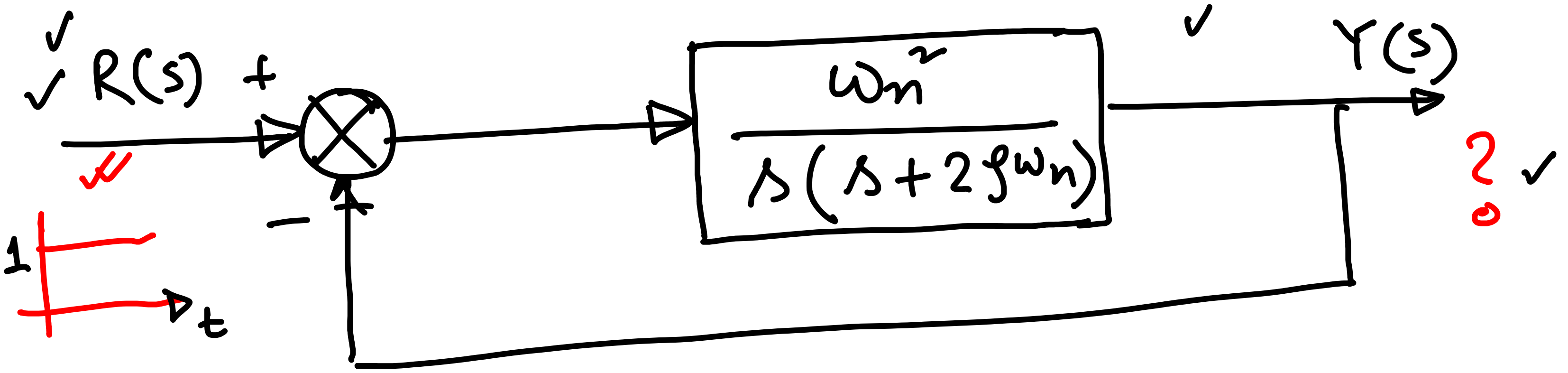
Time-domain analysis

Second-order system



$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

normalized second-order TF.

ω_n – natural frequency (rad/s)

ζ – damping ratio $= \dfrac{\text{Actual damping factor}}{\text{Critical damping factor}}$

$\zeta\omega_n$

$\zeta = 1 \quad \sigma = \zeta\omega_n$

The dynamic behavior of second-order system is characterized by two parameters ζ and ω_n.

Ch. eq^n:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

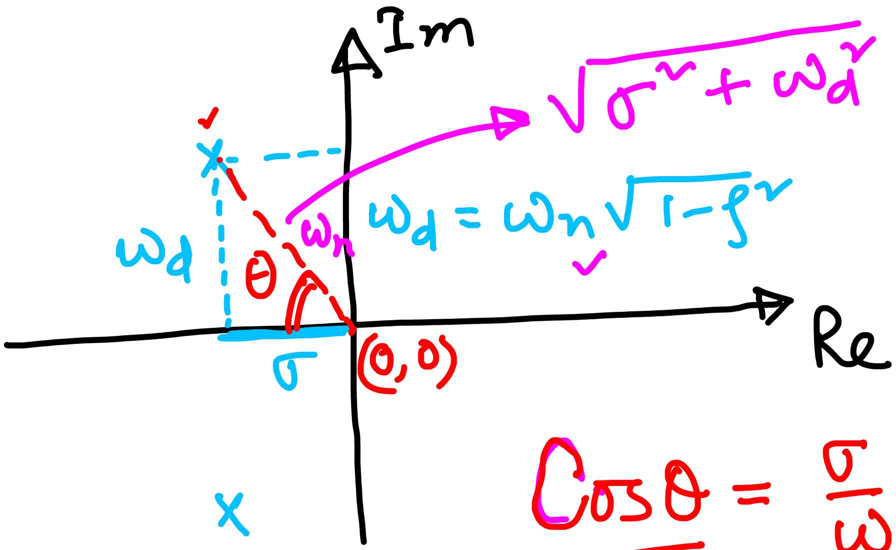
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$\sigma = \zeta\omega_n$ – damping factor.

