Assignment-2

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(a)

$$e(n) = d(n) - \sum_{k=0}^{M-1} w_k^* u(n-k)$$

where:

$u \rightarrow$ input

$e \rightarrow$ error

$d \rightarrow$ desired output

$w_k \rightarrow$ weights

$M \rightarrow$ order of filters

$$J = E\left[e(n) e^*(n')\right]$$

$$= E\left[|d(n)|^2\right] - \sum_{k=0}^{M-1} w_k^* E\left[u(n-k) d^*(n)\right]$$

$$- \sum_{k=0}^{M-1} w_k E\left[u^*(n-k) d(n)\right]$$

$$+ \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* w_i E\left[u(n-k) u^*(n-i)\right]$$

$\rightarrow E\left[|d(n)|^2\right] = \sigma_d^2$

$\rightarrow p(-k) = E\left[u(n-k) d^*(n)\right]$ } cross-correlation

$\rightarrow p^*(-k) = E\left[u^*(n-k) d(n)\right]$

$\rightarrow r(i-k) = E\left[u(n-k) u^*(n-i)\right]$ } auto-correlation

$$\boxed{J = \sigma_d^2 - \sum_{k=0}^{M-1} w_k^* p(-k) - \sum_{k=0}^{M-1} w_k p^*(-k) + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* w_i r(i-k)}$$

This represents the error performance surface of the FIR filter and the analytical expression is given here.

According to the given R & p, we substitute various terms.

$$\sigma_d^2 = E\left[w_0^H u(n) u^H(n) w_0\right]$$

$$= w_0^H R w_0$$

Also, $w_0 = R^{-1}p$

$$\sigma_d^2 = (R^{-1}p)^H R (R^{-1}p) = p^H R^{-1} p.$$

$$J(w) = \sigma_d^2 - w^H p - p^H w + w^H R w$$

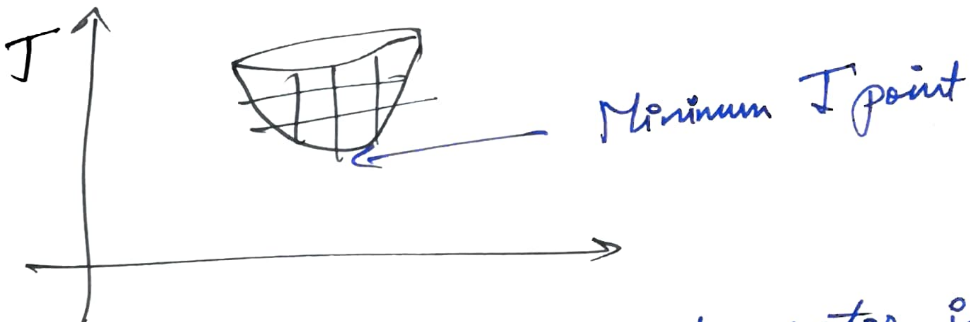
$$R = \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

we get,

$$J = 0.5 - \begin{bmatrix} 0. & 0.2939 \end{bmatrix} \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

$$+ \left(w - \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}\right)^H \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix} \left(w - \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.2939 \end{bmatrix} \right)$$

(b) The cost function J is bowl shaped curve characterized by filter tap weights $\omega_0, \omega_1, \ldots \omega_{M-1}$. Since the error surface is bowl shaped, it is characterized by a unique minimum.



At the minimum, the gradient vector is 0.

Here. $\nabla_k J = 0 \quad ; \quad k = 0, 1, \ldots M-1$

let $\omega_k = a_k + j b_k$.

$$\nabla_k J = \frac{\partial J}{\partial a_k} + j \frac{\partial J}{\partial b_k}$$

$$J = \sigma_d^2 - \sum_{k=0}^{M-1} \omega_k^* p(-k) - \sum_{k=0}^{M-1} \omega_k p^*(-k) + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} \omega_k^* \omega_i r(i-k)$$

$$\frac{\partial J}{\partial a_k} = 0 - p(-k) - p^*(-k) + \sum_{i=0}^{M-1} \omega_i r(i-k)$$

Similarly,

$$\frac{\partial J}{\partial b_k} = 0 + j\,p(-k) - j\,p^*(-k) - j\sum_{i=0}^{M-1} w_i\, r(i-k)$$

$$\frac{\partial J}{\partial a_k} + \frac{\partial J}{\partial b_k} = -p(-k) - p^*(-k) + \sum_{i=0}^{M-1} w_i\, r(i-k)$$

$$- p(-k) + p^*(-k) + \sum_{i=0}^{M-1} w_i\, r(i-k)$$

$$\boxed{\nabla_k J = -2p(-k) + 2\sum_{i=0}^{M-1} w_i\, r(i-k)}$$

↳ Gradient Equation

$$\nabla_k J = 0$$

$$\sum_{i=0}^{M-1} w_i\, r(i-k) = p(-k)$$

↳ Wiener-Hopf Equation equivalent for optimal filter weights

(c) $\boxed{R w_o = p}$ (Wiener Hopf eqn)

$$w_o = R^{-1} p$$

$$R = \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}$$

$$R^{-1} = \frac{1}{|R|}\begin{bmatrix} 1 & -0.4045 \\ -0.4045 & 1 \end{bmatrix}$$

$$|R| = 1 - (0.4045)^2 = 0.8363$$

$$R^{-1} = \frac{1}{0.8363}\begin{bmatrix} 1 & -0.4045 \\ -0.4045 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

$$w_0 = \frac{1}{0.8363}\begin{bmatrix} 1 & -0.4045 \\ -0.4045 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

$$\boxed{w_0 = \begin{bmatrix} -0.14215 \\ 0.3514 \end{bmatrix}}$$

↳ Wiener-Hopf Solution

(e) The stepsize parameter μ

$$0 < \mu < \frac{2}{\lambda_{max}} \quad (\text{from stability})$$

λ_{max} is the largest value of R matrix

$$R\alpha = \lambda\alpha \Rightarrow (R - \lambda I)\alpha = 0$$

$$\Rightarrow |R - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0.4045 \\ 0.4045 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = (0.4045)^2$$

