Q 1.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad -(1)$$

To prove:

$$V(r,\phi) = (A_1 \sin k\phi + A_2 \cos k\phi)(B_1 r^{k} + B_2 r^{-k})$$

Assuming linear behaviour, the solution $V(r,\phi)$ can be written as the product of separate solutions:

$$V(r,\phi) = R(r)\,\cancel{\Phi(\theta)}\,\Phi(\phi) \quad -(2)$$

~~Submitt~~ Substituting (2) into (1), we get

$$\frac{\Phi(\phi)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{R(r)}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0$$

Dividing equation by $V(r,\phi)$:

$$\frac{1}{r R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0$$

The equation is still not separable since both ~~terms~~ terms depend on r. To eliminate this independence, we multiply both sides by r^2,

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = 0$$

Choosing constants for the two independent terms,

$$\frac{r}{R(r)} \frac{d}{dr}\left(r \frac{dR(r)}{dr}\right) + k_r^2 = 0$$

$$\frac{1}{\phi(\phi)} \frac{d^2 \phi(\phi)}{d\phi^2} + k_\phi^2 = 0$$

$$\boxed{k_r^2 + k_\phi^2 = 0}$$

Now, The solution for $\phi(\phi)$ is

$$\boxed{\phi(\phi) = A_1 \sin(k\phi) + A_2 \cos(k\phi)}$$

where the ~~s~~ term k was used as $k^2 = k_\phi^2 = -k_r^2$.

The equation for $R(r)$ can now be written as:

$$r^2 \frac{d^2 R(r)}{dr^2} + r\frac{dR(r)}{dr} - k^2 R(r) = 0$$

Also, because the solution ~~is~~ for $\phi(\phi)$ is periodic, k must be an integer.

thus, $\boxed{R(r) = B_1 r^k + B_2 r^{-k}}$

Substituting these two solutions in eq ②,

$$\boxed{V(r,\phi) = (A_1 \sin k\phi + A_2 \cos k\phi)(B_1 r^k + B_2 r^{-k})}$$

Hence, proved.