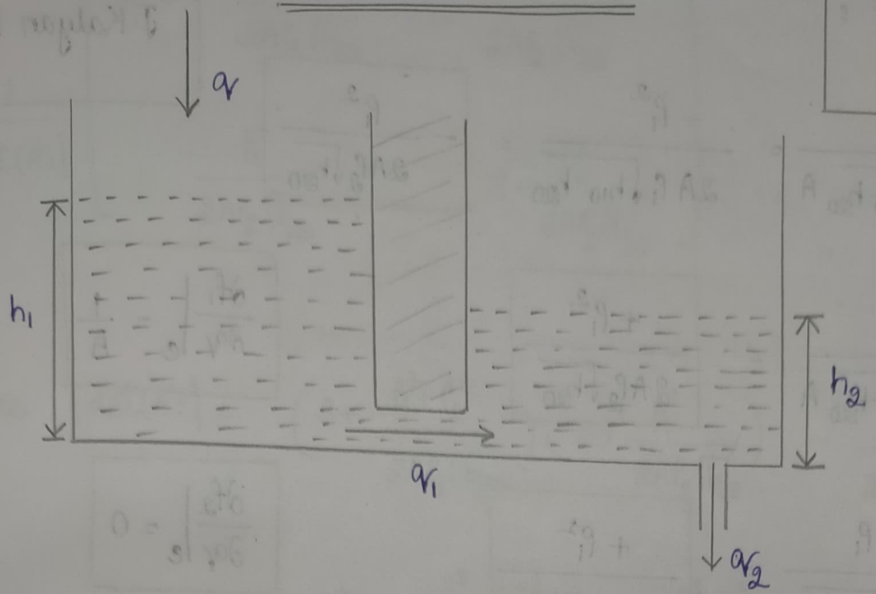
ASSIGNMENT-2

17EE35004
J.Kalyan Raman

1 (i)



$$q_1 = P_1\sqrt{h_1-h_2}$$

$$q_2 = P_2\sqrt{h_2}$$

$$A\frac{dh_1}{dt} = q - q_1 = q - P_1\sqrt{h_1-h_2}$$

$$A\frac{dh_2}{dt} = q_1 - q_2 = P_1\sqrt{h_1-h_2} - P_2\sqrt{h_2}$$

(a) State variables : h_1, h_2 , Input $= q$, Output $= h_2 = y$

(b)

EQUILIBRIUM : Let h_2 at equilibrium be h_{20}.
q at equilibrium be q_0

$$\frac{dh_2}{dt} = 0 \Rightarrow \quad P_1\sqrt{h_{10}-h_{20}} = P_2\sqrt{h_{20}}$$

$$P_1^2(h_{10}-h_{20}) = P_2^2 h_{20}^2$$

$$\boxed{h_{10} = \frac{P_1^2+P_2^2}{P_1^2}h_{20}}$$

$$\frac{dh_1}{dt} = 0 \Rightarrow \quad q_0 = P_1\sqrt{h_{10}-h_{20}} = P_2\sqrt{h_{20}} \Rightarrow \boxed{q_0 = P_2\sqrt{h_{20}}}$$

17EE 35004

J. Kalyan Raman

Incremental T.F :

$$\left.\frac{\partial f_1}{\partial h_1}\right|_e = \frac{-\rho_1}{2\sqrt{h_{10}-h_{20}}\,A} = \frac{-\rho_1^2}{2A\rho_1\sqrt{h_{10}-h_{20}}} = \boxed{-\frac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}}}$$

$$\left.\frac{\partial f_1}{\partial h_2}\right|_e = \frac{+\rho_1}{2\sqrt{h_{10}-h_{20}}\,A} = \boxed{\frac{+\rho_1^2}{2A\rho_2\sqrt{h_{20}}}} \qquad \boxed{\left.\frac{\partial f_1}{\partial q}\right|_e = \frac{1}{A}}$$

$$\left.\frac{\partial f_2}{\partial h_1}\right|_e = \frac{+\rho_1}{2A\sqrt{h_{10}-h_{20}}} = \boxed{\frac{+\rho_1^2}{2A\rho_2\sqrt{h_{20}}}} \qquad \boxed{\left.\frac{\partial f_2}{\partial q}\right|_e = 0}$$

$$\left.\frac{\partial f_2}{\partial h_2}\right|_e = \frac{-\rho_1}{2A\sqrt{h_{10}-h_{20}}} - \frac{\rho_2}{2A\sqrt{h_{20}}} = \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} - \frac{\rho_2}{2A\sqrt{h_{20}}} = \boxed{-\left(\frac{\rho_1^2+\rho_2^2}{\rho_2}\right)\cdot\frac{1}{2A\sqrt{h_{20}}}}$$

$$\frac{d}{dt}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & \frac{+\rho_1^2}{2A\rho_2\sqrt{h_{20}}}\\ \frac{+\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & \frac{-(\rho_1^2+\rho_2^2)}{2A\rho_2\sqrt{h_{20}}}\end{bmatrix}}_{\text{“A”}}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{1}{A}\\ 0\end{bmatrix}}_{\text{“B”}} q$$

$$\Delta y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\Delta h_2\\ \Delta h_1\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\end{bmatrix}}_{\text{“C”}}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix}$$

$$G(s) = C(sI-A)^{-1}B$$

$$(sI-A)^{-1} = \begin{bmatrix} s+\frac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}}\\ \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & s+\frac{\rho_1^2+\rho_2^2}{2A\rho_2\sqrt{h_{20}}}\end{bmatrix}^{-1}$$

