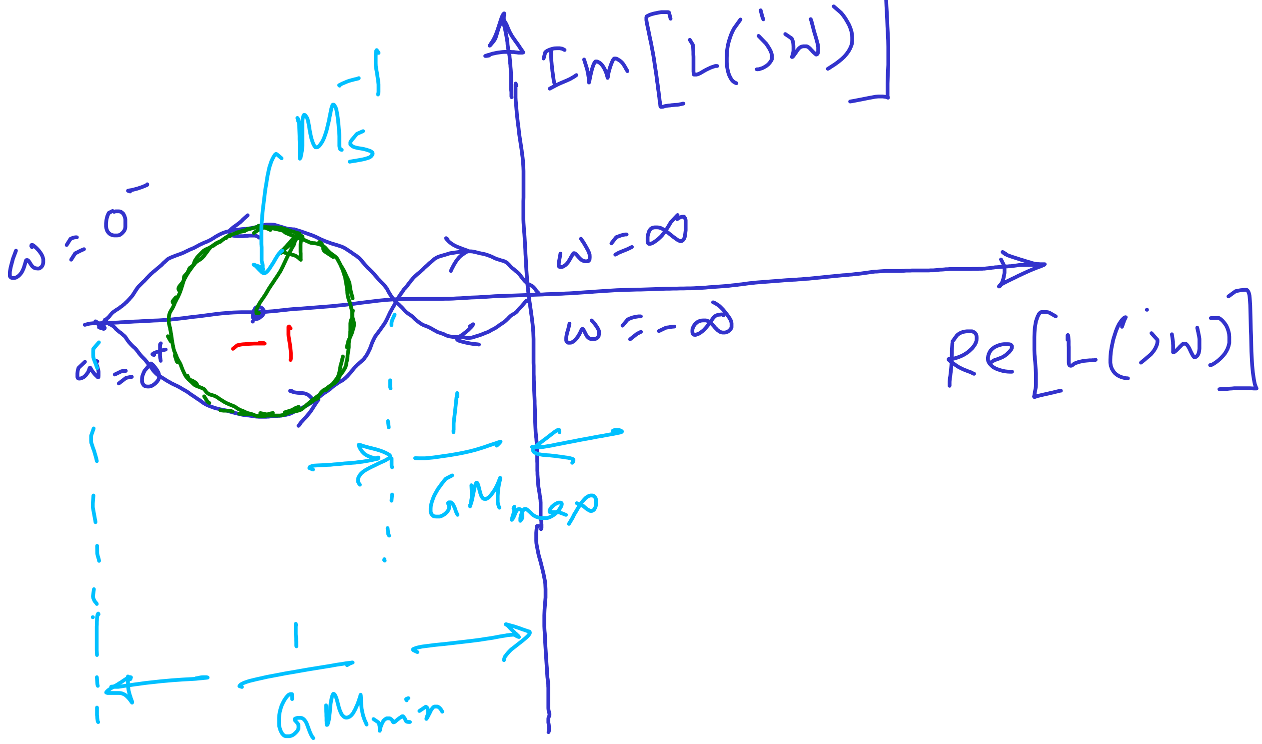
Relation among GM, PM, S

$$GM_{max} \geq \frac{M_s}{M_s - 1}$$ ✓

$$PM \geq 2\sin^{-1}\frac{1}{2M_s}$$ ✓

$$GM_{min} \leq \frac{M_s}{M_s + 1}$$



From the above figure

$$\frac{1}{GM_{min}} - 1 \geq M_s^{-1}$$

$$\Rightarrow \quad \frac{1}{GM_{min}} \geq M_s^{-1} + 1$$

$$\Rightarrow \quad \boxed{GM_{min} \leq \frac{M_s}{M_s + 1}}$$

Suppose you want $GM_{min} \leq 0.5$

then $$\frac{M_s}{M_s+1} \leq 0.5$$

$$\Rightarrow \quad M_s \leq 0.5M_s + 0.5$$

$$\Rightarrow \quad 0.5M_s \leq 0.5$$

$$\Rightarrow \quad \boxed{M_s \leq 1}$$

For a good design $M_s < 2$

$$\Rightarrow \quad GM_{min} \leq \frac{2}{3}$$

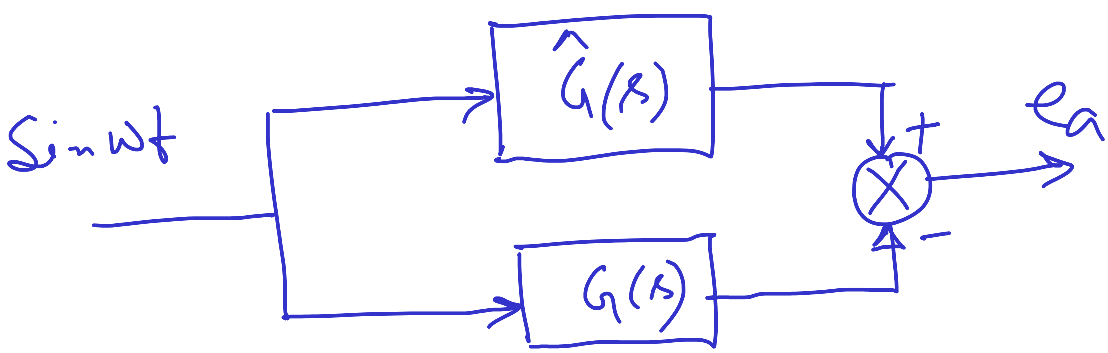
Robustness against unmodelled or neglected dynamics

