3.3 Hamiltonian Graphs

A **Hamiltonian path** in a graph G is a path which contains every vertex of G.

Since, by definition (see Section 1.6), no vertex of a path is repeated, this means that a Hamiltonian path in G contains every vertex of G once and only once.

A **Hamiltonian cycle** (or **Hamiltonian circuit**) in a graph G is a cycle which contains every vertex of G.

Since, by definition (again see Section 1.6), no vertex of a cycle is repeated apart from the final vertex being the same as the first vertex, this means that a Hamiltonian cycle in G with initial vertex v contains every other vertex of G precisely once and then ends back at v.

A graph G is called **Hamiltonian** if it has a Hamiltonian cycle.

By simply deleting the last edge of a Hamiltonian cycle we get a Hamiltonian path. However a non-Hamiltonian graph may possess a Hamiltonian path, i.e., Hamiltonian paths cannot always be used to form Hamiltonian cycles. For example, in Figure 3.21, G_1 has no Hamiltonian path (and so no Hamiltonian cycle), G_2 has the Hamiltonian path $a\ b\ c\ d$ but has no Hamiltonian cycle, while G_3 has the Hamiltonian cycle $a\ b\ d\ c\ a$.

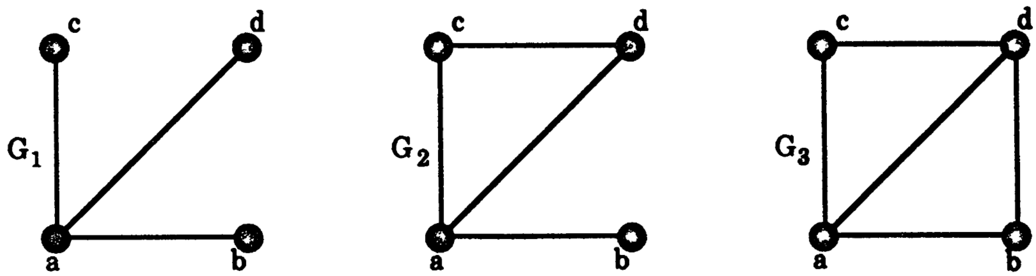

Figure 3.21: G_1 has no Hamiltonian path, G_2 has a Hamiltonian path but no Hamiltonian cycle, while G_3 has a Hamiltonian cycle.

Hamiltonian graphs are named after Sir William Hamilton, an Irish mathematician (1805–1865), who invented a puzzle, called the Icosian game, which he sold for 25 guineas to a games manufacturer in Dublin. The puzzle involved a dodecahedron on which each of the 20 vertices was labelled by the name of some capital city in the world. The object of the game was to construct, using the edges of the dodecahedron, a tour of all the cities which visited each city exactly once, beginning and ending at the same city. In other words, one had essentially to form a Hamiltonian cycle in the graph corresponding to the dodecahedron. We show such a cycle, using bolder lines, in Figure 3.22.

Clearly the n-cycle C_n with n distinct vertices (and n edges) is Hamiltonian. Moreover, given any Hamiltonian graph G, then, if G' is a supergraph of G obtained by adding in new edges between vertices of G, G' will also be Hamiltonian, since any Hamiltonian cycle in G will also be a Hamiltonian cycle in G'. In particular since K_n, the complete graph on n vertices, is such a supergraph of an n-cycle, K_n is Hamiltonian.

A graph G will be Hamiltonian if and only if its underlying simple graph is Hamiltonian since if G is Hamiltonian then any Hamiltonian cycle in G will remain a Hamiltonian cycle in the underlying simple graph of G (provided we delete the appropriate parallel edges). Conversely, if the underlying simple graph of a graph G

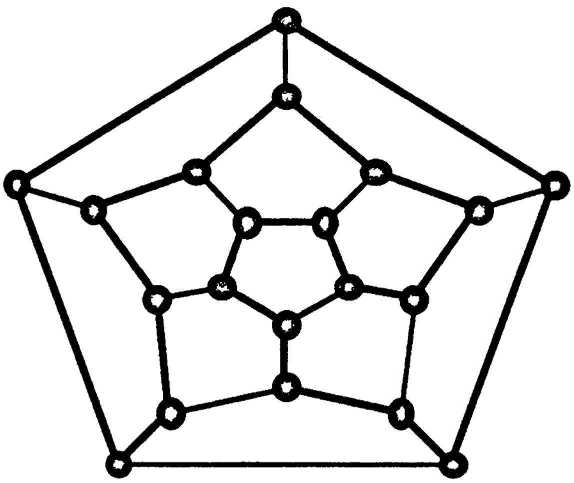

Figure 3.22: A Hamiltonian cycle in the graph of the dodecahedron.

is Hamiltonian then G will also be, because of the remarks of the previous paragraph. For this reason one usually only considers the Hamiltonian property for simple graphs.

