Assignment 2

Computational Geometry CS60064

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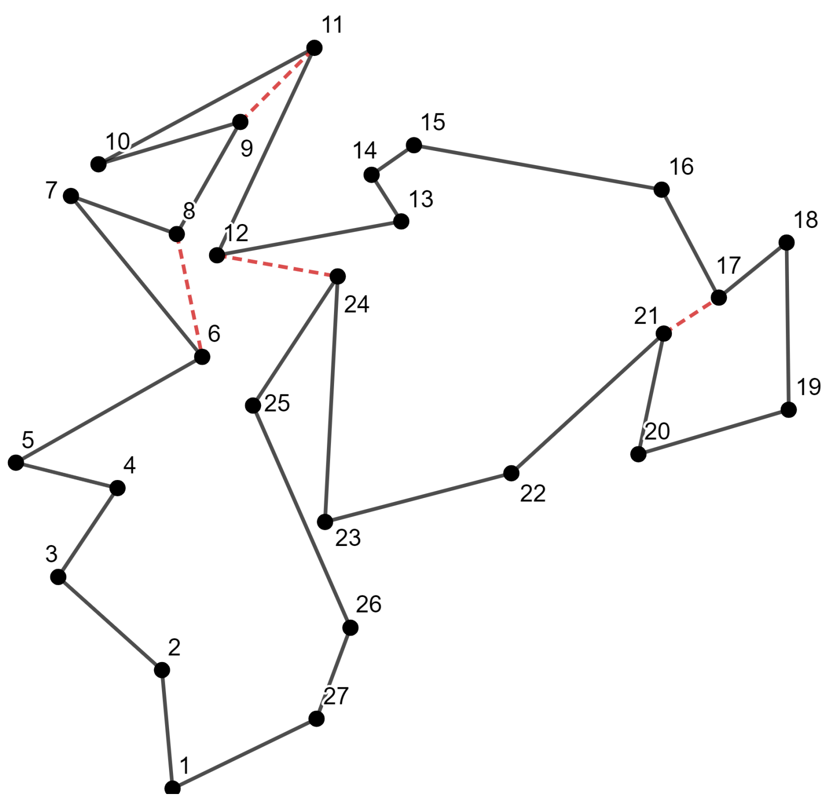
Question 1

Show a partition with minimum number of y-monotone polygons. Justify the minimality of partition.



Answer:

Minimum number of y-monotone polygons formed : 5

PROOF

1. It is known that a y-monotone polygon cannot have more than single top or a single bottom vertex, so there has to be at least 4 y-monotone polygons to accommodate all the top and bottom vertices.
2. There is only one top vertex (11) which is above the bottom vertex 10. Hence, we draw a diagonal from split vertex 9 to 11
3. Now, vertex 11 is top vertex of two different y-monotone polygons.
4. So, we need three more y-monotone polygons to accommodate rest of the top vertices. So, (2+3) the minimum number polygon is 5 as shown in the figure.
5. To avoid split or merger vertices, we add three more diagonals by joining vertices 5 to 6, 12 to 24 and 21 to 17 (Shown in red-dotted line)
6. After drawing these 4 more optimum diagonals, we get a total of 5 y-monotone polygons.

Question 2

Given n points in general positions in the $2D$-plane, sketch an $O(n \log n)$ time algorithm for determining the Tukey depth of a query point.

Answer:

Algorithm

1. Let's say the query point is Q, so form an axis (Let us call it $X - axis$) passing through Q, and find the angle subtended by the n points from that axis.
2. Sort the angles in increasing order, let's say the sorted order is α_1, α_2, α_3.... with $0 <= \alpha_i < 2\pi$ for i in $1, 2, 3, ..n$
3. Now, for every point P_i, we find the point P_j with the largest angle such that angle subtended by $P_j \leq \pi +$ angle subtended by P_i. We can say that we have two halves of plane, one containing $(j - i + 1)$ points and the other containing $(n - (j - i + 1)$. To get the tukey depth, we take minimum of $(j - i + 1)$ and $(n - (j - i + 1)$ across all $i \leq \pi$ and add 1 (to include Q).
4. To find the appropriate j, we can use two-pointer approach, since for every pair (i, j) and (i_1, j_1)
$$i_1 \geq i \implies j_1 > j$$
So, we can just find relevant j for i in $O(n)$

   TIME COMPLEXITY

(a) Calculating angles for every point takes $O(n)$ time

(b) Sorting the angles takes $O(n \log n)$

(c) Finding minimum across all points also takes $O(n)$ using two-pointer approach.

