Laplace Transform

[Pole with higher order]

Example: $G(s) = \dfrac{1}{s(s+1)^3(s+2)}$ $\qquad g(t)$?

Simple poles: 0, −2, pole at −1 of order 3.

$$G(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{K_1}{s+1} + \frac{K_2}{(s+1)^2} + \frac{K_3}{(s+1)^3}$$

$$A = sG(s)\Big|_{s=0} = \frac{1}{2}$$

$$B = (s+2)G(s)\Big|_{s=-2} = \frac{1}{2}$$

$$K_i = \frac{1}{(r-i)!}\,\frac{d^{r-i}}{ds^{r-i}}\left[(s+p)^r G(s)\right]\Big|_{s=-p}$$

$i = 3, 2, 1.$

$r = 3$

$r =$ order

$$K_3 = (s+1)^3 G(s)\Big|_{s=-1} = -1$$

$$K_2 = \frac{d}{ds}\left[(s+1)^3 G(s)\right]\Big|_{s=-1} = 0$$

$$K_1 = \frac{1}{2!}\,\frac{d^2}{ds^2}\left[(s+1)^3 G(s)\right]\Big|_{s=-1} = -1$$

$$G(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1} - \frac{1}{(s+1)^3}$$

Apply inverse Laplace Transform.

$$g(t) = \mathcal{L}^{-1} G(s)$$

$$\boxed{g(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t} - e^{-t} - \frac{1}{2}e^{-t}t^2, \quad t \geq 0}$$

Example: [Complex conjugate pole]

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$G(s) = \frac{\omega_n^2}{\underbrace{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$$g(t) = \mathcal{L}^{-1} G(s)$$

?

$$= \frac{K_1}{s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} + \frac{K_2}{s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}$$

$$K_1 = \left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right) G(s) \Big|_{s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{\omega_n}{2j\sqrt{1-\zeta^2}}$$

$$K_2 = -\frac{\omega_n}{2j\sqrt{1-\zeta^2}}$$

$$G(s) = \frac{\omega_n}{2j\sqrt{1-\zeta^2}} \left[\frac{1}{s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}} - \frac{1}{s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}\right]$$

Apply inverse Laplace Transform

$$g(t) = \mathcal{L}^{-1} G(s)$$

$$= \frac{\omega_n}{2j\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left(e^{j\omega_n\sqrt{1-\zeta^2}\,t} - e^{-j\omega_n\sqrt{1-\zeta^2}\,t}\right)$$

$$\boxed{g(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\,t, \quad t \geq 0}$$

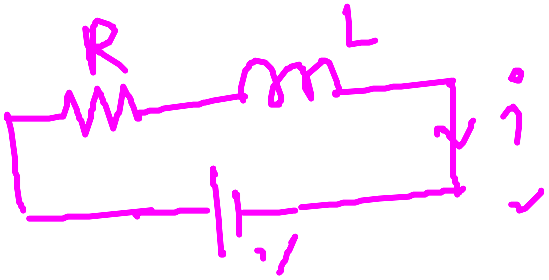
Solution of linear ordinary differential equation

Ex.1

$$\tau \frac{dy(t)}{dt} + y(t) = u(t), \quad y(0)=0.$$

Find $y(t)$. $u(t)$ is the input.

$u(t)$ is unit step.



Taking Laplace transform, we get

$$\tau s Y(s) - \tau \cancel{y(0)}^{\,0} + Y(s) = \frac{1}{s}$$

$y(0)=0$

$$\tau s Y(s) + Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(\tau s+1)} = \frac{A}{s} + \frac{B}{\tau s + 1}$$

$$v(t) = i(t)R + L\frac{di(t)}{dt}$$

$$\underbrace{\frac{L}{R}}_{=\tau} \frac{di(t)}{dt} + i(t) = \underbrace{\frac{1}{R}v(t)}_{u(t)}$$

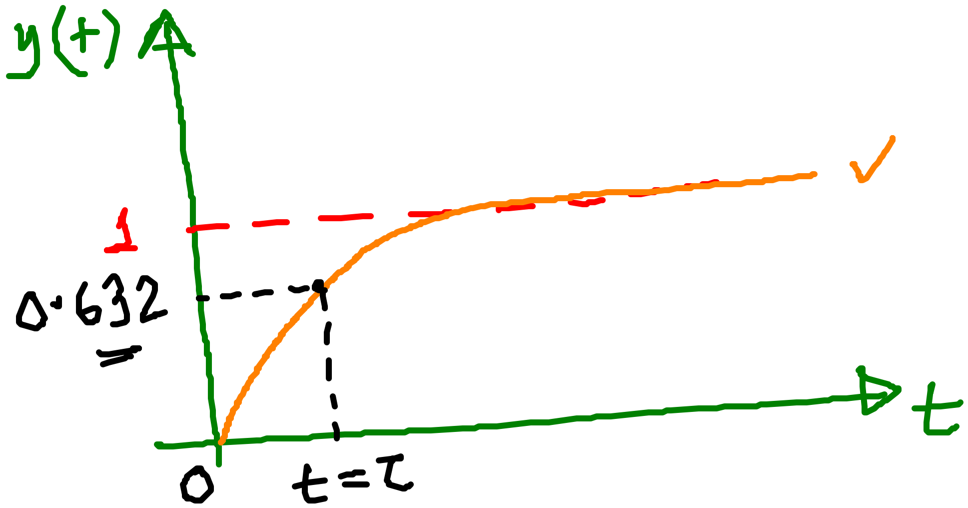
$$A = sY(s)\Big|_{s=0} = 1$$

$$B = (\tau s+1)\,Y(s)\Big|_{s=-\frac{1}{\tau}} = -\tau$$

$$Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}} \checkmark$$

Apply inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}Y(s) = \underline{1 - e^{-t/\tau}}, \quad t \geq 0.$$



At $t=\tau$,
$\underline{63.2\%}$ of the steady-state ($\underline{1}$)
$\underline{\tau = \text{time constant}}$

At $t=\tau$, $y(\tau) = 1 - e^{-1} = 1 - 0.368 = 0.632$

Ex2 $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 5u(t)$, $y(0) = -1$, $y'(0) = 2$

t?

input (unit step)

$$s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{5}{s}$$

$$s^2Y(s) + s - 2 + 3sY(s) + 3 + 2Y(s) = \frac{5}{s}$$

$$(s^2 + 3s + 2)Y(s) = \frac{5}{s} - s - 1$$

$$Y(s) = \frac{-s^2 - s + 5}{s(s^2 + 3s + 2)} = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

closed

$sY(s)$ analytic in right-half of 's'-plane

$$= \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$$

Apply inverse Laplace transform

$$\boxed{y(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t}, \quad t \geq 0}$$

When $t \to \infty$, $y(t) = \frac{5}{2}$. (Steady-state value)

If we apply final value theorem,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$