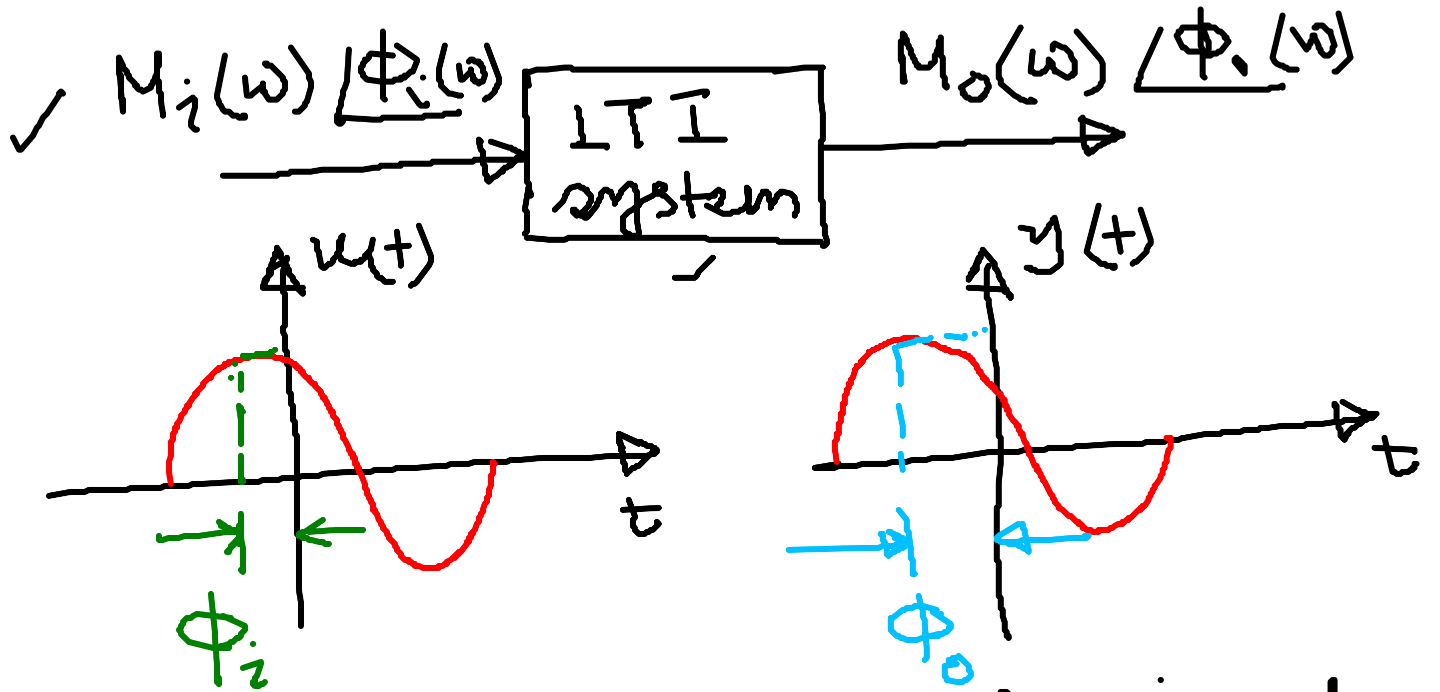
Frequency response of LTI systems

- Time domain
- Frequency domain



At steady-state, sinusoidal inputs to an LTI system generate sinusoidal responses of the same frequency but may be with different magnitude and phase.

If the LTI system is represented by the complex number, i.e., $M(\omega)\angle\Phi(\omega)$, then

$$M_o(\omega)\angle\Phi_o(\omega) = M_i(\omega)\,M(\omega)\angle\Phi_i(\omega)+\Phi(\omega)$$

Then, we have the following relationship

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)} \quad \text{and} \quad \angle\Phi(\omega) = \angle\Phi_o(\omega) - \angle\Phi_i(\omega)$$

$\Uparrow$ Magnitude frequency response ✓

$\Uparrow$ Phase frequency response ✓

$\underbrace{\text{Magnitude frequency response} \qquad \text{Phase frequency response}}_{\text{Frequency response}}$

