ASSIGNMENT – 4

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(1)

$$\frac{Y(s)}{R(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

(a) CCF :

$$\frac{Y(s)}{R(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} \cdot \frac{X(s)/s^2}{X(s)/s^2}$$

$$\frac{Y(s)}{R(s)} = \frac{(b_2 + b_1 s^{-1} + b_0 s^{-2})\, X(s)}{(1 + a_1 s^{-1} + a_0 s^{-2})\, X(s)}$$

$$\Rightarrow \quad R(s) = (1 + a_1 s^{-1} + a_0 s^{-2})\, X(s)$$

$$\boxed{X(s) = R(s) - (a_1 s^{-1} + a_0 s^{-2})\, X(s)}$$

$$\Rightarrow \quad Y(s) = (b_2 + b_1 s^{-1} + b_0 s^{-2})\, X(s)$$

$$Y(s) = b_2 X(s) + (b_1 s^{-1} + b_0 s^{-2})\, X(s)$$

$$Y(s) = b_2 \left[R(s) - (a_1 s^{-1} + a_0 s^{-2})\, X(s) \right] + (b_1 s^{-1} + b_0 s^{-2})\, X(s)$$

$$\boxed{Y(s) = b_2 R(s) + \left[(b_1 - b_2 a_1) s^{-1} + (b_0 - b_2 a_0) s^{-2} \right] X(s)}$$

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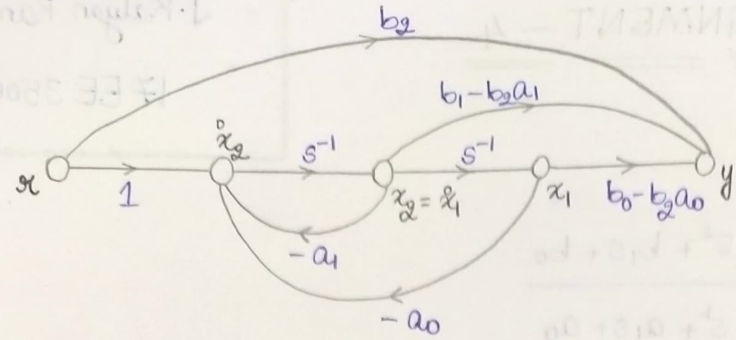
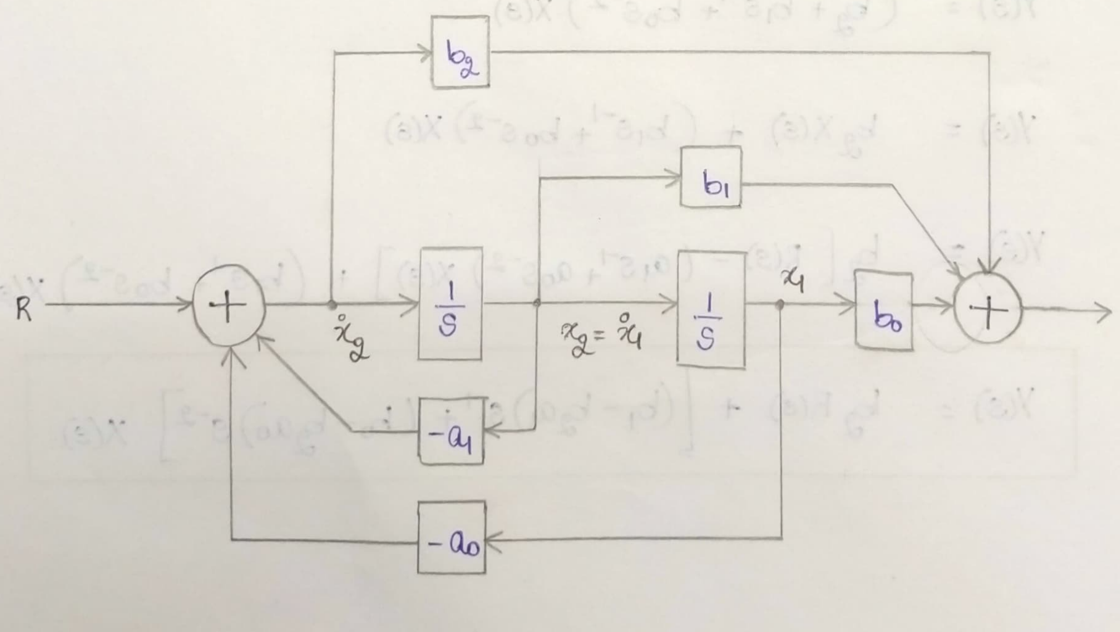


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} r$$

$$y = \underbrace{\begin{bmatrix} b_0 - b_2 a_0 & b_1 - b_2 a_1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} b_2 \end{bmatrix}}_{D} r$$

Controllability matrix : $S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -a_1 \end{bmatrix}$ Rank = 2

Observability matrix : $V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} b_0 - b_2 a_0 & b_1 - b_2 a_1 \\ -a_0(b_1 - b_2 a_1) & b_0 - b_2 a_0 - a_1(b_1 - b_2 a_1) \end{bmatrix}$ Rank = 2



(b) OCF :

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$$\frac{Y(s)}{R(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} \cdot \frac{1/s^2}{1/s^2}$$

$$\frac{Y(s)}{R(s)} = \frac{b_2 + b_1 s^{-1} + b_0 s^{-2}}{1 + a_1 s^{-1} + a_0 s^{-2}}$$

$$\Rightarrow \left(1 + a_1 s^{-1} + a_0 s^{-2}\right) Y(s) = \left(b_2 + b_1 s^{-1} + b_0 s^{-2}\right) R(s)$$

$$\Rightarrow \boxed{Y(s) = b_2 R(s) + s^{-1}\left(b_1 R(s) - a_1 Y(s)\right) + s^{-2}\left(b_0 R(s) - a_0 Y(s)\right)}$$

$$\dot{x}_1 = b_0 r - a_0 y$$

$$\dot{x}_2 = b_1 r - a_1 y + x_1$$

$$x_2 = y - b_2 r$$

$$\Rightarrow \dot{x}_1 = -a_0 x_2 + (b_0 - b_2 a_0) r$$

$$\dot{x}_2 = x_1 - a_1 x_2 + (b_1 - b_2 a_1) r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} b_0 - b_2 a_0 \\ b_1 - b_2 a_1 \end{bmatrix}}_{B} r$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} b_2 \end{bmatrix}}_{D} r$$

