Polygon Partitioning

Monotone Partitioning

Monotone Partitioning

- Define monotonicity
- Triangulate monotone polygons in linear time
- Partition a polygon into monotone pieces

Monotone polygons

- Definition
  - A polygonal chain C is **strictly monotone** with respect to L' if every line L orthogonal to L' meets C in at most one point.
  - A polygonal chain C is **monotone** if L has at most one connected component
    - Either empty, a single point, or a single line segment
  - A polygon P is said to be **monotone** with respect to a line L if ∂P can be split into two polygonal chains A and B such that each chain in monotone with respect to L

Monotone polygons

- The two chains should share a vertex at either end.
- Figure 2.1
  - A polygon monotone with respect to the vertical line
  - Two monotone chains
    - A = (v_0 ,…, v_{15}) and B= (v_{15},…, v_{24}, v_0)
  - Neither chain is strictly monotone (two horizontal edges - v_5v_6 and $v_{21}v_{22}$)

Monotone polygons

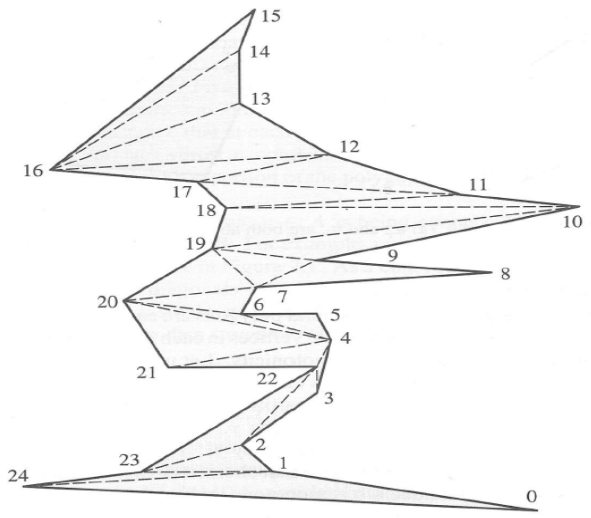


FIGURE 2.1 A polygon monotone with respect to the vertical.

Monotone polygons

- Properties of Monotone Polygons
  - The vertices in each chain of a monotone polygon are sorted with respect to the line of monotonicity
  - Let y axis be the fixed line of monotonicity
  - Then vertices can be sorted by y coordinate in linear time
  - Find a highest vertex, a lowest vertex and partition the boundary into two chains. (linear time)

Monotone polygons

- Define an **interior cusp** of a polygon as a reflex vertex v whose adjacent vertices v_- and v_+ are either both at or above, or both at or below, v. (See Figure 2.2)
- Interior cusp can't have both adjacent vertices with the same y coordinate as v
- Thus, d in Figure 2.2(b) is not an interior cusp

- Lemma 2.1.1
  - If a polygon P has no interior cups, then it is monotone.
  - Lack of interior cusp implies strict monotonicity

Monotone Partitioning

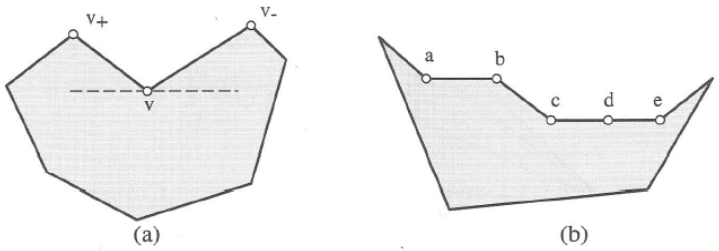


FIGURE 2.2 Interior cusps: (a) v_+ and v_- are both above v; (b) a, c, and e are interior cusps; b and d are not.

Triangulating a Monotone Polygon

- The shapes of monotone polygons are so special that they are easy to triangulate
- Linear time triangulation
- Hint of algorithm
  - Sort the vertices from top to bottom(in linear time)
  - Cut off triangles from the top in a "greedy" fashion

Triangulating a Monotone Polygon

- Algorithm
  - For each vertex v, connect v to all the vertices above it and visible via a diagonal
  - Remove the top portion of the polygon thereby triangulated
  - Continue with the next vertex below v.
- One can show that at any iteration, $v \in$ A is being connected to a chain of reflex vertices above it in the other chain B.

