Basic rules for sketching the root locus

1. **No. of branches:** The number of branches of the root locus equals to the number of closed-loop poles.

Ex.

$$1 + K\frac{(s+2)}{(s+3)(s+4)} = 0$$

Ch. eqn: $(s+3)(s+4) + K(s+2) = 0$ $\quad K=0$

$K \geq 0$

No. of closed-loop poles $= 2$

Open-loop poles: $-3, -4$

Zero: -2, ∞

✓ $K=0$

when $K=0$, $s=-3$ and $s=-4$ are two roots of the ch. eqn.

$K=\infty$

$$\underbrace{\frac{1}{K}}_{=0} + \frac{s+2}{(s+3)(s+4)} = 0$$

$$\frac{1}{K} + \frac{\underbrace{\frac{1}{s}}_{=0} + \underbrace{\frac{2}{s^2}}_{=0}}{\left(1+\underbrace{\frac{3}{s}}_{=0}\right)\left(1+\underbrace{\frac{4}{s}}_{=0}\right)} = 0$$

When $K \to \infty$, $s=-2$ is a root of ch. eqn.

$K \to \infty$, $s = \infty$ is a root of ch. eqn.

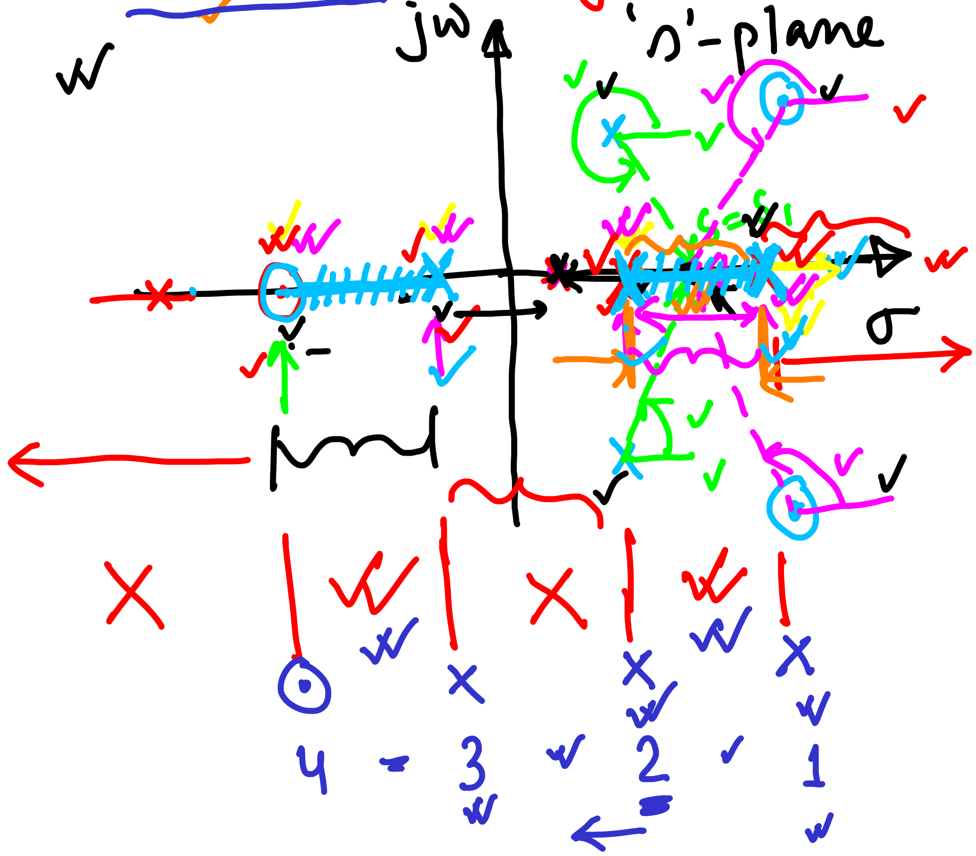
2. **Symmetry:** The root locus is symmetrical about the real-axis since the polynomial is a real co-efficient polynomial.

