Assignment 2

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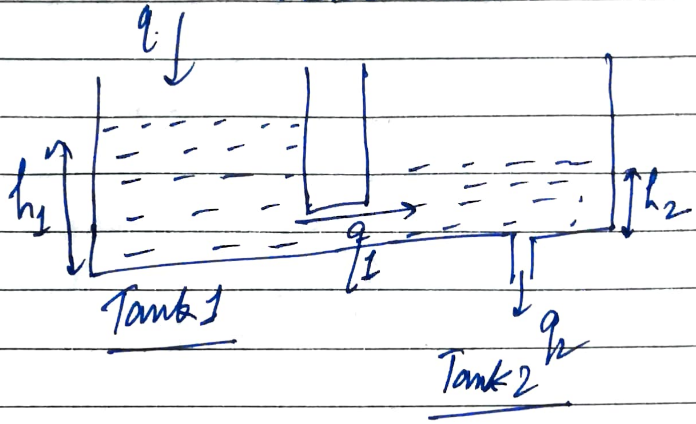
18EE35014

(1) (i)



$q_1 = P_1\sqrt{(h_1 - h_2)}$

$q_2 = P_2\sqrt{(h_2)}$

$\frac{A\,dh_1}{dt} = q - q_1 = q - P_1\sqrt{h_1 - h_2}$

$\frac{A\,dh_2}{dt} = q_1 - q_2 = P_1\sqrt{h_1 - h_2} - P_2\sqrt{h_2}$

(a)

State Variables: $h_1, h_2,$

Input: q, Output: $h_2 = y$.

Equilibrium Conditions:

h_{20} & q_0 are equilibrium states of h_2 & q.

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$$\frac{dh_2}{dt} = 0 \Rightarrow q_1 - q_2 = 0$$

$$\Rightarrow \ell_1\sqrt{h_1 - h_2} = \ell_2\sqrt{h_2}$$

$$\Rightarrow h_{10} - h_{20} = \frac{\ell_2^2}{\ell_1^2} h_{20}$$

$$\Rightarrow \boxed{h_{10} = \left(\frac{\ell_2^2}{\ell_1^2} + 1\right) h_{20}}$$

$$\frac{dh_1}{dt} = 0 \Rightarrow q - q_1 = 0$$

$$\Rightarrow q_0 = \ell_1\sqrt{h_{10} - h_{20}} = \ell_2\sqrt{h_{20}}$$

$$\boxed{q_0 = \ell_2\sqrt{h_{20}}}$$

Incremental Transfer function:

(b)

$$f_1 = \frac{dh_1}{dt}, \quad f_2 = \frac{dh_2}{dt}$$

$$\left.\frac{\partial f_1}{\partial h_1}\right|_e = \frac{-\ell_1}{2\sqrt{h_{10} - h_{20}}\, A} = \frac{-\ell_1^2}{2A\ell_1\sqrt{h_{10} - h_{20}}}$$

$$\boxed{\left.\frac{\partial f_1}{\partial h_1}\right|_e = \frac{-\ell_1^2}{2A\ell_2\sqrt{h_{20}}}}$$

$$\left.\frac{\partial f_1}{\partial h_2}\right|_e = \frac{\ell_1}{2A\sqrt{h_{10} - h_{20}}} = \frac{\ell_1^2}{2A\ell_2\sqrt{h_{20}}}$$

$$\boxed{\left.\frac{\partial f_1}{\partial h_2}\right|_e = \frac{\ell_1^2}{2A\ell_2\sqrt{h_{20}}}}$$

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$$\boxed{\left.\frac{\partial f_1}{\partial q}\right|_e = \frac{1}{A}}$$

$$\boxed{\left.\frac{\partial f_2}{\partial h_1}\right|_e = \frac{\rho_1}{2A\sqrt{h_1-h_2}} = \frac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}}}$$

$$\left.\frac{\partial f_2}{\partial h_2}\right|_e = \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} - \frac{\rho_2}{2\sqrt{h_{20}}\,A}$$

$$= \frac{\rho_2^2-\rho_1^2}{2A\rho_2\sqrt{h_{20}}}$$

State Space Rep.

$$\frac{d}{dt}\begin{bmatrix} h_2 \\ h_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & \frac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} \\ \frac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & \frac{\rho_2^2-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} \end{bmatrix}}_{A}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{A} \\ 0 \end{bmatrix}}_{B} q$$

$$y = \cancel{[1\ 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}\ \underbrace{[0\ \ 1]}_{C}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$G(S) = C(SI-A)^{-1}B$$

$$(SI-A)^{-1} = \begin{bmatrix} S+\frac{\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} \\ \frac{-\rho_1^2}{2A\rho_2\sqrt{h_{20}}} & S+\frac{\rho_1^2+\rho_2}{2A\rho_2\sqrt{h_{20}}} \end{bmatrix}^{-1}$$

