Assignment - 3

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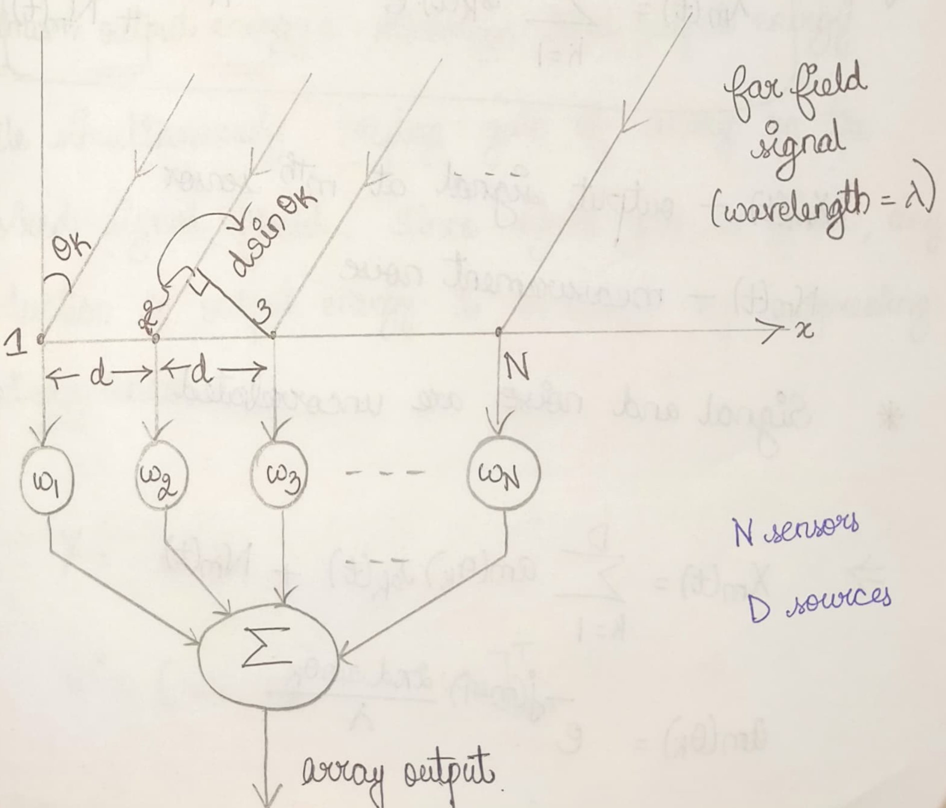
17EE35004

Capon's Minimum Variance Distortionless Response Algorithm :

Capon's MVDR algorithm finds maximum likelihood estimation of the power of the interested signals from their directions, which means to form a beam pointing towards the looking direction while nulls the directions of interference. This method uses array weights, which are obtained by minimizing output power subject to unity constraint in the looking direction.

Mathematical model:



Let warefront signal $S_K(t) = s_K(t) \cdot e^{j\omega_K t}$

$k = 1, 2, — D$

(Sources are point source, narrow band, far field),

$\therefore \; S_K(t - t_1) = s_K(t)\, e^{j\omega_K (t - t_1)}$

Taking first sensor as reference, induction signal of sensor "m" to "k^{th}" signal source is given as

$$s_K(t) \cdot e^{-j(m-1)\frac{2\pi d \sin\theta_K}{\lambda}}$$

path difference $\Rightarrow$ phase difference

$(m-1)\, d \sin\theta_K$ $\qquad$ $\frac{2\pi}{\lambda}(m-1)\, d\sin\theta_K$

$$\therefore \quad \boxed{X_m(t) = \sum_{K=1}^{D} s_K(t) \cdot e^{-j(m-1)\frac{2\pi d \sin\theta_K}{\lambda}} + N_m(t)}$$

$X_m(t)$ – output signal at m^{th} sensor

$N_m(t)$ – measurement noise

* Signal and noise are uncorrelated

$$\Rightarrow \quad X_m(t) = \sum_{K=1}^{D} a_m(\theta_K)\, s_K(t) + N_m(t)$$

$$a_m(\theta_K) = e^{-j(m-1)\frac{2\pi d \sin\theta_K}{\lambda}}$$

Collecting output signals of all sensors

$$X = AS + N$$

$$X = [x_1(t) \quad x_2(t) \quad \cdots \quad x_N(t)]^T_{N\times 1}$$

$$S = [s_1(t) \quad s_2(t) \quad \cdots \quad s_D(t)]^T_{D\times 1}$$

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\phi_1} & e^{-j\phi_2} & \cdots & e^{-j\phi_D} \\ \vdots & & & \\ e^{-j(N-1)\phi_1} & e^{-j(N-1)\phi_2} & \cdots & e^{-j(N-1)\phi_D} \end{bmatrix}_{N\times D} \qquad \phi_k = \frac{2\pi d}{\lambda}\sin\theta_k$$

$$N = [N_1(t) \quad N_2(t) \quad \cdots \quad N_N(t)]^T_{N\times 1}$$

Minimum output energy: minimize total output energy while simultaneously keeping gain of array on the desired signal fixed. Since signal gain is fixed, any reduction in output energy is obtained by suppressing interference.

