DSP: Part II

Lecture.4.1

Infinite Impulse Response (IIR)
(Filter Design)

Methodology for IIR

1) What is the characteristics of an IIR Filter
   - Its impulse response, its difference equation, its H(z), and stability

2) Analysis: From difference equation to H(z), to pole/zero plot and then to frequency response H(ω)

3) IIR Filters are usually designed from analog filters, so lets examine H(s) the Laplace domain.

4) Design an analog filter
   a) Simple RC analog filter
   b) Butterworth analog filter
5) Design an IIR filter by converting from H(s) to H(z)--Bilinear transform method
6) Designing of IIR Filters using Matlab.

3) Designing IIR Filters H(z) from Analog Filter Design H(s)

- An IIR filter can achieve a much sharper transition region than an FIR filter of the same order.
- H(z): designed from an analog filter, from which we can get its H(s) (Laplace transform of the impulse response in continuous time).
- We then transform H(s) to H(z): bilinear transformation.
  - By such an approach, only magnitude response is the focus, no control of phase response.
  - Advance techniques which take care of both magnitude and phase response is not covered in this course.

IIR Digital filter design

Design analog lowpass filter → Apply freq. band transformation $s \rightarrow s$ → Apply filter transformation $s \rightarrow z$ → Desired IIR filter

Idea: the initial design is ALWAYS a lowpass filter and its design can then be transformed to a high-pass filter, band-pass filter or band-stop filter in frequency band transformation.

3) What is Laplace transform

1) Laplace Transform, like continuous time Fourier Transform, operates on a continuous time signal to transform it into the frequency domain.

2) Laplace Transform is a 'super' Fourier transform:

Fourier transform Pair

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt,$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

Laplace transform

$$X(s) = \int_{0}^{\infty} x(t)e^{-\sigma t}e^{-j\Omega t}dt = \int_{0}^{\infty} x(t)e^{-(\sigma+j\Omega)t}dt$$

$$= \int_{0}^{\infty} x(t)e^{-st}dt,$$

$$s = \sigma + j\Omega$$

If σ is equals to zero, then X(s) = X(Ω)!

Laplace Transform and its relationship to Z-transform

- The Laplace transform is used to analyze continuous time system. Its discrete-time counterpart is the z-transform

$$X_d(z) \triangleq \sum_{n=0}^{\infty} x_d(nT) z^{-n}$$

- If we define $z = e^{sT}$, the z transform → the Laplace transform of a sampled continuous-time signal:

$$X_d(e^{sT}) \triangleq \sum_{n=0}^{\infty} x_d(nT) e^{-snT}$$

- As the sampling interval T goes to zero, we have

$$\lim_{T \to 0} X_d(e^{sT}) T = \lim_{\Delta t \to 0} \sum_{n=0}^{\infty} \left[\frac{x_d(t_n)}{\Delta t}\right] e^{-st_n} \Delta t$$

$$= \int_0^{\infty} x_d(t) e^{-st} dt \triangleq X(s)$$

where $t_n \triangleq nT$ and $\Delta t \triangleq t_{n+1} - t_n = T$

- In summary

the z transform (times the sampling interval T)of a discrete time signal $x_d(nT)$ approaches, as $T \to 0$, the Laplace Transform of the underlying continuous-time signal $x_d(t)$.

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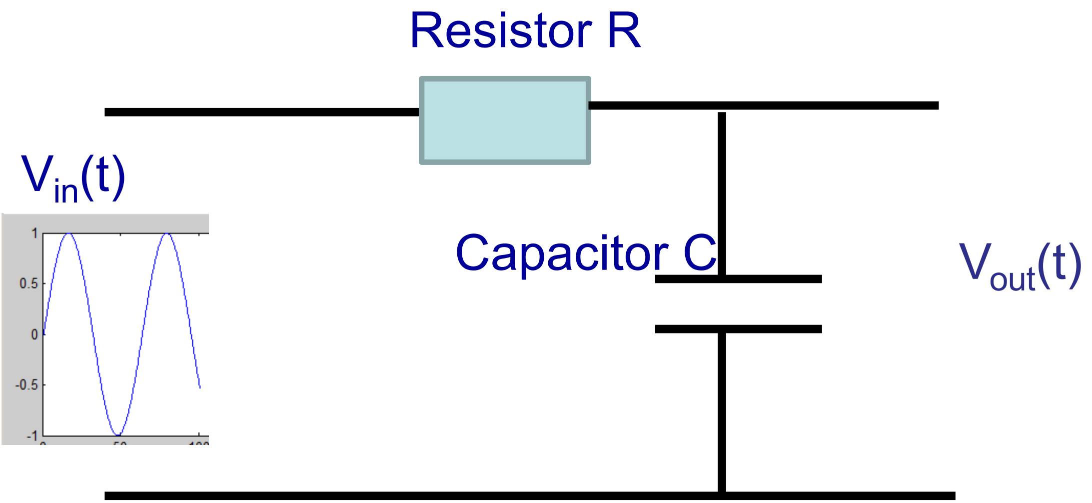
A simple RC analog filter



A simple RC circuit.
Is this a low-pass? Or a high-pass filter?