3. **Real-axis segment:**

$360^\circ + 180^\circ$, 0°

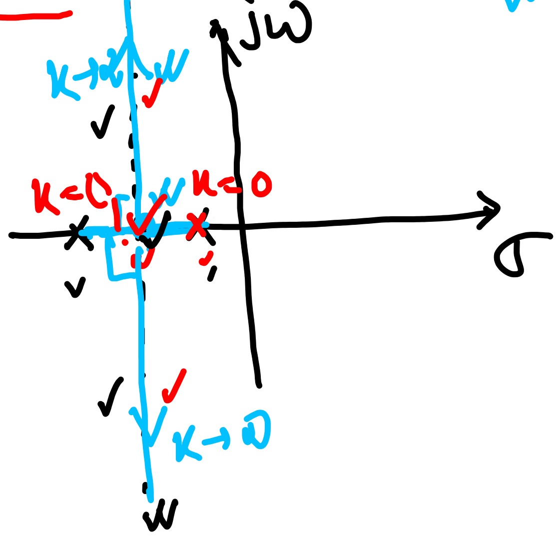
$$\angle KG(s)H(s) = (2k+1)180^\circ = 180^\circ$$

180°



- Resultant contribution of complex pole and/or zero is $360^\circ/0^\circ$.
- On the real-axis, for $k>0$, the root locus exists to the left of an odd number of real axis finite open-loop poles and/or finite open-loop zeros.

4. **Centroid and asymptots:**



The root locus approaches straight lines as asymptotes as the locus approaches infinity.

The real-axis intercept

$$\sigma = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{(\text{no. of finite poles}) - (\text{no of finite zeros})}$$

Angle of asymptotes

$$\theta = \frac{(2k+1)\pi}{(\text{no. of finite poles}) - (\text{no. of finite zeros})}$$

$k = 0, \pm 1, \pm 2, \dots$

It is needed when some zeros are at infinity.

Ex.

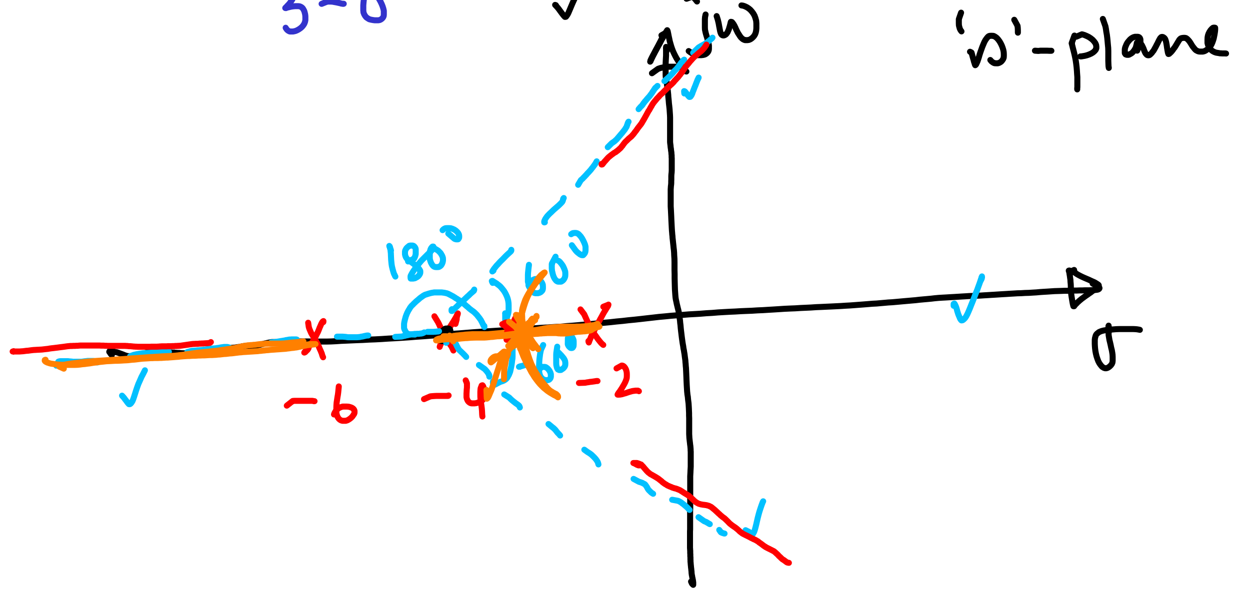
$$G(s) = \frac{k}{(s+2)(s+4)(s+6)} = kG(s)H(s)$$

