Digital Signal Processing

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Mid Semester Examination

Q6. Given z-transform,

$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}$$

$$= \frac{A}{(1-2z^{-1})} + \frac{B}{(3-z^{-1})}$$

$$A = X(z)(1-2z^{-1})\Big|_{z=2}$$

$$= \frac{5/2}{3-\frac{1}{2}} = 1$$

$$B = X(z)(3-z^{-1})\Big|_{z=\frac{1}{3}}$$

$$= \frac{5\times 3}{1-2\times 3} = -3$$

$$\therefore X(z) = \frac{1}{1-2z^{-1}} - \frac{3}{(3-z^{-1})}$$

Q6 The possible cases are:
Continued.

$x_1[n] = 2^n u[n] - \left(\frac{1}{3}\right)^n u[n]$ for ROC $|z| > 2$

$x_2[n] = -2^n u[-n-1] - \left(\frac{1}{3}\right)^n u[n]$ for ROC $\frac{1}{3} < |z| < 2$

$x_3[n] = -2^n u[-n-1] + \left(\frac{1}{3}\right)^n u[-n-1]$ with ROC $|z| < \frac{1}{3}$

Q4.

$$y[n] = \begin{cases} \alpha^n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \qquad |\alpha| < 1$$

$$x[n] = \begin{cases} \alpha^n & n \ge 0 \\ 0 & n < 0 \end{cases} \qquad |\alpha| < 1$$

DTFT of $x[n]$

$$X(e^{j\omega}) = \sum_{n \ge 0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}} =$$

Now, $y[n]$, using linearity & Time Shifting Props.

$$y[n] = x[n] - \alpha^M x[n-M]$$

$$= \frac{1}{1 - \alpha e^{-j\omega}} - \alpha^M X[n-M]$$

$$x[n-M] \xleftrightarrow{DTFT} e^{-j\omega M} X(e^{j\omega})$$

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$$y[n] = \frac{1}{1-\alpha e^{-j\omega}} - \frac{\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1-\alpha^M e^{-j\omega M}}{1-\alpha e^{-j\omega}} \quad |\alpha|<1$$

$$= \frac{1-(\alpha e^{-j\omega})^M}{1-\alpha e^{-j\omega}}$$

Q7. (a) $$y(n) = \frac{1}{M}\sum_{k=0}^{M-1} x(n-k)$$

for step response, input is

$$u[n] = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

$$y_s[n] = \frac{1}{M}\sum_{k=0}^{M-1} u[n-k]$$

$$= \frac{1}{M}\left[u[n] + u[n-1] + \dots + u[n-(M-1)]\right]$$

Step response

$$y_s[n] = \begin{cases} 1 & n \geq M-1 \\ 0 & n < 0 \\ \frac{(n+1)}{M} & \text{otherwise} \end{cases}$$

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(b) $$x[n] = \sum_{k=-\infty}^{\infty} [x(k) - x(k-1)]\, u[n-k)$$

$$\because u[n-k] = \begin{cases} 0 & n-k<0 \Rightarrow n<k \\ 1 & n \geq k \end{cases}$$

On Right hand Side,

RHS $$\Rightarrow \sum_{k=-\infty}^{n} x(k) - x(k-1) \Rightarrow \text{has to be proven.}$$

$$\Rightarrow \left[\sum_{k=-\infty}^{n} x(k)\right] - \sum_{k=-\infty}^{n} x(k-1)$$

Take $k-1 = t$

$$\Rightarrow x(n) + \sum_{k=-\infty}^{n-1} x(k) - \sum_{t=-\infty}^{n-1} x(t)$$

$$\Rightarrow x(n) - \sum_{k=-\infty}^{n-1} x(k) - \sum_{k=-\infty}^{n-1} x(k)$$

$$\Rightarrow x(n)$$

$\therefore$ RHS $= x(n)$

Proved.

Q2.

$y[n] = 0.25\, y[n-2] + x[n]$

Take Z-transform on both sides,

$Y[z] = \cancel{0.25 Y}\, 0.25 z^{-2} Y[z] + X[z]$

$$\boxed{\frac{Y[z]}{X[z]} = \frac{1}{1-0.25z^{-2}}} \rightarrow \text{Transfer function in z-domain}$$