Actual plant $\hat{G}(s) = \dfrac{10}{(s+1)(s+10)}$

$$\approx \frac{1}{s+1}$$

Approximated or Simplified plant

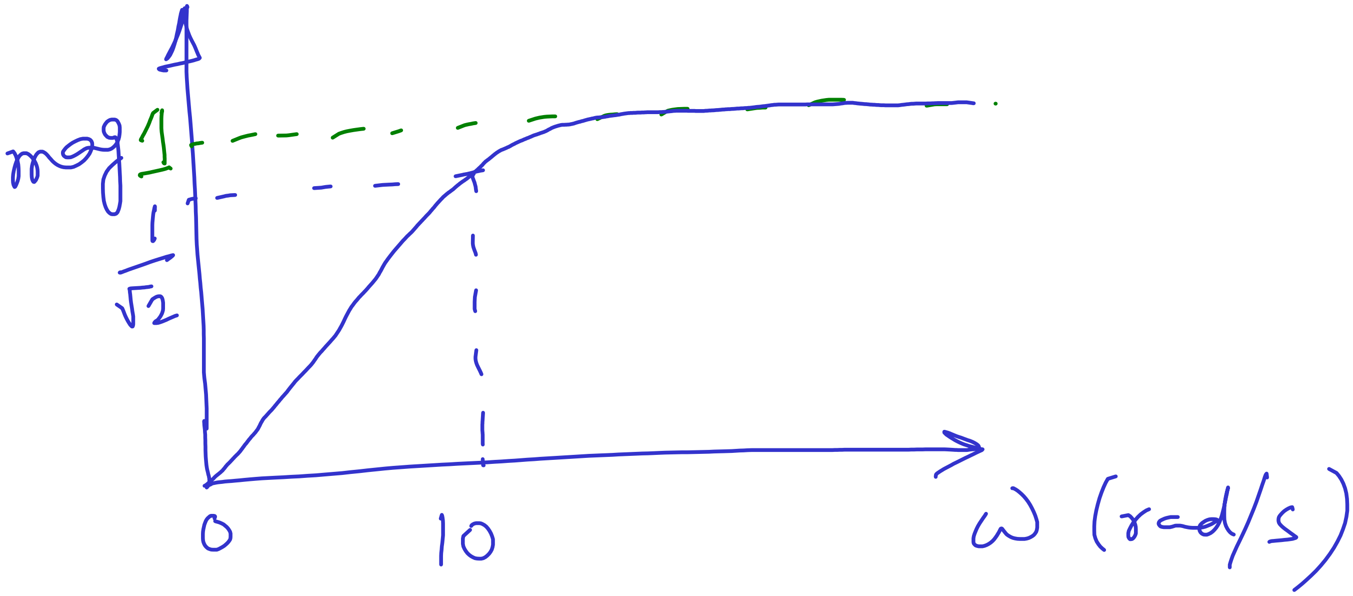
$$G(s) = \frac{1}{s+1}$$



$$|e_a(j\omega)| = |\hat{G}(j\omega) - G(j\omega)|$$

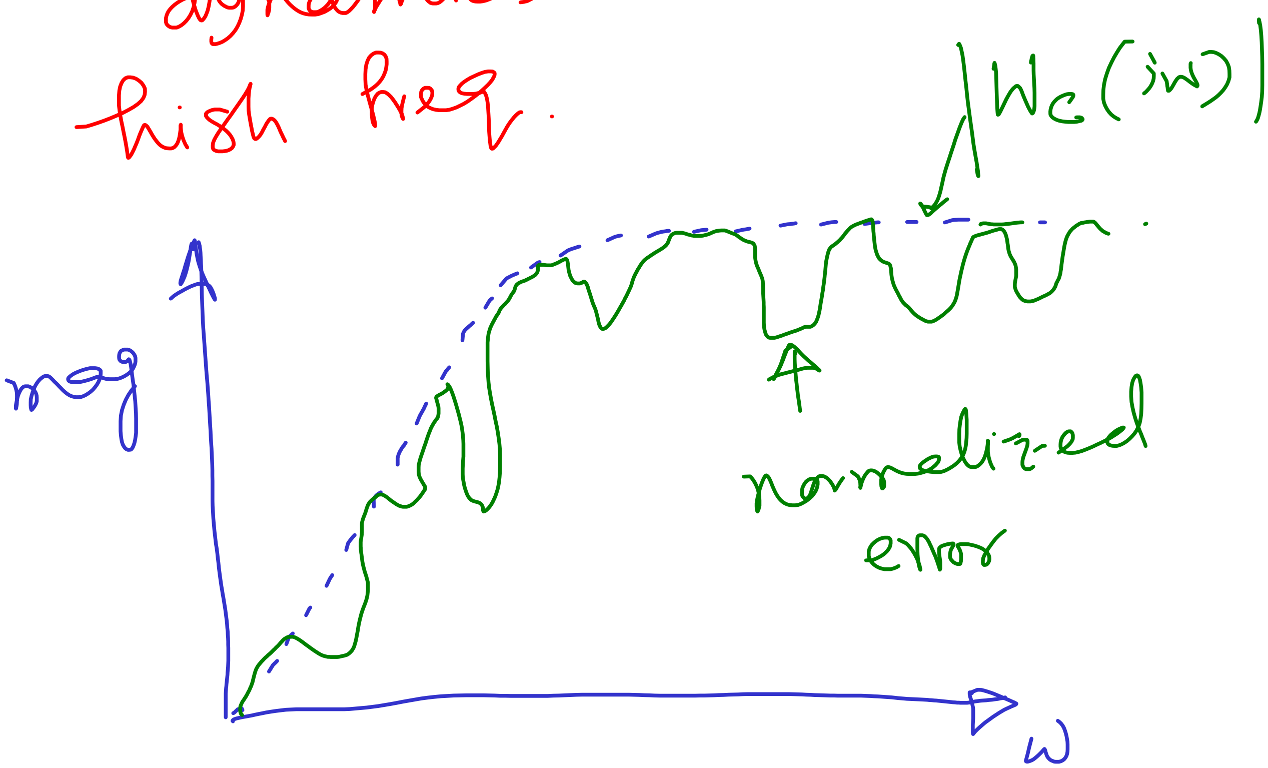
$$\Rightarrow \left|\frac{e_a(j\omega)}{G(j\omega)}\right| = \left|\frac{\hat{G}(j\omega)}{G(j\omega)} - 1\right|$$

$$\Rightarrow \frac{|e_a(j\omega)|}{|G(j\omega)|} = \left|\frac{-j\omega}{j\omega+10}\right| = \frac{\omega}{\sqrt{\omega^2+1}}$$



Normalized error is small at low freq & large at high freq.

