SSP

Test, 22nd March, 2022

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Q3:

$$R = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

a) from the Wiener-Hopf equation, we have

$$w = R^{-1} P \quad — (1)$$

Inverse of R is

$$R^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{(1-0.5^2)} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

Using equation (1),

$$w = \frac{1}{0.75} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \frac{1}{0.75} \begin{bmatrix} 0.375 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

(b) The minimum mean-square error is

$$J_{min} = \sigma_d^2 - p^H w$$

$$= \sigma_d^2 - \begin{bmatrix} 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$= \sigma_d^2 - 0.25$$

If $\sigma_d^2 = 1$ (assume)

$$J_{min} = 0.75$$

(c) The eigen-values of the matrix R are the roots of the characteristic equation:

$$(1-\lambda)^2 - 0.5^2 = 0 \quad \Rightarrow (0.5-\lambda)(1.5-\lambda) = 0$$

That is, the two roots are:

$\lambda_1 = 0.5$ and $\lambda_2 = 1.5$

The associated eigenvectors are defined by

$$Rq = \lambda q$$

for $\lambda_1 = 0.5$, we have

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0.5 \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

$$q_{11} + 0.5 q_{12} = 0.5 q_{11} \quad \text{— (ii)}$$

$$0.5 q_{11} + q_{12} = 0.5 q_{12} \quad \text{— (iii)}$$

From (ii) & (iii)

$q_{11} = -q_{12}$

Normalizing the eigenvectors q_1 to unit length, we therefore have

$$q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for the eigen value $d_2 = 1.5$, we have.

$$q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Accordingly we express the Wiener filter in terms of its eigenvalues & eigenvectors as follows:

$$w = \left(\sum_{i=1}^{n} \frac{1}{d_i} q_i q_i^H\right) P$$

$$= \left(\frac{1}{d_1} q_1 q_1^H + \frac{1}{d_2} q_2 q_2^H\right) P$$

$$= \left[\frac{1}{0.5} \times \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}\right) + \frac{1}{1.5} \times \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}\right)\right]\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \frac{1}{\cancel{2} \times 0.5}\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\right]\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \frac{1 \times 2}{\cancel{2}\,\cancel{2}}\left[\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\right]\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{6} - \frac{1}{6} \\ -\frac{1}{3} + \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \Rightarrow \text{Verified with parts (a) \& (b)}$$

~~If we don't consider the normalized version, it is equivalent to.~~

So, it is hereby verified with (a) & (b).

Q6

(a) Filter impulse response $= \omega$.

let x be the input vector, y be the output vector.

Average power $= \frac{1}{N} \sum_{N=-\infty}^{\infty} |y|^2$

$= E[y y^*]$

$= E[\omega^H x (\omega^H x)^*]$

we know that $(a^H b)^* = b^H a$

Hence, $E[yy^*] = E[\omega^H x x^H \omega]$

$= \omega^H E[x x^H] \omega$

By definition, $E[x x^H] = R$

$$\boxed{E[yy^*] = \omega^H R \omega}$$

Proved.

(b) If the input x is a zero mean white gaussian noise with ~~variance~~ with σ^2 variance

we know,

$R = \sigma^2 I$

Hence,

$E[yy^*] = E[y(n)^2] = \omega^H \sigma^2 I \omega$

$= \boxed{\sigma^2 \omega^H \omega}$

Q5: (a) The summarized version of the RMS algorithm:

$$y(n) = \hat{w}^H(n)\, u(n)$$
$$e(n) = d(n) - y(n)$$
$$w(n+1) = w(n) + \mu u(n) e^*(n)$$

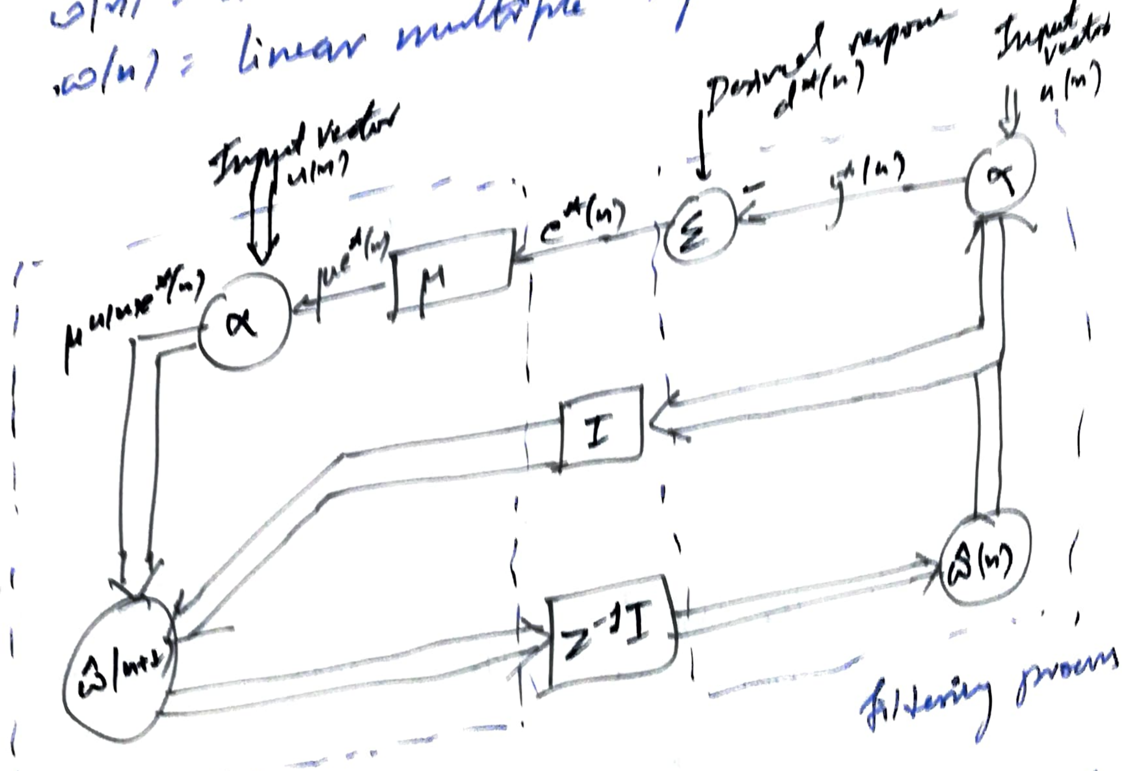
Where:

$u(n)$ is the input vector

$d(n)$ is the corresponding desired response

$\hat{w}(n)$: is an estimate of the tap-weight vector

$w(n)$ = linear multiple regression model



Adaptation process

Signal flow graph representation of the RMS algorithm.

b) The differences are the following:

→ Comparing the control mechanism for the LMS algorithm, with the method of steepest decent, we observe that the LMS algorithm uses the product $u(n-k)e^*(n)$ as an estimate of decent is the gradient vector $\nabla J(n)$ that characterizes the method of steepest decent.

- Now there is no expectation operator in all the paths. Therefore the recursive computation of each tap weight in the LMS algorithm suffers from a gradient noise.

For such an environment, we know that the method of steepest decent computes a tap vector $w(n)$ that moves down the ensemble average error performance surface along a deterministic trajectory that terminate on the wiener solution w_o.

Q2. (a). $x(n) = 0.5x(n-1) + w(n)$.

Hence,

$w_1 = 0.5$

and the AR parameters equal

$a_1 = -0.5$.

Accordingly we write Yule-Walker Equations,

$$\begin{bmatrix} r(0) \end{bmatrix} \begin{bmatrix} w_1 \end{bmatrix} = \begin{bmatrix} r(1) \end{bmatrix}$$

$$r(1) = 0.5 [r(0)]$$

Since the noise $w(n)$ has zero mean, so, will the AR process $x(n)$. Hence,

$$\text{Var}[x(n)] = E[(x^2(n))]$$

$$= r(0)$$

Also,

$$\sigma_\omega^2 = \sum_{k=0}^{1} a_k r(k)$$

$$= r(0) + a_1 r(1)$$

$$1 = r(0) - 0.5 r(1)$$

$$1 = r(0) - 0.5 \times (0.5 r(0))$$

$$1 = 0.75 r(0).$$

$$\boxed{r(0) = \frac{4}{3} = 1.33.}$$

$$\boxed{r(1) = \frac{2}{3} = 0.667}$$

(b). $x(n) = 1.5x(n-1) - 0.6x(n-2) + w(n)$

Hence,

$w_1 = 1.5$

$w_2 = -0.6.$

& then AR parameter equal.

$a_1 = -1.5$

$a_2 = 0.6.$

We write the Yule-Walker Equation, as.

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1.5 \\ -0.6 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix}$$

$$1.5r(0) - 0.6r(1) = r(1)$$

$$1.5r(1) - 0.6(r(0)) = r(2)$$

from here,

$$r(1) = \frac{1.5}{1.6} r(0)$$

$$r(2) = \frac{129}{160} r(0).$$

$var(u(n)) = E[u^2(n)]$

$= r(0).$

$\sigma_v^2 = \sum_{k=0}^{2} a_k r(k)$

~~$1 = r(0) + \frac{15}{16} r(0) + \frac{129}{160} (r(0))$~~

~~$r(0) = 0.8744$~~

$1 = r(0) - 1.5 \times \frac{15}{16} r(0) + 0.6 \times \frac{129}{160} r(0).$

$r(0) = 12.903.$

$r(1) = 12.096$

$r(2) = 10.403$

Q4: $r_y(l) = a^{|l|}$ $\quad -1 < a < 1$

$$x(n) = y(n) + v(n)$$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_{yx}(0) \\ r_{yx}(1) \end{bmatrix}$$

$$r_{dx}(l) = E[y(n)\, x^*(n-l)]$$

$$= r_y(l) + 0 = a^{|l|}$$

$$r_x(l) = r_y(l) + r_v(l)$$

$$= a^{|l|} + \sigma_v^2 \delta(l)$$

$$\begin{bmatrix} 1+\sigma_v^2 & a \\ a & 1+\sigma_v^2 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

$$\begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \frac{1}{(1+\sigma_v^2)^2 - a^2} \begin{bmatrix} 1+\sigma_v^2 - a^2 \\ a\sigma_v^2 \end{bmatrix}$$

MMSE

$$J_{min} = \sigma_y^2 - p^H w_o$$

$$= r_y(0) - w(0)\, r_{yx}(0) - w(1)\, r_{yx}(1)$$

$$= 1 - \frac{(1-a^2) + \sigma_v^2(1+a^2)}{(1+\sigma_v^2)^2 - a^2} = \sigma_v^2 \left[\frac{1+\sigma_v^2 - a^2}{(1+\sigma_v^2)^2 - a^2} \right]$$

$$= \sigma_v^2 \left[\frac{1}{1+\sigma_v^2 + a^2} \right]$$