Lecture 5

Controller design using Bode plots

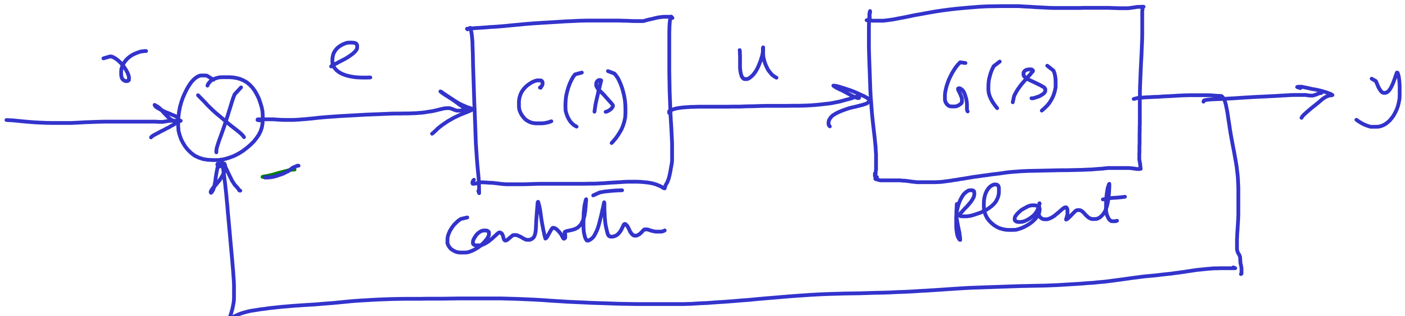
- lead, lag, lag-lead controllers

Design objectives:

- steady-state error specifications
- phase margin (PM)
- gain cross-over freq (gcf)

Note: As PM ↑, ξ ↑ $\left(\xi \approx \frac{PM}{100}\right)$

As gcf ↑, rise time ↓ ie the system becomes faster.



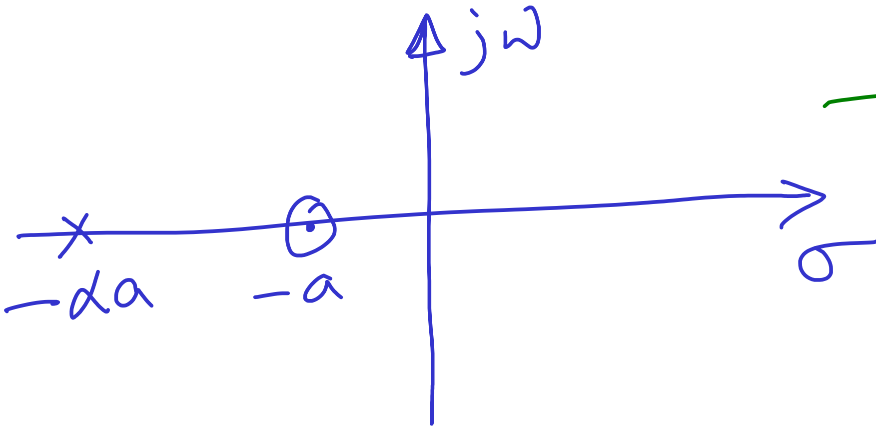
$C(s) = K H(s)$, K = servo gain used to meet steady-state error specific

$H(s)$ has normalized structure (ie $H(0) = 1$).

→ used to meet PM & g.c.f specifications.

Lead Controller

$$H(s) = \frac{1 + \frac{s}{a}}{1 + \frac{s}{\alpha a}}, \quad \alpha > 1, \quad a > 0$$



— pole-zero plot of $H(s)$

- Lead Controller is used — to provide phase lead along with gain amplification at a desired freq.

