ALL IN ONE

MATHEMATICS CHEAT SHEET

V2.6

Euler's Identity:

$$e^{i\pi} + 1 = 0$$

CONTAINING FORMULAE FOR ELEMENTARY, HIGH SCHOOL AND UNIVERSITY MATHEMATICS

COMPILED FROM MANY SOURCES BY ALEX SPARTALIS
2009-2012

REVISION HISTORY

2.1. 08/06/2012
UPDATED: Format
NEW: Multivariable Calculus
UPDATED: Convergence tests
UPDATED: Composite Functions
2.2. 10/07/2012
NEW: Three Phase – Delta & Y
NEW: Electrical Power
2.3. 14/08/2012
NEW: Factorial
NEW: Electromagnetics
NEW: Linear Algebra
NEW: Mathematical Symbols
NEW: Algebraic Identities
NEW: Graph Theory
UPDATED: Linear Algebra
UPDATED: Linear Transformations
2.4. 31/08/2012
NEW: Graphical Functions
NEW: Prime numbers
NEW: Power Series Expansion
NEW: Inner Products
UPDATED: Pi Formulas
UPDATED: General Trigonometric Functions Expansion
UPDATED: Linear Algebra
UPDATED: Matrix Inverse
2.5. 10/09/2012
NEW: Machin-Like Formulae
NEW: Infinite Summations To Pi
NEW: Classical Mechanics
NEW: Relativistic Formulae
NEW: Statistical Distributions
NEW: Logarithm Power Series
NEW: Spherical Triangle Identities
NEW: Bernoulli Expansion
UPDATED: Pi Formulas
UPDATED: Logarithm Identities
UPDATED: Riemann Zeta Function
UPDATED: Eigenvalues and Eigenvectors
2.6. 3/10/2012
NEW: QR Factorisation
NEW: Jordan Forms
NEW: Macroeconomics
NEW: Golden Ratio & Fibonacci Sequence
NEW: Complex Vectors and Matrices
NEW: Numerical Computations for Matrices
UPDATED: Prime Numbers
UPDATED: Errors within Matrix Formula
2.7. 2012
TO DO: USV Decomposition

CONTENTS

PART 1: PHYSICAL CONSTANTS

1.1 SI PREFIXES:

Prefix	Symbol	1000^m	10^n	Decimal	Scale
yotta	Y	1000^8	10^{24}	1000000000000000000000000	Septillion
zetta	Z	1000^7	10^{21}	1000000000000000000000	Sextillion
exa	E	1000^6	10^{18}	1000000000000000000	Quintillion
peta	P	1000^5	10^{15}	1000000000000000	Quadrillion
tera	T	1000^4	10^{12}	1000000000000	Trillion
giga	G	1000^3	10^9	1000000000	Billion
mega	M	1000^2	10^6	1000000	Million
kilo	k	1000^1	10^3	1000	Thousand
hecto	h	$1000^{2⁄3}$	10^2	100	Hundred
deca	da	$1000^{1⁄3}$	10^1	10	Ten
		1000^0	10^0	1	One
deci	d	$1000^{-1⁄3}$	10^{-1}	0.1	Tenth
centi	c	$1000^{-2⁄3}$	10^{-2}	0.01	Hundredth
milli	m	1000^{-1}	10^{-3}	0.001	Thousandth
micro	µ	1000^{-2}	10^{-6}	0.000001	Millionth
nano	n	1000^{-3}	10^{-9}	0.000000001	Billionth
pico	p	1000^{-4}	10^{-12}	0.000000000001	Trillionth
femto	f	1000^{-5}	10^{-15}	0.000000000000001	Quadrillionth
atto	a	1000^{-6}	10^{-18}	0.000000000000000001	Quintillionth
zepto	z	1000^{-7}	10^{-21}	0.000000000000000000001	Sextillionth
yocto	y	1000^{-8}	10^{-24}	0.000000000000000000000001	Septillionth

1.2 SI BASE UNITS:

Quantity	Unit	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

1.3 SI DERIVED UNITS:

Quantity	Unit	Symbol	Expression in terms of other SI units
angle, plane	radian*	rad	m/m = 1
angle, solid	steradian*	sr	$m^2/m^2 = 1$
Celsius temperature	degree Celsius	°C	K
electric capacitance	farad	F	C/V
electric charge, quantity of electricity	coulomb	C	A·s
electric conductance	siemens	S	A/V
electric inductance	henry	H	Wb/A
electric potential difference, electromotive force	volt	V	W/A
electric resistance	ohm	Ω	V/A
energy, work, quantity of heat	joule	J	N·m
force	newton	N	$kg\cdot m/s^2$
frequency (of a periodic phenomenon)	hertz	Hz	1/s
illuminance	lux	lx	lm/m^2
luminous flux	lumen	lm	cd·sr
magnetic flux	weber	Wb	V·s
magnetic flux density	tesla	T	Wb/m^2
power, radiant flux	watt	W	J/s
pressure, stress	pascal	Pa	N/m^2
activity (referred to a radionuclide)	becquerel	Bq	1/s
absorbed dose, specific energy imparted, kerma	gray	Gy	J/kg
dose equivalent, ambient dose equivalent, directional dose equivalent, personal dose equivalent, organ dose equivalent	sievert	Sv	J/kg
catalytic activity	katal	kat	mol/s

1.4 UNIVERSAL CONSTANTS:

Quantity	Symbol	Value	Relative Standard Uncertainty
speed of light in vacuum	c	299 792 458 $\mathrm{m \cdot s^{-1}}$	defined
Newtonian constant of gravitation	G	$6.67428(67) \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$	1.0×10^{-4}
Planck constant	h	$6.626\ 068\ 96(33) \times 10^{-34}\ \mathrm{J \cdot s}$	5.0×10^{-8}
reduced Planck constant	$\hbar = h/(2\pi)$	$1.054\ 571\ 628(53) \times 10^{-34}\ \mathrm{J \cdot s}$	5.0×10^{-8}

1.5 ELECTROMAGNETIC CONSTANTS:

Quantity	Symbol	Value (SI units)	Relative Standard Uncertainty
magnetic constant (vacuum permeability)	μ_0	$4\pi \times 10^{-7}\ \mathrm{N \cdot A^{-2}} = 1.256\ 637\ 061... \times 10^{-6}\ \mathrm{N \cdot A^{-2}}$	defined
electric constant (vacuum permittivity)	$\varepsilon_0 = 1/(\mu_0 c^2)$	$8.854\ 187\ 817... \times 10^{-12}\ \mathrm{F \cdot m^{-1}}$	defined
characteristic impedance of vacuum	$Z_0 = \mu_0 c$	376.730 313 461... Ω	defined
Coulomb's constant	$k_e = 1/4\pi\epsilon_0$	$8.987\ 551\ 787... \times 10^{9}\ \mathrm{N \cdot m^2 \cdot C^{-2}}$	defined
elementary charge	e	$1.602\ 176\ 487(40) \times 10^{-19}\ \mathrm{C}$	2.5×10^{-8}
Bohr magneton	$\mu_B = e\hbar/2m_e$	$927.400\ 915(23) \times 10^{-26}\ \mathrm{J \cdot T^{-1}}$	2.5×10^{-8}
conductance quantum	$G_0 = 2e^2/h$	$7.748\ 091\ 7004(53) \times 10^{-5}\ \mathrm{S}$	6.8×10^{-10}
inverse conductance quantum	$G_0^{-1} = h/2e^2$	12 906.403 7787(88) Ω	6.8×10^{-10}
Josephson constant	$K_J = 2e/h$	$4.835\ 978\ 91(12) \times 10^{14}\ \mathrm{Hz \cdot V^{-1}}$	2.5×10^{-8}
magnetic flux quantum	$\phi_0 = h/2e$	$2.067\ 833\ 667(52) \times 10^{-15}\ \mathrm{Wb}$	2.5×10^{-8}
nuclear magneton	$\mu_N = e\hbar/2m_p$	$5.050\ 783\ 43(43) \times 10^{-27}\ \mathrm{J \cdot T^{-1}}$	8.6×10^{-8}
von Klitzing constant	$R_K = h/e^2$	25 812.807 557(18) Ω	6.8×10^{-10}

1.6 ATOMIC AND NUCLEAR CONSTANTS:

Quantity	Symbol	Value (SI units)	Relative Standard Uncertainty
Bohr radius	$a_0 = \alpha/4\pi R_\infty$	5.291 772 108(18) × 10^{-11} m	3.3 × 10^{-9}
classical electron radius	$r_e = e^2/4\pi\epsilon_0 m_e c^2$	2.817 940 2894(58) × 10^{-15} m	2.1 × 10^{-9}
electron mass	m_e	9.109 382 15(45) × 10^{-31} kg	5.0 × 10^{-8}
Fermi coupling constant	$G_F/(\hbar c)^3$	1.166 39(1) × 10^{-5} GeV^{-2}	8.6 × 10^{-6}
fine-structure constant	$\alpha = \mu_0 e^2 c/(2h) = e^2/(4\pi\epsilon_0 \hbar c)$	7.297 352 537 6(50) × 10^{-3}	6.8 × 10^{-10}
Hartree energy	$E_h = 2R_\infty hc$	4.359 744 17(75) × 10^{-18} J	1.7 × 10^{-7}
proton mass	m_p	1.672 621 637(83) × 10^{-27} kg	5.0 × 10^{-8}
quantum of circulation	$h/2m_e$	3.636 947 550(24) × 10^{-4} m^2 s^{-1}	6.7 × 10^{-9}
Rydberg constant	$R_\infty = \alpha^2 m_e c/2h$	10 973 731.568 525(73) m^{-1}	6.6 × 10^{-12}
Thomson cross section	$(8\pi/3)r_e^2$	6.652 458 73(13) × 10^{-29} m^2	2.0 × 10^{-8}
weak mixing angle	$\sin^2\theta_W = 1 - (m_W/m_Z)^2$	0.222 15(76)	3.4 × 10^{-3}

1.7 PHYSICO-CHEMICAL CONSTANTS:

Quantity	Symbol	Value (SI units)	Relative Standard Uncertainty
atomic mass unit (unified atomic mass unit)	$m_u = 1\ u$	1.660 538 86(28) × 10^{-27} kg	1.7 × 10^{-7}
Avogadro's number	N_A, L	6.022 141 5(10) × 10^{23} mol^{-1}	1.7 × 10^{-7}
Boltzmann constant	$k = k_B = R/N_A$	1.380 6504(24) ×	1.8 × 10^{-6}

			10^{-23} J·K^{-1}	
Faraday constant		$F = N_A e$	96 485.3383(83)C·mol^{-1}	8.6×10^{-8}
first radiation constant		$c_1 = 2\pi hc^2$	3.741 771 18(19) × 10^{-16} W·m²	5.0×10^{-8}
	for spectral radiance	c_{1L}	1.191 042 82(20) × 10^{-16} W·m² sr^{-1}	1.7×10^{-7}
Loschmidt constant	at T=273.15 K and p=101.325 kPa	$n_0 = N_A/V_m$	2.686 777 3(47) × 10^{25} m^{-3}	1.8×10^{-6}
gas constant		R	8.314 472(15) J·K^{-1}·mol^{-1}	1.7×10^{-6}
molar Planck constant		$N_A h$	3.990 312 716(27) × 10^{-10} J·s·mol^{-1}	6.7×10^{-9}
molar volume of an ideal gas	at T=273.15 K and p=100 kPa	$V_m = RT/p$	2.2710 981(40) × 10^{-2} m³·mol^{-1}	1.7×10^{-6}
	at T=273.15 K and p=101.325 kPa		2.2413 996(39) × 10^{-2} m³·mol^{-1}	1.7×10^{-6}
Sackur-Tetrode constant	at T=1 K and p=100 kPa	$S_0/R = \frac{5}{2} + \ln\left[(2\pi m_u kT/h^2)^{3/2} kT/p\right]$	−1.151 704 7(44)	3.8×10^{-6}
	at T=1 K and p=101.325 kPa		−1.164 867 7(44)	3.8×10^{-6}
second radiation constant		$c_2 = hc/k$	1.438 775 2(25) × 10^{-2} m·K	1.7×10^{-6}
Stefan–Boltzmann constant		$\sigma = (\pi^2/60)k^4/\hbar^3 c^2$	5.670 400(40) × 10^{-8} W·m^{-2}·K^{-4}	7.0×10^{-6}
Wien displacement law constant		$b = (hc/k)/$4.965 114 231...	2.897 768 5(51) × 10^{-3} m·K	1.7×10^{-6}

1.8 ADOPTED VALUES:

Quantity	Symbol	Value (SI units)	Relative Standard Uncertainty
conventional value of	K_{J-90}	4.835 979 ×	defined

Josephson constant		10^{14} Hz·V^{-1}	
conventional value of von Klitzing constant	R_{K-90}	25 812.807 Ω	defined
molar mass constant	$M_u = M(^{12}\mathrm{C})/12$	1×10^{-3} kg·mol^{-1}	defined
molar mass of carbon-12	$M(^{12}\mathrm{C}) = N_A m(^{12}\mathrm{C})$	1.2×10^{-2} kg·mol^{-1}	defined
standard acceleration of gravity (gee, free-fall on Earth)	g_n	9.806 65 m·s^{-2}	defined
standard atmosphere	atm	101 325 Pa	defined

1.9 NATURAL UNITS:

Name	Dimension	Expression	Value (SI units)
Planck length	Length (L)	$l_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}}$	$1.616\,252(81) \times 10^{-35}$ m
Planck mass	Mass (M)	$m_\mathrm{P} = \sqrt{\frac{\hbar c}{G}}$	$2.176\,44(11) \times 10^{-8}$ kg
Planck time	Time (T)	$t_\mathrm{P} = \sqrt{\frac{\hbar G}{c^5}}$	$5.391\,24(27) \times 10^{-44}$ s
Planck charge	Electric charge (Q)	$q_\mathrm{P} = \sqrt{4\pi\varepsilon_0 \hbar c}$	$1.875\,545\,870(47) \times 10^{-18}$ C
Planck temperature	Temperature (Θ)	$T_\mathrm{P} = \sqrt{\frac{\hbar c^5}{G k^2}}$	$1.416\,785(71) \times 10^{32}$ K

PART 2: MATHEMTAICAL SYMBOLS

2.1 BASIC MATH SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
=	equals sign	equality	5 = 2+3
≠	not equal sign	inequality	5 ≠ 4
>	strict inequality	greater than	5 > 4
<	strict inequality	less than	4 < 5
≥	inequality	greater than or equal to	5 ≥ 4
≤	inequality	less than or equal to	4 ≤ 5
( )	parentheses	calculate expression inside first	2 × (3+5) = 16
[ ]	brackets	calculate expression inside first	[(1+2)*(1+5)] = 18
+	plus sign	addition	1 + 1 = 2
−	minus sign	subtraction	2 − 1 = 1
±	plus - minus	both plus and minus operations	3 ± 5 = 8 and -2
∓	minus - plus	both minus and plus operations	3 ∓ 5 = -2 and 8
*	asterisk	multiplication	2 * 3 = 6
×	times sign	multiplication	2 × 3 = 6
·	multiplication dot	multiplication	2 · 3 = 6
÷	division sign / obelus	division	6 ÷ 2 = 3
/	division slash	division	6 / 2 = 3
–	horizontal line	division / fraction	$\frac{6}{2} = 3$
mod	modulo	remainder calculation	7 mod 2 = 1
.	period	decimal point, decimal separator	2.56 = 2+56/100
a^b	power	exponent	$2^3 = 8$
a^*b*	caret	exponent	2 ^ 3 = 8
$\sqrt{a}$	square root	$\sqrt{a} \cdot \sqrt{a} = a$	$\sqrt{9} = \pm 3$
$\sqrt[3]{a}$	cube root		$\sqrt[3]{8} = 2$
$\sqrt[4]{a}$	forth root		$\sqrt[4]{16} = \pm 2$
$\sqrt[n]{a}$	n-th root (radical)		for *n*=3, $\sqrt[n]{8} = 2$
%	percent	1% = 1/100	10% × 30 = 3
‰	per-mille	1‰ = 1/1000 = 0.1%	10‰ × 30 = 0.3
ppm	per-million	1ppm = 1/1000000	10ppm × 30 = 0.0003
ppb	per-billion	1ppb = 1/1000000000	10ppb × 30 = 3×10^{-7}
ppt	per-trillion	1ppb = 10^{-12}	10ppb × 30 = 3×10^{-10}