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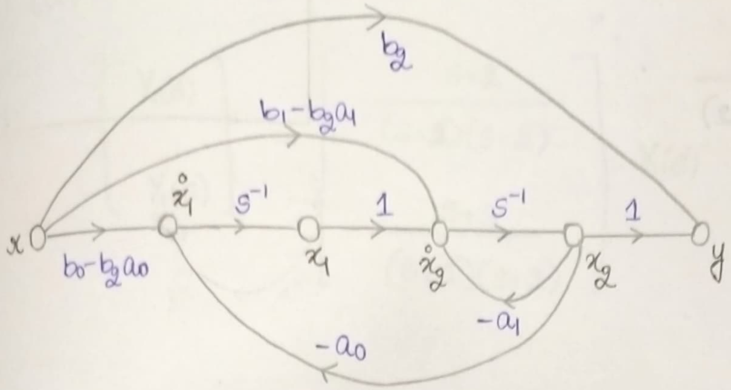
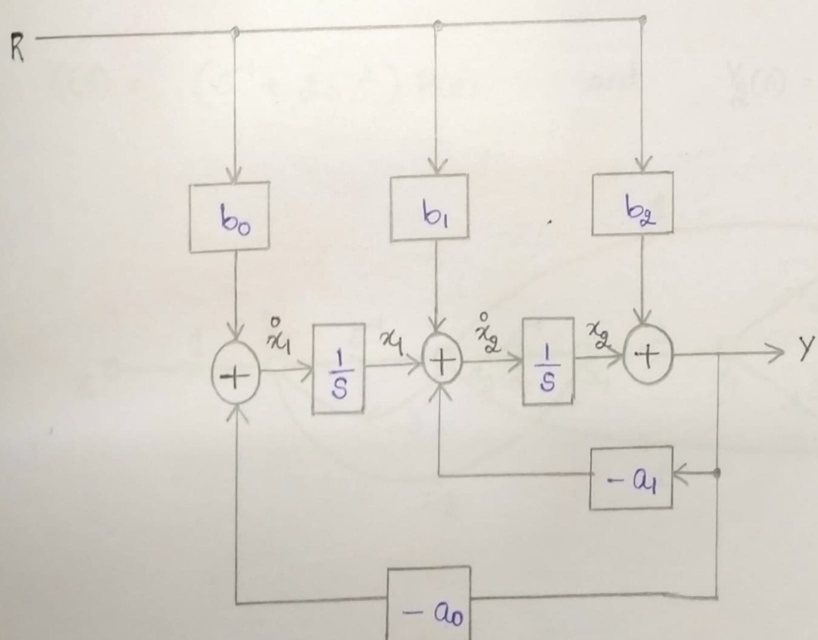


Controllability matrix : $[B \quad AB] = \begin{bmatrix} b_0 - b_2 a_0 & -a_0(b_1 - b_2 a_1) \\ b_1 - b_2 a_1 & b_0 - b_2 a_0 - a_1(b_1 - b_2 a_1) \end{bmatrix}$

Rank = 2

Observability matrix : $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -a_1 \end{bmatrix}$ Rank = 2



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2 (a)

(i) $$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{(s+1)(s+2)} \\ \dfrac{s+3}{(s+1)(s+2)} \end{bmatrix} X(s)$$

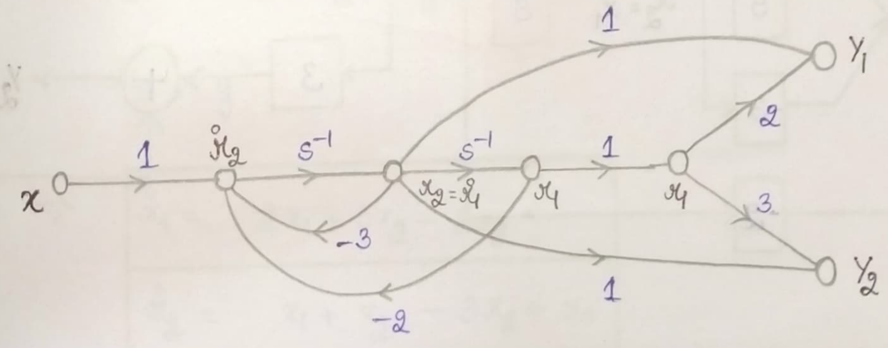
$$\frac{Y_1(s)}{X(s)} = \frac{s+2}{(s+1)(s+2)} = \frac{(s^{-1}+2s^{-2})\,R(s)}{(1+3s^{-1}+2s^{-2})\,R(s)}$$

Similarly $$\frac{Y_2(s)}{X(s)} = \frac{(s^{-1}+3s^{-2})\,R(s)}{(1+3s^{-1}+2s^{-2})\,R(s)}$$

$$\Rightarrow \quad X(s) = (1+3s^{-1}+2s^{-2})\,R(s)$$

$$\boxed{R(s) = X(s) - (3s^{-1}+2s^{-2})\,R(s)}$$

$Y_1(s) = (s^{-1}+2s^{-2})\,R(s)$ and $Y_2(s) = (s^{-1}+3s^{-2})\,R(s)$



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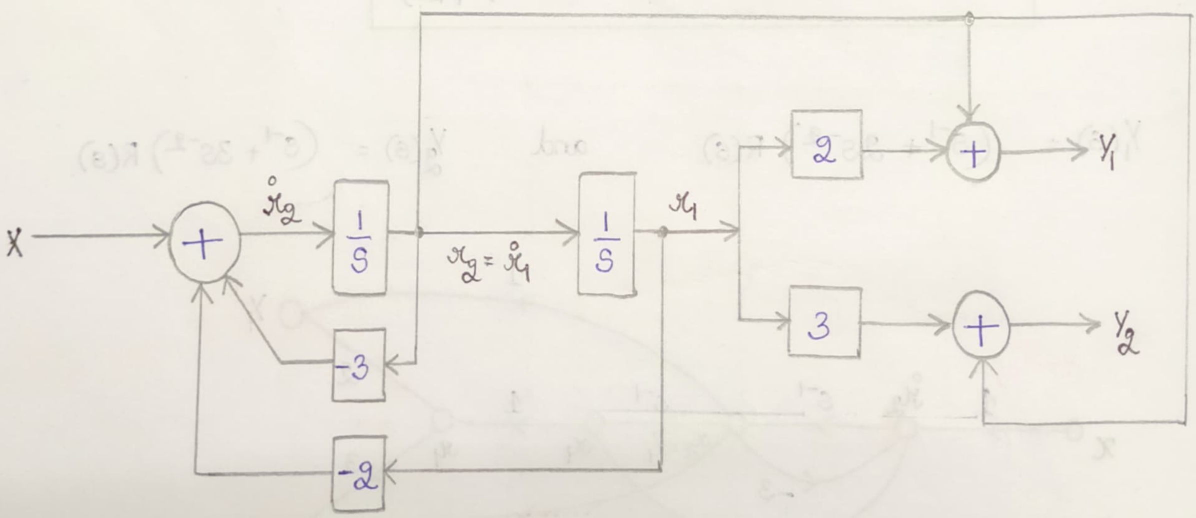
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} x$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Controllability matrix : $S = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$ Rank = 2

Observability matrix : $V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ -2 & -1 \\ -2 & 0 \end{bmatrix}$ Rank = 2