Recall-- CE2004 – CIRCUITS AND SIGNAL ANALYSIS

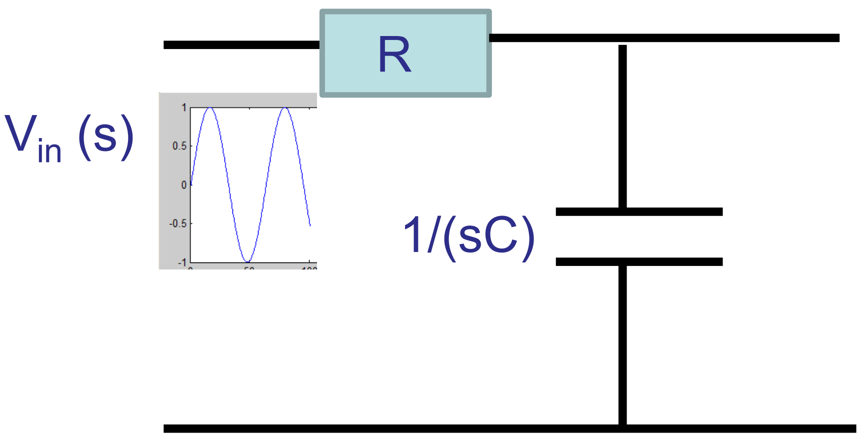
- As we vary the input continuous time signal's frequency, the output signal will undergo amplitude change as well as phase shift.
- It does not introduce new signal, only modifying the input signal's amplitude and phase.
  - The resistor's impedance is un-affected by frequency.
  - The capacitor's impedance is affected by frequency.

How do we analyse this filter?

Device	Time domain	s-domain	Impedance	Impedance wrt frequency increase
Resistor	$V(t) = R\,I(t)$	$V(s) = R\,I(s)$	$Z = R$	No Change
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$V(s) = \frac{1}{C}\frac{I(s)}{s}$	$Z = \frac{1}{sC}$	Impedance reduces
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$V(s) = Ls\,I(s)$	$Z = Ls$	Impedance increases

$$S = \sigma + j\,\Omega$$

As frequency increases, the capacitor becomes a short circuit,
And the inductor becomes an open circuit.



$V_{out}(s) = \frac{1/(sC)}{R+1/(sC)} V_{in}(s)$

This is therefore a lowpass filter!

How do we analyse this filter?

$$V_{out}(s) = \frac{1/(sC)}{R+1/(sC)} V_{in}(s)$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1/(sC)}{R+1/(sC)}$$

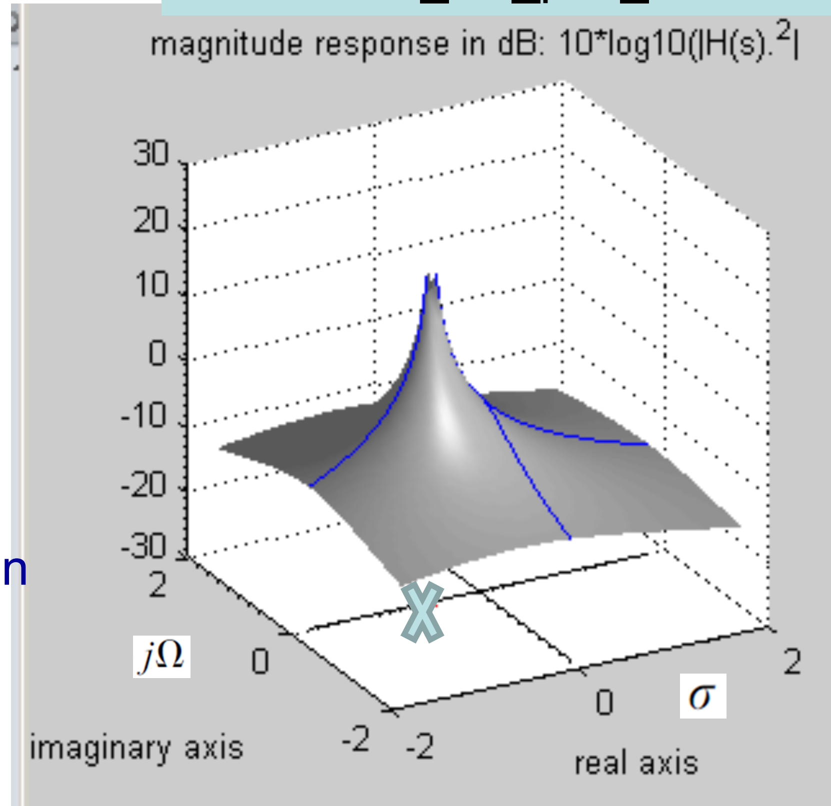
$$= \frac{1}{1+sRC}$$

$$= \frac{1/(RC)}{s+1/(RC)}$$

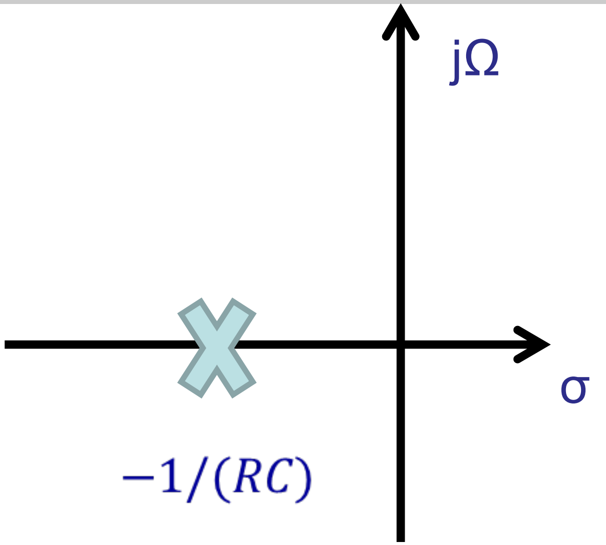
H(s) is a surface plot on the complex plane s

