Lecture 6 - part 1

Lag Compensation

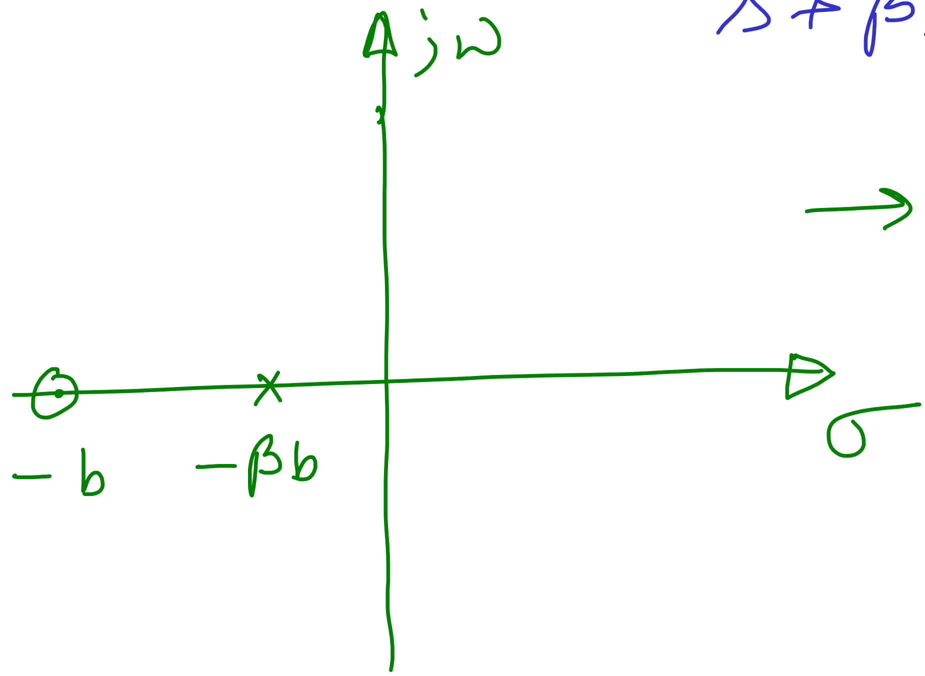
- Used to provide gain attenuation at a particular freq.
- Also used to provide small amount of phase lag.

* If an uncompensated system suffers from lack of PM then lag compensation can be used to compensate them but the g.c.f or speed of the response is reduced. Also lag compensators are used to make the system response low.