$\Rightarrow$ Neglected or unmodelled dynamics are dominant at high freq.



$$\left|\frac{e_a(j\omega)}{G(j\omega)}\right| = \left|\frac{\hat{G}(j\omega)}{G(j\omega)} - 1\right|$$

$$= \left|\Delta(j\omega)\, W_c(j\omega)\right|$$

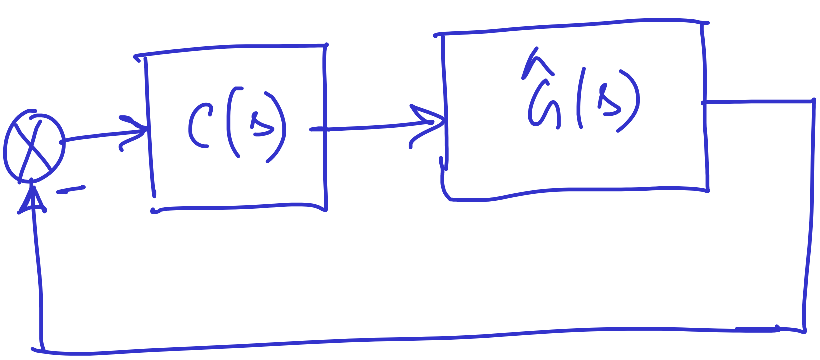
where $|\Delta(j\omega)| \leq 1 \quad \forall\, \omega$

True plant $\hat{G}(s) = G(s)\left[1 + W_c(s)\Delta(s)\right]$

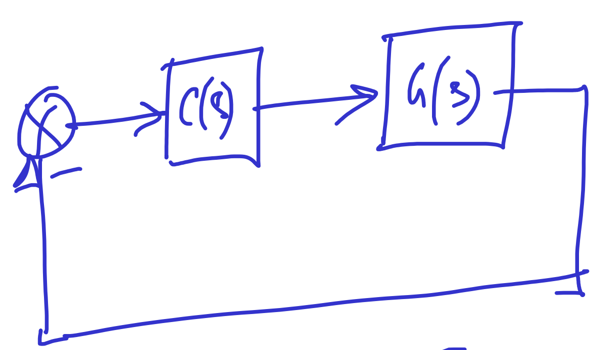
where $|\Delta(j\omega)| < 1 \quad \forall\, \omega$

