Digital Signal Processing
End Semester Exam

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Q 1. (a) $\left(x[n] - x[n] * h_1[n]\right) * h_2[n] = y[n]$

($*$ denotes convolution)

$$y[n] = x[n] * \left(h_2[n] - h_1[n] * h_2[n]\right)$$

Impulse Response of System =

$$h_2[n] - h_1[n] * h_2[n]$$

Frequency Response = $\mathcal{F}$ (Impulse Response)

$= h_2(e^{j\omega}) - \mathcal{F}(h_1[n] * \mathcal{F}(h_2[n]))$

$= H_2(e^{jw}) - \mathcal{F}(h_1[n]) * H_2(e^{jw})$

Here, $h_1(n) = \delta(n-1) \Rightarrow \mathcal{F}(h_1[n]) = e^{-j\omega}$.

& $H_2(e^{jw}) = \begin{cases} 1 & |\omega| \leq \pi/2 \\ 0 & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}$

$\therefore$ Overall frequency Response

$= \begin{cases} 1 - e^{-j\omega} & |\omega| \leq \pi/2 \\ 0 & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}$

$\therefore H(e^{j\omega}) = \begin{cases} 1 - e^{-j\omega} & |\omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}$

Unit Sample Response

$h[n] = \begin{cases} \delta[n] - \delta[n-1] & : |\omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}$

Q1: (b)

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega\times 10}}$$

$$x[n] \xrightarrow{DTFT} \frac{1}{1-\frac{1}{3}e^{-j10\omega}}$$

$$x[-n] \xrightarrow{DTFT} \frac{1}{1-\frac{1}{3}e^{-j10\omega}}$$

$$\text{if } x'[-n] \xrightarrow{DTFT} \frac{1}{1-\frac{1}{3}e^{j\omega}}$$

$x[-n]$ is 10 times unsampled $x'[-n]$.

$$\therefore x'[-n] = \left(\frac{1}{3}\right)^n u[n]$$

$$x'[n] = 3^n u[-n]$$

$$x[n] = \begin{cases} 3^{n/10} u\left[-\frac{n}{10}\right] & \text{when } \frac{n}{10} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Q2: (a) frequencies to be moved

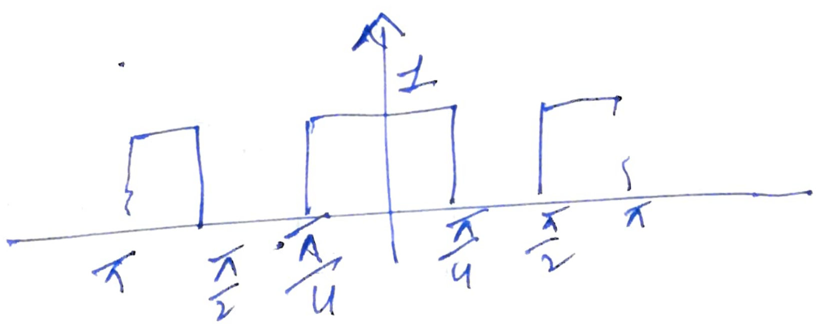
$5kHz \leq f \leq 10kHz$.

Maximum frequency in signal $\Rightarrow 20kHz$.

Sampling frequency as per the Nyquist criteria should be more than twice of signal to avoid aliasing.

$\therefore$ $\boxed{\text{frequency} \geq 40kHz}$

A bandstop is required at frequencies $w = \frac{w_0}{4}, \frac{w_0}{2}$, here $w_0 = 20kHz$ (max freq. in signal)



Q2: (b)

$f_s = 2kHz, \Rightarrow \omega_s = 4000\pi$

$x_a(t) = \sin(1000\pi t)$. $\omega_s > 2\times(1000\pi)$

$$H(e^{j\omega}) = \frac{2(\frac{1}{3} - e^{-j\omega})}{1-\frac{1}{3}e^{-j\omega}}$$

Since sampling rate is ~~lower~~ more than the nyquist rate the system will behave as if $H(e^{j\omega})$ is discretely being applied to the input signal in continuous domain.

$$|H(e^{j\omega})| = \frac{2|\frac{1}{3}-e^{-j\omega}|}{|1-\frac{1}{3}e^{-j\omega}|}$$

$$= \frac{2\sqrt{(\frac{1}{3}-\cos\omega)^2+\sin^2\omega}}{\sqrt{(1-\frac{1}{3}\cos\omega)^2+\frac{\sin^2\omega}{9}}}$$

at $\omega = 1000\pi$

$$|H(e^{j\omega})| = \frac{2\sqrt{(\frac{1}{3}-1)^2+0}}{\sqrt{(1-\frac{1}{3})^2}} = \frac{2\times\frac{2}{3}}{\frac{2}{3}} = 2$$

