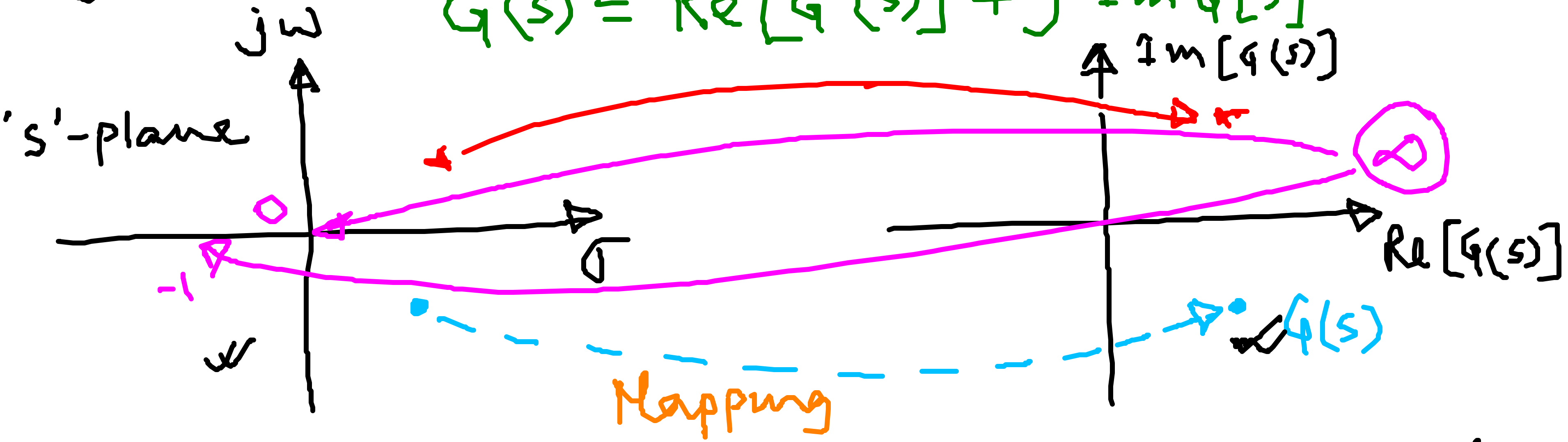
Mathematical Preliminaries

Function of a complex variable

Let s be a complex variable. A function of the complex variable G(s) is represented by its real and imaginary parts.

$$G(s) = Re[G(s)] + j\, Im\, G[s]$$



- If for every value of s, there is only one corresponding value of G(s), then it is called single-valued function.
- If the mapping from G(s) to s-plane is also single valued, it is called one-to-one mapping.

Example:

$G(s) = \infty$

$$G(s) = \frac{1}{s\,(s+1)}$$

s to G(s)-plane   single-valued mapping, but the reverse mapping is not single-valued mapping. $G(s) = \infty$ is mapped onto $s=0$ and $s=-1$.

Analytic function

A function of complex variable $G(s)$ is said to be analytic in a region of 's'-plane if the function and all its derivatives exist in the region.

Example: $G(s) = \dfrac{s+1}{s(s+2)}$, analytic except $s=0$ and $s=-2$.

$s=0$ $\quad$ $s=-2$

$G(s) = s+1$, analytic in finite s-plane.

Pole and zero of a function

Let $G(s)$ be an analytic and single-valued function in the neighbourhood of point p. It is said to have a pole of order r at $s=p$, if

$$\lim_{s \to p} (s-p)^{r} G(s)$$

is finite and non-zero.

$r=1$
$s=p$ (simple pole)

$r=2$
at $s=p$ (order 2)

Example:

$$G(s) = \frac{s+1}{s(s+2)^2}$$

Simple pole at $s=0$

pole of order 2 at $s=-2$.

G(s) is analytic in s-plane except at poles.

Zeros of a function

Let G(s) be analytic at s = z. It is said to have a zero of order r at s = z if

$$\lim_{s \to z} (s-z)^{-r} G(s)$$

is finite and nonzero.

or, G(s) has a zero of order r at s = z if $\frac{1}{G(s)}$ has an r-th order pole at s = z.

