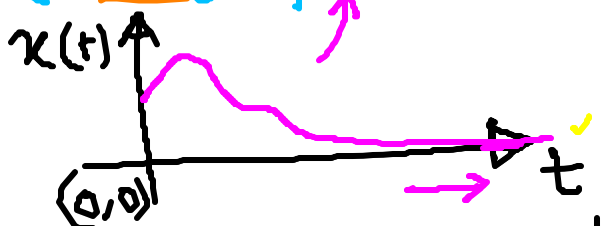
Stability of Linear time-invariant Systems

There are many definitions of stability depending upon the type of systems and the point of view considered.

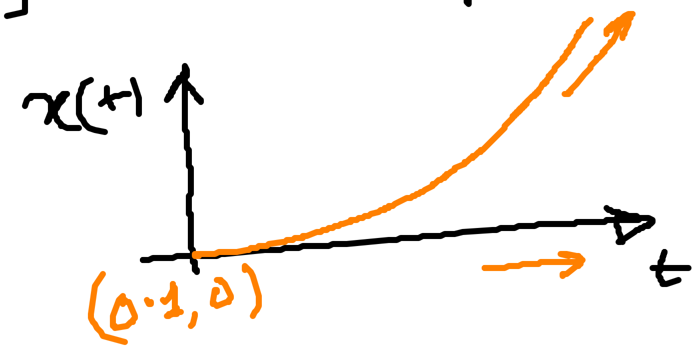
Total response of LTI system = Natural response (due to initial condition) + Forced response (due to forcing function)

From natural response point of view

- stable if the response approaches zero as $t \to \infty$ (asymptotic stability)

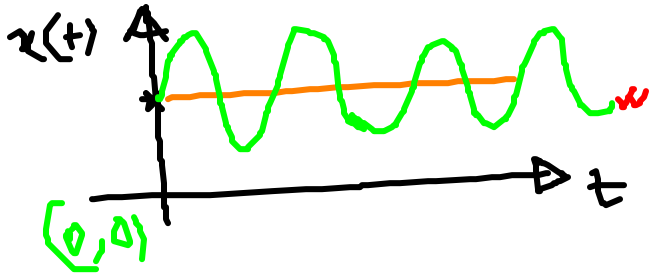


- unstable if the response tends to ∞ as $t \to \infty$.



- marginally stable if response neither decays nor grows but remains

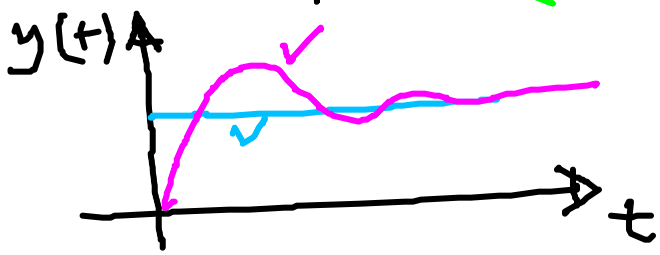
Constant or oscillates.



From force response point of view:

- stable if every bounded input yields a bounded output (BIBO stability)

BIBO - Bounded input bounded output



- Unstable if any bounded input yields an unbounded output

(1) $G(s) = \frac{1}{s^2+1} = \frac{Y(s)}{U(s)}$ $\qquad r(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$

Input $r(t) = \sin(t)\, u(t)$, $\quad U(s) = \frac{1}{s^2+1}$.

$$Y(s) = \frac{1}{(s^2+1)^2}$$

$$\boxed{y(t) = t \sin t}$$

Although $\sin(t)$ is a bounded signal, $y(t)$ is unbounded as $t \to \infty$. Hence $G(s)$ is not BIBO stable. The marginal stability in view of natural response is unstable from BIBO perspective.

