Realization

$$G(s) \longrightarrow (A, B, C, D)$$

Improper TF cannot be realized.

Example

$$G(s) = \frac{1}{s^2+5s+6} \;\cancel{=}\; \frac{Y(s)}{U(s)} \qquad \frac{1}{(s+2)(s+3)}$$

$$s^2 Y(s) + 5sY(s) + 6Y(s) = U(s)$$

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = u(t)$$

Let $y(t) = x_1$ and $x_2 = \dot{x}_1$.

Phase variable: one state-variable is the ~~time~~ derivative of the other state-variable.

$$\dot{x}_2 = -5x_2 - 6x_1 + u$$

$$\dot{x}_1 = x_2$$

Poles are $-2, -3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D \cdot u \qquad D = 0$$

Eigenvalues of matrix A:

$$\det(\lambda I - A) = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \right| = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda+5 \end{vmatrix} = 0$$

$$\lambda(\lambda+5) + 6 = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = -2$$

$$\boxed{\text{Eigenvalues of A are the poles of } G(s).}$$

Example

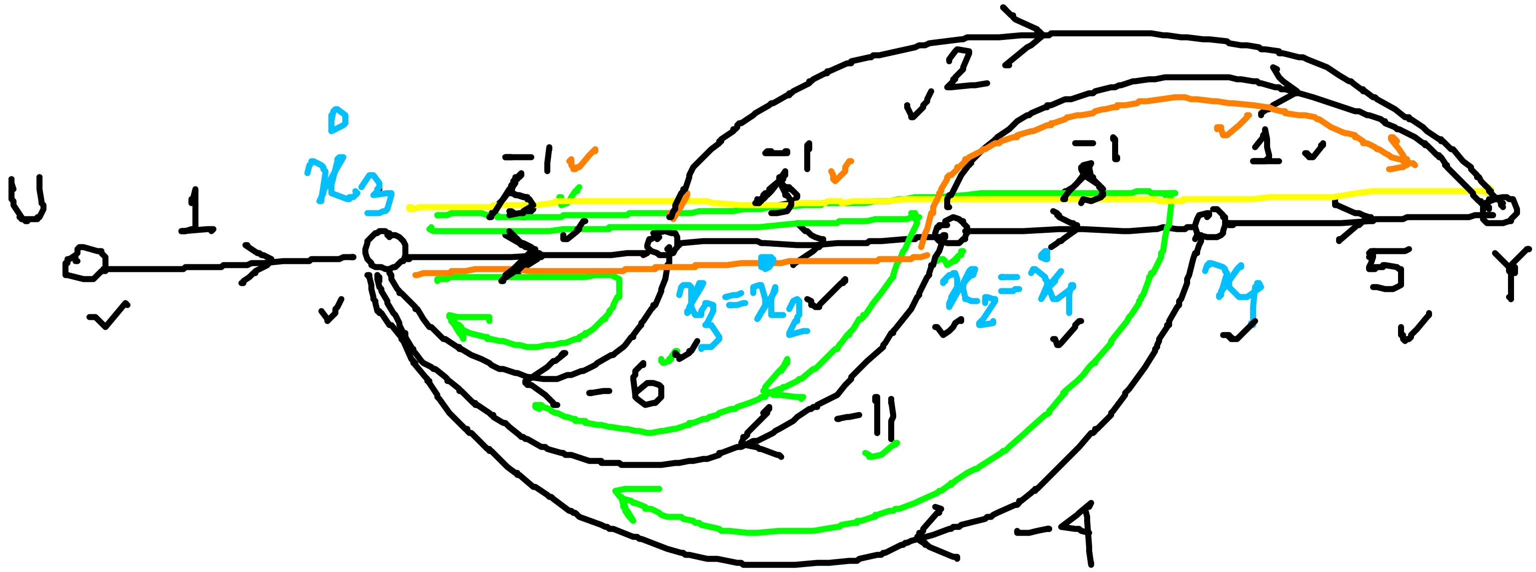
$$\frac{Y(s)}{U(s)} = \frac{2s^2 + s + 5 \,/ s^3}{s^3 + 6s^2 + 11s + 4 \,/ s^3}$$

$$= \frac{(2s^{-1} + s^{-2} + 5s^{-3})\, X(s)}{(1 + 6s^{-1} + 11s^{-2} + 4s^{-3})\, X(s)}$$

$$Y(s) = (2s^{-1} + s^{-2} + 5s^{-3})\, X(s)$$

$$U(s) = (1 + 6s^{-1} + 11s^{-2} + 4s^{-3})\, X(s)$$

$$X(s) = U(s) - (6s^{-1} + 11s^{-2} + 4s^{-3})\, X(s)$$



$$\dot{x}_3 = -6x_3 - 11x_2 - 4x_1 + u \qquad y = 5x_1 + x_2 + 2x_3$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_1 = x_2$$