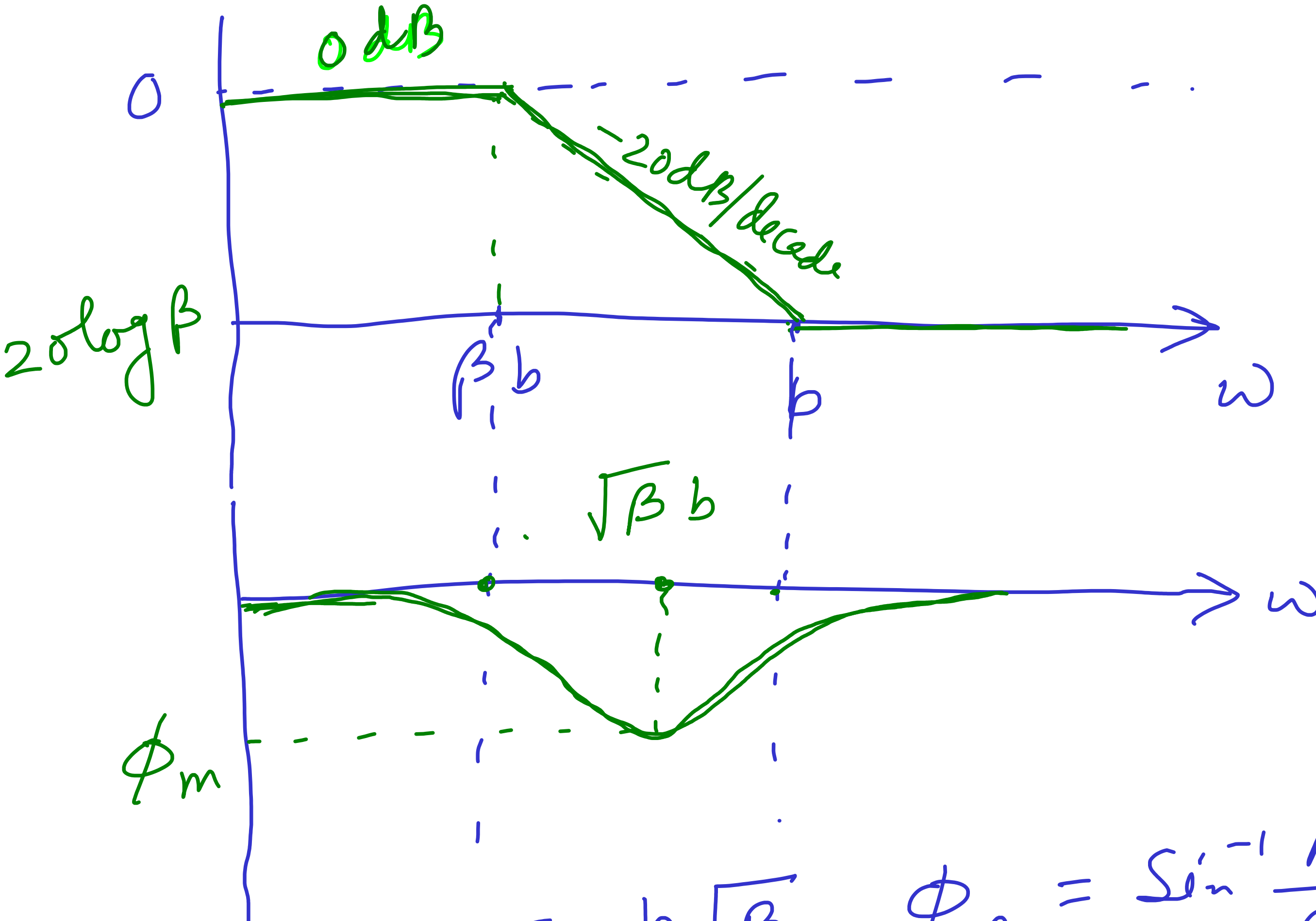
TF

$$H(s) = \frac{1 + \frac{s}{b}}{1 + s/\beta b}, \quad 0 < \beta < 1 \quad (b > 0)$$

$$\Rightarrow \quad H(s) = \frac{\beta(s+b)}{s+\beta b}$$



→ PZ-map of H(s)



$$W_m = b\sqrt{\beta}, \quad \phi_m = \sin^{-1}\frac{\beta-1}{\beta+1}$$

$$= -\sin^{-1}\frac{1-\beta}{1+\beta}$$

(-ve)

Algebraic method to design lag controllers.

objective: Given phase lag ϕ_c (-ve) and attenuation M_c (dB) (-ve) at freq ω_c, obtain β, b.

Soln: The quadratic eqn now becomes

$$\boxed{(q^2 - c + 1)\beta^2 + 2q^2 c\beta + c(q^2 c + c - 1) = 0} \quad \text{--- (1)}$$

As $q^2 c > 0$, $(\because c > 0)$ for a +ve soln of β to exist,

$$(q^2 - c + 1)(q^2 c + c - 1)c < 0$$

Now $c = 10^{M_c/10} < 1$ ($\because M_c$ negative in dB)

$$q = \tan\phi_c < 0 \quad (\because \phi_c = -ve)$$

Then $q^2 - c + 1 > 0$. Therefore for a +ve soln of β to exist,

$$q^2 c + c - 1 < 0 \quad [\text{Since } c > 0]$$

$$\Rightarrow \boxed{c < \frac{1}{q^2+1}}$$ — N & S condn for +ve soln of β

From ①,

$$q^2(\beta + c)^2 = (c-1)(\beta^2 - c)$$

$$\Rightarrow \beta^2 - c < 0 \quad [\because c - 1 < 0]$$

$$\Rightarrow \beta^2 < c < 1$$

$\Rightarrow$ If a +ve β exists then it will be < 1. Therefore

$$\boxed{c < \frac{1}{q^2+1}}$$ → is N & S condition for existence of lag compensators.

