Assignment - 3

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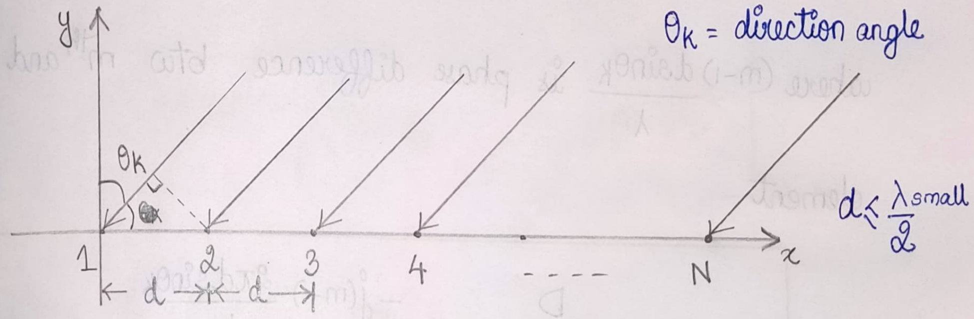
MUltiple SIgnal Classification (MUSIC) Algorithm :

Basic idea of MUSIC algorithm is to conduct characteristic decomposition for the covariance matrix of any array output data, resulting in a signal subspace orthogonal with a noise subspace corresponding to signal components.

Mathematical model of DOA estimation :



Let N be the number of sensors and D be the number of sources, (provided $N > D$).

Given sources are point sources, narrow band and are far field, therefore we can assume that wavefronts are planar.

Let wavefront signal be $S_k(t) = s_k(t) \cdot e^{j\omega_k(t)}$; $k = 1, 2, \ldots D$.

where $s_k(t)$ is complex envelope of $S_k(t)$

signal strength

$\omega_k(t)$ is angular frequency of $S_k(t)$.

Let t_1 be time required by EM antenna array dimension:

$$s_K(t-t_1) \approx s_K(t) \quad \text{(narrowband)}$$

$$S_K(t-t_1) = s_K(t-t_1)\, e^{j\omega_K(t-t_1)}$$

(delay signal)

$$= s_K(t) \cdot e^{j\omega_K(t-t_1)}$$

using first array element as reference point, at the moment "t", the induction signal of array element "m" to the "K^{th}" signal source is given as.

$$s_K(t) \cdot e^{-j(m-1)\frac{2\pi d \sin\theta_K}{\lambda}}$$

where $(m-1)\frac{d \sin\theta_K}{\lambda}$ is phase difference btw m^{th} and 1^{st} element

$$\therefore \quad \boxed{X_m(t) = \sum_{k=1}^{D} s_k(t) \cdot e^{-j(m-1)\cdot\frac{2\pi d \sin\theta_K}{\lambda}} + N_m(t).}$$

where X_m is output signal of m^{th} element

N_m is measurement noise.

Let $a_m(\theta_K) = e^{-j(m-1)\cdot\frac{2\pi d \sin\theta_K}{\lambda}}$

(response function)

* Signal and noise are uncorrelated.

$$\boxed{X_m(t) = \sum_{k=1}^{D} a_m(\theta_k) \cdot S_k(t) + N_m(t)}$$

Collect output signal recieved by all elements in a matrix.

$$\boxed{X = AS + N}$$

$$X = \begin{bmatrix} X_1(t) & X_2(t) & \cdots & X_N(t) \end{bmatrix}^T_{N\times 1}$$

$$S = \begin{bmatrix} S_1(t) & S_2(t) & \cdots & S_D(t) \end{bmatrix}^T_{D\times 1}$$

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\phi_1} & e^{-j\phi_2} & \cdots & e^{-j\phi_D} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j(N-1)\phi_1} & e^{-j(N-1)\phi_2} & \cdots & e^{-j(N-1)\phi_D} \end{bmatrix}_{N\times D}$$

with $\phi_k = \frac{2\pi d}{\lambda} \sin\theta_k$.

$$N = \begin{bmatrix} N_1(t) & N_2(t) & \cdots & N_N(t) \end{bmatrix}^T_{N\times 1}$$

Covariance matrix : $R_X = E[XX^H]$

$$R_X = E[(AS+N)\cdot(AS+N)^H]$$

$$R_X = A\,E[SS^H]\,A^H + E[NN^H]$$

$$\boxed{R_X = A R_S A^H + R_N.}$$

$R_N = \sigma^2 I_{N\times N}$ — noise correlation matrix.

we have $\text{Rank}(AR_SA^H) = D$: $AR_SA^H \to D$ positive & $(M-D)$ zero eigenvalues.

and $\sigma^2 > 0$

$\therefore$ R_X is full rank matrix – $\lambda_1, \lambda_2, \ldots \lambda_M$ are eigenvalues and $v_1, v_2, \ldots v_M$ are eigenvectors.