Given a simple graph G with n vertices, since G is a subgraph of the complete graph K_n, we can construct step-by-step simple supergraphs of G to eventually get K_n, simply by adding in an extra edge at each step between two vertices that are not already adjacent. We illustrate this in Figure 3.23.

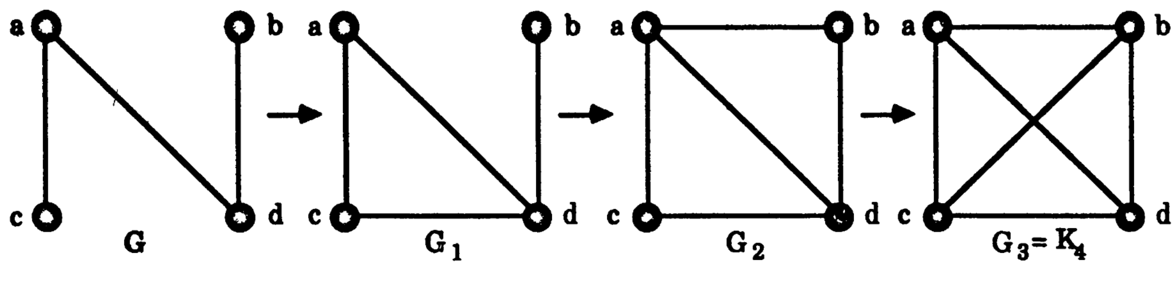

Figure 3.23: A build-up to K_4.

If, moreover, we start with a graph G that is not Hamiltonian, then, since the final outcome of the procedure is the Hamiltonian graph K_n, at some stage during the procedure we change from a non-Hamiltonian graph to a Hamiltonian graph. For example, the non-Hamiltonian graph G_1 above is followed by the Hamiltonian graph G_2. Notice that since supergraphs of Hamiltonian graphs are Hamiltonian, once a Hamiltonian supergraph is reached in the procedure, all the subsequent supergraphs are Hamiltonian. This discussion leads to the following definition.

> A simple graph G is called **maximal non-Hamiltonian** if it is not Hamiltonian but the addition to it of any edge connecting two non-adjacent vertices forms a Hamiltonian graph.

For example, G_1 of Figure 3.21 is maximal non-Hamiltonian since the addition of the edge ab gives the Hamiltonian G_2 as shown, while the other possibility, the addition

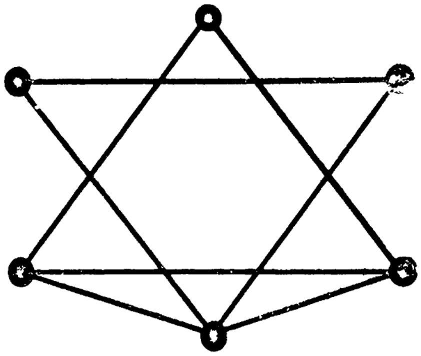

Figure 3.24: A maximal non-Hamiltonian graph.

of the edge bd, also gives a Hamiltonian graph (with Hamiltonian cycle $b\ d\ a\ c\ b$). Similarly the graph of Figure 3.24 is also maximal non-Hamiltonian:

Because of the stepwise procedure described above any non-Hamiltonian graph with n vertices will be a subgraph of a maximal non-Hamiltonian graph with n vertices. We use this to prove the following theorem, due to Dirac [18].

Theorem 3.6 (Dirac, 1952) *If G is a simple graph with n vertices, where $n \geq 3$, and the degree $d(v) \geq n/2$ for every vertex v of G, then G is Hamiltonian.*

Proof We suppose that the result is false. Then, for some value $n \geq 3$, there is a non-Hamiltonian graph in which every vertex has degree at least $n/2$. Any spanning supergraph, i.e., with precisely the same vertex set, also has every vertex with degree at least $n/2$, since any proper supergraph of this form is obtained by introducing more edges. Thus there will be a maximal non-Hamiltonian graph G with n vertices and $d(v) \geq n/2$ for every v in G. Using this G we obtain a contradiction.