2.2 GEOMETRY SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
∠	angle	formed by two rays	∠ABC = 30º
∡	measured angle		∡ABC = 30º
∢	spherical angle		∢AOB = 30º
∟	right angle	= 90º	α = 90º
º	degree	1 turn = 360º	α = 60º
´	arcminute	1º = 60´	α = 60º59'

′′	arcsecond	1′ = 60′′	α = 60º59'59"
AB	line	line from point A to point B	
$\overrightarrow{AB}$	ray	line that start from point A	
⊥	perpendicular	perpendicular lines (90º angle)	AC ⊥ BC
||	parallel	parallel lines	AB || CD
≅	congruent to	equivalence of geometric shapes and size	ΔABC ≅ ΔXYZ
~	similarity	same shapes, not same size	ΔABC ~ ΔXYZ
Δ	triangle	triangle shape	ΔABC ≅ ΔBCD
$\|x-y\|$	distance	distance between points x and y	$\|x-y\| = 5$
π	pi constant	$\pi = 3.141592654...$ is the ratio between the circumference and diameter of a circle	$c = \pi \cdot d = 2 \cdot \pi \cdot r$
rad	radians	radians angle unit	360º = 2π rad
grad	grads	grads angle unit	360º = 400 grad

2.3 ALGEBRA SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
x	x variable	unknown value to find	when $2x = 4$, then $x = 2$
≡	equivalence	identical to	
≜	equal by definition	equal by definition	
:=	equal by definition	equal by definition	
~	approximately equal	weak approximation	11 ~ 10
≈	approximately equal	approximation	$sin(0.01) \approx 0.01$
∝	proportional to	proportional to	$f(x) \propto g(x)$
∞	lemniscate	infinity symbol	
≪	much less than	much less than	1 ≪ 1000000
≫	much greater than	much greater than	1000000 ≫ 1
( )	parentheses	calculate expression inside first	2 * (3+5) = 16
[ ]	brackets	calculate expression inside first	[(1+2)*(1+5)] = 18
{ }	braces	set	
$\lfloor x \rfloor$	floor brackets	rounds number to lower integer	$\lfloor 4.3 \rfloor = 4$
$\lceil x \rceil$	ceiling brackets	rounds number to upper integer	$\lceil 4.3 \rceil = 5$
$x!$	exclamation mark	factorial	4! = 1*2*3*4 = 24
$\|x\|$	single vertical bar	absolute value	$\|-5\| = 5$
$f(x)$	function of x	maps values of x to f(x)	$f(x) = 3x+5$
$(f \circ g)$	function composition	$(f \circ g)(x) = f(g(x))$	$f(x)=3x$, $g(x)=x-1 \Rightarrow (f \circ g)(x)=3(x-1)$
(a,b)	open interval	$(a,b) \triangleq \{x \mid a < x < b\}$	$x \in (2,6)$
$[a,b]$	closed interval	$[a,b] \triangleq \{x \mid a \le x \le b\}$	$x \in [2,6]$
Δ	delta	change / difference	$\Delta t = t_1 - t_0$
Δ	discriminant	$\Delta = b^2 - 4ac$	
∑	sigma	summation - sum of all values in range of series	$\sum x_i = x_1+x_2+...+x_n$

∑∑	sigma	double summation	$\sum_{j=1}^{2}\sum_{i=1}^{8} x_{i,j} = \sum_{i=1}^{8} x_{i,1} + \sum_{i=1}^{8} x_{i,2}$
∏	capital pi	product - product of all values in range of series	$\prod x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_n$
e	e constant / Euler's number	e = 2.718281828...	$e = \lim (1+1/x)^x$, $x \to \infty$
γ	Euler-Mascheroni constant	γ = 0.527721566...	
φ	golden ratio	golden ratio constant	

2.4 LINEAR ALGEBRA SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
·	dot	scalar product	$a \cdot b$
×	cross	vector product	$a \times b$
$A \otimes B$	tensor product	tensor product of A and B	$A \otimes B$
$\langle x, y \rangle$	inner product		
[ ]	brackets	matrix of numbers	
( )	parentheses	matrix of numbers	
$\| A \|$	determinant	determinant of matrix A	
det(A)	determinant	determinant of matrix A	
$\|\| x \|\|$	double vertical bars	norm	
A^{T}	transpose	matrix transpose	$(A^{T})_{ij} = (A)_{ji}$
$A^{\dagger}$	Hermitian matrix	matrix conjugate transpose	$(A^{\dagger})_{ij} = (A)_{ji}$
A^{*}	Hermitian matrix	matrix conjugate transpose	$(A^{*})_{ij} = (A)_{ji}$
A^{-1}	inverse matrix	$A A^{-1} = I$	
rank(A)	matrix rank	rank of matrix A	rank(A) = 3
dim(U)	dimension	dimension of matrix A	rank(U) = 3

2.5 PROBABILITY AND STATISTICS SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
$P(A)$	probability function	probability of event A	$P(A) = 0.5$
$P(A \cap B)$	probability of events intersection	probability that of events A and B	$P(A \cap B) = 0.5$
$P(A \cup B)$	probability of events union	probability that of events A or B	$P(A \cup B) = 0.5$
$P(A \| B)$	conditional probability function	probability of event A given event B occured	$P(A \| B) = 0.3$
$f(x)$	probability density function (pdf)	$P(a \le x \le b) = \int f(x)\, dx$	
$F(x)$	cumulative distribution function (cdf)	$F(x) = P(X \le x)$	
μ	population mean	mean of population values	$\mu = 10$
$E(X)$	expectation value	expected value of random variable X	$E(X) = 10$
$E(X \| Y)$	conditional expectation	expected value of random variable X given Y	$E(X \| Y=2) = 5$

$var(X)$	variance	variance of random variable X	$var(X) = 4$
σ^2	variance	variance of population values	$\sigma^2 = 4$
$std(X)$	standard deviation	standard deviation of random variable X	$std(X) = 2$
σ_X	standard deviation	standard deviation value of random variable X	$\sigma_X = 2$
$\tilde{x}$	median	middle value of random variable x	$\tilde{x} = 5$
$cov(X,Y)$	covariance	covariance of random variables X and Y	$cov(X,Y) = 4$
$corr(X,Y)$	correlation	correlation of random variables X and Y	$corr(X,Y) = 3$
$\rho_{X,Y}$	correlation	correlation of random variables X and Y	$\rho_{X,Y} = 3$
$\sum$	summation	summation - sum of all values in range of series	$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4$
$\sum\sum$	double summation	double summation	$\sum_{j=1}^{2} \sum_{i=1}^{8} x_{i,j} = \sum_{i=1}^{8} x_{i,1} + \sum_{i=1}^{8} x_{i,2}$
Mo	mode	value that occurs most frequently in population	
MR	mid-range	$MR = (x_{max}+x_{min})/2$	
Md	sample median	half the population is below this value	
Q_1	lower / first quartile	25% of population are below this value	
Q_2	median / second quartile	50% of population are below this value = median of samples	
Q_3	upper / third quartile	75% of population are below this value	
x	sample mean	average / arithmetic mean	$x = (2+5+9) / 3 = 5.333$
s^2	sample variance	population samples variance estimator	$s^2 = 4$
s	sample standard deviation	population samples standard deviation estimator	$s = 2$
z_x	standard score	$z_x = (x-x) / s_x$	
$X \sim$	distribution of X	distribution of random variable X	$X \sim N(0,3)$
$N(\mu,\sigma^2)$	normal distribution	gaussian distribution	$X \sim N(0,3)$
$U(a,b)$	uniform distribution	equal probability in range a,b	$X \sim U(0,3)$
$exp(\lambda)$	exponential distribution	$f(x) = \lambda e^{-\lambda x}$, $x \geq 0$	
$gamma(c, \lambda)$	gamma distribution	$f(x) = \lambda\, c\, x^{c-1} e^{-\lambda x} / \Gamma(c)$, $x \geq 0$	
$\chi^2(k)$	chi-square distribution	$f(x) = x^{k/2-1} e^{-x/2} / (2^{k/2}\, \Gamma(k/2))$	
$F(k_1, k_2)$	F distribution		
$Bin(n,p)$	binomial distribution	$f(k) = {}_nC_k\, p^k (1-p)^{n-k}$	
$Poisson(\lambda)$	Poisson distribution	$f(k) = \lambda^k e^{-\lambda} / k!$	
$Geom(p)$	geometric distribution	$f(k) = p\,(1-p)^{k}$	
$HG(N,K,n)$	hyper-geometric distribution		
$Bern(p)$	Bernoulli distribution		

2.6 COMBINATORICS SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
$n!$	factorial	$n! = 1\cdot2\cdot3\cdot\ldots n$	$5! = 1\cdot2\cdot3\cdot4\cdot5 = 120$
${}_nP_k$	permutation	${}_nP_k = \dfrac{n!}{(n-k)!}$	${}_5P_3 = 5! / (5\text{-}3)! = 60$
${}_nC_k$ $\binom{n}{k}$	combination	${}_nC_k = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$	${}_5C_3 = 5!/[3!(5\text{-}3)!]=10$

2.7 SET THEORY SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
{ }	set	a collection of elements	A={3,7,9,14}, B={9,14,28}
$A \cap B$	intersection	objects that belong to set A and set B	$A \cap B = \{9,14\}$
$A \cup B$	union	objects that belong to set A or set B	$A \cup B = \{3,7,9,14,28\}$
$A \subseteq B$	subset	subset has less elements or equal to the set	$\{9,14,28\} \subseteq \{9,14,28\}$
$A \subset B$	proper subset / strict subset	subset has less elements than the set	$\{9,14\} \subset \{9,14,28\}$
$A \not\subset B$	not subset	left set not a subset of right set	$\{9,66\} \not\subset \{9,14,28\}$
$A \supseteq B$	superset	set A has more elements or equal to the set B	$\{9,14,28\} \supseteq \{9,14,28\}$
$A \supset B$	proper superset / strict superset	set A has more elements than set B	$\{9,14,28\} \supset \{9,14\}$
$A \not\supset B$	not superset	set A is not a superset of set B	$\{9,14,28\} \not\supset \{9,66\}$
2^A	power set	all subsets of A	
$\mathcal{P}(A)$	power set	all subsets of A	
A = B	equality	both sets have the same members	A={3,9,14}, B={3,9,14}, A=B
A^c	complement	all the objects that do not belong to set A	
A \ B	relative complement	objects that belong to A and not to B	A={3,9,14}, B={1,2,3}, A-B={9,14}
A - B	relative complement	objects that belong to A and not to B	A={3,9,14}, B={1,2,3}, A-B={9,14}
$A \Delta B$	symmetric difference	objects that belong to A or B but not to their intersection	A={3,9,14}, B={1,2,3}, $A \Delta B$={1,2,9,14}
$A \ominus B$	symmetric difference	objects that belong to A or B but not to their intersection	A={3,9,14}, B={1,2,3}, $A \ominus B$={1,2,9,14}
$a \in A$	element of	set membership	A={3,9,14}, $3 \in A$
$x \notin A$	not element of	no set membership	A={3,9,14}, $1 \notin A$
(a,b)	ordered pair	collection of 2 elements	
$A \times B$	cartesian product	set of all ordered pairs from A and B	
|A|	cardinality	the number of elements of set A	A={3,9,14}, |A|=3
#A	cardinality	the number of elements of set A	A={3,9,14}, #A=3

ℵ	aleph	infinite cardinality	
Ø	empty set	Ø = { }	C = {Ø}
U	universal set	set of all possible values	
$\mathbb{N}_0$	natural numbers set (with zero)	$\mathbb{N}_0 = \{0,1,2,3,4,...\}$	$0 \in \mathbb{N}_0$
$\mathbb{N}_1$	natural numbers set (without zero)	$\mathbb{N}_1 = \{1,2,3,4,5,...\}$	$6 \in \mathbb{N}_1$
$\mathbb{Z}$	integer numbers set	$\mathbb{Z} = \{...-3,-2,-1,0,1,2,3,...\}$	$-6 \in \mathbb{Z}$
$\mathbb{Q}$	rational numbers set	$\mathbb{Q} = \{x \mid x=a/b,\ a,b\in\mathbb{N}\}$	$2/6 \in \mathbb{Q}$
$\mathbb{R}$	real numbers set	$\mathbb{R} = \{x \mid -\infty < x < \infty\}$	$6.343434 \in \mathbb{R}$
$\mathbb{C}$	complex numbers set	$\mathbb{C} = \{z \mid z=a+bi,\ -\infty<a<\infty,\ -\infty<b<\infty\}$	$6+2i \in \mathbb{C}$

2.8 LOGIC SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
·	and	and	$x \cdot y$
^	caret / circumflex	and	x ^ y
&	ampersand	and	x & y
+	plus	or	$x + y$
∨	reversed caret	or	$x \vee y$
|	vertical line	or	$x \mid y$
x'	single quote	not - negation	x'
x	bar	not - negation	x
¬	not	not - negation	$\neg x$
!	exclamation mark	not - negation	$!\ x$
⊕	circled plus / oplus	exclusive or - xor	$x \oplus y$
~	tilde	negation	$\sim x$
⇒	implies		
⇔	equivalent	if and only if	
∀	for all		
∃	there exists		
∄	there does not exists		
∴	therefore		
∵	because / since		

2.9 CALCULUS & ANALYSIS SYMBOLS

Symbol	Symbol Name	Meaning / definition	Example
$\lim_{x \to x0} f(x)$	limit	limit value of a function	
ε	epsilon	represents a very small number, near zero	$\varepsilon \to 0$
e	e constant / Euler's number	e = 2.718281828...	$e = \lim (1+1/x)^x$, $x \to \infty$
y'	derivative	derivative - Leibniz's notation	$(3x^3)' = 9x^2$
y''	second derivative	derivative of derivative	$(3x^3)'' = 18x$
$y^{(n)}$	nth derivative	n times derivation	$(3x^3)^{(3)} = 18$
$\frac{dy}{dx}$	derivative	derivative - Lagrange's notation	$d(3x^3)/dx = 9x^2$
$\frac{d^2y}{dx^2}$	second derivative	derivative of derivative	$d^2(3x^3)/dx^2 = 18x$
$\frac{d^ny}{dx^n}$	nth derivative	n times derivation	
$\dot{y}$	time derivative	derivative by time - Newton notation	
$\ddot{y}$	time second derivative	derivative of derivative	
$\frac{\partial f(x,y)}{\partial x}$	partial derivative		$\partial(x^2+y^2)/\partial x = 2x$
$\int$	integral	opposite to derivation	
$\iint$	double integral	integration of function of 2 variables	
$\iiint$	triple integral	integration of function of 3 variables	
$\oint$	closed contour / line integral		
∯	closed surface integral		
∰	closed volume integral		
$[a,b]$	closed interval	$[a,b] = \{x \mid a \le x \le b\}$	
(a,b)	open interval	$(a,b) = \{x \mid a < x < b\}$	
i	imaginary unit	$i \equiv \sqrt{-1}$	$z = 3 + 2i$
z^*	complex conjugate	$z = a+bi \to z^*=a-bi$	$z^* = 3 + 2i$
z	complex conjugate	$z = a+bi \to z = a-bi$	$z = 3 + 2i$
∇	nabla / del	gradient / divergence operator	$\nabla f(x,y,z)$
$\overrightarrow{x}$	vector		
$\widehat{x}$	unit vector		
$x * y$	convolution	$y(t) = x(t) * h(t)$	
$\mathcal{L}$	Laplace transform	$F(s) = \mathcal{L}\{f(t)\}$	
$\mathcal{F}$	Fourier transform	$X(\omega) = \mathcal{F}\{f(t)\}$	
δ	delta function		

PART 3: AREA, VOLUME AND SURFACE AREA

3.1 AREA

Triangle: $A = \frac{1}{2}bh = \frac{1}{2}ab\sin C = \frac{a^2 \sin B \sin C}{2\sin A} = \sqrt{s(s-a)(s-b)(s-c)}$