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(ii)

$$Y(s) = \begin{bmatrix} \frac{s+2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

$$Y(s) = \frac{(s+2)\,X_1(s)}{s^2+3s+2} + \frac{(s+3)\,X_2(s)}{s^2+3s+2}$$

$$Y(s) = \frac{(s^{-1}+2s^{-2})\,X_1(s) + (s^{-1}+3s^{-2})\,X_2(s)}{1+3s^{-1}+2s^{-2}}$$

$$(1+3s^{-1}+2s^{-2})\,Y(s) = s^{-1}(X_1(s)+X_2(s)) + s^{-2}(2X_1(s)+3X_2(s))$$

$$\boxed{Y(s) = s^{-1}(X_1(s)+X_2(s)-3Y(s)) + s^{-2}(2X_1(s)+3X_2(s)-2Y(s))}$$

$$\dot{x}_1 = 2x_1 + 3x_2 - 2y$$

$$\dot{x}_2 = x_1 + x_2 - 3y + x_1$$

$$x_2 = y$$

$$\Rightarrow \boxed{\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 - 2x_2 \\ \dot{x}_2 &= x_1 + x_2 - 3x_2 + x_1 \end{aligned}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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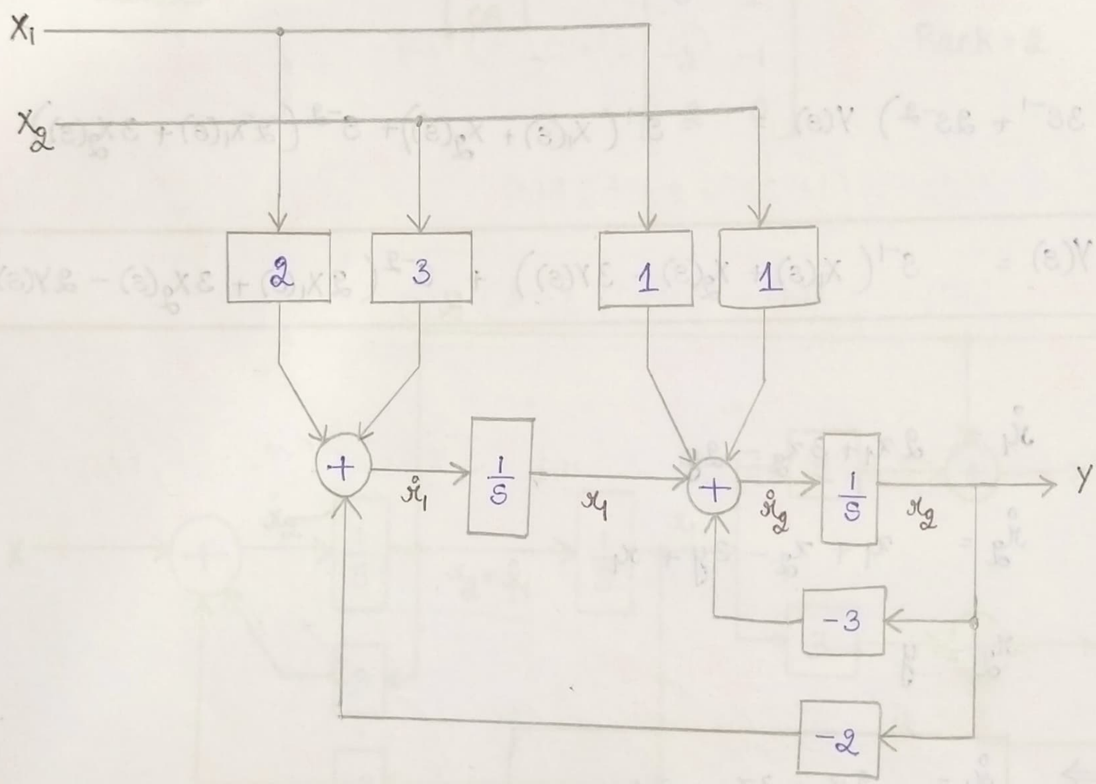
Controllability matrix : $S = \begin{bmatrix} B & AB \end{bmatrix}$

$$= \begin{bmatrix} 2 & 3 & -2 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$ Rank = 2

Observability matrix : $V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$ Rank = 2



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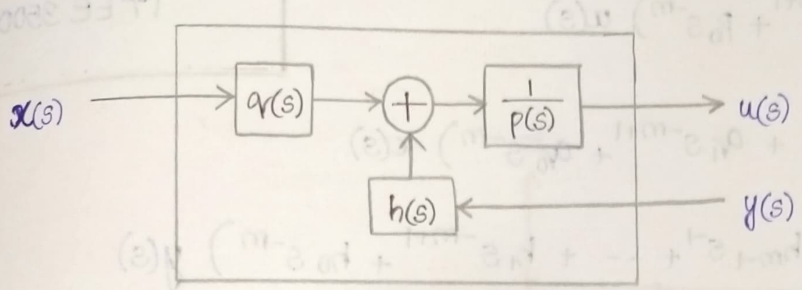
(b)



We desire m^{th} order controller:

$$\boxed{u(s) = \frac{q(s)}{p(s)} \cdot x(s) + \frac{h(s)}{p(s)} y(s)}$$

$$\frac{q(s)}{p(s)} = \frac{q_m s^m + \cdots + q_1 s + q_0}{s^m + p_{m-1} s^{m-1} + \cdots + p_1 s + p_0} \cdot \frac{1/s^m}{1/s^m}$$

and

$$\frac{h(s)}{p(s)} = \frac{h_m s^m + h_{m-1} s^{m-1} + \cdots + h_1 s + h_0}{s^m + p_{m-1} s^{m-1} + \cdots + p_1 s + p_0} \times \frac{1/s^m}{1/s^m}$$

$$u(s) = \frac{\left(q_m + q_{m-1} s^{-1} + \cdots + q_1 s^{-m+1} + q_0 s^{-m}\right) x(s)}{\left(1 + p_{m-1} s^{-1} + \cdots + p_1 s^{-m+1} + p_0 s^{-m}\right)}$$

$$+ \frac{\left(h_m + h_{m-1} s^{-1} + \cdots + h_1 s^{-m+1} + h_0 s^{-m}\right) y(s)}{1 + p_{m-1} s^{-1} + \cdots + p_1 s^{-m+1} + p_0 s^{-m}}$$

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$\Rightarrow$

$$\left(1 + p_{m-1} s^{-1} + \cdots + p_1 s^{-m+1} + p_0 s^{-m}\right) u(s)$$

$$= \left(a_m + a_{m-1} s^{-1} + \cdots + a_1 s^{-m+1} + a_0 s^{-m}\right) x(s)$$

$$+ \left(h_m + h_{m-1} s^{-1} + \cdots + h_1 s^{-m+1} + h_0 s^{-m}\right) y(s)$$

$$\boxed{\begin{aligned} u(s) = \; & a_m x(s) + h_m y(s) + s^{-1}\left(a_{m-1} x(s) + h_{m-1} y(s) - p_{m-1} u(s)\right) \\ & + \cdots + s^{-m+1}\left(a_1 x(s) + h_1 y(s) - p_1 u(s)\right) \\ & + s^{-m}\left(a_0 x(s) + h_0 y(s) - p_0 u(s)\right) \end{aligned}}$$

