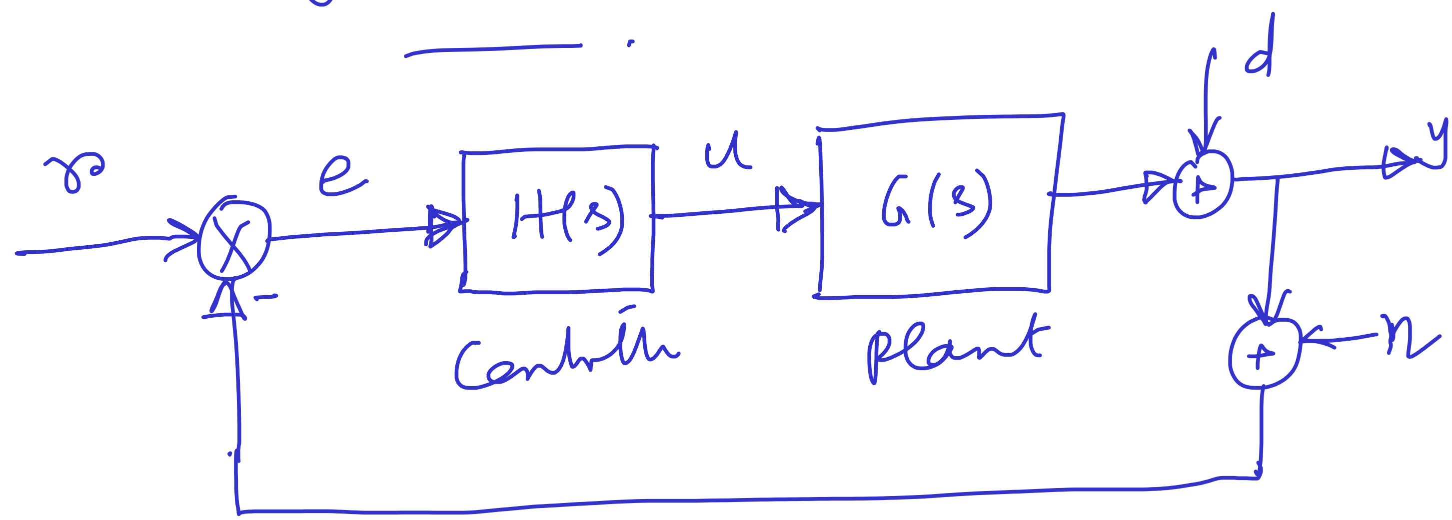
Lecture - 8

Pole-placement based Controller design

1-DOF dynamic o/p feedback control:



Note: The independent i/p to the controller is tracking error $\Rightarrow$ 1-DOF control

$\boxed{\text{Plant:}}$

$$G(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$:= \frac{c(s)}{a(s)} \longrightarrow \text{general strictly proper SISO plant}$$

$\boxed{\text{Controller:}}$

$$H(s) = \frac{h_m s^m + h_{m-1}s^{m-1} + \cdots + h_1 s + h_0}{s^m + k_{m-1}s^{m-1} + \cdots + k_1 s + k_0}$$

$$:= h(s)/k(s)$$

$\longrightarrow$ general bi-proper m-th order controller

Note: The controller in general is chosen as a bi-proper TF so as to have max^m number of tuneable parameters

Choice of controller order

The order of closed loop pole polyn is $m+n$ $\Rightarrow$ $m+n$ closed-loop poles have to be placed.

Now the number of gains of the controller $2m+1$

$$\Rightarrow 2m+1 \geq m+n$$

$$\Rightarrow \boxed{m \geq n-1}$$

$\Rightarrow$ The minimal controller order needed $n-1$.

- Naturally using first order controller like lead lag, one can not achieve desired performance & robustness.