17EE35004

J. Kalyan Raman

$$= \frac{1}{|\det(A)|}\begin{bmatrix} s+\dfrac{\rho_1^2+\rho_2^2}{2A\rho_2\sqrt{h_{20}}} & \dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} \\ \dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & s+\dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} \end{bmatrix}$$

where $|\det(A)| = \left(s+\dfrac{\rho_1^2+\rho_2^2}{2A\rho_2\sqrt{h_{20}}}\right)\left(s+\dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}}\right) - \dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}}\cdot\dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}}$

$$= s^2+\frac{\rho_1^2+2\rho_2^2}{2A\rho_2\sqrt{h_{20}}}+\frac{\rho_2^2\rho_1^2}{4A\rho_2^2 h_{20}}$$

$$= s^2+\frac{\rho_1^2+2\rho_2^2}{2A\rho_2\sqrt{h_{20}}}+\frac{\rho_1^2}{4Ah_{20}}.$$

$$C(sI-A)^{-1}B = \frac{1}{|\det(A)|}\cdot\begin{bmatrix}0 & 1\end{bmatrix}\cdot A\cdot\begin{bmatrix}\frac{1}{A}\\ 0\end{bmatrix}$$

$$= \frac{\dfrac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}}\cdot\dfrac{1}{A}\cdot 1}{|\det(A)|}.$$

$$\boxed{G(s) = \frac{\dfrac{\rho_1^2}{2A^2\rho_2\sqrt{h_{20}}}}{s^2+\dfrac{\rho_1^2+2\rho_2^2}{2A\rho_2\sqrt{h_{20}}}+\dfrac{\rho_1^2}{4Ah_{20}}}}$$

17EE35004

J. Kalyan Raman

(c)

$$A \frac{d}{dt}(\Delta h_1) = q_0 - \frac{\rho_1^2}{2\rho_2\sqrt{h_{20}}}(\Delta h_1 - \Delta h_2)$$

$$A \frac{d}{dt}(\Delta h_2) = \frac{\rho_1^2}{2\rho_2\sqrt{h_{20}}}(\Delta h_1 - \Delta h_2) - \frac{\rho_2}{2\sqrt{h_{20}}}(\Delta h_2).$$

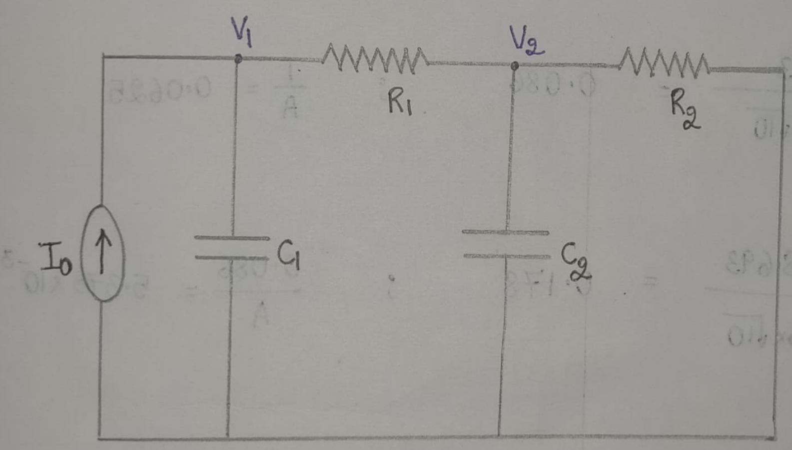
Let $\Delta h_1 = V_1$ $\quad q_0 = I_0$ $\quad \frac{2\rho_2\sqrt{h_{20}}}{\rho_1^2} = R_1$

$\Delta h_2 = V_2$ $\quad A = C_1 = C_2$

$\frac{2\sqrt{h_{20}}}{\rho_1} = R_2$

$\Rightarrow$

$$\boxed{\begin{array}{l} C_1 \frac{dV_1}{dt} = I_0 - \frac{V_1 - V_2}{R_1} \\ C_2 \frac{dV}{dt} = \frac{V_1 - V_2}{R_1} - \frac{V_2}{R_2} \end{array}}$$



17EE 35004
J. Kalyan Raman

(ii) Given $h_{20} = 10\,cm$ $\quad d_o = 0.5\,cm$ $\quad \rho = 1 g/cc$

$A = 16 cm^2$ $\quad g = 980\,cm/s^2$

$$a_1 = a_2 = \frac{\pi d_o^2}{4} = \frac{\pi \times 0.25}{4} cm^2 = 0.1963\,cm^2$$