once β is obtained by solving ①, one can determine

$$\boxed{b = \frac{\omega_c}{\beta}\sqrt{\frac{\beta^2 - c}{c-1}}}$$

EX Design a lag controller that will provide a phase lag of 50° & attenuation of 15 dB at 2 rad/s.

soln Here $\phi_c = -50^\circ$, $M_c = -15\,dB$

$\omega_c = 2$ rad/s

$$\Rightarrow q = -\tan 50^\circ = -1.192$$

$$c = 10^{M_c/10} = 10^{-15/10} = 0.0316$$

Now the quadratic eqn becomes

$$(q^2 - c + 1)\beta^2 + 2q^2 c\beta + (q^2 c + c - 1)c = 0$$

one can check that the N & S condn is satisfied. $\Rightarrow$ lag controller can be designed.

$$\Rightarrow 2.3887\beta^2 + 0.0898\beta - 0.0292 = 0$$

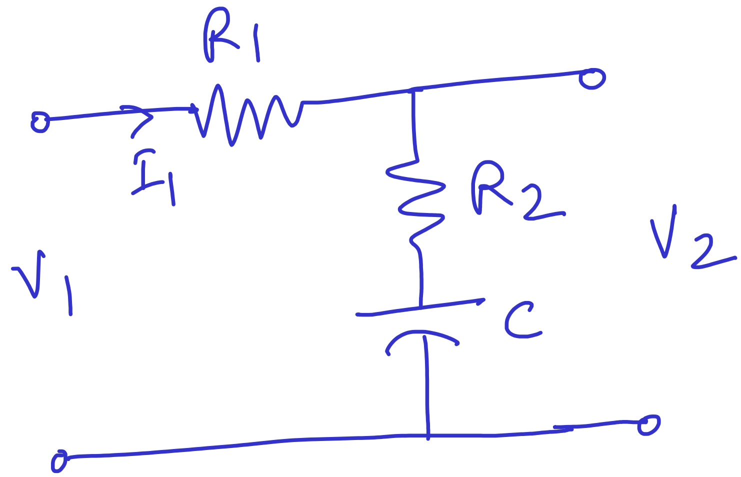
$$\Rightarrow \beta = 0.09336$$

$$b = \frac{\omega_c}{\beta}\sqrt{\frac{\beta^2 - c}{c - 1}} = 3.295$$

So the lag compensator is

$$H(s) = \frac{\beta(s+b)}{s+\beta b} = \frac{0.09336(s+3.295)}{s+0.3076}$$

An implementation of lag controller



$$I_1(s) = \frac{V_1(s)}{R_1 + R_2 + \frac{1}{Cs}}$$

$$\Rightarrow V_2(s) = I_1\left(R_2 + \frac{1}{Cs}\right)$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{sR_2C + 1}{sC(R_1+R_2)+1}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{R_2 \cancel{C}\left[s + \frac{1}{R_2C}\right]}{(R_1+R_2)\cancel{C}\left[s + \frac{1}{(R_1+R_2)C}\right]}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\beta(s+b)}{s+\beta b}$$

where $\beta = \frac{R_2}{R_1+R_2}$ (ie < 1)

& $b = \frac{1}{CR_2}$

note: No feedback amplifier is needed here.

Lag-lead controller

- used to provide phase lead as well as gain attenuation at a particular freq.