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$$= \frac{1}{|\det(A)|}\begin{bmatrix} s+\frac{l_1^2+l_2^2}{2Al_2\sqrt{h_{20}}} & \frac{l_1^2}{2Al_2\sqrt{h_{20}}} \\ \frac{l_1^2}{2Al_2\sqrt{h_{20}}} & s+\frac{l_1^2}{2Al_2\sqrt{h_{20}}} \end{bmatrix}$$

where, $\det(A) =$

$$\left(s+\frac{l_1^2+l_2^2}{2Al_2\sqrt{h_{20}}}\right)\left(s+\frac{l_1^2}{2Al_2\sqrt{h_{20}}}\right)-\frac{l_1^2}{2Al_2\sqrt{h_{20}}}\cdot\frac{l_1^2}{2Al_2\sqrt{h_{20}}}$$

$$= s^2+\frac{l_1^2+2l_2^2}{2Al_2\sqrt{h_{20}}}+\frac{l_2^2 l_1^2}{4A^2l_2^2 h_{20}}$$

$$= s^2+\frac{l_1^2+2l_2^2}{2Al_2\sqrt{h_{20}}}+\frac{l_1^2}{4A^2h_{20}}$$

$G(s) = C(sI-A)^{-1}B =$

$$\boxed{G(s) = \frac{\frac{l_1^2}{2l_2\sqrt{h_{20}}\times A^2}}{s^2+\frac{l_1^2+2l_2^2}{2Al_2\sqrt{h_{20}}}s+\frac{l_1^2}{4A^2h_{20}}}}$$

Transfer function

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(c)

$$A\frac{dh_1}{dt} = q_0 - \frac{\rho_1^2}{2\rho_2\sqrt{h_{20}}}(h_1 - h_2)$$

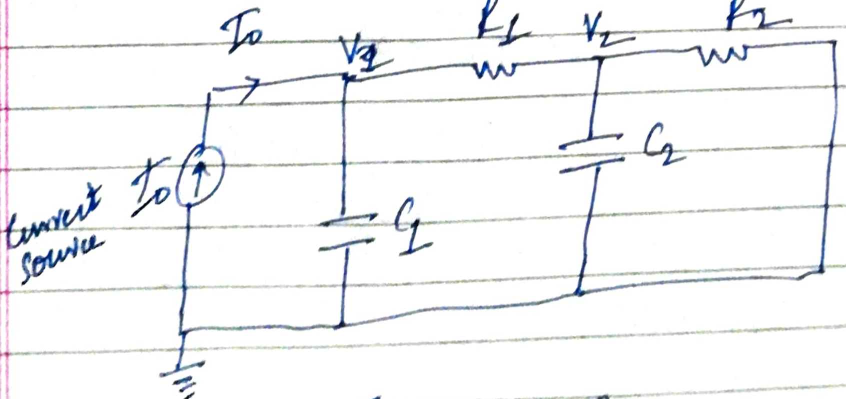
$$A\frac{dh_2}{dt} = \frac{\rho_1^2}{2\rho_2\sqrt{h_{20}}}(h_1 - h_2) - \frac{\rho_2}{2\sqrt{h_{20}}}\Delta h_2$$

If $h_1 = V_1$ & $q_0 = I_0$
$h_2 = V_2$ $\quad A = C_1 = C_2$

$$\frac{\rho_1^2}{2\rho_2\sqrt{h_{20}}} = \frac{1}{R_1}, \quad \frac{2\sqrt{h_{20}}}{\rho_2} = \frac{1}{R_2}$$

$$C_1\frac{dV_1}{dt} = I_0 - \cancel{R}\frac{(V_1 - V_2)}{R_1}$$

$$C_2\frac{dV_2}{dt} = \cancel{I_0} + \frac{V_1 - V_2}{R_1} - \frac{V_2}{R_2}$$



Electrical Equivalent System.