$q = K_m v$: $\quad K_m = 5 cm^3/s/V$

$$P_1 = a_1\sqrt{2g} = 0.1963\sqrt{2\times980} = 8.693\ cm^{5/2}s^{-1}$$

$$P_2 = a_2\sqrt{2g} = 0.1963\sqrt{2\times980} = 8.693\ cm^{5/2}s^{-1}$$

$$h_{10} = \frac{P_1^2+P_2^2}{P_1^2} h_{20} = 2h_{20} = 20\,cm$$

$$q_o = P_2\sqrt{h_{20}} = 8.693\sqrt{10}\ cm^3 s^{-1} = 27.49\ cm^3/s$$

$$v_o = \frac{q_o}{K_m} = \frac{27.49}{5} V = 5.498\,V$$

$$\frac{P_1^2}{2AP_2\sqrt{h_{20}}} = \frac{8.693}{2\times16\times\sqrt{10}} = 0.086 \quad ; \quad \frac{1}{A} = 0.0625$$

$$\frac{P_1^2+P_2^2}{2AP_2\sqrt{h_{20}}} = \frac{2\times8.693}{2\times16\times\sqrt{10}} = 0.178 \quad ; \quad \frac{0.086}{A} = 5.375\times10^{-3}$$

$$A = \begin{bmatrix} -0.086 & +0.086 \\ +0.086 & -0.178 \end{bmatrix} ; \quad B = \begin{bmatrix} 0.0625 \\ 0 \end{bmatrix} ; \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$G(s) = \frac{5.375\times10^{-3}}{s^2 + 0.258s + 0.0074}$$

17EE35004

J. Kalyan Raman

Non-Linear model:

$$A \frac{dh_1}{dt} = q - P_1\sqrt{h_1 - h_2}$$

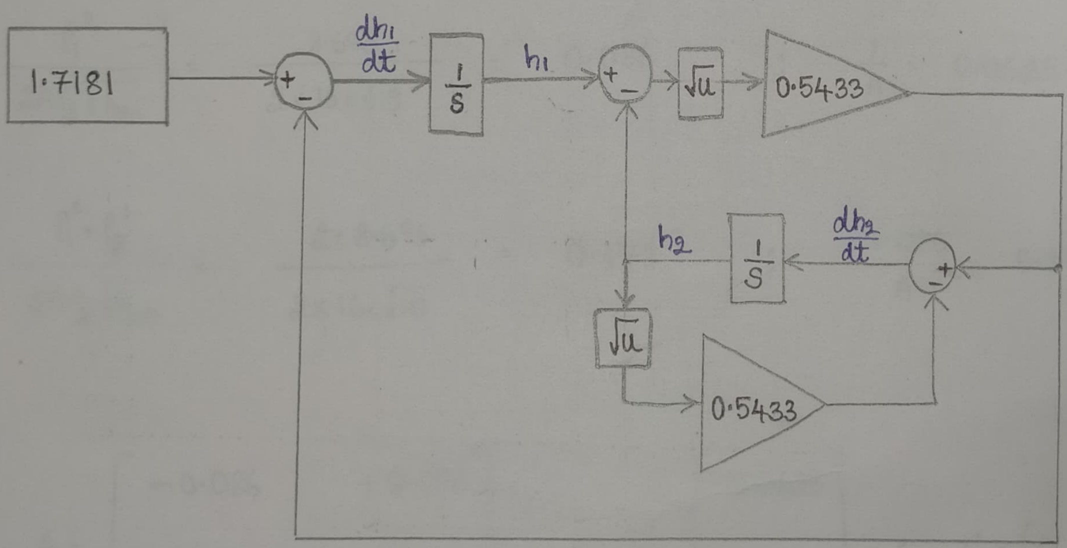
$$16 \frac{dh_1}{dt} = 27.49 - 8.693\sqrt{h_1 - h_2}$$

$$\Rightarrow \boxed{\frac{dh_1}{dt} = 1.7181 - 0.5433\sqrt{h_1 - h_2}}$$

$$A \frac{dh_2}{dt} = P_1\sqrt{h_1 - h_2} - P_2\sqrt{h_2}$$

$$16 \frac{dh_2}{dt} = 8.693\sqrt{h_1 - h_2} - 8.693\sqrt{h_2}$$

$$\Rightarrow \boxed{\frac{dh_2}{dt} = 0.5433\sqrt{h_1 - h_2} - 0.5433\sqrt{h_2}}$$



17EE 35004

J. Kalyan Raman

Linear model :