Frequency response

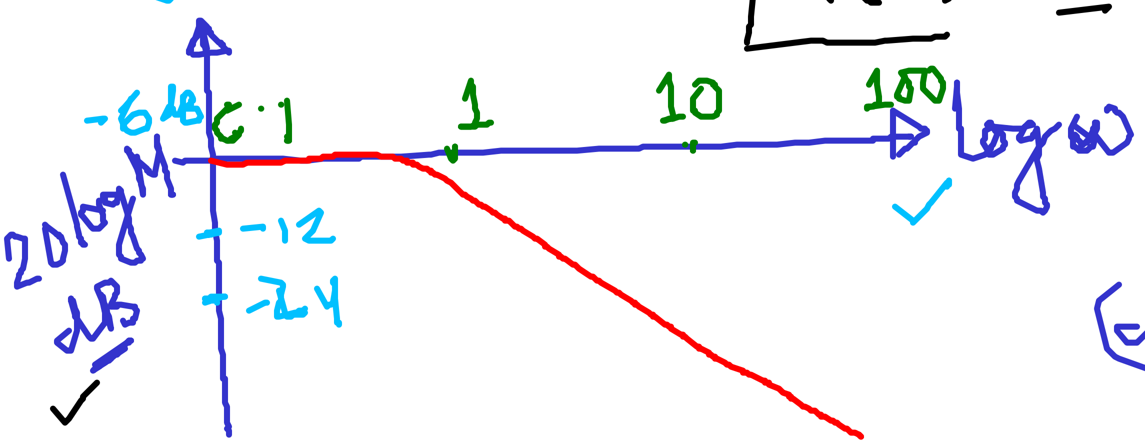
Ex

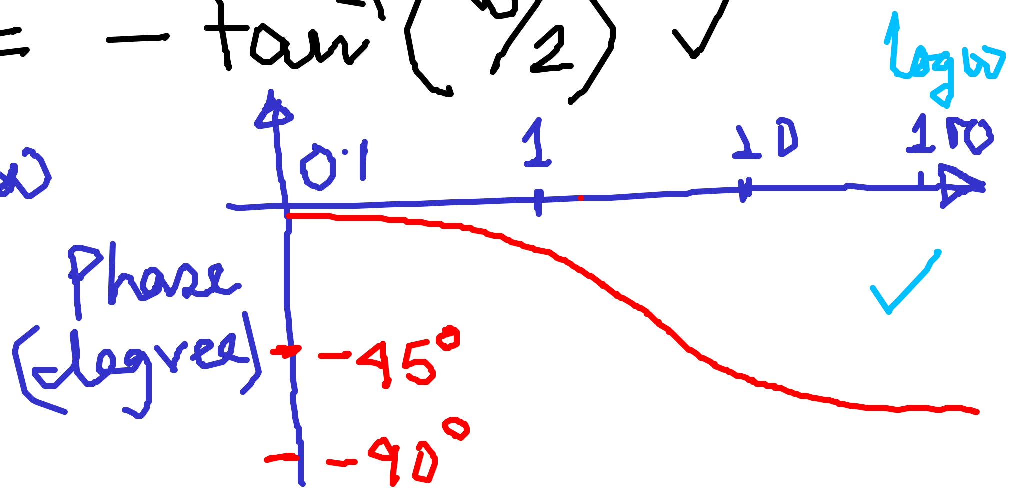
$$G(s) = \frac{1}{s+2} \qquad G(j\omega) = \frac{1}{j\omega + 2}$$

$$M = |G(j\omega)| = \frac{1}{\sqrt{4+\omega^2}} \quad ✓$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right) \quad ✓$$

Semi-log





Asymptotic approximation: Bode Plot

- log-magnitude and phase vs. $\log \omega$ plots together is known Bode plot.
- The plots can be approximated as a sequence of straight lines.

Let us consider a transfer function

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^n(s+p_1)(s+p_2)\cdots(s+p_m)}$$

The magnitude frequency response is

$$|G(j\omega)| = \frac{K\,|j\omega+z_1|\,|j\omega+z_2|\cdots|j\omega+z_k|}{|(j\omega)^n|\,|j\omega+p_1|\cdots|j\omega+p_m|}$$

Taking log, we have

$$20\log|G(j\omega)| = 20\log K + 20\log|j\omega+z_1| + 20\log|j\omega+z_2| + \cdots$$

$+ 20 \log |j\omega + z_k| - 20 \log |(j\omega)^n|$

$- 20 \log |j\omega + p_1| - \cdots - 20 \log |j\omega + p_m|$

Now

$\angle G(j\omega) = \angle j\omega + z_1 + \angle j\omega + z_2 \cdots \angle j\omega + z_k$

$- \angle (j\omega)^n - \angle j\omega + p_1 \cdots - \angle j\omega + p_m$

Plots for $G(s) = (s+a)$

$M = |G(j\omega)|$ $\quad G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$

At low frequency $(\omega \to 0)$, $G(j\omega) \approx a$.

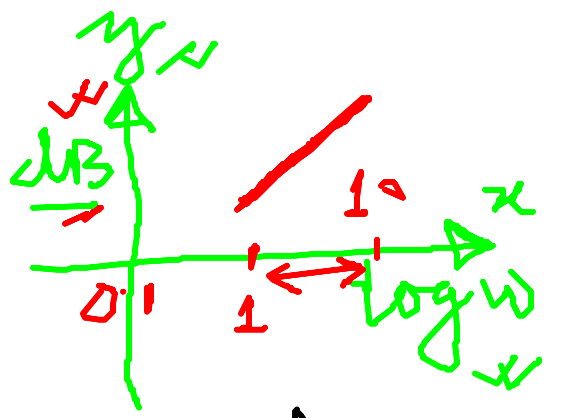
In dB, $20 \log M = 20 \log a$, where $M = |G(j\omega)|$.

It remains constant from $0.1a$ to a.

At high frequency $(\omega \gg a)$

$G(j\omega) \approx j\omega = \omega \angle 90^\circ$

$y = 20 \log M = 20 \log \omega$ (x)



If we plot $20 \log M$ vs. $\log \omega$, it is $y = 20x$, i.e., slope is 20 dB/decade

At $\omega = a$, $\angle G(j\omega) = \tan^{-1}\frac{\omega}{a} = 45°$. At low frequency $\angle G(j\omega) \approx 0°$. At high frequency $\angle G(j\omega) \approx 90°$.

'a' is known as Corner frequency or break frequency

Bode plot of $G(s) = \frac{1}{s+a}$

$$G(s) = \frac{1}{a\left(s/a + 1\right)}$$

$$G(j\omega) = \frac{1}{a\left(j\omega/a + 1\right)}$$

when $\omega << a$ (low frequency), $|G(j\omega)| \approx \frac{1}{a}$.

So, $20\log|G(j\omega)| = 20\log(1/a) = -20\log a$.

The Bode plot is constant until the corner frequency 'a' rad/s.

when $\omega >> a$ [high frequency],

$$G(j\omega) \approx \frac{1}{a\left(j\frac{\omega}{a}\right)} = \frac{1}{j\omega} = \frac{1}{\omega}\angle -90^\circ$$

in dB, $20\log|G(j\omega)| = 20\log\left(\frac{1}{\omega}\right)$

$$= -20\log\omega$$

This is similar to $(s+a)$, but with -ve slope.

$$\angle G(j\omega) = -\tan^{-1}\frac{\omega}{a}$$

when $\omega = a$, $\angle G(j\omega) = -45^\circ$ ✓

$\omega << a$ (low frequency), $\angle G(j\omega) \approx 0^\circ$

$\omega >> a$ (high frequency). $\angle G(j\omega) \approx -90^\circ$ ✓

Bode Plot for $G(s) = s$ ✓

$$G(j\omega) = j\omega$$

$$\underbrace{20\log|G(j\omega)|}_{y} = 20\underbrace{\log\omega}_{x}$$

in dB, slope is 20 dB/decade ✓

$\omega = 1, \quad 20 \log |G(j\omega)| = 0$

$\angle G(j\omega) = \tan^{-1}(\infty) = 90^\circ$ ✓

Bode plot for $G(s) = \frac{1}{s}$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} \angle -90^\circ$$

$$20 \log |G(j\omega)| = 20 \log\left(\frac{1}{\omega}\right) = -20 \log \omega$$

slope is -20 dB/decade.

$$\angle G(j\omega) = -90^\circ$$

Bode plot for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ ✓

$$G(s) = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1\right)$$

At low frequency, $G(s) \approx \omega_n^2 = \omega_n^2 \angle 0^\circ$

$$20 \log |G(j\omega)| = 20 \log \omega_n^2$$ ✓

At high frequency, $G(s) \approx s^2$

$$G(j\omega) \approx -\omega^2 = \omega^2 \angle 180^\circ$$

$$20\log|G(j\omega)| = 20\log\omega^2 = 40\log\omega$$

slope = 40 dB/decade

When $\omega = \omega_n$, i.e., $\frac{\omega}{\omega_n} = 1$, low and high frequency asymptotes are same.