* If a system has poor PM or sluggish response then we use lead controller.

phase lead $\phi = \tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{\alpha a}$

$$\Rightarrow \phi = \tan^{-1}\frac{\frac{\omega}{a} - \frac{\omega}{\alpha a}}{1 + \frac{\omega^2}{\alpha a^2}}$$

$$\Rightarrow \phi = \tan^{-1}\frac{\omega a(\alpha - 1)}{\omega^2 + \alpha a^2}$$

$$\Rightarrow \phi = \tan^{-1}\frac{\alpha - 1}{\frac{\omega}{a} + \frac{\alpha a}{\omega}}$$

For $\phi = \phi_m$, $\frac{\omega}{a} + \frac{\alpha a}{\omega}$ should be minized $\Rightarrow \frac{d}{d\omega}\left(\frac{\omega}{a} + \frac{\alpha a}{\omega}\right) = 0$

$$\Rightarrow \frac{1}{a} - \frac{\alpha a}{\omega^2} = 0$$

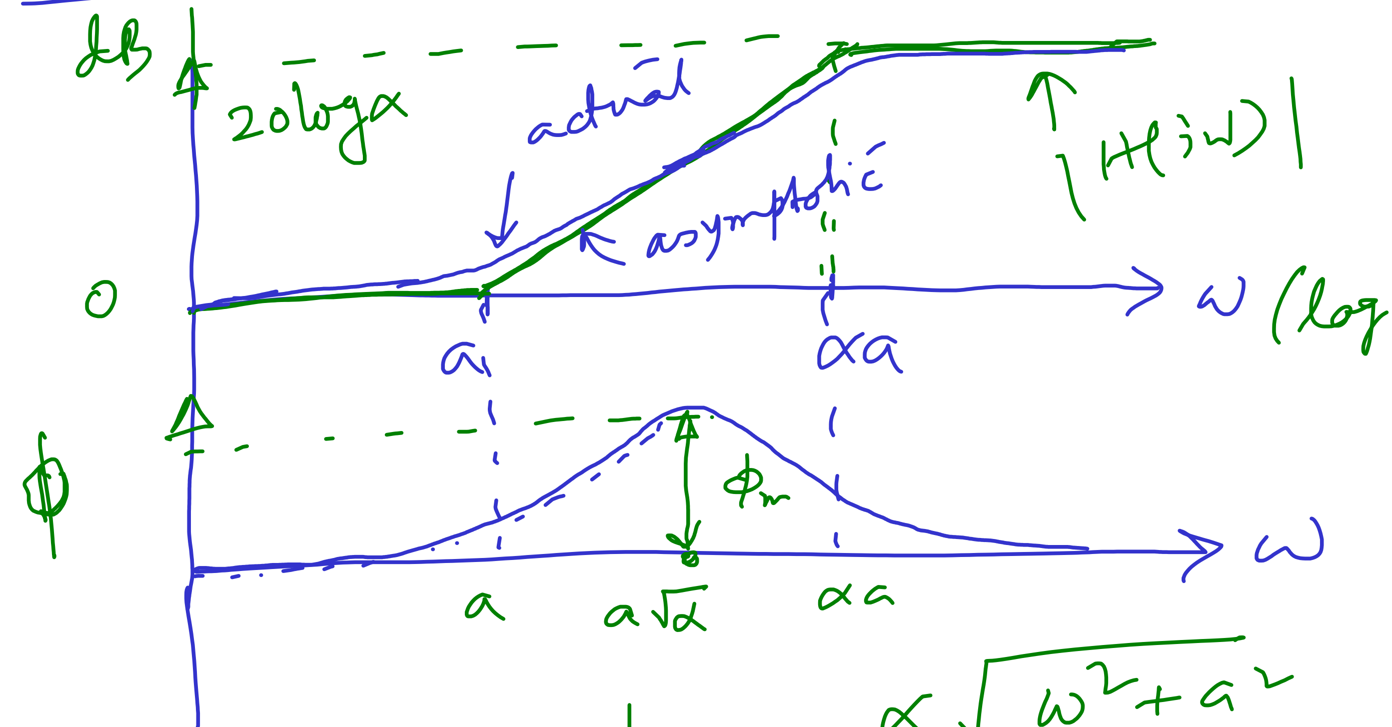
$$\Rightarrow \boxed{\omega_m = a\sqrt{\alpha}} \rightarrow \text{freq}$$

at which $\phi = \phi_m$ ~~t~~ true

$$\boxed{\phi_m = \tan^{-1}\frac{\alpha - 1}{2\sqrt{\alpha}}} \rightarrow \text{Max}^m \text{ phase lead}$$

$$\Rightarrow \boxed{\phi_m = \sin^{-1}\frac{\alpha-1}{\alpha+1}}, \quad \alpha > 1$$

Bode plot of lead controller



gain $$|H(j\omega)| = \frac{\alpha\sqrt{\omega^2+a^2}}{\sqrt{\omega^2+\alpha^2 a^2}}$$

In practice $\boxed{\alpha \leq 20}$ (ie $\phi_m < 64.8^\circ$)

Algebraic method to design lead controllers.

objective: Given phase lead ϕ_c, gain M_c (in dB) at freq

ω_c, obtain α, a.