$$S = \sigma + j\,\Omega$$

Matlab: Hs_3D_plot_ex1.m



Therefore in the s-domain, we see that the circuit has a pole when
s = - 1/(RC)



Example: RC = 2, => 1/(RC) = 0.5

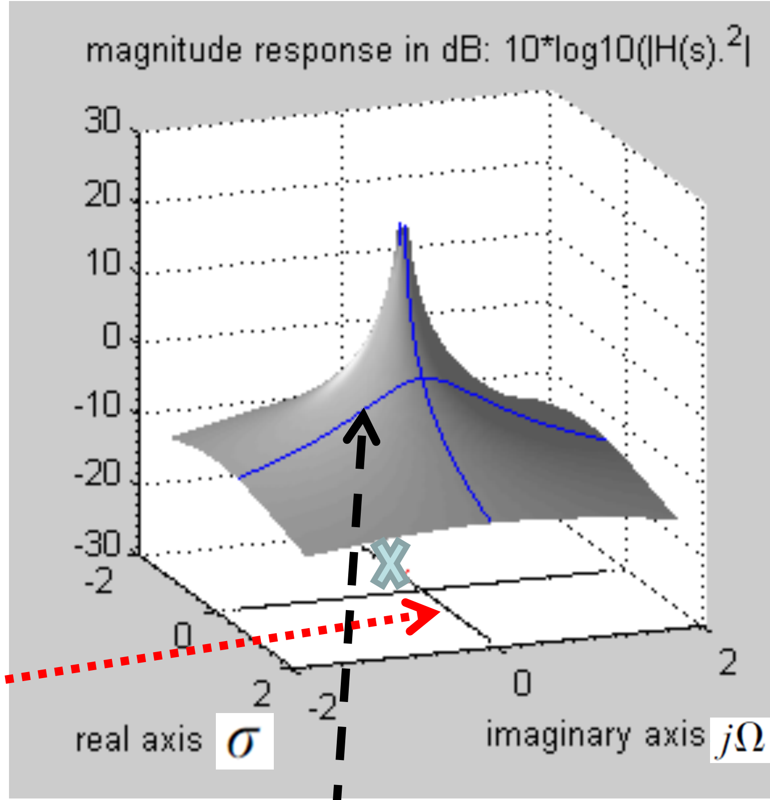
How do we 'see' that it is a low-pass filter?

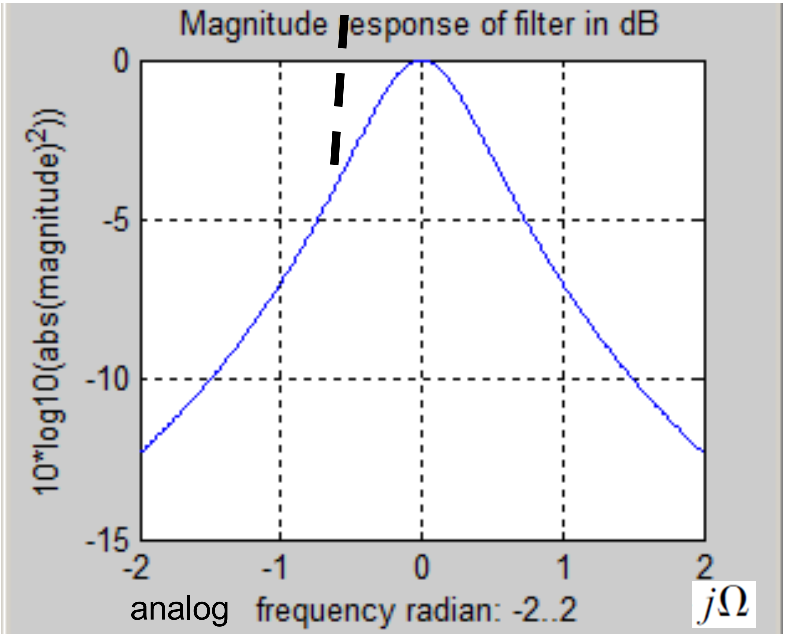
H(s)

