Linearizatoin of nonlinear systems

Smooth nonlinearity

Let a nonlinear system be given as

$$\dot{x} = f(x, u), \quad y = h(x).$$

The operating point is x_0, y_0 corresponding to input u_0. Expanding the nonlinear state equations into Taylor series

$$\dot{x} = f(x_0, u_0) + \left.\frac{\partial f(x,u)}{\partial x}\right|_{x_0,u_0} (x - x_0) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{x_0,u_0} (u - u_0) + \text{higher order terms}$$

$$y = h(x_0) + \left.\frac{\partial h(x)}{\partial x}\right|_{x=x_0} (x - x_0) + \text{higher order terms}$$

$\Delta\dot{x} = \dot{x} - \dot{x}_0$

$= \dot{x} - f(x_0, u_0)$

Let $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$.

$h(x_0)$

Since $\dot{x}_0 = f(x_0, u_0)$, we have

$$\boxed{\begin{aligned} \Delta\dot{x} &\approx \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} \Delta u \\ \Delta y &\approx \left.\frac{\partial h}{\partial x}\right|_{x_0} \Delta x \end{aligned}}$$

[neglecting the higher order terms]

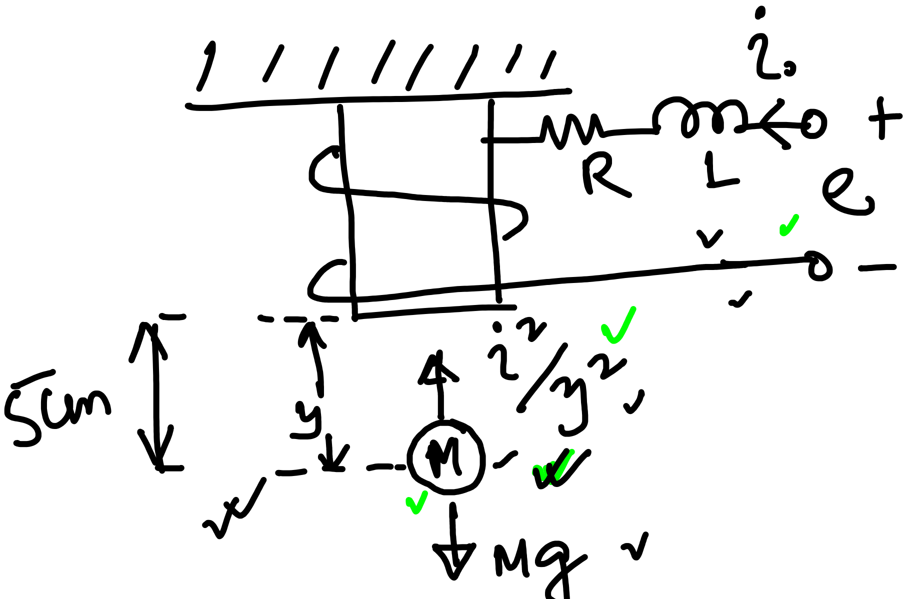
If $\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$, the linearized model is

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{\text{System matrix}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}}_{\text{Input matrix}} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_m \end{bmatrix}$$

Jacobian matrix

$$\Delta y = \underbrace{\begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix}}_{\text{output matrix}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

Modelling of attractive-type magnetic levitation system



$$M\frac{d^2y}{dt^2} = Mg - \frac{i^2(t)}{y^2} \quad (2)$$

$$e(t) = Ri(t) + L\frac{di(t)}{dt} \quad (1)$$

Let $x_1(t) = y(t)$, $x_2(t) = \dot{x}_1(t) = \dot{y}$ and $x_3(t) = i(t)$.

$$\left\{\begin{aligned} \dot{x}_2 &= \underbrace{g - \frac{1}{M}\frac{x_3^2}{x_1^2}}_{} = f_2 \\ \dot{x}_1 &= x_2 = f_1 \\ \dot{x}_3 &= \underbrace{-\frac{R}{L}x_3 + \frac{1}{L}e(t)}_{} = f_3 \end{aligned}\right\} \text{Nonlinear state equations}$$

Equilibrium point: $\dot{x}_1 = 0 \Rightarrow x_{20} = 0$

Let $x_{10} = y$ be the equilibrium distance.