$\therefore y_a(t) = 2\sin(1000\pi t + \phi)$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{\frac{1}{3}-\cos\omega}\right) - \tan^{-1}\left(\frac{\frac{1}{3}\sin\omega}{1-\frac{1}{3}\cos\omega}\right)$$

at $\omega = 1000\pi$

$\angle H(e^{j\omega}) = \tan^{-1}0 - \tan^{-1}0 = 0$

$$\boxed{\therefore y_a(t) = 2\sin 1000\pi t}$$

Q2 (a).

$$x_1[n] = \left(\frac{1}{6}\right)^n u[n]$$

$$y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n\right] u[n]$$

$$X_1[z] = \frac{z}{z-\frac{1}{6}}$$

$$Y_1[z] = \frac{az}{z-\frac{1}{2}} + \frac{10z}{z-\frac{1}{3}}$$

$$H(z) = \frac{Y_1[z]}{X_1[z]}$$

$$Y_1(z) = \frac{2az}{2z-1} + \frac{30z}{3z-1}$$

$$= \frac{6z^2(a+10) - 2z(a+15)}{6z^2 - 5z + 1}$$

$$H(z) = \frac{6z^2(a+10) - 2z(a+15)}{6z^2-5z+1} \times \frac{6z-1}{6z}$$

$$= \frac{[3z(a+10) - (a+15)](6z-1)}{3(6z^2-5z+1)}$$

for,

$h_2 = (-1)^n$; ROC : $|z| = 1$

$$H(1) = \frac{5\times[3(a-10) - (a+15)]}{3\times 2} = 7/4$$

$$\Rightarrow \frac{5(3a-30-a-15)}{3\times 2} = 7/4$$

$$\Rightarrow (2a-45)\times\frac{5}{6} = 7/4$$

$$\Rightarrow 2a-45 = \frac{21}{10}.$$

$$\Rightarrow 2a = \frac{471}{10}$$

$$\Rightarrow a = \frac{471}{20}.$$

$\cancel{H(z) =}$

Q. 3. (b) $\quad x_2[n] = \left(\frac{1}{2}\right)^n u[n]$

$$y_2[n] = \delta[n] + a\left(\frac{1}{4}\right)^n u[n]$$

$$X_2[z] = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}.$$

$$Y_2(z) = 1 + \frac{az}{z-\frac{1}{4}}.$$

$$= \frac{(4+4a)z-1}{4z-1}.$$

$$H(z) = \frac{Y_2(z)}{X_2(z)} = \frac{(4+4a)z-1}{4z-1} \times \frac{2z-1}{2z}$$

$$= \frac{(4(1+a)z-1)(2z-1)}{2z(4z-1)}$$

when, $x_1[n] = (-2)^n$

$$|z| \in \left(\frac{1}{2}, 2\right).$$

$$X_1(z) = \frac{z}{z+2} - \frac{z}{z+\frac{1}{2}}$$

$$= \frac{1-4z}{(z+2)(2z+1)}$$

$$Y_1[z] = H(z)\cdot X_1[z]$$

$$0 = \frac{(4(1+a)z-1)(2z-1)}{2z(4z-1)} \times \frac{(1-4z^{-1})}{(z+2)(2z+1)}$$

Q 40(a)(i).

$$H_{cap}(z) = e^{j\beta \frac{N}{k}} \frac{(z^{-1} - p_k^{*})}{1 - \alpha z^{-1}}$$

$$\tau_{gd}(\omega) = \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha \cos(\omega - \phi_k)}$$

Since real pole $|\alpha| < 1$ & $\phi_k = 0$.

$$\tau_{gd} = \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos\omega}$$

(ii) Numerator is positive.

For denominator let it is less than 0 for some, ω.

then,

$$\cos\omega \geq \frac{1+\alpha^2}{2\alpha}.$$

from AM-GM inequality

$AM \geq GM$

$$\frac{1+\alpha^2}{2} \geq \sqrt{\alpha^2}$$

$$1+\alpha^2 \geq 2\alpha$$

$$\frac{1+\alpha^2}{2\alpha} \geq 1$$

then,

$\cos\omega > 1$

which is not possible.

as $\boxed{-1 \leq \cos\omega \leq 1}$

So, the ~~all~~ group delays is non-negative for all ω.

(b). $y[n] = \alpha^n x[n]$.

$$Y(z) = X\left(\frac{z}{\alpha}\right)$$

$$= \frac{\left(1 - \frac{3\alpha}{2} z^{-1}\right)\left(1 + \frac{\alpha}{3} z^{-1}\right)\left(1 + \frac{5\alpha}{2} z^{-1}\right)}{\left(1 - \alpha z^{-1}\right)^2 \left(1 - \frac{\alpha}{4} z^{-1}\right)}$$

all poles & zeros should lie inside unit circle.