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(ii) Given

$d_0 = 0.5cm$

$$a_1 = a_2 = \frac{\pi d_0^2}{4} = \frac{\pi \times 0.25 \ cm^2}{4}$$

$$= 0.1963 \ cm^2$$

$A = 16cm^2$

$h_{20} = 10cm$

$g = 980 \ cm/s^2$

$k_m = 5 \ cm^3/s/v$

$q = k_m v$

$$R_1 = a_1\sqrt{2g} = 0.1963\sqrt{2\times 980} = 8.693 \ cm^{2.5}/s$$

$$R_2 = a_2\sqrt{2g} = 0.1963\sqrt{2\times 980}$$

$$= 8.693 \ cm^{2.5}/s.$$

$$h_{10} = \frac{R_1^2 + R_2^2}{R_1^2} h_{20} = 2h_{20} = 20 \ cm$$

$$q_0 = R_2\sqrt{h_{20}} = 8.693\sqrt{10} = 27.49 cm^3/s.$$

$$V_0 = \frac{q_0}{k_m} = \frac{27.49}{5} V = 5.498 V$$

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$$\frac{\beta_1^2}{2A\beta_2\sqrt{h_{20}}} = \frac{8.693}{2\times 16\sqrt{10}} = 0.086$$

$$\frac{1}{A} = \frac{1}{0.0625}$$

$$\frac{\beta_1^2+\beta_2^2}{2A\beta_2\sqrt{h_{20}}} = \frac{2\times 8.693}{2\times 16\sqrt{10}} = 0.178$$

$$\frac{0.086}{A} = 5.375\times 10^{-3}.$$

$$A = \begin{bmatrix} -0.086 & 0.086 \\ 0.086 & -0.178 \end{bmatrix} \quad B = \begin{bmatrix} 0.0625 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$G(s) = \frac{5.375\times 10^{-3}}{s^2+0.258s+0.0074}$$

Rough

$$\omega_n^2 = 0.0074$$

$$2\xi\omega_n = 0.258$$

$$\xi = \frac{\pi - \sin^{-1}(1-\xi^2)}{\sqrt{\omega_n^2 - \frac{0.258^2}{4}}}$$

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Non linear Model:

$$A\frac{dh_1}{dt} = q - C_1\sqrt{h_1-h_2}$$

$$16\frac{dh_1}{dt} = 27.49 - 8.693\sqrt{h_1-h_2}$$

$$\boxed{\frac{dh_1}{dt} = 1.7181 - 0.5433\sqrt{h_1-h_2}}$$

$$A\frac{dh_2}{dt} = C_1\sqrt{h_1-h_2} - C_2\sqrt{h_2}$$

$$16\frac{dh_2}{dt} = 8.693\sqrt{h_1-h_2} - 8.693\sqrt{h_2}$$

$$\boxed{\frac{dh_2}{dt} = 0.5433\left(\sqrt{h_1-h_2} - \sqrt{h_2}\right)}$$

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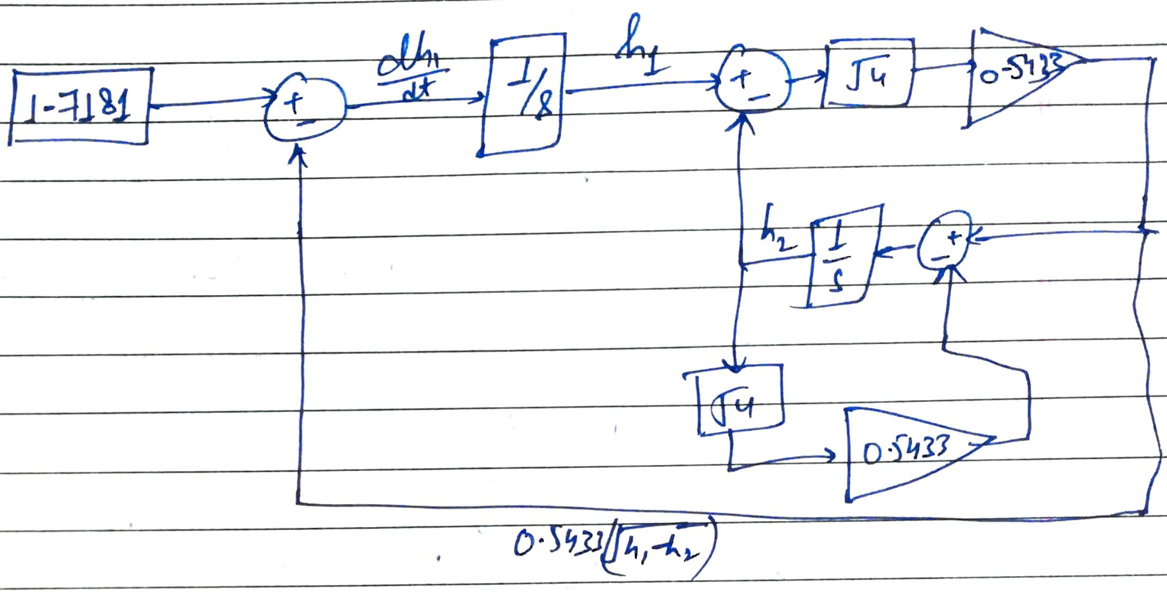
Non linear Model