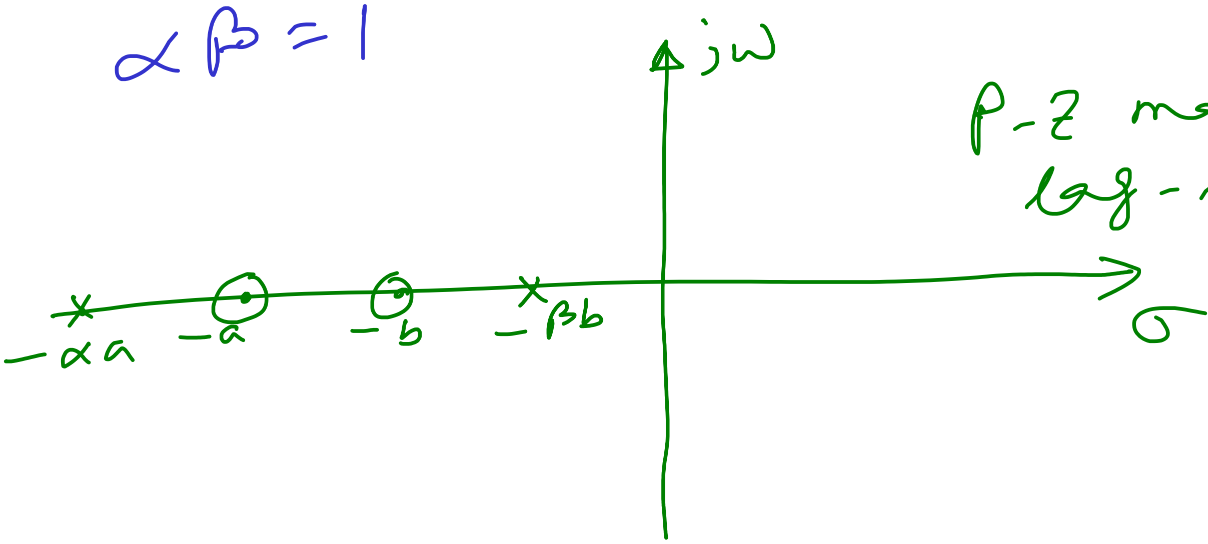
* used to trade-off the performance between lag & lead controllers.

<u>TF</u>:

$$H(s) = \underbrace{\frac{1+\frac{s}{a}}{1+\frac{s}{\alpha a}}}_{\text{lead}} \cdot \underbrace{\frac{1+\frac{s}{b}}{1+\frac{s}{\beta b}}}_{\text{lag}}$$

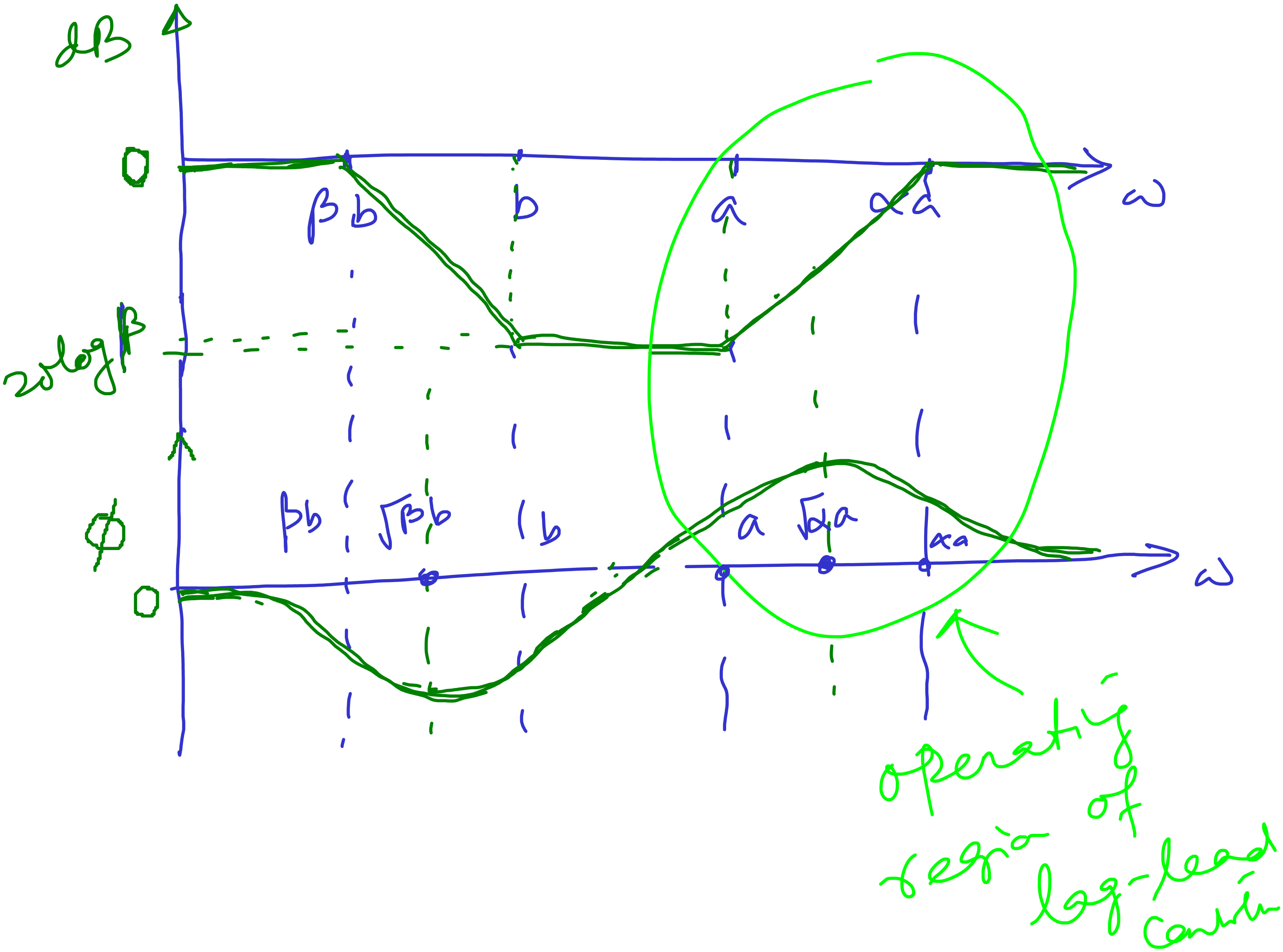
where $\alpha > 1$, $(\beta < 1)$, $a > b$,