Rectangle: $A = lw$

Square: $A = a^2$

Parallelogram: $A = bh = ab\sin A$

Rhombus: $A = a^2 \sin A$

Trapezium: $A = h\left(\frac{a+b}{s}\right)$

Quadrilateral: $A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \times \cos^2\left(\frac{\angle AB + \angle CD}{2}\right)}$

$A = \frac{d_1 d_2 \sin I}{2}$

Rectangle with rounded corners: $A = lw - r^2(4-\pi)$

Regular Hexagon: $A = \frac{3\sqrt{3} \times a^2}{2}$

Regular Octagon: $A = 2\left(1+\sqrt{2}\right) \times a^2$

Regular Polygon: $A = \frac{na^2}{4\tan\left(\frac{180}{n}\right)}$

3.2 VOLUME

Cube: $V = a^3$

Cuboid: $V = abc$

Pyramid: $V = \frac{1}{3} \times A(b) \times h$

Tetrahedron: $V = \frac{\sqrt{2}}{12} \times a^3$

Octahedron: $V = \frac{\sqrt{2}}{3} \times a^3$

Dodecahedron: $V = \frac{15+7\sqrt{5}}{4} \times a^3$

Icosahedron: $V = \frac{5\left(3+\sqrt{5}\right)}{12} \times a^3$

3.3 SURFACE AREA:

Cube: $SA = 6a^2$

Cuboids: $SA = 2(ab + bc + ca)$

Tetrahedron: $SA = \sqrt{3} \times a^2$

Octahedron: $SA = 2\times\sqrt{3}\times a^2$

Dodecahedron: $SA = 3\times\sqrt{25+10\sqrt{5}}\times a^2$

Icosahedron: $SA = 5\times\sqrt{3}\times a^2$

Cylinder: $SA = 2\pi r(h+r)$

3.4 MISELANIOUS

Diagonal of a Rectangle $d = \sqrt{l^2 + w^2}$

Diagonal of a Cuboid $d = \sqrt{a^2 + b^2 + c^2}$

Longest Diagonal (Even Sides) $= \dfrac{a}{\sin\left(\dfrac{180}{n}\right)}$

Longest Diagonal (Odd Sides) $= \dfrac{a}{2\sin\left(\dfrac{90}{n}\right)}$

Total Length of Edges (Cube): $= 12a$

Total Length of Edges (Cuboid): $= 4(a+b+c)$

Circumference $C = 2\pi r = \pi d$

Perimeter of rectangle $P = 2(a+b)$

Semi perimeter $s = \dfrac{P}{2}$

Euler's Formula $Faces + Verticies = Edges + 2$

3.5 ABBREVIATIONS (3.1, 3.2, 3.3, 3.4)

A=area
a=side 'a'
b=base
b=side 'b'
C=circumference
C=central angle
c=side 'c'
d=diameter
d=diagonal
d_1=diagonal 1
d_2=diagonal 2
E=external angle
h=height
I=internal angle
l=length
n=number of sides
P=perimeter
r=radius
r_1=radius 1

s=semi-perimeter
SA=Surface Area
V=Volume
w=width

PART 4: ALGEBRA

4.1 POLYNOMIAL FORMULA:

Qudaratic: Where $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic: Where $ax^3 + bx^2 + cx + d = 0$,

Let, $x = y - \frac{b}{3a}$

$$\therefore a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$$

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0$$

$$y^3 + \frac{\left(c - \frac{b^2}{3a}\right)}{a}y + \frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a} = 0$$

$$y^3 + \frac{\left(c - \frac{b^2}{3a}\right)}{a}y = -\frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a}$$

Let, $A = \frac{\left(c - \frac{b^2}{3a}\right)}{a} = 3st \ldots (1)$

Let, $B = -\frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a} = s^3 - t^3 \ldots (2)$

$$\therefore y^3 + Ay = B$$

$$y^3 + 3sty = s^3 - t^3$$

Solution to the equation = $s - t$

Let, $y = s - t$

$$\therefore (s-t)^3 + 3st(s-t) = s^3 - t^3$$

$$\left(s^3 - 3s^2t + 3st^2 - t^3\right) + \left(3s^2t - 3st^2\right) = s^3 - t^3$$

Solving (1) for *s* and substituting into (2) yields:

$$\left(\frac{A}{3t}\right)^3 - t^3 = B.$$

$$t^6 + Bt^3 - \frac{A^3}{27} = 0,$$

Let, $u = t^3$

$$\therefore u^2 + Bu - \frac{A^3}{27} = 0$$

$$ie: \alpha u^2 + \beta u + \gamma = 0$$
$$\alpha = 1$$
$$\beta = B$$
$$\gamma = -\frac{A^3}{27}$$
$$u = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$
$$u = \frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}$$
$$\therefore t = \sqrt[3]{u} = \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}$$

Substituting into (2) yields:

$$s^3 = B + t^3 = B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3$$

$$\therefore s = \sqrt[3]{B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3}$$

Now, $y = s - t$

$$\therefore y = \sqrt[3]{B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3} - \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}$$

Now, $x = y - \frac{b}{3a}$

$$x = \left(\sqrt[3]{B + \left(\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right)^3} - \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}\right) - \frac{b}{3a}$$

Where, $A = \frac{\left(c - \frac{b^2}{3a}\right)}{a}$ & $B = -\frac{\left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right)}{a}$

4.2 ALGEBRAIC EXPANSION:

Babylonian Identity:

$$\left(\frac{1}{2}\left(x-\frac{1}{x}\right)\right)^2+1=\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)^2;$$ (c1800BC)

Common Products And Factors:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$(x \pm y)^2 = x^2 \pm 2xy + y^2$$
$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$$
$$(x \pm y)^4 = x^4 \pm 4x^3y + 6x^2y^2 \pm 4xy^3 + y^4$$

Binomial Theorem:

For any value of n, whether positive, negative, integer or non-integer, the value of the nth power of a binomial is given by:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots + b^n$$

Binomial Expansion:

For any power of n, the binomial (a + x) can be expanded

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \ldots + x^n$$

This is particularly useful when x is very much less than a so that the first few terms provide a good approximation of the value of the expression. There will always be n+1 terms and the general form is:

$$(a+x)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} x^k$$

Note that the factorial is given by

$$n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$$

$$0! = 1$$

Difference of two squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Brahmagupta–Fibonacci Identity:

$$\begin{aligned}\left(a^2+b^2\right)\left(c^2+d^2\right) &= (ac-bd)^2 + (ad+bc)^2 && (1)\\ &= (ac+bd)^2 + (ad-bc)^2. && (2)\end{aligned}$$

Also,

$$\begin{aligned}\left(a^2+nb^2\right)\left(c^2+nd^2\right) &= (ac-nbd)^2 + n\,(ad+bc)^2 && (3)\\ &= (ac+nbd)^2 + n\,(ad-bc)^2, && (4)\end{aligned}$$

Degen's eight-square identity:

$$\begin{aligned}
&(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2+a_8^2)(b_1^2+b_2^2+b_3^2+b_4^2+b_5^2+b_6^2+b_7^2+b_8^2) =\\
&(a_1b_1-a_2b_2-a_3b_3-a_4b_4-a_5b_5-a_6b_6-a_7b_7-a_8b_8)^2+\\
&(a_1b_2+a_2b_1+a_3b_4-a_4b_3+a_5b_6-a_6b_5-a_7b_8+a_8b_7)^2+\\
&(a_1b_3-a_2b_4+a_3b_1+a_4b_2+a_5b_7+a_6b_8-a_7b_5-a_8b_6)^2+\\
&(a_1b_4+a_2b_3-a_3b_2+a_4b_1+a_5b_8-a_6b_7+a_7b_6-a_8b_5)^2+\\
&(a_1b_5-a_2b_6-a_3b_7-a_4b_8+a_5b_1+a_6b_2+a_7b_3+a_8b_4)^2+\\
&(a_1b_6+a_2b_5-a_3b_8+a_4b_7-a_5b_2+a_6b_1-a_7b_4+a_8b_3)^2+\\
&(a_1b_7+a_2b_8+a_3b_5-a_4b_6-a_5b_3+a_6b_4+a_7b_1-a_8b_2)^2+\\
&(a_1b_8-a_2b_7+a_3b_6+a_4b_5-a_5b_4-a_6b_3+a_7b_2+a_8b_1)^2
\end{aligned}$$

Note that:

$$\begin{aligned}
&(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2) =\\
&(a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2+\\
&(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2+\\
&(a_1b_3-a_2b_4+a_3b_1+a_4b_2)^2+\\
&(a_1b_4+a_2b_3-a_3b_2+a_4b_1)^2
\end{aligned}$$

and,

$$\begin{aligned}
&(a_5^2+a_6^2+a_7^2+a_8^2)(b_1^2+b_2^2+b_3^2+b_4^2) =\\
&(a_5b_1+a_6b_2+a_7b_3+a_8b_4)^2+
\end{aligned}$$

$$(a_5b_2 - a_6b_1 + a_7b_4 - a_8b_3)^2 +$$
$$(a_5b_3 - a_6b_4 - a_7b_1 + a_8b_2)^2 +$$
$$(a_5b_4 + a_6b_3 - a_7b_2 - a_8b_1)^2$$

4.3 LIMIT MANIPULATIONS:

$$\lim_{n\to\infty}(a_n \pm b_n) = \left(\lim_{n\to\infty}(a_n)\right) \pm \left(\lim_{n\to\infty}(b_n)\right)$$
$$\lim_{n\to\infty}(ka_n) = k\left(\lim_{n\to\infty}(a_n)\right)$$
$$\lim_{n\to\infty}(a_nb_n) = \left(\lim_{n\to\infty}(a_n)\right)\left(\lim_{n\to\infty}(b_n)\right)$$
$$\lim_{n\to\infty}(f(a_n)) = f\left(\lim_{n\to\infty}(a_n)\right)$$

4.4 SUMATION MANIPULATIONS:

$$\sum_{n=s}^{t} C \cdot f(n) = C \cdot \sum_{n=s}^{t} f(n)$$, where *C* is a constant

$$\sum_{n=s}^{t} f(n) + \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) + g(n)]$$
$$\sum_{n=s}^{t} f(n) - \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) - g(n)]$$
$$\sum_{n=s}^{t} f(n) = \sum_{n=s+p}^{t+p} f(n-p)$$
$$\sum_{n=s}^{j} f(n) + \sum_{n=j+1}^{t} f(n) = \sum_{n=s}^{t} f(n)$$
$$\left(\sum_{i=k_0}^{k_1} a_i\right)\left(\sum_{j=l_0}^{l_1} b_j\right) = \sum_{i=k_0}^{k_1}\sum_{j=l_0}^{l_1} a_ib_j$$
$$\sum_{i=k_0}^{k_1}\sum_{j=l_0}^{l_1} a_{i,j} = \sum_{j=l_0}^{l_1}\sum_{i=k_0}^{k_1} a_{i,j}$$
$$\sum_{n=0}^{t} f(2n) + \sum_{n=0}^{t} f(2n+1) = \sum_{n=0}^{2t+1} f(n)$$
$$\sum_{n=0}^{t}\sum_{i=0}^{z-1} f(z \cdot n + i) = \sum_{n=0}^{z \cdot t+z-1} f(n)$$
$$\sum_{n=s}^{t} \ln f(n) = \ln \prod_{n=s}^{t} f(n)$$
$$c^{\left[\sum_{n=s}^{t} f(n)\right]} = \prod_{n=s}^{t} c^{f(n)}$$

4.5 COMMON FUNCTIONS:

Constant Function:

$y=a$ or $f(x)=a$

Graph is a horizontal line passing through the point $(0,a)$

$x=a$

Graph is a vertical line passing through the point $(a,0)$

Line/Linear Function:

$y = mx + c$

Graph is a line with point $(0,c)$ and slope *m*.

Where the gradient is between any two points (x_1, y_1) & (x_2, y_2)

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

Also, $y = y_1 + m(x - x_1)$

The equation of the line with gradient *m* .and passing through the point (x_1, y_1) .

Parabola/Quadratic Function:

$y = a(x-h)^2 + k$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at (h,k).

$y = ax^2 + bx + c$

The graph is a parabola that opens up if $a > 0$ or down if $a < 0$ and has a vertex at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

$x = ay^2 + by + c$

The graph is a parabola that opens right if $a > 0$ or left if $a < 0$ and has a vertex at $\left(g\left(\frac{-b}{2a}\right), \left(\frac{-b}{2a}\right)\right)$. This is not a function.

Circle:

$(x-h)^2 + (y-k)^2 = r^2$

Graph is a circle with radius *r* and center (h,k).

Ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Graph is an ellipse with center (h,k) with vertices *a* units right/left from the center and vertices *b* units up/down from the center.

Hyperbola:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Graph is a hyperbola that opens left and right, has a center at (h,k), vertices a units left/right of center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

Graph is a hyperbola that opens up and down, has a center at (h,k), vertices b units up/down from the center and asymptotes that pass through center with slope $\pm\frac{b}{a}$.

4.6 LINEAR ALGEBRA:

Vector Space Axioms:

Let *V* be a set on which addition and scalar multiplication are defined (this means that if u and v are objects in *V* and *c* is a scalar then we've defined $\mathbf{u}+\mathbf{v}$ $\mathbf{u}+\mathbf{v}$ and *c*u in some way). If the following axioms are true for all objects u, v, and w in *V* and all scalars *c* and *k* then *V* is called a vector space and the objects in *V* are called vectors.

(a) $\mathbf{u}+\mathbf{v}$ $\mathbf{u}+\mathbf{v}$ is in *V* – This is called closed under addition.

(b) *c*u is in *V* – This is called closed under scalar multiplication.

(c) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(d) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(e) There is a special object in *V*, denoted 0 and called the zero vector, such that for all u in *V* we have $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$.

(f) For every u in *V* there is another object in *V*, denoted $-\mathbf{u}$ $-\mathbf{u}$ and called the negative of u, such that $\mathbf{u}-\mathbf{u}=\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

$\mathbf{u}-\mathbf{u}=\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

(g) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(h) $(c+k)\mathbf{u}=c\mathbf{u}+k\mathbf{u}$ $(c+k)\mathbf{u}=c\mathbf{u}+k\mathbf{u}$

(i) $c(k\mathbf{u})=(ck)\mathbf{u}$ $c(k\mathbf{u})=(ck)\mathbf{u}$

(j) $1\mathbf{u}=\mathbf{u}$ $1\mathbf{u}=\mathbf{u}$

Subspace: When the subspace is a subset of another vector space, only axioms (a) and (b) need to be proved to show that the subspace is also a vector space.

Common Spaces:

Real Numbers	$\Re, \Re^2, \Re^3, ..., \Re^n$ (n denotes dimension)
Complex Numbers:	$C, C^2, C^3, ..., C^n$ (n denotes dimension)
Polynomials	$P_1, P_2, P_3, ..., P_n$ (n denotes the highest order of x)
All continuous functions	$C[a, b]$(a & b denote the interval) (This is never a vector space as it has infinite dimensions)

Rowspace of a spanning set in R^n

Stack vectors in a matrix in rows
Use elementary row operations to put matrix into row echelon form
The non zero rows form a basis of the vector space

Columnspace of a spanning set in R^n

Stack vectors in a matrix in columns
Use elementary row operations to put matrix into row echelon form
Columns with leading entries correspond to the subset of vectors in the set that form a basis

Nullspace:

Solutions to $A\underline{x} = \underline{0}$ A

Using elementary row operations to put matrix into row echelon form, columns with no leading entries are assigned a constant and the remaining variables are solved with respect to these constants.

Nullity:

The dimension of the nullspace

$Columns(A) = Nullity(A) + Rank(A)$

Linear Dependence:

$c_1 r_1 + c_2 r_2 + ... + c_n r_n = 0$

Then, $c_1 = c_2 = c_n = 0$

If the trivial solution is the only solution, $r_1, r_2, ... r_n$ are independent.