Linear Model:

$I_0 = \Delta q$ $\qquad$ $C_1 = C_2 = A = 16$

$V_1 = \Delta h_1$ $\qquad$ $R_1 = R_2 = \frac{2q_0}{f_1^2} = \frac{2 \times 27.49}{8.693^2}$

$V_2 = \Delta h_2$

$= 0.7276.$

$$C_1 \frac{dV_1}{dt} = I_0 - \frac{V_1 - V_2}{R}$$

$$16 \frac{dV_1}{dt} = I_0 - \frac{V_1 - V_2}{0.7276}$$

$$\Rightarrow \boxed{\frac{dV_1}{dt} = 0.0625 I_0 - 0.0859 (V_1 - V_2)}$$

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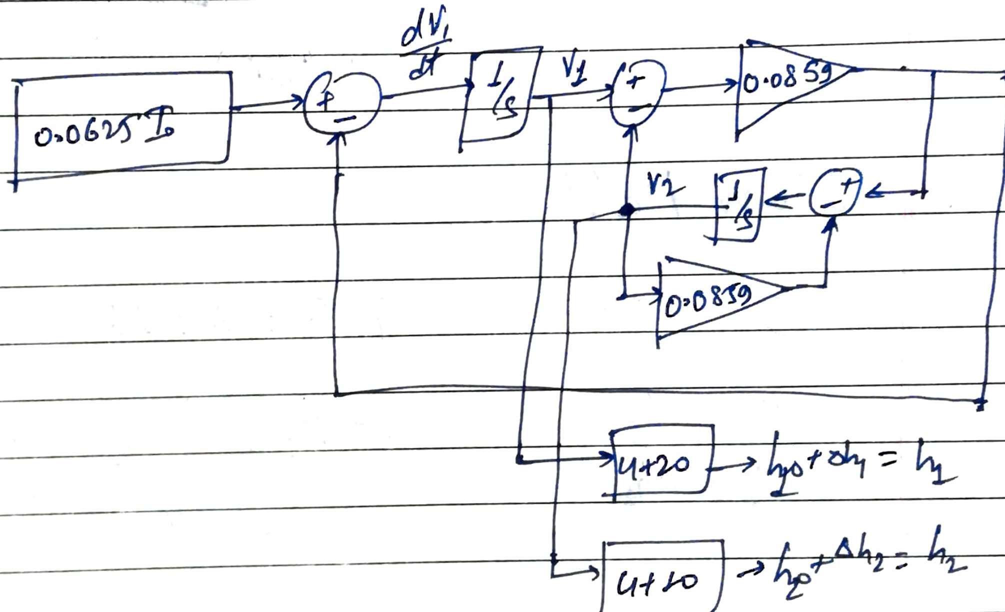
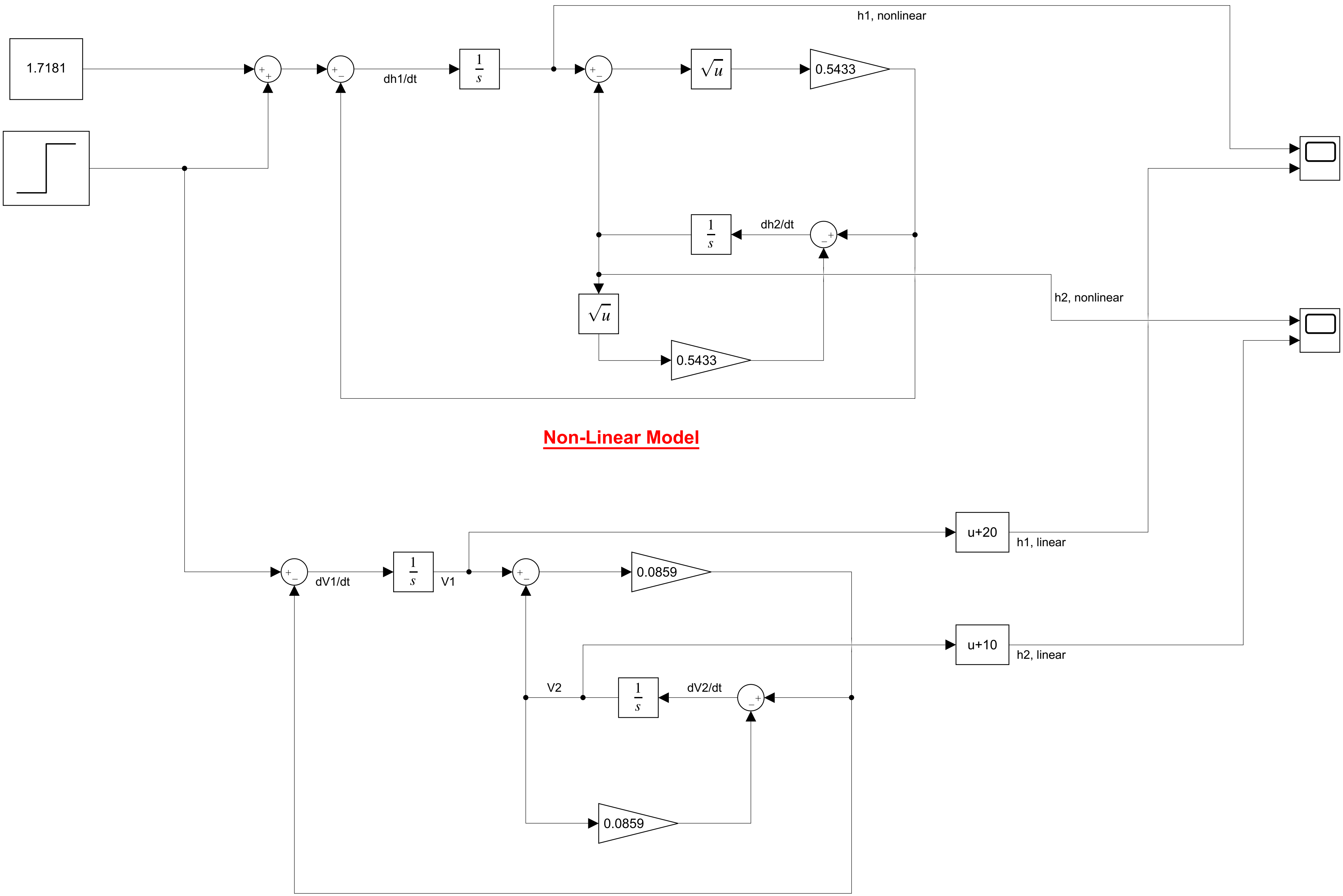
$$C_2 \frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1} - \frac{v_2}{R_2}$$