No. of finite poles $= 3$ $(-2, -4, -6)$

No. of finite zeros $= 0$

$$\sigma = \frac{(-2-4-6)-(0)}{3-0} = \frac{-12}{3} = -4$$

$$\theta = \frac{180^\circ}{3-0} = 60^\circ, 180^\circ, 300^\circ$$



5. Real-axis breakaway or break-in points:

$$1 + K\,G(s)H(s) = 0$$

$$K = -\frac{1}{G(s)H(s)}$$

Calculate $\frac{dK}{ds}$ and find the value of s so that $\frac{dK}{ds} = 0$.

Ex.

$$1 + \frac{K(s-3)(s-5)}{(s+1)(s+2)} = 0$$

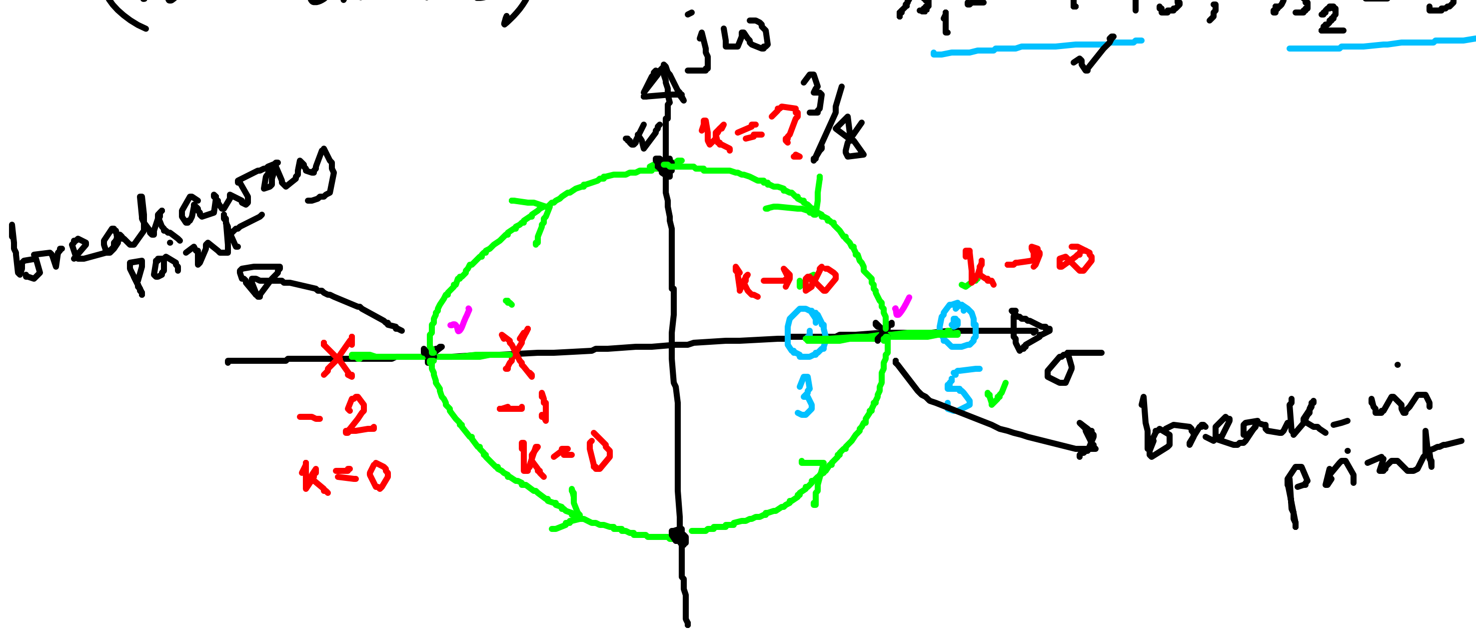
$G(s)H(s)$

$$\Rightarrow \frac{K(s-3)(s-5)}{(s+1)(s+2)} = -1$$

$$\Rightarrow K = -\frac{(s^2+3s+2)}{(s^2-8s+15)}$$

$$\frac{dK}{ds} = \frac{11s^2-26s-61}{(s^2-8s+15)^2} = 0$$

$s_1 = -1.45$, $s_2 = 3.82$



6. The $j\omega$-axis crossing:

The Routh-Hurwitz criterion is used to find the $j\omega$-axis crossing point and the corresponding gain K.