$r(A) \neq r(A | b)$: No Solution
$r(A) = r(A | b) = n$: Unique Solution
$r(A) = r(A | b) < n$: Infinite Solutions

Basis: S is a basis of V if:

S spans V
S is linearly dependant

$S = \{u_1, u_2, u_3, ..., u_n\}$

The general vector within the vector space is: $w = \begin{bmatrix} x \\ y \\ z \\ ... \end{bmatrix}$

$$w = c_1u_1 + c_2u_2 + c_3u_3 + ... + c_nu_n$$

Therefore, $[w] = \begin{bmatrix} u_{11} & u_{21} & u_{31} & ... & u_{n1} \\ u_{12} & u_{22} & u_{32} & ... & u_{n2} \\ u_{13} & u_{23} & u_{33} & ... & u_{n3} \\ ... & ... & ... & ... & ... \\ u_{1n} & u_{2n} & u_{3n} & ... & u_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ ... \\ c_n \end{bmatrix}$

If the determinant of the square matrix is not zero, the matrix is invertible. Therefore, the solution is unique. Hence, all vectors in w are linear combinations of S. Because of this, S spans w.

Standard Basis:

Real Numbers $S(\Re^n) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ ... \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ ... \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ 0 \\ ... \\ 1 \end{bmatrix} \right\}$

Polynomials $S(P_n) = \{1, x, x^2, x^3, ..., x^n\}$

Any set the forms the basis of a vector space must contain the same number of linearly independent vectors as the standard basis.

Orthogonal Complement:

$W^{\perp}$ is the nullspace of A, where A is the matrix that contains $\{v_1, v_2, v_3, ..., v_n\}$ in rows.

$$\dim(W^{\perp}) = nullity(A)$$

Orthonormal Basis:

A basis of mutually orthogonal vectors of length 1. Basis can be found with the Gram-Schmidt process outline below.

$$< v_i, v_j >= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In an orthonormal basis:

$$u =< u, v_1 > v_1 + < u, v_2 > v_2 + < u, v_3 > v_3 + ... + < u, v_n > v_n)$$

$$u = c_1v_1 + c_2v_2 + c_3v_3 + ... + c_nv_n)$$

Gram-Schmidt Process:

This finds an orthonormal basis recursively.

In a basis $B = \{u_1, u_2, u_3, ..., u_n\}$

$$q_1 = u_1$$

$$v_1 = \hat{q}_1 = \frac{q_1}{\|q_1\|}$$

Next vector needs to be orthogonal to v_1,

$$q_2 = u_2 - < u_2, v_1 > v_1$$

Similarly

$$q_3 = u_3 - < u_3, v_1 > v_1 - < u_3, v_2 > v_2$$

$$q_n = u_n - < u_3, v_1 > v_1 - < u_3, v_2 > v_2 - ... - < u_3, v_n > v_n$$

$$v_n = \hat{q}_n = \frac{q_n}{\|q_n\|}$$

Coordinate Vector:

If $\underline{v} = c_1 e_1 + c_2 e_2 + ... + c_n e_n$

$$\underline{v}_B = \begin{bmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{bmatrix}$$

For a fixed basis (usually the standard basis) there is 1 to 1 correspondence between vectors and coordinate vectors.

Hence, a basis can be found in R^n and then translated back into the general vector space.

Dimension:

Real Numbers	$\dim(\Re^n) = n$
Polynomials	$\dim(P_n) = n + 1$
Matricis	$\dim(M_{p,q}) = p \times q$

If you know the dimensions and you are checking if a set forms a basis of the vector space, only Linear Independence or Span needs to be checked.

4.7 COMPLEX VECTOR SPACES:

Form: $C^n = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ ... \\ a_n + ib_n \end{bmatrix}$

Dot Product:

$$u \bullet v = \bar{u}_1 v_1 + \bar{u}_2 v_2 + ... + \bar{u}_n v_n$$

Where:

$$u \bullet v = \overline{v \bullet u} \neq v \bullet u$$
$$(u+v) \bullet w = u \bullet w + v \bullet w$$
$$su \bullet v = \bar{s}(u \bullet v), s \in C$$
$$u \bullet u \geq 0$$
$$u \bullet u = 0 \text{ iff } u = 0$$

Inner Product:

$$\|u\| = \sqrt{u \bullet u} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}$$
$$d(u,v) = \|u - v\|$$

Orthogonal if $u \bullet v = 0$
Parallel if $u = sv, s \in C$

4.8 LINEAR TRANSITIONS & TRANSFORMATIONS:

Transition Matrix: From 1 vector space to another vector space

$$T(u) = T(c_1u_1 + c_2u_2 + c_3u_3 + \ldots + c_nu_n)$$
$$T(u) = c_1T(u_1) + c_2T(u_2) + c_3T(u_3) + \ldots + c_nT(u_n)$$

Nullity(T)+Rank(T)=Dim(V)=Columns(T)

Change of Basis Transition Matrix:

$$v_{B'} = M_{B'}{}^{-1}M_Bv_B$$
$$v_{B'} = C_{BB'}v_B$$

For a general vector space with the standard basis: $S = \{s_1, s_2, \ldots, s_n\}$

$$M_B = [(v_1)_S \mid \ldots \mid (v_n)_S]$$
$$M_{B'} = [(u_1)_S \mid \ldots \mid (u_m)_S]$$

Transformation Matrix: From 1 basis to another basis

$$V = span(\{v_1, v_2, v_3, \ldots, v_n\})$$
$$B_1 = \{v_1, v_2, v_3, \ldots, v_n\}$$

$$U = span(\{u_1, u_2, u_3, \ldots, u_m\})$$
$$B_2 = \{u_1, u_2, u_3, \ldots, u_m\}$$

$$A = [(T(v_1))_{B2} \mid (T(v_2))_{B2} \mid \ldots \mid (T(v_n))_{B2}]$$
$$A' = C_{B'B}{}^{-1}AC_{B'B}$$

4.9 INNER PRODUCTS:

Definition: An extension of the dot product into a general vector space.

Axioms:

1. $< u, v >=< v, u >$
2. $< u, v + w >=< u, v > + < u, w >$

3. $< ku, v >= k < u, v >$

4. $< u, u > \geq 0$
   $< u, u >= 0 \text{ iff } u = 0$

Unit Vector: $\hat{u} = \dfrac{u}{\|u\|}$

Cavchy-Schuarz Inequality: $< u, v >^2 \leq < u, u > \times < v, v >$

Inner Product Space:

$$\|u\| = < u, u >^{\frac{1}{2}} = \sqrt{< u, u >}$$

$$\|u\|^2 = < u, u >$$

$$< u, v >^2 \leq \|u\|^2 \times \|v\|^2 \Rightarrow \left(\frac{< u, v >}{\|u\|\|v\|} \right)^2 \leq 1 \Rightarrow -1 \leq \frac{< u, v >}{\|u\|\|v\|} \leq 1$$

$$\|u\| \geq 0, \|u\| = 0 \text{ iff } u = 0$$

$$\|ku\| = |k|\|u\|$$

$$\|u + v\| = \|u\| + \|v\|$$

Angle between two vectors:

As defined by the inner product,

$$\cos(\theta) = \frac{< u, v >}{\|u\|\|v\|}$$

Orthogonal if: $< u, v >= 0$

Distance between two vectors:

As defined by the inner product,

$$d(u, v) = \|u - v\|$$

Generalised Pythagoras for orthogonal vectors:

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

4.10 PRIME NUMBERS:

Determinate: $\Delta(N) = \left\lfloor \dfrac{1 + \left\lfloor \dfrac{3}{N} \right\rfloor}{1 + \displaystyle\sum_{k=1}^{\left\lfloor \frac{\sqrt{N}+1}{2} \right\rfloor} \left\lfloor \dfrac{2k+1}{N} \times \left\lfloor \dfrac{N}{2k+1} \right\rfloor \right\rfloor} \right\rfloor = \begin{cases} 1 & \text{if N is odd and prime} \\ 0 & \text{if N is odd and composite} \end{cases}$

List of Prime Numbers:

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349	353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463	467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601	607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733	739	743	751	757	761	769	773	787	797	809

811	821	823	827	829	839	853	857	859	863	877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013	1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151	1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291	1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451	1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583	1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733	1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889	1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053	2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213	2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357	2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531	2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687	2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819	2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999	3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	3119	3121	3137	3163	3167	3169	3181	3187	3191	3203	3209	3217	3221	3229	3251	3253	3257
3259	3271	3299	3301	3307	3313	3319	3323	3329	3331	3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511	3517	3527	3529	3533	3539	3541	3547	3557	3559	3571

Perfect Numbers: A perfect number is a positive integer that is equal to the sum of its proper positive divisors, excluding the number itself. Even perfect numbers are of the form $2^{p-1}(2^p-1)$, where (2^p-1) is prime and by extension p is also prime. It is unknown whether there are any odd perfect numbers.

List of Perfect Numbers:

Rank	*p*	Perfect number	Digits	Year	Discoverer
1	2	6	1	Known to the Greeks	
2	3	28	2	Known to the Greeks	
3	5	496	3	Known to the Greeks	
4	7	8128	4	Known to the Greeks	
5	13	33550336	8	1456	First seen in the medieval manuscript, Codex Lat. Monac.
6	17	8589869056	10	1588	Cataldi
7	19		12	1588	Cataldi
8	31		19	1772	Euler
9	61		37	1883	Pervushin
10	89		54	1911	Powers
11	107		65	1914	Powers
12	127		77	1876	Lucas
13	521		314	1952	Robinson
14	607		366	1952	Robinson
15	1279		770	1952	Robinson
16	2203		1327	1952	Robinson
17	2281		1373	1952	Robinson
18	3217		1937	1957	Riesel
19	4253		2561	1961	Hurwitz
20	4423		2663	1961	Hurwitz
21	9689		5834	1963	Gillies
22	9941		5985	1963	Gillies
23	11213		6751	1963	Gillies

24	19937		12003	1971	Tuckerman
25	21701		13066	1978	Noll & Nickel
26	23209		13973	1979	Noll
27	44497		26790	1979	Nelson & Slowinski
28	86243		51924	1982	Slowinski
29	110503		66530	1988	Colquitt & Welsh
30	132049		79502	1983	Slowinski
31	216091		130100	1985	Slowinski
32	756839		455663	1992	Slowinski & Gage
33	859433		517430	1994	Slowinski & Gage
34	1257787		757263	1996	Slowinski & Gage
35	1398269		841842	1996	Armengaud, Woltman, et al.
36	2976221		1791864	1997	Spence, Woltman, et al.
37	3021377		1819050	1998	Clarkson, Woltman, Kurowski, et al.
38	6972593		4197919	1999	Hajratwala, Woltman, Kurowski, et al.
39	13466917		8107892	2001	Cameron, Woltman, Kurowski, et al.
40	20996011		12640858	2003	Shafer, Woltman, Kurowski, et al.
41	24036583		14471465	2004	Findley, Woltman, Kurowski, et al.
42	25964951		15632458	2005	Nowak, Woltman, Kurowski, et al.
43	30402457		18304103	2005	Cooper, Boone, Woltman, Kurowski, et al.
44	32582657		19616714	2006	Cooper, Boone, Woltman, Kurowski, et al.
45	37156667		22370543	2008	Elvenich, Woltman, Kurowski, et al.
46	42643801		25674127	2009	Strindmo, Woltman, Kurowski, et al.
47	43112609		25956377	2008	Smith, Woltman, Kurowski, et al.

Amicable Numbers: Amicable numbers are two different numbers so related that the sum of the proper divisors of each is equal to the other number.

List of Amicable Numbers:

Amicable Pairs		Amicable Pairs		Amicable Pairs	
220	284	1,328,470	1,483,850	8,619,765	9,627,915
1,184	1,210	1,358,595	1,486,845	8,666,860	10,638,356
2,620	2,924	1,392,368	1,464,592	8,754,130	10,893,230
5,020	5,564	1,466,150	1,747,930	8,826,070	10,043,690
6,232	6,368	1,468,324	1,749,212	9,071,685	9,498,555
10,744	10,856	1,511,930	1,598,470	9,199,496	9,592,504
12,285	14,595	1,669,910	2,062,570	9,206,925	10,791,795
17,296	18,416	1,798,875	1,870,245	9,339,704	9,892,936
63,020	76,084	2,082,464	2,090,656	9,363,584	9,437,056
66,928	66,992	2,236,570	2,429,030	9,478,910	11,049,730
67,095	71,145	2,652,728	2,941,672	9,491,625	10,950,615
69,615	87,633	2,723,792	2,874,064	9,660,950	10,025,290
79,750	88,730	2,728,726	3,077,354	9,773,505	11,791,935
100,485	124,155	2,739,704	2,928,136	10,254,970	10,273,670
122,265	139,815	2,802,416	2,947,216	10,533,296	10,949,704
122,368	123,152	2,803,580	3,716,164	10,572,550	10,854,650
141,664	153,176	3,276,856	3,721,544	10,596,368	11,199,112
142,310	168,730	3,606,850	3,892,670	10,634,085	14,084,763
171,856	176,336	3,786,904	4,300,136	10,992,735	12,070,305
176,272	180,848	3,805,264	4,006,736	11,173,460	13,212,076

185,368	203,432	4,238,984	4,314,616	11,252,648	12,101,272
196,724	202,444	4,246,130	4,488,910	11,498,355	12,024,045
280,540	365,084	4,259,750	4,445,050	11,545,616	12,247,504
308,620	389,924	4,482,765	5,120,595	11,693,290	12,361,622
319,550	430,402	4,532,710	6,135,962	11,905,504	13,337,336
356,408	399,592	4,604,776	5,162,744	12,397,552	13,136,528
437,456	455,344	5,123,090	5,504,110	12,707,704	14,236,136
469,028	486,178	5,147,032	5,843,048	13,671,735	15,877,065
503,056	514,736	5,232,010	5,799,542	13,813,150	14,310,050
522,405	525,915	5,357,625	5,684,679	13,921,528	13,985,672
600,392	669,688	5,385,310	5,812,130	14,311,688	14,718,712
609,928	686,072	5,459,176	5,495,264	14,426,230	18,087,818
624,184	691,256	5,726,072	6,369,928	14,443,730	15,882,670
635,624	712,216	5,730,615	6,088,905	14,654,150	16,817,050
643,336	652,664	5,864,660	7,489,324	15,002,464	15,334,304
667,964	783,556	6,329,416	6,371,384	15,363,832	16,517,768
726,104	796,696	6,377,175	6,680,025	15,938,055	17,308,665
802,725	863,835	6,955,216	7,418,864	16,137,628	16,150,628
879,712	901,424	6,993,610	7,158,710	16,871,582	19,325,698
898,216	980,984	7,275,532	7,471,508	17,041,010	19,150,222
947,835	1,125,765	7,288,930	8,221,598	17,257,695	17,578,785
998,104	1,043,096	7,489,112	7,674,088	17,754,165	19,985,355
1,077,890	1,099,390	7,577,350	8,493,050	17,844,255	19,895,265
1,154,450	1,189,150	7,677,248	7,684,672	17,908,064	18,017,056
1,156,870	1,292,570	7,800,544	7,916,696	18,056,312	18,166,888
1,175,265	1,438,983	7,850,512	8,052,488	18,194,715	22,240,485
1,185,376	1,286,744	8,262,136	8,369,864	18,655,744	19,154,336
1,280,565	1,340,235				

Sociable Numbers: Sociable numbers are generalisations of amicable numbers where a sequence of numbers each of whose numbers is the sum of the factors of the preceding number, excluding the preceding number itself. The sequence must be cyclic, eventually returning to its starting point

.

