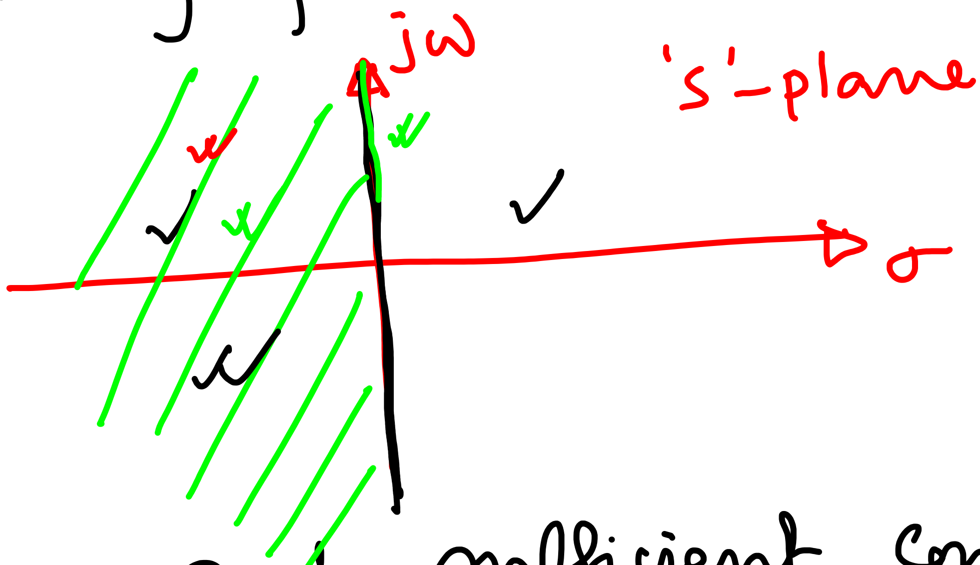
Routh-Hurwitz Criterion

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- A method of determining the location of zeros of a polynomial with constant real co-efficients with respect to the left and right-half of the 's'-plane without actually solving for the zeros.



- The necessary and sufficient condition that all roots of a constant co-efficient polynomial lie in the left-half of the 's'-plane is that the equation's Hurwitz determinant, D_k, $k = 1, 2, \ldots, n$ must all be positive.

$D_1 = a_{n-1}$, $D_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}$, $D_3 = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}$

$$D_n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & a_0 \\ 0 & 0 & 0 & & \end{vmatrix}$$

where the co-efficients with indices larger than n or negative need to be replaced by zero.

Routh table

$$a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$A = \dfrac{a_5a_4 - a_3a_6}{a_5}$	$B = \dfrac{a_5a_2 - a_1a_6}{a_5}$	$C = \dfrac{a_5a_0 - 0}{a_5} = a_0$	0
s^3	$D = \dfrac{Aa_3 - Ba_5}{A}$	$E = \dfrac{Aa_1 - a_0a_5}{A}$	0	
s^2	$F = \dfrac{DB - EA}{D}$	a_0		

$$\begin{matrix} A & B \\ C & D \end{matrix} \qquad \frac{CB - DA}{C}$$

s^1 $G = \dfrac{FE - a_0 D}{F}$ 0

$\rightarrow s^0$ a_0

For identification purpose

- The roots of the equation are all in the left half of the s-plane if all the elements of the first column are of the same sign.

- The number of changes of sign in the first column is equal to the number of roots with positive real part.

$(s-2)(s+1)(s-3)=0$

Ex.

$s^3 - 4s^2 + s + 6 = 0$ $\Rightarrow$ necessary condition fails $\Rightarrow$ unstable polynomial

s^3	1	1
s^2	-4	6
s	$\dfrac{-4-6}{-4} = 2.5$	0
s^0	$\dfrac{2.5\times 6 - 0\times -4}{2.5} = 6$	