$I_0 = \Delta q$ $\qquad C_1 = C_2 = A = 16$

$V_1 = \Delta h_1$

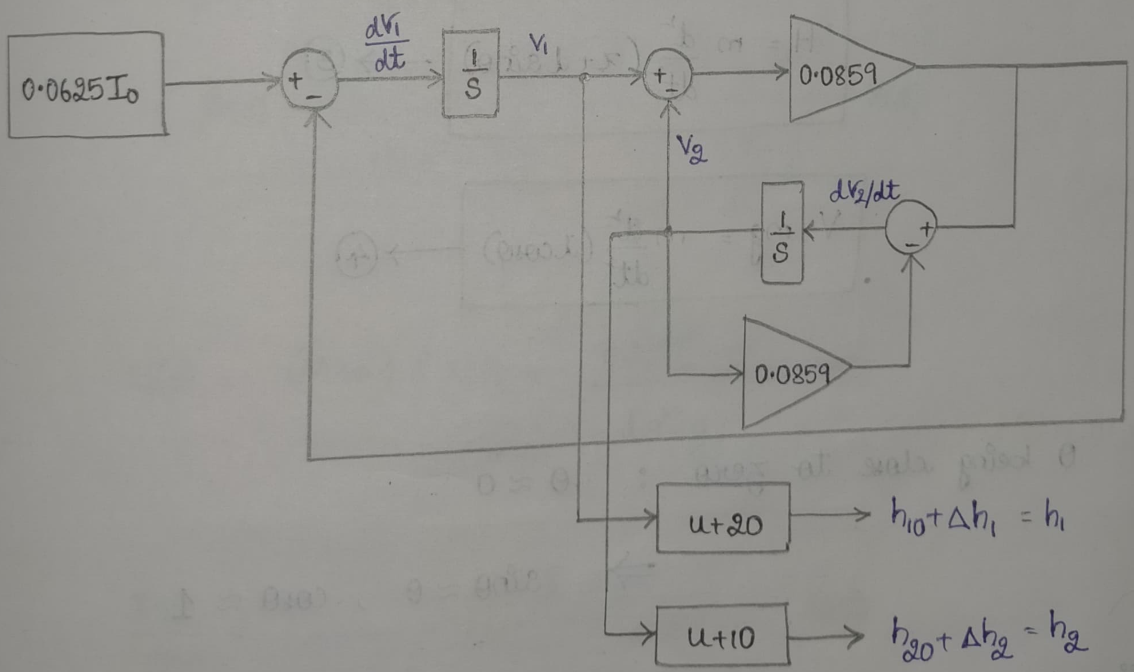
$V_2 = \Delta h_2$ $\qquad R_1 = R_2 = \dfrac{2q_0}{P_1^2} = \dfrac{2 \times 27.49}{8.693^2} = 0.7276$

$$C_1 \frac{dV_1}{dt} = I_0 - \frac{V_1 - V_2}{R}$$

$$16 \frac{dV_1}{dt} = I_0 - \frac{V_1 - V_2}{0.7276} \Rightarrow \boxed{\frac{dV_1}{dt} = 0.0625\, I_0 - 0.0859\,(V_1 - V_2)}$$

$$C_2 \frac{dV_2}{dt} = \frac{V_1 - V_2}{R_1} - \frac{V_2}{R_2}$$

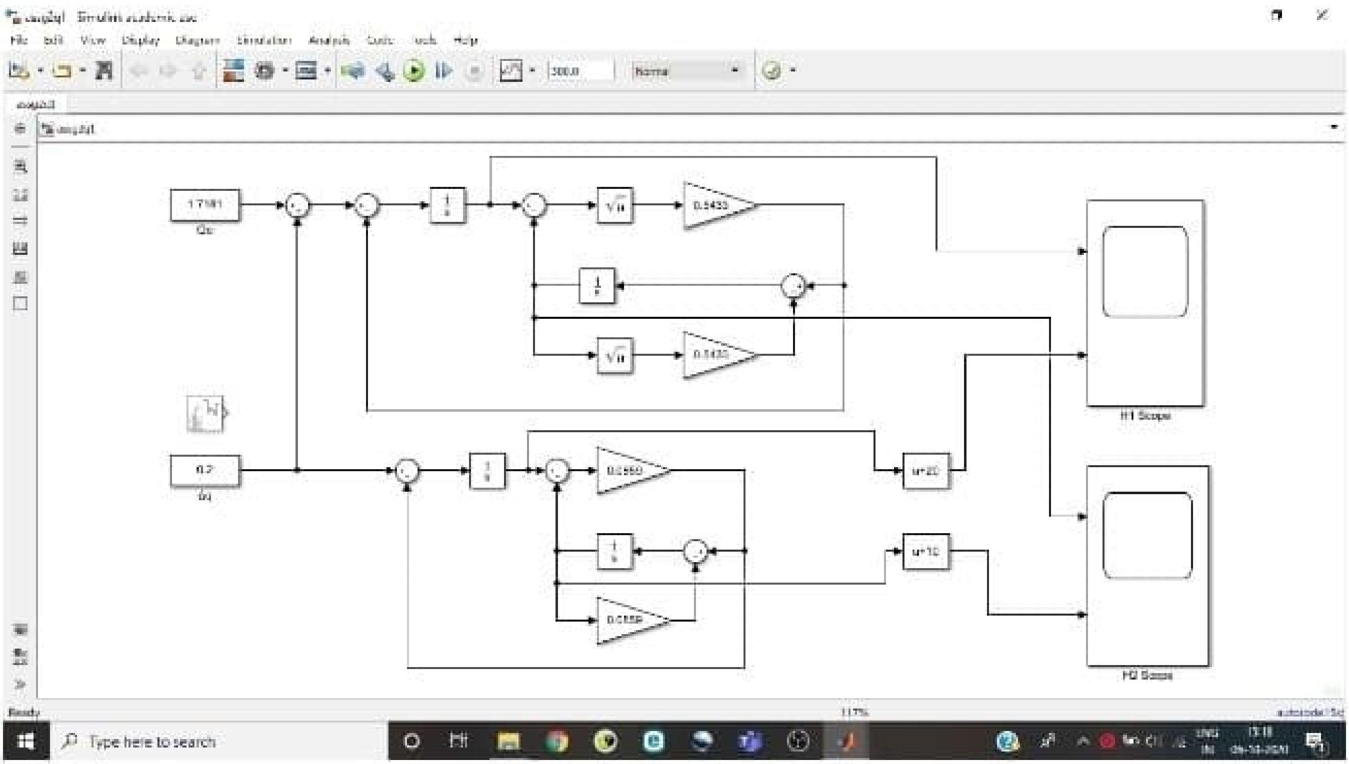
$$16 \frac{dV_2}{dt} = \frac{V_1 - V_2}{0.7276} - \frac{V_2}{0.7276} \Rightarrow \boxed{\frac{dV_2}{dt} = 0.0859(V_1 - V_2) - 0.0859\, V_2}$$