ω_n is the corner frequency.

Normalized and scaling

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$= \omega_n^2\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)$$

$$\left(\frac{G(s_1)}{\omega_n^2}\right) = s_1^2 + 2\zeta s_1 + 1, \quad s_1 = \frac{s}{\omega_n} = 1$$

Low frequency asymptotes is 0 dB and corner frequency is 1.

$$G(j\omega) = -\omega^2 + j2\zeta\omega_n\omega + \omega_n^2$$

$$= (\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega$$

$$\angle G(j\omega) = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$

$= 0$ $\quad \omega = \omega_n$

At $\omega = \omega_n$, $\angle G(j\omega) = 90^\circ$

$$|G(j\omega)| = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

At $\omega = \omega_n$, $20\log|G(j\omega)| = 20\log 2\zeta\omega_n^2$.

This difference depends on ζ.

Bode plot for $G(s) = \dfrac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Slope is -40 dB/decade

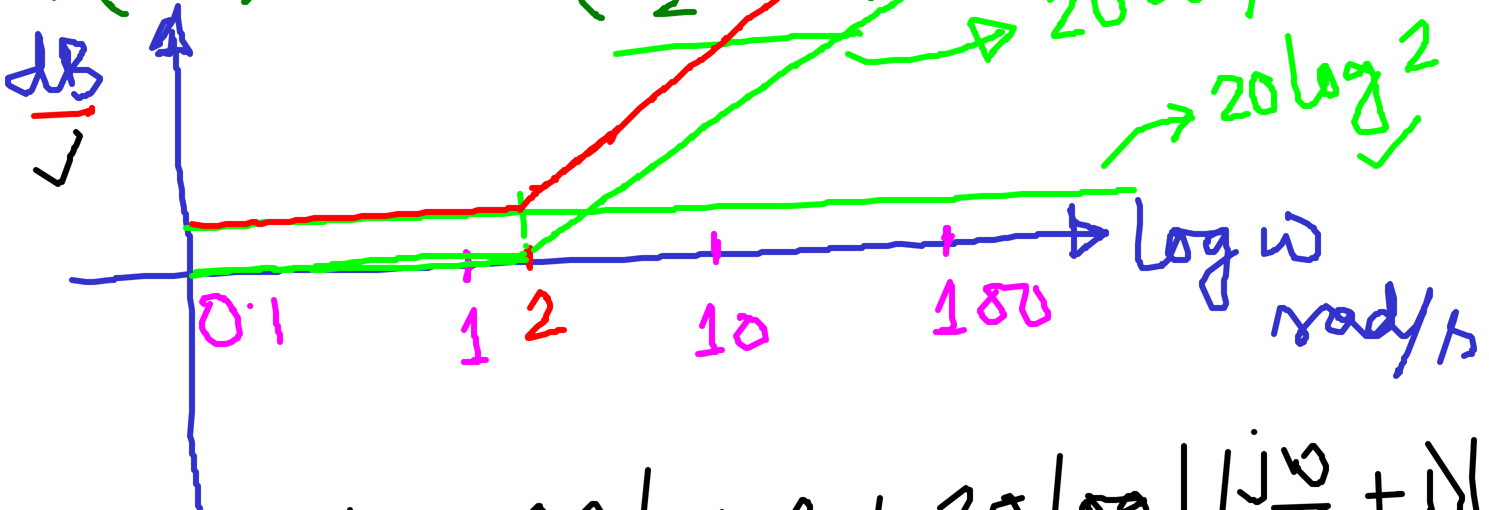
Slope is -90°/decade.

Ex

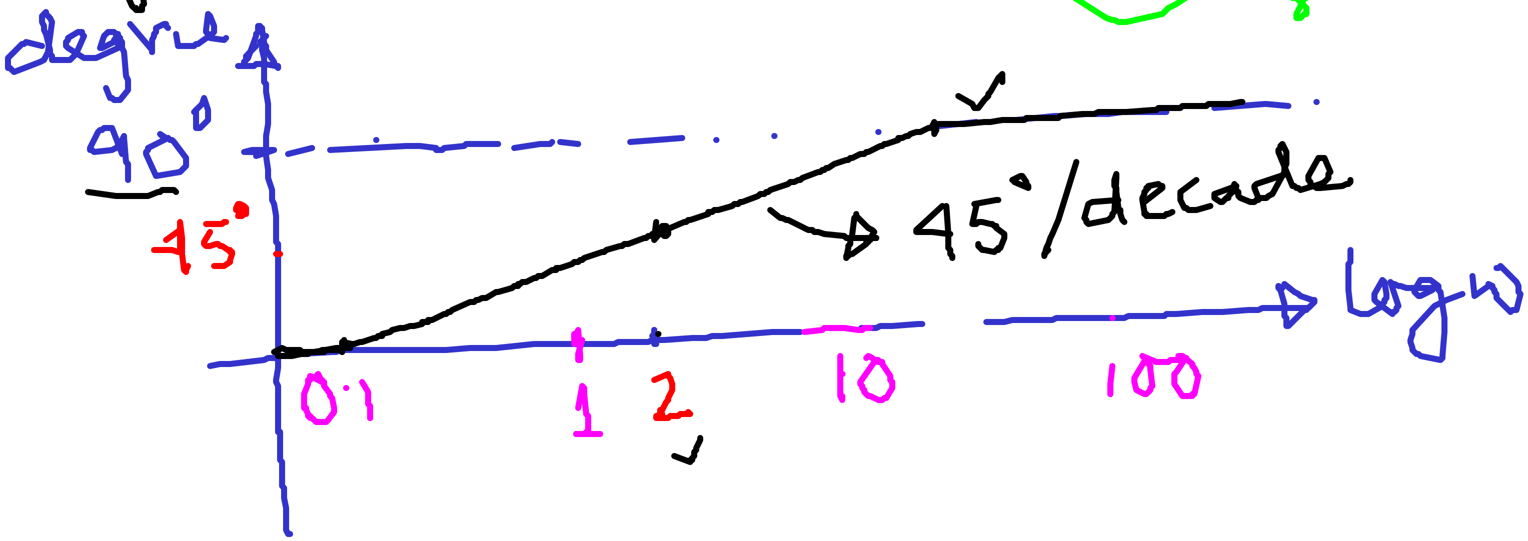
$$G(s) = (s+2) = 2\left(\frac{s}{2}+1\right)$$

