Introduction to Probability
Chapter 6 Poisson Process

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Outline

1. Poisson Process
2. Properties of Poisson Process
3. Poisson Distribution
4. Binomial Approximation to Poisson Distribution

References

1. Probability and statistics in engineering by Hines et al (2003) Wiley.
2. Mathematical Statistics by Richard J. Rossi (2018) Wiley.
3. Probability and Statistics with reliability, queuing and computer science applications by K. S. Trivedi (1982) Prentice Hall of India Pvt. Ltd.

- $f = o(h)$ if $Lim_{h\to 0}\frac{f(h)}{h} = 0$
- Example $f(x) = x^2$. Here f is $o(h)$.
- If $f_i = o(h)$, then $f = \sum_{i=1}^{n} a_i f_i = o(h)$
- ,i.e., linear combination of $o(h)$ functions is $o(h)$.
- Example $f_1(x) = x^2$ and $f_2(x) = x^5$. Both f_1 and f_2 are $o(h)$. Hence $f_1 - f_2$ is also $o(h)$.

$$\underbrace{\lim_{h\to 0}\frac{f_i(h)}{h} = 0}_{\text{given}} \qquad \lim_{h\to 0}\frac{1}{h}\sum_{i=1}^{n} a_i f_i(h) = \sum_i a_i \underbrace{\lim_{h\to 0}\frac{f_i(h)}{h}}_{=0}$$

$$= 0$$

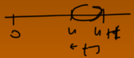
Poisson Process

Let events are occuring in time and $N(t)$ counts the number of events occuring in the interval $(0, t]$. Then $N(t)$ is said to follow Poisson Process with rate λ, i.e., $N(t) \sim PP(\lambda)$ if following assumptions holds:

1. $N(t)$ has independent increment, i.e., events occuring in disjoint time interval are independent.
2. $N(t)$ has stationary increment,i.e., the distribution of $N(t)$ depends on length of interval not on where it is situated.
3. If h is small

$P(N(t) \leq x) = P(N(u+t) - N(u) \leq x)$

$N(t) \overset{d}{=} N(t+u) - N(u)$



$$P(N(h) = 1) = \lambda h + o(h)$$
$$P(N(h) \geq 2) = o(h).$$

$N(t) \in \{0, 1, 2, \ldots\}$

Using assumption (3) $\boxed{P(N(h) = 0) = 1 - \lambda h + o(h).}$

$1 - P(N(h) = 1) - P(N(h) \geq 2) = 1 - \lambda h - o(h) - o(h)$

Poisson Process

Statement: Under the assumptions (1), (2) and (3) of Poisson Process,

$$P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \; n = 0, 1, \ldots$$

Example

Suppose that in a bank customers are arriving according to the Poisson process with rate $\lambda = 2$ customers per hour. The probability that from 10:00 am to 10:30 am no customer arrive is

$$\begin{aligned} P(N(1/2) = 0) &= e^{-2 \times \frac{1}{2}} \\ &= e^{-1}, \end{aligned}$$

here $N(t)$ is number of customer arriving in the interval $(0, t)$.

Properties of Poisson Process

- Sum of independent Poisson Process is a Poisson Process. That is if $N_1(t) \sim PP(\lambda_1)$ and $N_2(t) \sim PP(\lambda_2)$; and $N_1(t)$ and $N_2(t)$ are independent, then $N_1(t) + N_2(t) \sim PP(\lambda_1 + \lambda_2)$.
- Suppose each time an event occurs is classified as type-I or type-II. Each event is classified as type-I with probability p and is classified as type-II with probability $q = 1 - p$. Let $N(t) \sim PP(\lambda)$, where $N(t)$ is number of events occuring in the interval $(0, t]$. Let $N_1(t)$ and $N_2(t)$ denote, respectively, the number of type-I and type-II event occuring in the interval $(0, t]$. Here $N(t) = N_1(t) + N_2(t)$. Then $N_1(t) \sim PP(\lambda p)$ and $N_2(t) \sim PP(\lambda q)$; also $N_1(t)$ and $N_2(t)$ are independent.

Example

Example

A radioactive source emits particles (either with redish or with white light) at a rate of 6 per minute in accordance with Poisson process. Particles that are emitted with redish light has a probability $1/3$ and those emitted with white light has probability $2/3$. Find the probability that 5 particles emit with white light in 7 minute period.

Solution: $N(t)$ is the number of particles emitted with white light in interval $(0, t]$. Here $N(t) \sim PP(6 \times \frac{2}{3})$. Hence required probability

$$P(N(7) = 5) = \frac{e^{-6 \times \frac{2}{3} \times 7}(6 \times \frac{2}{3} \times 7)^5}{5!}$$

Poisson Distribution

- Fix $\lambda t = \mu$ then the Poisson Process $N(t) = Y$ becomes Poisson distribution such that PDF is,

$$P(Y = n) = \frac{e^{-\mu}(\mu)^n}{n!}, \ n = 0, 1, \dots$$

- MGF is $M(t) = e^{\mu(e^t - 1)}$.
- $E(X) = \mu$ and $Var(X) = \mu$.

Binomial Approximation to Poisson Distribution

- $Lim_{n \to \infty; np = \lambda} B(n, p) = Poiss(\lambda)$
- If n is large and p is small then Binomial distribution can be approximated by Poisson distribution.

Example

Consider a situation where due to certain infection persons are dying with probability 0.001. Consider that the population has 10000 persons in totallity. Find the probability that more than 5 persons will die due to this infection. Consider X denote the number of persons dying due to infection. $X \sim Bin(n, p)$, where $n = 10000$ and $p = 0.001$. X can be approximated by Possion distribution, i.e., $X \sim Poiss(\lambda)$, here $\lambda = np = 10$. Hence required probability $P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{i=0}^{5} \frac{e^{-10}(10)^i}{i!}$

Summary

Poisson process was introduced in this chapter. Properties of Poisson process were presented with illustrated examples. Then Poisson distribution was introduced. In last Binomial approximation to Poisson distibution was studied.