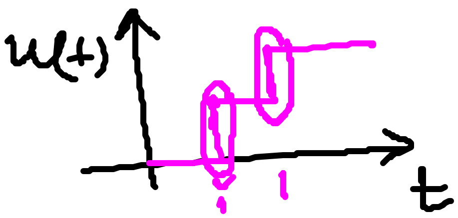
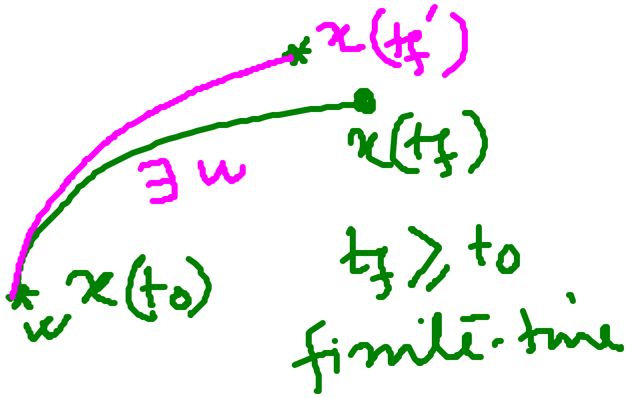
Controllability and Observability

Controllability:





The state $x(t)$ is said to be controllable at $t = t_0$ if there exists a piecewise continuous input $u(t)$ that will drive the state to any final state $x(t_f)$ for a finite time $t_f \geq 0$.

Let a system (LTI) be described as

$$\begin{cases} \dot{x}(t) = \underset{n\times n}{A}\, \underset{n\times 1}{x(t)} + \underset{n\times r}{B}\, \underset{r\times 1}{u(t)}, & x(0) \\ y(t) = C\, x(t) + D\, u(t) \end{cases}$$

$x(t) \in \mathbb{R}^n$

n is the number of state-variables

The pair (A, B) is controllable if and only if

$$S = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

is of rank n.

Controllability matrix

Example:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{B} u$$

Two state-variables

$n = 2$

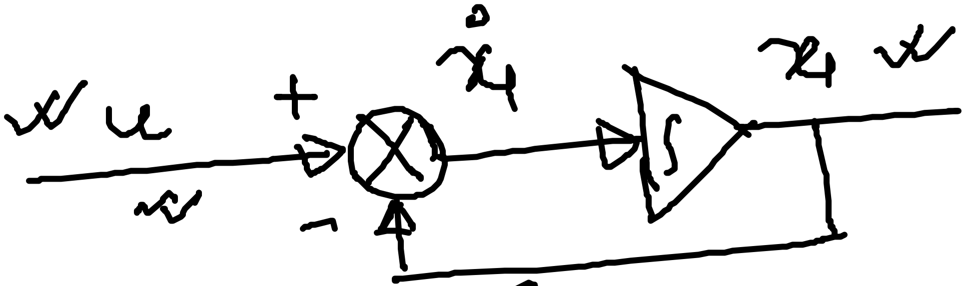
$$AB = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}-1\\ 0\end{bmatrix}$$

$$S = \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

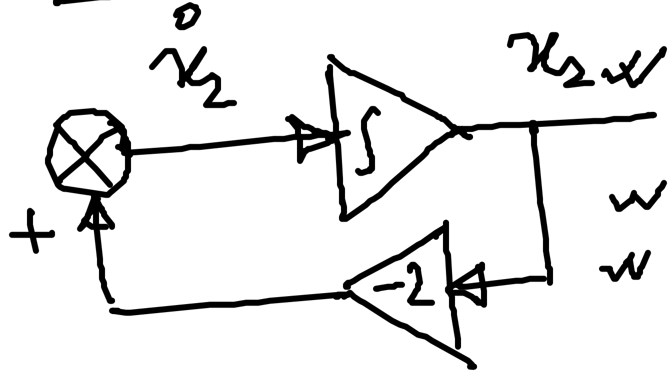
rank of $S = 1$.

NOT controllable.