List of Sociable Numbers:

C4s	
1264460	
1547860	
1727636	
1305184	
2115324	
3317740	
3649556	
2797612	
2784580	
3265940	
3707572	

3370604	
4938136	
5753864	
5504056	
5423384	
7169104	
7538660	
8292568	
7520432	
C5 Poulet 1918 5D	
12496	2^4*11*71
14288	2^4*19*47
15472	2^4*967
14536	2^3*23*79
14264	2^3*1783
C6 Moews&Moews 1992 11D	
21548919483	3^5*7^2*13*19*17*431
23625285957	3^5*7^2*13*19*29*277
24825443643	3^2*7^2*13*19*11*20719
26762383557	3^4*7^2*13*19*27299
25958284443	3^2*7^2*13*19*167*1427
23816997477	3^2*7^2*13*19*218651
C6 Moews&Moews 1995 11D/12D	
90632826380	2^2*5*109*431*96461
101889891700	2^2*5^2*31*193*170299
127527369100	2^2*5^2*31*181*227281
159713440756	2^2*31*991*1299709
129092518924	2^2*31*109*9551089
106246338676	2^2*17*25411*61487
C6 Needham 2006 13D	
1771417411016	2^3*11*20129743307
1851936384424	2^3*7*1637*20201767
2118923133656	2^3*7*863*43844627
2426887897384	2^3*59*5141711647
2200652585816	2^3*43*1433*4464233
2024477041144	2^3*253059630143
C6 Needham 2006 13D	
3524434872392	2^3*7*17*719*5149009
4483305479608	2^3*89*6296777359
4017343956392	2^3*13*17*3019*752651
4574630214808	2^3*607*6779*138967
4018261509992	2^3*31*59*274621481
3890837171608	2^3*61*22039*361769

C6 Needham 2006 13D	
4773123705616	2^4*7*347*122816069
5826394399664	2^4*101*3605442079
5574013457296	2^4*53*677*1483*6547
5454772780208	2^4*53*239*2971*9059
5363145542992	2^4*307*353*3093047
5091331952624	2^4*318208247039
C8 Flammenkamp 1990 Brodie ? 10D	
1095447416	2^3*7*313*62497
1259477224	2^3*43*3661271
1156962296	2^3*7*311*66431
1330251784	2^3*43*3867011
1221976136	2^3*41*1399*2663
1127671864	2^3*11*61*83*2531
1245926216	2^3*19*8196883
1213138984	2^3*67*2263319
C8 Flammenkamp 1990 Brodie ? 10D	
1276254780	2^2*3*5*1973*10781
2299401444	2^2*3*991*193357
3071310364	2^2*767827591
2303482780	2^2*5*67*211*8147
2629903076	2^2*23*131*218213
2209210588	2^2*13^2*17*192239
2223459332	2^2*131*4243243
1697298124	2^2*907*467833
C9 Flammenkamp 1990 9D/10D	
805984760	2^3*5*7*1579*1823
1268997640	2^3*5*17*61*30593
1803863720	2^3*5*103*367*1193
2308845400	2^3*5^2*11544227
3059220620	2^2*5*2347*65173
3367978564	2^2*841994641
2525983930	2*5*17*367*40487
2301481286	2*13*19*4658869
1611969514	2*805984757
C28 Poulet 1918 5D/6D	
14316	2^2*3*1193
19116	2^2*3^4*59
31704	2^3*3*1321
47616	2^9*3*31
83328	2^7*3*7*31
177792	2^7*3*463
295488	2^6*3^5*19
629072	2^4*39317
589786	2*294893
294896	2^4*7*2633

358336	2^6*11*509
418904	2^3*52363
366556	2^2*91639
274924	2^2*13*17*311
275444	2^2*13*5297
243760	2^4*5*11*277
376736	2^5*61*193
381028	2^2*95257
285778	2*43*3323
152990	2*5*15299
122410	2*5*12241
97946	2*48973
48976	2^4*3061
45946	2*22973
22976	2^6*359
22744	2^3*2843
19916	2^2*13*383
17716	2^2*43*103
This list is exhaustive for known social numbers where C>4	

4.11 GOLDEN RATIO & FIBONACCI SEQUENCE:

Relationship:

$$\frac{a+b}{a} = \frac{a}{b} \equiv \varphi,$$

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi},$$

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1}.$$

Infinite Series:

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

$$\varphi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}(2n+1)!}{(n+2)!n!4^{(2n+3)}}.$$

Continued Fractions:

$$\varphi = [1; 1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

$$\varphi^{-1} = [0;1,1,1,\dots] = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

Trigonometric Expressions:

$$\varphi = 1 + 2\sin(\pi/10) = 1 + 2\sin 18^\circ$$
$$\varphi = \frac{1}{2}\csc(\pi/10) = \frac{1}{2}\csc 18^\circ$$
$$\varphi = 2\cos(\pi/5) = 2\cos 36^\circ$$
$$\varphi = 2\sin(3\pi/10) = 2\sin 54^\circ.$$

Fibonacci Sequence:

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \varphi.$$

$$\sum_{n=1}^{\infty} |F(n)\varphi - F(n+1)| = \varphi.$$

$$\lim_{n\to\infty} \frac{F(n+a)}{F(n)} = \varphi^a,$$

4.12 FERMAT'S LAST THEOREM:

$a^n + b^n \neq c^n$ for integers $a, b \,\&\, c$ and $n > 2$

Proposed by Fermat in 1637 and proved by Andrew Wiles in 1994. The proof is too long to be written here. See: http://www.cs.berkeley.edu/~anindya/fermat.pdf

PART 5: COUNTING TECHNIQUES & PROBABILITY

5.1 2D

Triangle Number $T_n = \dfrac{n(n+1)}{2}$

$n^2 = T_n + T_{n-1}$

Square Number $T_n = n^2$

Pentagonal Number $T_n = \dfrac{n(3n-1)}{2}$

5.2 3D

Tetrahedral Number $T_n = \dfrac{n^3 + 3n^2 + 2n}{6}$

Square Pyramid Number $T_n = \dfrac{2n^3 + 3n^2 + n}{6}$

5.3 PERMUTATIONS

Permutations: $= n!$

Permutations (with repeats): $= \dfrac{n!}{(groupA)! \times (groupB)! \times \ldots}$

5.4 COMBINATIONS

Ordered Combinations: $= \dfrac{n!}{(n-p)!}$

Unordered Combinations: $= \dbinom{n}{p} = \dfrac{n!}{p!(n-p)!}$

Ordered Repeated Combinations: $= n^p$

Unordered Repeated Combinations: $= \dfrac{(p+n-1)!}{p! \times (n-1)!}$

Grouping: $= \dbinom{n}{n_1}\dbinom{n-n_1}{n_2}\dbinom{n-n_1-n_2}{n_3}\ldots = \dfrac{n!}{n_1!n_2!n_3!\ldots n_r!}$

5.5 MISCELLANEOUS:

Total Number of Rectangles and Squares from a a x b rectangle:

$\sum = T_a \times T_b$

Number of Interpreters: $= T_{L-1}$

Max number of pizza pieces: $= \dfrac{c(c+1)}{2} + 1$

Max pieces of a crescent: $= \dfrac{c(c+3)}{2} + 1$

Max pieces of cheese: $= \dfrac{c^3 + 5c}{6} + 1$

Cards in a card house: $= \frac{l(3l+1)}{2}$

Different arrangement of dominos: $= 2^{n-d} \times n!$

Unit Fractions: $\frac{a}{b} = \frac{1}{INT\left[\frac{b}{a}\right]+1} + \frac{a - MOD\left[\frac{b}{a}\right]}{b\left(INT\left[\frac{b}{a}\right]+1\right)}$

Angle between two hands of a clock: $\theta = 5.5m - 30h$

Winning Lines in Noughts and Crosses: $= 2(a+1)$

Bad Restaurant Spread: $= \frac{P}{1-s}$

Fibonacci Sequence: $= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

ABBREVIATIONS (5.1, 5.2, 5.3, 5.4, 5.5)

a=side ‘a’
b=side ‘b’
c=cuts
d=double dominos
h=hours
L=Languages
l=layers
m=minutes
n= nth term
n=n number
P=Premium/Starting Quantity
p=number you pick
r=number of roles/turns
s=spread factor
T=Term
θ=the angle

5.6 FACTORIAL:

Definition: $n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$

Table of Factorials:

0!	1 (by definition)		
1!	1	11!	39916800
2!	2	12!	479001600
3!	6	13!	6227020800
4!	24	14!	87178291200
5!	120	15!	1307674368000
6!	720	16!	20922789888000
7!	5040	17!	355687428096000

8!	40320	18!	6402373705728000
9!	362880	19!	121645100408832000
10!	3628800	20!	2432902008176640000

Approximation: $n! = \sqrt{2\pi} \times n^{n+\frac{1}{2}} \times e^{-n}$ (within 1% for n>10)

5.7 THE DAY OF THE WEEK:

This only works after 1753

$$= MOD7\left(d + y + \left[\frac{31m}{12}\right] + \left[\frac{y}{4}\right] - \left[\frac{y}{100}\right] + \left[\frac{y}{400}\right]\right)$$

d=day
m=month
y=year
SQUARE BRAKETS MEAN INTEGER DIVISION
INT=Keep the integer
MOD=Keep the remainder

5.8 BASIC PROBABILITY:

$$\sum P = 1$$

5.9 VENN DIAGRAMS:

Complementary Events: $1 - P(A) = P(\overline{A})$

Totality: $P(A) = \sum_{i=1}^{m} P(A \mid B_i)P(B_i)$

$P(A) = P(A \cap B) + P(A \cap B')$

Conditional Probability: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(B) \cdot P(A \mid B)$

Union : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Independent Events: $P(A \cap B) = P(A) \cdot P(B)$

$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

$P(B \mid A) = P(B)$

Mutually Exclusive: $P(A \cap B) = 0$

$P(A \cap B') = P(A)$

$P(A \cup B) = P(A) + P(B)$

$P(A \cup B') = P(B')$

Baye's Theorem:

$$P(B|A)=\frac{P(A|B)P(B)}{P(A)}=\frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|B')P(B')}$$

Event's Space: $P(A)=\sum_{i=1}^{m}P(A\cap B_i)$

5.11 BASIC STATISTICAL OPERATIONS:

Variance: $v=\sigma^2$

Mean: $\mu=\frac{\sum x_i}{n_s}$

Standardized Score: $z=\frac{x_i-\mu}{\sigma}$

Confidence Interval:

5.12 DISCRETE RANDOM VARIABLES:

Standard Deviation: $\sigma=\sqrt{\frac{\sum\left(x_i-\bar{x}\right)^2}{n_s}}$

Expected Value:

$$E[X]=\sum_{1}^{i}P(x_i)\times x_i$$

$$E[aX+b]=aE[X]+b$$

Variance:

$$v=\frac{\sum\left(x_i-\bar{x}\right)^2}{n_s}$$

$$v=\left(E\left[x-E[x]\right]\right)^2$$

$$v=E[x^2]-\left(E[x]\right)^2$$

$$\mathrm{var}[aX+b]=a^2\,\mathrm{var}[X]$$

Probability Mass Function: $P(x)=f(x)=P(X=x)$

Cumulative Distribution Function: $F(x)=P(X\le x)$

5.13 COMMON DRVs:

Bernoulli Trial:

Definition: 1 trial, 1 probability that is either fail or success

Outcomes: $S_X=\{0,1\}$

Probability: $P_X(x)=\begin{cases}p & x=1\\ 1-p & x=0\end{cases}$

Expected Value: $E[X]=p$

Variance: $Var[X]=p-p^2=p(1-p)$

Binomial Trial:

Definition: Repeated Bernoulli Trials

Outcomes: $S_X=\{0,1,2,3,...n\}$

Probability: $P_X(x) = \binom{n}{x} \cdot (p)^x \cdot (1-p)^{n-x}$

Expected Value: $E[X] = np$

Variance: $Var[X] = np(1-p)$

n=number to choose from
p=probability of x occurring
x=number of favorable results

Geometric Trial:

Definition: Number of Bernoulli Trials to get 1st Success.

Outcomes: $S_X = \{0,1,2,3,...\}$

Probability: $P_X(x) = p(1-p)^{x-1}$

Negative Binomial Trial:

Definition: Number to 1st get to n success.

Probability: $P_X(x) = \binom{x-1}{n-1} p^x (1-p)^{n-x}$

5.14 CONTINUOUS RANDOM VARIABLES:

Probability Density Function: $= f(x)$

$$\text{If } \int_{-\infty}^{\infty} f(x)dx = 1 \;\&\; f(x) \geq 0 \; for -\infty \leq x \leq \infty$$

Cumulative Distribution Function: $= F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$

Interval Probability: $P(a \leq X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx$

Expected Value: $E(x) = \int_{-\infty}^{\infty} x \times f(x)dx$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) \times f(x)dx$$

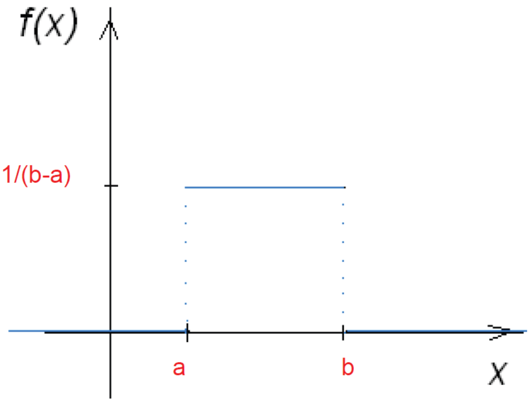
Variance: $Var(X) = E(X^2) - (E(X))^2$

5.15 COMMON CRVs:

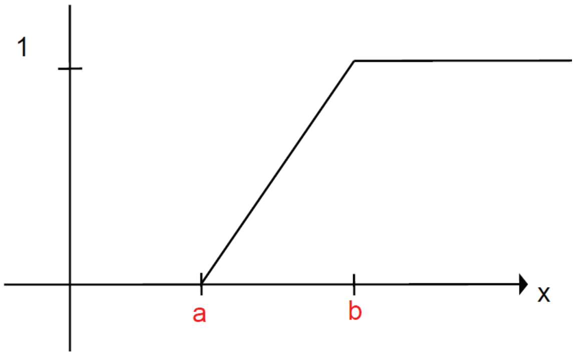
Uniform Distribution:

Declaration: $X \sim Uniform(a,b)$

PDF: $f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & otherwise \end{cases}$



CDF: $$F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



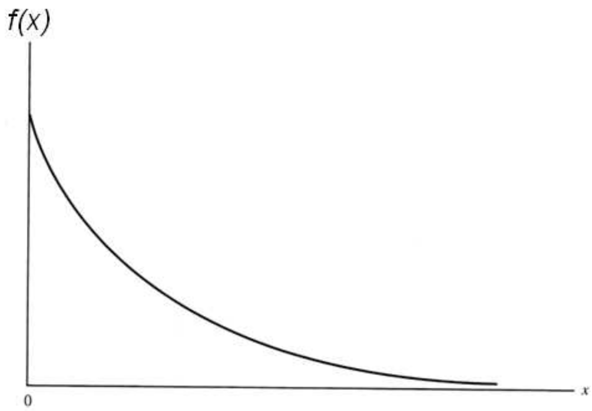
Expected Value: $$= \frac{a+b}{2}$$

Variance: $$= \frac{(b-a)^2}{12}$$

Exponential Distribution:

Declaration: $X \sim Exponential(\lambda)$

PDF: $$f(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \ge 0 \end{cases}$$



CDF: $F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$

Expected Value: $= \frac{1}{\lambda}$

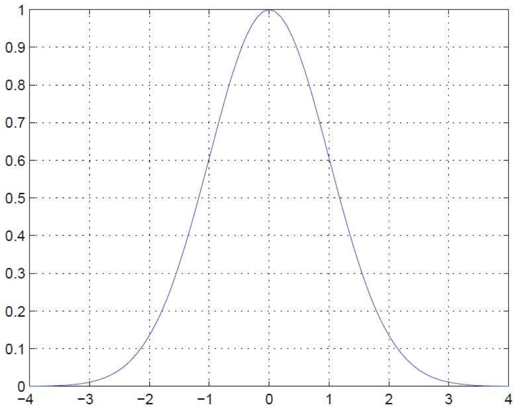
Variance: $= \frac{1}{\lambda^2}$

Normal Distribution:

Declaration: $X \sim Normal(\mu, \sigma^2)$

Standardized Z Score: $Z = \frac{x - \mu}{\sigma}$

PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}z^2}$



CDF: $\Phi(Z)$ (The integration is provided within statistic tables)

Expected Value: $= \mu$

Variance: $= \sigma^2$

5.16 MULTIVARIABLE DISCRETE:

Probability:

$$P(X = x, Y = y) = f(x, y)$$

$$P(X \le x, Y \le y) = \sum f(x, y) \text{ over all values of x \& y}$$

Marginal Distribution:

$$f_X(x) = \sum_y f(x_i, y)$$

$$f_Y(y) = \sum_x f(x, y_i)$$

Expected Value:

$$E[X] = \sum_x x \times f_X(x)$$

$$E[Y] = \sum_y y \times f_Y(y)$$

$$E[X, Y] = \sum_x \sum_y x \times y \times f_{X,Y}(x, y)$$

Independence: $f(x, y) = f_X(x) \times f_Y(y)$

Covariance: $Cov = E[X,Y] - E[X] \times E[Y]$

5.17 MULTIVARIABLE CONTINUOUS:

Probability:

$$P(X \le x, Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)dxdy$$

$$P(Y < y) = P(-\infty < X < \infty, Y < y) = \int_{-\infty}^{y} f_Y(y)dy$$

Marginal Distribution:

$$f_X(x) = \int_{a}^{b} f(x,y)dy$$ where a & b are bounds of y

$$f_Y(y) = \int_{a}^{b} f(x,y)dx$$ where a & b are bounds of x

Expected Value:

$$E[X] = \int_{-\infty}^{\infty} x \times f_X(x)dx$$

$$E[Y] = \int_{-\infty}^{\infty} y \times f_Y(y)dy$$

$$E[X,Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \times y \times f_{X,Y}(x,y)dxdy$$

