Assignment-2

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(a)

$$e(n) = d(n) - \sum_{k=0}^{M-1} w_k^* u(n-k)$$

where:

$u \rightarrow$ input

$e \rightarrow$ error

$d \rightarrow$ desired output

$w_k \rightarrow$ weights

$M \rightarrow$ order of filters

$$J = E[e(n) e^*(n)]$$

$$= E[|d(n)|^2] - \sum_{k=0}^{M-1} w_k^* E[u(n-k) d^*(n)]$$

$$- \sum_{k=0}^{M-1} w_k E[u^*(n-k) d(n)]$$

$$+ \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* w_i E[u(n-k) u^*(n-i)]$$

$\rightarrow E[|d(n)|^2] = \sigma_d^2$

$\rightarrow p(-k) = E[u(n-k) d^*(n)]$ } cross-correlation

$\rightarrow p^*(-k) = E[u^*(n-k) d(n)]$

$\rightarrow r(i-k) = E[u(n-k) u^*(n-i)]$ } auto-correlation

$$\boxed{J = \sigma_d^2 - \sum_{k=0}^{M-1} w_k^* \, p(-k) - \sum_{k=0}^{M-1} w_k \, p^*(-k) + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* \, w_i \, r(i-k)}$$

This represents the error performance surface of the FIR filter and the analytical expression is given here.

According to the given R & p, we substitute various terms.

$$\sigma_d^2 = E\left[w_o^H u(n) u^H(n) w_o\right]$$
$$= w_o^H R w_o$$

Also, $w_o = R^{-1}p$

$$\sigma_d^2 = (R^{-1}p)^H R (R^{-1}p) = p^H R^{-1} p.$$

$$J = \sigma_d^2 - w^H p - p^H w + w^H R w$$

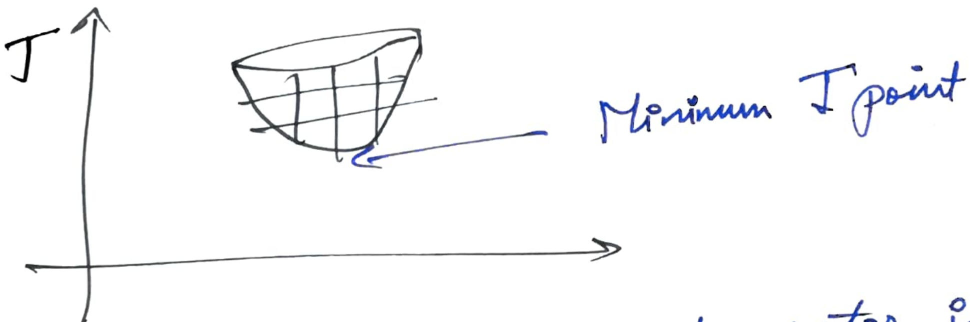
$$R = \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

we get,

$$J = 0.5 - \begin{bmatrix} 0. & 0.2939 \end{bmatrix} \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

$$+ \left(w - \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix} \right)^H \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix} \left(w - \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.2939 \end{bmatrix} \right)$$

(b) The cost function J is bowl shaped curve characterized by filter tap weights $w_0, w_1, \dots w_{M-1}$.
Since the error surface is bowl shaped, it is characterized by a unique minimum.



At the minimum, the gradient vector is 0.

Here. $\nabla_k J = 0 \quad ; \quad k = 0, 1, \dots M-1$

let $w_k = a_k + j b_k$.

$$\nabla_k J = \frac{\partial J}{\partial a_k} + j \frac{\partial J}{\partial b_k}$$

$$J = \sigma_d^2 - \sum_{k=0}^{M-1} w_k^* p(-k) - \sum_{k=0}^{M-1} w_k p^*(-k) + \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k^* w_i r(i-k)$$

$$\frac{\partial J}{\partial a_k} = 0 - p(-k) - p^*(-k) + \sum_{i=0}^{M-1} w_i r(i-k)$$

Similarly,

$$\frac{\partial J}{\partial b_k} = 0 + j\,p(-k) - j\,p^*(-k) - j\sum_{i=0}^{M-1} w_i\, r(i-k)$$

$$\frac{\partial J}{\partial a_k} + \frac{\partial J}{\partial b_k} = -p(-k) - p^*(-k) + \sum_{i=0}^{M-1} w_i\, r(i-k)$$

$$- p(-k) + p^*(-k) + \sum_{i=0}^{M-1} w_i\, r(i-k)$$

$$\boxed{\nabla_k J = -2p(-k) + 2\sum_{i=0}^{M-1} w_i\, r(i-k)}$$

↳ Gradient Equation

$$\nabla_k J = 0$$

$$\sum_{i=0}^{M-1} w_i\, r(i-k) = p(-k)$$

↳ Wiener-Hopf Equation equivalent for optimal filter weights

(c) $\boxed{R w_0 = p}$ (Wiener Hopf eqn)

$$w_0 = R^{-1} p$$

$$R = \begin{bmatrix} 1 & 0.4045 \\ 0.4045 & 1 \end{bmatrix}$$

$$R^{-1} = \frac{1}{|R|}\begin{bmatrix} 1 & -0.4045 \\ -0.4045 & 1 \end{bmatrix}$$

$$|R| = 1 - (0.4045)^2 = 0.8363$$

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$$R^{-1} = \frac{1}{0.8363}\begin{bmatrix} 1 & -0.4045 \\ -0.4045 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

$$w_0 = \frac{1}{0.8363}\begin{bmatrix} 1 & -0.4045 \\ -0.4045 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.2939 \end{bmatrix}$$

$$\boxed{w_0 = \begin{bmatrix} -0.14215 \\ 0.3514 \end{bmatrix}}$$

↳ Wiener-Hopf Solution

(e) The stepsize parameter μ

$$0 < \mu < \frac{2}{\lambda_{max}} \quad \text{(from stability)}$$

λ_{max} is the largest value of R matrix

$$R\alpha = \lambda\alpha \Rightarrow (R - \lambda I)\alpha = 0$$

$$\Rightarrow |R - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0.4045 \\ 0.4045 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = (0.4045)^2$$

$$\Rightarrow \lambda = 1.4045,\ 0.5955.$$

So, $\lambda_{max} = 1.4045$

$$0 < \mu < \frac{2}{1.4045}$$

$$\Rightarrow \boxed{0 < \mu < 1.4239}$$