$\dot{x}_1 = -x_1 + u, \quad \dot{x}_2 = -2x_2$



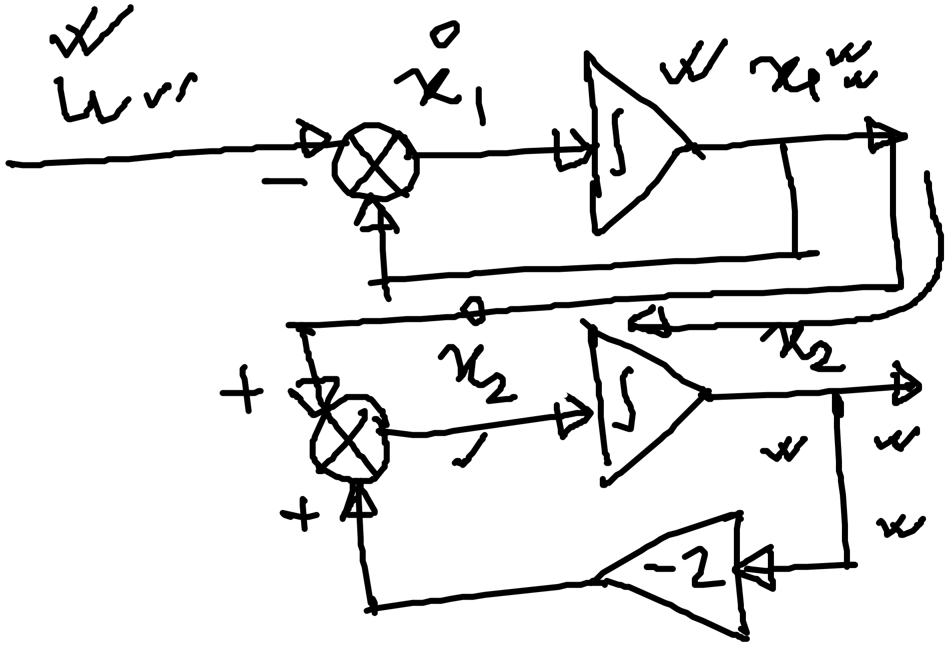
x_2 is not connected to u.



Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u \quad \checkmark$$

$$\dot{x}_1 = -x_1 + u, \qquad \dot{x}_2 = x_1 - 2x_2$$



$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \checkmark$$

$$S = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Rank $S = 2$

Controllable

Observability

$x(t_0)$ $u(t)$ A, B, C, D $\quad x(t) = e^{A(t-t_0)}x(t_0)$

t_0 $y(t)$ t_f

The state $x(t_0)$ is said to be observable if, given any input $u(t)$, there exists a finite time $t_f \geq t_0$ such that the knowledge of $u(t)$ for $t_0 \leq t \leq t_f$, matrices A, B, C and D, and the output $y(t)$ for $t_0 \leq t \leq t_f$ are sufficient to determine $x(t_0)$.

Observability matrix

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is of rank n.

n is the number of state variables.