$\dot{x}_2 = 0 \Rightarrow g - \frac{1}{M}\frac{x_{30}^2}{x_{10}^2} = 0$

$\dot{x}_3 = 0 \Rightarrow -\frac{R}{L}x_{30} + \frac{1}{L}e_0$

$$\frac{1}{L}e_0 = \frac{R}{L}x_{30}$$

$$e_0 = R x_{30}$$

$$x_{30} = \sqrt{x_{10}^2 g M}$$

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 1 \qquad \frac{\partial f_1}{\partial x_3} = 0 \qquad \frac{\partial f_1}{\partial e} = 0$$

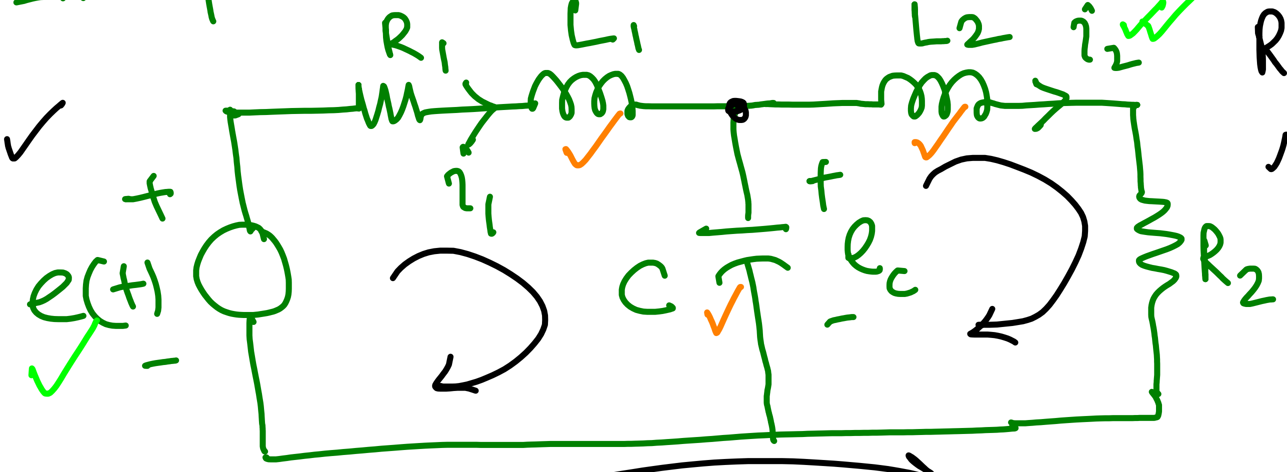
$$\frac{\partial f_2}{\partial x_1} = \frac{2}{M}\frac{x_3^2}{x_1^3}\bigg|_{x_{10},x_{30}} = \frac{2}{M}\frac{x_{30}^2}{x_{10}^3} = \frac{2x_{10}^2 g M}{M x_{10}^3} = \frac{2g}{x_{10}}$$

$$\frac{\partial f_2}{\partial x_2} = 0, \quad \frac{\partial f_2}{\partial x_3} = -\frac{2}{M}\frac{x_3}{x_1^2}\bigg|_{x_{30},\, x_{10}} = -\frac{2}{M}\frac{x_{30}}{x_{10}^2} = -\frac{2}{M}\frac{\sqrt{Mg\,x_{10}^2}}{x_{10}^2}$$

$$\frac{\partial f_3}{\partial x_1} = 0, \quad \frac{\partial f_3}{\partial x_2} = 0, \quad \frac{\partial f_3}{\partial x_3} = -\frac{R}{L}, \quad \frac{\partial f_3}{\partial e} = \frac{1}{L}$$

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ \frac{2g}{x_{10}} & 0 & -\frac{2}{M}\frac{\sqrt{Mg\,x_{10}^2}}{x_{10}^2}\\ 0 & 0 & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}\Delta e$$

Example:



Represent the ckt. in state-space form. Also draw the SFG and find TF from $E(s)$ to $I_2(s)$.