$\left|\frac{3\alpha}{2}\right| < 1$; $\left|\frac{\alpha}{3}\right| < 1$; $\left|\frac{5\alpha}{2}\right| < 1$; $|\alpha| < 1$; $\left|\frac{\alpha}{4}\right| < 1$

$\Rightarrow |\alpha| < \frac{2}{3}$; $|\alpha| < 3$; $|\alpha| < \frac{2}{5}$; $|\alpha| < 1$; $|\alpha| < 4$

Intersection is $\boxed{|\alpha| < \frac{2}{3}}$

Q5: (i) $H(z) = 1 - z^{-1} + z^{-2}$

let $t = z^{-1}$

$t^2 - t + 1 = 0$

$t = \frac{1 \pm \sqrt{-3}}{2}$

$= \frac{1 \pm j\sqrt{3}}{2}$

$z^{-1} = \frac{1 \pm j\sqrt{3}}{2} = e^{j\pi/3}, e^{-j\pi/3}$

$\boxed{w_0 = \frac{\pi}{3}}$

The notch frequency $w_0 = \frac{\pi}{3}$

It will ~~suppress~~ supress frequencies of $\frac{\pi}{3}$.

$\cos\left(\frac{\pi}{3} n\right) u[n] = \frac{e^{j\frac{\pi}{3} n} + e^{-j\pi/3\, n}}{2}$

$\cos\left(\frac{\pi}{3} n\right) u[n] = \frac{1 - \cos\frac{\pi}{3} z^{-1}}{1 - 2\cos\left(\frac{\pi}{3}\right) z^{-1} + z^{-2}}$

$= \frac{1 - \frac{z^{-1}}{2}}{1 - z^{-1} + z^{-2}}$

$X[z] = \frac{1 - \frac{z^{-1}}{2}}{1 - z^{-1} + z^{-2}}$

(ii) $Y[z] = X[z]H[z]$

$$= \left(1 - \frac{z^{-1}}{2}\right)$$

final value theorem of causal system,

$$\lim_{n\to\infty} y[n] = \lim_{z\to 1} (z-1)\,Y(z)$$

$$= \lim_{z\to 1} (z-1)\left(1 - \frac{z^{-1}}{2}\right)$$

$$= 0$$

Since $\lim_{n\to\infty} y[n] = 0$, the steady state response is 0, when sinusoidal sequence is applied.

Q6. (a). Using matrix approach, we can write $h[n]$ as ~~an~~ 4x4 matrix form & $g[n]$ as a 1x4 column matrix.

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

$\therefore \; y_c(n) = x(n) \; (N) \; h(n) = \{6, 7, 6, 5\}$

(b). Verification by forward & Inverse DFT.

As we know

$Y[k] = H(k) G(k)$

$G(k) = \sum_{n=0}^{N-1} g(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots N-1$

Given $g(n) = \{1, 2, 0, 1\}, \; N = 4$

$G(0) = \sum_{n=0}^{3} g(n) = 4$

$G(1) = \sum_{n=0}^{3} g(n) e^{-\frac{n\pi j}{2}} = 1 - 2j + 0 + 1j$

$= \cancel{2-2j} \; (1-j).$

$G(2) = \sum_{n=0}^{3} g(n) e^{-n\pi j} = 1 - 2 + 0 + (-1)$

$= -2$

$G(3) = \sum_{n=0}^{3} g(n) e^{-\frac{3jn\pi}{2}} = 1 + 2j + 0 - 1j$

$= 1 + j.$

$4,\ 1-j,\ -2,\ 1+j.$

$$G(k) = \{ \cancel{4,\ 2-2j,\ -2,\ 2+2j} \}$$

Given $h(n) = \{2, 2, 1, 1\}$

$$H(0) = \sum_{n=0}^{3} h(n) = 6.$$

$$H(1) = \sum_{n=0}^{3} h(n) e^{-\frac{n\pi j}{2}} = 2 - 2j - 1 + 1j$$

$$= \cancel{2-2j}.\ (1-j).$$

$$H(2) = \sum_{n=0}^{3} h(n) e^{-n\pi j} = 2 - 2 + 1 - 1$$

$$= 0$$

$$H(3) = \sum_{n=0}^{3} h(n) e^{-\frac{3jn\pi}{2}} = 2 + 2j - 1 - 1j$$

$$= (1+j).$$

$$G(k) = \cancel{\{6,\ 2-2j,\ 0,\ 2+2j\}}\ \{6,\ 1-j,\ 0,\ 1+j\}.$$

$-\pi j$

NO.,

$$Y[k] = X[k] \cdot H[k]$$

$$= (24, -2j, 0, 2j)$$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] e^{j2\pi nk/N}, \quad n = 0, 1, \ldots N-1$$