a

$$G(j\omega) = 2\left(\frac{j\omega}{2}+1\right)$$

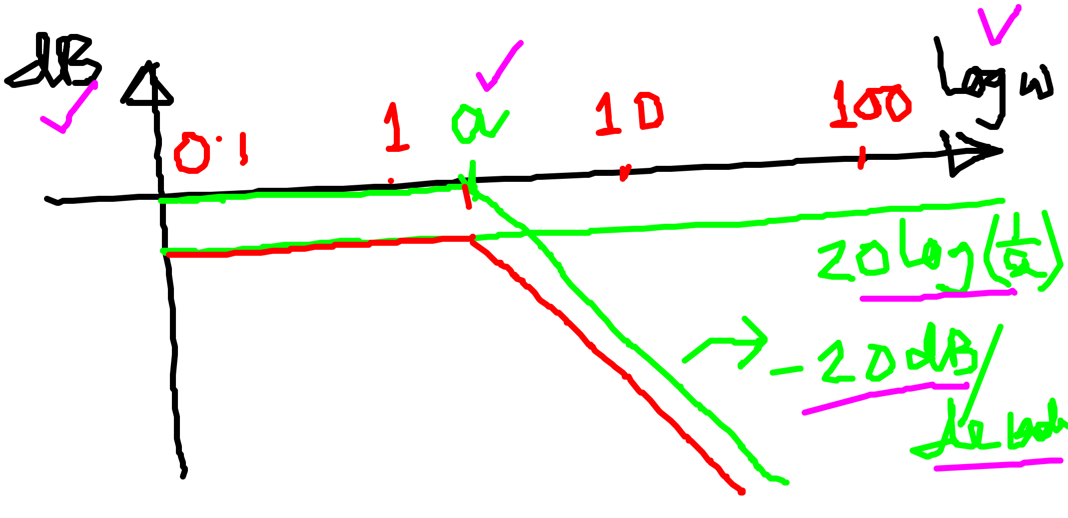


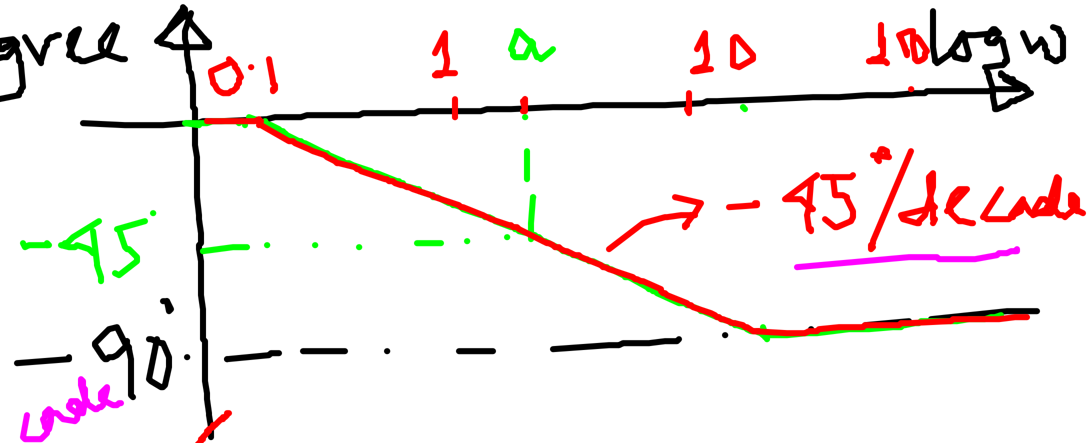
$$20\log|G(j\omega)| = 20\log 2 + 20\log\left|\left(\frac{j\omega}{2}+1\right)\right|$$