$s = \sigma + j\,\Omega$

Matlab: Hs_3D_plot_ex1.m

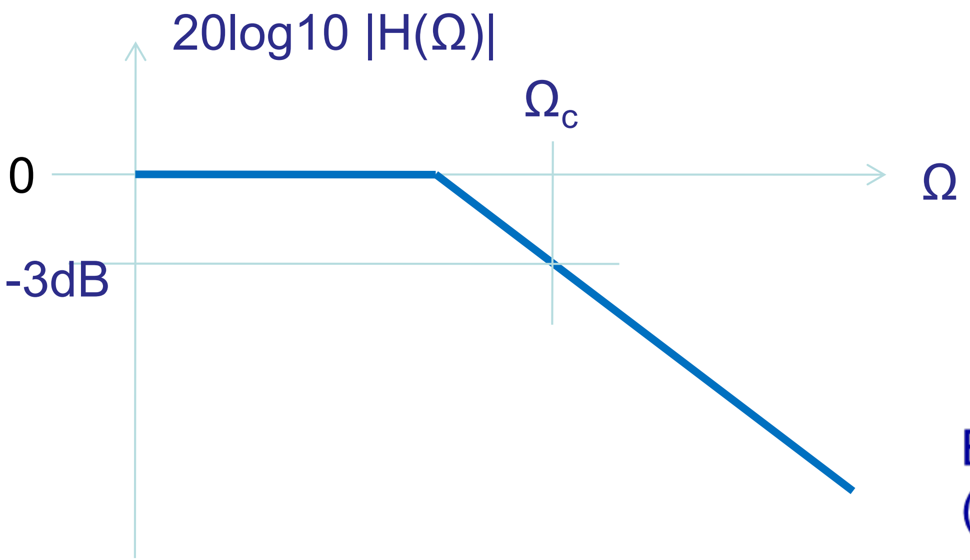
When σ = 0, the 's' depends only on jΩ (the unit circle), the frequency. In other words, examine the value of H(s) for σ = 0, the y-axis. The frequency response: a low pass characteristic.





H(s) = H(σ+jΩ); if σ = 0, then H(j Ω) the continuous time Fourier Transform!

Single pole analysis: Magnitude and Phase Response



$$V_{out} / V_{in} = \frac{1}{RC}\left(\frac{1}{s+\frac{1}{RC}}\right)$$

$s = \sigma + j\Omega$

Evaluate H(s) for $s = j\Omega$
(evaluate H(s) only along the y-axis) .

$$|H(\Omega)| = \left|\frac{1}{1+j\Omega RC}\right|$$

$$\angle H(\Omega) = tan^{-1}(-\Omega RC)$$

Important points: $\Omega = 0$, Ω_c, and ∞

when $\Omega = 0$, $|H(\Omega)| = 1$, 20log10(1) = 0dB
when $\Omega = 1/RC$, $|H(\Omega)| = 0.707$, 20*log10(0.707) = -3dB
And when $\Omega = 10/RC$, 20*log10|H(Ω)| approx -20dB

Bode Plot (to represent the gain and phase of a system as a function of frequency) **with Matlab commands**

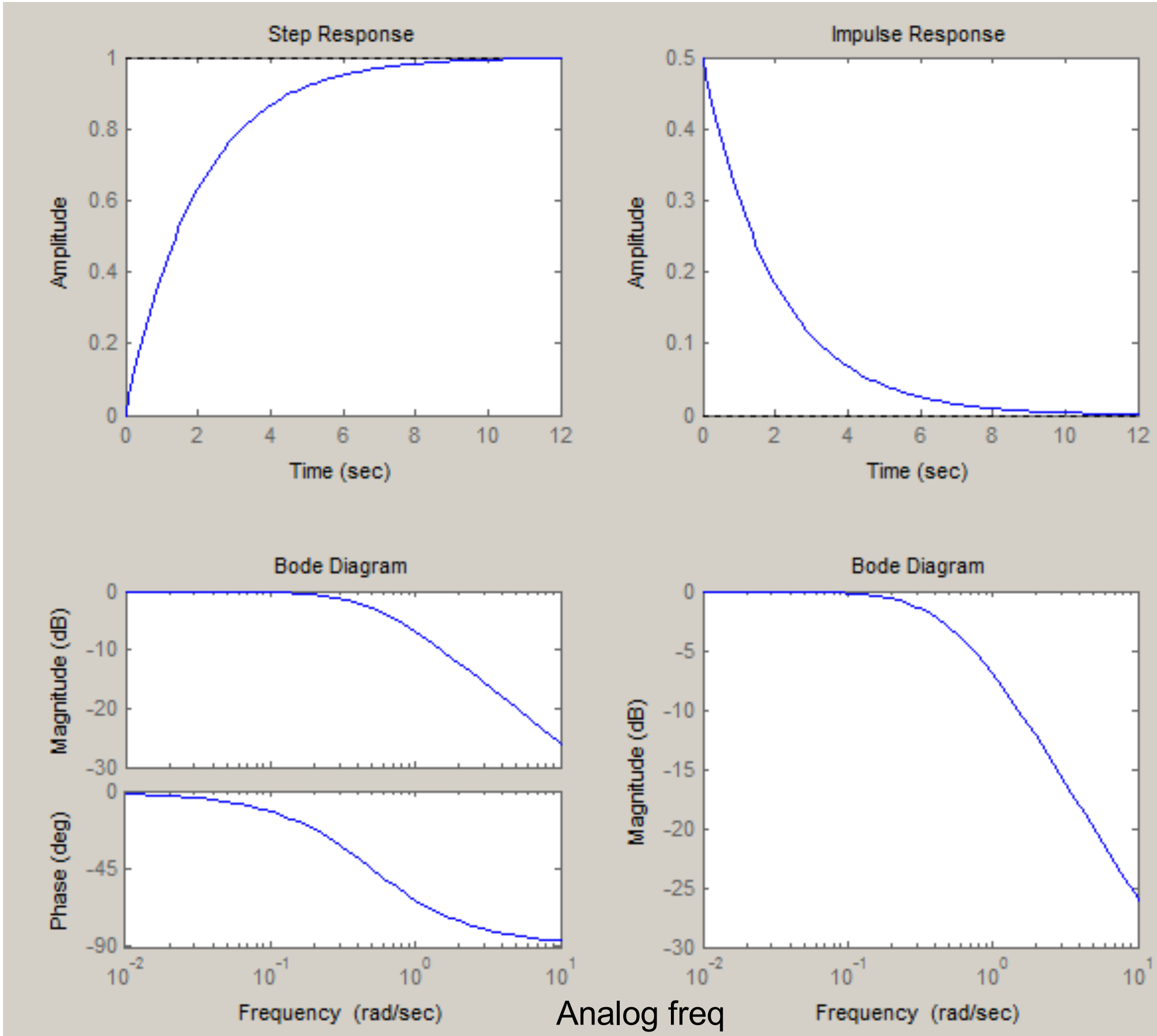
```
px = 0.5;
%(a pole at real axis = -0.5)

tfx = tf([px],[1 px]);

%alternatively
%freqs([1],[1 px]);
ltiview(tfx)
```

```
>> tfx

Transfer function:
  0.5
-------
s + 0.5
```