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y[k] = \frac{24}{4} = 6$$

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{\frac{j\pi k}{2}}$$

$$= \frac{1}{4} \left(24 - 2j(j) + 0 - 2j(j) \right)$$

$$= \frac{1}{4} \left(24 + 2 + 2 \right) = 7$$

$$y(2) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\pi k} = \frac{1}{4} \left(24 + 2j + 0 - 2j \right)$$

$$= 6$$

$$y(3) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{\frac{3j\pi k}{2}} = \frac{1}{4} \left(24 - (2j)j + 0 + (2j)j \right)$$

$$= 5$$

$$\therefore \quad y(n) = (6, 7, 6, 5)$$

Verified from above obtained result.

Q7. (a). DFT matrix

$$W = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 \cdots & \omega^{2(N-1)} \\ \vdots & & & \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{bmatrix}$$

Prove $WW^* = W^*W = I$

for $i = j$ $\quad w_{ij}$ of W^*W

$$= \frac{1}{N}\sum_{k=0}^{N-1} \omega^{ik}\omega^{-ik} = \frac{1}{N}\sum_{0}^{N-1} 1 = \frac{N}{N} = 1$$

So, diagonal elements are 1.

for $i \neq j$

$$w_{ij} = \sum_{k=0}^{N-1} \omega^{(j-i)k}$$

let $\omega^{j-i} = \alpha \quad j-i \neq 0.$

$\boxed{\omega^{N} = 1}$

So, $\omega^{(j-i)N} = (1)^{j-i} = 1$

$= \alpha^N.$

$$w_{ij} = \sum_{k=0}^{N-1} \alpha^k = \frac{1-\alpha^N}{1-\alpha} \quad \left[\because \text{sum of GP} = \frac{a(1-r^n)}{1-r}\right]$$

$$= \frac{1-1}{1-\alpha} = 0$$

$\therefore$ Matrix is unitary.

That is why DFT matrix said to be a unitary matrix.

Q 7 (b). $X_N \rightarrow$ DFT of x_N

$x_N \rightarrow$ discrete signal

$X_N = W_N x_N$

$\cancel{x_N = W_N^{-1} X_N}$

$x_N = W_N^{-1} X_N$

In general the inverse of matrix requires $O(N^3)$ operations hence the IDFT should be $O(N^3)$

But we use the unitary property of DFT matrix W_N.

$$W_N^H W_N^{-1} = N I_N$$

$$W_N^{-1} = \frac{W_N^H}{N}$$

$$W_N^{-1} = \frac{1}{N} W_N^H$$

Since W_N is symmetrical $\boxed{W_N^T = W_N}$

$$\boxed{W_N^{-1} = \frac{1}{N} W_N^*}$$

So, the inverse is not directly calculated rather by taking conjugate of the terms.

Hence the IDFT goes to $O(N^2)$ operations.

Q7: (c) Assuming X & x related as,

$$X[n] = \sum_{k=0}^{N-1} x[k]\, e^{j2\pi kn/N}$$

Then,

$$|X[n]|^2 = \sum_{k=0}^{N-1} x[k] \cdot \sum_{k'=0}^{N-1} x^*[k']\, e^{j2\pi(k-k')n/N}.$$

$$\sum_{n=0}^{N-1} |X[n]|^2 = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} x[k] \sum_{k'=0}^{N-1} x^*[k']\, e^{j2\pi(k-k')n/N}$$

$$= \sum_{k=0}^{N-1} x[k] \sum_{k'=0}^{N-1} x^*[k'] \sum_{n=0}^{N-1} e^{j2\pi(k-k')n/N}.$$

$$=$$

Here, the 2nd term is a geometric series and can be written as.

$$\sum_{n=0}^{N-1} e^{j2\pi(k-k')n/N} = \frac{e^{j2\pi(k-k')} - 1}{e^{j2\pi(k-k')/N} - 1}$$

$$= \frac{\cos(2\pi(k-k')) - 1}{\cos(2\pi(k-k')/N) - 1}$$

Here, if $k \neq k'$, => RHS to be zero.

In that case we should be able to see that sum is simply N.

We can write.

$$\sum_{n=0}^{N-1} e^{j2\pi(k-k')n/N} = N\delta_{kk'}$$

where $\delta_{kk'} = 0$ when $k \neq k'$ & 1 when $k = k'$. Therefore.

$$\sum_{n=0}^{N-1} |x[n]|^2 = N\sum_{k=0}^{N-1} |X[k]|^2$$

and Parseval's theorem follows.