$$s_{1,2} = -\sigma \pm j\omega_d$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ – damping frequency (rad/s)



$$\sqrt{\sigma^2+\omega_d^2} = \sqrt{\zeta^2\omega_n^2+\omega_n^2(1-\zeta^2)} = \omega_n$$

$$\cos\theta = \frac{\sigma}{\omega_n} = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

Unit step response

(1) Underdamped Case $(0<\zeta<1)$

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

$$R(s) = \frac{1}{s}$$

$$Y(s) = \frac{\omega_n^2}{s(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right),\ t \geq 0$$

(2) Undamped Case ($\zeta = 0$)

$$y(t) = 1 - \cos\omega_n t\ ,\quad t \geq 0$$

(3) Critically damped Case ($\zeta = 1$)

Since $$\lim_{\zeta\to 1}\frac{\sin\omega_n\sqrt{1-\zeta^2}\,t}{\sqrt{1-\zeta^2}} = \omega_n t$$

$$y(t) = 1 - e^{-\omega_n t}(1+\omega_n t)\ ,\quad t \geq 0.$$

(4) Overdamped Case ($\zeta > 1$)

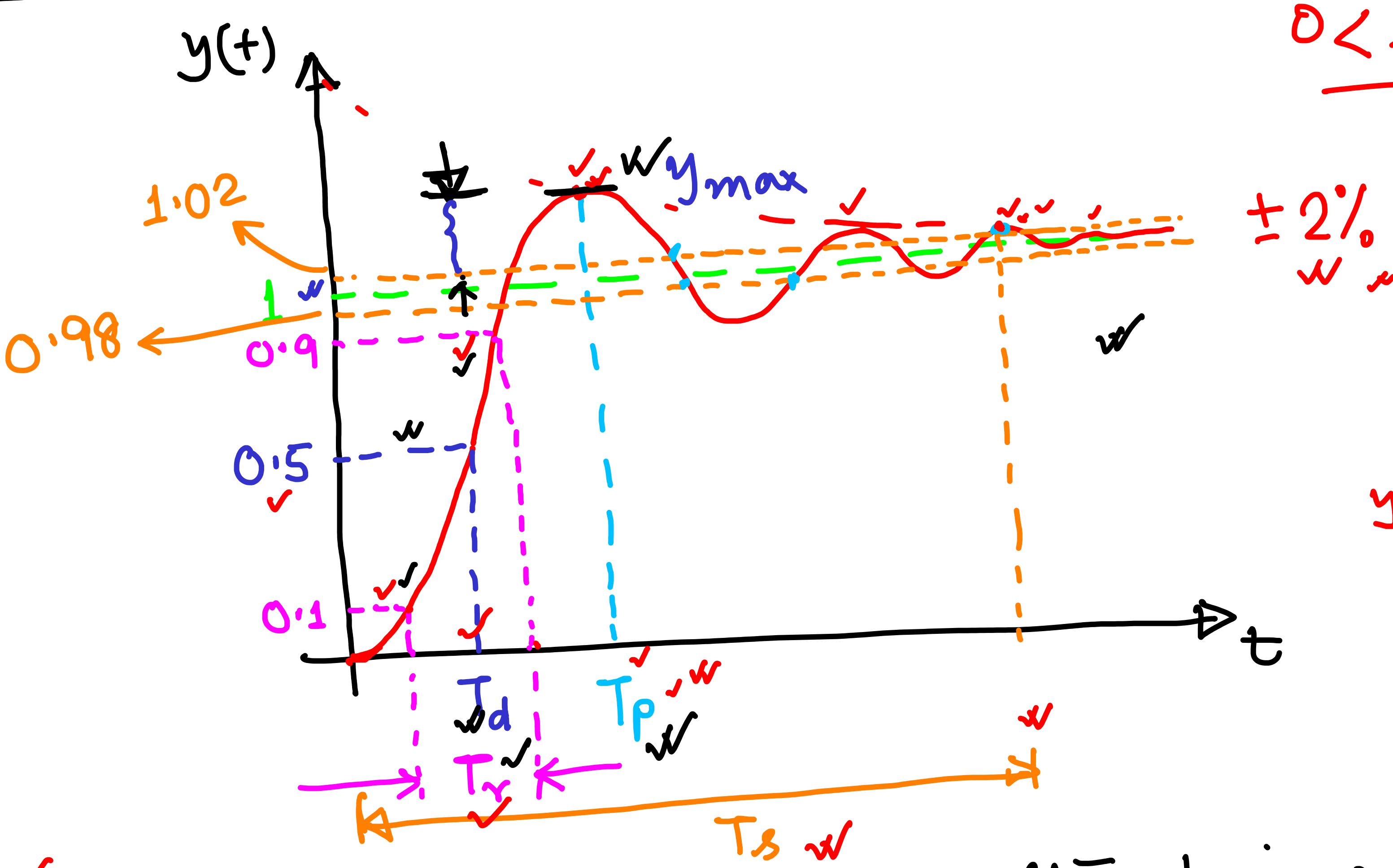
$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right),\ t \geq 0$$

