Modeling in State-space

State variables: The smallest set of variables that completely determines the behavior of the dynamical system for any time $t \geq t_0$ with the knowledge of these variables at $t = t_0$ together with the information of the input for $t \geq t_0$.

- The concept of state is not limited to only physical system - it is also applicable to biological systems, economics, social system, etc..
- State variables need not be physically measurable or observable quantities.

State vector: If n state variables are needed, then these n state variables are considered as the n-components of a vector x. This vector is called state vector.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

State-space: The n dimensional space whose coordinate axes are x_1-axis, x_2-axis, ..., x_n-axis is called a state-space.

Example:

order

$$\frac{d^3y(t)}{dt^3} + 9\frac{d^2y(t)}{dt^2} + 26\frac{dy(t)}{dt} + 24y(t) = 24u(t)$$

Input: $u(t)$

$$\boxed{\text{Output } y(t) = x_1(t)}$$

$$x_2(t) = \frac{dx_1(t)}{dt} = \frac{dy(t)}{dt}$$

$$x_3(t) = \frac{dx_2(t)}{dt} = \frac{d^2y(t)}{dt^2}$$

Order = 3 (Three state-variable are required)

$$\begin{cases} \dot{x}_3(t) = -9x_3(t) - 26x_2(t) - 24x_1(t) + 24u(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_1(t) = x_2(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_3(t) \\ \dot{x}_2(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_3(t) \\ x_2(t) \\ x_1(t) \end{bmatrix} + \begin{bmatrix} 24 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3(t) \\ x_2(t) \\ x_1(t) \end{bmatrix} + 0 \cdot u(t)$$

✓

$$\frac{d^3y}{dt^3} + 9\frac{d^2y}{dt^2} + 26\frac{dy}{dt} + 24y(t) = 24u(t)$$

What is the TF of the system?

Taking Laplace transform, we get

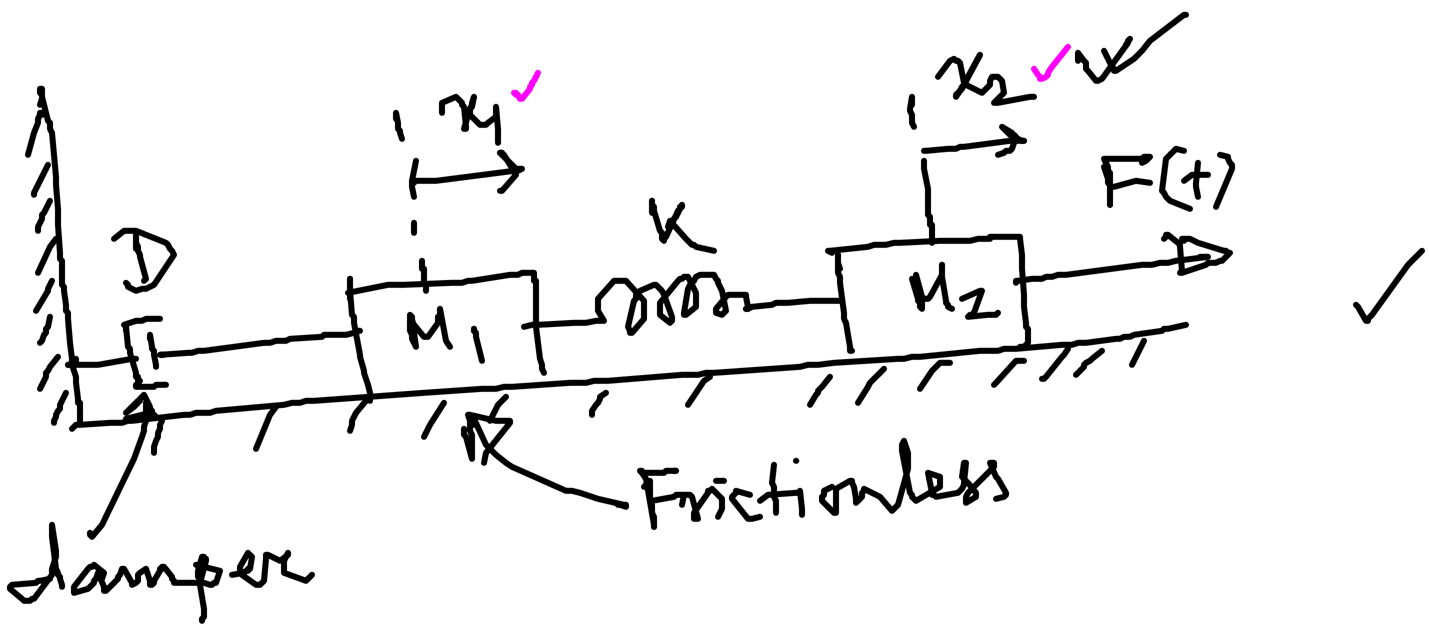
$$s^3 Y(s) + 9s^2 Y(s) + 26sY(s) + 24Y(s) = 24U(s)$$

[All initial condition are zero]

$$\boxed{\frac{Y(s)}{U(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}}$$

3rd order system.

