EE60101: Advanced Sensing Techniques

Assignment-1

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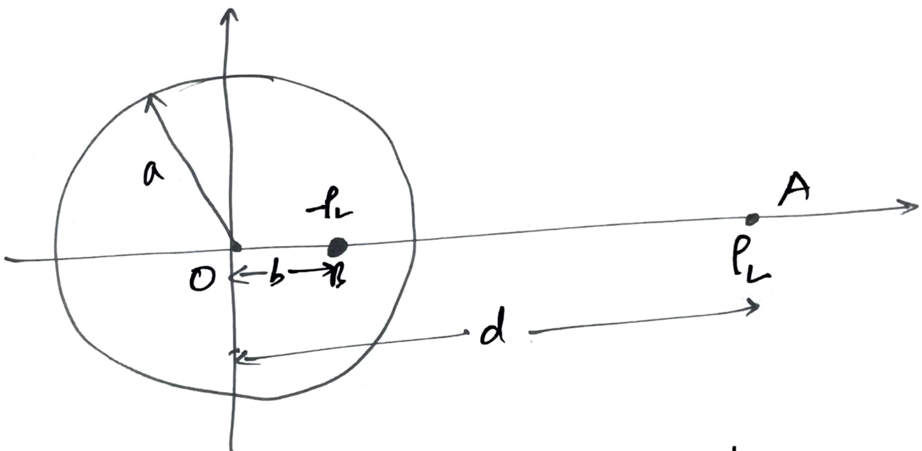
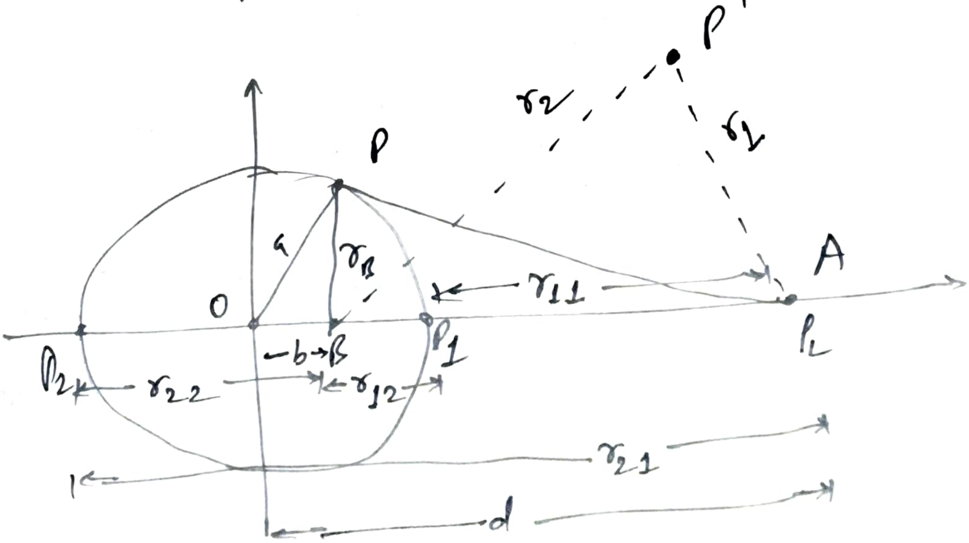
Q.1. We will be using the Image Method to solve this question. Before solving this question, we will be solving a case of a wire of charge ρ_L per unit length at distance b from a grounded conducting cylinder of radius a.

a.





To solve the problem, we must place the image at a distance b such that the potential at the location of the surface is constant. Here we specifically require a constant potential rather than a zero potential.

Now, the total potential at point P' is (using the potential due to a line charge)

$$V_{P'} = V^{+} + V^{-}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0}\left(\ln\frac{r_2}{r_0} - \ln\frac{r_1}{r_0}\right) = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1}$$

where, r_0 is a distance taken for reference for calculating potential.

This is a generalized expression of potential of any point in the system.

then,

$$V_{P1} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{r_{12}}{r_{11}} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{a-b}{d-a} \quad -①$$

$$V_{P2} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{r_{22}}{r_{21}} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{a+b}{a+d} \quad -②$$

Since, the potential on the cylinder cannot, in fact, be calculated unless we know the exact location of the line charges, which we do not. However we must know that whatever the potential on this surface, it must be constant.