Let R_X eigenvalues are sorted such that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_M > 0$

where largest D corresponds to signal and $(M-D)$ to noise

$$\Rightarrow \quad R_X v_j = \sigma^2 v_j \quad ; \quad j > D,\ j \leq M$$

$$(AR_SA^H + \sigma^2 I)\, v_j = \sigma^2 v_j$$

$$\boxed{AR_SA^H v_j = 0.}$$

multiply $R_S^{-1}(A^HA)^{-1}A^H$ on both sides

$$R_S^{-1}(A^HA)^{-1}(A^HA)R_SA^H v_j = 0$$

$$\boxed{A^H v_j = 0} \quad ; \quad j = D+1, D+2, \ldots M.$$

Let E_n be a noise subspace constructed

$$E_n = [v_{D+1}, v_{D+2}, \ldots, v_M].$$

to define spacital spectrum $P_{mu}(\theta)$.

$$P_{mu}(\theta) = \frac{1}{a^H(\theta) E_n E_n^H a(\theta)} = \frac{1}{\| E_n^H a(\theta) \|^2}$$

Why ?

when $a(\theta)$ is orthogonal with each column of E_n, the value of this denominator is zero. ; but due to noise it becomes minimum $\Rightarrow$ $P_{mu}(\theta)$ has a peak.

as θ varies, we get peaks at DOA's.

Improvement for coherent signals :

→ When source is a correlated signal (or coherent) or a signal with low SNR ; estimated performance of MUSIC algorithm deteriorates.

→ J: Transformation matrix is a N^{th} order anti-matrix.

$$J(i,j) = \begin{cases} 1 & \text{if } i+j = N+1 \\ 0 & \text{otherwise} \end{cases}$$

Let ~~R_x~~ $Y = JX^*$

$$R_Y = E[YY^H] = JR_xJ^*$$

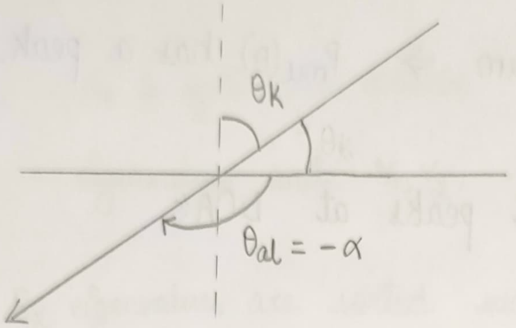
and $R = R_x + R_y = R_x + JR_xJ^*$

(modified R_x) $R = AR_SA^H + J[AR_SA^H] + 2\sigma^2 I_{N\times N}$

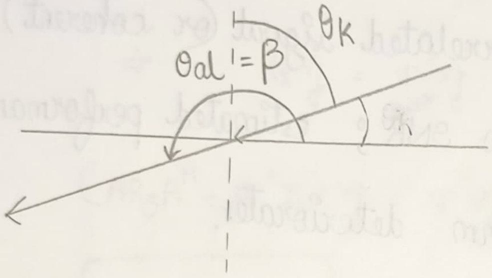
$\therefore$ R, R_x, R_y have same noise subspace.

The modified "R" must be used to obtain DOA value by finding the peak.

* Conversion of azimuthal to θ_K.



~~$\theta_K = 180 - \alpha$~~



~~$\theta_K = \beta - 180$~~

general form or function : $\theta_K = \cos^{-1}\left(\sin(-\theta_{al})\right)$

* The results contain both un-modified and modified ones.

MUSIC algorithm characteristics :

→ Ability to simultaneosly measure multiple signals

→ High precision measurement

→ High resolution for antenna beam signals

→ Applicable to short data circumstances

→ Achieve real time processing after using high-speed processing technology.

* NOTE : more number of array elements, more difference between incident angle, more number of snapshots ; the higher resolution the MUSIC algorithm has. If $d > \frac{\lambda}{2}$, then algorithm reports a false peak.

MUSIC applications :

→ Frequency estimation

→ Radio direction finding

→ Time-Reversal imaging

→ Fast detection of DTMF frequencies.