Independence: $f(x,y) = f_X(x) \times f_Y(y)$

Covariance: $Cov = E[X,Y] - E[X] \times E[Y]$

Correlation Coefficient: $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

ABBREVIATIONS

σ = Standard Deviation

μ = mean

n_s = number of scores

p = probability of favourable result

v = variance

x_i = Individual x score

$\bar{x}$ = mean of the x scores

z = Standardized Score

PART 6: FINANCIAL

6.1 GENERAL FORMUALS:

Profit: $p = s - c$

Profit margin: $m = \frac{p}{c} \times 100$

Simple Interest: $= P(1 + tr)$

Compound Interest: $= P(1 + r)^t$

Continuous Interest: $= Pe^{rt}$

ABBREVIATIONS (6.1):

c=cost
I=interest
m=profit margin (%)
p=profit
P=premium
r=rate
s=sale price
t=time

6.2 MACROECONOMICS:

GDP: y = AE = AD = C + I + G + NX
y = Summation of all product quantities multiplied by cost

RGDP: RGDP = Summation of all product quantities multiplied by base year cost

NGDP: NGDP = Summation of all product quantities multiplied by current year cost

Growth: $\text{Growth} = \frac{RGDP_{CURRENT} - RGDP_{BASE}}{RGDP_{BASE}} \times 100$

Net Exports: NX = X - M

Working Age Population: WAP = Labor Force + Not in Labor Force

Labor Force: LF = Employed + Unemployed

Unemployment: UE = Frictional + Structural + Cyclical

Natural Unemployment: NUE = Frictional + Structural

Unemployment Rate: $\Delta\text{UE\%} = \frac{UE}{LF} \times 100$

Employment Rate: $\Delta\text{E\%} = \frac{E}{LF} \times 100$

Participation Rate: $\Delta\text{P\%} = \frac{LF}{WAP} \times 100 = \frac{UE + E}{WAP} \times 100$

CPI: CPI = Indexed Average Price of all Goods and Services

Inflation Rate: $\text{Inflation Rate} = \frac{CPI_{CURRENT} - CPI_{BASE}}{CPI_{BASE}} \times 100$

ABBREVIATIONS (6.2)

AD=Aggregate Demand

AE=Aggregate Expenditure
C=Consumption
CPI=Consumer Price Index
E=Employed
G=Government
I=Investment
LF=Labor Force
M=Imports
NGDP=Nominal GDP
NUE=Natural Unemployment
NX=Net Export
P=Participation
RGDP=Real GDP (Price is adjusted to base year)
UE=Unemployed
WAP=Working Age Population
X=Exports
Y=GDP

PART 7: PI

7.1 AREA:

Circle: $A = \pi r^2 = \frac{\pi d^2}{4} = \frac{Cd}{4}$

Cyclic Quadrilateral: $\sqrt{(s-a)(s-b)(s-c)(s-d)}$

Area of a sector (degrees) $A = \frac{Q}{360} \times \pi r^2$

Area of a sector (radians) $A = \frac{1}{2} r^2 \theta$

Area of a segment (degrees) $A = \frac{r^2}{2}\left(\frac{Q}{180} \times \pi - \sin Q\right)$

Area of an annulus: $A = \pi\left(r_2^{\ 2} - r_1^{\ 2}\right) = \pi\left(\frac{w}{2}\right)^2$

Ellipse : $A = \frac{\pi}{4} lw = \pi r_1 r_2$

7.2 VOLUME:

Cylinder: $V = \pi r^2 h$

Sphere: $V = \frac{4}{3}\pi r^3$

Cap of a Sphere: $V = \frac{1}{6}\pi h\left(3r_1^{\ 2} + h^2\right)$

Cone: $V = \frac{1}{3}\pi r^2 h$

Ice-cream & Cone: $V = \frac{1}{3}\pi r^2 (h + 2r)$

Doughnut: $V = 2\pi^2 r_2 r_1^{\ 2} = \frac{\pi^2}{4}(b+a)(b-a)^2$

Sausage: $V = \frac{\pi w^2}{4}\left(l - \frac{w}{3}\right)$

Ellipsoid: $V = \frac{4}{3}\pi r_1 r_2 r_3$

7.3 SURFACE AREA:

Sphere: $SA = 4\pi r^2$

Hemisphere: $SA = 3\pi r^2$

Doughnut: $SA = 4\pi^2 r_2 r_1 = \pi^2\left(b^2 - a^2\right)$

Sausage: $SA = \pi w l$

Cone: $SA = \pi r\left(r + \sqrt{r^2 + h^2}\right)$

7.4 MISELANIOUS:

Length of arc (degrees) $l = \frac{Q}{360} \times C = \frac{Q}{180} \times \pi r$

Length of chord (degrees) $l = 2r \times \sin\left(\frac{Q}{2}\right) = 2\sqrt{r^2 - h^2}$

Perimeter of an ellipse $P \approx \pi(r_1 + r_2)\left(\frac{1 + \frac{3(r_1 - r_2)^2}{(r_1 + r_2)^2}}{10 + \sqrt{4 - \frac{3(r_1 - r_2)^2}{(r_1 + r_2)^2}}}\right)$

7.6 PI:

$$\pi \approx 3.14159265358979323846264338327950288...$$

$$\pi = \frac{C}{d}$$

John Wallis: $\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \times ... = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}$

Isaac Newton: $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3 \times 2^3}\right) + \frac{1 \times 3}{2 \times 4}\left(\frac{1}{5 \times 2^5}\right) + \frac{1 \times 3 \times 6}{2 \times 4 \times 6}\left(\frac{1}{7 \times 2^7}\right) + ...$

James Gregory: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} ...$

Leonard Euler: $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

$$\frac{\pi}{4} = \frac{3}{4} \times \frac{5}{4} \times \frac{7}{8} \times \frac{11}{12} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{20} \times \frac{23}{24} \times \frac{29}{28} \times \frac{31}{32} \times ...$$

where the numerators are the odd primes; each denominator is the multiple of four nearest to the numerator.

$$`\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} - \frac{1}{13} + ...$$

If the denominator is a prime of the form 4m - 1, the sign is positive; if the denominator is 2 or a prime of the form 4m + 1, the sign is negative; for composite numbers, the sign is equal the product of the signs of its factors.

Jozef Hoene-Wronski: $\pi = \lim_{n \to \infty} \frac{4\,n\left((1+i)^{\left(\frac{1}{n}\right)} - (1-i)^{\left(\frac{1}{n}\right)}\right)}{i}$

Franciscus Vieta: $\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times ...$

Integrals:

$$\int_{-\infty}^{\infty} \operatorname{sech}(x)\,dx = \pi$$

$$\int_{-1}^{1} \sqrt{1 - x^2}\,dx = \frac{\pi}{2}$$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi$$

$$\int_{0}^{1} \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$$

Infinite Series:

$$\sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!} = \frac{\pi}{2}$$

$$12\sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}} = \frac{1}{\pi}$$

$$\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}} = \frac{1}{\pi}$$

$$\frac{\sqrt{3}}{6^5}\sum_{k=0}^{\infty} \frac{((4k)!)^2(6k)!}{9^{k+1}(12k)!(2k)!}\left(\frac{127169}{12k+1} - \frac{1070}{12k+5} - \frac{131}{12k+7} + \frac{2}{12k+11}\right) = \pi$$

$$\sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) = \pi$$

$$\frac{1}{2^6}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1} - \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{2^2}{10n+7} + \frac{1}{10n+9}\right) = \pi$$

See also: Zeta Function within Part 17

Continued Fractions:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ddots}}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}$$

7.7 CIRCLE GEOMETRY:

Radius of Circumscribed Circle for Rectangles: $r = \frac{\sqrt{a^2 + b^2}}{2}$

Radius of Circumscribed Circle for Squares: $r = \frac{a}{\sqrt{2}}$

Radius of Circumscribed Circle for Triangles: $r = \frac{a}{2\sin A}$

Radius of Circumscribed Circle for Quadrilaterals:

$$r = \frac{1}{4} \times \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

Radius of Inscribed Circle for Squares: $r = \frac{a}{2}$

Radius of Inscribed Circle for Triangles: $r = \frac{A}{s}$

Radius of Circumscribed Circle: $r = \frac{a}{2\sin\left(\frac{180}{n}\right)}$

Radius of Inscribed Circle: $r = \frac{a}{2\tan\left(\frac{180}{n}\right)}$

7.8 ABBREVIATIONS (7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7):

A=Angle ‘A’

A=Area

a=side 'a'
B=Angle 'B'
b=side 'b'
B=Angle 'B'
c=side 'c'
C=circumference
d=diameter
d=side 'd'
h=shortest length from the center to the chord
r=radius
r_1=radius 1 ($r_1 < r_2$)
r_2=radius 2 ($r_2 < r_3$)
r_3=radius 3
l=length
n=number of sides
P=perimeter
Q=central angle
s=semi-perimeter
w=width
w=length of chord from r_1

7.9 CRESCENT GEOMETRY:

Area of a lunar crescent: $A = \frac{1}{4}\pi cd$

Area of an eclipse crescent:

$$A = w^2\left(\pi - \frac{2\pi\left(\cos^{-1}\left(\frac{w^2 + l^2 - b^2}{2wl}\right)\right)}{360} + \frac{\sin 2\left(\cos^{-1}\left(\frac{w^2 + l^2 - b^2}{2wl}\right)\right)}{2}\right)$$

$$-b^2\left(\pi - \frac{2\pi\left(\cos^{-1}\left(\frac{w^2 + l^2 - b^2}{2wl}\right)\right)}{360} + \frac{\sin 2\left(\cos^{-1}\left(\frac{w^2 + l^2 - b^2}{2wl}\right)\right)}{2}\right)$$

7.10 ABBREVIATIONS (7.9):

A=Area
b=radius of black circle
c=width of the crescent
d=diameter
l=distance between the centres of the circles
w=radius of white circle

PART 8: PHYSICS

8.1 MOVEMENT:

Stopping distance: $$s = \frac{v^2}{-2a}$$

Centripetal acceleration: $$a = \frac{v^2}{r}$$

Centripetal force: $$F_C = ma = \frac{mv^2}{r}$$

Dropping time : $$t = \sqrt{\frac{2h}{g}}$$

Force: $$F = \frac{ma}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

Kinetic Energy: $$E_k = \frac{1}{2}mv^2$$

Maximum height of a cannon: $$h = \frac{(u \sin\theta)^2}{g}$$

Pendulum swing time: $$t = 2\pi\sqrt{\frac{l}{g}}$$

Potential Energy: $$E_p = mgh$$

Range of a cannon: $$s = t(u\cos\theta) = \frac{2u\sin\theta}{g} \times (u\cos\theta)$$

Time in flight of a cannon: $$t = \frac{2u\sin\theta}{g}$$

Universal Gravitation: $$F = G\frac{m_1 m_2}{r^2}$$

ABBREVIATIONS (8.1):

a=acceleration (negative if retarding)
c=speed of light (3×10^8 ms^{-1})
E_k=Kinetic Energy
E_p=potential energy
F=force
g=gravitational acceleration (≈9.81 on Earth)
G=gravitational constant = 6.67×10^{-11}
h=height
l=length of a pendulum
m=mass
m_1=mass 1
m_2=mass 2

r=radius
r=distance between two points
s=distance
t=time
u=initial speed
v=final speed
θ=the angle

8.2 CLASSICAL MECHANICS:

Newton's Laws:

First law: If an object experiences no net force, then its velocity is constant; the object is either at rest (if its velocity is zero), or it moves in a straight line with constant speed (if its velocity is nonzero).

$$\sum \mathbf{F} = 0 \;\Rightarrow\; \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0.$$

Second law: The acceleration a of a body is parallel and directly proportional to the net force F acting on the body, is in the direction of the net force, and is inversely proportional to the mass m of the body, i.e., F = ma.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}.$$

$$\mathbf{J} = \int_{\Delta t} \mathbf{F}\,\mathrm{d}t.$$

$$\mathbf{J} = \Delta\mathbf{p} = m\Delta\mathbf{v}.$$

$$\mathbf{F} + \mathbf{u}\frac{\mathrm{d}m}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

$$\mathbf{F} = m\mathbf{a}.$$

Third law: When two bodies interact by exerting force on each other, these forces (termed the action and the reaction) are equal in magnitude, but opposite in direction.

$$\sum \mathbf{F}_{a,b} = -\sum \mathbf{F}_{b,a}$$

Inertia:

$$p = mv$$

$$p = p_1 + p_2$$

$$= m_1 v_1 + m_2 v_2 .$$

$$\Delta p = F\Delta t .$$

$$\Delta p = \int_{t_1}^{t_2} F(t)dt .$$

$$\frac{dp_1}{dt} = -\frac{dp_2}{dt},$$

$$\frac{d}{dt}(p_1 + p_2) = 0.$$

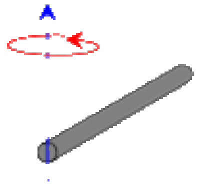
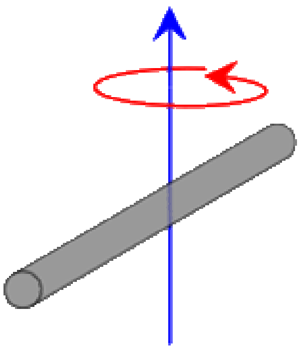
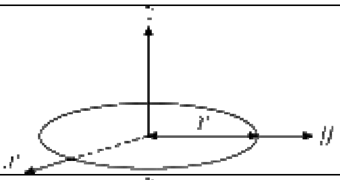
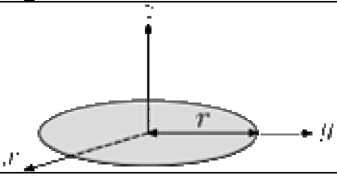
$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

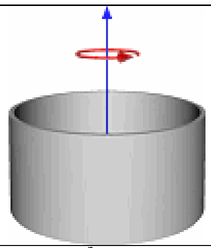
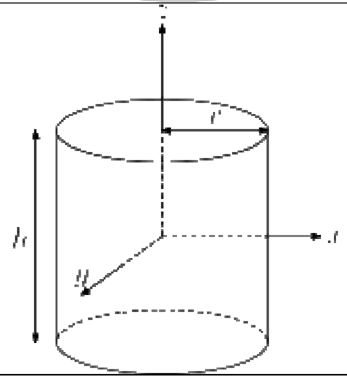
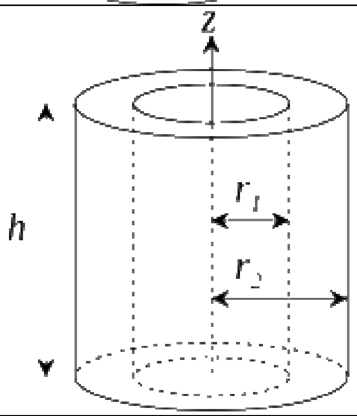
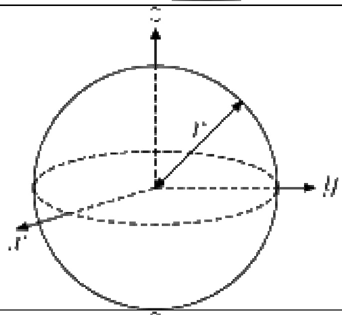
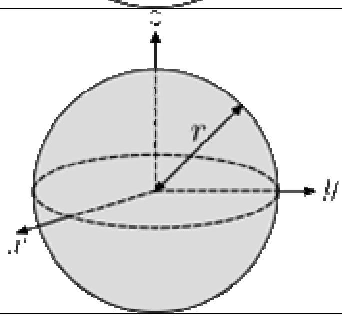
$$\tfrac{1}{2}m_1 u_1^2 + \tfrac{1}{2}m_2 u_2^2 = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 .$$

