Prob-Stats Workshop

Section 1

Random Experiment

Outcomes, Events, Sample Space

Probability

Random Experiment

A **random experiment** is a process characterized by the following **properties**:

(i) It can be repeated arbitrarily often under same conditions,

(ii) The outcome of each experiment depends on chance and hence cannot be predicted with certainty.

Example: Rolling a dice, tossing a coin etc. etc.

Outcomes, Events and Sample Space

- Outcome: An **outcome** is the result of a single trial of an experiment.
- Sample Space (Ω): The universal set of all possible outcomes.
- Event ($\mathcal{E}$): An **event** is one or more outcomes of an experiment.
- We are interested in $\mathcal{E} \subseteq \Omega$, so we perform the experiment N times and see how many times $\mathcal{R}$ actually Occurred and call is relative frequency.

$$\# \, \mathcal{E} = f/N$$

$$Probability(\mathcal{E}) = \lim_{n \to \infty} f/N$$

Example

There are 5 chits for the five corresponding pools at IITK, i.e., Mughals, Mauryans, Rajputs, Marathas, and Veeras. A person will come and choose one of the chits. The name on the choosen chit will be declared as a winner.

Example (Continued..)

In the example before,

- What is the sample space?
- What are outcomes?
- Give example of an event which has more than one outcomes.

Probability

Probability is the measure of the likelihood that an event will occur.

Axioms of Probability:

1: For any event A, $P(A) \geq 0$.

2: Probability of the sample space SS is $P(S)=1$.

3: If $A1,A2,A3,\cdots A1,A2,A3,\cdots$ are disjoint events, then

$P(A1 \cup A2 \cup A3 \ldots) = P(A1) + P(A2) + P(A3) + \ldots\ldots$

Example

In the random experiment, mentioned before what is the probability that Mughals will win?

In the random experiment, what is the probability that any of the five pool will win?

In the random experiment, mentioned before what is the probability that either Mughals or Mauryans will win?

Puzzle

Three ants are sitting at the three corners of an equilateral triangle. Each ant starts randomly picks a direction and starts to move along the edge of the triangle. What is the probability that none of the ants collide? Assume that they move at same rate.

Another Puzzle

You have a biased coin. You need to create a random experiment such that the experiment behaves like a random coin toss experiment, which means, there will be 2 outcomes and there will be equal probability for occurrence of both.

Another puzzle

A rod of length L is cut into three halves. What is the probability that the three cut pieces of rod will form a triangle?

Section 2

Random Variable

Probability Mass Function

Probability Density Function

Cumulative Density Function

Random Variable

A Random Variable X is a function which maps the Sample Space to Real Line.

$$X: \Omega \ \rightarrow \mathcal{R}$$

Typically, random variables are denoted using using upper case letters X(w) or more simply X.

Discrete Random Variable

If the random variable, X(w) takes only a finite or countable infinite values.

Example,

- w = Mughals implies X = 1
- w = Mauryans implies X = 2
- w = Marathas implies X = 3
- w = Rajputs implies X = 4
- w = Veeras implies X = 5

Continuous Random Variable

If X(w) takes uncountable infinite number of possible values, then it’s called Continuous Random Variable.

Example,

Height of Students in a class,

Time required to reach the office

Probability Mass Function (PMF)

It is a function that gives the probability that a discrete random variable X exactly equals to value x.

$$P\ (X = x)$$

In our example before, if we assume equally likely probabilities, then

$P(X = 1) = 0.2$ implies 0.2 probability that Mughals are the winners

$P(X = 2) = 0.2$ implies 0.2 probability that Mauryans are the winners

$P(X = 3) = 0.2$ implies 0.2 probability that Marathas are the winners

$P(X = 4) = 0.2$ implies 0.2 probability that Rajputs are the winners

$P(X = 5) = 0.2$ implies 0.2 probability that Veeras are the winners

Probability Density Function (PDF)

It is a function analogous to PMF but it is defined for a continuous random variable. The functional value at any given point can be interpreted as providing a **RELATIVE LIKELIHOOD** that the value of the random variable would equal that sample.