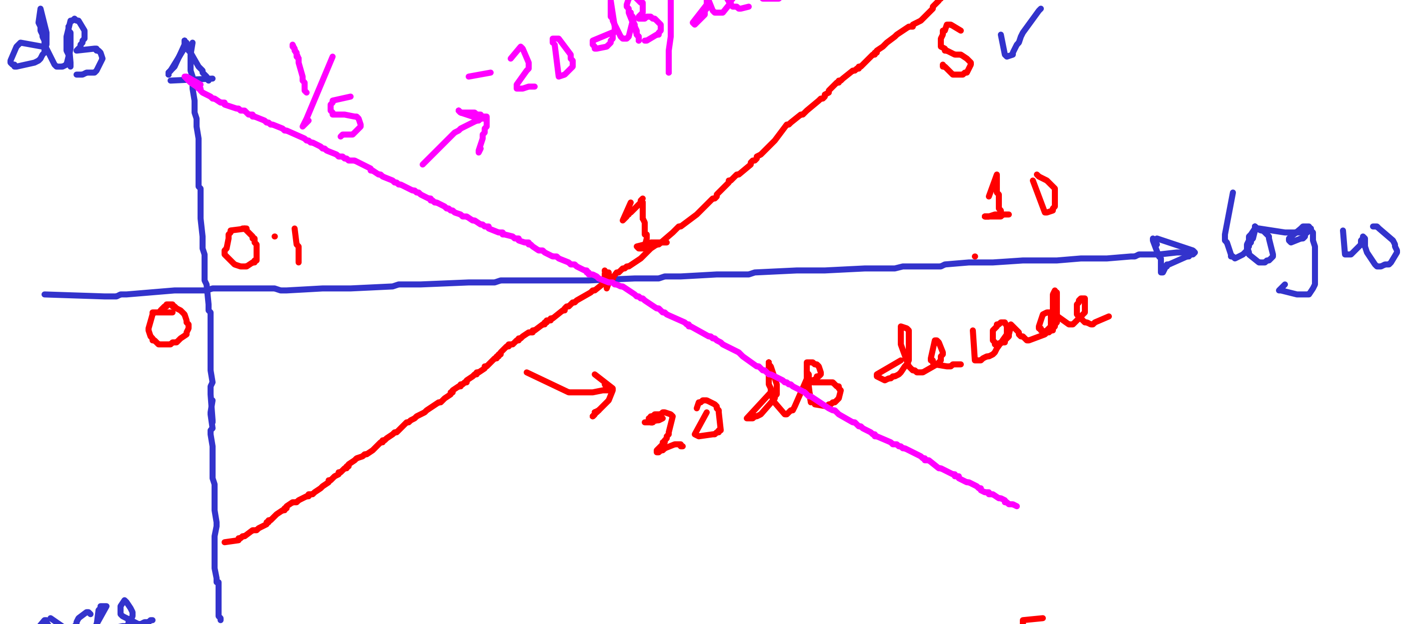
Ex

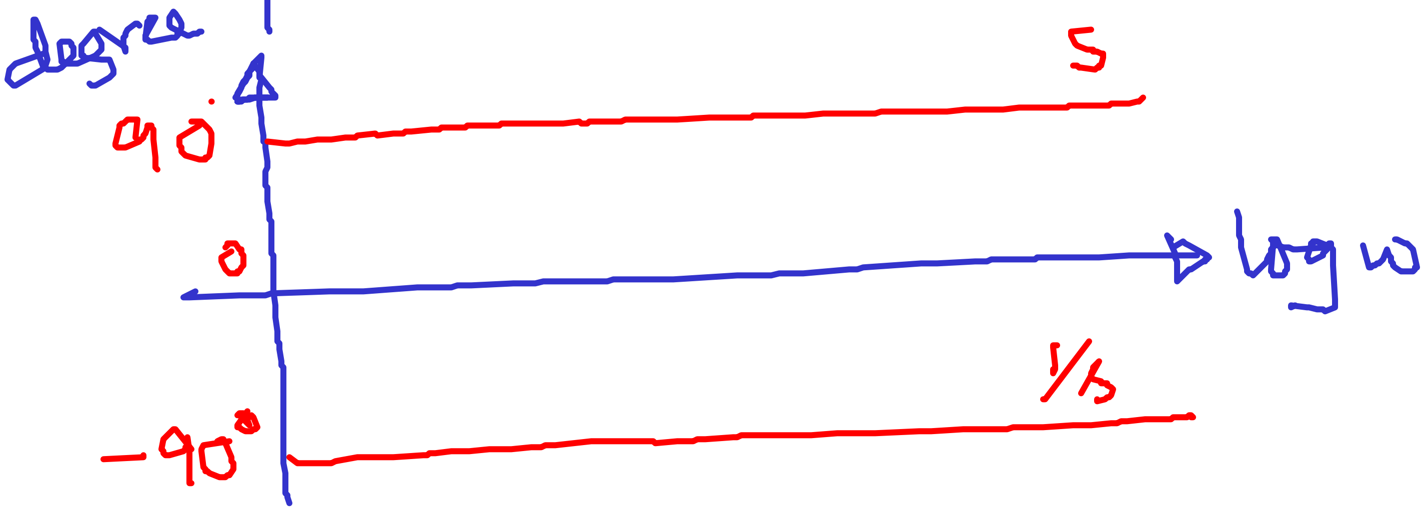
$$G(s) = \frac{1}{s+a}$$




degree
0.1
1
a
10
100
log w
-45
-90°
-45°/decade

Ex

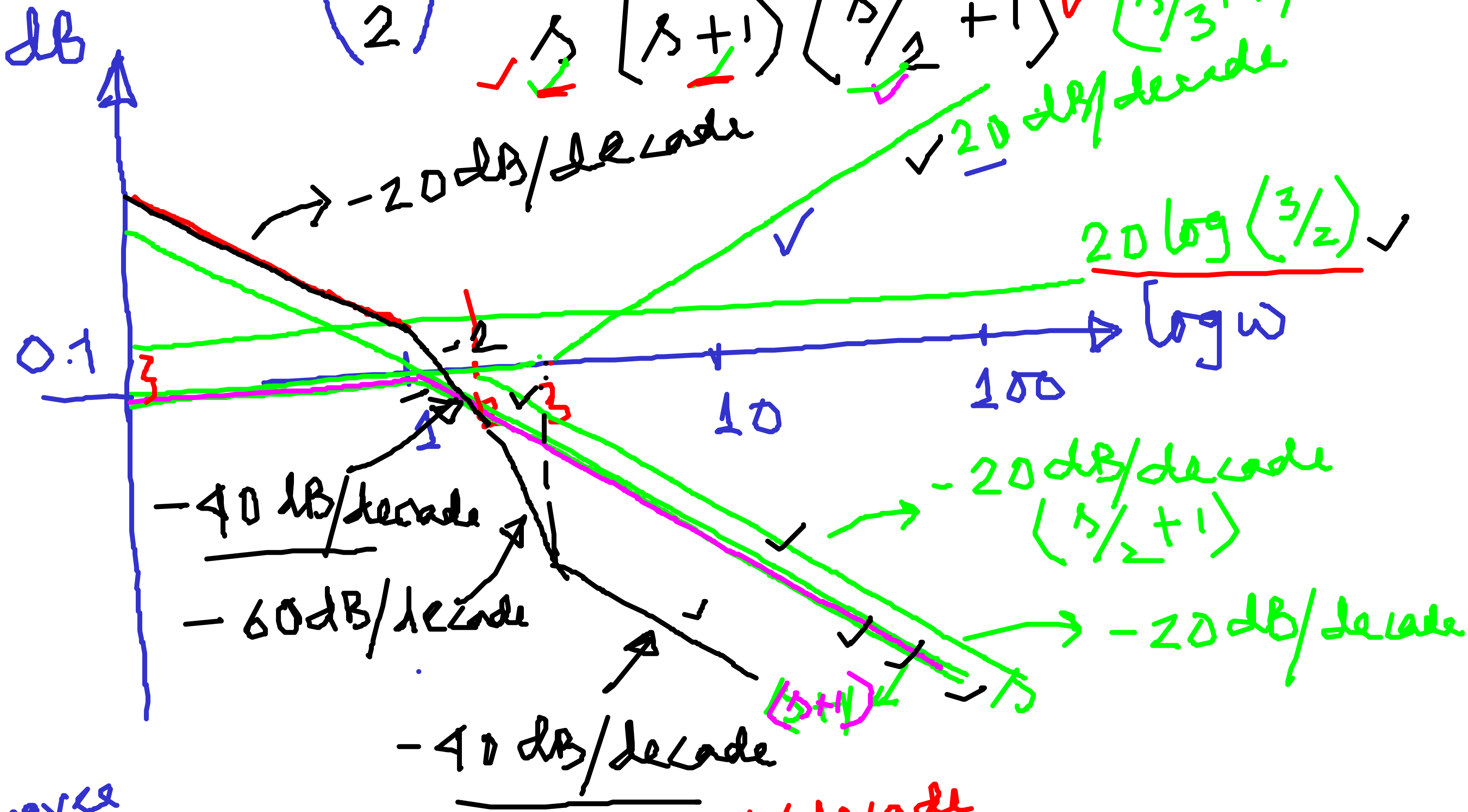

dB
1/s
-20 dB/decade
s
0.1
1