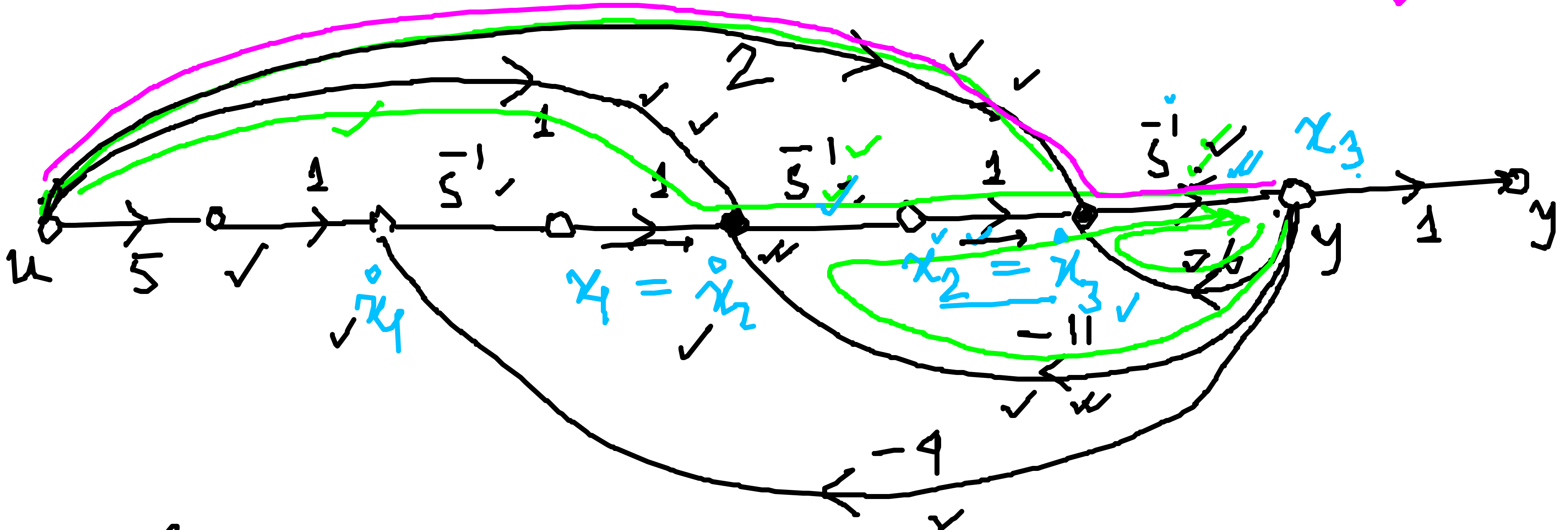
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Controllable Canonical Form (CCF).

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + s + 5}{s^3 + 6s^2 + 11s + 4} = \frac{2s^{-1} + s^{-2} + 5s^{-3}}{1 + \underbrace{6s^{-1} + 11s^{-2} + 4s^{-3}}}$$

$$Y(s) = \left(2s^{-1} + s^{-2} + 5s^{-3}\right)U(s) - \left(6s^{-1} + 11s^{-2} + 4s^{-3}\right)Y(s)$$



$$\dot{x}_1 = 5u - 4x_3$$
$$\dot{x}_2 = u - 11x_3 + x_1$$
$$\dot{x}_3 = x_2 - 6x_3 + 2u$$

$$y = x_3$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 0 & -4\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}5\\ 1\\ 2\end{bmatrix}}_{B}u$$

$$y = \underbrace{\begin{bmatrix}0 & 0 & 1\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Observable Canonical Form (OCF)

Linearization of nonlinear system

Let $\dot{x}(t) \overset{=0}{=} f(t, x(t))$, $x(t_0) = x_0$, $t \in [t_0, t_1]$. A point $x_e \in D$ is said to be an equilibrium point at time t_e if $f(t, x_e) = 0 \quad \forall\, t \geq t_e$.

Ex

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\epsilon x_2 - \frac{g}{l}\sin x_1 \end{cases}$$

Find equilibrium pt.

$$\dot{x}_1 = 0 = x_2, \qquad \dot{x}_2 = 0 = -\epsilon x_2 - \frac{g}{l}\sin x_1$$

$$x_1 = n\pi$$

$$n = 0, \pm 1, \pm 2 \cdots$$

Many equilibrium points.

Ex.

$$\dot{x}_1 = 2 + \sin(x_1 + x_2)$$

$$\dot{x}_2 = 3 - \sin x_1$$

$\dot{x}_1 = 0$.

$\dot{x}_2 = 0$

No equilibrium point exist.

Ex

$$\dot{x}_1 = -x_1 + \underbrace{2x_1x_2}_{=0}$$

$$\dot{x}_2 = -\underbrace{2x_1x_2}_{=0}$$

$\dot{x}_1 = 0$

$\dot{x}_2 = 0$

$\underline{x_1 = 0}$

$\overset{=0}{2x_1}x_2 = 0$

$x_2 \rightarrow$ anything

Many equilibrium points (x_2-axis) but they are not isolated.