+ 1

+ve to -ve

- -4

-ve to +ve

+ 2.5

+ve to +ve

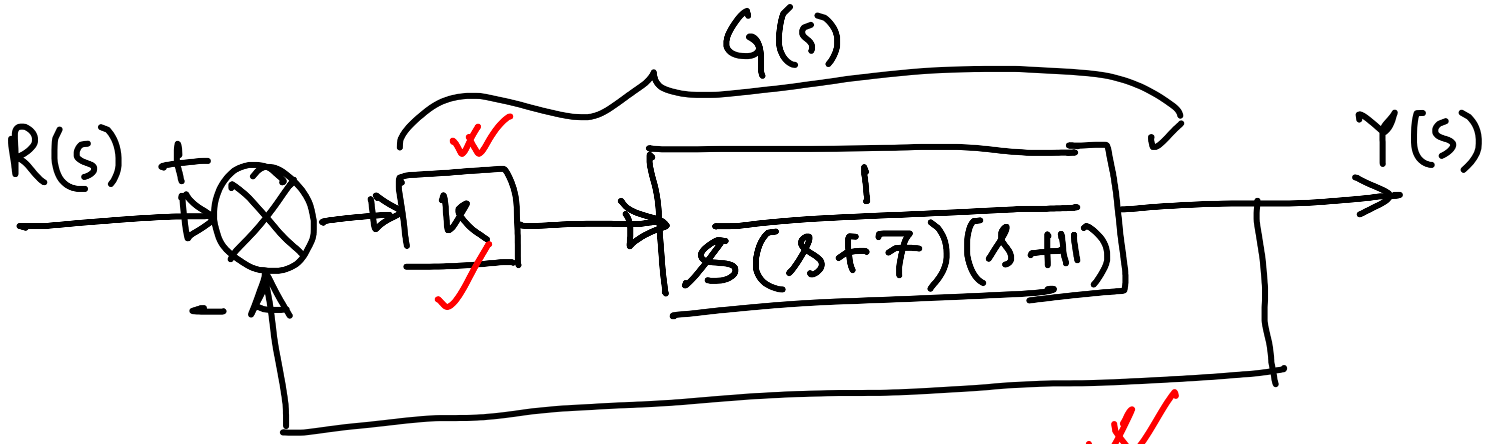
+ 6

First column

no. of Changes of sign $= 2$

Two roots are in RHP

Ex



Assume $K > 0$.

$$\text{Closed-Loop TF} = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{K}{\underbrace{s^3 + 18s^2 + 77s + K}_{\text{stable}}}$$

$$1s^3 + 18s^2 + 77s + K = 0 \rightarrow 1386$$

s^3	1	77
s^2	18	K
s	$\frac{1386-K}{18}$	0
s^0	K	

The closed-loop is to be stable, if $1386 - K > 0$

$\Rightarrow K < 1386$

$$\boxed{0 < K < 1386}$$

If $K = 1386$, then

$$s^3 + 18s^2 + 77s + 1386 = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 77 \\ s^2 & 18 & 1386 \\ s & \frac{1386-1386}{18} = 0 \;\; 36 & \\ s^0 & 1386 & \end{array}$$

no change of sign

all elements in a row become zero.

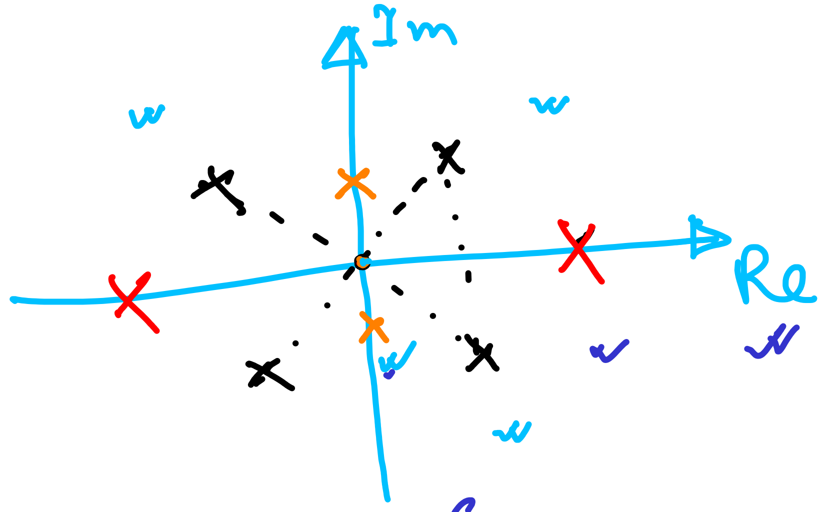
$A(s) = 18s^2 + 1386$ (Auxiliary polynomial)

$$\frac{dA(s)}{ds} = 36s$$

An even polynomial (i.e. $s^4 + 2s^2 + 5$) is a factor of the original polynomial.

↳ when a row is completely zero

Even polynomial has roots symmetric to the origin.



- No roots in the RHP. So no roots in the LHP since they should be symmetric w.r.t. origin.

– Two roots on the imaginary axis.

$$18s^2 + 1386 = 0$$

$$s^2 = -\frac{1386}{18} = -77$$

$$s_{1,2} = \pm 8.77j$$

$$\omega = \frac{2\pi}{T} = 8.77 \text{ rad/s.}$$

$$T = \frac{2\pi}{8.77} = 0.716\ s.$$

Ex. Determine the stability of the closed-loop TF

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\ \epsilon$	$7/2$	
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	
s	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	
s^0	3		

[Zero appears only in the first column]

Label	$\epsilon = +ve$	$\epsilon = -ve$
s^5	+ve	+ve
s^4	+ve	+ve
s^3	+ve	–ve
s^2	–ve	+ve
s	+ve	+ve
s^0	+ve	+ve

Change

Change

Change

Change

Two poles in RHP.

ϵ - method $\rightarrow$ when the element in the first column becomes zero.