Question: What is the probability that $X = 0$, X follows a standard normal distribution.

Some Standard Distributions

Distribution	PMF/PDF and Support	Expected Value	Variance	MGF
Bernoulli Bern(p)	$P(X=1)=p$ $P(X=0)=q=1-p$	p	pq	$q+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^k q^{n-k}$ $k \in \{0,1,2,\dots n\}$	np	npq	$(q+pe^t)^n$
Geometric Geom(p)	$P(X=k)=q^k p$ $k \in \{0,\ 1,\ 2,\ \dots\}$	q/p	q/p^2	$\frac{p}{1-qe^t},\ qe^t<1$
Negative Binomial NBin(r,p)	$P(X=n)=\binom{r+n-1}{r-1}p^r q^n$ $n \in \{0,\ 1,\ 2,\ \dots\}$	rq/p	rq/p^2	$(\frac{p}{1-qe^t})^r,\ qe^t<1$
Hypergeometric HGeom(w,b,n)	$P(X=k)=\binom{w}{k}\binom{b}{n-k}/\binom{w+b}{n}$ $k \in \{0,1,2,\dots,n\}$	$\mu=\frac{nw}{b+w}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}(1-\frac{\mu}{n})$	messy
Poisson Pois(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k \in \{0,\ 1,\ 2,\ \dots\}$	λ	λ	$e^{\lambda(e^t-1)}$

Uniform Unif(a,b)	$f(x)=\frac{1}{b-a}$ $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x\,-\,\mu)^2/(2\sigma^2)}$ $x\in(-\infty,\infty)$	μ	σ^2	$e^{t\mu+\frac{\sigma^2t^2}{2}}$
Exponential Expo(λ)	$f(x)=\lambda e^{-\lambda x}$ $x\in(0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}, t<\lambda$
Gamma Gamma(a,λ)	$f(x)=\frac{1}{\Gamma(a)}(\lambda x)^a e^{-\lambda x}\frac{1}{x}$ $x\in(0,\infty)$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^a, t<\lambda$
Beta Beta(a,b)	$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$ $x\in(0,1)$	$\mu=\frac{a}{a+b}$	$\frac{\mu(1-\mu)}{(a+b+1)}$	messy
Log-Normal $\mathcal{LN}(\mu,\sigma^2)$	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-(\log x-\mu)^2/(2\sigma^2)}$ $x\in(0,\infty)$	$\theta=e^{\mu+\sigma^2/2}$	$\theta^2(e^{\sigma^2}-1)$	doesn't exist
Chi-Square χ^2_n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$ $x\in(0,\infty)$	n	$2n$	$(1-2t)^{-n/2}, t<1/2$
Student-t t_n	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)}(1+x^2/n)^{-(n+1)/2}$ $x\in(-\infty,\infty)$	0 if $n>1$	$\frac{n}{n-2}$ if $n>2$	doesn't exist

Cumulative Density Function (CDF)

It is a function which maps the value of a Random Variable to [0,1], i.e. it specifies a probability measure for that value. It is defined as

