Laplace Transform

Some important signals:

Unit step: $u(t)$

$$f(t) = 1 \quad t > 0$$
$$= 0 \quad t < 0$$

Impulse (unit)

$$f(t) = \delta(t) = 1 \quad t = 0$$
$$= 0 \quad t \neq 0$$

Unit Ramp $r(t)$

$$f(t) = t \quad t \geq 0$$
$$= 0 \quad t < 0$$

Parabolic: $p(t)$

$$f(t) = \frac{1}{2}t^2 \quad t \geq 0$$
$$= 0 \quad t < 0$$

$$\mathcal{L}[\delta(t)] = 1, \quad \mathcal{L}[u(t)] = \frac{1}{s}, \quad \mathcal{L}[r(t)] = \frac{1}{s^2}, \quad \mathcal{L}[p(t)] = \frac{1}{s^3}$$

$$\mathcal{L}[e^{-\alpha t}] = \frac{1}{s+\alpha}, \quad \mathcal{L}[t e^{-\alpha t}] = \frac{1}{(s+\alpha)^2}$$

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}, \quad \mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots f^{(n-1)}(0)$$

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

$$\mathcal{L}\left[f_1(t) * f_2(t)\right] = F_1(s)\, F_2(s)$$

$$\mathcal{L}\left[f(t-T)\, u(t-T)\right] = e^{-sT} F(s)$$

Initial Value Theorem

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$ if the limit exists.

Final Value Theorem

If $sF(s)$ is analytic on the imaginary axis and in the right-half of the s-plane

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

- If $sF(s)$ has any pole with real part zero or positive, the final value theorem is NOT applicable.

Example: $f(t) \xrightarrow{\mathcal{L}} F(s) = \dfrac{5}{s(s^2+s+2)}$ ✓

$$sF(s) = \frac{5}{s^2+s+2}$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5}{s^2+s+2} = \frac{5}{2}$$

Let $F_1(s) = \frac{\omega}{s^2+\omega^2}$ ✓ , $F_2(s) = \frac{s+1}{s(s-2)}$ ✓

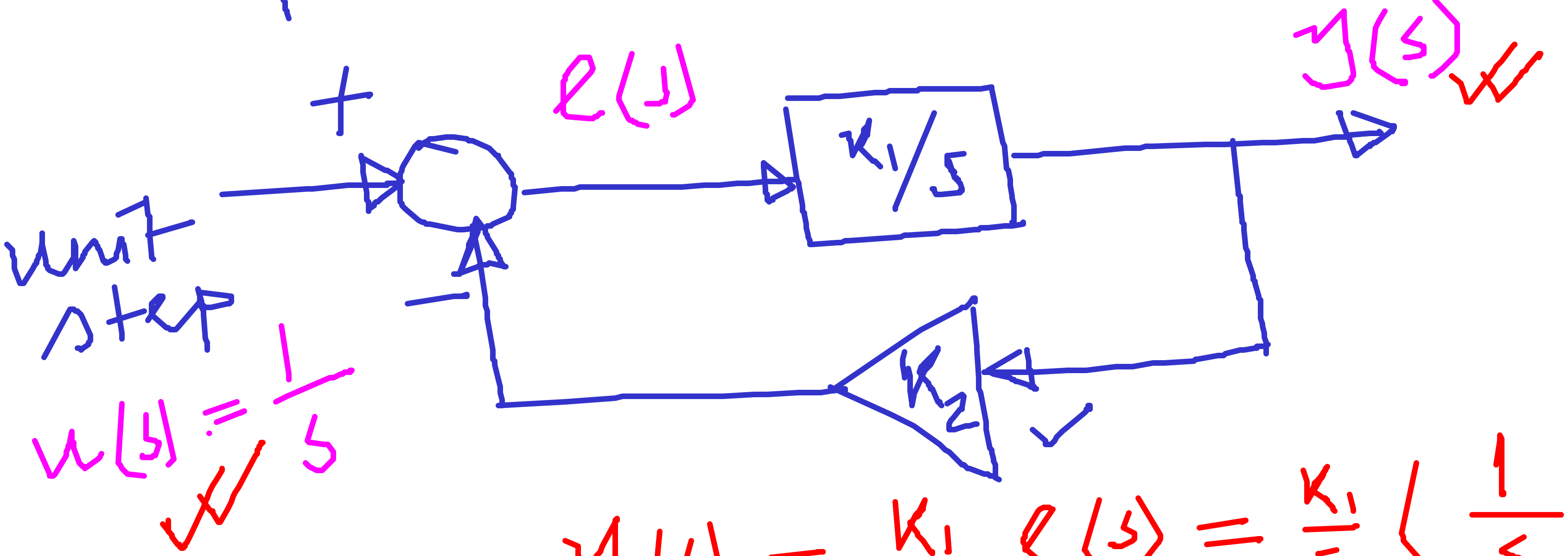
$$sF_1 = \frac{s\omega}{s^2+\omega^2} \qquad sF_2 = \frac{s+1}{s-2}$$

sF_1 is not analytic on the imaginary axis, and sF_2 is not analytic in the right-half of the s-plane.

— So Final value theorem cannot be applied.

Example:

$t \to \infty$



$$y(s) = \frac{K_1}{s} e(s) = \frac{K_1}{s}\left(\frac{1}{s} - K_2 y(s)\right)$$

$$\left(1 + \frac{K_1 K_2}{s}\right) Y(s) = \frac{K_1}{s^2}$$

$$y(s) = \frac{K_1/s^2}{\underline{s + K_1 K_2}} = \frac{K_1}{s(s + K_1 K_2)}$$
$$s$$

$sY(s) = \dfrac{K_1}{\underline{s + K_1 K_2}}$ is analytic in the closed right-half of the s-plane.

$K_1, K_2 > 0$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{K_1}{s + K_1 K_2} = \frac{1}{K_2}$$

$K_2 = 1, \quad y(t) = 1, \quad t \to \infty$

$K_2 = 0.25, \quad y(t) = 4, \quad t \to \infty$

Inverse Laplace Transform

$G(s)$ is given, find $g(t)$

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

Example:

$$G(s) = \frac{5s+3}{(s+1)(s+2)(s+3)}$$

Find $g(t)$

Oliver Heaviside

$$G(s) = \frac{5s+3}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

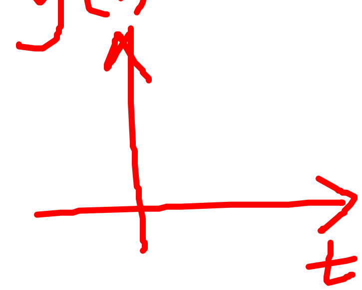
$$A = (s+1)\,G(s)\Big|_{s=-1} = -1$$

$$B = (s+2)\,G(s)\Big|_{s=-2} = 7$$

$$C = (s+3)\,G(s)\Big|_{s=-3} = -6$$

$$G(s) = -\frac{1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3}$$

$$\boxed{g(t) = -e^{-t} + 7e^{-2t} - 6e^{-3t}}$$



$t \to \infty$

$g(t) = 0$

The pole -1 is the dominating pole.