$$L_1\frac{di_1}{dt} + R_1 i_1 + e_c(t) = e(t)$$

$$L_2\frac{di_2}{dt} + R_2 i_2 = e_c(t)$$

$$C\frac{de_c}{dt} + i_2 = i_1$$

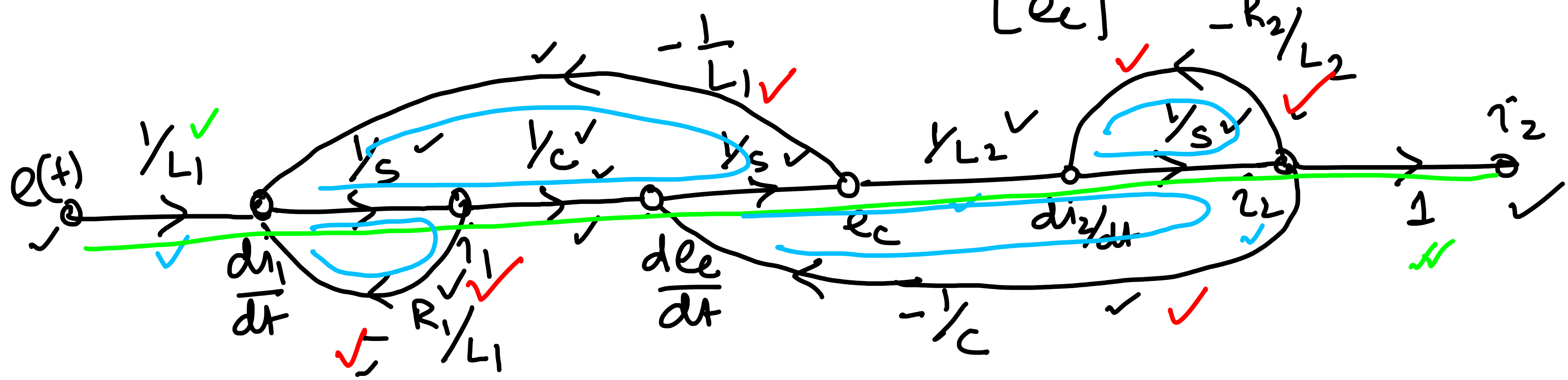
$$\frac{di_1}{dt} = -\frac{R_1}{L_1} i_1 - \frac{1}{L_1} e_c(t) + \frac{1}{L_1} e(t)$$

$$\frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 + \frac{1}{L_2} e_c(t)$$

$$\frac{de_c}{dt} = \frac{1}{C}(i_1 - i_2) = \frac{1}{C} i_1 - \frac{1}{C} i_2$$

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{de_c}{dt} \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -\frac{1}{L_1} \\ 0 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ e_c \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} e(t)$$

$$i_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ e_c \end{bmatrix}$$

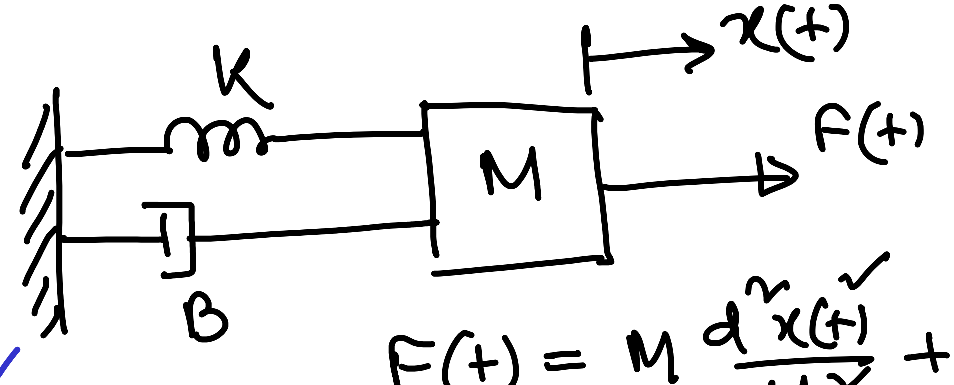


Apply Mason's gain formula:

$$\frac{I_2(s)}{E(s)} = \frac{\frac{1}{L_1} \times \frac{1}{s} \times \frac{1}{C} \times \frac{1}{s} \times \frac{1}{L_2} \times \frac{1}{s}}{1 - \left(-\frac{R_1}{L_1 s} - \frac{1}{L_1 C s^2} - \frac{1}{L_2 C s^2} - \frac{R_2}{L_2 s}\right) + \left(\frac{R_1 R_2}{L_1 L_2 s^2} + \frac{R_1}{L_1 L_2 C s^3} + \frac{R_2}{L_1 L_2 C s^3}\right)}$$

$$= \frac{1}{L_1 L_2 C s^3 + (R_1 L_2 + R_2 L_1) C s^2 + (L_1 + L_2 + R_1 R_2 C) s + R_1 + R_2}$$

Example



$$F(t) = M\frac{d^2x(t)}{dt^2} + Kx + B\frac{dx}{dt}$$

Let $x_1 = x(t) = x$ and $\frac{dx_1}{dt} = x_2$ (velocity) $= v$
(position)

$$\frac{dx_2}{dt} = -\frac{B}{M}x_2 - \frac{K}{M}x_1 + \frac{1}{M}F(t)$$

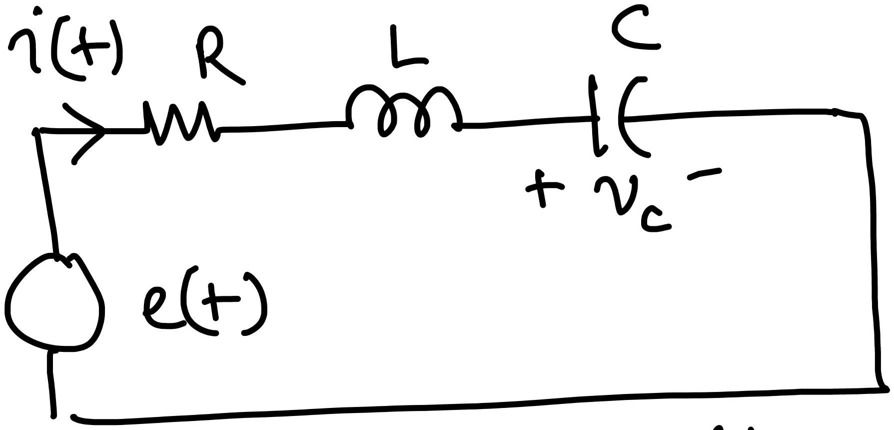
$$\boxed{\frac{dv}{dt} = -\frac{B}{M}v - \frac{Kx}{M} + \frac{1}{M}F(t)}$$

Let f_s be the force acting on the spring.

$$f_s = Kx.$$

$$\frac{df_s}{dt} = K\frac{dx}{dt} = Kv$$

$$\boxed{\frac{1}{K}\frac{df_s}{dt} = v}$$



$$e(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

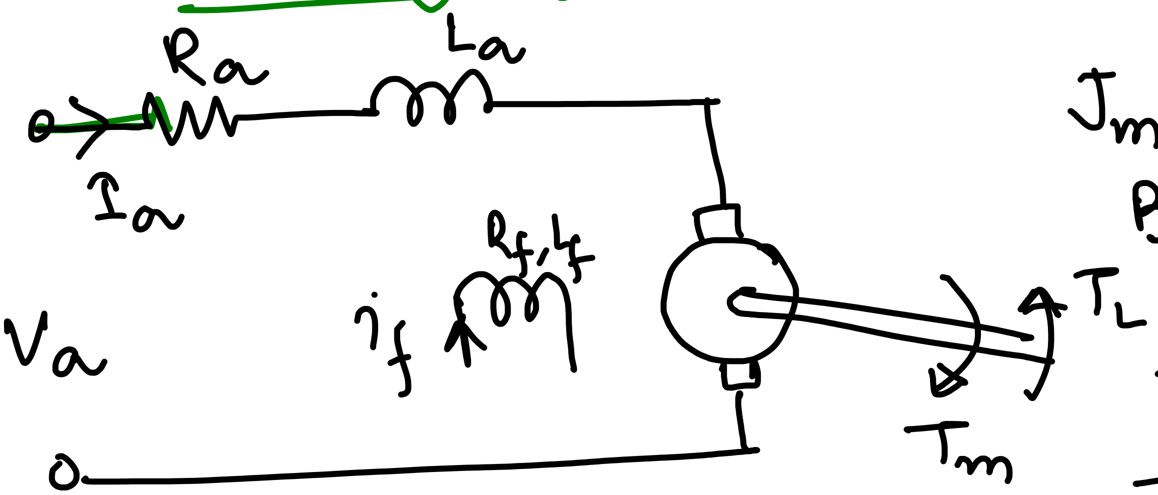
$$v_c(t) = \frac{1}{C}\int_0^t i(t)\,dt \Rightarrow \boxed{i(t) = C\frac{dv_c}{dt}}$$