$$F_X(x) = P(X<=x).$$

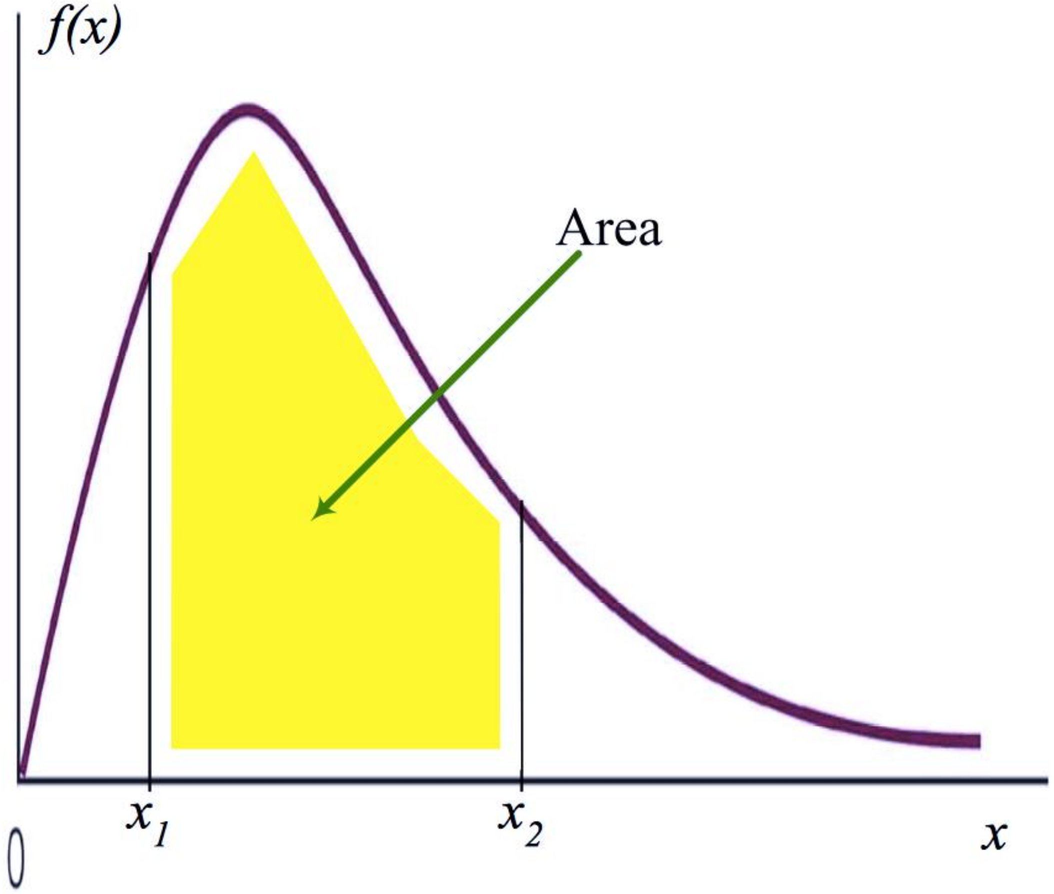
For discrete random variable, it is defined as

$$F(x) = \mathrm{P}(X \leq x) = \sum_{x_i \leq x} \mathrm{P}(X = x_i) = \sum_{x_i \leq x} p(x_i).$$

For continuous random variable, it is defined as

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

Interpreting CDF



Properties of CDF

Properties:

- $0 <= F_X(x) <= 1$
- $\lim_{x \rightarrow -\text{inf}} F_X(x) = 0$
- $\lim_{x \rightarrow \text{inf}} F_X(x) = 1$
- $x <= y$ =) $F_X(x) <= F_X(y)$.

Example

From our example, it was found out that

$$
F_x(x) = \begin{cases} 0 & \text{for } x < 1 \\ 0.2 & \text{for } 1 \le x < 2 \\ 0.4 & \text{for } 2 \le x < 3 \\ 0.6 & \text{for } 3 \le x < 4 \\ 0.8 & \text{for } 4 \le x < 5 \\ 1 & \text{for } x \ge 5 \end{cases}
$$

Plot this function.

How do you interpret $F_x(2)$?

Section 3

Conditional Probability

Bayes Theorem

Independence of Events

Conditional Probability

Conditional probability formalizes the notion of “having information” about the outcome of a probabilistic experiment.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Example

There are five pools Mughals, Mauryans, Marathas, Rajputs and Veeras. There is a person whose identity is unknown and he is equally likely to be in any of the pool. What’s the probability that he is a Mauryan?