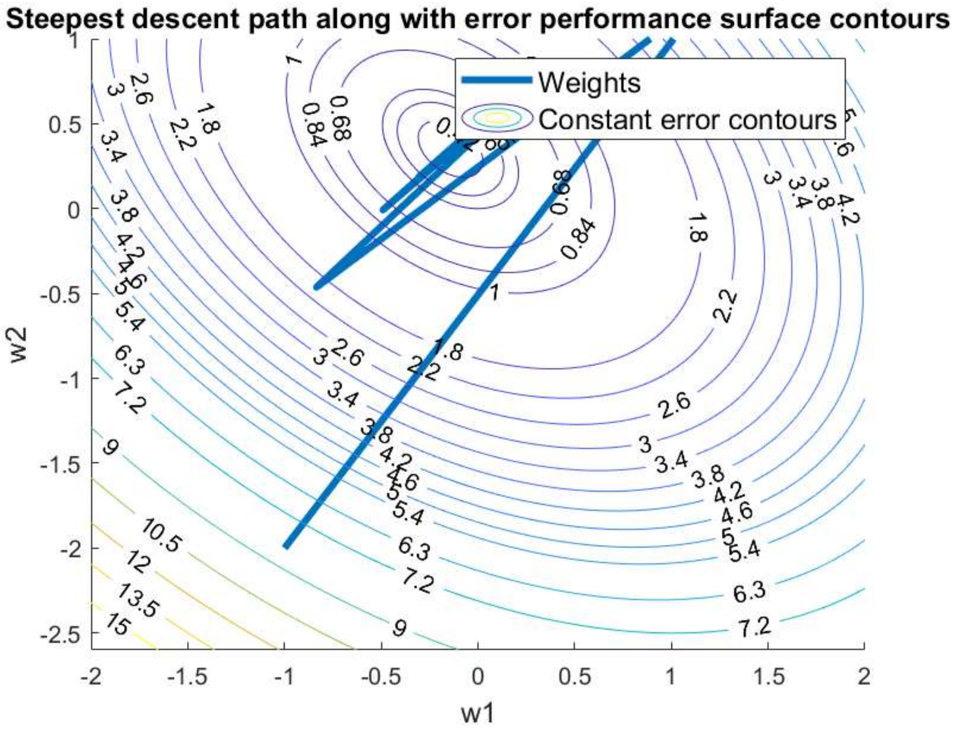
$$\Rightarrow \lambda = 1.4045,\ 0.5955.$$

So, $\lambda_{max} = 1.4045$

$$0 < \mu < \frac{2}{1.4045}$$

$$\Rightarrow \boxed{0 < \mu < 1.4239}$$

```matlab
R = [ 1  0.4045;0.4045  1];

p = [0; 0.2939];
var_d = 0.5;

wo = R\p;

numIter = 30;
mu = 1.2;

J_hist = zeros(numIter+1, 1);
w_hist = zeros(numIter+1, 2);

w = [-1; -2];
J = var_d - conj(w).'*p - conj(p).'*w + conj(w).'*R*w;

J_hist(1, 1) = J;
w_hist(1,:) = w.';

for k=1:numIter
    w = w + (p - R*w).*mu;
    J = var_d - conj(w).'*p - conj(p).'*w + conj(w).'*R*w;

    J_hist(k+1, 1) = J;
    w_hist(k+1,:) = w.';
end

xl = -2;
xr = 2;
yl = -2.6;
yr = 1;
x = linspace(xl, xr, 1000);
y = linspace(yl, yr, 1000);
[X, Y] = meshgrid(x,y);
J = 0.5 - 2.*p(1).*X - 2.*p(2).*Y + R(1,1).*(X.^2)+R(2,1).*(Y.*X)+R(1,2).*(X.*Y)+R(2,2).*(Y.^2);
levels = [0.38:0.04:0.5, 0.68:0.16:1, 1.8:0.4:5, 5.4:0.9:8, 9:1.5:15];
linewidth = 3;
fontsize = 12;

figure(1)
hold on;
plot(w_hist(:,1), w_hist(:,2), 'LineWidth', linewidth, 'DisplayName','Weights')
contour(X,Y,J,levels,'ShowText','on', 'DisplayName', 'Constant error contours')
legend('FontSize', fontsize);
title('Steepest descent path along with error performance surface contours', 'FontSize', fontsize)
xlabel('w1', 'FontSize',fontsize)
ylabel('w2', 'FontSize',fontsize)
xlim([xl,xr])
ylim([yl,yr])
hold off;
```




```
figure(2)
hold on;
plot(0:numIter, J_hist, 'LineWidth', linewidth)
title('Error vs iteration', 'FontSize',fontsize)
xlabel('Iteration', 'FontSize',fontsize)
ylabel('Error', 'FontSize',fontsize)
hold off;
```