Ex



Order = 2

$$M_1 \frac{d^2 x_1(t)}{dt^2} + D \frac{dx_1(t)}{dt} + K(x_1 - x_2) = 0$$

order = 2

$$M_2 \frac{d^2 x_2(t)}{dt^2} + K(x_2 - x_1) = F(t)$$

Independent?

YES

Two independent differential equations; Order of the system = sum of the order of the differential eqns = 2+2; No of state variables = 4.

$$\frac{dv_1}{dt} = \frac{d^2x_1}{dt^2}$$

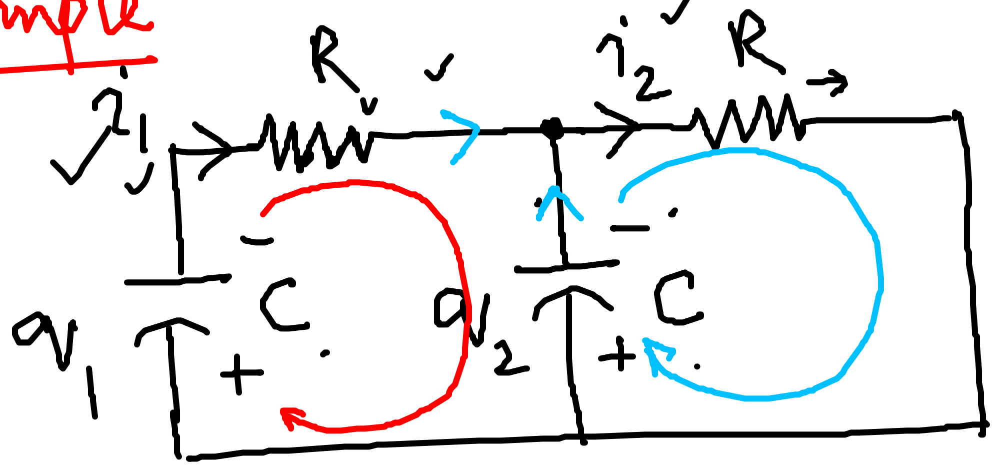
Let $v_1 = \frac{dx_1}{dt}$ and $v_2 = \frac{dx_2}{dt}$.

$$\left.\begin{aligned} \frac{dv_1}{dt} &= -\frac{D}{M_1} v_1 - \frac{K}{M_1} x_1 + \frac{K}{M_1} x_2 \\ \frac{dv_2}{dt} &= -\frac{K}{M_2} x_2 + \frac{K}{M_2} x_1 + \frac{1}{M_2} F(t) \end{aligned}\right\}$$

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{D}{M_1} & 0 & -\frac{K}{M_1} & \frac{K}{M_1} \\ 0 & 0 & \frac{K}{M_2} & -\frac{K}{M_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_2} \\ 0 \\ 0 \end{bmatrix} F(t)$$

$$x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ x_1 \\ x_2 \end{bmatrix}$$ ✓

Example



$$\left.\begin{aligned} \frac{v_1}{C} + i_1 R - \frac{v_2}{C} &= 0 \\ \frac{v_2}{C} + i_2 R &= 0 \end{aligned}\right\}$$

$i_2 = i_1 + \frac{dv_2}{dt}$

$\Rightarrow \frac{dv_2}{dt} = i_2 - i_1$

$$i_1 = \frac{dv_1}{dt} = \dot{v}_1$$

State variables are v_1, v_2

$$i_2 - i_1 = \frac{dv_2}{dt} = \dot{v}_2$$

$$i_2 = i_1 + \dot{v}_2 = \underline{\dot{v}_1 + \dot{v}_2}$$

✓ $R\dot{v}_1 = -\frac{1}{C}v_1 + \frac{1}{C}v_2$

✓ $R(\dot{v}_1 + \dot{v}_2) = -\frac{1}{C}v_2$

$$\underline{\begin{bmatrix} R & 0 \\ R & R \end{bmatrix}} \underline{\underline{\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}}} = \begin{bmatrix} -\frac{1}{C} & \frac{1}{C} \\ 0 & -\frac{1}{C} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix}\dot{v}_1 \\ \dot{v}_2\end{bmatrix} = \begin{bmatrix} R & 0 \\ R & R\end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{C} & \frac{1}{C} \\ 0 & -\frac{1}{C}\end{bmatrix}\begin{bmatrix} v_1 \\ v_2\end{bmatrix}$$