$$s^2+3s+2+K(s^2-8s+15)=0$$

$$s^2(K+1)+s(3-8K)+(2+15K)=0$$

s^2	$(K+1)$	$(2+15K)$
s	$(3-8K)$	
s^0	$(2+15K)$	

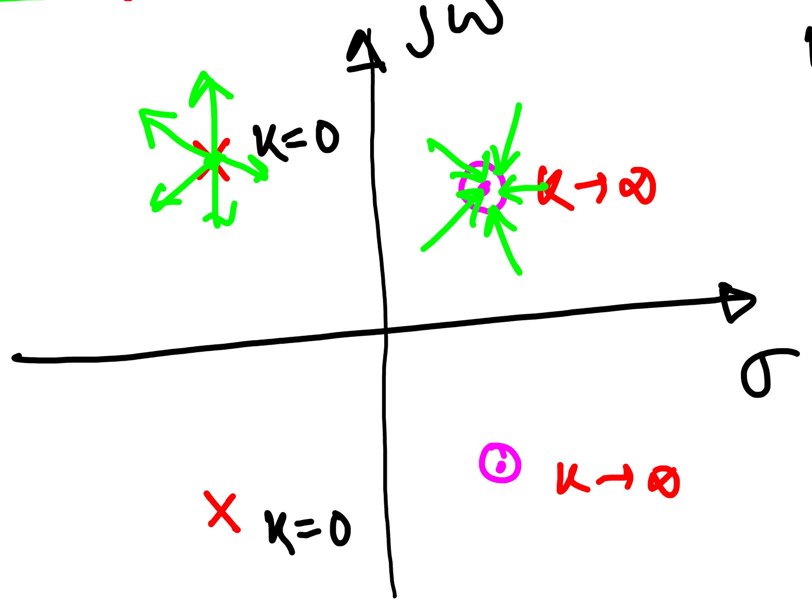
$$3-8K=0$$

$$\Rightarrow K=\frac{3}{8}$$

Auxiliary eqn:

$$A(s)=\frac{11}{8}s^2+\frac{61}{8}=0$$

7. Angle of departure and Arrival:

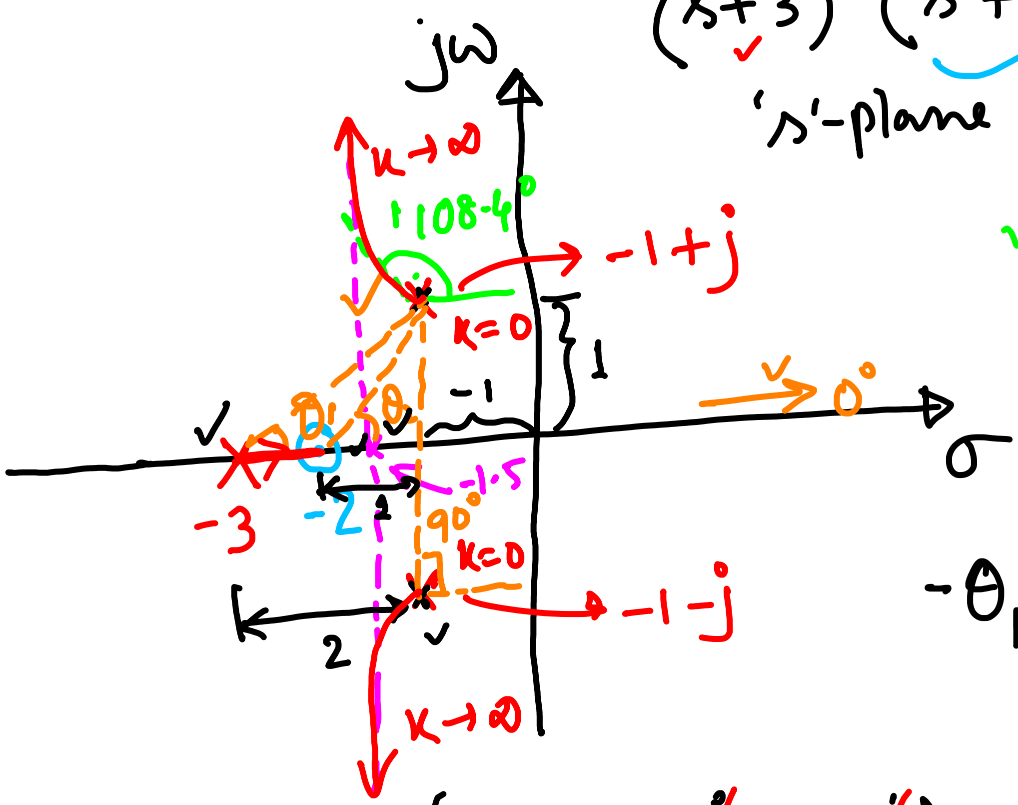


When complex conjugate pole and/or zero exist, we need to find angle of departure or arrival.

Ex

$$KG(s)H(s) = \frac{K(s+2)}{(s+3)\underbrace{(s^2+3s+2)}}$$

$s_{1,2} = -1 \pm j$



A point on root locus must satisfies

$$\underline{\angle KG(s)H(s)} = (2K+1)180^\circ$$

$$-\theta_1 + \theta_2 - 90^\circ - \theta = 180^\circ$$

Centroid $\sigma = \dfrac{(-3-1+\cancel{j}-1-\cancel{j})-(-2)}{3-1} = -\dfrac{3}{2} = -1.5$

Angle of asymptotes $\theta = \dfrac{180^\circ}{2} = 90^\circ,\ 270^\circ$

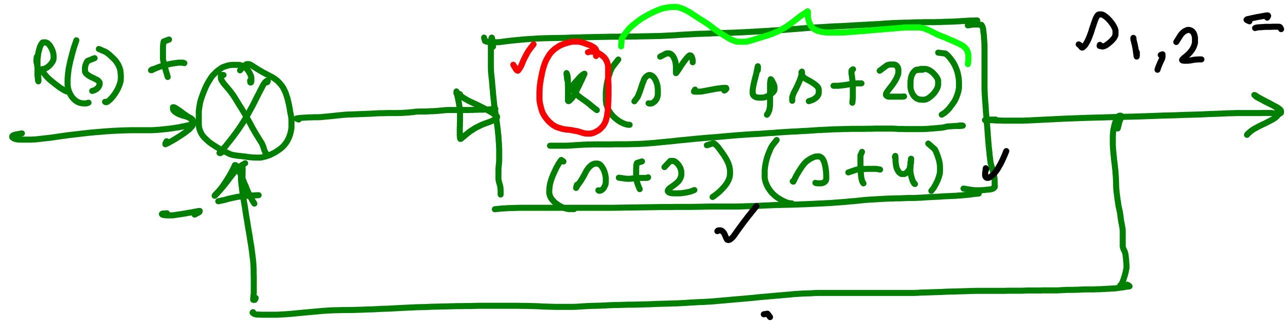
$$\theta_1 = \tan^{-1}(1/2) \quad \theta_2 = \tan^{-1}(1/1)$$

$$-\theta = 180^\circ + 90^\circ + \theta_1 - \theta_2 = 270^\circ + 26.56^\circ - 45^\circ$$