$V_{P1} = V_{P2}$

$$\frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{a-b}{d-a} = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{a+b}{a+d}.$$

$$\Rightarrow \frac{a-b}{d-a} = \frac{a+b}{a+d}.$$

$$\Rightarrow a^2 - ab + ad - bd = ad - a^2 + bd - ab$$

$$\Rightarrow \boxed{b = \frac{a^2}{d}}$$

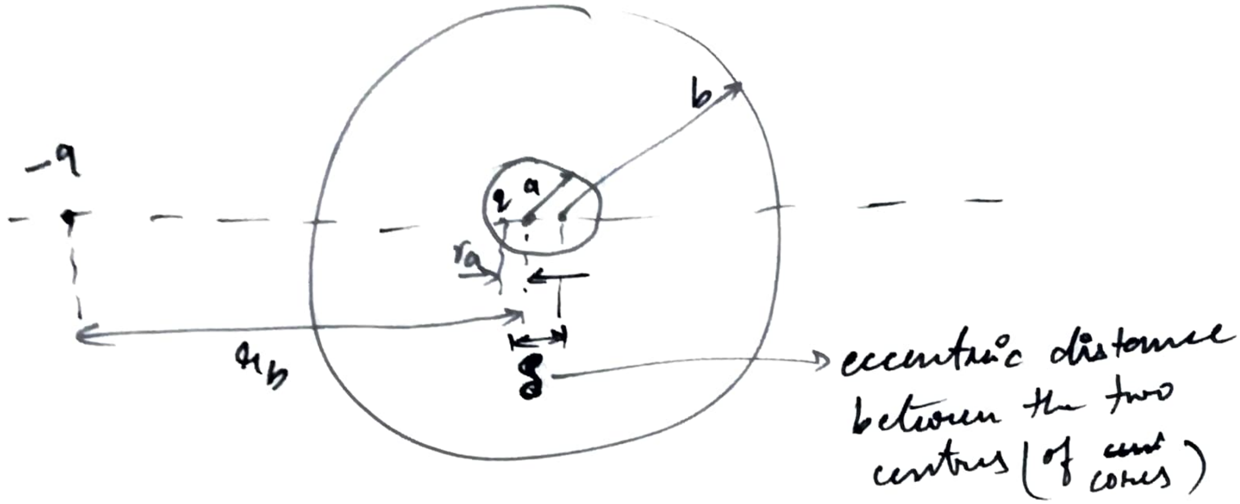
Location of the image charge.

Now, the ~~location~~ actual potential on the surface of the cylinder

$$V_{P1} = V_{P2} = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{a}{d} \text{ or}$$

$$\boxed{V_P = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{r_B}{r_A}}$$

To apply to this present problem, in the figure below, note that the image wires of charge $\pm q$ per unit length are both located to the left of the centre of the inner conductor, say at distances r_a & r_b.



for the inner cylinder to be an equipotential, we must have.

$$\boxed{r_b = \frac{a^2}{r_a}} \quad \text{—(1)}$$

→ {from above proved theory}

& for the outer cylinder to be an equipotential,

$$\boxed{r_b + \delta = \frac{b^2}{r_a + \delta}} \quad \text{—(2)}$$

Substituting value of r_b from ① into ②,

$$\frac{a^2}{r_a} + \delta = \frac{b^2}{r_a + \delta}$$

$$\Rightarrow (a^2 + \delta r_a)(r_a + \delta) = b^2 r_a$$

$$\Rightarrow \delta r_a^2 + (a^2 + \delta^2 - b^2) r_a + \delta a^2 = 0.$$

$$r_a = \frac{(b^2 - a^2 - \delta^2) - \sqrt{(a^2 + \delta^2 - b^2)^2 - 4\delta^2 a^2}}{2\delta} \quad — ③$$