$$16 \frac{dv_2}{dt} = \frac{v_1 - v_2}{\cancel{R_1}\ 0.7276} - \frac{v_2}{0.7276}$$

$$\Rightarrow \boxed{\frac{dv_2}{dt} = 0.0859(v_1 - v_2) - 0.0859 v_2}$$

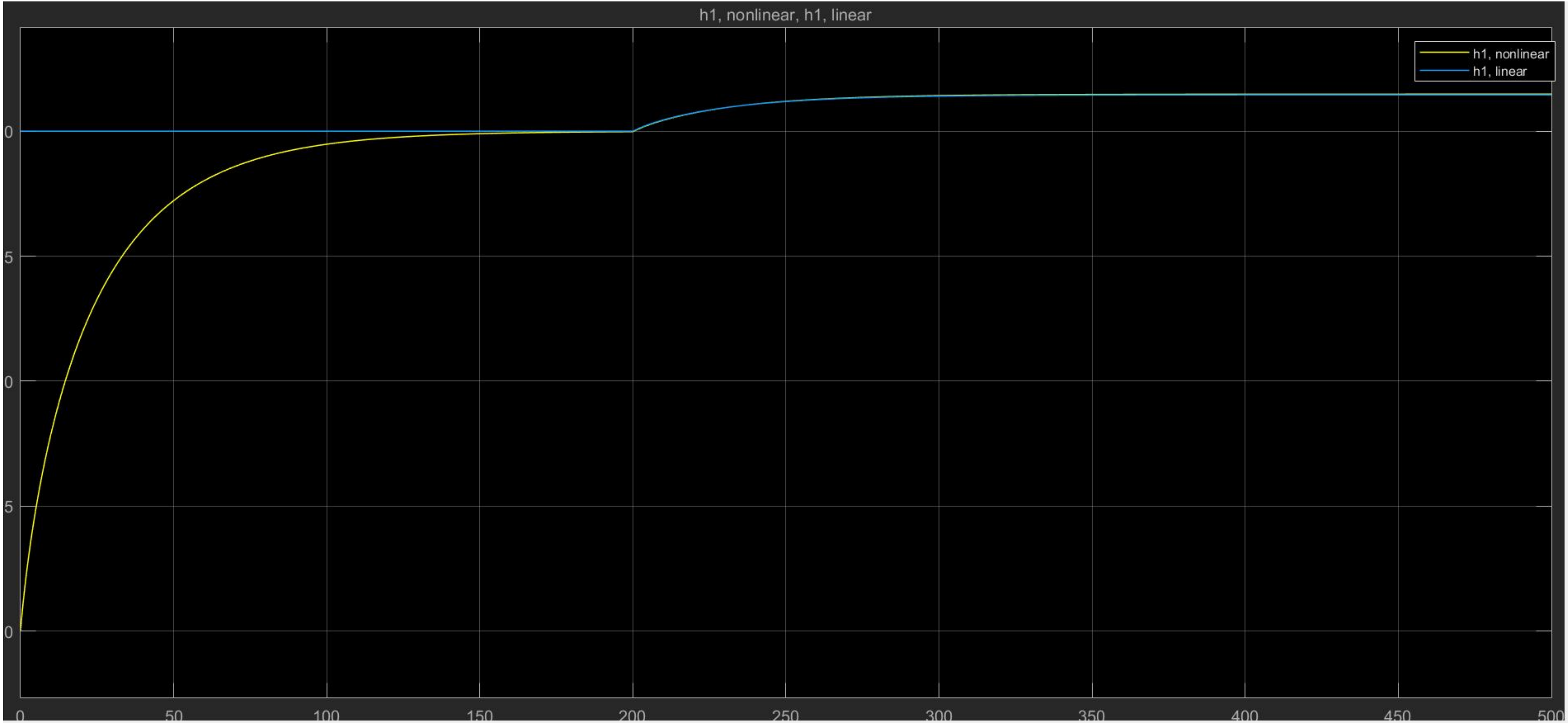