Total time complexity : $O(n \log n)$

Question 3

You are given a simple polygon P with n sides and two points s and t in P, and let T denote a triangulation of P. Show that the Euclidean Shortest Path (ESP) between s and t is unique. Also show that the minimum set of triangles containing the ESP forms a path in the dual tree of T.

Answer:

Euclidean Shortest Path (ESP) between s and t is unique

Let us denote the **ESP** and shortest euclidean distance between s and t be $\mu_0(s,t)$ and $d(s,t)$ respectively. Since this is the ESP for the specified pair of points, it can be assumed that path turns (at the Polygon vertices) will be having concave angles.

If $\mu(s,t)$ is not unique, then let $\mu_1(s,t)$ and $\mu_2(s,t)$ be two distinct shortest paths from s to t. Let α and β be two points of $\mu_1(s,t) \cap \mu_2(s,t)$ such that the two paths are disjoint between α and β; that is, $\mu_1(s,t) \cap \mu_2(s,t) = \alpha, \beta$. We have $|\mu_1(\alpha,\beta)| = |\mu_2(\alpha,\beta)| = d(\alpha,\beta)$. The two paths $\mu_1(\alpha,\beta)$ and $\mu_2(\alpha,\beta)$ enclose some region of P's interior, free of obstacles, since P is simple. At least one of the two paths has a convex corner; cutting off the corner shortens the path, which is a contradiction, since the path was chosen to have minimum length.

Hence it is shown that the Euclidean Shortest Path (ESP) between s and t is unique.

Minimum set of triangles containing the ESP forms a path in the dual tree of T

Let $\tau(s)$ be the triangle containing s, and $\tau(t)$ be the triangle containing t. We define $\tau(\mu(s,t))$ to be the minimal set of triangles containing $\mu(s,t)$. $\mu(s,t)$ crosses each diagonal of the triangulation at most once else it could be shortened. Therefore, it crosses only the diagonals that separate s and t, each exactly once which leads to the conclusion that the triangles $\tau(\mu(s,t))$ form a path in the dual tree of T.

Question 4

4(a)

a) Let L be an arbitrary line segment interior to a convex polygon P with n vertices. Does there exist a triangulation such that the number of intersections of L with all diagonals become $O(\log n)$? If so, provide a method for constructing such a triangulation.

Answer:

Yes, there exists a triangulation such that the number of intersections of L with all diagonals become $O\log(n)$. We will be using the concept of **Ear- Clipping Algorithm**. Let us say the number of vertices in the convex polygon be n.

Algorithm

- Add diagonals being formed between the alternate vertices and chop off the triangle formed and store the new reduced polygon formed. It can be also seen that the new polygon formed after one iteration will also be convex and will be having $\frac{n}{2}$ or one more if odd number of vertices.
- Repeat the above step until whole polygon is triangulated.

Since the number of vertices in every iteration is decreasing by half, the number of diagonals will also decrease by half. So we conclude that the number of iterations will be $O(\log n)$.

Correctness of Proposed Algorithm Let us consider an iteration having d diagonals going to be formed (edges for new polygon) for the next iteration. Then it is obvious that the line segment L will be intersecting maximum of two such diagonals because once it gets out of the convex region, it won't be re-entering that again as this is a convex polygon. And we know that the total number of iterations will be $O(\log n)$ making the overall intersections, $O(\log n)$.

4(b)

b) Are there any polygons such that for any triangulation, such a line L will have $\Omega(n)$ intersections with diagonals? If so, show an example.

Answer: If we take the line segment L which lies completely inside a triangle of the triangulation formed then the number of intersections will be zero for that triangulation. **So, there doesn't exist any such polygon such that for any triangulation, such a line L will have $\Omega(\log n)$ intersections with diagonals**.