The capacitance is related by $C = q/\Delta V$ where $\Delta V = V_b - V_a$. Also, we know that potential at distance r from a wire of charge q per unit length is $2kq \ln r$ + constant. where $\left(k = \frac{1}{4\pi\epsilon_0}\right)$

$$V_a = 2kq \ln(a - r_a) - 2q \ln(r_b - a)$$

$$= 2qk \ln \frac{a - r_a}{r_b - a}$$

$$= 2kq \ln \frac{a - r_a}{\frac{a^2}{r_a} - a} = 2kq \ln \frac{r_a}{a} \quad — ④$$

$$V_b = 2kq \ln \frac{b - \delta - r_a}{r_b - b + \delta} = 2kq \ln \frac{b - r_a - \delta}{b^2/(r_a + \delta) - b}$$

$$= 2kq \ln \frac{r_a + \delta}{b} \quad — ⑤$$

Then

$$\Delta V = V_b - V_a = 2kq \ln\left[\frac{a}{b}\left(1 + \frac{\delta}{r_a}\right)\right] \quad — ⑥$$

from eq (3)

$$\frac{d}{r_a} = \frac{2\delta^2}{(b^2-a^2-\delta^2) - \sqrt{(b^2-a^2-\delta^2)^2 - 4a^2\delta^2}}$$

Here, $\boxed{b-a >> \delta.}$

then.

$\frac{\delta}{r_a} \to \frac{0}{0}$

Differentiating once,

$$\frac{d}{r_a} \simeq \frac{4\delta}{-2\delta - \frac{-2\times 2\delta\times(b^2-a^2-\delta^2) - 8a^2\delta}{2\sqrt{(b^2-a^2-\delta^2)^2 - 4a^2\delta^2}}}$$

$$= \frac{\cancel{2\delta}\ 2}{-1 + \frac{(b^2-a^2-\delta^2) + 2a^2}{\sqrt{(b^2-a^2-\delta^2)^2 - 4a^2\delta^2}}}$$

$$= \frac{2\sqrt{(b^2-a^2-\delta^2)^2 - 4a^2\delta^2}}{\left((b^2-a^2-\delta^2) - \sqrt{(b^2-a^2-\delta^2)^2 - 4a^2\delta^2}\right) + 2a^2}$$

$$\simeq \frac{2(b^2-a^2-\delta^2)\sqrt{1 - \frac{4a^2\delta^2}{(b^2-a^2-\delta^2)^2}}}{2a^2}$$

$$\simeq \frac{2(b^2-a^2)\times\left(1 - \frac{4a^2\delta^2}{2(b^2-a^2)^2}\right)}{2a^2}$$

$\left\{ \because \text{Using } (1-x^2)^{1/2} = 1 - \frac{x^2}{2} \right\}$
$x \to 0$

Since, $b-a >> \delta.$

$$\simeq \frac{2(b^2-a^2)}{2a^2} = \frac{b^2-a^2}{a^2}$$

Then,

$$\Delta V = 2kq \ln\left[\frac{a}{b}\left(1 + \frac{b^2 - a^2}{a^2}\right)\right]$$

$$= 2kq \ln\frac{b}{a}.$$

$$C = \frac{q}{\Delta V} = \boxed{\frac{1}{2k \ln\frac{b}{a}}}$$

Capacitance per unit length

Here, $\varepsilon_r = 0.5$, $a = 2cm$, $b = 10cm$; $L = 10m$

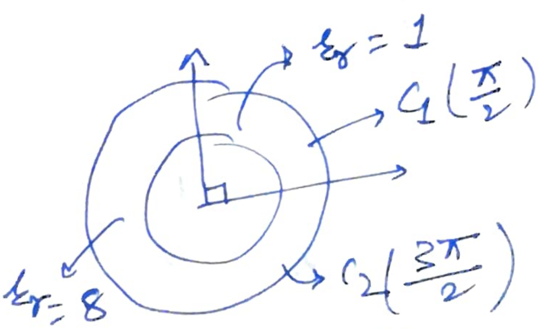
$$\text{Total Capacitance} = C = \frac{1}{2 \times \frac{1}{4\pi\varepsilon_0 \times \varepsilon_r} \times \ln\frac{b}{a}} \times L$$

$$= \frac{(9 \times 10^9)^{-1} \times 0.5}{2 \times \ln\left(\frac{10}{2}\right)} F/m \times 10\,m$$

$$\simeq 1.7259 \times 10^{-11}\, F/m \times 10\,m$$

$$\simeq 17.26\, pF/m \times 10m = 172.6\, pF$$

Q2.