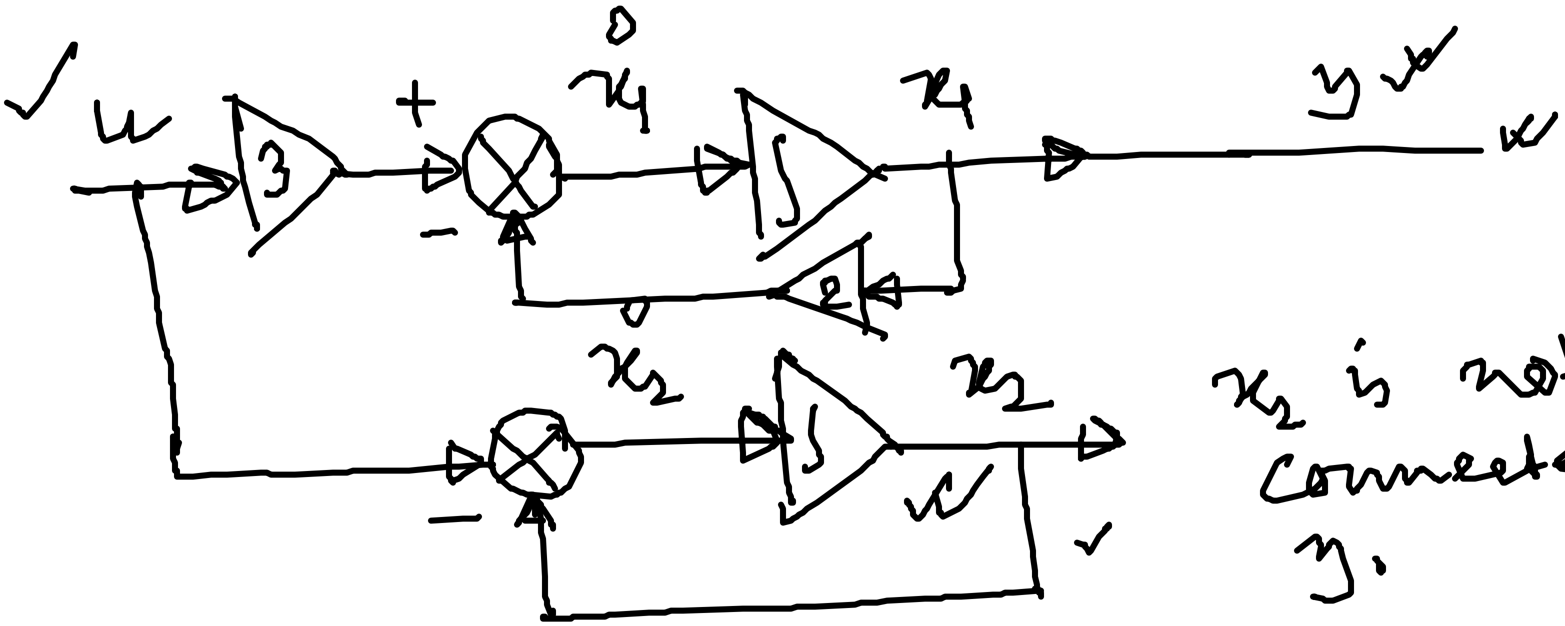
Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}_1 = -2x_1 + 3u, \quad \dot{x}_2 = -x_2 + u$$

$$y = x_1$$



x_2 is not connected to y.

$$V = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$CA = \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}} = \underset{1\times 2}{\begin{bmatrix} -2 & 0 \end{bmatrix}}$$