(2) With zero initial condition, the system is BIBO stable if its output $y(t)$ is bounded to a bounded input $u(t)$.

$$y(t) = \int_0^{\infty} u(t-\tau)\, g(\tau)\, d\tau$$

$g(\tau)$ is the impulse response.

$$|y(t)| = \left| \int_0^{\infty} u(t-\tau)\, g(\tau)\, d\tau \right| \le \int_0^{\infty} |u(t-\tau)|\,|g(\tau)|\, d\tau$$

(Let $|u(t)| \le M$)
bounded input

$$\le M \int_0^{\infty} |g(\tau)|\, d\tau$$

If the system's output $y(t)$ is bounded,

$$|y(t)| \le N < \infty.$$

If $\int_0^\infty |g(\tau)|\, d\tau \le N < \infty$, i.e., $\exists$ a finite positive Q such that

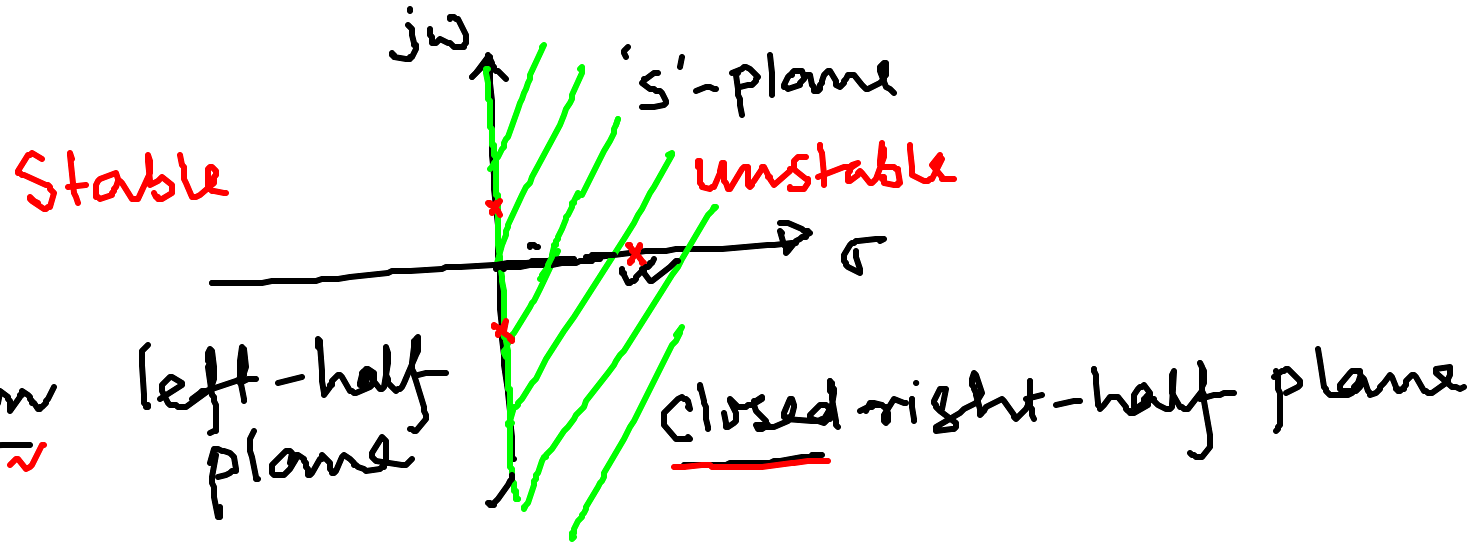
$$\boxed{\int_0^\infty |g(\tau)|\, d\tau \le Q < \infty}$$

then the system is BIBO stable.

(3) Asymptotic stability: $y(t)$ reaches zero as $t \to \infty$ when $u(t) \equiv 0$.

Asymptotic stability implies BIBO stability.