Triangulating a Monotone Polygon

- For example, v_{16} is connected to (v_{14}, v_{13}, v_{12}) in the first iteration.
- As a consequence, no visibility check is required
- The algorithm can be implemented with a single stack holding the reflex chain above.
- Between the linear sorting and this simple data structure, O(n) is achieved.

Polygon Partitioning

Trapezoidalization

Trapezoidalization

A **horizontal trapezoidalization** of a polygon is obtained by drawing a horizontal line through every vertex of the polygon.

Pass through each vertex v the maximal horizontal segment s such that $s \subset P$ and $s \cap \partial P = v$

See Figure 2.3

We only consider polygons whose vertices have unique y coordinates.

Trapezoidalization

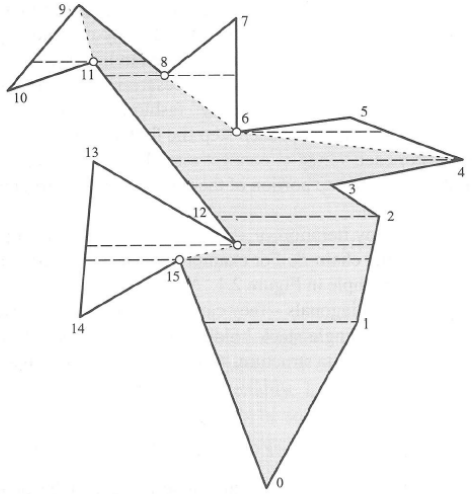


FIGURE 2.3 Trapezoidalization. Dashed lines show trapezoid partition lines; dotted diagonals resolve interior cusps (circled). The shaded polygon is one of the resulting monotone pieces.

Trapezoidalization

A **trapezoid** is a quadrilateral with two parallel edges

Call the vertices through which the horizontal lines are drawn **supporting vertices**.

Every trapezoid has exactly two supporting vertices

The connection to monotone polygons

- If a supporting vertex is on the interior of an upper or lower trapezoid edge, then it is an interior cusp

Trapezoidalization

Monotone partition

- Connect every interior vertex v to the opposing supporting vertex of the trapezoid v supports
- Then, these diagonals partition P into monotonic pieces
- Recall Lemma 2.1.1
- For example, diagonal v_6v_4, $v_{15}v_{12}$, and so on. (Figure 2.3)

Plane Sweep

Useful in many geometric algorithms

Main idea is to "sweep" a line over the plane maintaining some type of data structure along the line

Sweep a horizontal line L over the polygon, stopping at each vertex

Sorting the vertices by y coordinate $O(n\log n)$ time

Plane Sweep

Fine the edge immediately to the left and immediately to the right of v along L

- A sorted list $\mathcal{L}$ of polygon edges pierced by L is maintained at all times
- For example, for the sweep line in the position shown in Figure 2.4 $\mathcal{L} = (e_{19}, e_{18}, e_{17}, e_6, e_8, e_{10})$
- How to determine that v lies between e_{17} and e_6 in Figure 2.4?

Plane Sweep

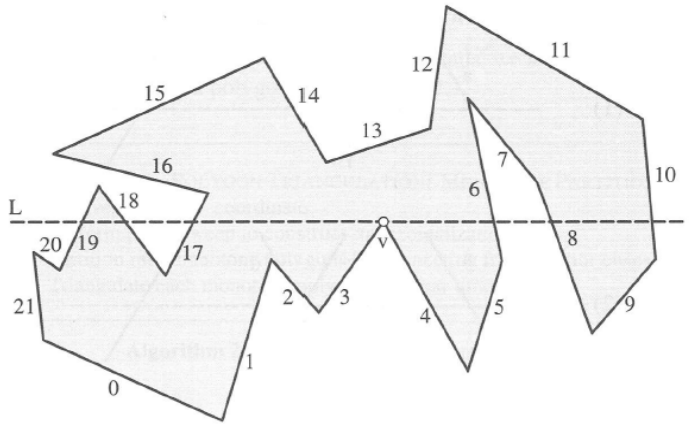


FIGURE 2.4 Plane sweep. Labels index edges.

Plane Sweep

Assume that e_i is a pointer to an edge and the vertical coordinate of v is y.

