Statistical Signal Processing

Mid-Semester Test, Spring 2021-22

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Q1a Here it is given that $F_X(x)$ is a distribution function.
Since F is a distribution function, it is non decreasing.

Let $Y = F_X(X)$.

then we can find the distribution function of Y by,

$$G_Y(y) = P(Y \leq y) = P[F(X) \leq y]$$
$$= P[X \leq F^{-1}(y)]$$

Since, F is non-decreasing & given to be continuous, the inverse exists.

$$\therefore G_Y(y) = F[F^{-1}(y)]$$
$$\Rightarrow \boxed{G_Y(y) = y}$$

$\therefore$ the probability distribution function of $Y = F(X)$

$$g_Y(y) = \frac{d}{dy}[G_Y(y)] = 1$$

Since F is a distribution function its range is always $[0,1]$.

Hence $g_Y(y) = 1, 0 \leq y \leq 1 \Rightarrow Y$ is a uniform variate on $[0,1]$.

Q2.

Here, using the moment generating functions (mgf) of X & Y random variables.

$$M_X(t) = E(e^{tX}) = e^{t^2/2}$$

$$M_Y(t) = E(e^{tY}) = e^{t^2/2}$$

$$M_Z(t) = E(e^{tZ}) = E\{e^{t(X+Y)}\}$$

$$= E(e^{tX} e^{tY})$$

$$= M_X(t)\, M_Y(t)$$ since X & Y are independent.

$$= e^{\frac{1}{2}(2)t^2}$$

The above is the moment generating function of a normal random variable with mean 0 & variance 2.

By using the uniqueness theorem of the mgf $Z \sim N(0,2)$.

So, the pdf of $Z = N(0,2)$.

Q2.

Given,

$$V(t) = X\cos(\omega t) + Y\sin\omega t$$

$$N(0,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-x^2}{2\sigma^2}} = X = Y$$

(a)

$$V(t) = \frac{e^{\frac{-x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \cancel{e^{\frac{-x^2}{2\sigma^2}}} [\cdot \cos\omega t + \sin\omega t]$$

$$= \frac{e^{-x^2/2\sigma^2}}{\sqrt{\pi}\,\sigma} \cos(\omega t - \pi/4)$$

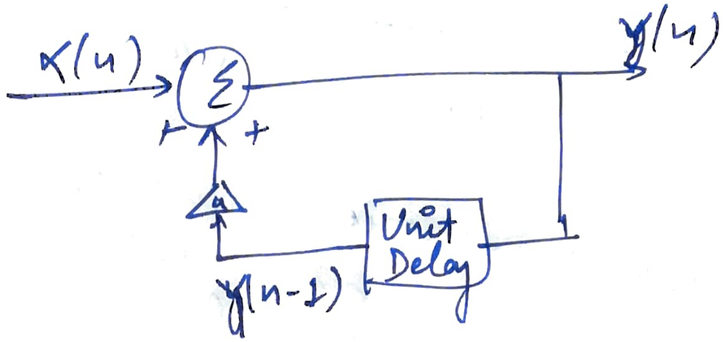
(b) By comparision,

$$R = \frac{e^{-x^2/2\sigma^2}}{\sqrt{\pi}\,\sigma}$$

$$\theta = \frac{\pi}{4}$$

$\therefore$ 'R' & 'θ' are independent.

Q 4.



Here,

$$Y(n) = aY(n-1) + x(n).$$

The impulse response of the system =

$$h(n) = a\,h(n-1) + \delta(n).$$

$$\Rightarrow h(n) = a^n u(n)$$

where $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$.

unit step sequence.

Taking fourier transform, we obtain

$$H(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \frac{1}{1 - ae^{-j\Omega}} \quad a < 1, |\Omega| < \pi$$

$$S_{xx}(\Omega) = \sigma^2 \quad |\Omega| < \pi$$

↳ power spectrum

$\therefore$ The power spectral density of the output $Y(n)$

$$S_{YY}(\Omega) = |H(\Omega)|^2 S_{xx}(\Omega)$$

$$= H(\Omega) H(-\Omega) S_{xx}(\Omega)$$

$$= \frac{\sigma^2}{(1-ae^{-j\Omega})(1-ae^{j\Omega})}$$

$$= \frac{\sigma^2}{1+a^2-2a\cos\Omega} \qquad |\Omega| < \pi$$

Taking inverse fourier transform,

$$R_{YY}(k) = \frac{\sigma^2}{1-a^2} a^{|k|}$$

Thus, the average power of $Y(n)$ is

$$E[Y^2(n)] = R_{YY}(0) = \frac{\sigma^2}{1-a^2}$$

Q6. Proof:

Let a be an arbitrary (nonzero) M-by-1 complex valued vector. We define the scalar random variable y as the inner product of a and the observation vector $u(n)$, by

$$y = a^H u(n).$$

Taking the Hermitian transpose of both sides & assuming y is a scalar, we get

$$y^* = u^H(n) \cdot a$$

{$*$ → complex conjugate}

The mean-square value of the random variable y is

$$E[|y^2|] = E[yy^*]$$

$$= E[a^H u(n) u^H(n) a]$$

$$= a^H E[u(n) u^H(n)] a$$

$$= a^H R a$$

where R is the correlation matrix, $R = E[u(n) u^H(n)]$

The expression $a^H R a$ is known as Hermitian form.

Since $E[|y^2|] \geq 0$

$$a^H R a \geq 0$$

So, a hermitian form that satisfies this condition for every nonzero a is said to be non-negative definite or positive semidefinite.

Accordingly we may state that the correlation matrix of a discrete-time stochastic process is always non-negative definite.