$$\dot{r}_1 = a_0 x + h_0 y - p_0 u$$

$$\dot{r}_2 = a_1 x + h_1 y - p_1 u + r_1$$

$$\dot{r}_3 = a_2 x + h_2 y - p_2 u + r_2$$

$$\vdots$$

$$\dot{r}_m = a_{m-1} x + h_{m-1} y - p_{m-1} u + r_{m-1}$$

$$r_m = u - a_m x - h_m y$$

$$\Rightarrow \quad u = r_m + a_m x + h_m y.$$

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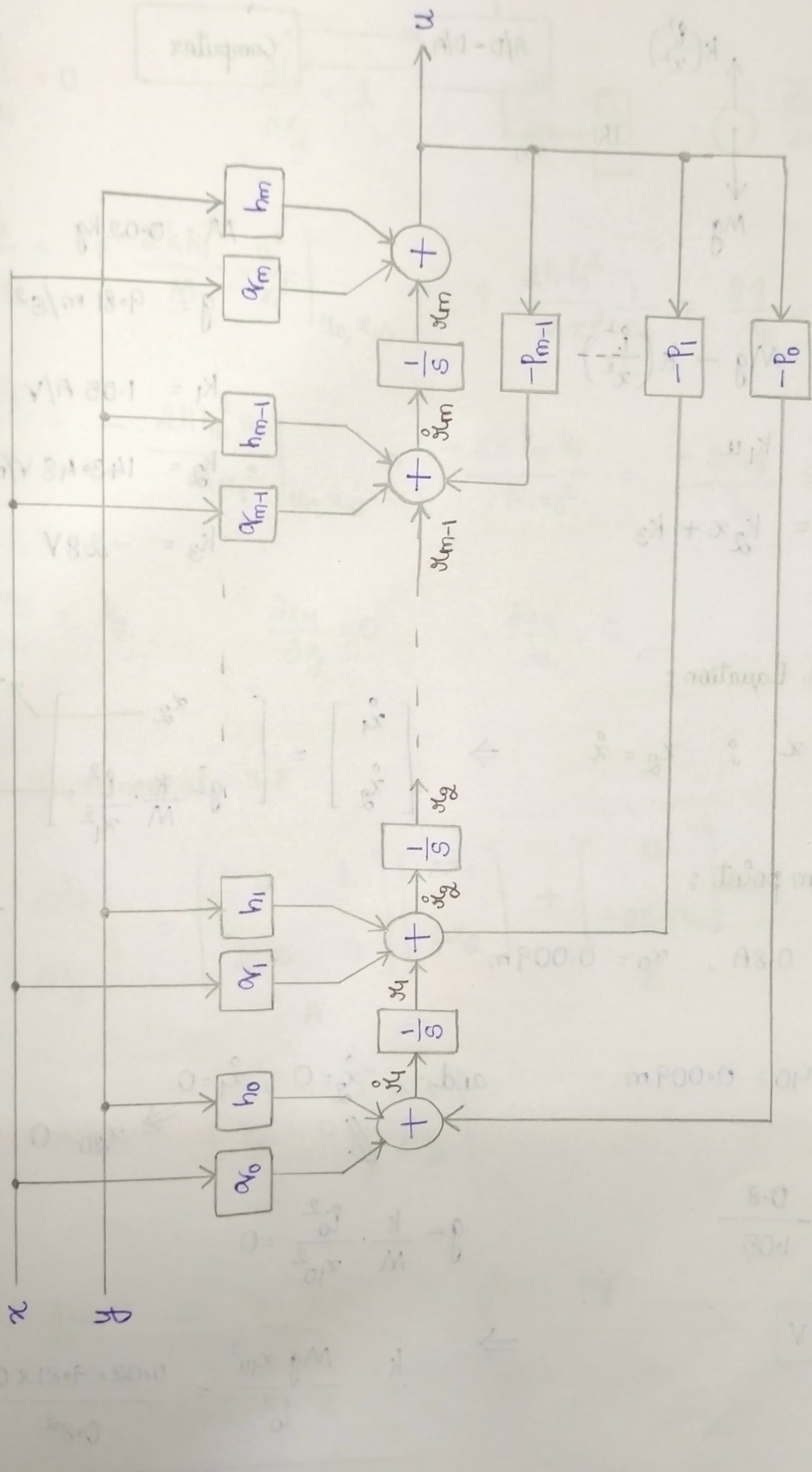

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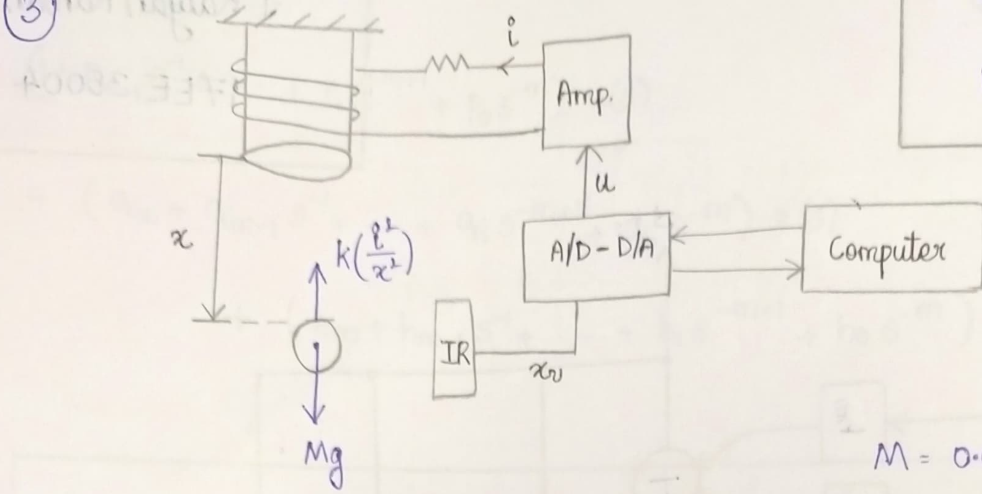

$M = 0.02\,kg$

$g = 9.81\ m/s^2$

$k_1 = 1.05\ A/V$

$k_2 = 143.48\ V/m$

$k_3 = -2.8\,V$

$$M\ddot{x} = Mg - k\left(\frac{i^2}{x^2}\right)$$

$$i = k_1 u$$

$$x_v = k_2 x + k_3$$

State Space Equation:

$x_1 = x$; $x_2 = \dot{x}$ $\Rightarrow$ $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g - \frac{k}{M}\cdot\frac{i^2}{x_1^2} \end{bmatrix}$ $\begin{array}{l} \rightarrow f_1 \\ \rightarrow f_2 \end{array}$

Equilibrium point:

$i_0 = 0.8A$, $x_0 = 0.009\,m$

$\Rightarrow$ $x_{10} = 0.009\,m$ and $\dot{x}_2 = 0$, $\dot{x}_1 = 0 \Rightarrow x_{20} = 0$.