$$\begin{bmatrix} R & 0 \\ R & R\end{bmatrix}^{-1} = \frac{1}{R^2}\begin{bmatrix} R & -R \\ 0 & R\end{bmatrix}^{T} = \begin{bmatrix} \frac{1}{R} & 0 \\ -\frac{1}{R} & \frac{1}{R}\end{bmatrix}$$

Describe the ckt. in terms of v_1 and v_2.

$$\begin{bmatrix}\dot{v}_1 \\ \dot{v}_2\end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{CR} & \frac{1}{CR} \\ \frac{1}{CR} & -\frac{2}{CR}\end{bmatrix}}_{\bar{A}}\begin{bmatrix} v_1 \\ v_2\end{bmatrix}$$

$$\boxed{\dot{v} = \bar{A} v}$$

Describe the ckt in terms of i_1 and i_2.

$$\frac{1}{C}\frac{dv_1}{dt} + R\frac{di_1}{dt} - \frac{1}{C}\frac{dv_2}{dt} = 0$$

$$\frac{1}{C}\frac{dv_2}{dt} + R\frac{di_2}{dt} = 0$$

i_1 ; $i_2 - i_1$

$$\begin{cases} \dfrac{di_1}{dt} = -\dfrac{1}{CR} i_1 + \dfrac{1}{CR}(i_2 - i_1) = -\dfrac{2}{RC} i_1 + \dfrac{1}{RC} i_2 \\[2ex] \dfrac{di_2}{dt} = \dfrac{1}{RC} i_1 - \dfrac{1}{RC} i_2 \end{cases}$$

$$\begin{bmatrix}\dot{i}_1\\ \dot{i}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-\frac{2}{RC} & \frac{1}{RC}\\ \frac{1}{RC} & -\frac{1}{RC}\end{bmatrix}}_{A}\begin{bmatrix}i_1\\ i_2\end{bmatrix}$$

$$\boxed{\frac{d}{dt} i = A\, i}$$

Relationship between $v = \begin{bmatrix}v_1\\ v_2\end{bmatrix}$ and $i = \begin{bmatrix}i_1\\ i_2\end{bmatrix}$

$$i_1 = -\frac{1}{RC} v_1 + \frac{1}{RC} v_2$$

$$i_2 = -\frac{1}{RC} v_2$$

$$\begin{bmatrix}i_1\\ i_2\end{bmatrix} = \underbrace{\begin{bmatrix}-\frac{1}{RC} & \frac{1}{RC}\\ 0 & -\frac{1}{RC}\end{bmatrix}}_{T}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

T is a nonsingular matrix.

$$\boxed{i = T v}$$

$$z = T_1 v$$

Any nonsingular matrix T_1, we have a new state vector $\underline{\underline{z}}$.

$$\frac{di}{dt} = T \frac{dv}{dt}$$

$$A\, i = T \frac{dv}{dt}$$

$$\underline{T \frac{dv}{dt} = A T v} \Rightarrow \frac{dv}{dt} = \underbrace{T^{-1} A T}_{\bar{A}}\, v$$

$$\boxed{\bar{A} = T^{-1} A T}$$

$$A = T \bar{A} T^{-1} \qquad\qquad T_1 \bar{A} T_1^{-1} = A'$$

Eigenvalues of A and $T^{-1} A T = \bar{A}$ are same.

State variables are not unique. We can have infinite number of representation by choosing different nonsingular matrix T. This is called similarity Transformation.