Rank of $V = 1$

NOT observable

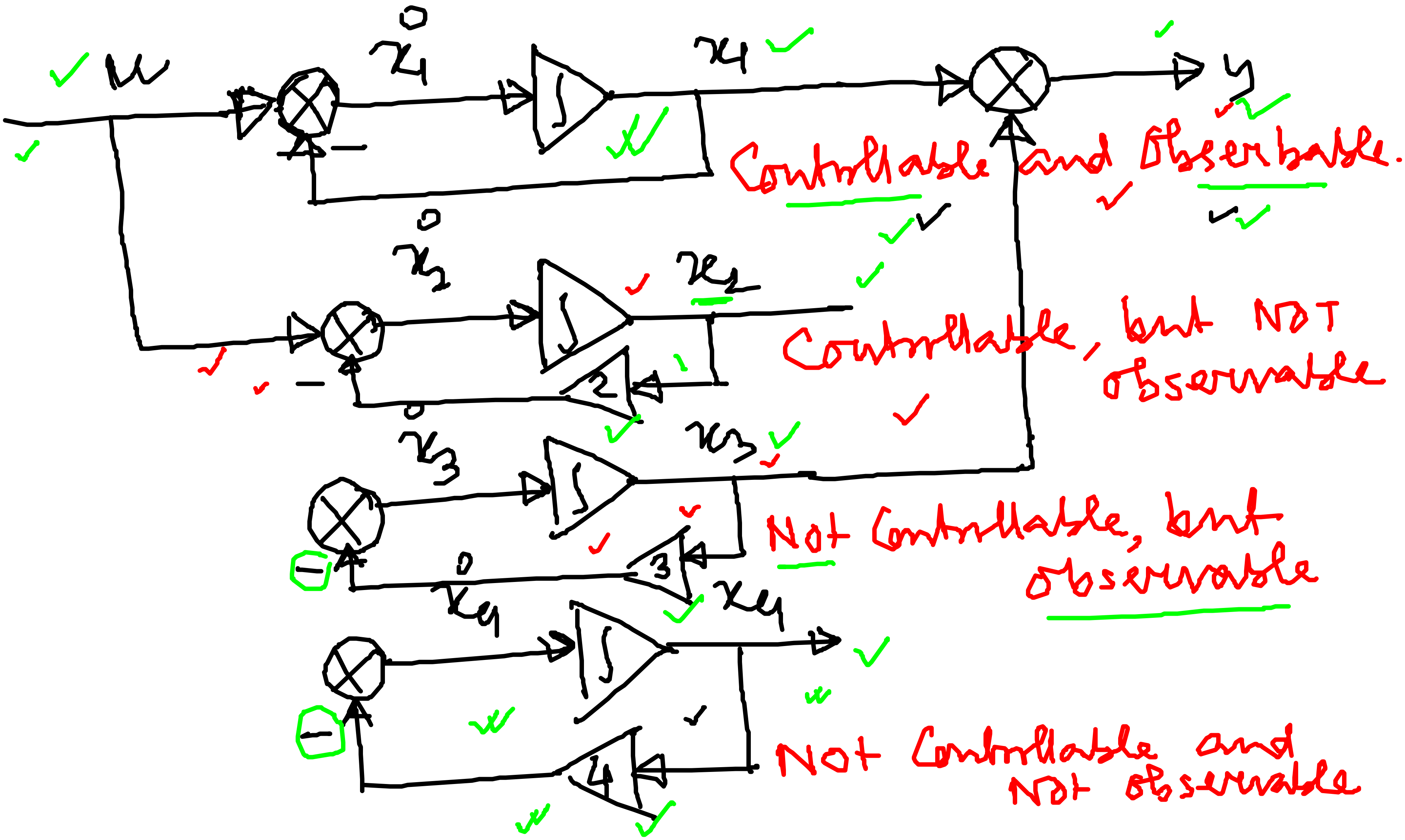
Example

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix} = \underbrace{\begin{bmatrix}-1&0&0&0\\0&-2&0&0\\0&0&-3&0\\0&0&0&-4\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \underbrace{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}_{B}u$$

$$y = \underbrace{\begin{bmatrix}1&0&1&0\end{bmatrix}}_{C}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s+1}$$

$$Y(s) = \boxed{C(sI-A)^{-1}B}\,U(s)$$



Realization

Transfer function to state-space

$G(s) \longrightarrow (A, B, C, D)$

Improper TF = numerator's degree is greater than denominator's degree

$$G(s) = C(sI - A)^{-1}B + D$$

$$A \in \mathbb{R}^{n \times n}$$

$$(sI - A)^{-1} = \frac{\mathrm{Adj}(sI - A)}{\det(sI - A)}$$

$\Downarrow$

degree $\underline{n}$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}_{\underline{\underline{2\times 2}}}$$

$$\det(sI - A) = \left| \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \right| = \begin{vmatrix} s-1 & -2 \\ 0 & s-3 \end{vmatrix}$$

$$= (s-1)(s-3)$$

$$= s^2 - 4s + 3$$

degree 1

$$\mathrm{Adj}\begin{pmatrix} s-1 & -2 \\ 0 & s-3 \end{pmatrix}$$

Since, $(sI - A)^{-1} = \dfrac{\mathrm{Adj}(sI - A)}{\det(sI - A)}$, degree of the element of $\mathrm{Adj}(sI - A)$ is less than the degree of $\det(sI - A)$. So an improper TF cannot be realized.