Problem I

Numerical Methods

(Exercise 6.1)

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A. Derive the 4th order Adams-Bashforth formula

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

Sol: The Adams-Bashforth method integrates the differential equation,

$$\frac{dy}{dx} = y'(x) = f(x,y) \text{ over the integral } [x_n, x_{n+1}].$$

Approximating $f(x)$ over $[x_{n-1}, x_n]$ using linear interpolation,

$$f(x) \approx f_n + \frac{f_n - f_{n-1}}{h}(x - x_n)$$

Here, x_n and x_{n-1} are evenly spaced with step size h.

$$\therefore \int_{x_n}^{x_{n+1}} f(x)dx = \int_{x_n}^{x_{n+1}} f_n + (x-x_n)\frac{f_n - f_{n-1}}{h}\, dx = y_{n+1} - y_n$$

Let, $x - x_n = u$ $\quad \therefore \frac{du}{dx} = 1 \Rightarrow du = dx.$

When $x = x_n$, $u = 0$

& $x = x_{n+1}$, $u = h$.

$$\text{So, } \int_{x_n}^{x_{n+1}} f(x)\, dx = \int_0^h f_n + \frac{f_n - f_{n-1}}{h}.u \, du = I \text{ (let.)}$$

$$\Rightarrow I = f_n \cdot h + \frac{f_n - f_{n-1}}{h} \cdot \frac{h^2}{2}$$

$$\Rightarrow I = hf_n + \frac{h}{2}(f_n - f_{n-1})$$

$$\text{So, } y_{n+1} = y_n + hf_n + \frac{hf_n}{2} - \frac{h}{2}f_{n-1}$$

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})} \text{ (Proved.)}$$

6. Derive the fourth-order Adams - Moulton formula

$$y_{n+1} = y_n + \frac{h}{24}\left[9f_{n+1} + 19 f_n - 5f_{n-1} + f_{n-2}\right]$$

Sol: We try to derive the formula similar to the previous approach, but instead using Lagranges' interpolation over $f_{n+1}, f_n, f_{n-1}, f_{n-2}$ to approximate $f(x)$. The four points are uniformly spaced [step size $=h$]

$$f(x) \approx f_{n+1}\, l_{n+1}(x) + f_n l_n(x) + f_{n-1} l_{n-1}(x) + f_{n-2}\, l_{n-2}(x).$$

Here, $l_k(x)$ are Lagranges' basis polynomials.

- $$l_{n+1}(x) = \frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})}{(x_{n+1}-x_n)(x_{n+1}-x_{n-1})(x_{n+1}-x_{n-2})} = \frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})}{6h^3}$$
- $$l_n(x) = \frac{(x-x_{n+1})(x-x_{n-1})(x-x_{n-2})}{(x_n-x_{n+1})(x_n-x_{n-1})(x_n-x_{n-2})} = \frac{(x-x_{n+1})(x-x_{n-1})(x-x_{n-2})}{-2h^3}$$
- $$l_{n-1}(x) = \frac{(x-x_{n+1})(x-x_n)(x-x_{n-2})}{(x_{n-1}-x_{n+1})(x_{n-1}-x_n)(x_{n-1}-x_{n-2})} = \frac{(x-x_{n+1})(x-x_n)(x-x_{n-2})}{2h^3}$$
- $$l_{n-2}(x) = \frac{(x-x_{n+1})(x-x_n)(x-x_{n-1})}{(x_{n-2}-x_{n+1})(x_{n-2}-x_n)(x_{n-2}-x_{n-1})} = \frac{(x-x_{n+1})(x-x_n)(x-x_{n-1})}{-6h^3}$$

Now, $$\int_{x_n}^{x_{n+1}} f(x)\,dx = \int_{x_n}^{x_{n+1}} f_k\, l_k(x)\,dx \quad \text{for } k = n+1, n, n-1, n-2.$$

So, $$\int_{x_n}^{x_{n+1}} f_{n+1}\, l_{n+1}(x)\,dx = \frac{f_{n+1}}{6h^3}\int_{x_n}^{x_{n+1}} (x-x_n)(x-x_{n-1})(x-x_{n-2})\,dx$$

$$\Rightarrow \underset{(\text{let})}{I_1} = \frac{9}{24} h f_{n+1}$$

Similarly, $$\int_{x_n}^{x_{n+1}} f_n\, l_n(x)\,dx = \frac{f_n}{-2h^3}\int_{x_n}^{x_{n+1}} (x-x_{n+1})(x-x_{n-1})(x-x_{n-2})\,dx$$

$$\Rightarrow I_2 = \frac{19}{24} h f_n.$$

$$\int_{x_n}^{x_{n+1}} f_{n-1} l_{n-1}(x)\,dx = \frac{f_{n-1}}{2h^3}\int_{x_n}^{x_{n+1}} (x-x_{n+1})(x-x_n)(x-x_{n-2})\,dx$$

Let, $x-x_n = u$. $\therefore x-x_{n+1} = u-h$, $x-x_{n-2} = u+2h$

$$\therefore I_3 = \frac{f_{n-1}}{2h^3}\int_0^h (u-h)u(u+2h)\,du$$

$$= \frac{f_{n-1}}{2h^3}\int_0^h \left(u^3+2u^2h-u^2h-2uh^2\right)du \Rightarrow I_3 = f_{n-1}\left(\frac{-5}{24}h\right)$$

Finally, $$\int_{x_n}^{x_{n+1}} f_{n-2} l_{n-2}(x)\,dx = \frac{f_{n-2}}{-6h^3}\int_{x_n}^{x_{n+1}} (x-x_{n+1})(x-x_n)(x-x_{n-1})\,dx$$

$$\Rightarrow I_4 = \frac{f_{n-2}}{-6h^3}\int_0^h u^3-uh^2\,du \Rightarrow I_4 = \frac{f_{n-2}}{24}h.$$

So, $$\int_{x_n}^{x_{n+1}} f(x)\,dx = \frac{h}{24}\left[9f_{n+1}+19f_n-5f_{n-1}+f_{n-2}\right]$$

$$\therefore \boxed{y_{n+1} = y_n + \frac{h}{24}\left[9f_{n+1}+19f_n-5f_{n-1}+f_{n-2}\right]}$$ (Proved.)

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