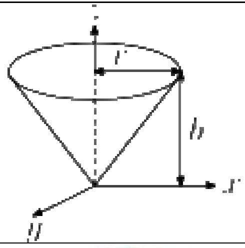
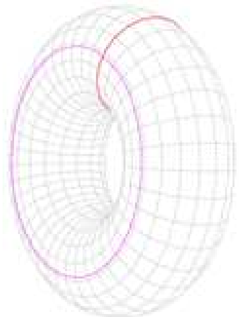
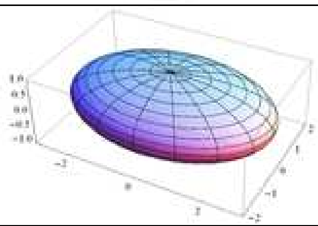
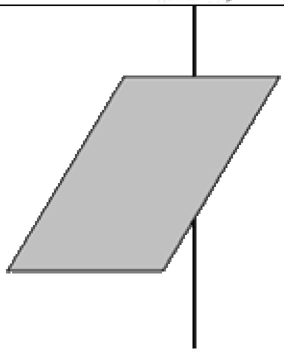
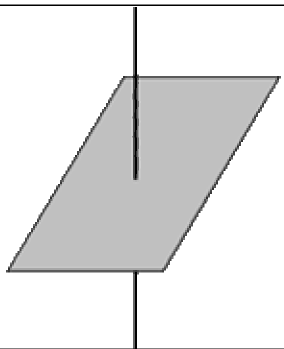
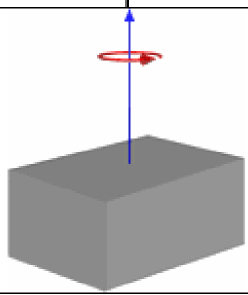
$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2$$
$$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 + \left(\frac{2m_1}{m_1 + m_2}\right) u_1 \,.$$
$$p_x = mv_x$$
$$p_y = mv_y$$
$$p_z = mv_z.$$

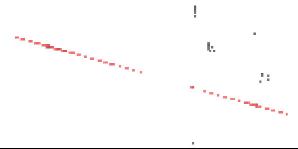
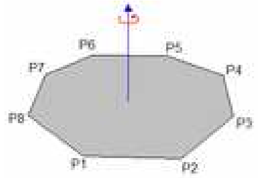
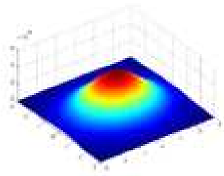
Moments of Inertia:

$$I_{\text{net}} = \sum_j I_j$$
$$I_d = I_{\text{com}} + Md^2$$
$$I_k \leq I_i + I_j$$

Description	Diagram	Formulae
Two point masses, *M* and *m*, with reduced mass μ and separated by a distance, *x*.		$I = mr^2$
Rod of length *L* and mass *m* (Axis of rotation at the end of the rod)		$I = \frac{Mm}{M+m}x^2 = \mu x^2$
Rod of length *L* and mass *m*		$I_{\text{end}} = \frac{mL^2}{3}$
Thin circular hoop of radius *r* and mass *m*		$I_{\text{center}} = \frac{mL^2}{12}$
Thin circular hoop of radius *r* and mass *m*		$I_z = mr^2$ $I_x = I_y = \frac{mr^2}{2}$
Thin, solid disk of radius *r* and mass *m*		$I_z = \frac{mr^2}{2}$ $I_x = I_y = \frac{mr^2}{4}$

Thin cylindrical shell with open ends, of radius r and mass m		$$I = mr^2$$
Solid cylinder of radius r, height h and mass m		$$I_z = \frac{mr^2}{2}$$ $$I_x = I_y = \frac{1}{12}m\left(3r^2 + h^2\right)$$
Thick-walled cylindrical tube with open ends, of inner radius r_1, outer radius r_2, length h and mass m		$$I_z = \frac{1}{2}m\left({r_1}^2 + {r_2}^2\right)$$ $$I_x = I_y = \frac{1}{12}m\left[3\left({r_2}^2 + {r_1}^2\right) + h^2\right]$$ or when defining the normalized thickness $t_n = t/r$ and letting $r = r_2$, then $$I_z = mr^2\left(1 - t_n + \frac{1}{2}{t_n}^2\right)$$
Sphere (hollow) of radius r and mass m		$$I = \frac{2mr^2}{3}$$
Ball (solid) of radius r and mass m		$$I = \frac{2mr^2}{5}$$

Right circular cone with radius *r*, height *h* and mass *m*		$I_z = \frac{3}{10}mr^2$ $I_x = I_y = \frac{3}{5}m\left(\frac{r^2}{4} + h^2\right)$
Torus of tube radius *a*, cross-sectional radius *b* and mass *m*.		About a diameter: $\frac{1}{8}\left(4a^2 + 5b^2\right)m$ About the vertical axis: $\left(a^2 + \frac{3}{4}b^2\right)m$
Ellipsoid (solid) of semiaxes *a*, *b*, and *c* with axis of rotation *a* and mass *m*		$I_a = \frac{m(b^2 + c^2)}{5}$
Thin rectangular plate of height *h* and of width *w* and mass *m* (Axis of rotation at the end of the plate)		$I_e = \frac{mh^2}{3} + \frac{mw^2}{12}$
Thin rectangular plate of height *h* and of width *w* and mass *m*		$I_c = \frac{m(h^2 + w^2)}{12}$
Solid cuboid of height *h*, width *w*, and depth *d*, and mass *m*		$I_h = \frac{1}{12}m\left(w^2 + d^2\right)$ $I_w = \frac{1}{12}m\left(h^2 + d^2\right)$ $I_d = \frac{1}{12}m\left(h^2 + w^2\right)$

Solid cuboid of height *D*, width *W*, and length *L*, and mass *m* with the longest diagonal as the axis.		$$I = \frac{m\,(W^2D^2 + L^2D^2 + L^2W^2)}{6\,(L^2 + W^2 + D^2)}$$
Plane polygon with vertices $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$, ..., $\vec{P}_N$ and mass m uniformly distributed on its interior, rotating about an axis perpendicular to the plane and passing through the origin.		
	$$I = \frac{m}{6}\frac{\sum\limits_{n=1}^{N-1} \Vert\vec{P}_{n+1} \times \vec{P}_n\Vert((\vec{P}_{n+1} \cdot \vec{P}_{n+1}) + (\vec{P}_{n+1} \cdot \vec{P}_n) + (\vec{P}_n \cdot \vec{P}_n))}{\sum\limits_{n=1}^{N-1} \Vert\vec{P}_{n+1} \times \vec{P}_n\Vert}$$	
Infinite disk with mass normally distributed on two axes around the axis of rotation (i.e. $\rho(x,y) = \frac{m}{2\pi ab} e$ Where : $\rho(x,y)$ is the mass-density as a function of x and y).		$I = m(a^2 + b^2)$

Velocity and Speed:

$$v_{AVE} = \frac{\Delta P}{\Delta t}$$

$$\mathbf{V} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{P}}{\Delta t} = \frac{d\mathbf{P}}{dt} = \dot{\mathbf{P}} = \dot{x}_p\vec{i} + \dot{y}_P\vec{j} + \dot{z}_P\vec{k}.$$

$$|\mathbf{V}| = |\dot{\mathbf{P}}| = \frac{ds}{dt},$$

$$\mathbf{V}(t) = \int_0^t \mathbf{A}dt = \mathbf{A}t + \mathbf{V}_0.$$

$$v^2 = v_x^2 + v_y^2 + v_z^2.$$

Acceleration:

$$a_{AVE} = \frac{\Delta V}{\Delta t}$$

$$\mathbf{A} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{V}}{\Delta t} = \frac{d\mathbf{V}}{dt} = \dot{\mathbf{V}} = \ddot{\mathbf{P}} = \ddot{x}_p\vec{i} + \ddot{y}_P\vec{j} + \ddot{z}_P\vec{k}.$$

Trajectory (Displacement):

$$\mathbf{P}(t) = \int_0^t \mathbf{V}(t)dt = \int(\mathbf{A}t + \mathbf{V}_0)dt = \tfrac{1}{2}\mathbf{A}t^2 + \mathbf{V}_0 t + \mathbf{P}_0.$$

$$\mathbf{P}(t) = \mathbf{P}_0 + \left(\frac{\mathbf{V} + \mathbf{V}_0}{2}\right) t.$$

$$(\mathbf{P} - \mathbf{P}_0) \cdot \mathbf{A}t = (\mathbf{V} - \mathbf{V}_0) \cdot \frac{\mathbf{V} + \mathbf{V}_0}{2} t \; .$$

Kinetic Energy:

$$E_k = \tfrac{1}{2} m v^2$$

$$E_k = \frac{p^2}{2m}$$

$$E_k = \int \mathbf{F} \cdot d\mathbf{x} = \int \mathbf{v} \cdot d(m\mathbf{v}) = \int d\left(\frac{mv^2}{2}\right) = \frac{mv^2}{2}.$$

$$E_k = E_i + \frac{MV^2}{2}.$$

$$E_r = \int \frac{v^2 dm}{2} = \int \frac{(r\omega)^2 dm}{2} = \frac{\omega^2}{2} \int r^2 dm = \frac{\omega^2}{2} I = \tfrac{1}{2} I \omega^2$$

Centripetal Force:

$$F = ma_c = \frac{mv^2}{r}$$

$$F = mr\omega^2 \, .$$

$$F = mr\frac{4\pi^2}{T^2} \, .$$

Circular Motion:

$$\omega = \frac{2\pi}{T} = 2\pi f = \frac{|v|}{|r|} \, ;$$

$$\omega_{cyc} = \omega_{rad}/2\pi$$

$$\omega_{cyc} = \omega_{deg}/360$$

$$v = \frac{2\pi r}{T} = \omega r$$

$$\theta = 2\pi \frac{t}{T} = \omega t$$

$$a = \frac{v^2}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$, or

$$\alpha = \frac{a_T}{r} \, ,$$

$$\alpha = \frac{\tau}{I} \, .$$

Angular Momentum:

$$\mathbf{L} = I\boldsymbol{\omega} \, .$$

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} \, .$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\mathbf{L} = \sum_i \mathbf{R}_i \times m_i \mathbf{V}_i$$
$$\mathbf{L} = \sum_i \mathbf{R}_i \times m_i \mathbf{V}_i$$

Torque:

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0 + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$
$$\tau = rF \sin\theta$$
$$\boldsymbol{\tau}_{\mathrm{net}} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}(I\boldsymbol{\omega})}{\mathrm{d}t} = I\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = I\boldsymbol{\alpha},$$
$$|\tau| = (\text{moment arm})(\text{force}).$$

Work:

$$W = \int_{\theta_1}^{\theta_2} \tau \ \mathrm{d}\theta,$$

Laws of Conservation:

Momentum: $\frac{d}{dt}(p_1 + p_2) = 0.$

Energy: $\sum E_{IN} = \sum E_{OUT}$

Force: $\sum F_{NET} = 0 \Rightarrow \sum F_{UP} = \sum F_{DN}, \sum F_L = \sum F_R, \sum cm = \sum acm$

ABBREVIATIONS (8.2)

a=acceleration
E_K=Kinetic Energy
E_r=rotational kinetic energy
F=force
I=mass moment of inertia
J=impulse
L=angular momentum
m=mass
P=path
p=momentum
t=time
v=velocity
W=work
τ=torque

8.3 RELATIVISTIC EQUATIONS:

Kinetic Energy:

$$E_k = \frac{mc^2}{\sqrt{1-(v/c)^2}} - mc^2$$

Momentum:

$$p = \frac{mv}{\sqrt{1-(v/c)^2}}$$

Time Dilation:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}}$$

Length Contraction:

$$L = L_0 \cdot \sqrt{1-\frac{v^2}{c^2}}.$$

Relativistic Mass:

$$m = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

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PART 9: TRIGONOMETRY

9.1 CONVERSIONS:

Degrees	30°	60°	120°	150°	210°	240°	300°	330°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
Grads	33⅓ grad	66⅔ grad	133⅓ grad	166⅔ grad	233⅓ grad	266⅔ grad	333⅓ grad	366⅔ grad
Degrees	45°	90°	135°	180°	225°	270°	315°	360°
Radians	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Grads	50 grad	100 grad	150 grad	200 grad	250 grad	300 grad	350 grad	400 grad

9.2 BASIC RULES:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Sin Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cos Rule: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

Tan Rule:

$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}$$

$$\frac{\tan\frac{A-C}{2}}{\tan\frac{A+C}{2}} = \frac{a-c}{a+c}$$

$$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{b-c}{b+c}$$

Auxiliary Angle:

$$a\sin x \pm b\cos x = R\sin(x \pm \alpha), \quad 0 < \alpha < \tfrac{\pi}{2}$$

$$\text{where } R^2 = a^2 + b^2, \quad \tan\alpha = \frac{b}{a}$$

Pythagoras Theorem: $a^2 + b^2 = c^2$

9.3 RECIPROCAL FUNCTIONS

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

9.4 BASIC IDENTITES:

Pythagorean Identity: $\cos^2\theta + \sin^2\theta = 1$

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = +\cos\theta$$
$$\tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\sec(-\theta) = +\sec\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sin(\tfrac{\pi}{2} - \theta) = +\cos\theta$$
$$\cos(\tfrac{\pi}{2} - \theta) = +\sin\theta$$
$$\tan(\tfrac{\pi}{2} - \theta) = +\cot\theta$$
$$\csc(\tfrac{\pi}{2} - \theta) = +\sec\theta$$
$$\sec(\tfrac{\pi}{2} - \theta) = +\csc\theta$$
$$\cot(\tfrac{\pi}{2} - \theta) = +\tan\theta$$
$$\sin(\pi - \theta) = +\sin\theta$$
$$\cos(\pi - \theta) = -\cos\theta$$
$$\tan(\pi - \theta) = -\tan\theta$$
$$\csc(\pi - \theta) = +\csc\theta$$
$$\sec(\pi - \theta) = -\sec\theta$$
$$\cot(\pi - \theta) = -\cot\theta$$

9.5 IDENTITIES (SINΘ):

- $\sin\theta$
- $\pm\sqrt{1 - \cos^2\theta}$
- $\pm\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}$
- $\frac{1}{\csc\theta}$
- $\pm\frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}$

- $\pm\frac{1}{\sqrt{1+\cot^2\theta}}$

9.6 IDENTITIES (COSΘ):

- $\pm\sqrt{1-\sin^2\theta}$
- $\cos\theta$
- $\pm\frac{1}{\sqrt{1+\tan^2\theta}}$
- $\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$
- $\frac{1}{\sec\theta}$
- $\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$

9.7 IDENTITIES (TANΘ):

- $\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$
- $\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$
- $\tan\theta$
- $\pm\frac{1}{\sqrt{\csc^2\theta-1}}$
- $\pm\sqrt{\sec^2\theta-1}$
- $\frac{1}{\cot\theta}$

9.8 IDENTITIES (CSCΘ):

- $\frac{1}{\sin\theta}$
- $\pm\frac{1}{\sqrt{1-\cos^2\theta}}$
- $\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$
- $\csc\theta$
- $\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$
- $\pm\sqrt{1+\cot^2\theta}$

9.9 IDENTITIES (COTΘ):

- $\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$

- $\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$
- $\frac{1}{\tan\theta}$
- $\pm\sqrt{\csc^2\theta-1}$
- $\pm\frac{1}{\sqrt{\sec^2\theta-1}}$
- $\cot\theta$

9.10 ADDITION FORMULAE:

Sine: $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

Cosine: $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

Tangent: $\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

Arcsine:

$$\arcsin\alpha\pm\arcsin\beta=\arcsin(\alpha\sqrt{1-\beta^2}\pm\beta\sqrt{1-\alpha^2})$$

Arccosine:

$$\arccos\alpha\pm\arccos\beta=\arccos(\alpha\beta\mp\sqrt{(1-\alpha^2)(1-\beta^2)})$$