$$\boxed{\frac{di(t)}{dt} = -\frac{R}{L}i(t) - \frac{1}{L}v_c(t) + \frac{1}{L}e(t)}$$

Analogy

- Force $(F(t))$ → e.m.f. $(e(t))$
- Mass (M) → Inductance (L)
- Damper (B) → Resistor (R)
- Spring (K) → Inverse of capacitance $\left(\frac{1}{C}\right)$
- Force acting on spring (f_s) → Voltage across C (V_c)
- Velocity (v) → Current through inductor (i)

Example: Modeling of DC Motor



J_m = Motor inertia

B = Viscous-friction co-efficient

T_m = Motor torque

T_L = Load torque

θ = rotor displacement

ω_m = rotor angular speed

$$V_a = I_a R_a + L_a \frac{di_a}{dt} + e_b \quad \text{(Back e.m.f.)}$$

$$e_b = K_b \omega_m$$

$$T_m = J_m \frac{d\omega_m}{dt} + B\omega_m + T_L, \qquad T_m = K_T I_a$$

$$\omega_m = \frac{d\theta}{dt}$$

$$\begin{bmatrix} \frac{dI_a}{dt} \\ \frac{d\omega_m}{dt} \\ \frac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} -R_a/L_a & -K_b/L_a & 0 \\ K_T/J_m & -B/J_m & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_a \\ \omega_m \\ \theta \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} V_a + \begin{bmatrix} 0 \\ -1/J_m \\ 0 \end{bmatrix} T_L .$$

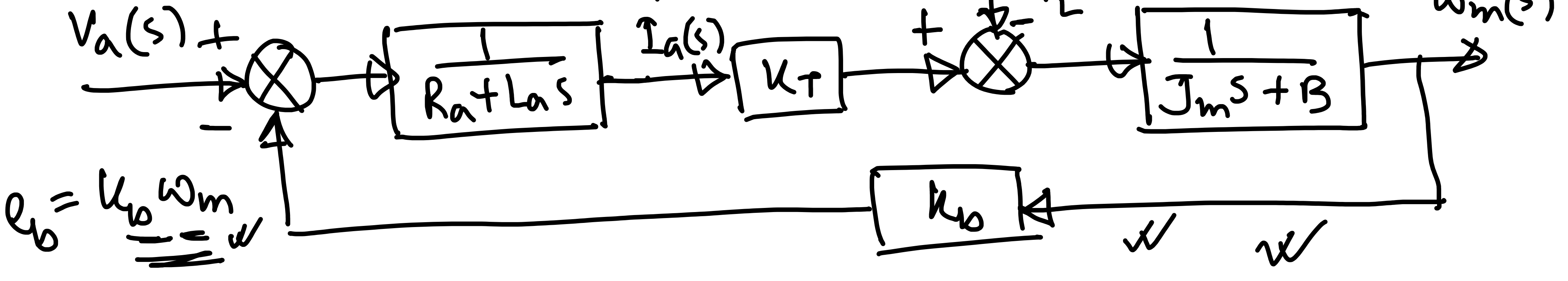
$$\omega_m = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_a \\ \omega_m \\ \theta \end{bmatrix}$$