G can not be complete, since K_n is Hamiltonian. Thus there are two nonadjacent vertices u and v in G. Let $G+uv$ denote the supergraph of G obtained by introducing an edge from u to v. Then, since G is maximal non-Hamiltonian, $G+uv$ must be Hamiltonian. Also, if C is a Hamiltonian cycle of $G+uv$, then it must contain the edge uv (since otherwise it would be a Hamiltonian cycle in G). Thus, choosing such a C, we may write $C = v_1v_2\ldots v_nv_1$ where $v_1 = u, v_n = v$ (and the edge v_nv_1 is just vu, i.e., uv). Now let

$$S = \{v_i \in C: \text{ there is an edge from } u \text{ to } v_{i+1} \text{ in } G\} \text{ and}$$

$$T = \{v_j \in C: \text{ there is an edge from } v \text{ to } v_j \text{ in } G\}.$$

Then $v_n \notin T$, since otherwise there would be an edge from v to $v_n = v$, i.e., a loop, impossible because G is simple. Also $v_n \notin S$ (interpreting v_{n+1} as v_1), since otherwise we would again get a loop, this time from u to $v_1 = u$. Thus $v_n \notin S \cup T$. Then, letting $|S|$, $|T|$ and $|S \cup T|$ denote the number of elements in S, T, and $S \cup T$ respectively, we get

$$|S \cup T| < n \tag{3.1}$$

Also, for every edge incident with u there corresponds precisely one vertex v_i in S. Thus

$$|S| = d(u) \tag{3.2}$$

Similarly

$$|T| = d(v) \tag{3.3}$$

Moreover, if v_k is a vertex belonging to both S and T, then there is an edge e joining u to v_{k+1} and an edge f joining v to v_k. This would give

$$C' = v_1\ v_{k+1}\ v_{k+2}\ \ldots\ v_n\ v_k\ v_{k-1}\ \ldots\ v_2\ v_1.$$

as a Hamiltonian cycle in G, (see Figure 3.25), a contradiction since G is non-Hamiltonian.

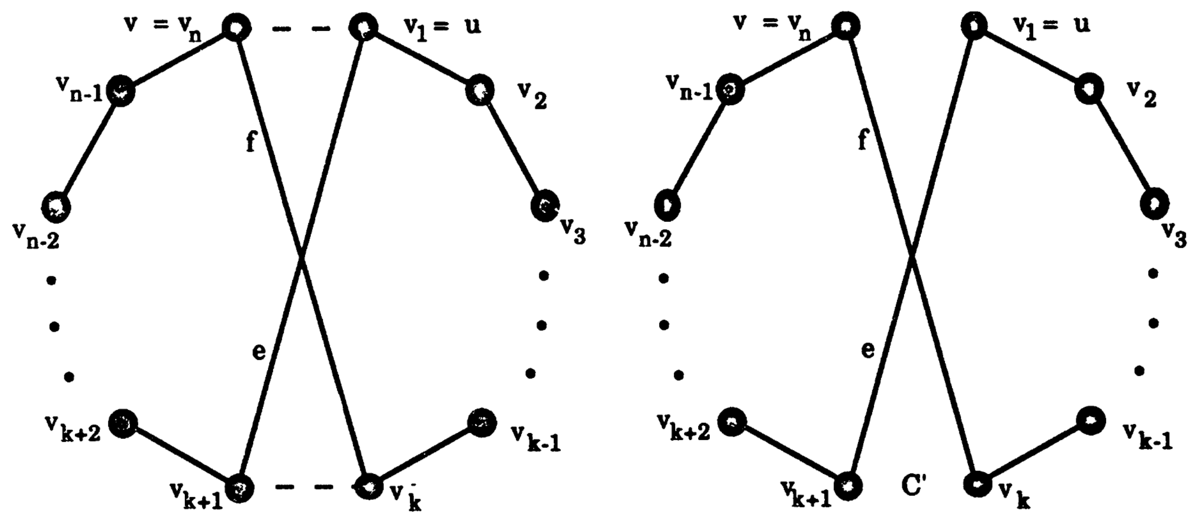

Figure 3.25

This shows that there is no vertex v_k in $S \cap T$, i.e., $S \cap T = \emptyset$. Thus $|S \cup T| = |S| + |T|$. Hence, by (3.1), (3.2) and (3.3) above,

$$d(u) + d(v) = |S| + |T| = |S \cup T| < n. \tag{3.4}$$

This is impossible since in G, $d(u) \geq n/2$ and $d(v) \geq n/2$, and so $d(u) + d(v) \geq n$. This contradiction tells us that we have wrongly assumed the result to be false. □

We now use the ideas of the above proof to present some results on Hamiltonian graphs by Bondy and Chvatal [8].

Theorem 3.7 *Let G be a simple graph with n vertices and let u and v be non-adjacent vertices in G such that*

$$d(u) + d(v) \geq n.$$

Let $G + uv$ denote the supergraph of G obtained by joining u and v by an edge. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof Suppose that G is Hamiltonian. Then, as noted earlier, the supergraph $G+uv$ must also be Hamiltonian.