$$y = w^H X$$

$$w = [w_1 \quad w_2 \quad \cdots \quad w_N]^T_{N\times 1}$$

$$Y = (w^H A) S + (w^H N)$$

In order for output of filter to be distortionless, gain corresponding to signal should be 1.

$$w^H A = I_{P \times P}$$

$$w^H a_\theta = 1 \quad , \quad a_\theta \text{ is a column of } A$$

$$\mathcal{L}(w;\lambda) = E\left[|Y|^2\right] = E\left[|w^H x|^2\right] + \lambda\,(w^H a_\theta - 1)$$

$$= w^H E[x x^H]\, w + \lambda\,(w^H a_\theta - 1)$$

$$= w^H R w + \lambda\,(w^H a_\theta - 1)$$

$$\frac{\partial \mathcal{L}}{\partial w^H} = R w + \lambda a_\theta = 0$$

$$w_{min} = -R^{-1}(\lambda a_\theta)$$

$$w_{min} = -\lambda\,(R^{-1} a_\theta)$$

$$w_{min}^H a_\theta = 1$$

$$(-\lambda)\, a_\theta^H R^{-1} a_\theta = 1$$

$$-\lambda = \frac{1}{a_\theta^H R^{-1} a_\theta}$$

$$\therefore \quad w_{min} = \frac{R^{-1} a_\theta}{a_\theta^H R^{-1} a_\theta}$$

$$\text{Energy } (P) = w_{min}^H R \, w_{min}$$

$$= \frac{a_\theta^H R^{-1} \cdot R \cdot R^{-1} a_\theta}{(a_\theta^H R^{-1} a_\theta)(a_\theta^H R^{-1} a_\theta)}$$

$$= \frac{a_\theta^H R^{-1} a_\theta}{(a_\theta^H R^{-1} a_\theta)^2}$$

$$= \frac{1}{a_\theta^H R^{-1} a_\theta}.$$

$$\therefore \quad P_{mvdr}(\theta) = \frac{1}{a_\theta^H R^{-1} a_\theta}$$

For θ at DOA, P_{mvdr} will be at peak.

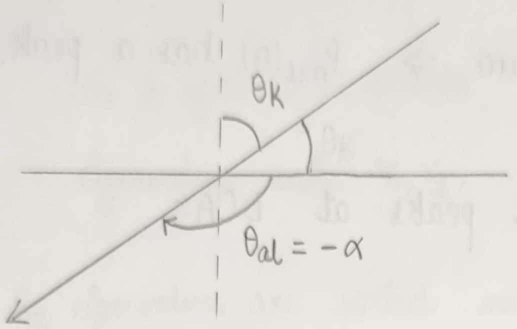
* Improvement for coherent signals.

$$R_{new} = A R_S A^H + J[A R_S A^H] + 2\sigma^2 I_{N \times N}$$

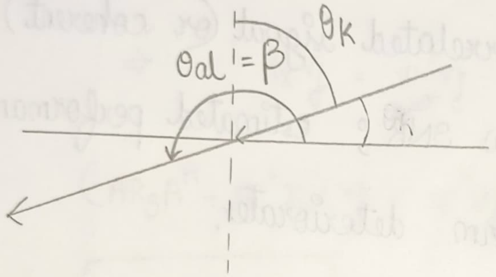
$J = N^{th}$ order anti matrix

The modified "R" must be used to obtain DOA value by finding the peak.

* <u>Conversion of azimuthal to θ_K.</u>



$\cancel{\theta_K = 180 - \alpha}$



$\cancel{\theta_K = \beta - 180}$

general form or function : $\theta_K = \cos^{-1}\left(\sin(-\theta_{al})\right)$

* The results contain both un-modified and modified ones.

→ MVDR maximizes output SNR while keeping amplitude of interested signals from known directions unchanged because of unity constraint.

Applications :

→ Smart antennas that automatically orient themselves towards direction of signal source to obtain maximum signal amplitude.

→ For directional audio capture in presence of multiple audio sources and noise simultaneously.

→ Used in SONAR arrays for underwater signal source DOA.

