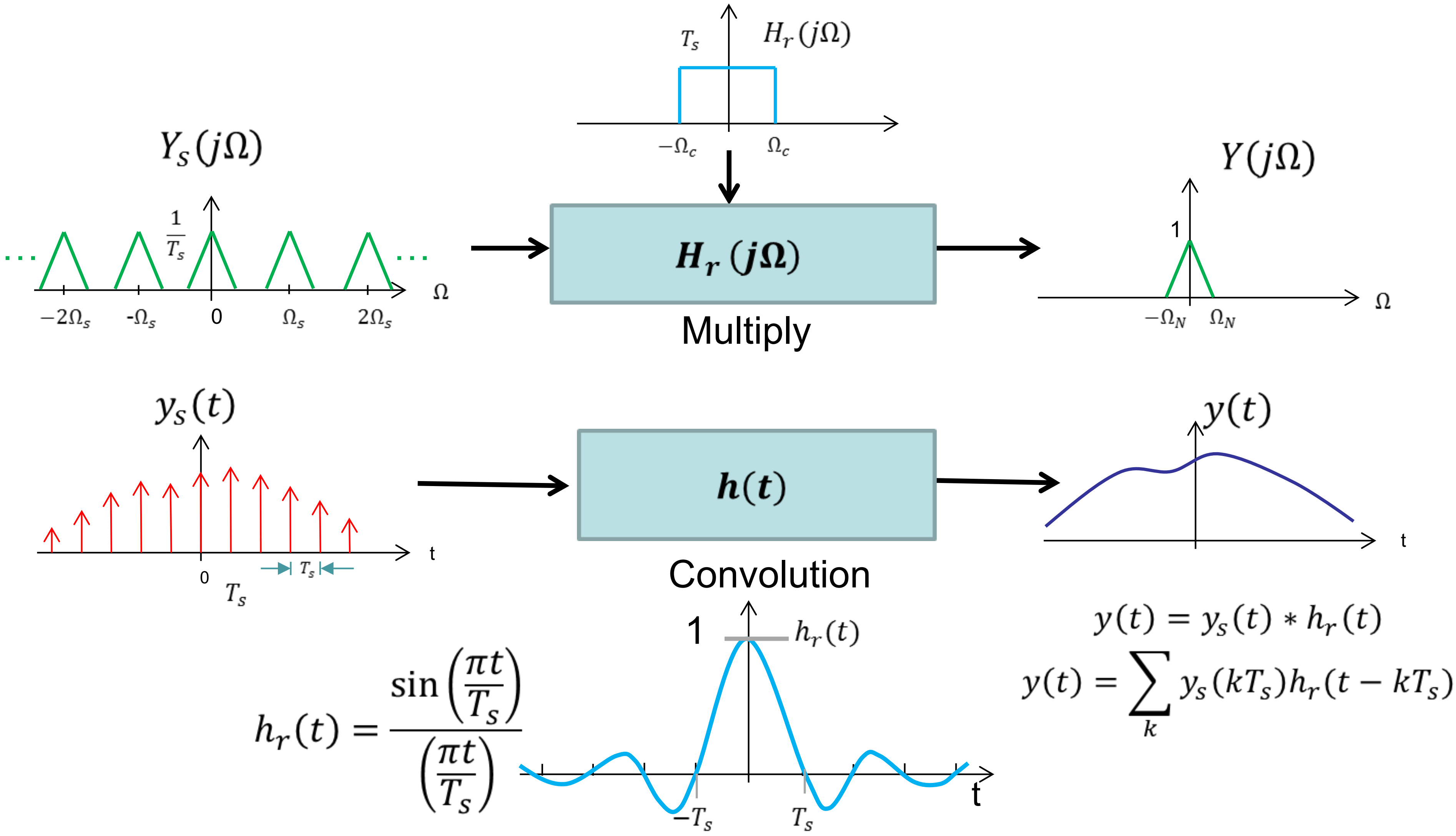
Ideal Low pass filter impulse response

Note: you should be able to derive these FT pair.