Linear Model



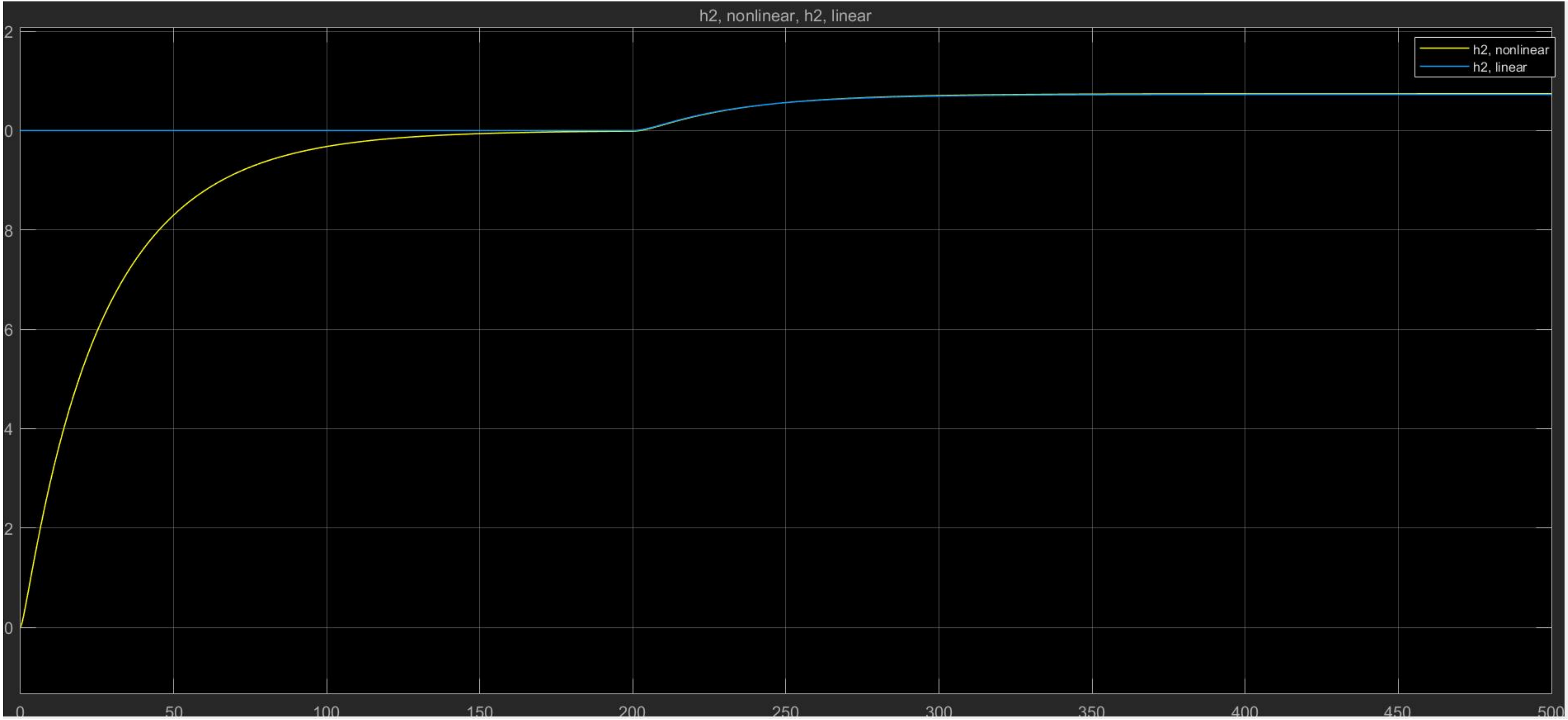


Non-Linear Model

Linear Model



h1, nonlinear, h1, linear
h1, nonlinear
h1, linear
0
5
0
5
0
0
50
100
150
200
250
300
350
400
450
500