tf(*num*,*den*): creates a continuous-time transfer function with numerator(s) and denominator(s) specified by num and den.

ltiview(sys1,sys2,...,sysn): opens an LTI Viewer containing the step response of the LTI models sys1,sys2,...,sysn.

A summary: what have we learnt and what is the next step?

- We can design a simple RC analog filter (with a single pole in its transfer function) by choosing the appropriate capacitor and resistor value and placing them in a voltage divider configuration.
- The position of the pole affects the shape of H(s) and ultimately affects the shape of H(s) at the y-axis, i.e., H(Ω).

- Attempt the tutorial question in IIR.doc (Q1) to see a different configuration.
- We will learn how to convert of H(s) to H(z) later, but now let's learn how to design more complex analog filters (with multi-poles) using Butterworth design…

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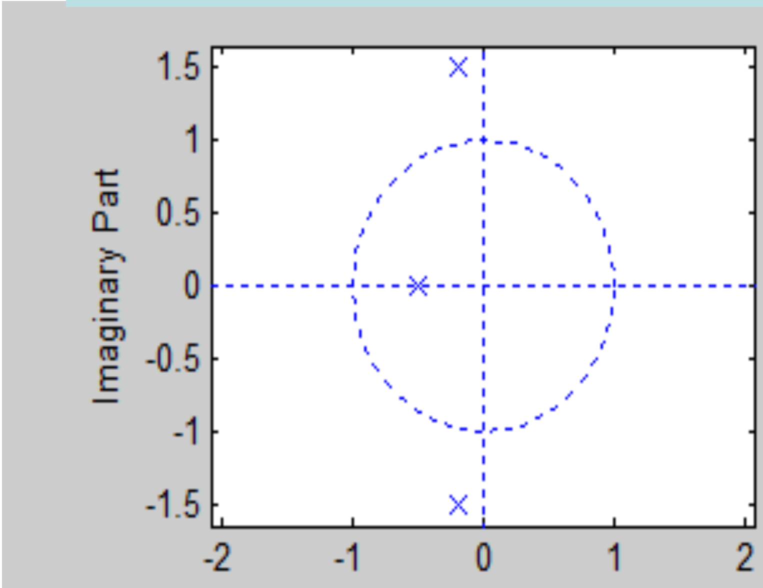
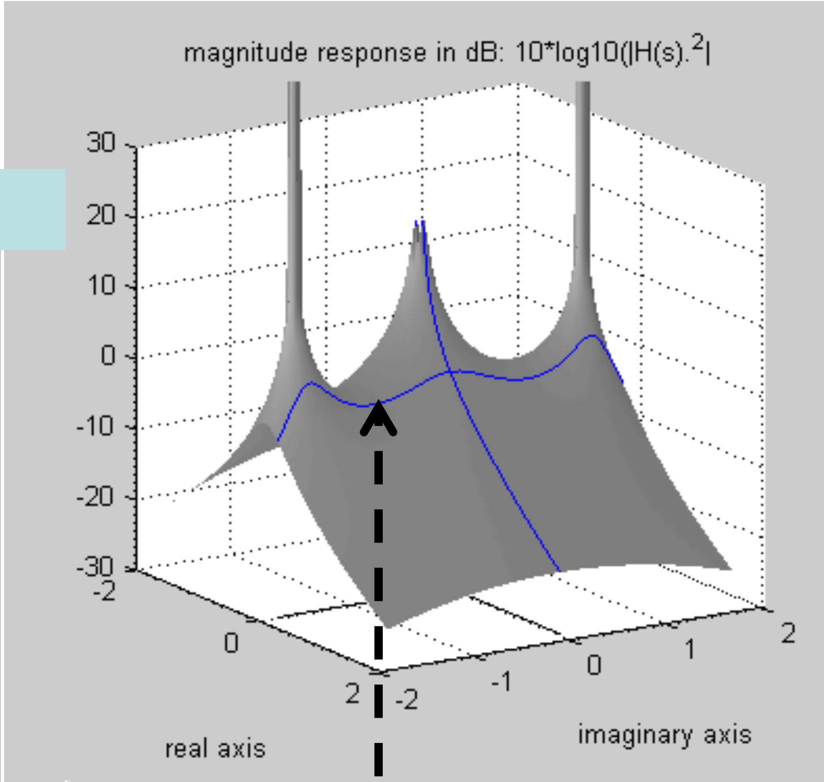
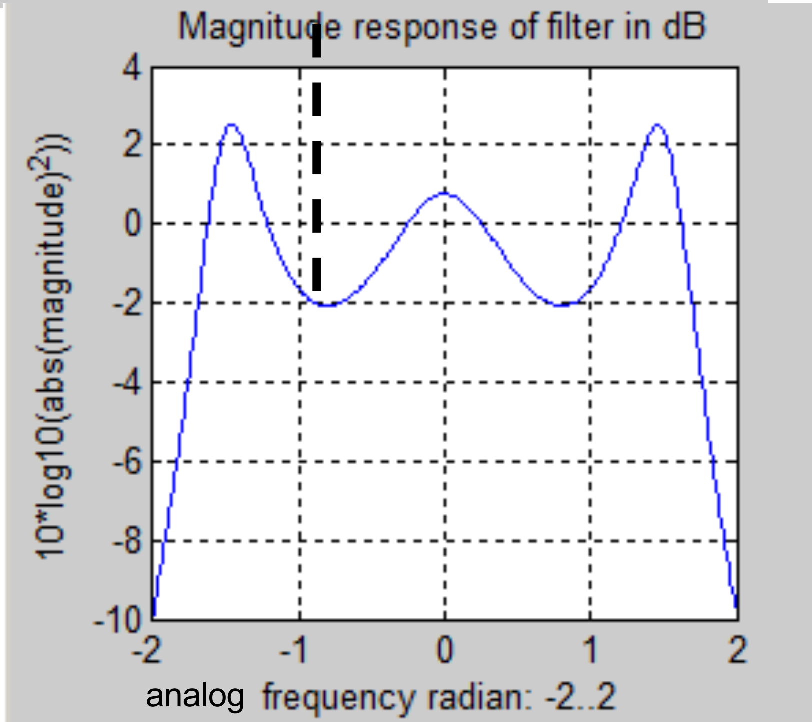
More complicated IIR filters: 3 poles

Matlab:Hs_3D_plot_ex2.m








```
% three poles
pp = [-0.5, -0.2+j*1.5, -0.2-j*1.5];
zz = []; K = 1.25;
% Convert these into numerator and
% denominator polynomial coefficients
[nn,dd]=zp2tf(zz',pp',K);
tfx = tf(nn,dd);
```

```
Transfer function:
             1.25
-----------------------------
s^3 + 0.9 s^2 + 2.49 s + 1.145