Conversely, suppose that $G+uv$ is Hamiltonian. Then, if G is not Hamiltonian, just as in the proof of Theorem 3.6 we obtain the inequality $d(u)+d(v) < n$. However, by hypothesis, $d(u) + d(v) \geq n$. Hence G must be Hamiltonian also, as required. □

Motivated by Theorem 3.7 we now define what we mean by the closure $c(G)$ of a simple graph G.

> Let G be a simple graph. If there are two nonadjacent vertices u_1 and v_1 in G such that $d(u_1) + d(v_1) \geq n$ in G, join u_1 and v_1 by an edge to form the supergraph G_1. Then, if there are two nonadjacent vertices u_2 and v_2 such that $d(u_2) + d(v_2) \geq n$ in G_1, join u_2 and v_2 by an edge to form the supergraph G_2. Continue in this way, recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains. The final supergraph thus obtained is called the **closure** of G and is denoted by $c(G)$.

We give an example of the closure operation in Figure 3.26. For this example, $c(G) = K_7$.

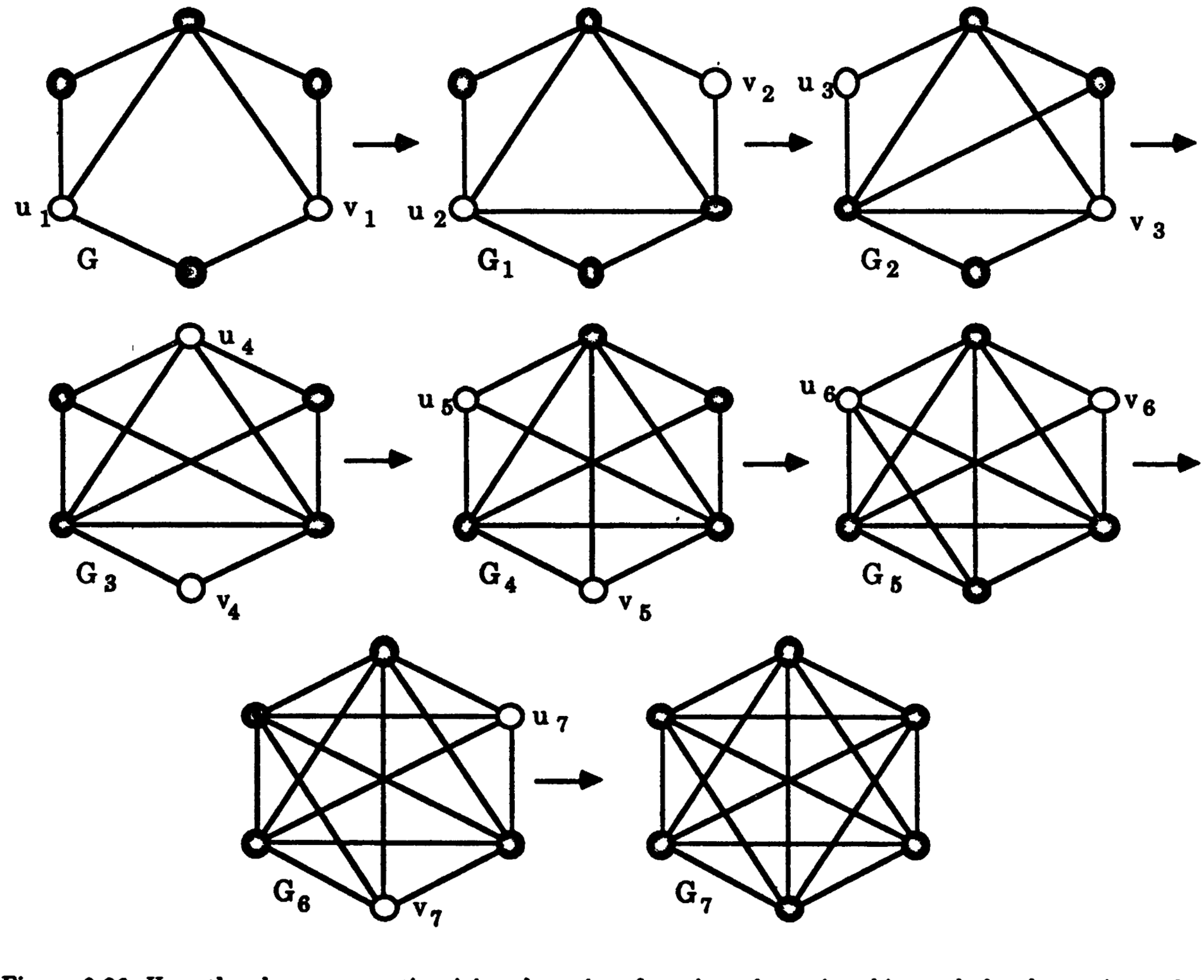

Figure 3.26: Here the closure operation joins the pairs of vertices shown in white and the closure is reached after seven such joins.