$$V_a(s) = I_a(s) R_a + L_a s I_a(s) + K_b \omega_m(s)$$

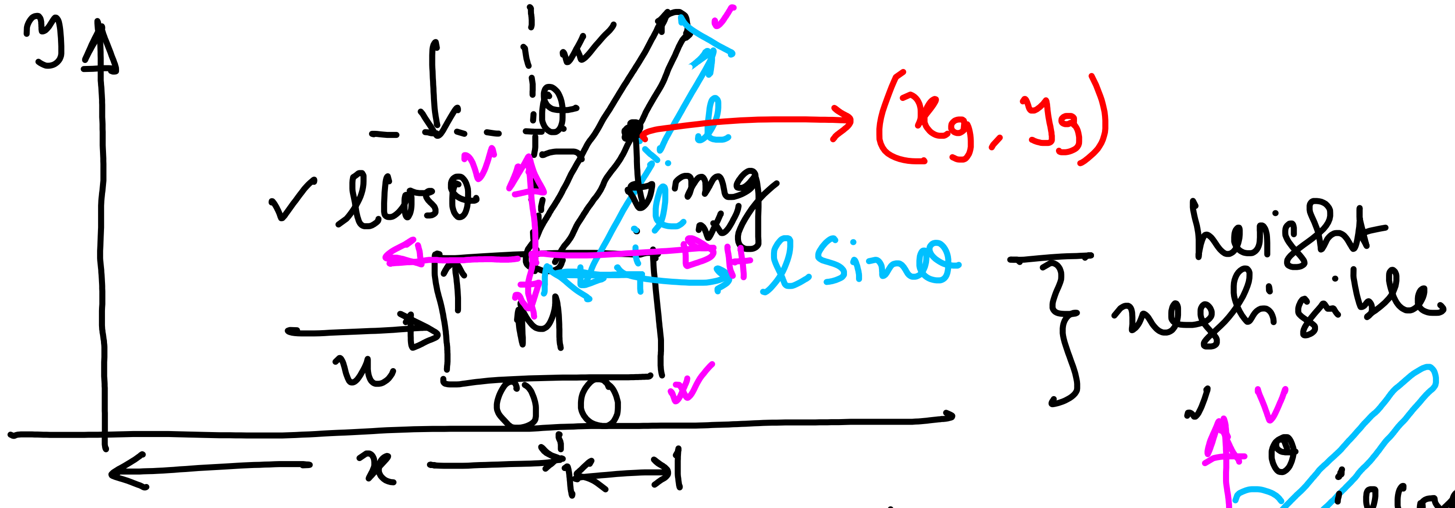
$$I_a(s) = \frac{1}{R_a + L_a s}\left[V_a(s) - K_b \omega_m(s)\right]$$

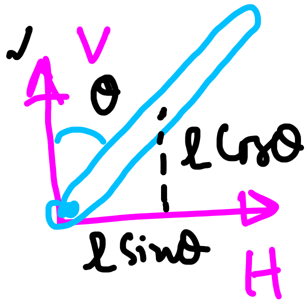
$$K_T I_a(s) = J_m s \omega_m(s) + B\omega_m(s) + T_L(s)$$

$$\omega_m(s) = \frac{1}{J_m s + B}\left[K_T I_a(s) - T_L(s)\right]$$



Example: Inverted Pendulum mounted on a motor-driven cart





m − mass of the pendulum rod

M − mass of the cart

The length of the pendulum rod $= 2l$

$$x_g = x + l\sin\theta, \quad y_g = l\cos\theta$$

If I is the moment of inertia of the rod about its CG,

$$I\ddot{\theta} = V l\sin\theta - H l\cos\theta \quad ✓ ✓$$

Along horizontal dir^n:

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \quad ✓ ✓$$

Along vertical dir^n:

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \quad ✓$$

The horizontal motion of the cart is described by

$$M \frac{d^2x}{dt^2} = u - H$$ ✓

Let θ be small, then $\sin\theta = \theta$ and $\cos\theta = 1$.

$$\left.\begin{aligned} I\ddot{\theta} &= Vl\theta - Hl \\ m(\ddot{x} + l\ddot{\theta}) &= H \\ V &= mg \\ M\ddot{x} &= u - H \end{aligned}\right\}$$

Hence we obtain,

$$\Rightarrow M\ddot{x} + m\ddot{x} + ml\ddot{\theta} = u$$ ✓

$$I\ddot{\theta} = mgl\theta - ml(\ddot{x} + l\ddot{\theta})$$

$$\Rightarrow (I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta$$ ✓