Suppose we know the endpoints of e_i

Then, we can compute the x coordinate of the intersection between L and e_i

We can determine v's position by computing x coordinates of each intersection

Time proportional to the length of $\mathcal{L}$ ($O(n)$)by a naive search from left to right

With an efficient data structure, a height-balanced tree, the search require $O(\log n)$ time

Plane Sweep

Updates at each event

There are three types of event (Figure 2.5)

Let v fall between edges a and b and v be shared by edges c and d

- c is above L and d is below. Then delete c from $\mathcal{L}$ and insert d :
  - $(\ldots, a, c, b, \ldots) \Rightarrow (\ldots, a, d, b, \ldots)$
- Both c and d are above L. Then delete both c and d from $\mathcal{L}$:
  - $(\ldots, a, c, d, b, \ldots) \Rightarrow (\ldots, a, b, \ldots)$
- Both c and d are below L. Then insert both c and d into $\mathcal{L}$:
  - $(\ldots, a, b, \ldots) \Rightarrow (\ldots, a, c, d, b, \ldots)$

Plane Sweep

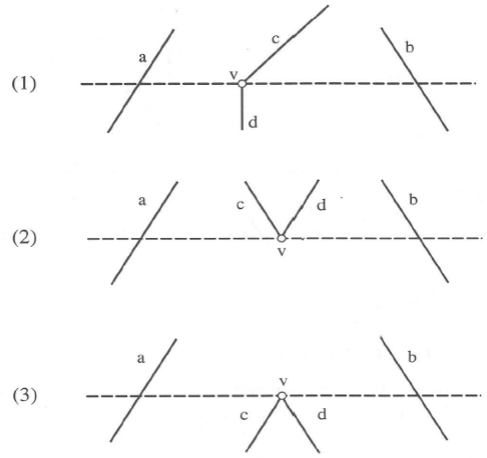


FIGURE 2.5 Sweep line events: (1) replace c by d; (2) delete c and d; (3) insert c and d.

Plane Sweep

In Figure 2.4, the list $\mathcal{L}$ of edges pierced by L is initially empty, when L is above the polygon

Then follows this sequence as it passes each event vertex

$$(e_{12}, e_{11})$$
$$(e_{15}, e_{14}, e_{12}, e_{11})$$
$$(e_{15}, e_{14}, e_{12}, e_6, e_7, e_{11})$$
$$(e_{15}, e_{14}, e_{13}, e_6, e_7, e_{10})$$
$$(e_{16}, e_{14}, e_{13}, e_6, e_7, e_{10})$$
$$(e_{16}, e_6, e_7, e_{10})$$
$$(e_{16}, e_6, e_8, e_{10})$$
$$(e_{19}, e_{18}, e_{16}, e_6, e_8, e_{10})$$
$$(e_{19}, e_{18}, e_{17}, e_6, e_8, e_{10}) .$$

Triangulation in $O(n \log n)$

Algorithm: POLYGON TRIANGULATION: MONOTONE PARTITION
Sort vertices by y coordinate.
Perform plane sweep to construct trapezoidalization.
Partition into monotone polygons by connecting from interior cusps.
Triangulate each monotone polygon in linear time.

Algorithm 2.1 $O(n \log n)$ polygon triangulation.

Polygon Partitioning

Partition into Monotone Mountains

Monotone Mountains

A **monotone mountain** is a monotone polygon with one of its two monotone chains a single segment, the **base**.

See Figure 2.6

Note that both end points of the base must be convex

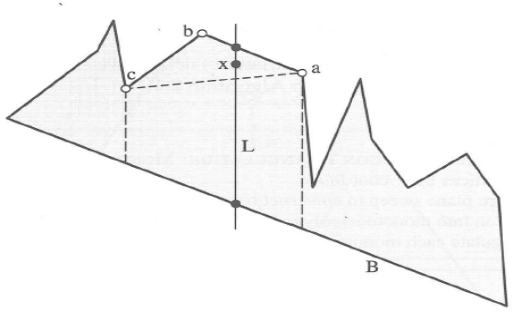


FIGURE 2.6 A monotone mountain with base B; b is an ear tip.