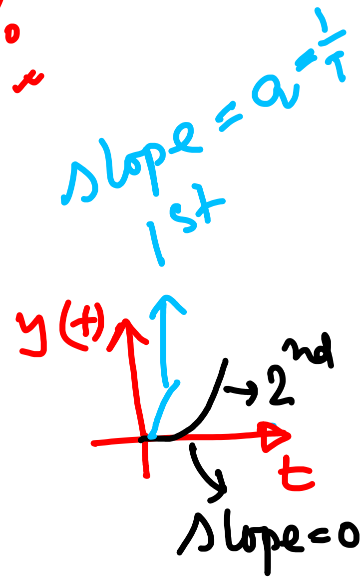
$$s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n \quad \text{and}$$

$$s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n.$$

Second-order underdamped response

$0 < \zeta < 1$





Delay time (T_d) is defined as the time required for the step response to reach 50% of its final value.

Rise time (T_r) is defined as the time required to go from 10% of the final value to 90% of the final value.

Peak time (T_p): The time required to reach the first peak.

Settling time (T_s) is defined as the time required for the transient's damped oscillation to reach and stay within $\pm 2\%$ of the steady-state value.

Evaluation of T_p:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\frac{dy(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\, t = 0$$

$$\Rightarrow \quad \omega_n \sqrt{1-\zeta^2}\, t = n\pi, \quad n = 0, 1, \dots$$

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

- At $t=0$, the slope is zero. This is the difference with first order step response.
- At $t = T_p$ ($n=1$), $\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$

% of overshoot (% OS): $\dfrac{y_{max} - 1}{1} \times 100 = \% OS$

$$y_{max} = y(T_p) = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\boxed{\% OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100}$$

Settling time $\boxed{T_s \approx \dfrac{4}{\zeta \omega_n}}$ $\pm 2\%$

Delay time $\boxed{T_d \approx \dfrac{1 + 0.7\zeta}{\omega_n}}$

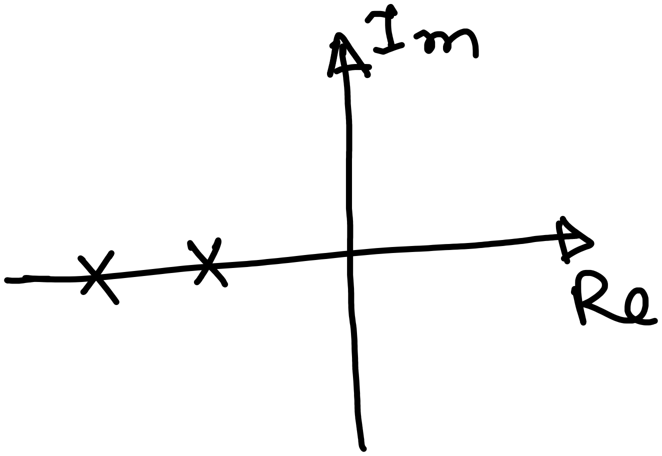
Rise time $\boxed{T_r \approx \dfrac{0.8 + 2.5\zeta}{\omega_n}}$

Step response corresponding to various pole location:

$$s_{1,2} = -\sigma \pm j\omega_d = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2}$$