State-space equations

Assume that a MIMO system involves n integrators; it has r inputs $u_1, u_2, \cdots u_r$; m outputs $y_1, y_2, \cdots, y_m$. Then the system can be represented as

$$\text{State equation}\left\{\begin{array}{l}
\text{System equation}\left\{\begin{array}{l}
\dot{x}_1(t) = f_1(x_1, \cdots, x_n; u_1, \cdots u_r), \quad x(0) \\
\quad\vdots \\
\dot{x}_n(t) = f_n(x_1, \cdots, x_n; u_1, \cdots u_r)
\end{array}\right. \\
\text{Output equation}\left\{\begin{array}{l}
y_1(t) = g_1(x_1, \cdots, x_n, u_1, \cdots u_r) \\
\quad\vdots \\
y_m(t) = g_m(x_1, \cdots, x_n; u_1, \cdots u_r)
\end{array}\right.
\end{array}\right.$$

Note: For time-varying system if in $\dot{x} = f(x, u, t)$

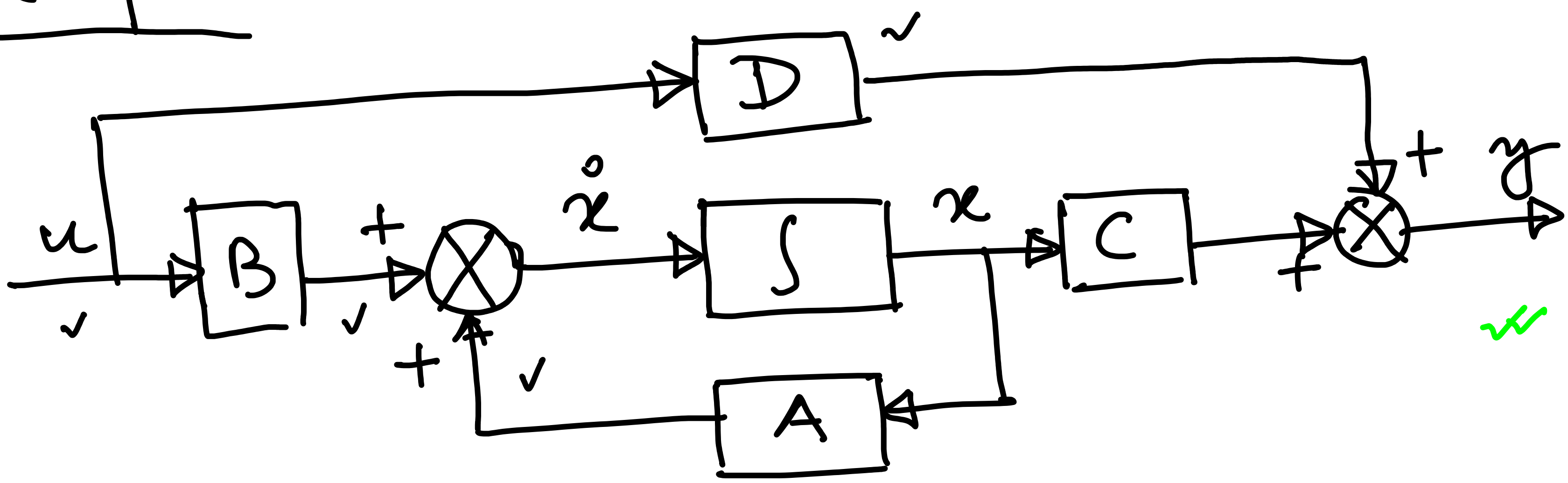
When the system is LTI, the state equation are as follow:

$$\underset{n\times 1}{\dot{x}(t)} = \underset{n\times n}{A}\,\underset{n\times 1}{x(t)} + \underset{n\times r}{B}\,\underset{r\times 1}{u(t)}, \quad x(0)$$

$$\underset{m\times 1}{y(t)} = \underset{m\times n}{C}\,\underset{n\times 1}{x(t)} + \underset{m\times r}{D}\,\underset{r\times 1}{u(t)}$$

$A \in \mathbb{R}^{n\times n}$ (system matrix) $\quad B \in \mathbb{R}^{n\times r}$ (input matrix) $\quad C \in \mathbb{R}^{m\times n}$ (output matrix) $\quad D \in \mathbb{R}^{m\times r}$ (feedforward matrix)

Block-diagram of LTI system represented in state-space



State-space to transfer function

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad x(0)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Taking Laplace transform, we get

$$s\,X(s) - x(0) = A\,X(s) + B\,U(s)$$
$$Y(s) = C\,X(s) + D\,U(s)$$

I = Identity matrix,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{n \times n}{(sI} - \underset{n \times n}{A)}\,X(s) = x(0) + B\,U(s)$$

$$\boxed{X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\,U(s)}$$

$$Y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}B\,U(s) + D\,U(s)$$

Assume all initial condition = 0.