Monotone Mountains

- Lemma 2.3.1
  - **Every strictly convex vertex of a monotone mountain M, with the possible exception of the base endpoints, is an ear tip**
  - Proof
    - Let a, b, c be three consecutive vertices of M, with b a strictly convex vertex not an endpoint of the base B.
    - The direction of monotonicity is horizontal
    - Aim to prove that ac is a diagonal by contradiction

Monotone Mountains

- Lemma 2.3.1
  - Proof (con't)
    - Assume that ac is not a diagonal
    - By Lemma 1.5.1, either it is exterior in the neighborhood of an endpoint or it intersects ∂M.
    - **Case 1**. Suppose that ac is locally exterior
    - If a(or c) is not an endpoint of B, contradiction by the assumption that b is convex.
    - If a(or c) is the right endpoint of B, the same contradiction.
    - Or ac is below B, it is impossible because c lies above B
    - Thus, ac is locally interior

Monotone Mountains

- Lemma 2.3.1
  - Proof (con't)
    - **Case 2**. Assume that ac intersects ∂M
    - This requires a reflex vertex x to be interior to $\triangle abc$(cf. Figure 1.12)
    - Because x is interior, it cannot lie on either chain C=(a,b,c) or B.
    - Thus, a vertical line L through x meets ∂M in at least three points(contradiction)

Triangulating a Monotone Mountain

Linear time algorithm

The base is identified in linear time

- The base endpoints are extreme along the direction of monotonicity
- Leftmost and rightmost vertices

The "next" convex vertex is found without a search in constant time

- Similar with updating the ear tip status in Section 1.4
- Instead, update the convexity status
- Use stored internal angles of vertices
- By subtracting from a and c's angles appropriately upon removal of $\triangle abc$
- Require linking the convex vertices into a list and updating the list with each ear clip

Triangulating a Monotone Mountain

Algorithm: TRIANGULATION OF MONOTONE MOUNTAIN
Identify the base edge.
Initialize internal angles at each nonbase vertex.
Link nonbase strictly convex vertices into a list.
while list nonempty do
    For convex vertex b, remove $\triangle abc$.
    Output diagonal ac.
    Update angles and list.

Algorithm 2.2 Linear-time triangulation of a monotone mountain.

Trapezoidalization to monotone mountains

Build a monotone mountain from trapezoids abutting on a particular base edge, for example $v_{11}v_{12}$ (Figure 2.7)

$T(i, j)$ represents the trapezoid with support vertices v_i and v_j, below and above

Trapezoidalization to monotone mountains

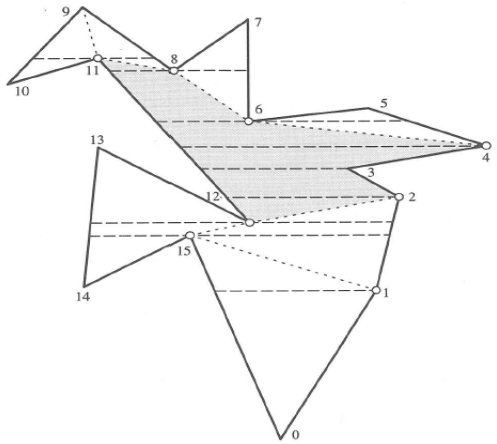


FIGURE 2.7 A partition into monotone mountains.

Trapezoidalization to monotone mountains

$T(12,2)$ is based on B but muse be cut by the diagonal $v_{12}v_2$

$T(2,3)$ and $T(3,4)$ may be included in their entirety

$T(4,6)$ must be cut to separate off the nonmonotonicity at v_6

Similarly $T(6,8)$ must be cut by v_6v_8

$T(8,11)$ must be cut by v_8v_{11}

The resulting union is a monotone mountain

Trapezoidalization to monotone mountains

- Lemma 2.3.2
  - In a trapezoidalization of a polygon P, connecting every pair of trapezoid-supporting vertices that do not lie on the same(left/right) side of their trapezoid partitions P into monotone mountains.
  - Proof
    - Lemma 2.1.1 guarantees that the pieces of the partition are monotone.
    - Thus only need to prove that each piece has one chain that is a single segment

Trapezoidalization to monotone mountains

- Lemma 2.3.2
  - Proof (con't)
    - Suppose to the contrary that both monotone chains A and B of one piece Q of the partition each contain at least two edges.
    - Let z be the topmost vertex of Q, adjacent on $Q = A \cup B$ to vertices a and b, with b below a (Figure 2.8)
    - In order for B to contain more than just the edge zb, b cannot be the endpoint
    - But the upper supporting vertex c of T(b, c) cannot lie on zb, for c must lie at or below a
    - Thus c is not on the same side of T(b, c) as b, and the diagonal cb is included in the partition
    - This contradicts the assumption that B extends below b.