$\longrightarrow$ multiplicative uncertainty model

Condition for R.S against unmodelled dynamics



True feedback system



Approximated feedback system

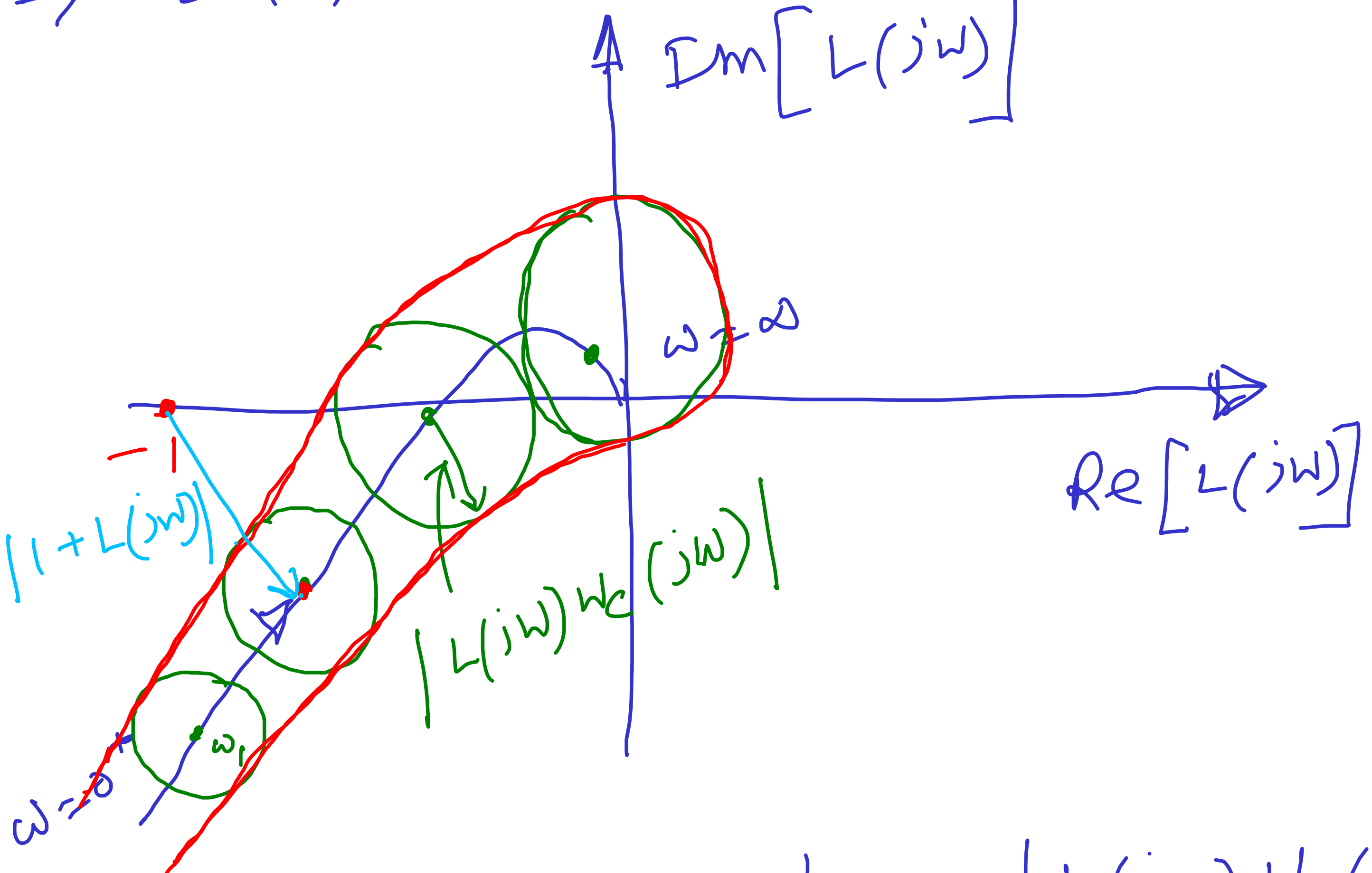
Actual loop TF

$$\hat{L}(s) = \hat{G}(s)\, C(s)$$

$$\Rightarrow \hat{L}(s) = G(s)\left[1 + W_c(s)\Delta(s)\right] C(s)$$

$$\Rightarrow \hat{L}(s) = L(s)\left[1 + W_c(s)\Delta(s)\right]$$

$$\Rightarrow \hat{L}(s) = L(s) + \Delta(s) L(s) W_c(s)$$



For R.S $\quad |1+L(j\omega)| > |L(j\omega) W_c(j\omega)|$