```

IIR filter design

We know: by placing more poles (or zeros) on the s-plane, we can shape the filter response.

- We MUST design poles only on the LHS (Left Hand Side) of the s-plane (i.e., $\sigma<0$)

We will discuss more on this in the next slide.

- We can put zeros anywhere in the s-plane.

Why are poles of H(s) on LHS?

The stability condition for a system can be represented in terms of its impulse response h(t) or its transfer function H(s). A system is stable if its impulse response is absolutely integrable,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

- Consider H(s) = 1/(s+a) and a > 0;
- Therefore there is a pole at s = -a (LHS).
- The inverse Laplace Transform of H(s) is

  h(t) = $e^{(-at)}u(t)$

For causal system to be stable, the poles must be on the left hand side and must include the imaginary axis: http://en.wikipedia.org/wiki/BIBO_stability

Table Basic Laplace transform pairs

$x(t),\ t \geq 0$	$X(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
c	$\frac{c}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \Omega_0 t$	$\frac{\Omega_0}{s^2+\Omega_0^2}$
$\cos \Omega_0 t$	$\frac{s}{s^2+\Omega_0^2}$

Butterworth filter design

- Butterworth analog Low Pass Filter: a design method to put the poles to obtain the necessary frequency response.
- The design of the Butterworth analog filter begins with the magnitude^2 response,

