Numerical methods (Assignment - 05) Proshmit Dasgupta

2) Prove the central difference relation

$$y_0'' = \frac{-y_2 + 16y_1 - 30y_0 + 16y_{-1} - y_{-2}}{12h^2} + O(h^4)$$

Try to calculate the explicit form of the error term $O(h^4)$.

Sol: We know that by Taylor's series expansion,

$$y_1 = y(x_0+h) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + \frac{h^5}{120}y_0^{(5)} + \frac{h^6}{720}y_0^{(6)} \quad \text{—(i)}$$

$$y_{-1} = y(x_0-h) = y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} - \frac{h^5}{120}y_0^{(5)} + \frac{h^6}{720}y_0^{(6)} \quad \text{—(ii)}$$

$$y_2 = y(x_0+2h) = y_0 + 2hy_0' + 2h^2y_0'' + \frac{4}{3}h^3y_0''' + \frac{2}{3}h^4y_0^{(4)} + \frac{4}{15}h^5y_0^{(5)} + \frac{4}{45}h^6y_0^{(6)} \quad \text{—(iii)}$$

$$y_{-2} = y(x_0-2h) = y_0 - 2hy_0' + 2h^2y_0'' - \frac{4}{3}h^3y_0''' + \frac{2}{3}h^4y_0^{(4)} - \frac{4}{15}h^5y_0^{(5)} + \frac{4}{45}h^6y_0^{(6)} \quad \text{—(iv)}$$

Proof: Adding (i), (ii) $\longrightarrow$ $2y_0 + h^2 y_0'' + \frac{h^4}{12} y_0^{(4)} + \frac{h^6}{360} y_0^{(6)}$ — (v) $= (y_1 + y_{-1})$

Adding (iii), (iv) $\longrightarrow$ $2y_0 + 4h^2 y_0'' + \frac{4}{3} h^4 y_0^{(4)} + \frac{8h^6}{45} y_0^{(6)}$ — (vi) $= (y_2 + y_{-2})$

$\therefore 16 \times (v) = 16y_1 + 16y_{-1} = 32y_0 + 16h^2 y_0'' + \frac{4}{3} h^4 y_0^{(4)} + \frac{2}{45} h^6 y_0^{(6)} + \cdots$ — (vii)

$$(vii) - (vi) = 16y_1 + 16y_{-1} - y_2 - y_{-2}$$

$$= \left[32y_0 + 16h^2 y_0'' + \frac{4}{3} h^4 y_0^{(4)} + \cdots\right] - \left(2y_0 + 4h^2 y_0'' + \frac{4}{3} h^4 y_0^{(4)} + \cdots\right)$$

$$\Rightarrow 16(y_1 + y_{-1}) - y_2 - y_{-2} = 30y_0 + 12h^2 y_0'' + - \frac{\cancel{6}\,2}{\cancel{45}\,15} h^6 y_0^{(6)}$$

$$\Rightarrow 12h^2 y_0'' = -y_2 + 16y_1 - 30y_0 + 16y_{-1} - y_{-2} + \frac{2}{15} h^6 y_0^{(6)}$$

$$\Rightarrow y_0'' = \frac{-y_2 + 16y_1 - 30y_0 + 16y_{-1} - y_{-2}}{12h^2} + \frac{2}{180} \frac{h^6}{h^2} y_0^{(6)}$$

$$\therefore y_0'' = \frac{-y_2 + 16y_1 - 30y_0 + 16y_{-1} - y_{-2}}{12h^2} + \underbrace{\frac{h^4}{90} y_0^{(6)}}_{O(h^4)} \quad [\text{Proved.}]$$

So, explicit form of the error term $\longrightarrow$ $\boxed{\frac{h^4}{90} y_0^{(6)} = O(h^4)}$.