$$\Rightarrow \left|\frac{L(j\omega)\, W_c(j\omega)}{1+L(j\omega)}\right| < 1 \quad \forall \omega \geq 0$$

$$\Rightarrow \left| W_c(j\omega)\, T(j\omega) \right| < 1 \quad \forall\ \omega \geq 0$$

where $T(s) = \dfrac{L(s)}{1+L(s)}$ = Complementary

$\underbrace{\text{Sensitivity function} = 1 - S}$

$\Rightarrow \quad S + T = 1$

So for R.S against neglected dynamics

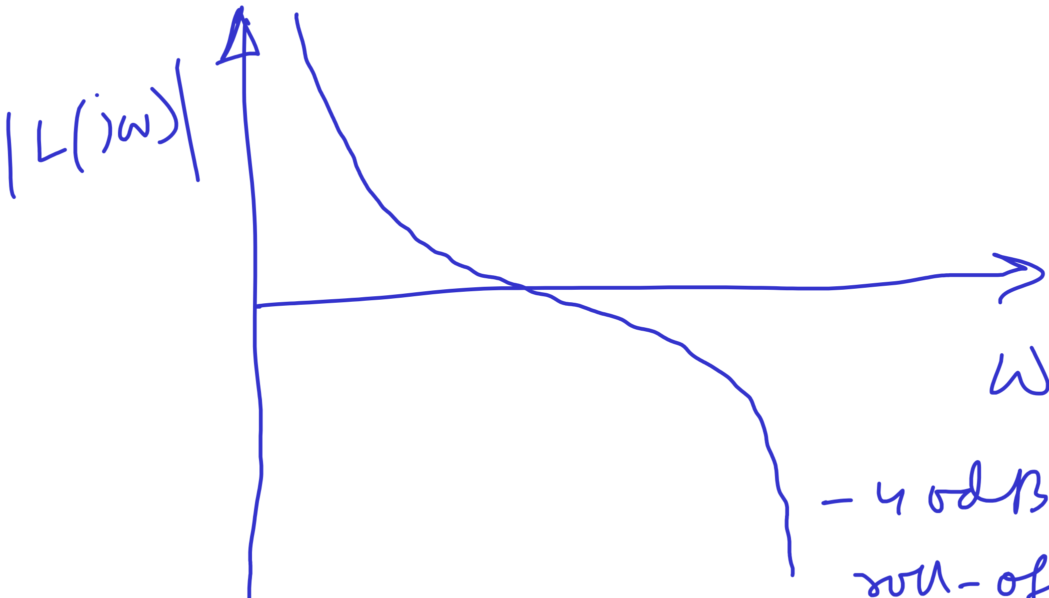
$$\boxed{|T(j\omega)| < \frac{1}{|W_c(j\omega)|} \quad \forall\ \omega}$$