On the other hand, for the graph G on 7 vertices of Figure 3.27, $d(u) + d(v) < 7$ for any pair u, v of nonadjacent vertices in G and so the closure operation does not get off the ground, i.e., $c(G) = G$.

Notice that in the example of Figure 3,26 there were often various choices available of pairs of nonadjacent vertices u, v with $d(u) + d(v) \geq n$. Thus the closure procedure

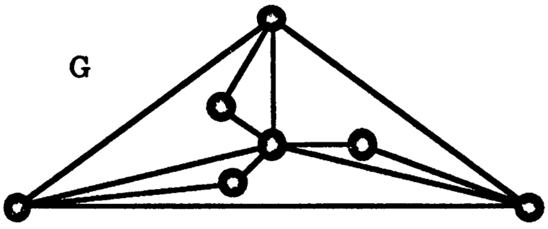

Figure 3.27: $c(G) = G$.

could have been carried out in several different ways. The question arises as to whether each different way gives the same result, i.e., do we always end with the same $c(G)$? Yes, we do — we omit the details here. (The interested reader is referred to Lemma 4.4.2 of Bondy and Murty [7].)

The importance of $c(G)$ is given in the following result:

Theorem 3.8 (Bondy and Chvatal, 1976) *A simple graph G is Hamiltonian if and only if its closure $c(G)$ is Hamiltonian.*

Proof Since $c(G)$ is a supergraph of G, if G is Hamiltonian then $c(G)$ must be Hamiltonian.

Conversely, suppose that $c(G)$ is Hamiltonian. Let G, G_1,$G_2,\ldots,G_{k-1}$, $G_k = c(G)$ be the sequence of graphs obtained by performing the closure procedure on G. Since $c(G) = G_k$ is obtained from G_{k-1} by setting $G_k = G_{k-1} + uv$, where u, v is a pair of nonadjacent vertices in G_{k-1} with $d(u)+d(v) \geq n$, it follows by Theorem 3.7 that G_{k-1} is Hamiltonian. Similarly G_{k-2}, so $G_{k-3},\ldots$, so G_1, and so G must be Hamiltonian, as required. □

Corollary 3.9 *Let G be a simple graph on n vertices, with $n \geq 3$. If $c(G)$ is complete, i.e., if $c(G) = K_n$, then G is Hamiltonian.*

Proof This is immediate from the Theorem since any complete graph is Hamiltonian. □

For example, for the graph G of Figure 3.26 we got $c(G)$ as the complete graph K_7 and so, by the Corollary, it follows that G is Hamiltonian.

Unfortunately, the closure operation is not always helpful in determining if a graph is Hamiltonian. For example, $c(G) = G$ for the graph G of Figure 3.27 so here the operation provides no additional information.

Although the closure operation tells us that the graph G of Figure 3.26 is Hamiltonian, this is obvious from the drawing of G. For this reason we look at a less obvious example G in Figure 3.28.

We perform the closure operation on G in Figures 3.28 and 3.29, illustrating the nonadjacent pair of vertices u, v with $d(u) + d(v) \geq n = 9$ we use at each step by white dots. Since the final graph G_{15} in our closure construction is the complete graph K_9, we may conclude from the Corollary that our initial graph G is Hamiltonian.