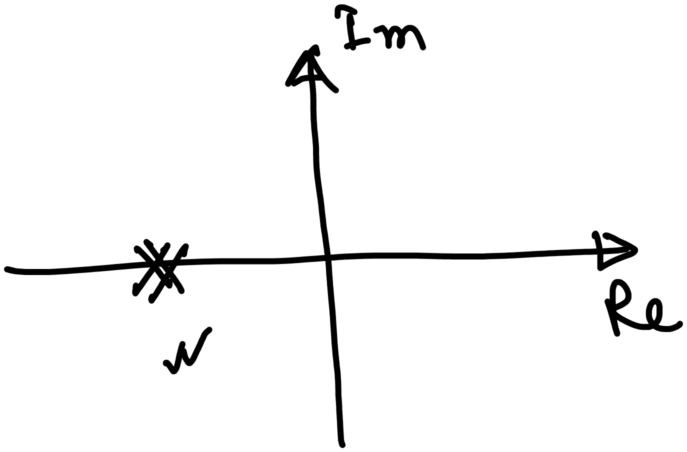
$\pm j\omega_n$

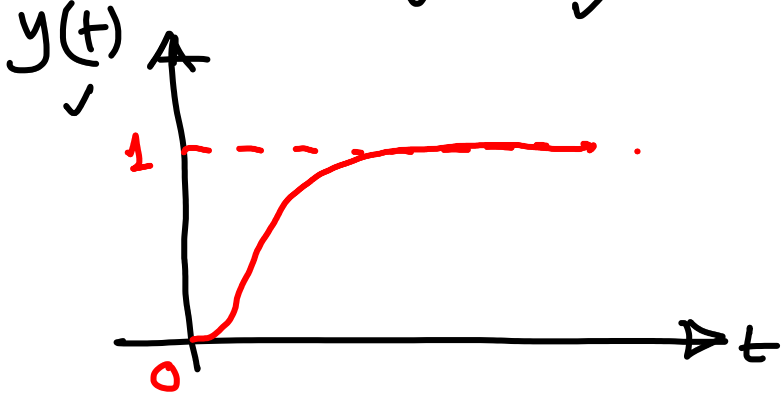
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}, \quad \zeta>1$$