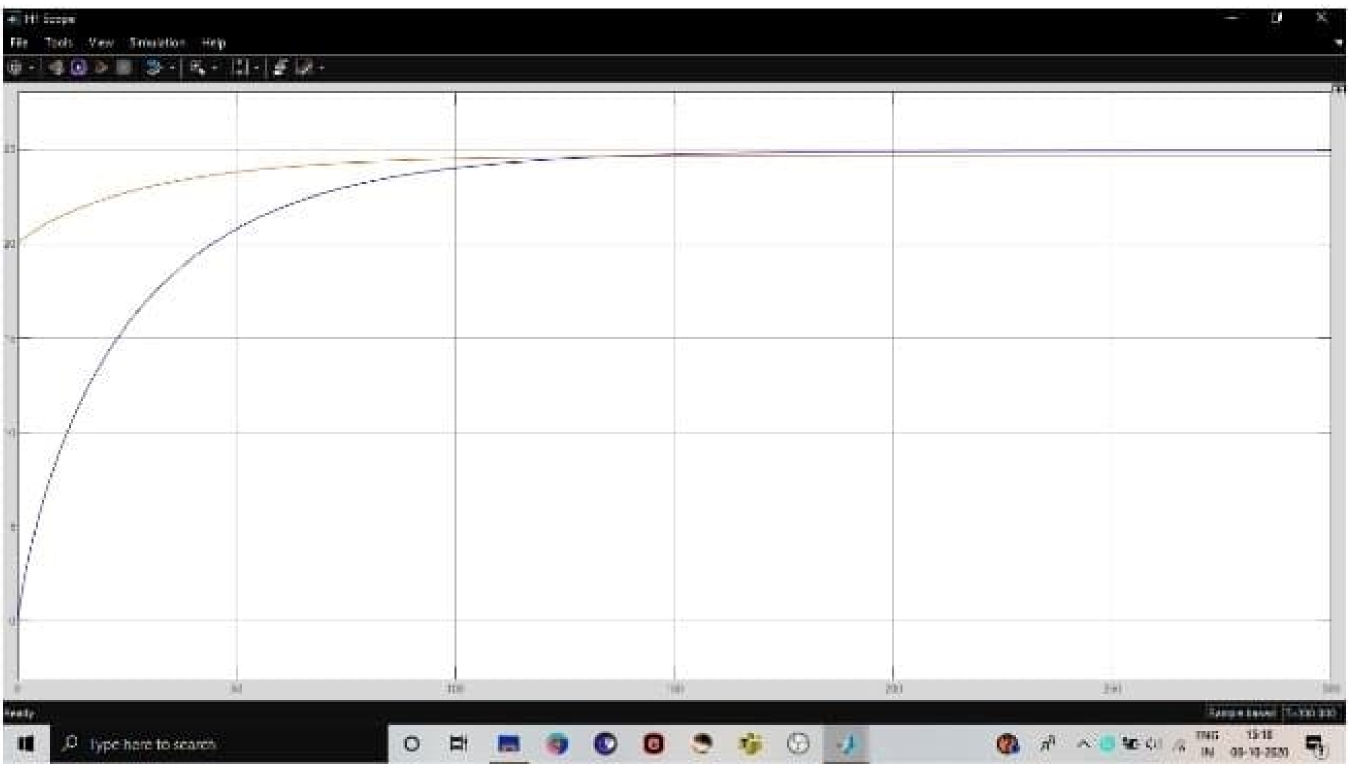
17EE35004
J. Kalyan Raman

Assignment 2 :