Now, $z = e^{j\omega}$

$$H(e^{j\omega}) = \left.\frac{z^2}{z^2-0.25}\right|_{z=e^{j\omega}}$$

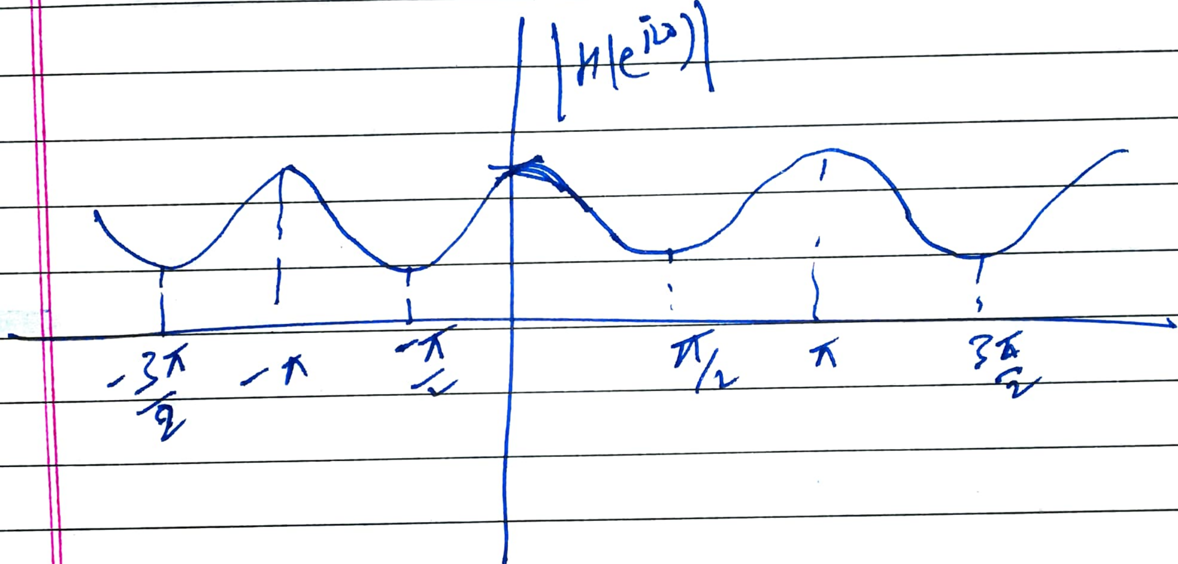
$$= \frac{(e^{j\omega})^2}{(e^{j\omega})^2 - 0.25}.$$

Poles at $z = \pm 0.5$

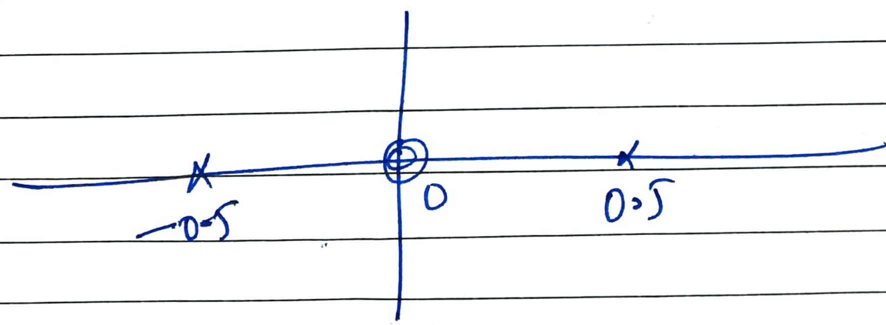
zeros at $z = 0$

$$H(e^{j\omega}) = \frac{(e^{j\omega})^2}{\cos 2\omega + j\sin 2\omega - 0.25}$$

$$|H(e^{j\omega})| = \frac{1}{(\cos 2\omega - 0.25)^2 + (\sin 2\omega)^2}$$



Poles and zeros



Since the system has two poles, the system will enhance the frequencies at $\omega = 0$ & $\omega = \pi$.

Impulse Response Calculation

$x[n] = \delta[n]$

$h[n] = 0.25\,h[n-2] + \delta[n].$

Since we are considering casual system.

$h[n] = 0$ for $n < 0$.

$n = 0$

$h[0] = 0.25\,h[\cancel{0-2}] + 1 = 1$

$h[1] = 0.25\,h[-1] + 0 = 0$

$h[2] = 0.25\,h[0] + 0 = 0.25$

$h[3] = 0.25\,h[1] + 0 = 0.25 \times 0 = 0$

$h[4] = 0.25\,h[2] + 0 = 0.25^2$

$h[5] = h[3] * 0.25 + 0 = 0$

$$h[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ 0 & n = 2k+1 \text{ for } k = 0,1,2,\ldots \\ (0.25)^{n/2} & n = 2k \text{ for } k = 1,2,3,\ldots \end{cases}$$

Q5.