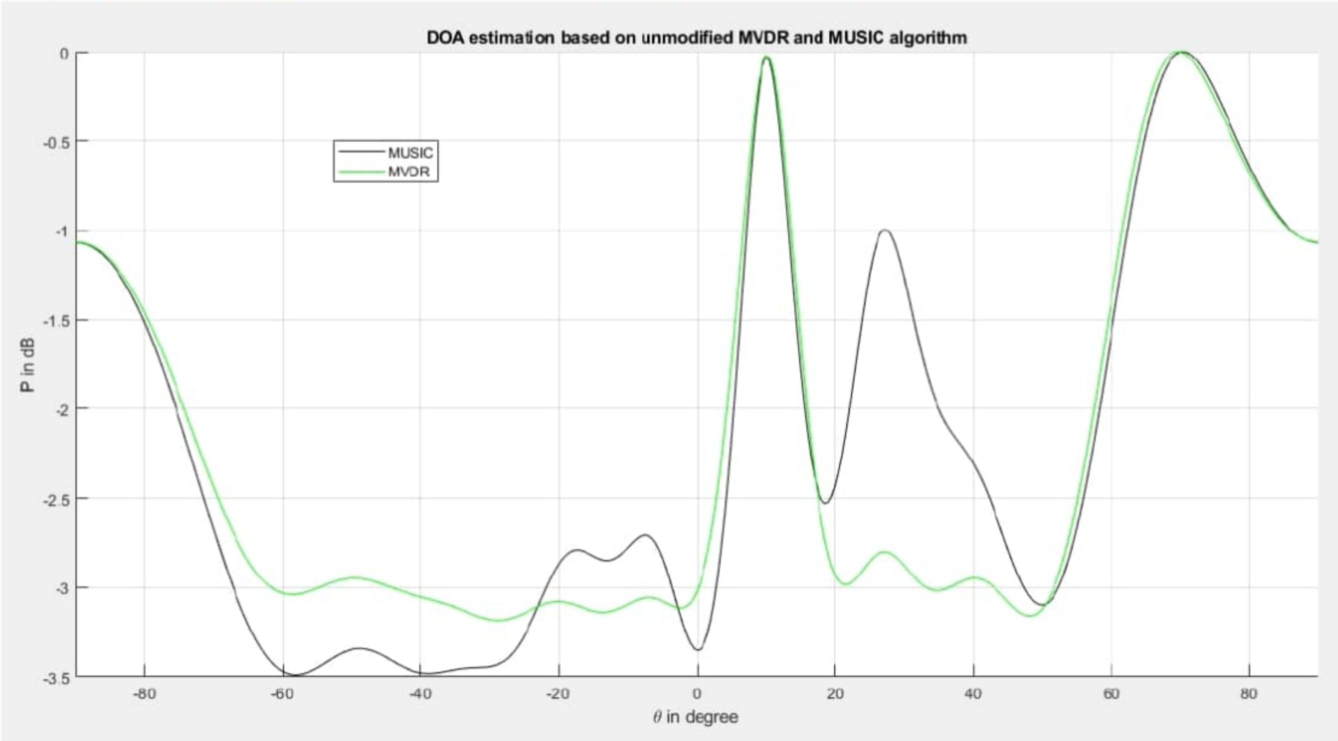
MATLAB CODE :

RTSP_Assg_3_17EE35004.m :

```
%% MVDR Algorithm for DOA : Initialization of parameters
clc;
clear;
azimuth = [-100 200]/180*pi;
doa = acos(sin((-azimuth)));      % Azimuth to direction of arrival conversion
N = 4500;                         % Number of Snapshots
f = 2*10^9;
w = 2*pi*f*[1 1]';                % Angular Frequency
M = 10;                           % Number of array elements
P = length(w);                    % The number of signal
lambda = 150/1000;                % Wavelength
d = lambda/2;                     % Element spacing
snr = 5;                          % SNR
D = zeros(P,M);                   % To create a matrix with P row and M column

for k=1:P
D(k,:) = exp(-1i*2*pi*d*sin(doa(k))/lambda*(0:M-1));
end
D=D';

%% Generating Signals and Noise
Xs = 2*exp(1i*(w*(1:N)));         % Generating signal
X = D*Xs;
X = awgn(X,snr);                  % Insert Gaussian white noise
R = X*X';                         % Data covarivance matrix

% Modification in MVDR algorithm for coherent sources
J = fliplr(eye(M));               % Anti-matrix
R = R+J*conj(R)*J;                % Modified R matrix
[N,V] = eig(R);                   % Find the eigenvalues and eigenvectors of R
NN = N(:,1:M-P);                  % Estimate noise subspace

%% Theta search for Peak finding
theta = -90:0.5:90;               % Peak search
Pmusic = zeros(length(theta),1);  % P_music function
Pmvdr = zeros(length(theta),1);   % P_mvdr function

for ii=1:length(theta)
    SS=zeros(1,length(M));

    for jj=0:M-1
        SS(1+jj)=exp(-1i*2*jj*pi*d*sin(theta(ii)/180*pi)/lambda);
    end

    PP=SS*(NN*NN')*SS';
    Pmusic(ii)=abs(1/ PP);

    PP=SS*(inv(R))*SS';

    Pmvdr(ii)=abs(1/PP);
end

%% Plotting the results of theta, Pmusic and Pmvdr function

figure;
hold on
Pmusic=10*log10(Pmusic/max(Pmusic)); % Spatial spectrum function (normalized)
plot(theta,Pmusic,'-k')
Pmvdr=10*log10(Pmvdr/max(Pmvdr));    % Spatial spectrum function (normalized)
plot(theta,Pmvdr,'-g')
xlabel('\theta in degree')
legend({'MUSIC','MVDR'});
ylabel('P in dB')
title('DOA estimation based on MVDR and MUSIC algorithm')
xlim([-90,90]);
grid on
```

Plot results :

(a) unmodified mvdr and unmodified music



(b) modified mvdr and modified music