Cantor Sets

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0.1 The Cantor Set and Cantor Function

[1]

This section discusses a little about Cantor sets and the Cantor functions by touching on a few topics in topology, measure theory countability and uncountability. It can be completely skipped as far as analysis is concerned.

0.1.1 Definition of the Cantor Set

Start with the set $[0, 1]$. Divide it into three sets of equal length. Remove the middle part. Repeat this procedure on the other two subintervals left. The set obtained after repeating this procedure infinitely many times is called as the Cantor set.

A few questions which immediately arise in our mind are:

- Is there anything left?
  Yes, at least the endpoints of the deleted middle-third subintervals. There are countably many such points.
- Are there any other points left?
  Yes, in some sense, a whole lot more. But in some other sense, just some "dust" - which in some ways is scattered, in some other ways it is bound together.

0.1.2 Ternary Representation of Cantor's Set

We can represent any real number in any base. We will use the ternary (base 3) representation because Cantor's set has a special representation in base 3.

$$\left(\frac{1}{3}\right)_3 = 1 \times 3^{-1} = 0.1000\ldots = 0.0222\ldots$$
$$\left(\frac{2}{3}\right)_3 = 2 \times 3^{-1} = 0.2000\ldots$$
$$\left(\frac{7}{9}\right)_3 = 1 \times 3^{-2} + 2 \times 3^{-1} = 0.2100\ldots = 0.2022\ldots$$
$$\left(\frac{8}{9}\right)_3 = 2 \times 3^{-2} + 2 \times 3^{-1} = 0.2200\ldots$$

Remark 0.1. *We observe that a number is in Cantor's set if and only if its ternary representation contains only the digits* 0 *and* 2 *(in other words, it has no ones). This ternary expansion is unique except if* x *is a rational number of the form* $\frac{p}{3^n}$ *for some integers* p, n *(these are called* ***triadic rationals****). For triadic rationals, there are two possible ternary expansions, a terminating one and a non-terminating one. For definiteness, for triadic rationals, we shall always take the non-terminating ternary expansion. With this preparation, the Cantor set is defined as the set of all* x *that do not have the digit one in their ternary expansion.*

Definition 0.1. ***(Cantor Set).*** *A set* $\mathcal{C}$ *called the cantor set is described as*

$$\mathcal{C} = \{x \in [0, 1] : x = 0.c_1c_2c_3\ldots c_n\ldots \textit{ in its terneary representation}$$
$$\textit{and for each positive natural } i,\ c_i \in \{0, 2\}\}$$

0.1.3 Properties of Cantor Sets

Cantor's Set is uncountable

Proposition 0.1. ***(Cantor Set is uncountable).*** *The cardinality of Cantor's set is the continuum. In other words, Cantor's set has the same cardinality as the interval* $[0, 1]$.

Proof. The function $f : \mathcal{C} \mapsto [0, 1]$ defined as $f(x) := x$ is an injection from $\mathcal{C}$ to $[0, 1]$. Define the function $g : [0, 1] \mapsto \mathcal{C}$ as $g(0.c_1c_2c_3\ldots) = 0.c'_1c'_2c'_3\ldots$ where for each i, $c'_i = c_i \bmod 3$. Then, g is an injection from $[0, 1]$ to $\mathcal{C}$. Apply the Cantor Schroder Bernstein theorem now to conclude that $\#\mathcal{C} = \#[0, 1]$. □

[1] Informal! This section can be skipped altogether without any loss of continuity.

Cantor's Set is negligible

Proposition 0.2. ***(Cantor's set is negligible).*** *Lebesgue measure of the Cantor Set is 0.*

Proof. At each step, the number of intervals doubles and their length decreases by 3. Therefore, the length of the Cantor Set is

$$1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = 0$$

□

Remark 0.2. *Cantor's set has no interior points, it is nowhere dense. In other words, it is just "dust". That's because its length is 0, so it contains no continuous parts (no intervals). Furthermore, Cantor's set is bounded because it is a subset of the set* $[0, 1]$.

Cantor's Set is a perfect set

Lemma 0.1. *Cantor's set is a closed and compact set.*

Proof. $\mathcal{C}$ is the complement relative to $[0, 1]$ of open intervals, the ones removed in its construction. Therefore, $\mathcal{C}$ is bounded as well as closed on the real line. Apply the Heine-Borel theorem now to conclude that $\mathcal{C}$ is a compact set. [2] □

Proposition 0.3. ***(Cantor's set has no isolated points).*** *In any neighbourhood of a point in Cantor's set, there is another point from Cantor's set.*

Proof. Since $\mathcal{C}$ is closed, it contains all of its limit points. Therefore, if x is an isolated point in $\mathcal{C}$, there must exist a neighbourhood $N(x)$ such that $N(x) \cap \mathcal{C} = \{x\}$. However, the Cantor set $\mathcal{C}$ is constructed by iteratively removing open intervals from the closed interval $[0, 1]$. Therefore, there are always points from $\mathcal{C}$ remaining. This property implies that no point in $\mathcal{C}$ can be isolated.