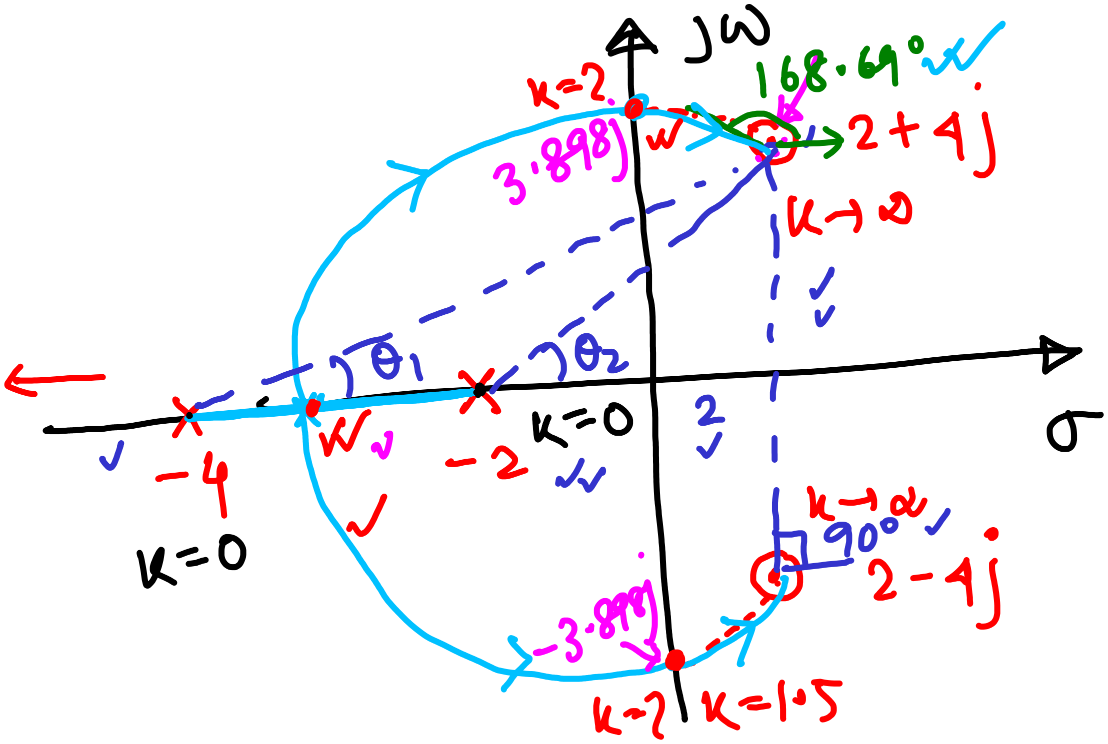
$$= 251.6^\circ$$

$$\underline{\theta = -251.6^\circ = 108.4^\circ}$$

Ex



$s_{1,2} = 2 \pm 4j$ (zeros)



$$-\theta_1 - \theta_2 + 90^\circ + \theta = 180^\circ$$

$$-\tan^{-1}(4/6) - 45^\circ + 90^\circ + \theta = 180^\circ$$

$$-33.69^\circ - 45^\circ + 90^\circ + \theta = 180^\circ$$

$$\theta = 168.69^\circ$$

Angle of arrival

$$K = -\frac{(s^2+6s+8)}{(s^2-4s+20)}$$

$$\frac{dK}{ds} = 0, \quad 5s^2 - 12s - 76 = 0$$

$$s_{1,2} = -2.86, \quad \cancel{5.28}$$

$$1 + \frac{K(s^2-4s+20)}{s^2+6s+8} = 0$$

$$s^2 + 6s + 8 + Ks^2 - 4Ks + 20K = 0$$

$$s^2(K+1) + (6-4K)s + (8+20K) = 0$$

$$\begin{array}{lll} s^2 & (K+1) & (8+20K) \\ s & (6-4K) & \\ s^0 & (8+20K) & \end{array}$$

$K > 0$

$6 - 4K = 0$

$\Rightarrow K = 6/4 = 1.5$

Auxiliary eq^n

$A(s) = 2.5s^2 + 38 = 0$

$\frac{dA(s)}{ds} = 5s$

$s_{1,2} = \pm 3.898j$

Root sensitivity (pole sensitivity)

The ratio of the fractional change in a closed-loop pole to the fractional change in a system parameter.

$$S = \frac{\delta s/s}{\delta K/K} = \frac{K}{s}\frac{\delta s}{\delta K}$$

$\frac{K}{s} \cdot \frac{\Delta s}{\Delta K} = S$

$\Rightarrow \Delta s = \frac{s}{K} S \Delta K$

Ex Let ch. eq^n be

$$s^2 + 10s + K = 0$$

$$\Rightarrow 2s\frac{ds}{dK} + 10\frac{ds}{dK} + 1 = 0$$

$$\Rightarrow \frac{ds}{dk} = -\frac{1}{2s+10}$$

$$S = \frac{k}{s} \times -\frac{1}{2s+10} = -\frac{k}{s(2s+10)}$$

- This is a complex quantity.