$$C_1 = \frac{4\pi\varepsilon_0\varepsilon_r}{\left(\frac{1}{a} - \frac{1}{b}\right)} \times \frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4k\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$C_2 = \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{1}{a} - \frac{1}{b}} \times \frac{\frac{3\pi}{2}}{2\pi} = \frac{3 \times 8}{4k\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{24}{4k\left[\frac{1}{a} - \frac{1}{b}\right]}.$$

Since, C_1 & C_2 are parallel,

$$C_{eq} = C_1 + C_2$$

$$= \frac{25}{4\times 9\times 10^9 \times \left[\frac{1}{0.03} - \frac{1}{0.05}\right]} F$$

$$= 5.2083 \times 10^{-11} F \simeq 52.1 pF$$

Q 3. For this cable, it has been proved that

$$R = \frac{\ln(b/a)}{2\pi\sigma L}.$$

$$= \frac{\ln\left(\frac{20}{10}\right)}{2\pi\times 4\times 10^4} \Omega \approx \cancel{2.76 \mu\Omega}$$

$$= 2.76 \mu\Omega$$

Q 4. $V(x,y,z) = 5xy + y^3z + 5kz^2$

If V satisfies Laplace's Equation

$$\nabla^2 V = 0$$

If we assume k is constant.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$0 + 6yz + 10k = 0$$

$$\Rightarrow \boxed{k = -\frac{3}{5}yz}$$

So, k is not a constant.

Taking K dependent of x, y and z.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$5\frac{\partial^2 k}{\partial x^2} z^2 + 5\frac{\partial^2 k}{\partial y^2} z^2 + 6yz + \frac{\partial^2}{\partial z^2}(5kz^2) = 0.$$

From the general solution of Laplace Equation, there are separation of variables, making solution of form $f \cdot g \cdot h$ where f, g & h are separate equations of independent variables.

Here, $\frac{\partial^2 k}{\partial x^2}$ & $\frac{\partial^2 k}{\partial y^2}$ have to be zero otherwise z^2 term will remain.

Following this,

$$6yz + \frac{\partial^2}{\partial z^2}(5kz^2) = 0$$

Double Integrating with constants.

$$yz^3 + 9z + C_2 + 5kz^2 = 0$$

$$\Rightarrow 5kz^2 = -yz^3 - 9z - C_2$$

$$\therefore V(x, y, z) = 5xy + y^3z - yz^3 + 9z + C_2.$$

Satisfies Laplace Equation.

Since there are no boundary condition, V doesn't have unique solution.

Q5.

Dividing the region into (4x4) grid

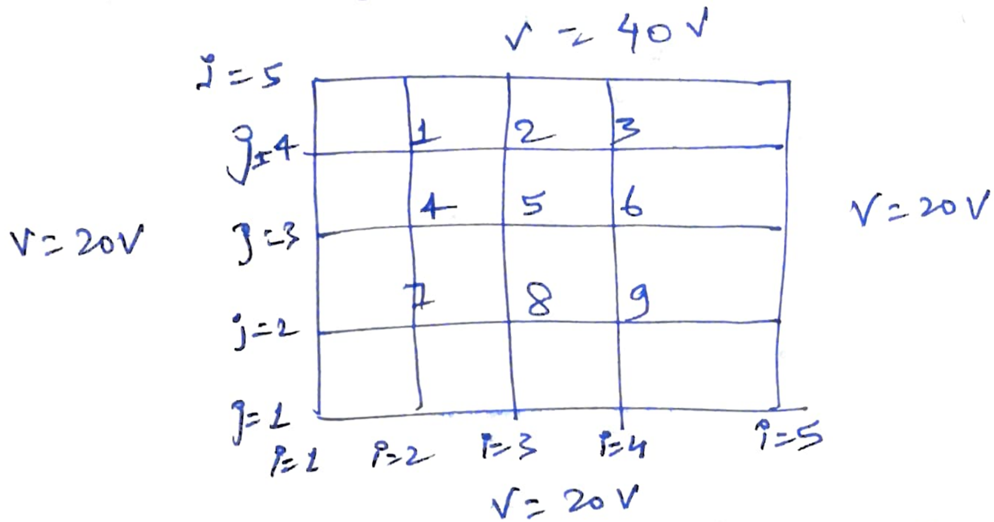


Fig:. Finite difference grid and node numbering for internal nodes.