Soln: Let $q = \tan\phi_c$ [ϕ_c given]

$$\Rightarrow q = \frac{\omega_c a(\alpha-1)}{\omega_c^2 + \alpha a^2}$$

$$\Rightarrow q = \frac{\frac{\omega_c}{a}(\alpha-1)}{\left(\frac{\omega_c}{a}\right)^2 + \alpha}$$

$$\Rightarrow q = \frac{\bar{\omega}_c(\alpha-1)}{\bar{\omega}_c^2 + \alpha} \quad \text{— (1)}$$

[where $\bar{\omega}_c = \frac{\omega_c}{a}$]

gain $\bar{M}_c = |H(j\omega_c)|$

$$\Rightarrow \bar{M}_c = \frac{\alpha\sqrt{\omega_c^2 + a^2}}{\sqrt{\omega_c^2 + \alpha^2 a^2}}$$

$$\Rightarrow \bar{M}_c = \frac{\alpha\sqrt{\bar{\omega}_c^2 + 1}}{\sqrt{\bar{\omega}_c^2 + \alpha^2}}$$

$$M_c = 20\log \bar{M}_c = 20\log \frac{\alpha\sqrt{\bar{\omega}_c^2 + 1}}{\sqrt{\bar{\omega}_c^2 + \alpha^2}}$$

$$\Rightarrow M_c = 10\log_{10}\left[\frac{\alpha^2(\bar{\omega}_c^2+1)}{\bar{\omega}_c^2+\alpha^2}\right]$$

Now let $c = 10^{M_c/10}$ [M_c given]

$$\Rightarrow c = \frac{\alpha^2(\bar{\omega}_c^2+1)}{\bar{\omega}_c^2+\alpha^2} \quad \text{--- (2)}$$

From (2), $\bar{\omega}_c^2 = \dfrac{\alpha^2(c-1)}{\alpha^2-c}$ --- (3)

From (1) $(\bar{\omega}_c^2+\alpha)^2 q^2 = \bar{\omega}_c^2(\alpha-1)^2$ --- (4)

Substituting (3) in (4), we get

$$q^2\left[\alpha^2(c-1)+\alpha(\alpha^2-c)\right]^2 = \alpha^2(c-1)(\alpha^2-c)(\alpha-1)^2$$

$$\Rightarrow q^2\cancel{\alpha^2}\cancel{(\alpha-1)^2}(\alpha+c)^2 = \cancel{\alpha^2}(c-1) \times (\alpha^2-c)\cancel{(\alpha-1)^2}$$

$$\Rightarrow q^2(\alpha+c)^2 = (c-1)(\alpha^2-c) \quad [\because \alpha>1]$$

$$\Rightarrow \boxed{(q^2-c+1)\alpha^2 + 2q^2c\alpha + (q^2c+c-1)c = 0}$$

$$-\text{(5)}$$

From (5), for existence of a positive solution of α

$$(q^2 - c + 1)(q^2 c + c - 1)c < 0$$

Now as $c > 1$, we have $(q^2 c + c - 1)c > 0$. So one should have

$$q^2 - c + 1 < 0$$

$$\Rightarrow \boxed{q^2 < c - 1} \quad - \text{ N \& S}$$

(necessary & sufficient) condition for existence of a valid solution to exist. [Note: if α +ve then $\alpha > 1$ follows from the eqn just before (5) since $c > 1$]

Then a can be determined from (3) as

$$\boxed{a = \frac{w_c}{\alpha}\sqrt{\frac{\alpha^2 - c}{c - 1}}}$$

Ex Determine the parameters of a lead controller $H(s)$ that will provide a phase lead of 45° & gain $= 10$ dB at $\omega_c = 8$ rad/s.