10
log w
0
20 dB decade

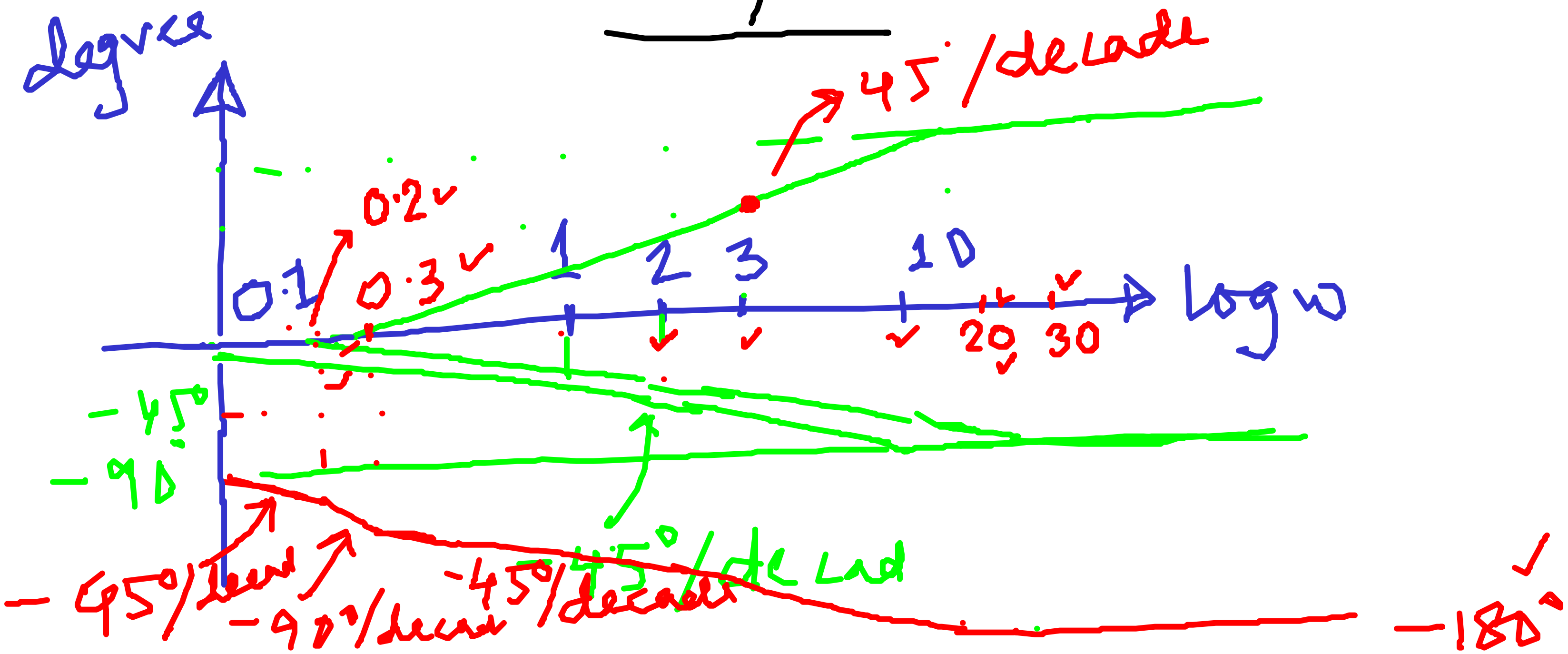

degree
90°
s
0
log w
1/s
-90°

Ex

$$G(s) = \frac{(s+3)}{s(s+1)(s+2)}$$

$$= \left(\frac{3}{2}\right) \frac{\left(\frac{s}{3}+1\right)}{s(s+1)\left(\frac{s}{2}+1\right)}$$