Example

We get the information that he is a boy. What’s the probability that he is a Mauryan?

Bayes Theorem

Bayes' theorem describes the probability of an event, based on prior knowledge of conditions that might be related to the event. Bayes Rule comes from the law of total probability.

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)} \cdot$$

$$P(B) = \sum_j P(B \mid A_j) P(A_j),$$

$$\Rightarrow P(A_i \mid B) = \frac{P(B \mid A_i)\, P(A_i)}{\sum_j P(B \mid A_j)\, P(A_j)} \cdot$$

For proposition A and evidence B,

- P (A), the prior, is the initial degree of belief in A.
- P (A | B) is the “posterior,” is the degree of belief having accounted for B.
- The quotient P(B |A)/P(B) represents the support B provides for A.

Independence of Events

Two events X and Y are called independent if $P(X|Y) = P(X)$.

Another way to write the above condition is $P(XY) = P(X)P(Y)$.

In simple language, knowledge of the event Y doesn’t help us in giving more information about event X.

Puzzles!

The random experiment of drawing the chit is taking place. The person who draws the chit decides that if the chit draws out “Mughals” or “Mauryans”, he will toss an unbiased coin and if head comes he will change the result to “Marathas”, otherwise he will not change the result.

It was later declared that the winners are Marathas.

i. What is the probability that “Marathas” are the true winners?

ii. What is the probability that the coin was tossed ?

Section 4

Expectation

Variance

Covariance

Correlation

Expectation

The expected value of a random variable, intuitively, is the long-run average value of repetitions of the experiment it represents.

For discrete random variables, it is defined as

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} x_i \, p_i$$

For continuous random variables, it is defined as

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) \, dx.$$

Expectation (More Precise Definition)

If X is discrete, then the expectation of $g(X)$ is defined as, then

$$E[g(X)] = \sum_{x \in \mathcal{X}} g(x) f(x),$$

where f is the probability mass function of X and $\mathcal{X}$ is the support of X.

If X is continuous, then the expectation of $g(X)$ is defined as,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx,$$

where f is the probability density function of X.

Linearity of Expectation

- This property will be exploited the most while doing probability puzzles.

$$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[aX] = a\,\mathrm{E}[X],$$

where X and Y are (arbitrary) random variables, and a is a scalar.

Variance

Variance is the expectation of the squared deviation of a random variable from its mean. Informally, it measures how far a set of (random) numbers are spread out from their average value.

Formally, it is defined as follows:

$$\begin{aligned}
\operatorname{Var}(X) &= \operatorname{E}\left[(X - \operatorname{E}[X])^2\right] \\
&= \operatorname{E}\left[X^2 - 2X\operatorname{E}[X] + \operatorname{E}[X]^2\right] \\
&= \operatorname{E}\left[X^2\right] - 2\operatorname{E}[X]\operatorname{E}[X] + \operatorname{E}[X]^2 \\
&= \operatorname{E}\left[X^2\right] - \operatorname{E}[X]^2
\end{aligned}$$

Standard Deviation of a random variable is the square root of Variance.

Covariance

Covariance is a measure of the joint variability of two random variables. The sign of the covariance therefore shows the tendency in the linear relationship between the variables. Formally, it is defined as,

$$
\begin{aligned}
\operatorname{cov}(X,Y) &= \mathrm{E}[(X-\mathrm{E}[X])\,(Y-\mathrm{E}[Y])] \\
&= \mathrm{E}[XY - X\,\mathrm{E}[Y] - \mathrm{E}[X]Y + \mathrm{E}[X]\,\mathrm{E}[Y]] \\
&= \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] - \mathrm{E}[X]\,\mathrm{E}[Y] + \mathrm{E}[X]\,\mathrm{E}[Y] \\
&= \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y].
\end{aligned}
$$

Covariance (… continued)

If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values, i.e., the variables tend to show similar behavior, the covariance is positive, and vice-versa.