Question 1 : Non-linear(above) and Linear(below) models

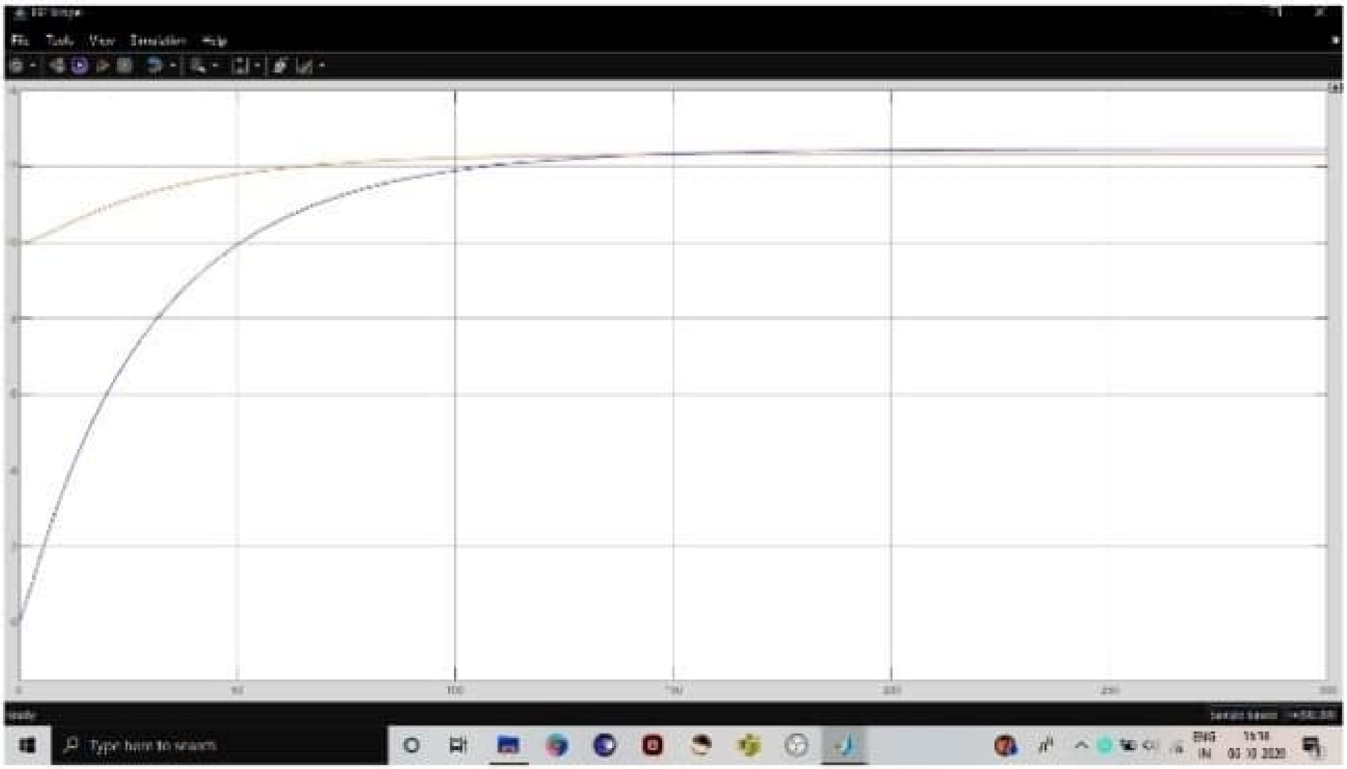


H1 scope

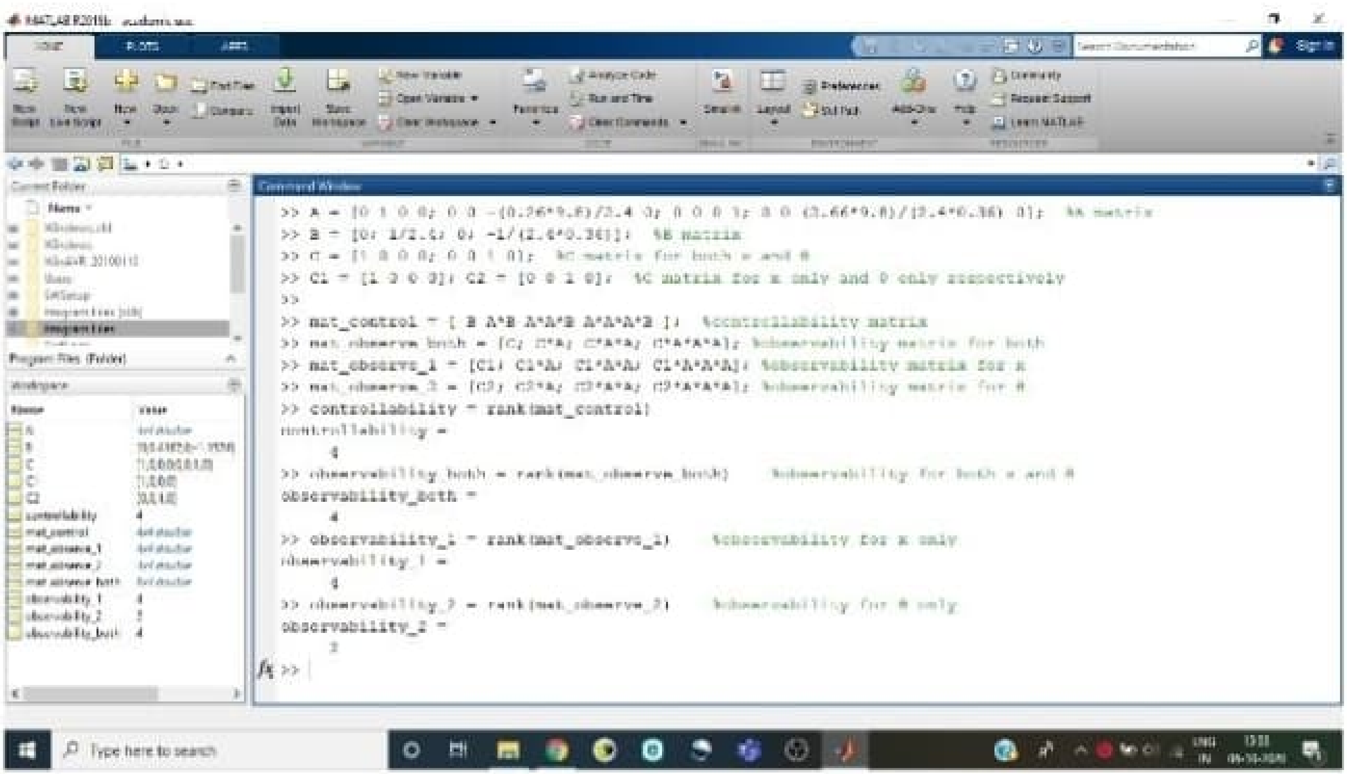


17EE35004
J. Kalyan Raman

H2 Scope :



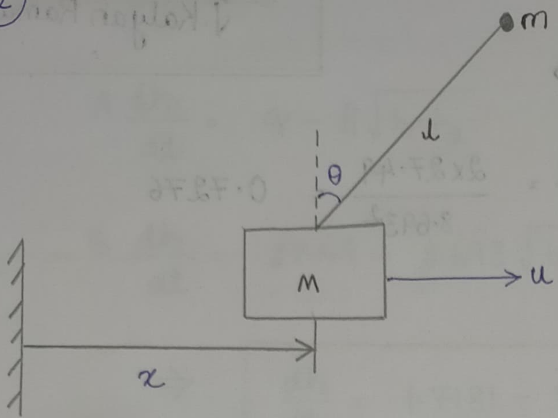
Question 2 : Controllability and Observability



17EE35004

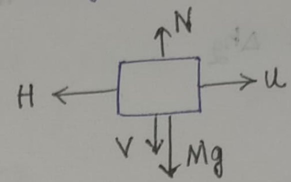
J. Kalyan Raman

(2)

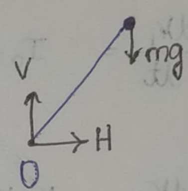


:

FBD of M.



FBD of (m + rod)



Centre of mass of m is at

$(x + l\sin\theta,\ l\cos\theta)$

moment of intertia of m abouts its CM be I.

Torque equation : $\boxed{I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta} \longrightarrow (1)$

Force equation : $\boxed{M\ddot{x} = u - H} \longrightarrow (2)$

$$\boxed{H = m\frac{d^2}{dt^2}(x + l\sin\theta)} \longrightarrow (3)$$

$$\boxed{V - mg = m\frac{d^2}{dt^2}(l\cos\theta)} \longrightarrow (4)$$

(a) θ being close to zero : $\theta \approx 0$