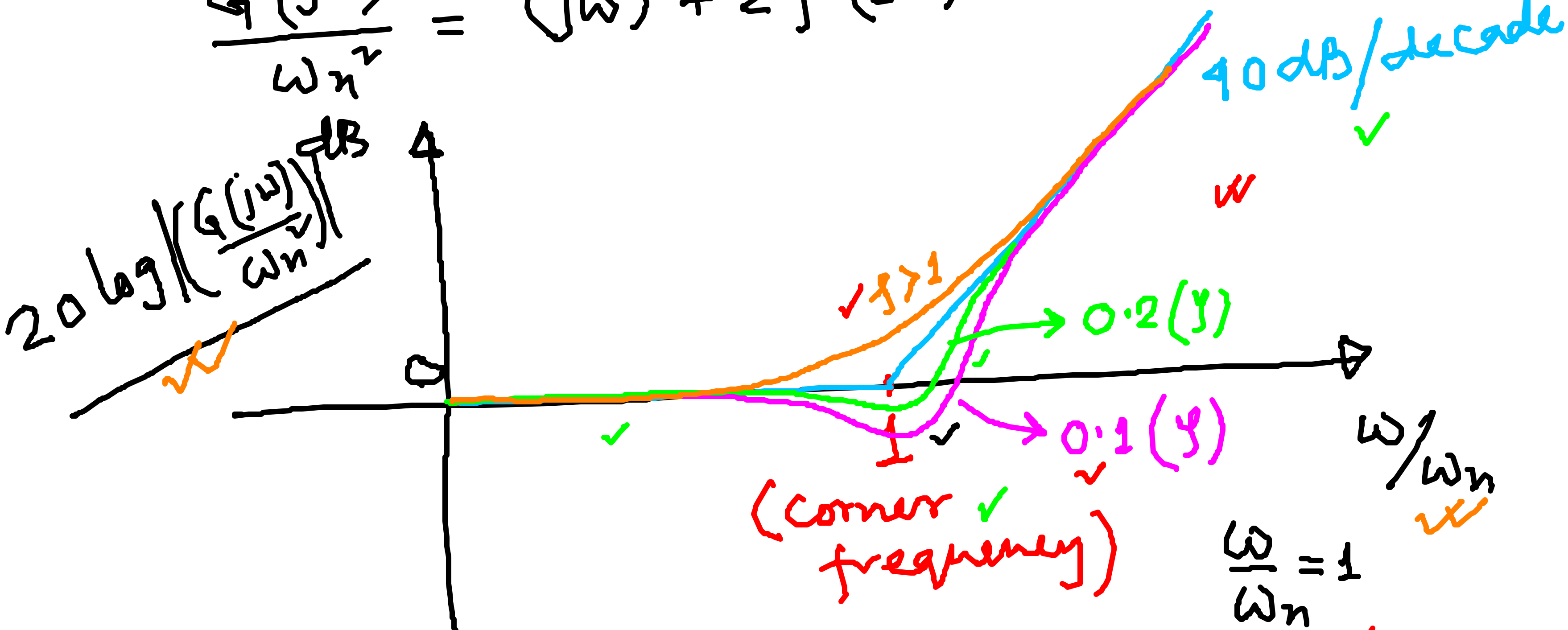


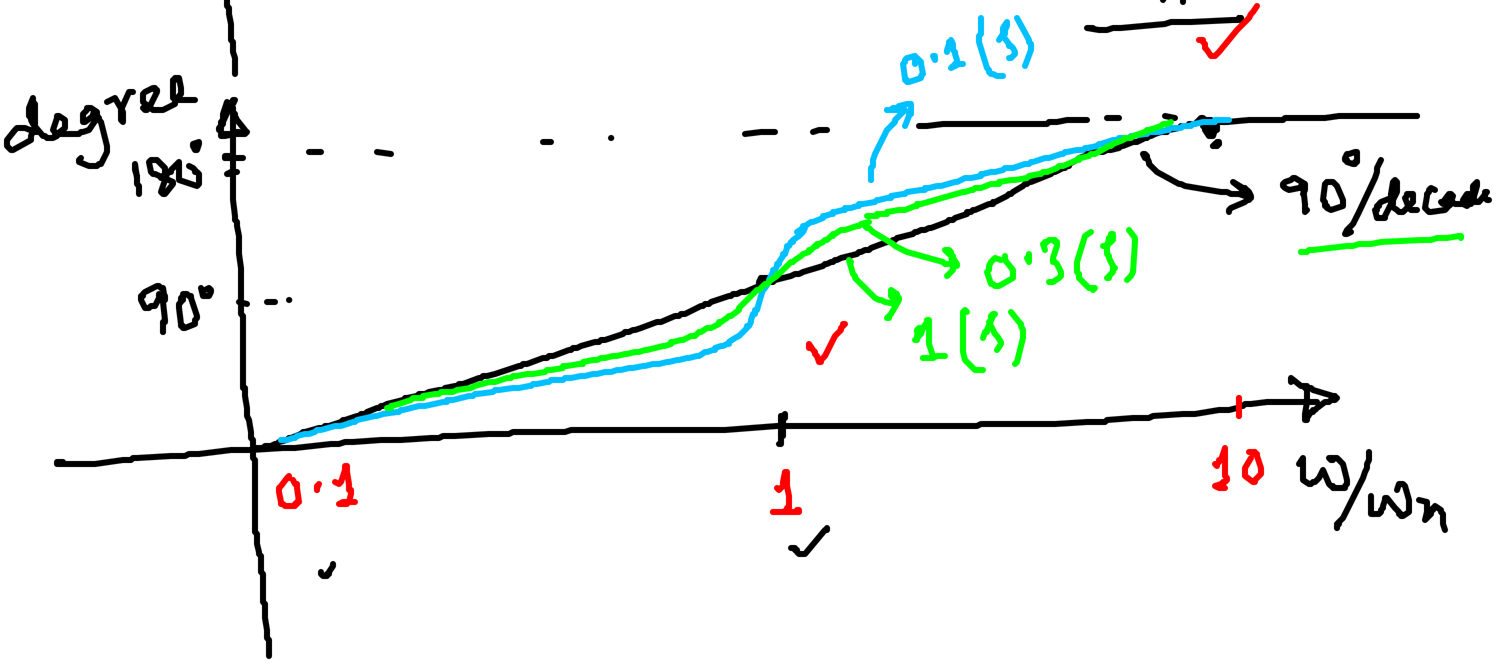
Ex

$$G(s) = s^2 + 2\zeta \omega_n s + \omega_n^2$$

$$\frac{G(s)}{\omega_n^2} = \left(\underbrace{\frac{s^2}{\omega_n^2}}_{s_1^2} + 2\zeta \underbrace{\frac{s}{\omega_n}}_{s_1} + 1 \right) \qquad \frac{s}{\omega_n} = s_1$$

$$\frac{G(j\omega)}{\omega_n^2} = (j\omega)^2 + 2\zeta (j\omega) + 1$$





Ex

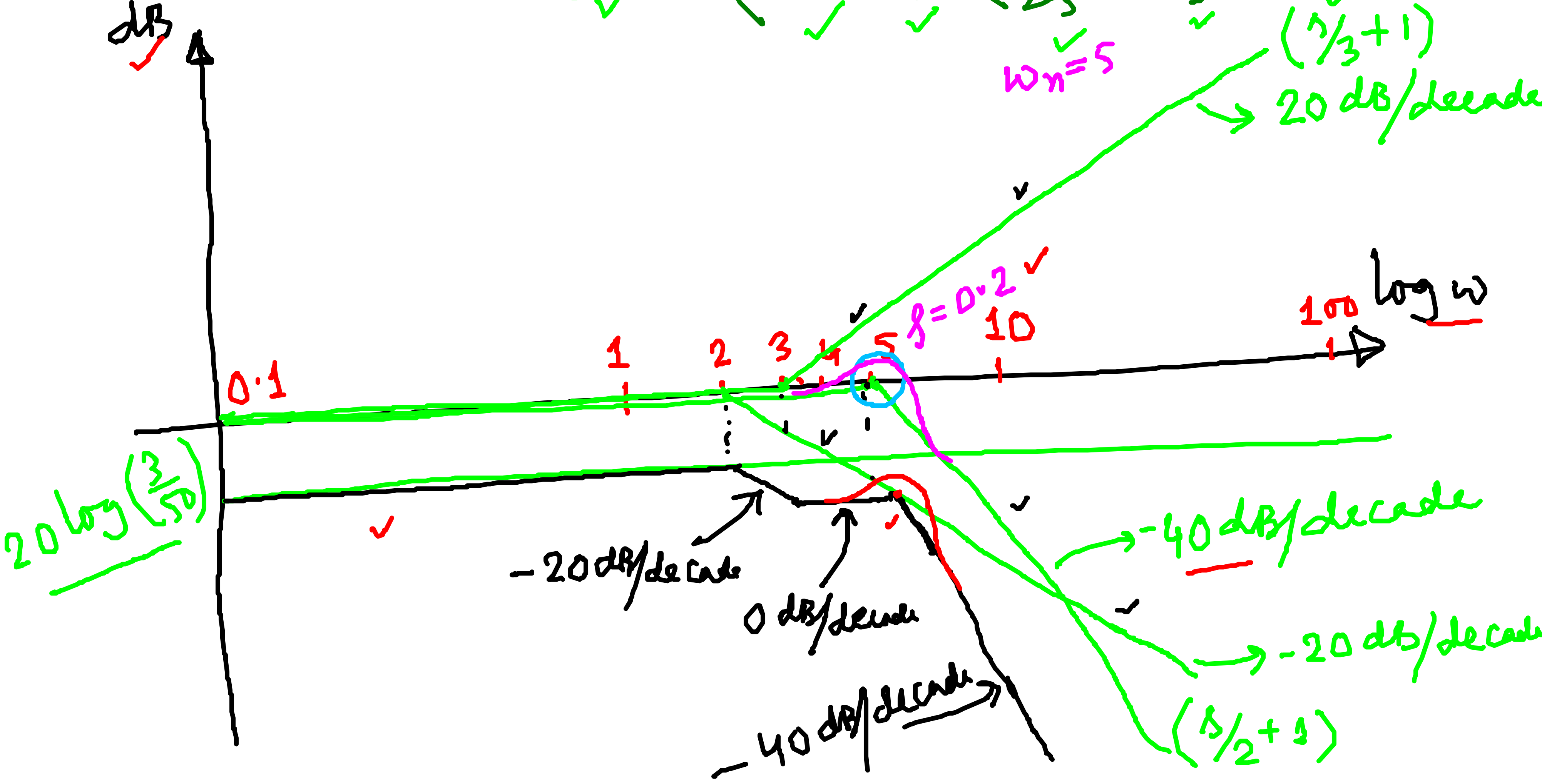
$$G(s) = \frac{(s+3)}{(s+2)(s^2+2s+25)}$$