Soln Here $\phi_c = 45^\circ$, $M_c = 10$ dB, $\omega_c = 8$

$\Rightarrow q = \tan\phi_c = 1$, $c = 10^{M_c/10} = 10$

so $c > q^2 + 1$ is satisfied
$\Rightarrow$ lead compensation exists

We have

$$(q^2 - c + 1)\alpha^2 + 2q^2 c\alpha + (q^2 c + c - 1)c = 0$$

$$\Rightarrow \quad -8\alpha^2 + 20\alpha + 190 = 0$$

$$\Rightarrow \boxed{\alpha = 6.2812}$$

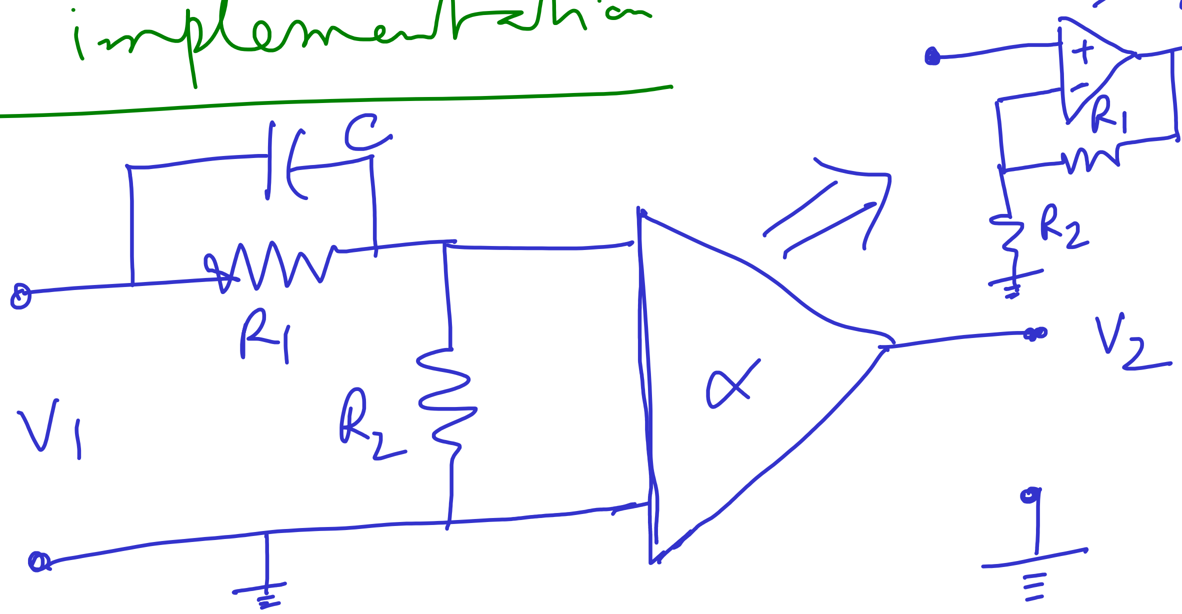
& $a = \dfrac{\omega_c}{a}\sqrt{\dfrac{\alpha^2 - c}{c-1}} = 2.304$

Then $$H(s) = \frac{\alpha(s+a)}{s+\alpha a}$$

$$= \frac{6.2812\,(s+2.304)}{s+14.472}$$

An implementation

non-inverting amplifier with gain $1+\frac{R_1}{R_2}$



$$\frac{V_2(s)}{V_1(s)} = \alpha\left[\frac{R_2}{R_2 + R_1 \,\|\, \frac{1}{Cs}}\right] V_1$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\alpha\, R_2\left(R_1+\frac{1}{Cs}\right)}{R_2\left(R_1+\frac{1}{Cs}\right)+\frac{R_1}{Cs}}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\alpha\left[R_2 + R_1 R_2 C s\right]}{R_1 + R_2 + R_1 R_2 C s}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\alpha\left(s + \frac{1}{R_1 C}\right)}{s + \frac{R_1 + R_2}{R_1 R_2 C}}$$

$$\Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{\alpha(s + a)}{s + \alpha a}$$

where $a = \frac{1}{R_1 C}$

$$\alpha = \frac{R_1 + R_2}{R} > 1$$