over-damped case

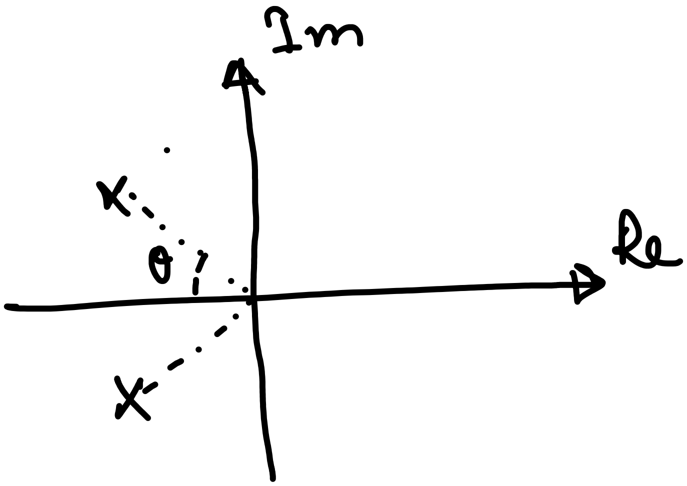
$\zeta = 1$

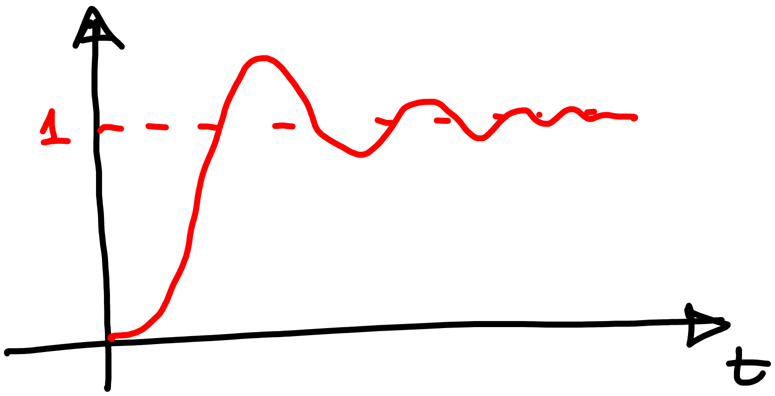




Critically-damped Case

$0 < \zeta < 1$

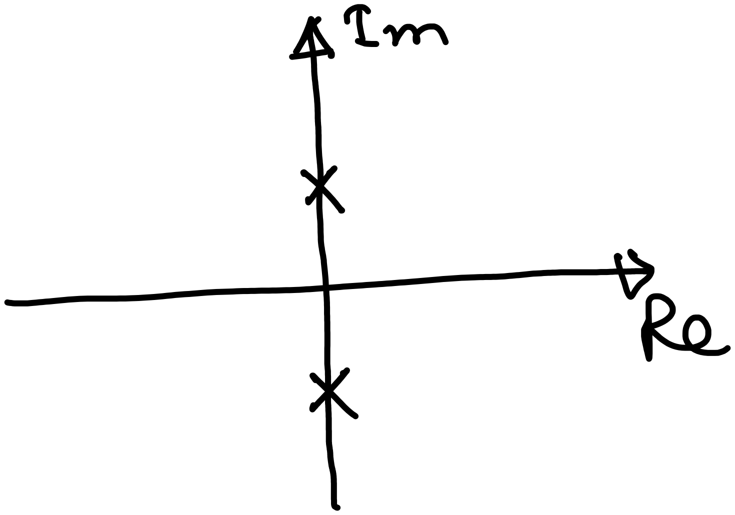


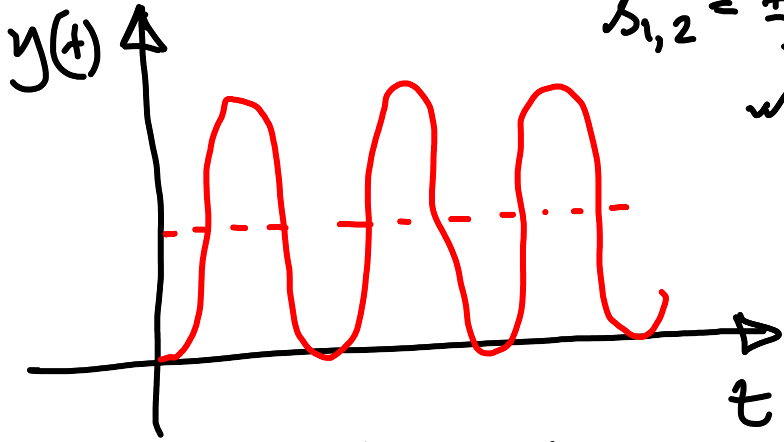


under-damped response