□

Remark 0.3. *In topology, a compact set that has no isolated points is called a* ***perfect set****.*

Proposition 0.4. ***(Cantor's set is totally disconnected).*** *Given any two elements, $a, b \in \mathcal{C}$, Cantor's set can be divided into two disjoint and closed neighbourhoods A and B, one containing a and the other containing b.*

Proof. Given say the numbers a and b from above differing at their ith position in ternary representation. Let a has 0 at its ith position, while b has 2 at its ith position.

$$\text{Neighbourhood } A := \text{all elements of } \mathcal{C} \text{ whose } ith \text{ digit is } 0.$$

$$\text{Neighbourhood } B := \text{all elements of } \mathcal{C} \text{ whose } ith \text{ digit is } 2.$$

This proves the claim that neighbourhoods of A and B are closed and disjoint containing a and b respectively. □

Cantor's Set is a Fractal

Proposition 0.5. *Cantor's set is self-similar. More accurately: magnify Cantor's set by* 3*, get* 2 *copies of itself.*

Proof. The proof is clear from the construction of the Cantor's set. □

[2] Morally compact means that every task that may theoretically require an infinite number of steps/infinite amount of data can be accomplished in a finite number of steps /with finite resources.

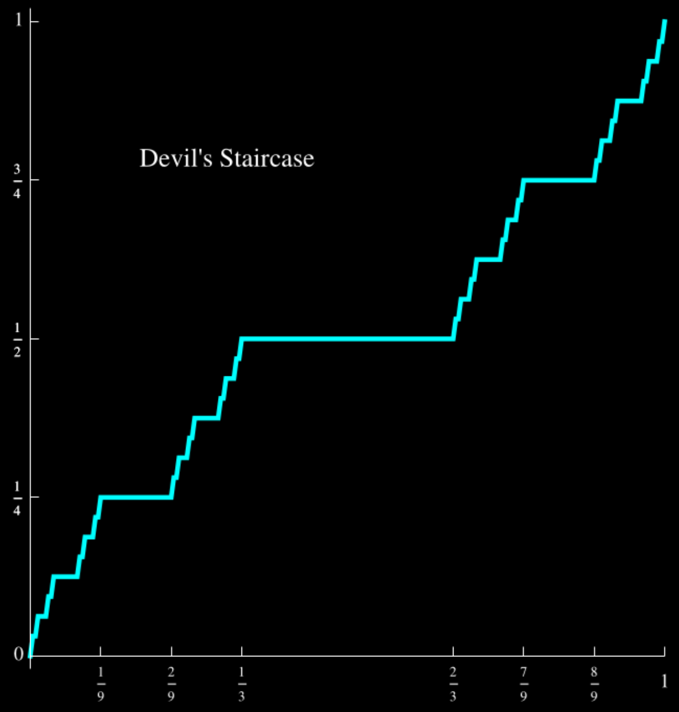

Figure 1:

0.1.4 The Cantor Function

The function $f : \mathcal{C} \to [0,1]$, defined as

$$f(0.c_1c_2\ldots) := 0.\frac{c_1}{2}\frac{c_1}{2}\ldots$$

has the following properties:

1. It is onto.
2. It is increasing.
3. It is not one-to-one. For instance:

$$f\left(\frac{1}{3}\right) = f\left(0.\overline{0222}_3\right) = 0.\overline{01111}_2 = 0.\overline{1}_2 = \frac{1}{2}$$

$$f\left(\frac{2}{3}\right) = f\left(0.\overline{2000}_3\right) = 0.\overline{10000}_2 = 0.\overline{1}_2 = \frac{1}{2}$$

Two inputs of f have the same outputs if and only if they are the endpoints of an interval removed - like $(\frac{1}{3},\frac{2}{3})$ or $(\frac{1}{9},\frac{2}{9})$ etc.

Extend f to the whole interval $[0,1]$ by making it constant on these removed intervals. The function obtained by this extension is called **Cantor's function**. The graph of the Cantor Function is shown in the figure 1. We observe the following properties of the Cantor's Function:

- Cantor's function is onto.
- Cantor's function is increasing, but constant almost everywhere (except on the "dust").
- Cantor's function is continuous.
- The derivative of Cantor's function is 0 almost everywhere.

[3]

[3] Cantor's function, also called the Devil's Staircase, makes a continuous finite ascent (from 0 to 1) in an infinite number

of steps (there are infinitely many intervals removed) while staying constant most of the time.

The following YouTube video click here resembles the musical illustration of this ascent (followed by a descent and by more playing along the staircase). It is called "L'Escalier du Diable" (i.e. "The Devil's Staircase"). It was composed in the 90's by the composer György Ligeti. This composition is harmonically self-similar, like Cantor's set. Its structure has the musical equivalent of dividing an interval into three parts, changing the middle, repeating the process and matching the pitch arrangements to this structure, just like with Cantor's set and function. But unlike its mathematical counterpart, the process is finite, its infinity being only suggested.