$$\max_{\omega} \left| W_c(j\omega)\, T(j\omega) \right| < 1 \rightarrow \text{for RS}$$

- For R.S against neglected dynamics $|T(j\omega)|$ should be as less as possible.
- we should make $|T(j\omega)|$ small

at high freq. This is possible if $|L(j\omega)| < 1$ at high freq

- For a good design $|L(j\omega)|$ should have -40 dB/decade or more roll-off rate at high freq



-40dB/decade roll-off rate

<u>EX 1</u>

$$\hat{G}(s) = \frac{e^{-\tau s}}{s^2}, \quad \tau = \text{delay}$$

$$G(s) = \frac{1}{s^2}$$

suppose a PD Controller is designed

for $G(s)$ to place the closed-loop poles at $s = -1, -2$. Determine the amount of delay τ that can be tolerated.

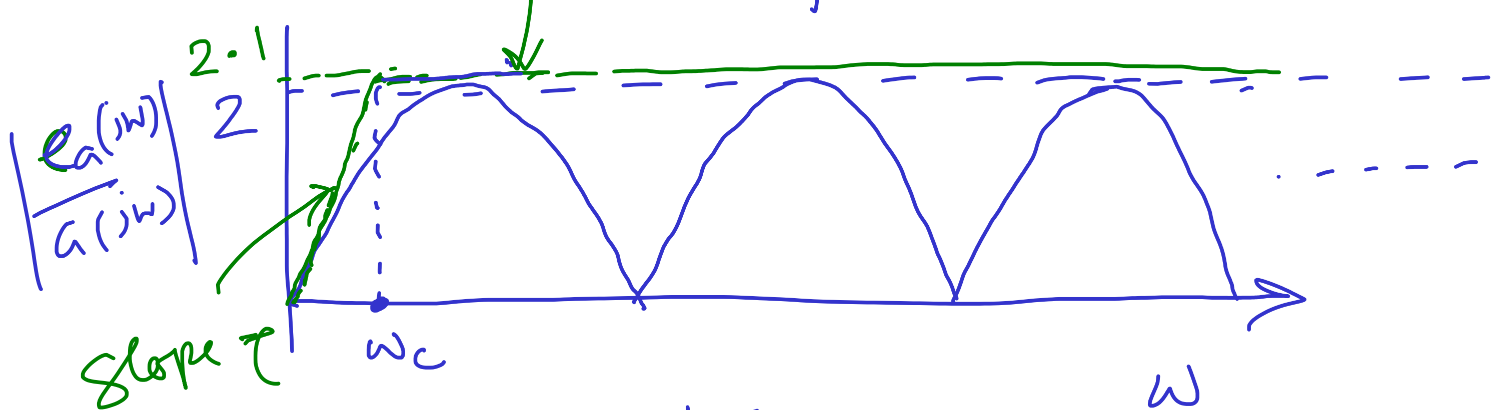
Soln

$$\left| \frac{\hat{G}(j\omega)}{G(j\omega)} - 1 \right| = \left| e^{-j\tau\omega} - 1 \right|$$