h2, nonlinear, h2, linear
h2, nonlinear
h2, linear
2
0
8
6
4
2
0
0
50
100
150
200
250
300
350
400
450
500

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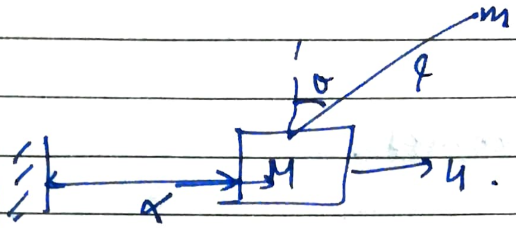
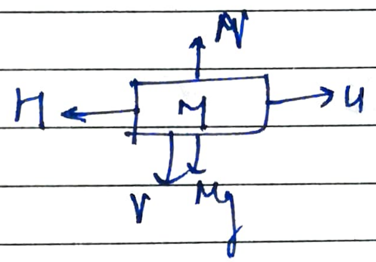
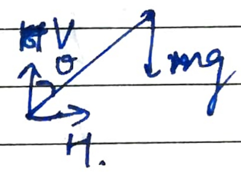
Q2.



FBD of M.



FBD of m + rod.



Force Equation:

$$u - H = M\frac{d^2x}{dt^2} \quad -(1)$$

$$V - mg = m\frac{d^2}{dt^2}(l\cos\theta) \quad -(2)$$

$$\& \quad H = m\frac{d^2}{dt^2}(l\sin\theta + x) \quad -(3)$$

Torque Equation,

Taking I as moment of Inertia of bob about its centre of mass, (also considering radius of bob)

$$(V\sin\theta - H\cos\theta)\,l = I\frac{d^2\theta}{dt^2} \quad -(4)$$

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(a) When θ being close to 0.

$\sin\theta \simeq \theta$, $\cos\theta \simeq 1$.

Then,
Equation (2) becomes

$V - mg = 0 \Rightarrow \boxed{V = mg}$

Eq (3) becomes.

$H = ml\ddot{\theta} + m\ddot{x}$

Eq (4) becomes

$Vl\theta - Hl = I\ddot{\theta}$

Substituting values from modified eq (2) & (3) into eq (4),

$mgl\theta - ml^2\ddot{\theta} - ml\ddot{x} = I\ddot{\theta}$

$\Rightarrow mgl\theta - ml\ddot{x} = (I + ml^2)\ddot{\theta}$

Neglecting I in comparision to ml^2.

$$\boxed{mgl\theta - ml\ddot{x} = ml^2\ddot{\theta}} \quad - \cancel{(5)}$$

$$\boxed{mg\theta - m\ddot{x} = ml\ddot{\theta}} \quad - (5)$$

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$$u - ml\ddot{\theta} - m\ddot{x} = M\ddot{x}$$

$$\boxed{u = ml\ddot{\theta} + (m+M)\ddot{x}} \quad - (6)$$

Assuming relaxed initial conditions, apply Laplace transformation,

$$U(s) = mls^2\theta(s) + (M+m)s^2 X(s)$$

$$\theta(s) = \frac{U(s) - (M+m)s^2 X(s)}{mls^2}$$

On Eqn (5)

$$mg\theta(s) - ms^2 X(s) = mls^2\theta(s)$$

$$\boxed{\theta(s) = \frac{ms^2 X(s)}{mg - mls^2}}$$

$$\frac{ms^2 X(s)}{mg - mls^2} = \frac{U(s) - (M+m)s^2 X(s)}{mls^2}$$

$$\Rightarrow U(s) = \frac{mls^2 \times ms^2 X(s)}{mg - mls^2} + (M+m)s^2 X(s)$$

$$\Rightarrow \boxed{\frac{X(s)}{U(s)} = \frac{ls^2 - g}{s^2\left[Mls^2 - (M+m)g\right]}} = \frac{0.26s^2}{s^2(0.6240s^2 - 26.0680)}$$

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$$\frac{\theta(s)}{U(s)} = \frac{ms^2}{mg - mgls^2} \times \frac{ls^2 - g}{s^2[Mls^2 - (M+m)g]}$$

$$\boxed{\frac{\theta(s)}{U(s)} = \frac{1}{(M+m)g - Mls^2}} = \frac{1}{26.068 - 0.6240 s^2}$$

(b). $(M+m)\ddot{x} + ml\ddot{\theta} = u$ — (7)

$\ddot{x} + l\ddot{\theta} = g\theta$ — (8)

~~Subtract~~

Multiply $m \times$ eq (8) & subtract from eq (7).