Trapezoidalization to monotone mountains

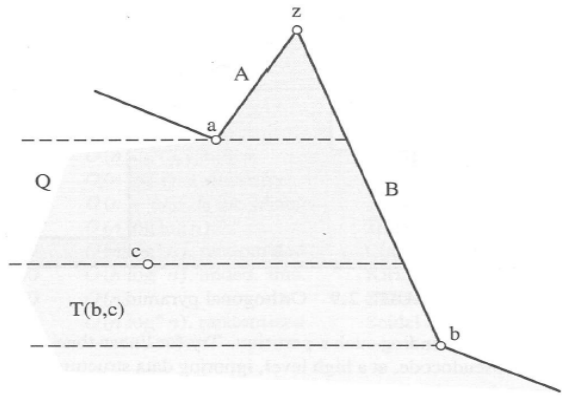


FIGURE 2.8 Proof of Lemma 2.3.2: Diagonal *cb* must be present.

Polygon Partitioning

Linear-Time Triangulation

Linear-Time Triangulation

Table 2.1. History of triangulation algorithms.

Year	Complexity	Reference
1911	$O(n^2)$	Lennes (1911)
1978	$O(n \log n)$	Garey et al. (1978)
1983	$O(n \log r)$, r reflex	Hertel & Mehlhorn (1983)
1984	$O(n \log s)$, s sinuosity	Chazelle & Incerpi (1984)
1988	$O(n + nt_0)$, t_0 int. triangs.	Toussaint (1990)
1986	$O(n \log \log n)$	Tarjan & Van Wyk (1988)
1989	$O(n \log^* n)$, randomized	Clarkson, Tarjan & Van Wyk (1989)
1990	$O(n \log^* n)$, bnded. ints.	Kirkpatrick, Klawe & Tarjan (1990)
1990	$O(n)$	Chazelle (1991)
1991	$O(n \log^* n)$, randomized	Seidel (1991)

Linear-Time Triangulation

Chazelle's algorithm ($O(n)$)

- Main structure: **visibility** map
- Generalization of a trapezoidalization to drawing horizontal chords toward both sides of each vertex in a polygonal chain
- Mimics merge sort, a common technique for sorting by divide-and-conquer
- The polygon of n vertices is partitioned into chains with $n/2$ vertices, and into chains of $n/4$ vertices, and so on
- This leads to an $O(n\log n)$ time complexity

Linear-Time Triangulation

Chazelle's algorithm ($O(n)$) (con't)

- Improves on this by dividing the process into two phases
- First, only coarse approximations of the visibility maps
- Coarse enough so that the merging can be accomplished in linear time
- Second, refines the coarse map into a complete visibility map, again in linear
- A triangulation is then produced from the trapezoidalization defined by the visibility map as before

Randomized Triangulation

Seidel's algorithm ($O(n\log^* n)$)

- Follows the trapezoidalization -> monotone mountains -> triangulation path
- His improvement is in building the trapezoidalization quickly
- Uses a "query structure" Q, a data structure that permits location of a point in its containing trapezoid in time proportional to the depth of the structure
- The depth of the structure could be $\Omega(n)$ for n segments, but
- By adding the segments in random

Randomized Triangulation

Randomized algorithms

- Can be expected to work well on all inputs
- Developed into a key technique for creating algorithms that are both efficient and simple

Seidel's algorithm

- The visibility map by inserting the segments in random order in $O(n\log n)$ time and $O(n)$ space
- Results in another $O(n\log n)$ algorithm

Randomized Triangulation

Seidel's algorithm (con't)

- The segments form the edges of a simple polygon
- This can be exploited by running the algorithm in $\log^* n$ phases
- In phase i, a subset of the segments is added in random order, producing Q_i
- The entire polygon is traced through Q_i, locating each vertex in a trapezoid of the current visibility map

Randomized Triangulation

Seidel's algorithm (con't)

- In phase $i+1$, more segments are added
- The knowledge of where they were in Q_i helps locate their endpoints more quickly
- The process is repeated until the entire visibility map is constructed

Analysis of the expected time

- Over all possible n! segment insertion orders
- It shows the expected time is $O(n\log^* n)$

Polygon Partitioning

Convex Partitioning

Convex Partitioning

Two goals

- Partition a polygon into as few convex pieces as possible
- Do so as fast as possible