Pole-placement for a second order plant:

$$G(s) = \frac{c_1 s + c_0}{s^2 + a_1 s + a_0}, \quad H(s) = \frac{h_1 s + h_0}{s + k_0}$$

Let the desired closed-loop pole-poly^n be

$$s^3 + \delta_2 s^2 + \delta_1 s + \delta_0 := \delta(s) \quad -(1)$$

The characteristic eqn is

$$1 + G(s)H(s) = 0$$

$$\Rightarrow \quad 1 + \frac{c(s)}{a(s)} \cdot \frac{h(s)}{k(s)} = 0$$

$$\Rightarrow \quad a(s)\,k(s) + c(s)\,h(s) = 0$$

$$\Rightarrow \quad (s^2 + a_1 s + a_0)(s + k_0) + (c_1 s + c_0)(h_1 s + h_0) = 0$$

$$\Rightarrow \quad s^3 + (k_0 + a_1 + h_1 c_1) s^2 + (a_0 + k_0 a_1 + h_0 c_1 + h_1 c_0) s + k_0 a_0 + h_0 c_0 = 0 \quad -(2)$$

Comparing (1) & (2),

$$k_0 a_0 + h_0 c_0 = \delta_0$$

$$a_0 + k_0 a_1 + h_0 c_1 + h_1 c_0 = \delta_1$$

$$k_0 + a_1 + h_1 c_1 = \delta_2$$

$$1 = 1$$

$$\Rightarrow \underbrace{\begin{bmatrix} a_0 & 0 & c_0 & 0 \\ a_1 & a_0 & c_1 & c_0 \\ 1 & a_1 & 0 & c_1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\substack{4\times 4 \\ S \,=\, \text{sylvester matrix}}} \underbrace{\begin{bmatrix} k_0 \\ 1 \\ h_0 \\ h_1 \end{bmatrix}}_{\substack{4\times 1 \\ W}} = \underbrace{\begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ 1 \end{bmatrix}}_{\substack{4\times 1 \\ \bar{\delta}}}$$

$$\Rightarrow \boxed{SW = \bar{\delta}} \rightarrow \text{sylvester matrix eqn}$$

$$\Rightarrow \boxed{W = S^{-1}\bar{\delta}} \rightarrow \text{To obtain the controller parameter vector}$$

Note: S is invertible if $c(s)$, $a(s)$ are co-prime ie has no common factor.

General Case

plant order n & the controller order $m = n-1$.

$$
S' = \begin{bmatrix}
a_0 & 0 & \cdots & 0 & c_0 & 0 & \cdots & 0 \\
a_1 & a_0 & \cdots & 0 & c_1 & c_0 & \cdots & 0 \\
a_2 & a_1 & \cdots & 0 & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & a_0 & c_{n-1} & c_{n-2} & & c_0 \\
a_{n-1} & \vdots & & a_1 & 0 & c_{n-1} & & c_1 \\
1 & a_{n-1} & & \vdots & 0 & 0 & & c_2 \\
0 & 1 & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & 0 & & a_{n-1} & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}_{2n \times 2n}
$$

— General Sylvester matrix

$$
W = \begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ \vdots \\ k_{n-2} \\ 1 \\ h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{bmatrix}_{2n \times 1}, \qquad
\delta = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{2n-2} \\ 1 \end{bmatrix}_{2n \times 1}
$$

Different TFs in 1-DOF Control

1) r - to - y TF

$$\frac{y}{r} = \frac{GH}{1+GH} = \frac{ch}{\delta}$$ [where $\delta = a\kappa + ch$

2) d - to - y TF

$$\frac{y}{d} = \frac{1}{1+GH} = \frac{a\kappa}{\delta}$$

3) η - to - y TF

$$\frac{y}{\eta} = \frac{GH}{1+GH} = \frac{ch}{\delta}$$

4) r - to - u TF

$$\frac{u}{r} = \frac{H}{1+GH} = \frac{ah}{\delta}$$

Limitations of 1-DOF Control

Once $\delta(s)$ is chosen, the controller polyn's $h(s)$ & $\kappa(s)$ get fixed. If this roots of these poly^{n}s are close to $j\omega$ axis then the following occur:

- roots of $h(s)$ close to $j\omega$ axis causes a) high p.o in $y(t)$ & b) more control effort & c) poor noise attenuation.
- roots of $K(s)$ close to $j\omega$ axis may cause peak in disturbance response.

Choice of desired closed-loop pole polyn

- First choose dominant pole pair based on p.o & settling time specification

$$\text{dominant pole locations } s = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}$$

$$p.o = e^{-\xi\pi/\sqrt{1-\xi^2}}, \quad t_s = \frac{4}{\xi\omega_n}$$

$\Rightarrow$ obtain ξ & ω_n

[neglecting additional overshoot due to the roots of $h(s)$]

- The nondominant poles can be

chosen 5-6 times away from
$-\xi\omega_n$ in LHP.

The above can be consider as a
starting phase for choice of Closed
-loop poles.