$M\ddot{x} = u - mg\theta$

$$\boxed{\ddot{x} = \frac{u}{M} - \frac{m}{M}g\theta}$$

(7) − (M+m) × (8)

$-Ml\ddot{\theta} = u - (M+m)g\theta$

$$\boxed{\ddot{\theta} = \frac{(M+m)g\theta}{Ml} - \frac{u}{Ml}}$$

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State Variables: $x, \dot{x}, \theta, \dot{\theta}$

Output: $\begin{bmatrix} x \\ \theta \end{bmatrix}$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u.$$

$m = 0.26$ kg

$M = 2.4$ kg

$l = 0.36$ m

from Matlab,

Rank of $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = 4.$

Rank of $\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = 4.$

$\Rightarrow$ Observability $= 4$

Controllability $= 4.$

```
m = 0.26;
M = 2.4;
l = 0.26;
g = 9.8;
A = [0 1 0 0
    0 0 -(m*g)/M 0
    0 0 0 1
    0 0 ((M+m)*g)/(M*l) 0];
B = [0
    1/M
    0
    -1/(M*l)];
C1 = [1 0 0 0];
C2 = [0 0 1 0];
D = [0
    0];
```

```
cont_mat = [B A*B A*A*B A*A*A*B];
controllability = rank(cont_mat)
```

```
controllability = 4
```

```
observability_mat_x = [C1; C1*A; C1*A*A; C1*A*A*A;];
obervaility_x = rank(observability_mat_x)
```

```
obervaility_x = 4
```

```
observability_mat_theta = [C2; C2*A; C2*A*A; C2*A*A*A;];
obervaility_theta = rank(observability_mat_theta)
```

```
obervaility_theta = 2
```

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$\therefore$ Pendulum can be balanced using x & θ feedback only x feedback.

Because for

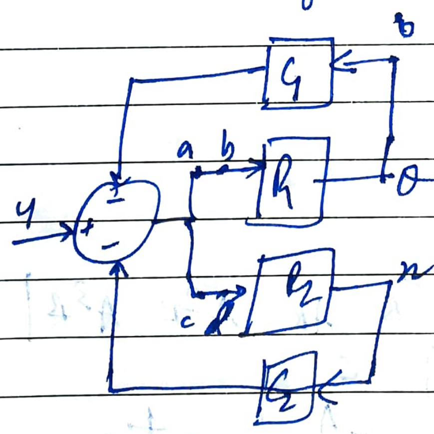
$C_1 = [1 \ 0 \ 0 \ 0] \Rightarrow$ observability = controllability = 4

& same when $C_2 = [0 \ 0 \ 1 \ 0] \Rightarrow$

observability = 2
Controllability = 4.

There is a pole zero cancellation in $\theta(s)$ which reduced a rank of 2.

(C)



$$P_1 = \frac{k_1}{s^2 - a^2}$$

$$P_2 = \frac{k_2 (s^2 - b^2)}{s^2 (s^2 - a^2)}$$

If we break the loop between a & b, a b

Loop $gain_1 = \dfrac{P_1 G_1}{1 + P_2 G_2}$ (θ)

& between c & d: c. .d

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$$\text{loop gain}_2 = \frac{P_2 C_2}{1+P_1 C_1}$$

loop gain$_1$

$$= \frac{\frac{k_1}{s^2-a^2} C_1}{1 + C_2 \frac{k_2(s^2-b^2)}{s^2-a^2}}$$

$$= \frac{k_1 C_1}{(s^2-a^2) + k_2 C_2 (s^2-b^2)}$$

loop gain$_2$

$$= \frac{\frac{k_2(s^2-b^2)}{s^2(s^2-a^2)} C_2}{1 + C_1 \frac{k_1}{s^2-a^2}}$$

$$= \frac{k_2(s^2-b^2) C_2}{s^2\left[s^2-a^2+k_1 C_1\right]}$$

~~It~~ is

Here loop gain$_2$ has a right hand plane zero: $s = +b$.

Therefore we have a robustness issue in case of x-feedback.

But in case of θ-feedback, there is no right hand plane zero, so there is no robustness issue.