Using Explicit Solution Method :.

We start with equation:

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} \approx \frac{V_{i-1,j} + V_{i+1,j} + V_{i,j-1} + V_{i,j+1} - 4V_{i,j}}{h^2} = 0$$

Here,

$$\boxed{V_{i,j} = \frac{V_{i-1,j} + V_{i+1,j} + V_{i,j-1} + V_{i,j+1}}{4}} \quad \text{--- (1)}$$

and assuming that all potentials are known, including at interior nodes, at any step of the solution.

Steps: Approximation:

all interior potentials are set to zero for lack of a better choice. This is the guess to start the solution.

Evaluation of the potential at node (i,j) consists of calculating the average of the four potentials as shown in eq(1). Most upto date value of points is always used.

$$V_1 = \frac{20 + V_2 + 40 + V_4}{4} = \frac{20+0+40+0}{4} = 15\,V$$

$$V_2 = \frac{V_1 + V_3 + 40 + V_5}{4} = \frac{15+0+40+0}{4} = 13.75\,V$$

$$V_3 = \frac{V_2 + 20 + 40 + V_6}{4} = \frac{13.75+20+40+0}{4}\,V$$

$$= 18.4375\,V$$

$$V_4 = \frac{20 + \cancel{V_1}\, V_5 + V_1 + V_7}{4} = \frac{20+0+15+0}{4}$$

$$= 8.75\,V$$

$$V_5 = \frac{V_4 + V_6 + V_2 + V_8}{4} = \frac{8.75+0+13.75+0}{4}$$

$$= 5.625\,V$$

$$V_6 = \frac{V_5 + 20 + V_3 + V_9}{4} = \frac{5.625+20+18.4375+0}{4}$$

$$= 11.015625\,V.$$

$$V_7 = \frac{20 + V_4 + V_8 + 20}{4} V = \frac{20 + 8.75 + 0 + 20}{4} V$$

$$= 12.1875 V$$

$$V_8 = \frac{V_7 + V_9 + V_5 + 20}{4} = \frac{12.1875 + 0 + 5.625 + 20}{4} V$$

$$= 9.453125 V$$

$$V_9 = \frac{V_8 + 20 + V_6 + 20}{4} = \frac{9.453125 + 20 + 11.015625 + 20}{4} V$$

$$= 15.1171875 V.$$

these are the internal nodes potential calculation :

Step 2: Solution:

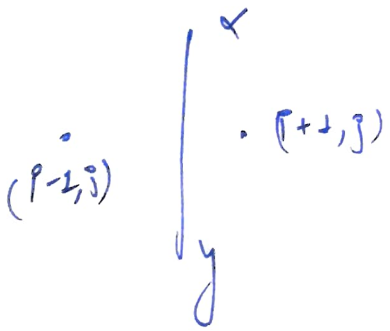
Since only 1 iteration is mentioned. The approximation will stop here. And this is the final solution.

The boundary node potentials will remain constant as given.

40	40	40	40	40
20	15	13.75	18.4375	20
20	8.75	5.625	11.015625	20
20	12.1875	9.453125	15.1172	20
20	20	20	20	20

Potential at the interior & boundary nodes.

Q6:



Since xy boundary follows neuman condition, $V_{i-1,j} = V_{i+1,j}$.
Because neuman boundary condition occurs whenever the normal component of the electric field intensity is zero on a boundary.
~~Since electric field direction in horizontal direction is zero.~~
So, Potentials at $(i-1, j)$ & $(i+1, j)$ will be same.

So, $V_P = V_B = 19.23\ V$

So, $$V_A = \frac{V_B + V_P + V_C + V_D}{4}$$ {Using Finite Diff. Modelling, Explicit Solution Method}.

$$= \frac{2 \times 19.23 + 21.28 + 26.52}{4}\ V$$

$$\boxed{V_A = 21.565\ V}$$