Note: The magnitude of the covariance is not easy to interpret because it is not normalized. So, it is not used generally for interpretation.

Correlation

Cor(X,Y) is a measure of the linear correlation between two variables X and Y. It has a value between +1 and −1, where 1 is total positive linear correlation, 0 is no linear correlation, and −1 is total negative linear correlation. This is a normalized version of covariance.

Correlation is represented using rho, it is calculated as the ratio of covariance to the standard deviation of X and Y.

$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

Property of Correlation

- If two variables X and Y are independent, then they have zero correlation but this DOESN’T hold other way round.
- It might happen that two variables have 0 correlation but they are still not independent.

Section 5

Chebyshev's Inequality

Weak Law of Large numbers

Central Limit Theorem

Chebyshev’s Inequality

- Let X be a random variable certain distribution with finite variance σ^2, then

$$P(|X-\mu| \geq k\sigma) \leq 1/k^2$$

Or, alternatively

$$P(|X-\mu| \geq \epsilon) \leq \sigma^2/\epsilon^2$$

Weak Law of Large numbers

- Let $\{X_n\}$ be a sequence of iid random variables with mean μ and variance $\sigma^2 < \infty$. And let, $\bar{X}_n = \frac{1}{n}\sum_i X_i$ Then,

$$P[|\bar{X}_n - \mu| \geq \epsilon] \leq \frac{\sigma^2}{n\epsilon^2}$$

We also say that $\bar{X}_n \longrightarrow \mu$ in probability

Central Limit Theorem

- Let $X_1, X_2, \cdots, X_n$ is a random sample from a distribution (**any**) that has mean μ and variance $\sigma^2 > 0$.
- Further let, $Y_n = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma}$
- Y_n converges in distribution to a standard normal distribution.

$$Y_n \rightarrow N(0,1)$$

Section 6

Puzzles!

Puzzle 1

There is a family in Haryana. They try to conceive until they receive a boy. What is the expected number of children will the family have?

Puzzle 2

There is a square table of side 5 inches. There is a coin whose diameter is half inch. The coin is tossed and it is known that it will land on table. What is the probability that it will land completely on table.

Puzzle 3

Suppose there are 3 curtains, behind one door is a grand prize and the other two are worthless prizes. A contestant is asked to chose one door and the Monte Hall opens one of other two doors which contains worthless prize. Hall provides the contestant to switch the curtain after he has revealed one worthless prize. Should he switch or not?

Puzzle 4

A coin is tossed till the time there are two consecutive heads. What is the expected number of tosses.

Puzzle 5

An urn contains n balls numbered $1,2,3,\ldots,n$. We remove k balls at random and add up their numbers. Find the expected value of this final number.

Puzzle 6

We throw m balls randomly, uniformly and independently into n bins.
What is the expected number of empty bins.

Puzzle 7 (GS Interview, 2016)

You have 50 red marbles, 50 blue marbles and 2 jars. One of the jars is chosen at random and then one marble will be chosen from that jar at random. How would you maximize the chance of drawing a red marble? What is the probability of doing so? All 100 marbles should be placed in the jars.

Puzzle 8 (Try this at home)

Player M has \$1, and Player N has \$2. Each play gives one of the players \$1 from the other. Player M is enough better than enough Player N that he wins 2/3 of the plays. They play until one is bankrupt. What is the chance that player M wins.

Resources

- Introduction to Mathematical Statistics- Hogg, Craig and Mckean
- Fifty challenging problems in probability by Frederick Mosteller
- https://static1.squarespace.com/static/54bf3241e4b0f0d81bf7ff36/t/55e9494fe4b011aed10e48e5/1441352015658/probability_cheatsheet.pdf
- http://www.madandmoonly.com/doctormatt/mathematics/dice1.pdf

Thank You!