Compromise on the number of pieces

Find a quick algorithm whose output size is bounded with respect to the optimum

Two types of partition may be distinguished

- By diagonals
- By segments

Optimum Partition

- Lemma 2.5.1 (Chazelle)
  - Let Φ be the fewest number of convex pieces into which a polygon may be partitioned. For a polygon of r reflex vertices, $\lceil r/2 \rceil + 1 \leq \Phi \leq r + 1$
  - Proof
    - Drawing a segment that bisects each reflex angle results in a convex partition
    - The number of pieces is $r + 1$ (Figure 2.10)
    - At most two reflex angles can be resolved by a single partition segment (Figure 2.11)
    - This results in $\lceil r/2 \rceil + 1$ convex pieces

Optimum Partition

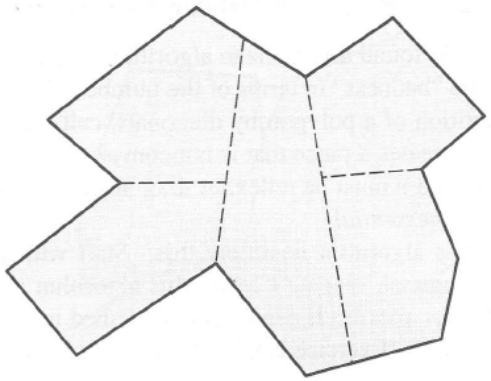


FIGURE 2.10 $r + 1$ convex pieces: $r = 4$; 5 pieces.

Optimum Partition

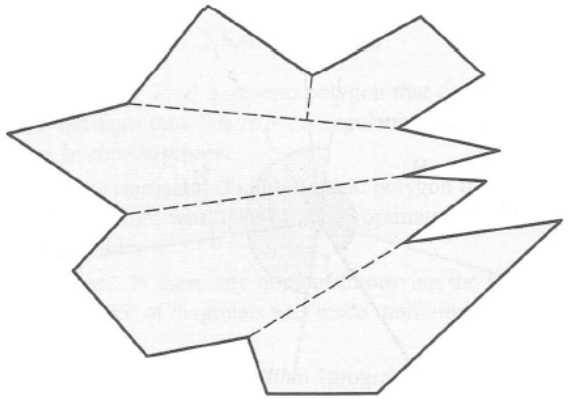


FIGURE 2.11 $\lceil r/2 \rceil + 1$ convex pieces: $r = 7$; 5 pieces.

Hertel and Mehlhorn Algorithm

A very clean algorithm that partitions with diagonals quickly

- has bounded "badness" in terms of the number of convex pieces

A diagonal d *is* **essential** for vertex v if removal of d makes v non-convex

The algorithm

- start with a triangulation of P
- remove an inessential diagonal
- repeat

Hertel and Mehlhorn Algorithm

- Lemma 2.5.2
  - There can be at most two diagonals essential for any reflex vertex
  - Proof
    - Let v be a reflex vertex and v_+ and v_- its adjacent vertices.
    - At most one essential diagonal in the halfplane H_+ to the left of vv_+
    - If there were two, the one closest to vv_+ could be removed (Figure 2.12)
    - Similarly, there can be at most one essential diagonal in the halfplane H_- to the left of v_-v
    - Together these cover the interior angle at v, and so there are at most two diagonals essential for v

Hertel and Mehlhorn Algorithm

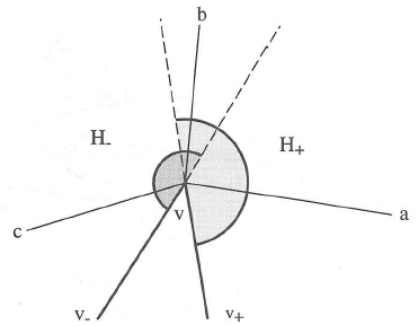


FIGURE 2.12 Essential diagonals. Diagonal a is not essential because b is also in H_+. Similarly c is not essential.

Hertel and Mehlhorn Algorithm

- Lemma 2.5.3
  - The *Hertel-Mehlhorn* algorithm is never worse than four-times optimal in the number of convex pieces
  - Proof
    - By Lemma 2.5.2, each reflex vertex can be "responsible for" at most two essential diagonals.
    - The number of essential diagonals can be no more than $2r$
    - Thus, the number of convex pieces M produced by the algorithm satisfies $2r + 1 \geq M$
    - Since $\Phi \geq \lceil r/2 \rceil + 1$ by Lemma 2.5.1,
    - $4\Phi \geq 2r + 4 > 2r + 1 \geq M$

Optimal Convex Partitions

Finding a convex partition optimal in the number of pieces is much more time consuming than finding a suboptimal one.