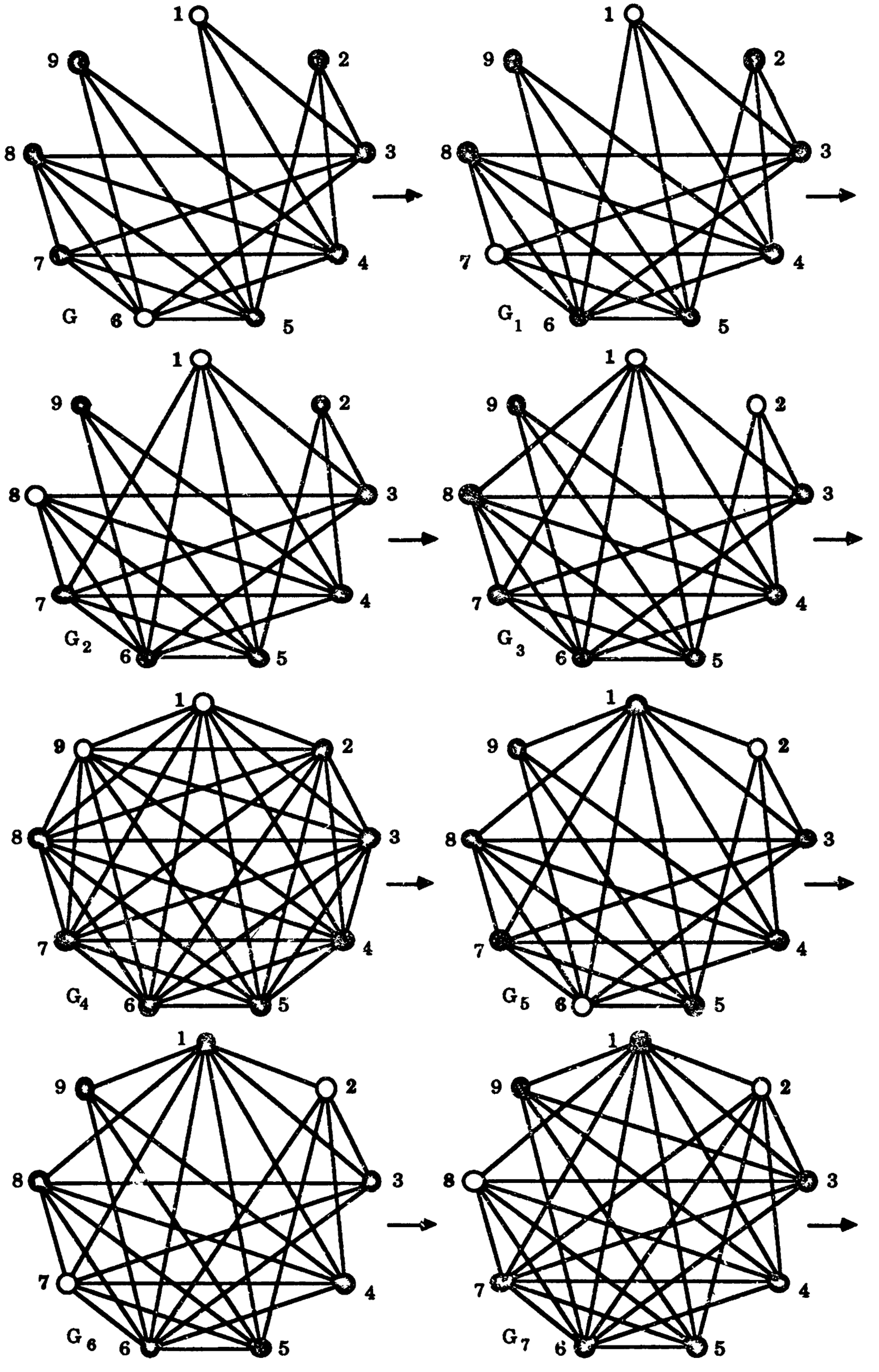

Figure 3.28: The closure operation on G. In G_5 vertex 1 has reached (maximum possible) degree 8, while vertex 6 reaches degree 8 in G_6.