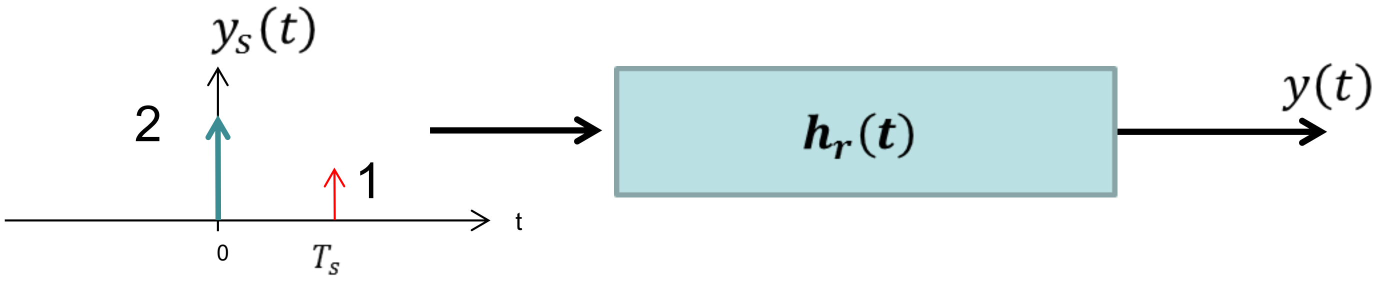
$$h_r(t) = \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\left(\frac{\pi t}{T_s}\right)}$$

D. More on the ideal reconstruction of a band-limited signal from its samples

- Multiply in frequency domain → Convolution in time



D. Let’s examine the reconstruction filter in some detail (2 samples in the input of D/C)

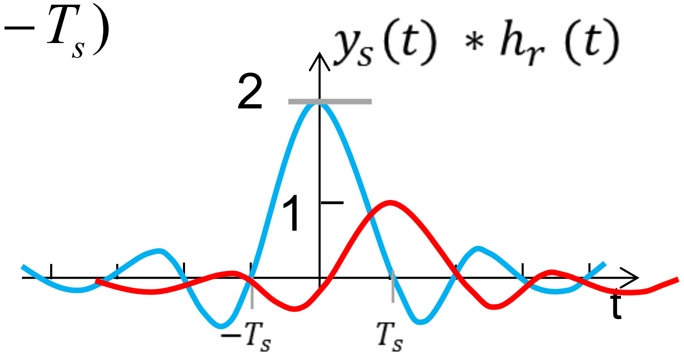


$$x(t) = 2\delta(t) + \delta(t - T_s)$$

$$y(t) = y_s(t) * h_r(t)$$

$$y(t) = \sum_k y_s(kT_s) h_r(t - kT_s)$$

$$y(t) = \sum_k y_s(kT_s) \frac{\sin\left(\frac{\pi(t - kT_s)}{T_s}\right)}{\left(\frac{\pi(t - kT_s)}{T_s}\right)}$$