(see the example we discussed in the first lecture on Root Locus)

* $s = -9.47$ $\quad k = 5$

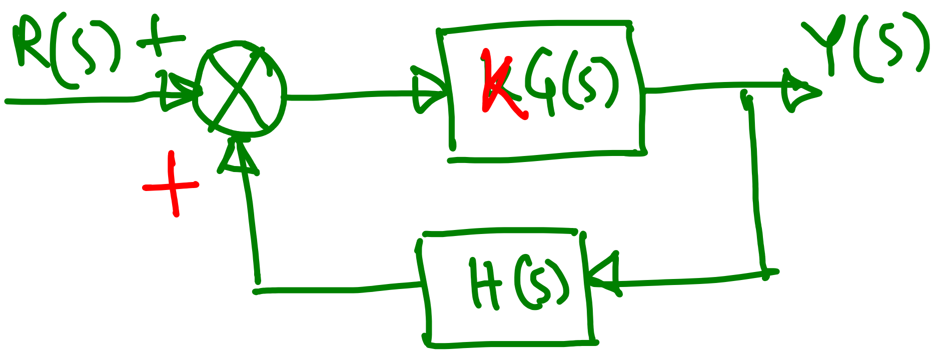
$$S = -\frac{5}{-9.47} \times \frac{1}{(-9.47\times 2+10)} = -0.059$$

$$\Delta s = S\,\frac{s}{k}\,\Delta k = -0.059 \times \frac{-9.47}{5} \times 0.5 = 0.056\angle 0^\circ$$

moves forwards right

[suppose we consider 10% change in k]

Root locus for positive feedback System

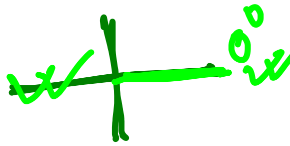


$$\text{Closed-loop TF} = \frac{K G(s)}{\underbrace{1 - K G(s) H(s)}}$$

Poles of closed-loop system satisfy

$$1 - K G(s) H(s) = 0.$$

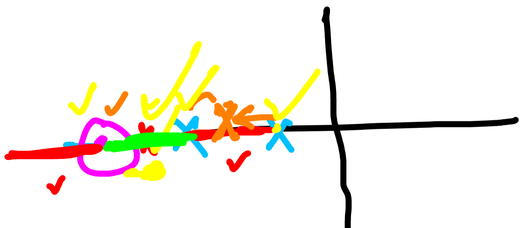
$$\Rightarrow \boxed{K G(s) H(s) = 1 = 1\angle 2k\pi = 1\angle k360^\circ} \quad k = 0, \pm 1, \pm 2$$



$$K = \frac{1}{|G(s)H(s)|}$$

$$\angle K G(s) H(s) = \angle k360^\circ, \quad k = 0, \pm 1, \cdots$$

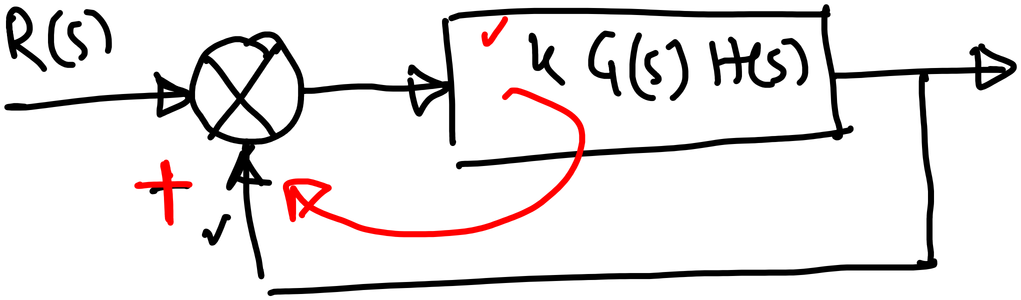
1. No. of branches: No change
2. Symmetry: No change
3. Real-axis segment: On the real-axis, the root locus exists to the left of an EVEN number of real-axis, finite open-loop poles and/or finite open-loop zeros.



4. Starting and ending: No change
5. Centroid and angle of asymptotes:

σ: No change

$$\theta = \frac{2\pi k}{(\text{no of finite poles}) - (\text{no finite zeros})}, \quad k = 0, \pm 1, \cdots$$



$k > 0$

$0 \le k < \infty$

If we consider $k < 0$.

$-\infty < k \le 0$