$\alpha\beta = 1$



P-Z map of lag-lead controller

Bode plot of lag-lead controller



Lecture - 6 - part 2

Design of lag-lead Controller

objective: Given a phase lead of ϕ_c & attenuation M_c (in dB) at a particular freq ω_c, determine a, b, α

soln :

$$M_c = \underbrace{20\log_{10}\beta}_{\text{attenuation by the lag part}} + \underbrace{20\log_{10}\frac{\alpha\sqrt{\omega_c^2+a^2}}{\sqrt{\omega_c^2+\alpha^2a^2}}}_{\text{gain of the lead part}}$$

$$\Rightarrow \frac{M_c}{10} = \log(\beta^2) + \log\frac{\alpha^2(\omega_c^2+a^2)}{\omega_c^2+\alpha^2a^2}$$

$$\Rightarrow \frac{M_c}{10} = \log\frac{(\omega_c^2+a^2)}{(\omega_c^2+\alpha^2a^2)}$$

$$\Rightarrow \frac{M_c}{10} = \log\frac{\bar{\omega}_c^2+1}{\bar{\omega}_c^2+\alpha^2} \quad \left[\text{where } \bar{\omega}_c = \frac{\omega_c}{a}\right]$$

Then $C = 10^{M_c/10}$

$$= \frac{\bar{\omega}_c^2 + 1}{\bar{\omega}_c^2 + \alpha^2} \quad \text{--- (1)}$$

Here $q = \tan(\underbrace{\phi_c + \phi_2})$

phase lead to be provided by the lead part

small $(1^\circ$–$5^\circ)$ phase lag by the lag part.

Now
$$q = \frac{\bar{\omega}_c(\alpha - 1)}{\alpha^2 + \bar{\omega}_c^2} \quad \text{--- (2)}$$
[Same as lead Controller]

From (1) & (2), by eliminating $\bar{\omega}_c$ one obtains

$$\boxed{(q^2 C + C - 1)C\alpha^2 + 2q^2 C\alpha + (q^2 - C + 1) = 0} \checkmark$$

As $q^2 C > 0$, for existence of a +ve soln $(q^2 C + C - 1)(q^2 - C + 1) < 0$

Now since $0 < c < 1 \Rightarrow c(q^2 - c + 1) > 0$

therefore $q^2 c + c - 1 < 0$

$$\Rightarrow \boxed{c < \frac{1}{q^2+1}}$$ → N & S condition for existence of log-lead cont.