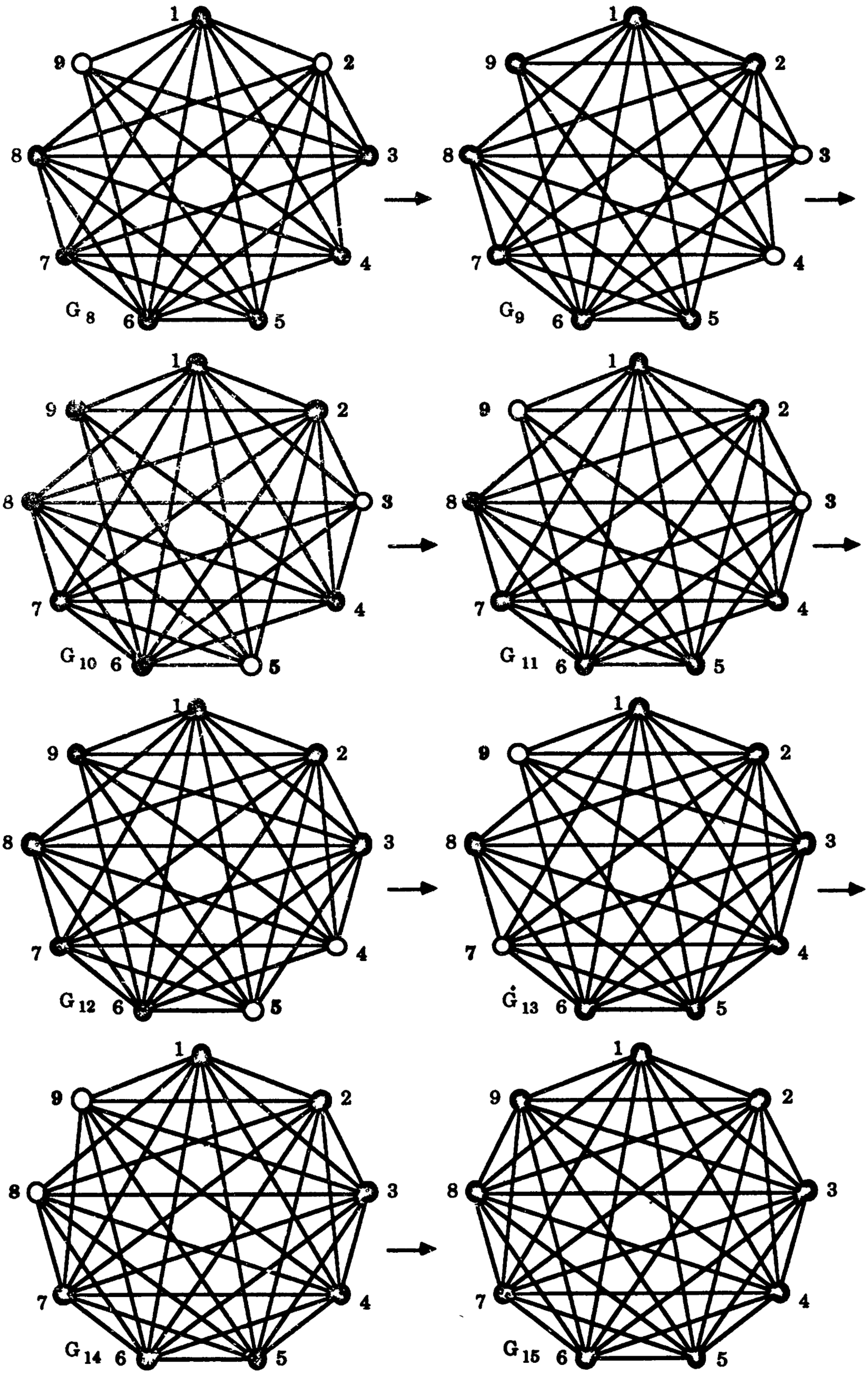

Figure 3.29: The closure operation continued. Vertices 2 and 3 reach degree 8 in G_9 and G_{12} respectively, vertices 4 and 5 reach degree 8 in G_{13}, vertex 7 reaches degree 8 in G_{14} and, finally, vertices 8 and 9 reach degree 8 in G_{15}.

Exercises for Section 3.3

3.3.1 Show that the graph G_1, of Figure 3.30 is Hamiltonian and that the graph G_2 has a Hamiltonian path but has not a Hamiltonian cycle. (Try looking at the vertices of degree two.)

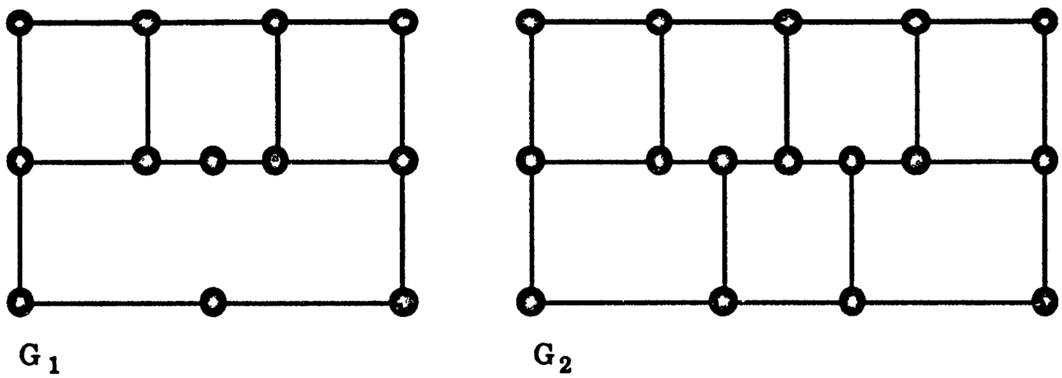

Figure 3.30

3.3.2 Characterise all simple Euler graphs having an Euler tour which is also a Hamiltonian cycle.

3.3.3 Let G be a bipartite graph with bipartition $V = X \cup Y$.

(a) Show that if G is Hamiltonian then $|X| = |Y|$.

(b) Show that if G is not Hamiltonian but has a Hamiltonian cycle then $|X| = |Y| \pm 1$.