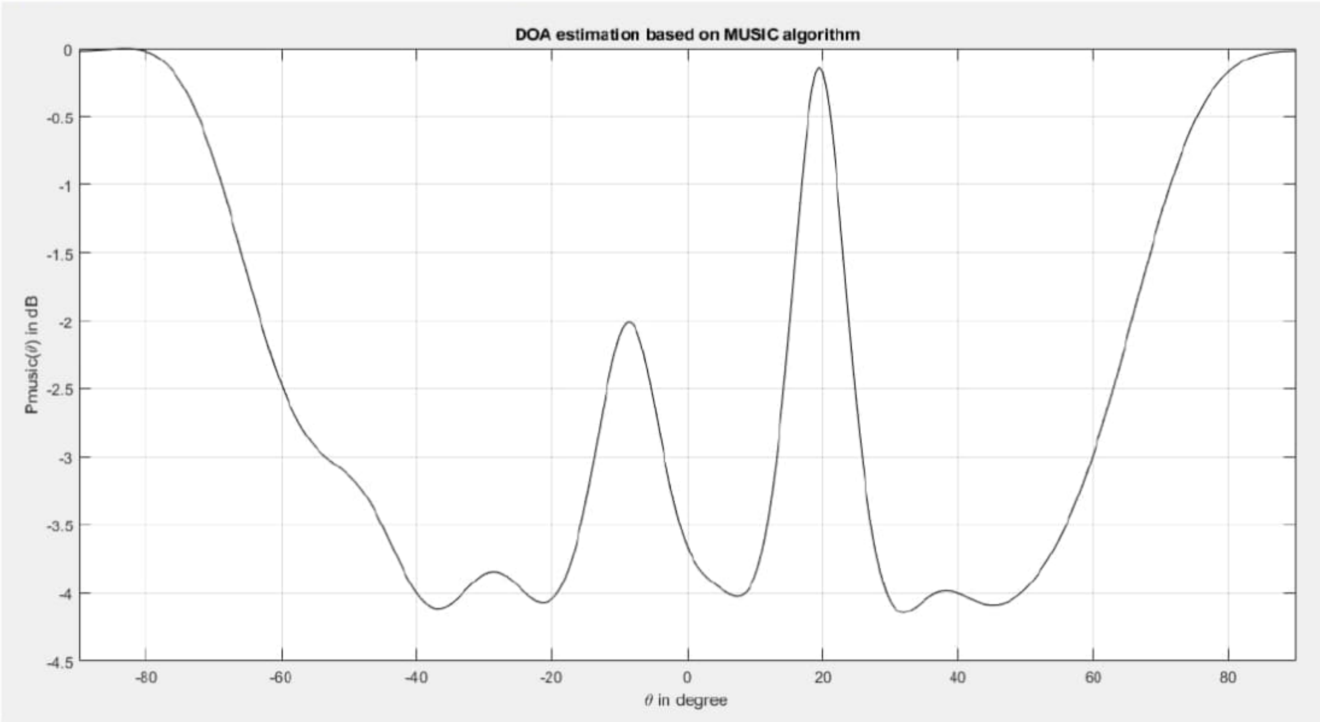
MATLAB CODE :

RTSP_Assg_2_1_17EE35004.m :

```
%% MUSIC Algorithm for DOA : Initialization of parameters
clc;
clear;

azimuth = [-100 200]/180*pi;
doa = asin(sin((-azimuth)));       % Azimuth to direction of arrival conversion
N = 4500;                          % Number of Snapshots
f = 2*10^9;
w = 2*pi*f*[1 1]';                 % Angular Frequency
M = 10;                            % Number of array elements
P = length(w);                     % The number of signal
lambda = 150/1000;                 % Wavelength
d = lambda/2;                      % Element spacing
snr = 5;                           % SNR
D = zeros(P,M);                    % To create a matrix with P row and M column

for k=1:P
D(k,:) = exp(-1i*2*pi*d*sin(doa(k))/lambda*(0:M-1));
end
D=D';

%% Generating Signals and Noise
Xs = 2*exp(1i*(w*(1:N)));          % Generating signal
X = D*Xs;
X = awgn(X,snr);                   % Insert Gaussian white noise
R = X*X';                          % Data covarivance matrix

% Modification in MUSIC algorithm for coherent sources
J = fliplr(eye(M));                % Anti-matrix
R = R+J*conj(R)*J;                 % Modified R matrix

[N,V] = eig(R);                    % Find the eigenvalues and eigenvectors of R
NN = N(:,1:M-P);                   % Estimate noise subspace

%% Theta search for Peak finding
theta = -90:0.5:90;                % Peak search
Pmusic = zeros(length(theta),1);   % P_music function

for ii=1:length(theta)
    SS=zeros(1,length(M));

    for jj=0:M-1
        SS(1+jj)=exp(-1i*2*jj*pi*d*sin(theta(ii)/180*pi)/lambda);
    end

    PP=SS*(NN*NN')*SS';
    Pmusic(ii)=abs(1/ PP);
end

%% Plotting the results of theta and Pmusic function
figure;
Pmusic=10*log10(Pmusic/max(Pmusic)); % Spatial spectrum function (normalized)
plot(theta,Pmusic,'-k')
xlabel('\theta in degree')
ylabel('P(\theta) in dB')
title('DOA estimation based on MUSIC algorithm')
xlim([-90,90]);
grid on
```

Plot Results :

(1) Unmodified MUSIC Algorithm :



(2) Modified MUSIC Algorithm : (sharp peaks)