$$G(z) = \frac{z^3}{(z-\frac{1}{2})(z+\frac{1}{3})^2}$$

$$\frac{G(z)}{z} = \frac{z^2}{(z-\frac{1}{2})(z+\frac{1}{3})^2}$$

$$= \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{3}} + \frac{C}{(z+\frac{1}{3})^2}$$

$$A = \frac{G(z)}{z}\left(z-\frac{1}{2}\right)\Big|_{z=\frac{1}{2}}$$

$$= \frac{(\frac{1}{2})^2}{(\frac{1}{2}+\frac{1}{3})^2} = \frac{\frac{1}{4}}{(\frac{5}{6})^2} = \frac{9}{25}.$$

$$C = \frac{G(z)}{z}\left(z+\frac{1}{3}\right)^2\Big|_{z=-\frac{1}{3}}$$

$$= \frac{(-\frac{1}{3})^2}{(-\frac{1}{3}-\frac{1}{2})} = \frac{-1\times6}{9\times5} = \frac{-2}{15}.$$

for B,

$$\frac{G(z)}{z} = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)^2}$$

$$= \frac{A\left(z+\frac{1}{3}\right)^2 + B\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right) + C\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)^2}$$

Comparing z^2 coefficient in numerator,

$$1 = A + B$$

$$B = 1 - A = 1 - \frac{9}{25} = \frac{16}{25}.$$

$$\frac{G(z)}{z} = \frac{9}{25} \cdot \frac{1}{\left(z-\frac{1}{2}\right)} + \frac{16}{25}\left(\frac{1}{z+\frac{1}{2}}\right) - \frac{2}{15}\left(\frac{1}{\left(z+\frac{1}{3}\right)^2}\right)$$

$$G(z) = \frac{9}{25}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{16}{25}\left(\frac{z}{z+\frac{1}{3}}\right) - \frac{2}{15}\left(\frac{z}{\left(z+\frac{1}{3}\right)^2}\right)$$

Since ROC $|z| > \frac{1}{2}$

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$$g[n] = \frac{9}{25}\left(\frac{1}{2}\right)^n u[n] + \frac{16}{25}\left(-\frac{1}{3}\right)^n u[n] - \frac{2}{15}\left(\frac{n}{3}\right)\left(-\frac{1}{3}\right)^n u[n]$$

$$= \left[\frac{9}{25}\left(\frac{1}{2}\right)^n + \frac{16}{25}\left(\frac{-1}{3}\right)^n - \frac{2}{45}n\left(-\frac{1}{3}\right)^n\right]u[n]$$

Q3:- Here we have to design an FIR filter it should be an all zero system i.e of the form

$$H(z) = \frac{1}{z^M}\sum_{k=0}^{M} b_k z^{M-k}$$

~~To block~~.

A zero at $\omega = 0.1$ on the unit circle is needed to block out low frequency component.

i.e one of the zeros $= e^{j(0.1)}$

Since complex zeros cannot exist without its conjugate so the other zero will be $e^{-j(0.1)}$.

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$\therefore$ Zeros $\Rightarrow$ $e^{\pm j(0.1)}$

or,

$z_1 = \cos(0.1) + j\sin(0.1)$

$z_2 = \cos(0.1) - j\sin(0.1)$

$$H(z) = \frac{(z-z_1)(z-z_2)}{z^2}$$

Trivial poles are added.

$$H(z) = \frac{z^2 - (z_1+z_2)z + z_1 z_2}{z^2}$$

$$= \frac{z^2 - 2\cos(0.1)z + 1}{z^2}$$

$$= \frac{1}{z^2}\left(1 \cdot z^{2-0} - 2\cos(0.1)z^{2-1} + 1 \cdot z^{2-2}\right)$$

$\therefore h[0] = h[2] = \alpha_0 = 1$

$h[1] = -2\cos(0.1) = -1.99 = \alpha_1$

$$\boxed{\begin{aligned}\alpha_0 &= 1\\ \alpha_1 &= -1.99\end{aligned}}$$

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Q1.

$$H(z) = \frac{z - z_1}{z - p_1}$$

$$H(e^{j\omega}) = \frac{e^{j\omega} - z_1}{e^{j\omega} - p_1}$$

$$|H(e^{j\omega})| = \frac{|e^{j\omega} - z_1|}{|e^{j\omega} - p_1|}$$

$p_1 = |p_1| e^{j\omega_1}$ where $|p_1| < 1$

$z_1 = |z_1| e^{j\omega_1}$ where $|z_1| > 1$

when $\omega = \omega_1$ $|z - p_1|$ is minimum

$\omega = \omega_1 + \pi$ $|z - p_1|$ is maximum

$\omega = \phi_1$ $|z - z_1|$ is minimum

$\omega = \phi_1 + \pi$ $|z - z_1|$ is maximum.