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)}\,U(s)$$

$G(s)$ Transfer function

Example:

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2u(t)$$

Taking Laplace transform and all initial condition are zero,

$$s^2 Y(s) + 5s Y(s) + 6 Y(s) = 2U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{2}{s^2+5s+6} \quad \checkmark$$

Let $x_1 = y$ and $x_2 = \dot{x}_1 = \dot{y}$.

$$\dot{x}_2 = -5x_2 - 6x_1 + 2u$$

$$\dot{x}_1 = x_2$$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \quad \checkmark$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + 0 \cdot u$$

$$A = \underset{2\times 2}{\begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

$$G(s) = C\underline{(sI - A)^{-1}}B + D$$

$$(sI-A) = s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix} = \begin{bmatrix}s+5 & 6\\ -1 & s\end{bmatrix}$$

$$\underline{(sI-A)^{-1}} = \frac{1}{\underbrace{s(s+5)+6}_{\Delta}}\begin{bmatrix}s & -6\\ 1 & s+5\end{bmatrix}$$

$$G(s) = \begin{bmatrix}0 & 1\end{bmatrix} \times \frac{1}{\Delta}\begin{bmatrix}s & -6\\ 1 & s+5\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix} + 0$$

$$= \frac{1}{\Delta}\begin{bmatrix}1 & s+5\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix} = \frac{2}{\Delta}$$

$$= \frac{2}{s^2+5s+6}$$

Solution of state equation

$$X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}B\,U(s)$$

$$x(t) = \mathcal{L}^{-1}X(s) = \underbrace{\mathcal{L}^{-1}(sI-A)^{-1}}_{e^{At}}x(0) + \mathcal{L}^{-1}\underbrace{(sI-A)^{-1}B\,U(s)}$$

$$\mathcal{L}^{-1}[g_1(s)\;\; g_2(s)]$$

Convolution

$$= g_1(t) * g_2(t)$$

$$\boxed{\mathcal{L}^{-1}(sI-A)^{-1} = e^{At}}$$

$$\boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B \underset{=0}{u(\tau)}\, d\tau}$$

$$y(t) = C\,x(t) + D\,u(t)$$

$$y(t) = C\,e^{At}x(0) + C\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau + D\,u(t)$$

State transition matrix

Suppose $u(t) \equiv 0$.

$$x(t) = e^{At}x(0)$$

$$\dot{x}(t) = A\,x(t)$$

$$sX(s) - x(0) = A\,X(s)$$

$$(sI-A)X(s) = x(0)$$

$$X(s) = (sI-A)^{-1}x(0)$$

$$x(t) = \mathcal{L}^{-1}\underbrace{(sI-A)^{-1}}\,x(0)$$

$$= e^{At}x(0)$$

$$\frac{d\,x(t)}{dt} = \frac{d}{dt}\left[e^{At}x(0)\right]$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$\frac{dx(t)}{dt} = A e^{At} x(0) = A x(t)$$

State transition matrix satisfies the system equation.

$$\dot{x}(t) = A x(t)$$

$$\phi(t) = e^{At}$$

$$\frac{d\phi(t)}{dt} = \frac{d}{dt}\left(e^{At}\right) = A e^{At} = A\phi(t)$$

$$\boxed{\frac{d\phi(t)}{dt} = A\,\phi(t)}$$

$$\boxed{x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau.}$$

$t_0 = 0$

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$u(t)$ is the unit step.

Find the state-transition matrix.

$$\mathcal{L}^{-1}(sI-A)^{-1} = e^{At}$$

$$(sI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)} = \frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{s(s+3)+2} = \frac{\begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}}{(s+2)(s+1)}$$

$$= \begin{pmatrix} \frac{s+3}{(s+2)(s+1)} & \frac{1}{(s+2)(s+1)} \\ -\frac{2}{(s+2)(s+1)} & \frac{s}{(s+2)(s+1)} \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{\frac{2}{s+1} - \frac{1}{s+2}} & \underbrace{\frac{1}{s+1} - \frac{1}{s+2}} \\ \underbrace{-\frac{2}{s+1} + \frac{2}{s+2}} & \underbrace{-\frac{1}{s+1} + \frac{2}{s+2}} \end{pmatrix}$$

$$\mathcal{L}^{-1}(sI-A)^{-1} = e^{At} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$= e^{At}x(0) + \int_0^t \overset{2\times 2}{\begin{pmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{pmatrix}} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \, d\tau.$$

$$= e^{At}x(0) + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At}\underline{\underline{x(0)}} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}}_{\text{Solution of state equation.}}$$