Example:

$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^2}$$

$G(s) = 0$
$\Rightarrow$ zero

$G(s) = \infty$
$\Rightarrow$ pole

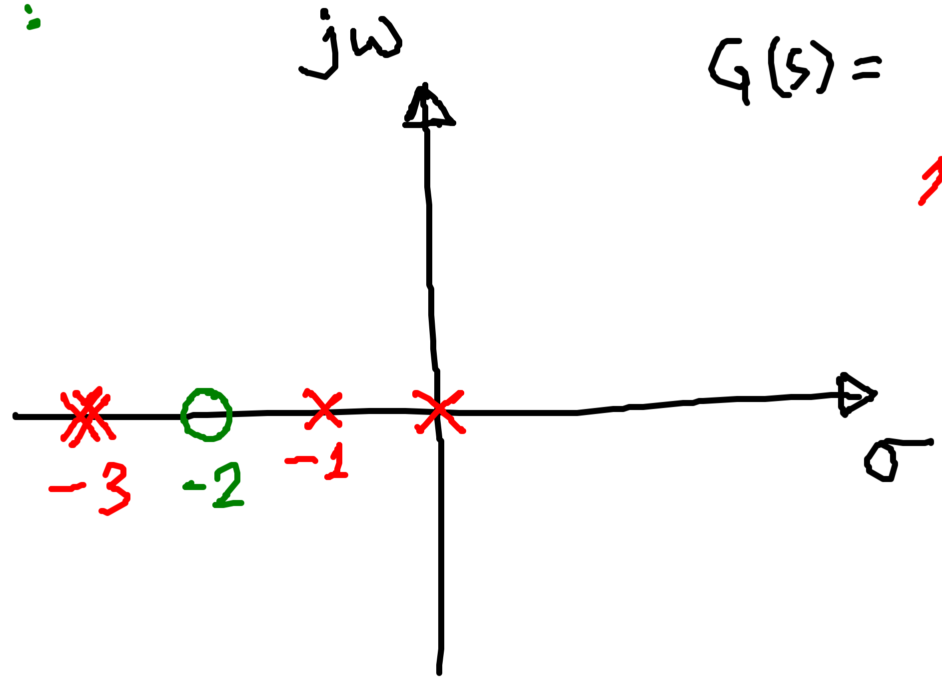
Poles: 0, −1, −3, −3

Zeros: −2, ∞, ∞, ∞ X

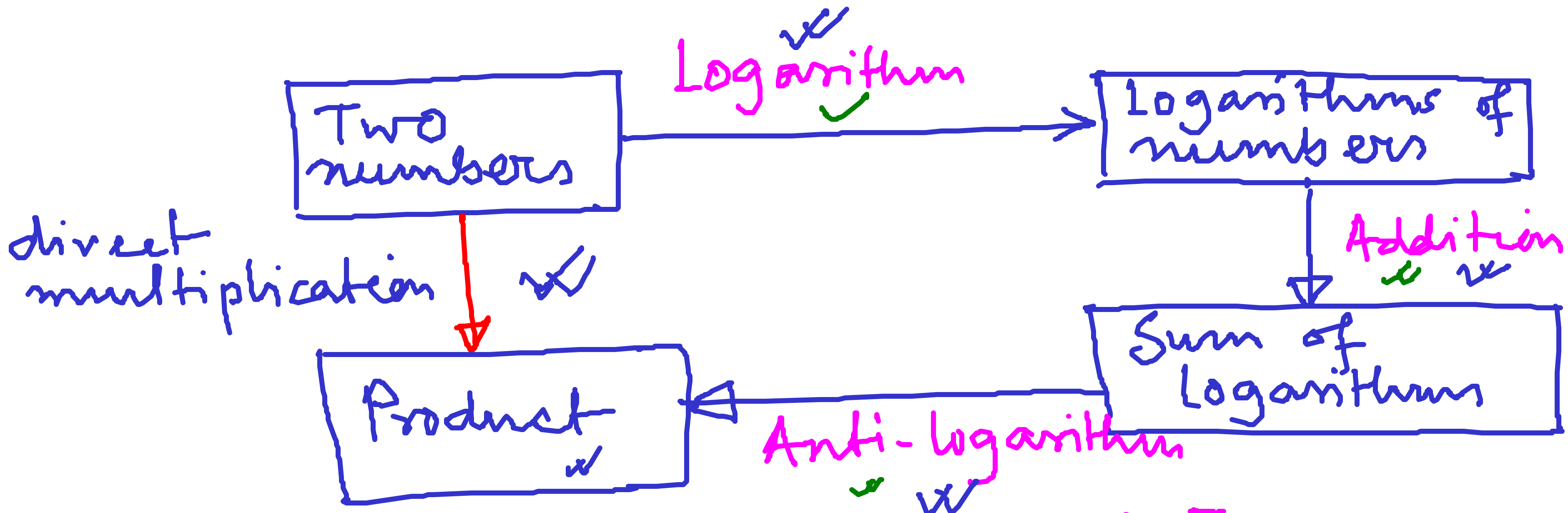
Pole-zero map:

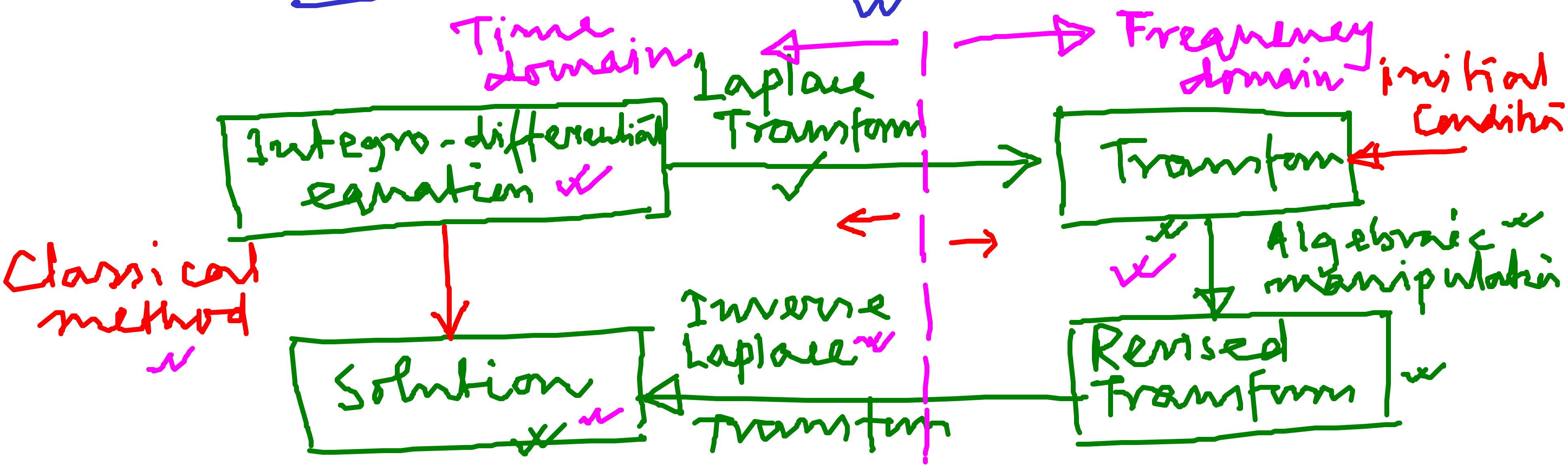
$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^2}$$

pole — ×
zero — O



Laplace Transform





Given a real function $f(t)$ that satisfies $\int_0^\infty |f(t)| e^{-\sigma t}\, dt < \infty$ for some finite real σ, then

$$F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = \mathcal{L}[f(t)]$$

- s is called Laplace variable.
- one sided Laplace transform, it makes sense since the system are causal.
- The integration limit is 0^- to ∞ in order to handle the jump discontinuity at origin.
- A mathematical tool used to solve differential equation (linear ordinary).
- The homogeneous equation and the particular integral of the solution are obtained in one operation.
- It converts into algebraic form in 's'-domain.