- Greene's algorithm runs in $O(r^2n^2) = O(n^4)$ time
- Keil's algorithm improved it to $O(r^2n\log n) = O(n^3\log n)$ time
- Both employ **dynamic programming**

The problem is even more difficult if the partition **may be formed with arbitrary segments**

- Figure 2.13
- Chazelle solve this problem in his thesis with an intricate $O(n + r^2) = O(n^3)$ algorithm

Optimal Convex Partitions

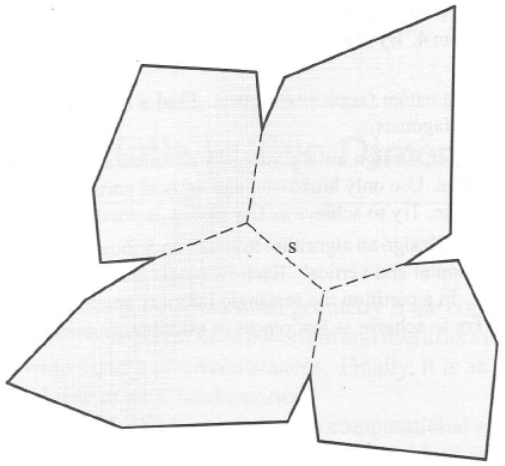


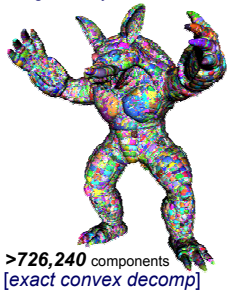
FIGURE 2.13 An optimal convex partition. Segment s does not touch ∂P.

Approximate Convex Decomposition (ACD)

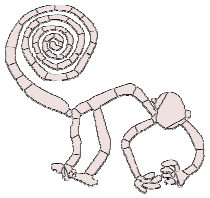
- **ACD**
  - All sub-models will have tolerable *concavity*
  - Convex decomposition is useful but
    - can be costly to construct
    - may result in unmanageable number of components
- **Benefits of ACD**

1. Number of sub-models is significantly less

54 components
[*ACD*] *0.04* concavity



>726,240 components
[*exact convex decomp*]



2. Reveal structural features

Approximate Convex Decomposition (ACD)

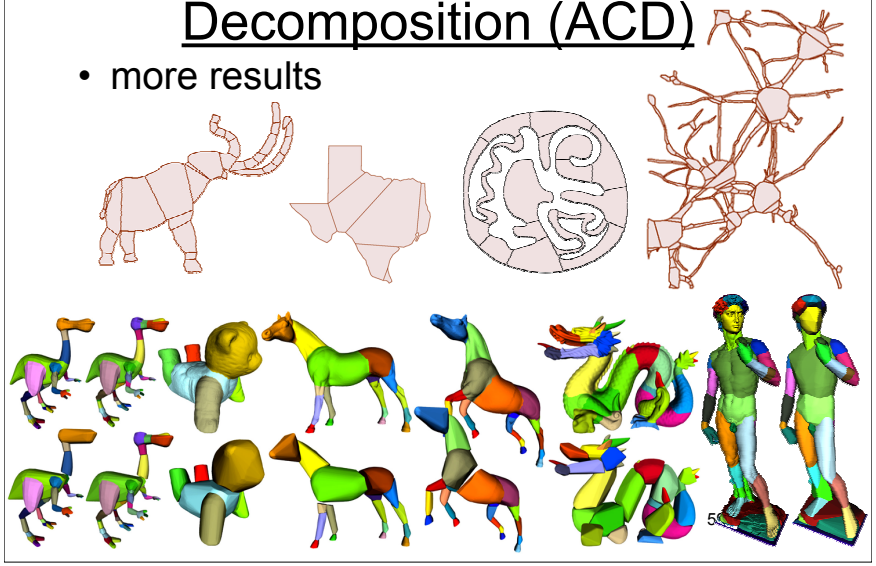
- How does it work?

Input model | Measure concavity | Shaded using its concavity | Decompose at areas with high concavity

- Sub-models form a nice **hierarchical representation** of the original model

Approximate Convex Decomposition (ACD)

- more results



Conclusion

- Polygon decomposition
  - decompose to monotonic polygon/ mountain
  - trapezoidal decomposition
  - convex decomposition
- Homework assignment
  - Ex: 2.2.3-2, 2.2.3-4, 2.5.4-7