$\Rightarrow \sin\theta \approx \theta$, $\cos\theta \approx 1$

$I\ddot{\theta} = Vl\theta - Hl$

$H = m\ddot{x} + ml\ddot{\theta}$

$V = mg + 0 = mg$

17EE35004

J. Kalyan Raman

$$\Rightarrow u = m\ddot{x} + ml\ddot{\theta} + M\ddot{x}$$

$$\boxed{u = (M+m)\ddot{x} + ml\ddot{\theta}} \longrightarrow (5)$$

$$\Rightarrow I\ddot{\theta} = mgl\theta - (m\ddot{x} + ml\ddot{\theta})\,l$$

$$(ml^2 + I)\ddot{\theta} = mgl\theta - ml\ddot{x}$$

here $I \ll ml^2$

$$\Rightarrow ml^2\ddot{\theta} = mgl\theta - ml\ddot{x}$$

$$\boxed{ml\ddot{\theta} = mg\theta - m\ddot{x}} \longrightarrow (6)$$

Applying Laplace Transform: assuming initial conditions are relaxed

$$U(s) = (M+m)s^2X(s) + mls^2\Phi(s)$$

$$s^2 ml\Phi(s) = mg\Phi(s) - ms^2X(s)$$

$$\Phi(s) = \frac{-ms^2X(s)}{mls^2 - mg} = \frac{-s^2}{ls^2 - g}X(s)$$

$$U(s) = (M+m)s^2X(s) + \frac{mls^2(-s^2)}{ls^2 - g}X(s)$$

$$\frac{X(s)}{U(s)} = \frac{ls^2 - g}{s^2\left[(M+m)(ls^2 - g) - mls^2\right]} \cancel{X(s)}$$