$$|H(\Omega)|^2 = H(s)H(-s)|_{s=j\Omega} = \frac{1}{1+(\Omega/\Omega_c)^{2n}}$$

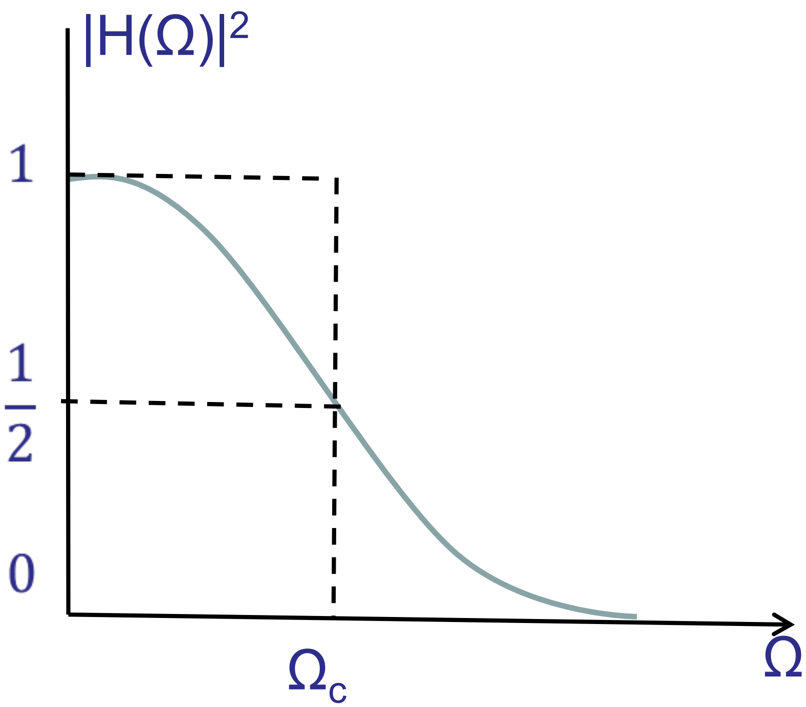
where n is the order of the filter. It is shown that $|H(0)| = 1$ and $|H(\Omega_c)| = 1/\sqrt{2}$ or equivalently $20\log_{10}|H(\Omega_c)| = -3\text{dB}$ for all values of n. Thus Ω_c is called the -3dB cut-off frequency

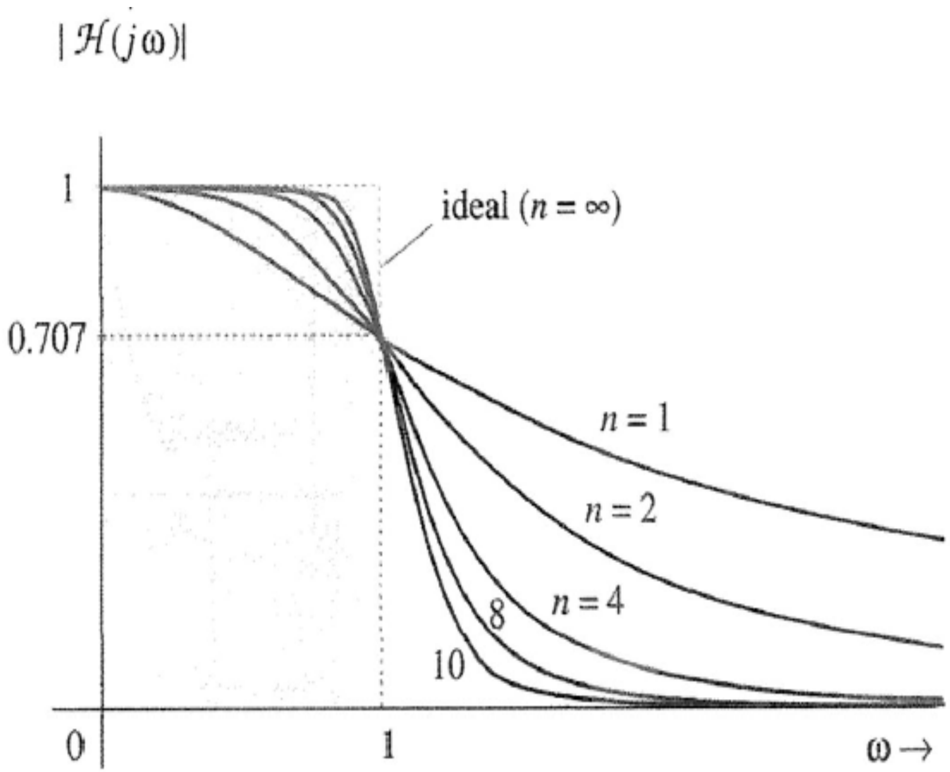
Butterworth filter design

The Butterworth magnitude^2 transfer function

$$|H(\Omega)|^2 = H(s)H(-s)|_{s=j\Omega} = \frac{1}{1+(\Omega/\Omega_c)^{2n}}$$

The magnitude response: monotonically decreasing.





Equation relating Butterworth analog lowpass filter

$$|H(\Omega)|^2 = \frac{1}{1+(\frac{j\Omega}{j\Omega_c})^{2n}}$$

$$|H(\Omega)|^2 = H(s)\ H(-s)|_{s=j\Omega} = \frac{1}{1+(\frac{s}{j\Omega_c})^{2n}}$$

The solution to the above equations are only poles. The poles are that the following locations:

$$1 + (\frac{s}{j\Omega_c})^{2n} = 0$$

$$(\frac{s}{j\Omega_c})^{2n} = -1 = 1e^{j((-\pi)+2k\pi)}, k = 0, 1, 2$$

$$(\frac{s}{j\Omega_c}) = 1e^{\frac{j(2k-1)\pi}{2n}}$$

$$s_k = \Omega_c e^{j\pi/2} e^{j(2k-1)\pi/(2n)}, k = 0, 1, 2 \ldots$$

The poles are distributed on the circle with radius Ω_c, and spaced at interval angle π/n radian

What is the interval angle when n=1,2,4,…?