[Note a +ve soln of α ensures $\alpha > 1$ as well as shown in lead controller case].

Once α is obtained, then from (1)

$$c\bar{\omega}_c^2 + c\alpha^2 = \bar{\omega}_c^2 + 1$$

$$\Rightarrow \bar{\omega}_c^2 (1 - c) = \alpha^2 c - 1$$

$$\Rightarrow \bar{\omega}_c = \sqrt{\frac{\alpha^2 c - 1}{1 - c}}$$

$$\Rightarrow \frac{\omega_c}{a} = \sqrt{\frac{\alpha^2 c - 1}{1 - c}}$$

$$\Rightarrow \boxed{a = \omega_c \sqrt{\frac{1-c}{\alpha^2 c - 1}}}$$ ✓

To find b if α is known

$$-\phi_2 = \tan^{-1}\frac{\omega_c}{b} - \tan^{-1}\frac{\omega_c}{\beta b}$$

$$\Rightarrow \tan(-\phi_2) = \frac{\frac{\omega_c}{b} - \frac{\omega_c}{\beta b}}{1 + \frac{\omega_c}{b}\cdot\frac{\omega_c}{\beta b}}$$

$$\Rightarrow \tan(-\phi_2) = \frac{\frac{\omega_c}{b}(1-\alpha)}{1 + \left(\frac{\omega_c}{b}\right)^2 \cdot \alpha}$$

$$\Rightarrow \boxed{\left(\frac{b}{\omega_c}\right)^2 + \frac{\alpha - 1}{\tan(-\phi_2)}\left(\frac{b}{\omega_c}\right) + \alpha = 0}$$

Take the smaller of two real +ve soln to obtain b, since $b << \omega_c$.

EX Determine the TF of a lag-lead controller that will provide a phase lead of 50° and attenuation of 15 dB at $\omega_c = 6$ rad/s

(consider $\phi_2 = 2^\circ$)

Soln Here $q = \tan(50^\circ + 2^\circ)$

$= 1.28$

$c = 10^{M_c/10} = 10^{-15/10} = 0.0316$

Note $c < \frac{1}{q^2+1} \Rightarrow$ lag-lead controller can be designed.

To solve α, a:

The quadratic eqn becomes

$(q^2c + c - 1)c\alpha^2 + 2q^2c\alpha + (q^2 - c + 1) = 0$

$\Rightarrow -0.02898\alpha^2 + 0.1036\alpha + 2.6066 = 0$

$\Rightarrow \boxed{\alpha = 11.4376}$

$\therefore a = w_c\sqrt{\frac{1-c}{\alpha^2 c - 1}}$

$\Rightarrow \boxed{a = 3.3337}$

To solve for b.

$$\left(\frac{b}{\omega_c}\right)^2 + \frac{\alpha - 1}{\tan(-\phi_2)}\left(\frac{b}{\omega_c}\right) + \alpha = 0$$

$$\Rightarrow \left(\frac{b}{\omega_c}\right)^2 - 298.893\left(\frac{b}{\omega_c}\right) + 11.4376 = 0$$