$$= \left| \cos\tau\omega - j\sin\tau\omega - 1 \right|$$

$$= \sqrt{(\cos\tau\omega - 1)^2 + \sin^2\tau\omega}$$

$$W_c(s) = \left| 2\sin\frac{\tau\omega}{2} \right|$$



Let $W_c(s) = \dfrac{Ks}{Ts+1}$

Then $W_c(\alpha) = \frac{K}{T}$

$\Rightarrow \frac{K}{T} = 2.1 \Rightarrow \boxed{K = 2.1T}$

From figure $\omega_c = \frac{2.1}{\tau}$

Now with the PD Control the closed-loop pole polyn becomes

$$1 + GC = 0$$

$$\Rightarrow 1 + \frac{1}{s^2}(K_p + K_d s) = 0$$

$$\Rightarrow s^2 + K_d s + K_p = 0 \quad \text{--- (1)}$$

On the other hand, the desired closed loop pole polyn

$$(s+1)(s+2) = 0$$

$$\Rightarrow s^2 + 3s + 2 = 0 \quad \text{--- (2)}$$

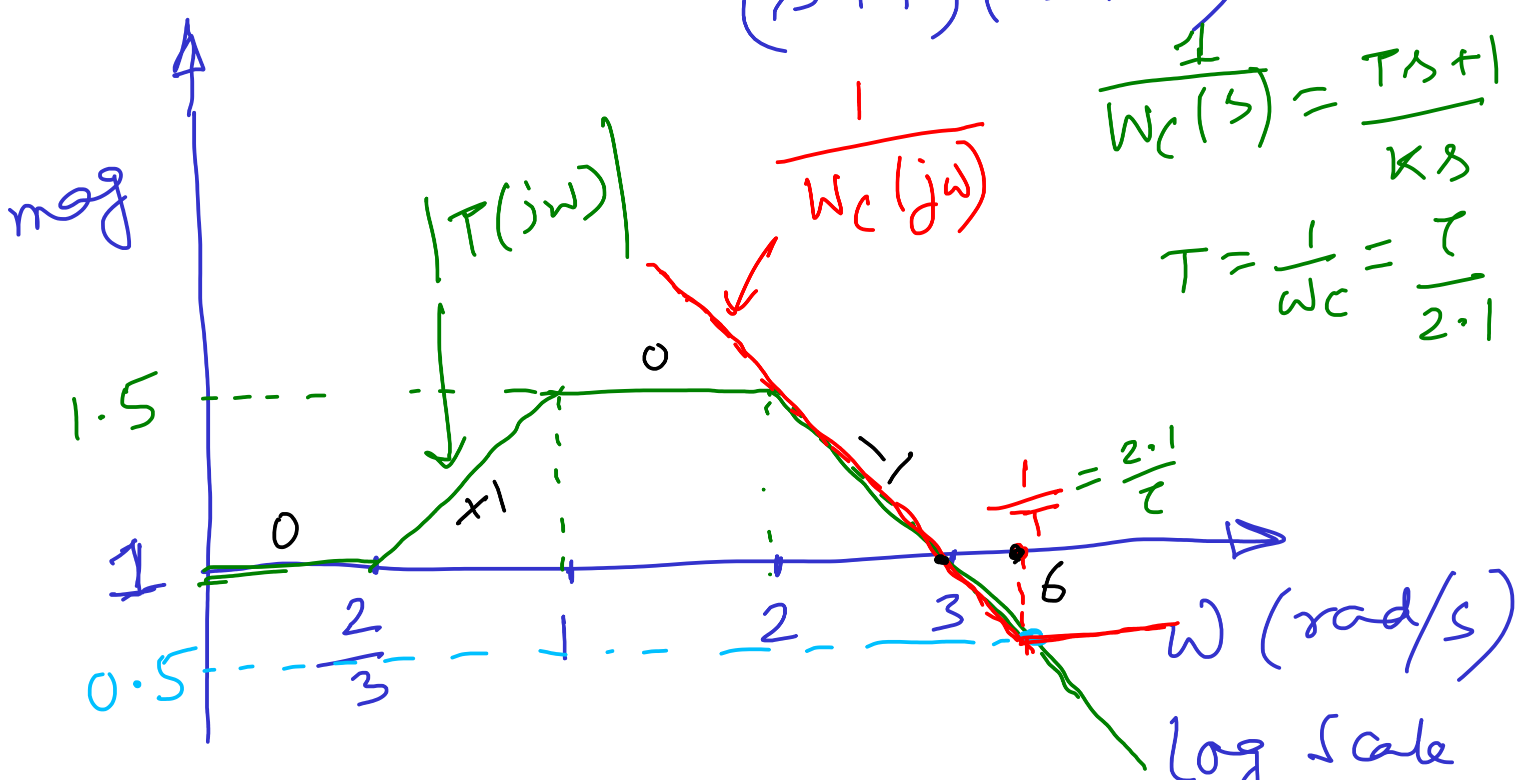
Comparing (1) & (2), $\boxed{K_p = 2,\ K_d = 3}$