3.3.4 Prove that the wheel W_n is Hamiltonian for every $n \geq 4$.

3.3.5 Prove that the n-cube Q_n is Hamiltonian for each $n \geq 2$.

3.3.6 Prove that the graphs (i) and (ii) of Figure 3.14 are Hamiltonian but (iii) and (iv) are not. Prove that graph (iii) does have a Hamiltonian path though, but graph (iv) does not.

3.3.7 Let G be a Hamiltonian graph. Show that G does not have a cut vertex.

3.3.8 Let G be a Hamiltonian graph and let S be a proper subset of vertices of G. Prove that $\omega(G - S) \leq |S|$. (This generalises the previous Exercise.)

3.3.9 Show, by giving an example, that the condition "$d(v) \geq n/2$" in Dirac's Theorem (Theorem 3.6) can not be changed to "$d(v) \geq (n-1)/2$".

3.3.10 Let H be an Euler graph and let $G = L(H)$, the line graph of H. (See Exercise 3.1.7.) Prove that G is Hamiltonian.

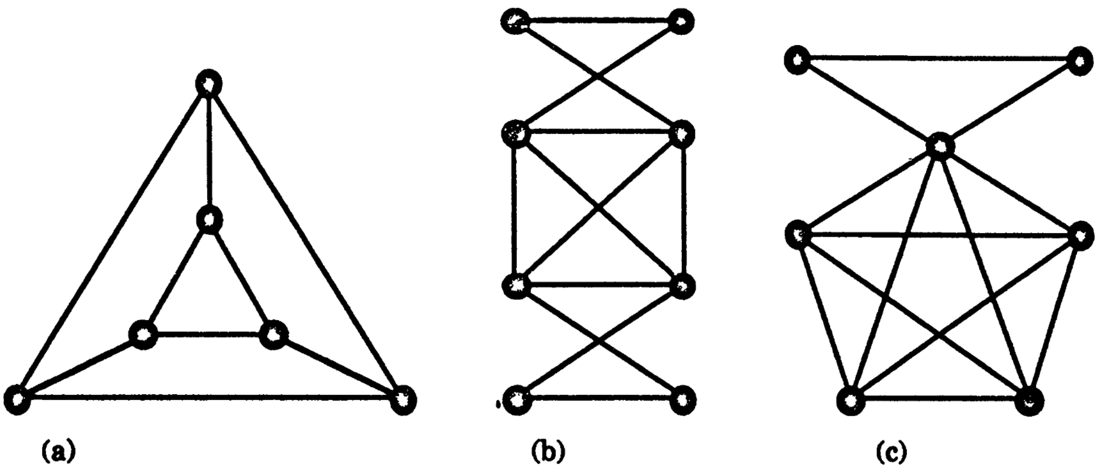

Figure 3.31: Find the closure of each of these graphs.

3.3.11 Find the closure $c(G)$ for each of the graphs of Figure 3.31. Which of these graphs are Hamiltonian?

3.3.12 Prove that, for each $n \geq 1$, the complete tripartite graph $K_{n,2n,3n}$ is Hamiltonian but $K_{n,2n,3n+1}$ is not Hamiltonian. (See Exercise 1.6.13 for the definition of $K_{r,s,t}$.)

3.3.13 In a Hamiltonian graph G two Hamiltonian cycles C and C' are considered to be the same if C is a cyclic rotation of C' or a cyclic rotation of the reverse of C'. Thus, for example, if $C = v_1 v_2 v_3 v_4 v_1$ is a Hamiltonian cycle in G we consider it to be the same as the Hamiltonian cycles $C_{(1)} = v_2 v_3 v_4 v_1 v_2$, $C_{(2)} = v_3 v_4 v_1 v_2 v_3$, $C_{(3)} = v_4 v_1 v_2 v_3 v_4$, $C_{(4)} = v_1 v_4 v_3 v_2 v_1$, $C_{(5)} = v_4 v_3 v_2 v_1 v_4$, $C_{(6)} = v_3 v_2 v_1 v_4 v_3$ and $C_{(7)} = v_2 v_1 v_4 v_3 v_2$.

(a) Prove that the complete graph K_n has $(n-1)!/2$ different Hamiltonian cycles.

(b) How many different Hamiltonian cycles does $K_{n,n}$ have?

3.3.14 There are n guests at a dinner party, where $n \geq 4$. Any two of these guests know, between them, all the other $n-2$. Prove that the guests can be seated round a circular table so that each one is sitting between two people they know.

3.3.15 Let G be a simple k-regular graph with $2k-1$ vertices. Prove that G is Hamiltonian.

3.3.16 Let G_1 and G_2 be two simple graphs and let G denote their join $G_1 + G_2$. (See Exercise 1.5.5.) Prove that if G_1 is Hamiltonian and has at least as many vertices as G_2 has then G is also Hamiltonian.