(4)



For LTI system, BIBO and zero input stability and asymptotic stability all have the requirement that the roots of the characteristic equation all be located in the open left-half of s-plane.

$$G(s) = \mathcal{L}[g(t)] = \int_0^{\infty} g(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} g(t)\, e^{-\sigma t} \cdot e^{-j\omega t}\, dt$$

$$|G(s)| \leq \int_0^{\infty} |g(t)|\, |e^{-\sigma t}|\, dt$$

Let 's' be a pole of $G(s)$. Then $G(s) = \infty$.

$$\infty \leq \int_0^{\infty} |g(t)|\, \underbrace{|e^{-\sigma t}|}_{\leq 1}\, dt$$

If any pole is on the closed right-half of 's'-plane, $|e^{-\sigma t}| \leq 1$ since $\sigma \geq 0$. Then, it implies

$$\infty \leq \int_0^{\infty} |g(t)|\, dt$$

The system is unstable.

(5) Poles of multiplicity greater than 1 on the imaginary axis lead to unstable response.

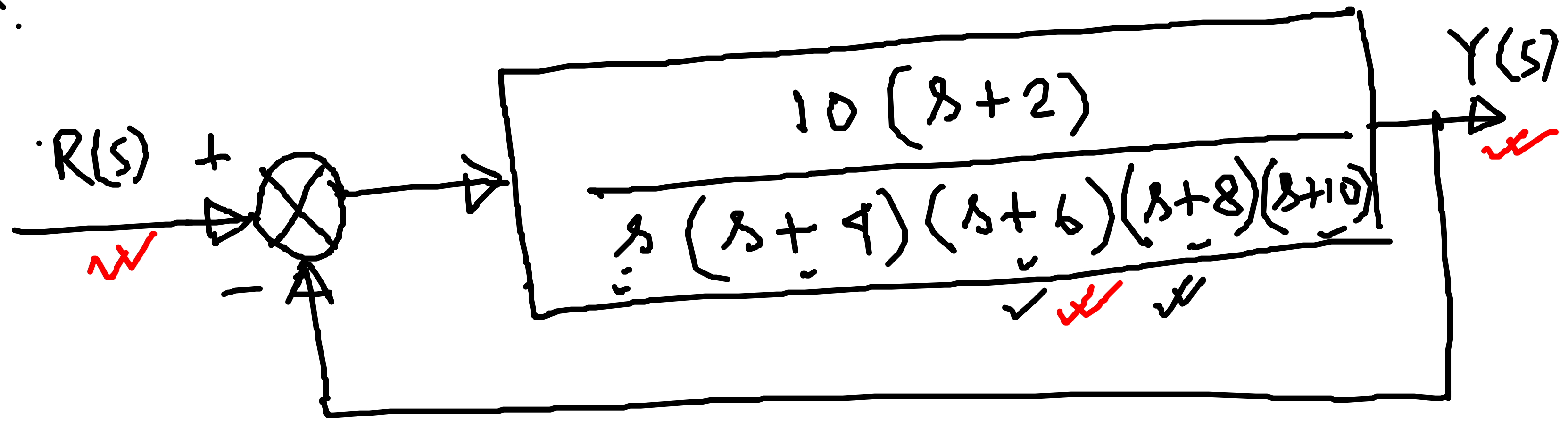
$$G(s) = \frac{1}{s} \qquad\qquad G(s) = \frac{1}{s^2}$$

multiplicity = 2

unstable

Unstable systems have closed-loop transfer functions with at least one pole in RHP and/or poles of multiplicity greater than 1 on the imaginary axis.

Ex.



$$\frac{Y(s)}{R(s)} = \frac{10(s+2)}{1 s^{5} + 28 s^{4} + 284 s^{3} + 1232 s^{2} + 1930 s + 20}$$

What are the locations of closed-loop poles?

Necessary condition for stability (asymptotic)

- All co-efficients are with same sign.
- All co-efficients should be present.

Ex.

Ch. eqn:

$$1s^3 - 2s^2 + 5s + 6 = 0 \quad \text{(unstable)}$$

$$s^4 + 5s^3 + 2s + 6 = 0 \quad \text{(unstable)}$$

$0 \cdot s^2$ is absent.

- If any of the above condition fails, the system is not stable.

$\Rightarrow$ sufficient condition for instability.