$\Downarrow$

$$g - \frac{k}{M}\cdot\frac{i_0^2}{x_{10}^2} = 0$$

$$u_0 = \frac{i_0}{k_1} = \frac{0.8}{1.05}$$

$$\boxed{u_0 = 0.762\,V}$$

$$\Rightarrow k = \frac{Mg\,x_{10}^2}{i_0^2} = \frac{0.02 \times 9.81 \times 0.009^2}{0.8^2}$$

$$\boxed{k = 24.83\ \mu N.m^2/A^2}$$

Incremental State Space :

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$f_1 = x_2$ and $f_2 = g - \frac{k}{M} \cdot k_1^2 \frac{u^2}{x_1^2}$

$\frac{\partial f_1}{\partial x_1} = 0$ $\quad \frac{\partial f_1}{\partial x_2} = 1$ $\quad \frac{\partial f_1}{\partial u} = 0$ $\quad \frac{\partial f_2}{\partial x_2} = 0$

$$\frac{\partial f_2}{\partial x_1} = + \frac{2kk_1^2}{M} \cdot \frac{u^2}{x_1^3}\Big|_{u_0, x_{10}} = + \frac{2k \cdot i_0^2}{M x_0^2} \cdot \frac{1}{x_0} = \frac{2g}{x_0} = 2180$$

$$\frac{\partial f_2}{\partial u} = - \frac{2kk_1^2 u}{M x_1^2}\Big|_{u_0, x_{10}} = - \frac{2k\, i_0 \cdot k_1}{M x_0^2} = - \frac{2k_1 g}{i_0} = -25.75$$

$\frac{\partial x_v}{\partial x_1} = k_2$ $\quad \frac{\partial x_v}{\partial x_2} = 0$ $\quad \frac{\partial x_v}{\partial u} = 0$

∴ Incremental T.F

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2180 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -25.75 \end{bmatrix}}_{B} \Delta u$$

$$\Delta x_v = \underbrace{\begin{bmatrix} k_2 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

T.F from Δu to Δx_v is given by

$$C(sI-A)^{-1}B.$$

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$$= \begin{bmatrix} 143.48 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -2180 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -25.75 \end{bmatrix}$$

$$= \begin{bmatrix} 143.48 & 0 \end{bmatrix} \cdot \frac{1}{s^2 - 2180} \begin{bmatrix} s & 1 \\ 2180 & s \end{bmatrix} \begin{bmatrix} 0 \\ -25.75 \end{bmatrix}$$

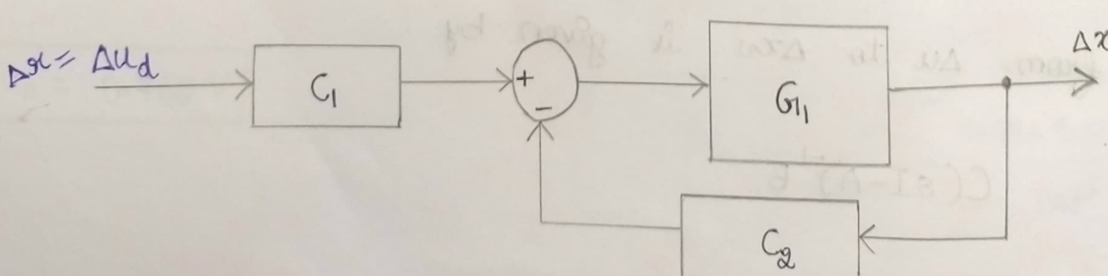
$$= \frac{143.48 \times (-25.75)}{s^2 - 2180} = \frac{-3694.61}{s^2 - 2180}$$

$$\frac{\Delta x_v(s)}{\Delta u(s)} = \frac{-36.9461 \times 100}{s^2 - 2180}$$

$$\boxed{\frac{\Delta x_v(s)}{\Delta u(s)} = \frac{-3694.61}{s^2 - 2180}}$$

$$G_1 = \frac{-3694.61}{s^2 - 2180}$$

We need to design a 2-DOF controller for the above plant, using pole placement method.



Choosing C_1 - PI and C_2 - PID controllers.

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$$\Rightarrow G_1(s) = P_1 + \frac{P_2}{s} \quad , \quad C_2(s) = \frac{K_I + K_P s + K_D s^2}{s}$$

$$\frac{\Delta x_r}{\Delta u_d} = \frac{G_1 C_1}{1 + G_1 C_2} = \frac{\frac{-3694.61}{s^2 - 2180} \cdot \left(\frac{P_1 s + P_2}{s}\right)}{1 + \frac{-3694.61}{s^2 - 2180} \cdot \frac{(K_I + K_P s + K_D s^2)}{s}}$$

$$\frac{\Delta x_r}{\Delta u_d} = \frac{-3694.67\,(P_1 s + P_2)}{s^3 - (2180 + 3694.61 K_D)s^2 - 3694.61 K_P s - 3694.61 K_I}$$

$\therefore$ Characteristic equation is given by

$$\boxed{s^3 - (2180 + 3694.61 K_D)s^2 - 3694.61 K_P s - 3694.61 K_I = 0}$$

Desired characteristic equation properties.

$T_s \leq 2\,sec$, $P.O \leq 5\%$, $|GM| \geq 6dB$ and $|PM| \geq 60^\circ$

$$P.O = 5\% \quad \Rightarrow \quad e^{\frac{-\rho\pi}{\sqrt{1-\rho^2}}} = 0.05$$

$$\Rightarrow \frac{\rho\pi}{\sqrt{1-\rho^2}} = 2.996$$

$$\Rightarrow \boxed{\rho = 0.6901}$$

$$T_s = 2s \quad \Rightarrow \quad \omega_n = \frac{-\ln(0.02\sqrt{1-\rho^2})}{\rho T_s}$$

$$\boxed{\omega_n = 3.0687}$$

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$\Rightarrow s^2 + 2 \times 0.6901 \times 3.0687\, s + (3.0687)^2 = 0$

$$\boxed{s^2 + 4.2354\, s + 9.4169 = 0}$$

Let the third pole be given by $s + a = 0$.

$\Rightarrow (s+a)(s^2 + 4.2354\, s + 9.4169) = 0$

$|GM| \geq 6\text{dB} \Rightarrow |GM| \geq 2$

$\Rightarrow \dfrac{\|S\|_\infty}{\|S\|_\infty - 1} \geq 2$

$\Rightarrow \|S\|_\infty \geq 2\|S\|_\infty - 2$

$\Rightarrow \|S\|_\infty \leq 2$

$|PM| \geq 60° \Rightarrow 2\sin^{-1}\left(\dfrac{1}{2\|S\|_\infty}\right) \leq 60°$

$\sin^{-1}\left(\dfrac{1}{2\|S\|_\infty}\right) \leq 30$

$\dfrac{1}{2\|S\|_\infty} \leq \dfrac{1}{2}$

$\Rightarrow \|S\|_\infty \geq 1$

$\therefore \quad 1 \leq \|S\|_\infty \leq 2$

Similarly $\quad \|T\|_\infty \leq 2$

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For ensuring good robustness and disturbance rejection :

$$a \gg P\omega_n$$

For a good design $a = 750 * P\omega_n$

$$a = 1588.282$$

$$\Rightarrow \quad (s + 1588.282)(s^2 + 4.2354 s + 9.4169) = 0$$

$$\boxed{s^3 + 1592.5174 s^2 + 6736.4265 s + 14956.6928 = 0.}$$

is the desired characteristic equation.