$$= \left(\frac{3}{2\times 25}\right)\frac{\left(s/3+1\right)}{\left(s/2+1\right)\left(\frac{s^2}{25}+\frac{2}{25}s+1\right)}$$

$2\zeta\omega_n = 2$

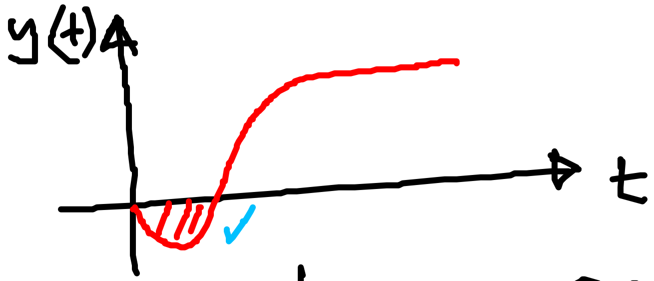
$\zeta = \frac{1}{\omega_n} = 0.2$

$\omega_n = 5$



Non-minimum phase systems

– Transfer functions those having poles and/or zeros in the right-half of the 's'-plane are known as non-minimum phase transfer function.



- Magnitude and phase are not uniquely related.

$$G_1(s) = \frac{s+2}{s+3} = \frac{\sqrt{\omega^2+4}}{\sqrt{\omega^2+9}} \left(\angle \tan^{-1}\omega/2 - \angle \tan^{-1}\omega/3 \right)$$

$$G_2(s) = \frac{s-2}{s+3} = \underbrace{\left(\frac{s+2}{s+3}\right)}_{G_1(s)} \times \left(\frac{s-2}{s+2}\right)$$

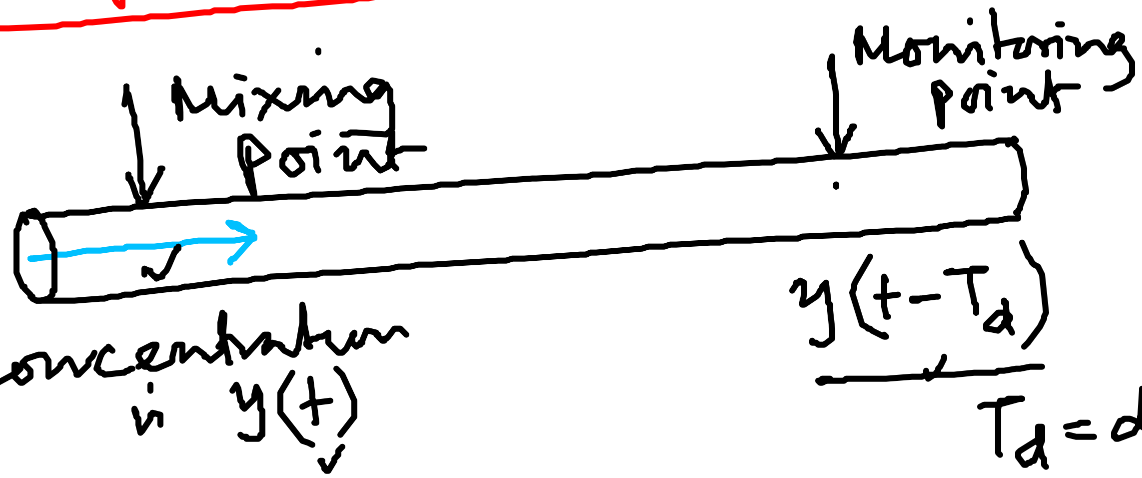
All pass TF

magnitude is 1

$$\left(\pi - \tan^{-1}\omega/2 - \tan^{-1}\omega/2\right)$$

contributing extra phase

Systems with transportation lag



T_d = delay

$$c(t) = y(t - T_d)$$

$$C(s) = e^{-T_d s}\, Y(s)$$

$$\boxed{\frac{C(s)}{Y(s)} = e^{-T_d s}} \approx \frac{\left(1 - \frac{T_d s}{2}\right)}{1 + \frac{T_d s}{2}}$$

[First order Pade approximation]

Transportation lag $e^{-sT_d} = G(s)$

$$G(j\omega) = e^{-j\omega T_d} = \cos(\omega T_d) - j \sin(\omega T_d)$$

$$|G(j\omega)| = 1$$

$$\angle G(j\omega) = \tan^{-1}\left(-\frac{\sin(\omega T_d)}{\cos(\omega T_d)}\right)$$

$= -\omega T_d$ radian

$= -57.3\, \omega T_d$ degree