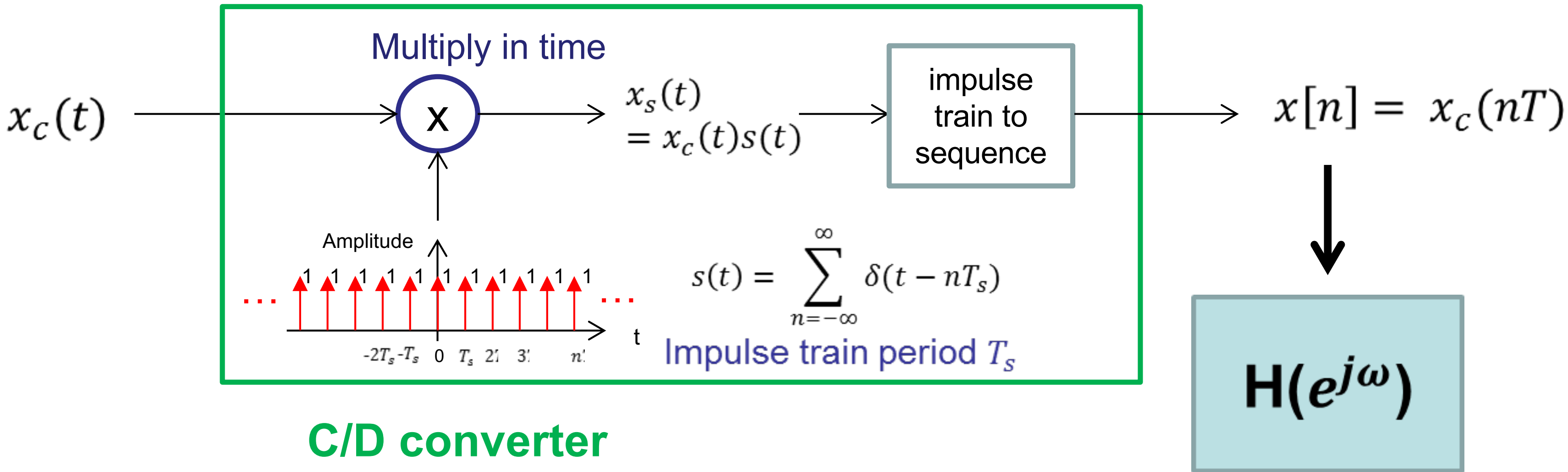


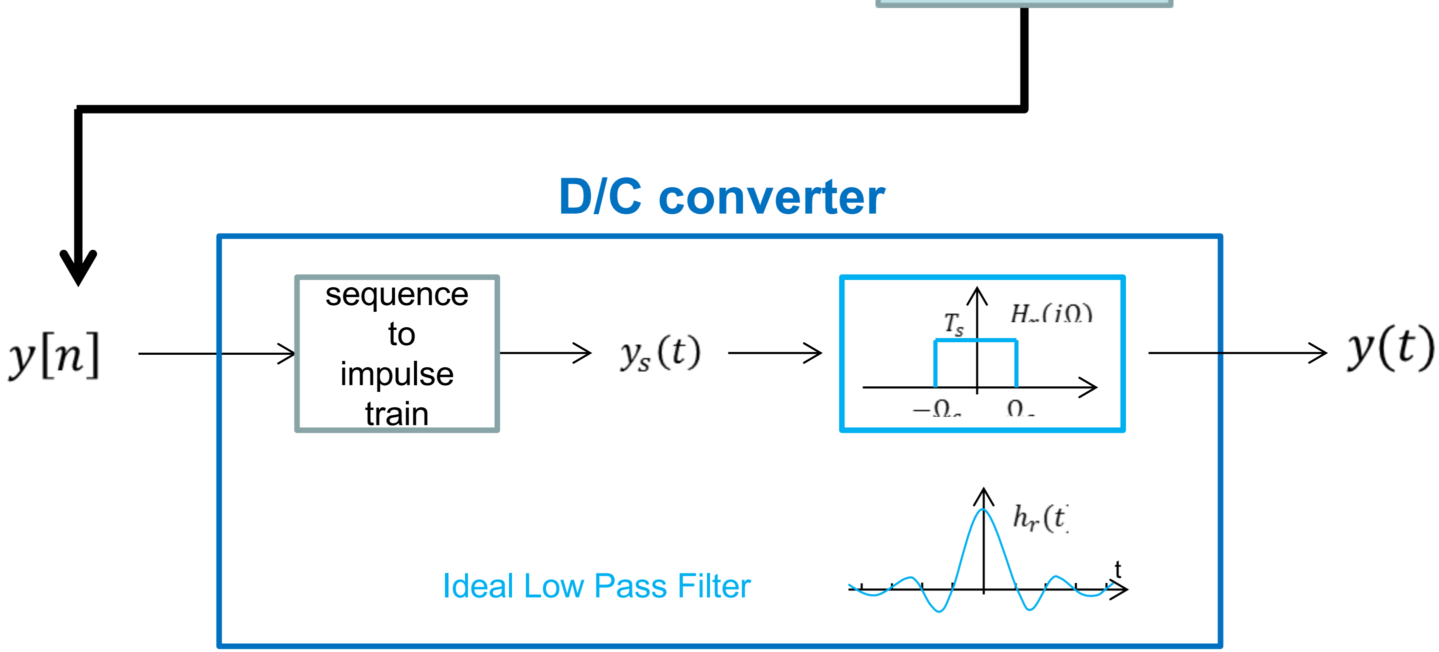
The above sketch shows (blue) response due to $2\delta(0)$ and (red) due to $1\ \delta(t - T_s)$

The red response is the sinc function centered at $(1\ T_s)$

$$h_r(t - 1T_s) = \frac{\sin\left(\frac{\pi(t - 1T_s)}{T_s}\right)}{\left(\frac{\pi(t - 1T_s)}{T_s}\right)}$$

E. Recap C/D and D/C blocks





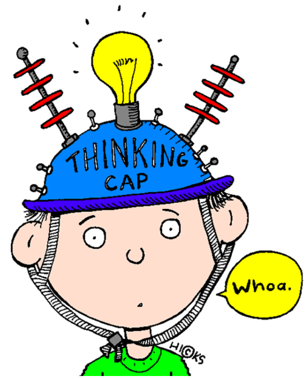
Notations

Time	Freq
$x_c(t)$	$X_c(j\Omega)$
$x_s(t)$	$X_s(j\Omega)$
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
$y_s(t)$	$Y_s(j\Omega)$
$y(t)$	$Y(j\Omega)$

Summary so far…

- We have studied Sampling Theorem
  - If we can sample at least twice the sampling frequency, then we may reconstruct the original signal (ignoring quantization at the moment)
- We showed why we need $F_S > 2*F_{max}$ in frequency domain. The key is to understand that sampling causes convolution of original signal spectrum with impulse train spectrum.
- To reconstruct, we pass it through an ideal LPF to remove the repeated images at the higher frequencies (multiple of F_s)

Helpful thinking…



- Sampling — to discrete domain

  By-product: repeated images at the higher frequencies (multiple of F_s)
- Reconstruction — to remove the repeated images at the higher frequencies; also to continuous domain