On comparing coefficients.

$$2180 + 3694.61 K_D = -1592.5174 \quad \Rightarrow \quad K_D = -1.0211$$

$$3694.61 K_P = -6736.4265 \quad \Rightarrow \quad K_P = -1.8233$$

$$3694.61 K_I = -14956.6928 \quad \Rightarrow \quad K_I = -4.0483$$

Given that tracking must be maintained.

For tracking $\left.\dfrac{\Delta x}{\Delta r}\right|_{s \to 0} = 1$

$$\frac{\Delta x}{\Delta r}(s=0) = \frac{P_2}{k_i} = 1 \quad \Rightarrow \quad P_2 = K_I = -4.0483$$

Let $P_1 = 0$.

$$\therefore \quad C_1 = \frac{-4.0483}{s} \quad ; \quad C_2 = \frac{-1.0211 s^2 - 1.8233 s - 4.0483}{s}$$

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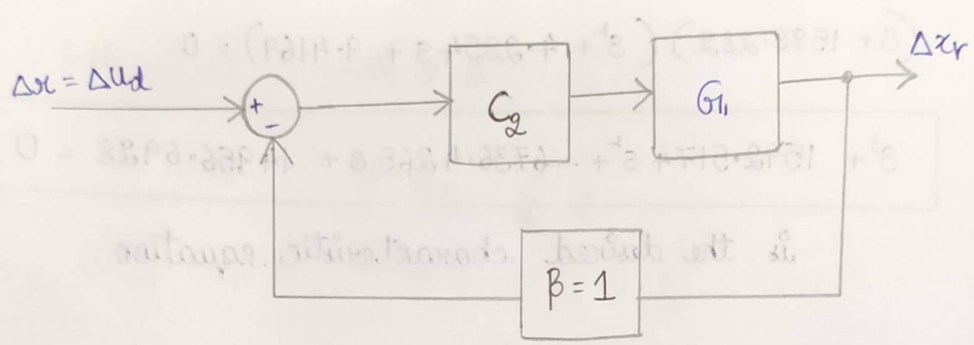
For the design : $|GM| = 9.78\ dB$

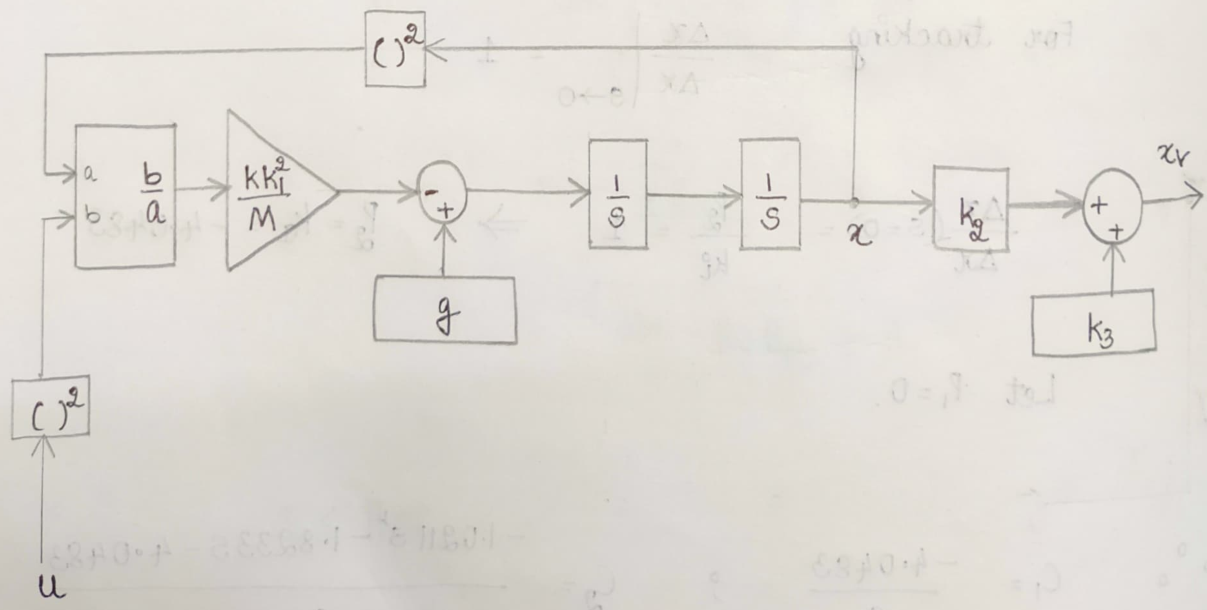
$|PM| = 90^\circ$

For 1 DOF controller implementation, no extra calculations are required, but block diagram changes.



Non-Linear Model :

$$\boxed{\ddot{x} = g - \frac{kK_1^2}{M} \cdot \frac{u^2}{x^2}} \quad \text{and} \quad \boxed{x_r = k_2 x + k_3}$$



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Problem 3 : Magnetic levitation controller design

Matlab code :

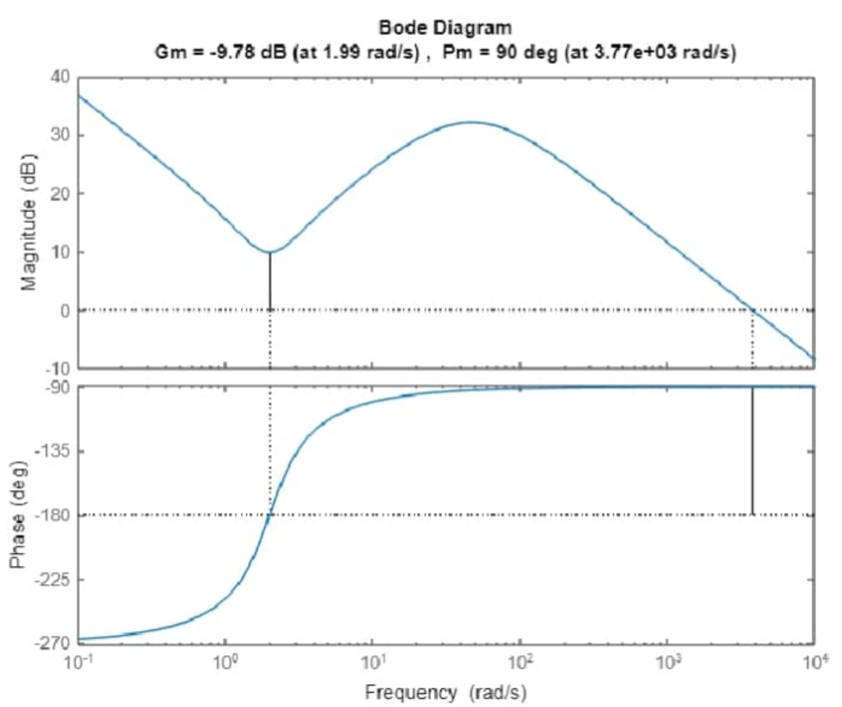
```
zeta = 0.6901;
w = 3.0687;
b1 = 2*zeta*w;
b2 = w*w;
N = -3694.61;
p = 750*zeta*w;

C = [p+b1; p*b1+b2; p*b2];
D = [-2180; 0; 0];

P = [0 0 N; N 0 0;0 N 0];  %X = [kp; ki; kd]
X = P\(C-D);
kp = X(1);
ki = X(2);
kd = X(3);

G = tf(N,[1 0 -2180]);
C = tf([kd kp ki],[1 0]);

CL_TF = G*C;
figure;
bode(CL_TF);
margin(CL_TF);
```