$$\Rightarrow \frac{b}{\omega_c} = 0.0383$$

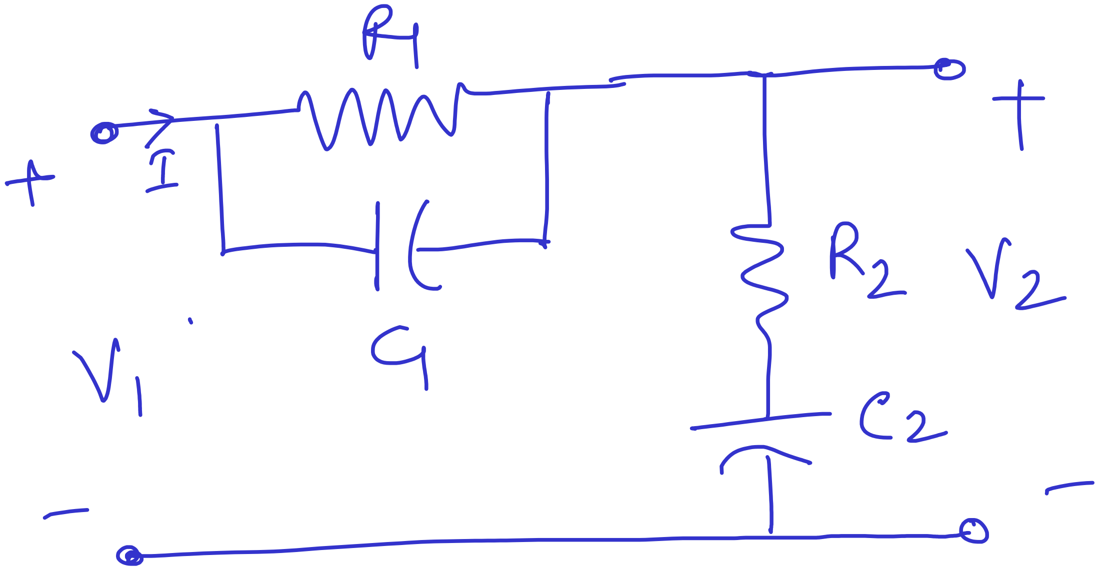
$$\Rightarrow \boxed{b = 0.2296}$$

Then the lag-lead controller becomes

$$H(s) = \frac{(s+a)(s+b)}{(s+\alpha a)(s+\beta b)}$$

$$= \frac{(s+3.3337)(s+0.2296)}{(s+38.1295)(s+0.0201)}$$

<u>Lag-lead controller</u> implementation.



Here $$I(s) = \frac{V_1}{R_1 \parallel \frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}}$$

$$\Rightarrow \quad V_2(s) = \left(R_2 + \frac{1}{C_2 s}\right) I$$

$$\Rightarrow \quad \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{C_2 s}}{R_1 \parallel \frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}}$$

$$\Rightarrow \quad \frac{V_2(s)}{V_1(s)} = \frac{\left(R_2 + \frac{1}{C_2 s}\right)\left(R_1 + \frac{1}{C_1 s}\right)}{R_1 \frac{1}{C_1 s} + \left(R_2 + \frac{1}{C_2 s}\right)\left(R_1 + \frac{1}{C_1 s}\right)}$$

$$\Rightarrow \quad \frac{V_2(s)}{V_1(s)} = \frac{\left(R_2 + \frac{1}{C_2 s}\right)\left(R_1 + \frac{1}{C_1 s}\right)}{R_1 R_2 + R_1/C_1 s + \frac{R_2}{C_1 s} + \frac{R_1}{C_2 s} + \frac{1}{C_1 C_2 s}}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\left(1+\frac{1}{R_2C_2s}\right)\left(1+\frac{1}{R_1C_1s}\right)}{1+\frac{1}{C_1R_2s}+\frac{1}{C_1R_1s}+\frac{1}{C_2R_2s}+\frac{1}{C_1C_2R_1R_2}\cdot\frac{1}{s^2}}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+s\left(\frac{1}{C_1R_1}+\frac{1}{C_2R_2}+\frac{1}{C_1R_2}\right)+\frac{1}{C_1C_2R_1R_2}}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{(s+a)(s+b)}{(s+\alpha a)(s+\beta b)}$$

Where $a = \frac{1}{C_1R_1}$, $b = \frac{1}{C_2R_2}$

$$\alpha a + \beta b = \frac{1}{C_1R_1} + \frac{1}{C_2R_2} + \frac{1}{C_1R_2}$$

$$\Rightarrow \alpha a + \beta b = a + b + \frac{1}{C_1R_2}$$

$$\Rightarrow \frac{1}{CR_2} = (\alpha-1)a + (\beta-1)b$$

[Note: No feedback amplifier is needed)