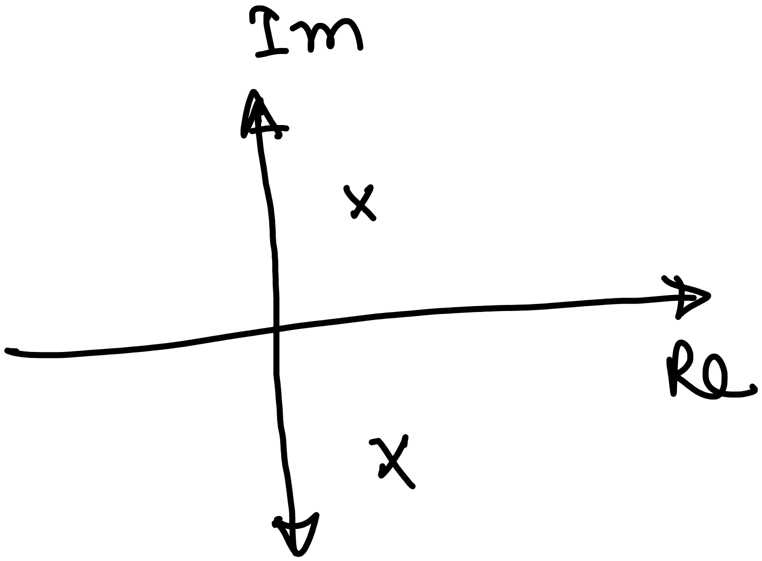
$\zeta = 0.$

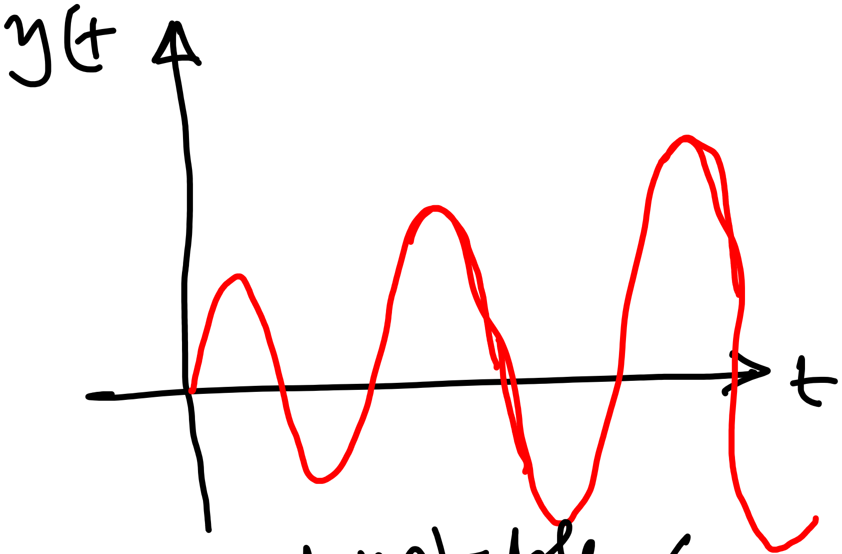
$s_{1,2} = \pm j\omega_n$





undamped case





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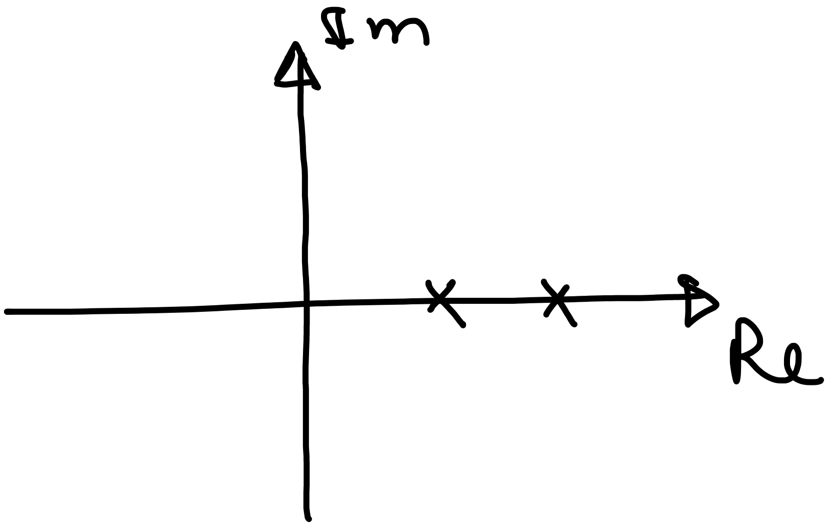

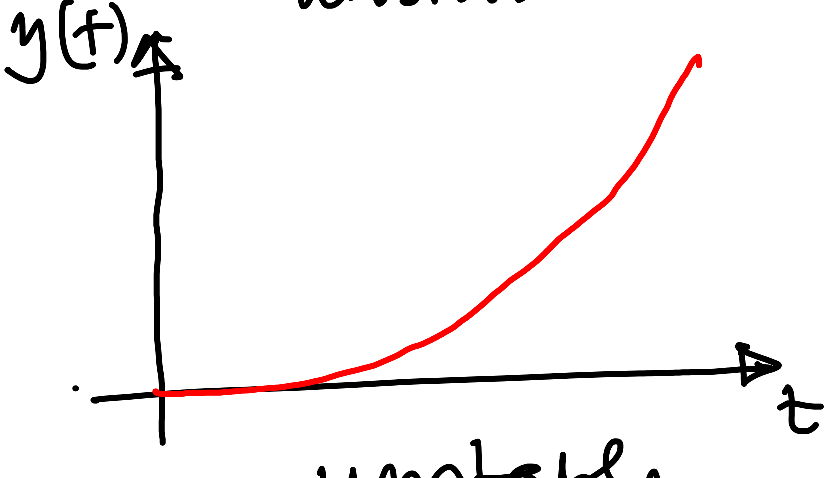

unstable