$$= \frac{ls^2 - g}{s^2\left[Mls^2 - (M+m)g\right]}$$

17EE 35004
J. Kalyan Raman

$$\boxed{\frac{X(s)}{U(s)} = \frac{\frac{1}{M}\left(s^2 - \frac{g}{l}\right)}{s^2\left(s^2 - \frac{(M+m)g}{Ml}\right)}}$$

$$\Rightarrow \quad \frac{\Phi(s)}{U(s)} = \frac{-s^2}{l\cancel{\left(s^2 - \frac{g}{l}\right)}} \cdot \frac{\frac{1}{M}\cancel{\left(s^2 - \frac{g}{l}\right)}}{s^2\left(s^2 - \frac{(M+m)g}{Ml}\right)}$$

$$\boxed{\frac{\Phi(s)}{U(s)} = \frac{-\frac{1}{Ml}}{s^2 - \frac{(M+m)g}{Ml}}}$$

(b)

$$(M+m)\ddot{x} + ml\ddot{\theta} = u \longrightarrow (7)$$

$$\ddot{x} + l\ddot{\theta} = g\theta. \longrightarrow (8)$$

(7) $- m \times$ (8) : $\quad M\ddot{x} = u - mg\theta.$

$$\boxed{\ddot{x} = -\frac{mg}{M}\theta + \frac{u}{M}}$$

(7) $- (M+m) \times$ (8) : $\quad -Ml\ddot{\theta} = u - (M+m)g\theta$

$$\boxed{\ddot{\theta} = \frac{(M+m)g}{Ml}\theta - \frac{u}{Ml}}$$

State Variables : $x, \dot{x}, \theta, \dot{\theta}$, Output $= \begin{bmatrix} x \\ \theta \end{bmatrix}$

17EE 35004

J. Kalyan Raman

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/Ml \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$m = 0.26\,kg$, $M = 2.4\,kg$, $l = 0.36\,m$

From matlab : Rank of $[B \;\; AB \;\; A^2B \;\; A^3B]$ is 4

Rank of $\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$ is 4.

$\Rightarrow$ Observability = 4

Controllability = 4.

$\therefore$ Pendulum can be balanced using • ~~only~~ x & θ feedback

• only x feedback.

Because for $C = [1 \;\; 0 \;\; 0 \;\; 0]$ $\Rightarrow$ observability = controllability = 4

$C = [0 \;\; 0 \;\; 1 \;\; 0]$ $\Rightarrow$ observability = 2

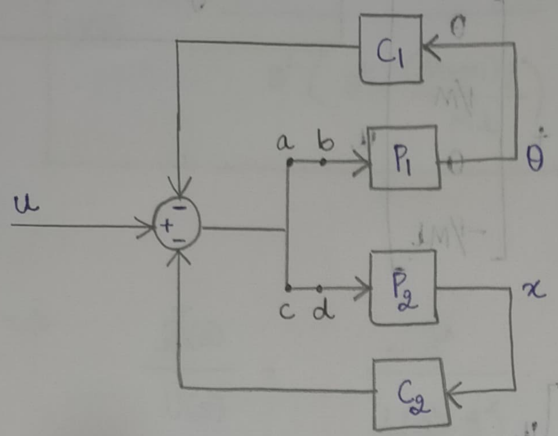
controllability = 4.

There is a pole zero cancellation in $\Phi(s)$ which reduced a rank of 2.

17EE35004

J. Kalyan Raman

(c)



$$P_1 = \frac{k_1}{s^2-a^2}$$

$$P_2 = \frac{k_2(s^2-b^2)}{s^2(s^2-a^2)}$$

Break the loop btw a & b : $\xrightarrow{a \quad b}$ and btw c & d : $\xrightarrow{c \quad d}$

$$\text{Loop gain}_1 = \frac{P_1C_1}{1+P_2C_2} \quad (\theta)$$

$$= \frac{\frac{k_1}{s^2-a^2} \cdot C_1}{1 + C_2 \cdot \frac{k_2(s^2-b^2)}{s^2-a^2}}$$

$$= \frac{k_1C_1}{(s^2-a^2)+k_2C_2(s^2-b^2)}$$

$$\text{Loop gain}_2 = \frac{P_2C_2}{1+P_1C_1} \quad (x)$$

$$= \frac{\frac{k_2(s^2-b^2)}{s^2(s^2-a^2)} C_2}{1+C_1\frac{k_1}{s^2-a^2}}$$

$$= \frac{k_2(s^2-b^2)C_2}{s^2[s^2-a^2+k_1C_1]}$$

It is evident from the expressions.

Loop gain_2 has a right hand ~~pole~~ plane zero : $s=+b$. Therefore we have a robustness issue in case of x feedback.

Whereas Loop gain_1 doesn't have any right half plane zero, so there is no robustness issue in case of θ feedback.