Butterworth poles

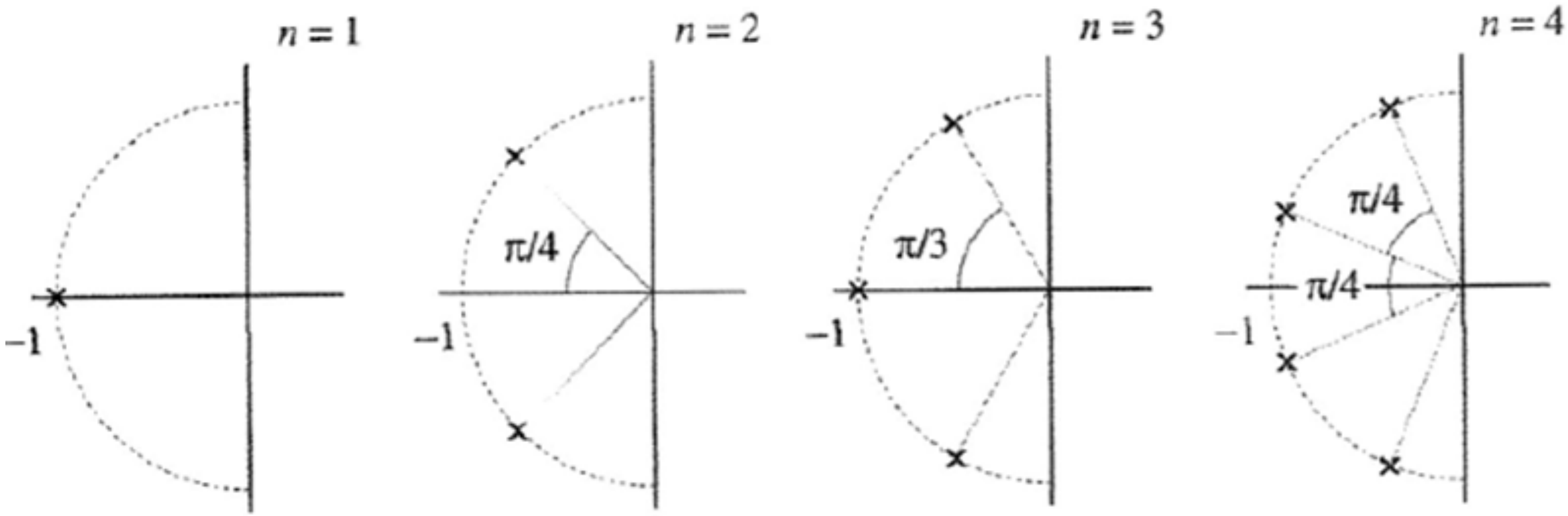
If we normalize the frequency with Ω_c, then $\Omega_c=1$ (as to be shown soon, with a simple frequency transformation will map the result to actual frequency)

Butterworth filters are a family of filters with poles distributed evenly around the Left-Hand Plane (LHP) unit circle such that the poles are given by ($\Omega_c=1$):

from the result from the previous slide

$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}$ where k = 1, 2, 3, …, n (k>n: repeating)

Here are the pole locations for Butterworth filters for orders n = 1 to 4:



LHP only

Normalized Butterworth transfer functions

Therefore the transfer function of Butterworth filter is defined as

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\ldots(s-s_n)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1}$$

Poles

Denominator D(s) for transfer function H(s) of the Butterworth filter (normalized)

n	D(s)
1	$\boxed{(s+1)}$
2	$(s^2 + 1.414s + 1)$
3	$(s+1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6810s + 1)(s^2 + 1.6810s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$

Frequency Band Transformation (to all filter types)

Start from the LP formula on the previous slide:

For lowpass/highpass, there is one critical frequency say Ω_c. For bandpass/bandstop there are two, Ω_u and Ω_l ($\Omega_u > \Omega_l$)

- Replace s in the transfer function H(s) using one of the transformations presented in the table below .

Transformations from a unit cutoff analogue frequency Ω_c

Lowpass to...	Use substitution	New cutoff(s)
Lowpass	$s \leftarrow s/\Omega_c$	Ω_c
Highpass	$s \leftarrow \Omega_c/s$	Ω_c
Bandpass	$s \leftarrow \left(s^2 + \Omega_l\Omega_u\right) / \left(s\left(\Omega_u - \Omega_l\right)\right)$	Ω_l, Ω_u
Bandstop	$s \leftarrow s\left(\Omega_u - \Omega_l\right) / \left(s^2 + \Omega_l\Omega_u\right)$	Ω_l, Ω_u

A summary: what have we learnt?

1) Placing poles/zeros for $H(s)$ in different places in s domain allows us to change the response of the transfer function $H(\Omega)$.
2) The poles must all be on the left hand side of $H(s)$ so that $h(t) \to 0$ as $t \to \infty$, and hence is stable.
3) We learn how to design Butterworth analog filters.