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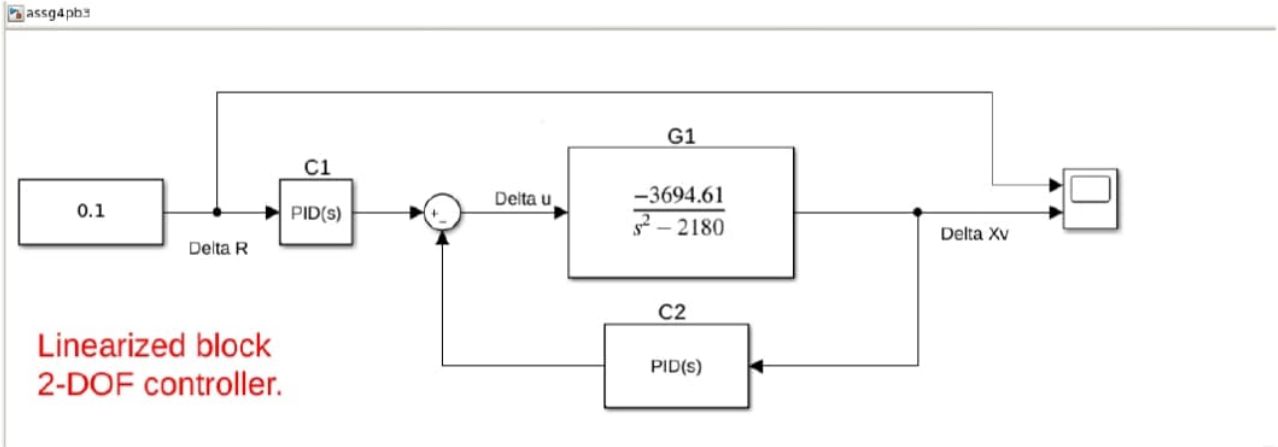
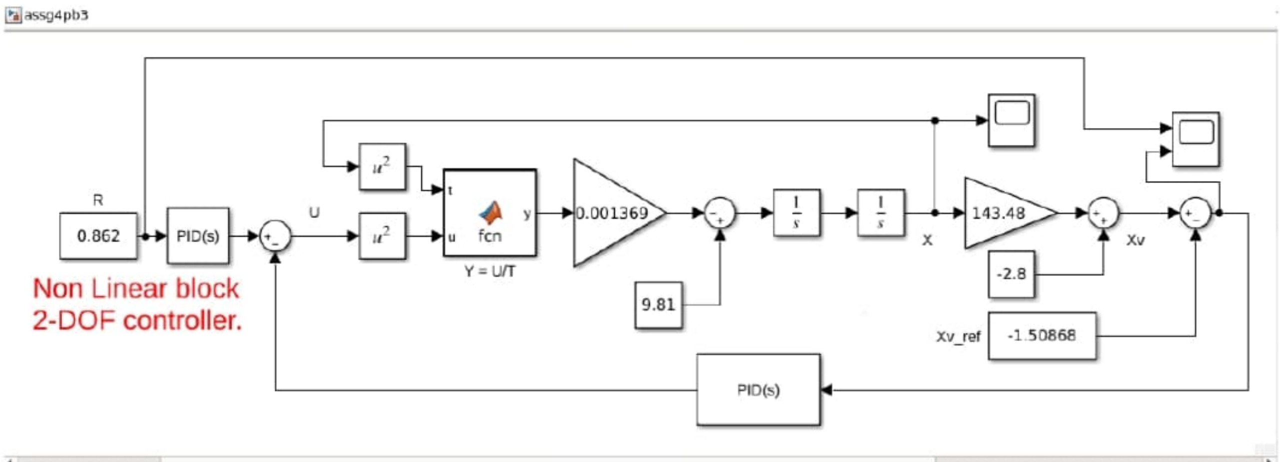
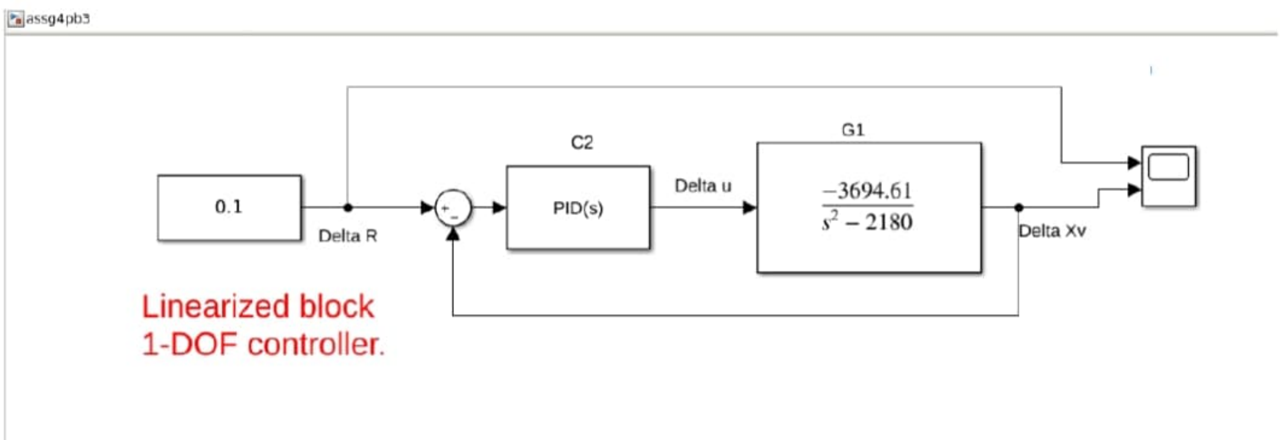
SIMULINK Diagram :