Then $T(s) = \dfrac{GC}{1+GC}$

$$= \frac{(K_p + K_d s)\cdot\frac{1}{s^2}}{1+(K_p+K_d s)\cdot\frac{1}{s^2}}$$

$$= \frac{K_p + K_d s}{s^2 + K_d s + K_p}$$

$$= \frac{2+3s}{(s+1)(s+2)}$$



$$\frac{1}{W_c(s)} = \frac{Ts+1}{Ks}$$

$$T = \frac{1}{\omega_c} = \frac{\tau}{2.1}$$

For R.S, $\dfrac{2.1}{\tau} > 6 \Rightarrow \boxed{\tau < 0.31\,\text{sec}}$

EX 2 Let $\hat{G}(s) = \frac{1}{(s-1)}\left[1 \pm \frac{0.2s}{s+10}\right]$

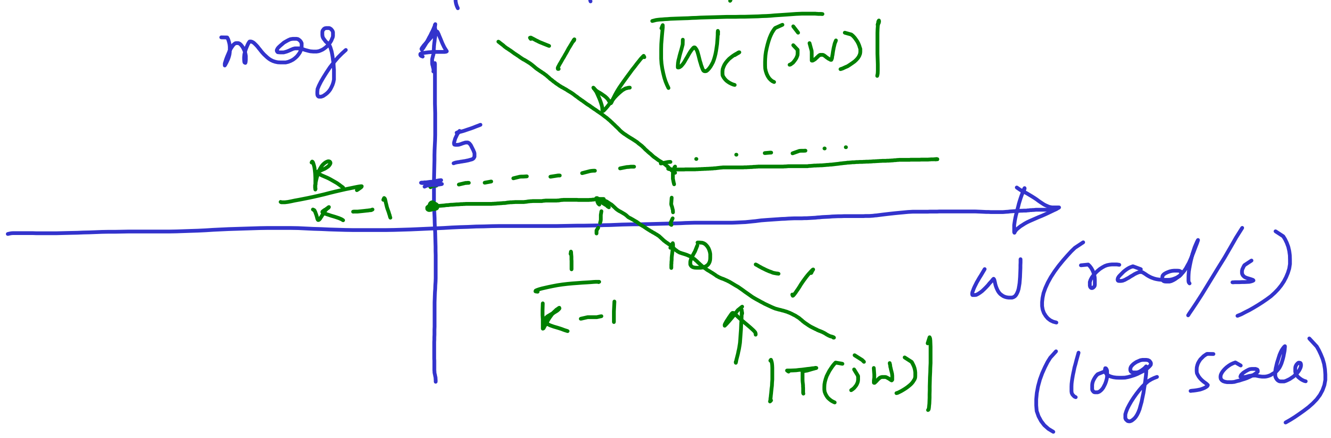
Consider a proportional controller $C(s) = K$ designed for the approximated plant $G(s) = \frac{1}{s-1}$. Find the range of K such that the true closed-loop system is stable.

Soln

Here $W_c(s) = \frac{0.2s}{s+10}$

$$\Rightarrow W_c(s) = \frac{0.02s}{0.1s+1}$$

$$T(s) = \frac{GC}{1+GC} = \frac{K}{s+K-1}$$



To guarantee that $|T(j\omega)|$ lies below $1/|W_e(j\omega)|$, one must have

$$\frac{1}{k-1} < 10 \quad \& \quad \frac{k}{k-1} < 5$$

$$\downarrow \qquad\qquad \downarrow$$

$$k > 1.1 \qquad\qquad k > 1.25$$

So to satisfy both $\boxed{k > 1.25}$