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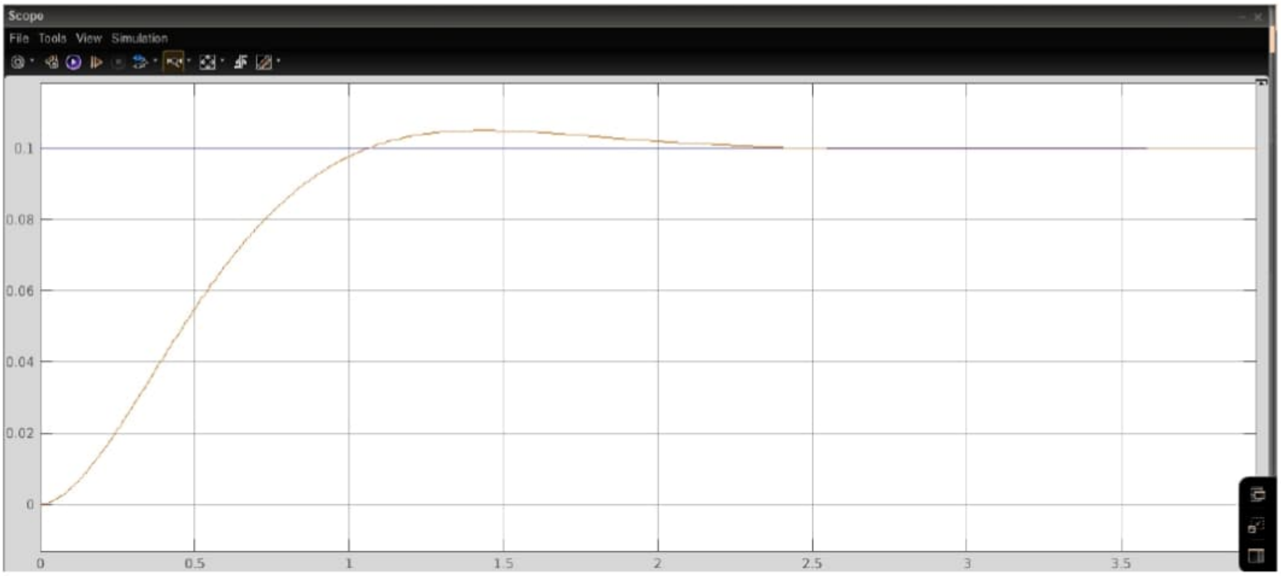
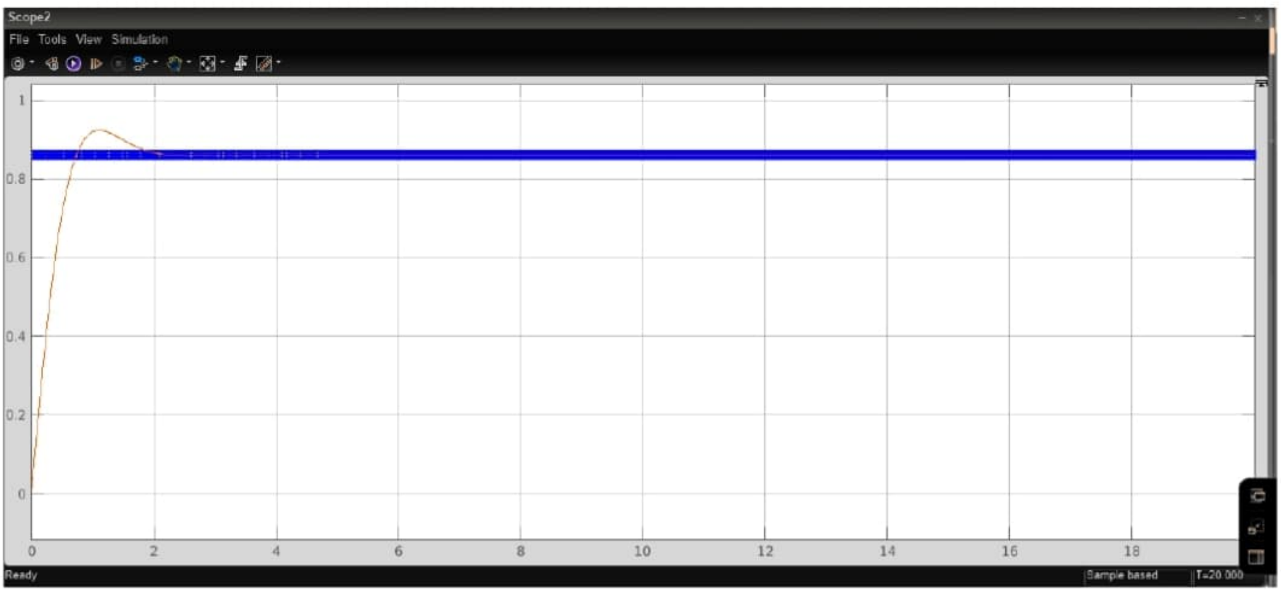
Waveforms :

(a) Linearized block with 2-DOF controller



(b) Non-linear block with 2-DOF controller



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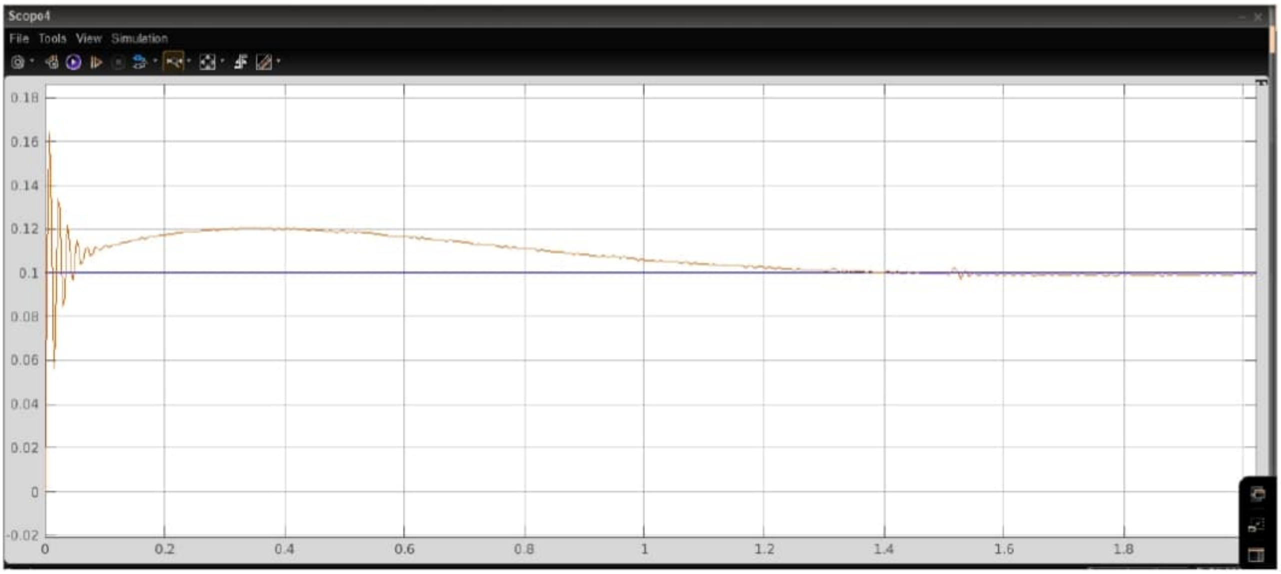
(c) Linearized block with 1-DOF controller



It is evident from the waveforms that 2-DOF controller provides better control action. In 1-DOF controller, there is a high oscillation before reaching steady state, which is highly undesirable. So 2-DOF is superior in control action over 1-DOF controller.