Arctangent: $\arctan\alpha\pm\arctan\beta=\arctan\left(\frac{\alpha\pm\beta}{1\mp\alpha\beta}\right)$

9.11 DOUBLE ANGLE FORMULAE:

Sine:

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

Generally,

$$\sin(nx)=\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\sin\left(\frac{1}{2}(n-k)\pi\right)$$

Cosine:

$$\cos 2\theta=\cos^2\theta-\sin^2\theta = 2\cos^2\theta-1 = 1-2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

Generally,

$$\cos(nx)=\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\cos\left(\frac{1}{2}(n-k)\pi\right)$$

Tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Generally.

$$\tan(nx) = \frac{\sin(nx)}{\cos(nx)} = \frac{\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\sin\left(\frac{1}{2}(n-k)\pi\right)}{\sum_{k=0}^{n}\binom{n}{k}\cos^k(x)\sin^{n-k}(x)\cos\left(\frac{1}{2}(n-k)\pi\right)}$$

Cot:

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

9.12 TRIPLE ANGLE FORMULAE:

Sine:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Cosine:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Tangent:

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Cot:

$$\cot 3\theta = \frac{3\cot\theta - \cot^3\theta}{1-3\cot^2\theta}$$

9.13 HALF ANGLE FORMULAE:

Sine:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Cosine:

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Tangent:

$$\begin{aligned}\tan\frac{\theta}{2} &= \csc\theta - \cot\theta \\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ &= \frac{\sin\theta}{1+\cos\theta} \\ &= \frac{1-\cos\theta}{\sin\theta}\end{aligned}$$

Cot:

$$\cot\frac{\theta}{2} = \csc\theta + \cot\theta$$
$$= \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$
$$= \frac{\sin\theta}{1-\cos\theta}$$
$$= \frac{1+\cos\theta}{\sin\theta}$$

9.14 POWER REDUCTION:

Sine:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$
$$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$
$$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$$

If n is even:

$$\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{(\frac{n}{2}-k)}\binom{n}{k}\cos((n-2k)\theta)$$

If n is odd:

$$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{(\frac{n-1}{2}-k)}\binom{n}{k}\sin((n-2k)\theta)$$

Cosine:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$$
$$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$$
$$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$$

If n is even:

$$\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos((n-2k)\theta)$$

If n is odd:

$$\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos((n-2k)\theta)$$

Sine & Cosine:

$$\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$$
$$\sin^3\theta\cos^3\theta = \frac{3\sin 2\theta - \sin 6\theta}{32}$$
$$\sin^4\theta\cos^4\theta = \frac{3-4\cos 4\theta + \cos 8\theta}{128}$$
$$\sin^5\theta\cos^5\theta = \frac{10\sin 2\theta - 5\sin 6\theta + \sin 10\theta}{512}$$

9.15 PRODUCT TO SUM:

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$$
$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$
$$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$$
$$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$$

9.16 SUM TO PRODUCT:

$$\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$$
$$\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$
$$\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

9.17 HYPERBOLIC EXPRESSIONS:

Hyperbolic sine:
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{e^{2x}-1}{e^{2x}+1}$$

Hyperbolic cotangent:
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\frac{1}{2}(e^x + e^{-x})}{\frac{1}{2}(e^x - e^{-x})} = \frac{e^{2x}+1}{e^{2x}-1}$$

Hyperbolic secant: $$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant: $$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

9.18 HYPERBOLIC RELATIONS:

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$
$$\tanh(-x) = -\tanh x$$
$$\coth(-x) = -\coth x$$
$$\operatorname{sech}(-x) = \operatorname{sech} x$$
$$\operatorname{csch}(-x) = -\operatorname{csch} x$$
$$\operatorname{arsech} x = \operatorname{arcosh} \frac{1}{x}$$
$$\operatorname{arcsch} x = \operatorname{arsinh} \frac{1}{x}$$
$$\operatorname{arcoth} x = \operatorname{artanh} \frac{1}{x}$$

9.19 MACHIN-LIKE FORMULAE:

Form:

$$\frac{\pi}{4} = \sum_n^N a_n \arctan \frac{1}{b_n}$$

Formulae:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$$
$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$
$$\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{7}$$
$$\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$$
$$\frac{\pi}{4} = 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}$$
$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110443}$$
$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943}$$

$$\frac{\pi}{4} = 183\arctan\frac{1}{239} + 32\arctan\frac{1}{1023} - 68\arctan\frac{1}{5832} + 12\arctan\frac{1}{110443} - 12\arctan\frac{1}{4841182} - 100\arctan\frac{1}{6826318}$$

$$\frac{\pi}{4} = 183\arctan\frac{1}{239} + 32\arctan\frac{1}{1023} - 68\arctan\frac{1}{5832} + 12\arctan\frac{1}{113021} - 100\arctan\frac{1}{6826318} - 12\arctan\frac{1}{33366019650} + 12\arctan\frac{1}{43599522992503626068}$$

Identities:

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy} \text{ for } xy < 1,$$

$$\arctan x - \arctan y = \arctan\frac{x-y}{1+xy} \text{ for } xy > -1,$$

$$\arctan\frac{a}{b} + \arctan\frac{c}{d} = \arctan\frac{ad+bc}{bd-ac} \text{ for } \frac{ac}{bd} < 1,$$

$$\arctan\frac{a}{b} - \arctan\frac{c}{d} = \arctan\frac{ad-bc}{bd+ac} \text{ for } \frac{ac}{bd} > -1.$$

9.20 SPHERICAL TRIANGLE IDENTITIES:

$$\frac{\sin\left(\frac{1}{2}(A-B)\right)}{\sin\left(\frac{1}{2}(A+B)\right)} = \frac{\tan\left(\frac{1}{2}(a-b)\right)}{\tan\left(\frac{1}{2}c\right)}$$

$$\frac{\sin\left(\frac{1}{2}(a-b)\right)}{\sin\left(\frac{1}{2}(a+b)\right)} = \frac{\tan\left(\frac{1}{2}(A-B)\right)}{\cot\left(\frac{1}{2}c\right)}$$

$$\frac{\cos\left(\frac{1}{2}(A-B)\right)}{\cos\left(\frac{1}{2}(A+B)\right)} = \frac{\tan\left(\frac{1}{2}(a+b)\right)}{\tan\left(\frac{1}{2}c\right)}$$

$$\frac{\cos\left(\frac{1}{2}(a-b)\right)}{\cos\left(\frac{1}{2}(a+b)\right)} = \frac{\tan\left(\frac{1}{2}(A+B)\right)}{\cot\left(\frac{1}{2}c\right)}$$

9.21 ABBREVIATIONS (9.1-9.19)

A=Angle ‘A’

a=side ‘a’
B=Angle ‘B’
b=side ‘b’
B=Angle ‘B’
c=side ‘c’

PART 10: EXPONENTIALS & LOGARITHIMS

10.1 FUNDAMENTAL THEORY:

$$e^{\ln(x)} = x \qquad \text{if } x > 0$$
$$\ln(e^x) = x.$$
$$\ln x = \int_1^x \frac{1}{t} dt$$
$$x = b^a \iff \log_b x = a$$

10.2 IDENTITIES:

$$\log_b(1) = 0$$
$$\log_b(b) = 1$$
$$\log_b(xy) = \log_b(x) + \log_b(y)$$
$$\log_b\left(\tfrac{x}{y}\right) = \log_b(x) - \log_b(y)$$
$$\log_b(x^d) = d\log_b(x)$$
$$\log_b(\sqrt[y]{x}) = \frac{\log_b(x)}{y}$$
$$x^{\log_b(y)} = y^{\log_b(x)}$$
$$c\log_b(x) + d\log_b(y) = \log_b(x^c y^d)$$
$$b^{\log_b(x)} = x \text{ because } \operatorname{antilog}_b(\log_b(x)) = x$$
$$\log_b(b^x) = x \text{ because } \log_b(\operatorname{antilog}_b(x)) = x$$
$$\log(\log(c^d)) = \log(\log(c)) + \log(d)$$
$$\log(\log(\sqrt[d]{c})) = \log(\log(c)) - \log(d)$$
$$\log_b(a+c) = \log_b a + \log_b\left(1 + \frac{c}{a}\right)$$
$$\log_b(a-c) = \log_b a + \log_b\left(1 - \frac{c}{a}\right)$$
$$x^{\frac{\log(\log(x))}{\log(x)}} = \log(x)$$
$$\int \log_a x\,dx = x(\log_a x - \log_a e) + C$$
$$\ln(a \times 10^n) = \ln a + n\ln 10.$$

10.3 CHANGE OF BASE:

$$\log_b x = \frac{\log_k x}{\log_k b}.$$

10.4 LAWS FOR LOG TABLES:

$$xy = b^{\log_b(x)}b^{\log_b(y)} = b^{\log_b(x)+\log_b(y)} \Rightarrow \log_b(xy) = \log_b(b^{\log_b(x)+\log_b(y)}) = \log_b(x)+\log_b$$
$$x^y = (b^{\log_b(x)})^y = b^{y\log_b(x)} \Rightarrow \log_b(x^y) = y\log_b(x)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(xy^{-1}) = \log_b(x) + \log_b(y^{-1}) = \log_b(x) - \log_b(y)$$

$$\log_b(\sqrt[y]{x}) = \log_b(x^{\frac{1}{y}}) = \frac{1}{y}\log_b(x)$$

10.5 COMPLEX NUMBERS:

$$\log(z) = \ln|z| + i\arg(z) = \ln(r) + i\,(\theta + 2\pi k)$$
$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = \operatorname{Log}(z_1 z_2) \pmod{2\pi i}$$
$$\operatorname{Log}(z_1) - \operatorname{Log}(z_2) = \operatorname{Log}(z_1/z_2) \pmod{2\pi i}$$
$${z_1}^{z_2} = e^{z_2 \operatorname{Log}(z_1)}$$
$$\operatorname{Log}\left({z_1}^{z_2}\right) = z_2 \operatorname{Log}(z_1) \pmod{2\pi i}$$

10.6 LIMITS INVOLVING LOGARITHMIC TERMS

$$\lim_{x\to 0^+} \log_a x = -\infty \quad \text{if } a > 1$$
$$\lim_{x\to 0^+} \log_a x = \infty \quad \text{if } a < 1$$
$$\lim_{x\to \infty} \log_a x = \infty \quad \text{if } a > 1$$
$$\lim_{x\to \infty} \log_a x = -\infty \quad \text{if } a < 1$$
$$\lim_{x\to 0^+} x^b \log_a x = 0 \quad \text{if } b > 0$$
$$\lim_{x\to \infty} \frac{1}{x^b} \log_a x = 0 \quad \text{if } b > 0$$

PART 11: COMPLEX NUMBERS

11.1 GENERAL:

Fundamental: $i^2 = -1$

Standard Form: $z = a + bi$

Polar Form: $z = rcis\theta = r(\cos\theta + i\sin\theta)$

Argument: $\arg(z) = \theta$, where $\tan\theta = \frac{b}{a}$

Modulus: $\mathrm{mod}(z) = r = |z| = |a + bi| = \sqrt{a^2 + b^2}$

Conjugate: $\bar{z} = a - bi$

Exponential: $z = r \cdot e^{i\theta}$

De Moivre's Formula:

$$z = rcis\theta$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} cis\left(\frac{\theta + 2k\pi}{n}\right), \text{ k=0,1,…,(n-1)}$$

Euler's Identity: $e^{i\pi} + 1 = 0$ (Special Case when n=2)

$$\sum_{k=0}^{n-1} e^{\frac{2i\pi k}{n}} = 0 \quad \text{(Generally)}$$

11.2 OPERATIONS:

Addition: (a + bi) + (c + di) = (a + c) + (b + d)i

Subtraction: (a + bi) - (c + di) = (a - c) + (b - d)i.

Multiplication: $(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i.$

Division:

$$\frac{(a + bi)}{(c + di)} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bci - adi + bd}{(c + di)(c - di)} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i.$$

Sum of Squares: $a^2 + b^2 = (a + bi)(a - bi)$

11.3 IDENTITIES:

Exponential: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Logarithmic: $\log(x + iy) = \frac{1}{2}\ln(x^2 + y^2) + i\arg(x + iy),$

Trigonometric:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{\sin(x)}{\cos(x)}$$

$$sin(a + bi) = sin(a)cosh(b) + cos(a)sinh(b)i$$

$$cos(a+bi) = cos(a)cosh(b) - sin(a)sinh(b)i$$

Hyperbolic:

$$e^{ix} = \cos x + i\ \sin x$$
$$e^{-ix} = \cos x - i\ \sin x$$
$$\cosh ix = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$$
$$\sinh ix = \frac{1}{2}(e^{ix} - e^{-ix}) = i \sin x$$
$$\tanh ix = i \tan x$$
$$\cosh x = \cos ix$$
$$\sinh x = -i \sin ix$$
$$\tanh x = -i \tan ix$$

PART 12: DIFFERENTIATION

For Differential Equations, see Functions

12.1 GENERAL RULES:

Plus Or Minus:

$$y = f_{(x)} \pm g_{(x)} \pm h_{(x)} \ldots$$
$$y' = f'_{(x)} \pm g'_{(x)} \pm h'_{(x)} \ldots$$

Product Rule:

$$y = uv$$
$$y' = u'v + uv'$$

Quotient Rule:

$$y = \frac{u}{v}$$
$$y' = \frac{u'v - uv'}{v^2}$$

Power Rule:

$$y = \left(f_{(x)}\right)^n$$
$$y' = n\left(f_{(x)}\right)^{n-1} \times f'_{(x)}$$

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

Blob Rule:

$$y = e^{f_{(x)}}$$
$$y' = f'_{(x)} \times e^{f_{(x)}}$$

Base A Log:

$$y = \log_a f_{(x)}$$
$$y' = \frac{f'_{(x)}}{f_{(x)} \times \ln(a)}$$

Natural Log:

$$y = a\ln\left(f_{(x)}\right)$$
$$y' = a \times \frac{f'_{(x)}}{f_{(x)}}$$

Exponential (X):

$$y = k^x$$
$$y' = \ln k \times k^x$$

First Principles:

$$f'_{(x)} = \lim_{h \longrightarrow 0}\left(\frac{f_{(x+h)} - f_{(x)}}{h}\right)$$

12.2 EXPONETIAL FUNCTIONS:

$$\frac{d}{dx}e^x = e^x.$$
$$\frac{d}{dx}a^x = (\ln a)a^x.$$

12.3 LOGARITHMIC FUNCTIONS:

$$\frac{d}{dx}\ln x = \frac{1}{x} = \frac{1}{x\ln e}, \qquad x > 0$$
$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}, \qquad b > 0, b \neq 1$$
$$\frac{d}{dx}\log_b(x) = \frac{d}{dx}\frac{\ln(x)}{\ln(b)} = \frac{1}{x\ln(b)} = \frac{\log_b(e)}{x}.$$

12.4 TRIGONOMETRIC FUNCTIONS:

$$(\sin(x))' = \cos(x)$$
$$(\cos(x))' = -\sin(x)$$
$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$
$$(\cot(x))' = \left(\frac{\cos(x)}{\sin(x)}\right)' = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = -(1 + \cot^2(x)) = -\csc^2(x)$$
$$(\sec(x))' = \left(\frac{1}{\cos(x)}\right)' = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}.\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$
$$(\csc(x))' = \left(\frac{1}{\sin(x)}\right)' = -\frac{\cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}.\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$
$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$
$$(\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$$
$$(\arctan(x))' = \frac{1}{x^2+1}$$

12.5 HYPERBOLIC FUNCTIONS:

$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \mathrm{sech}^2 x = 1/\cosh^2 x$$
$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\mathrm{csch}^2 x = -1/\sinh^2 x$$

$$\frac{d}{dx}\operatorname{csch} x = -\coth x \operatorname{csch} x$$

$$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1+x^2}}$$

12.5 PARTIAL DIFFERENTIATION:

First Principles:

$$\boxed{\frac{\partial f}{\partial x}(a,b) = \frac{d(f(x,b))}{dx}\Big|_{x=a} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}}$$

$$\boxed{\frac{\partial f}{\partial y}(a,b) = \frac{d(f(a,y))}{dy}\Big|_{y=b} = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}}$$

ie:

$$\boxed{\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}}$$

$$\boxed{\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}}$$

Gradient:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

Total Differential:

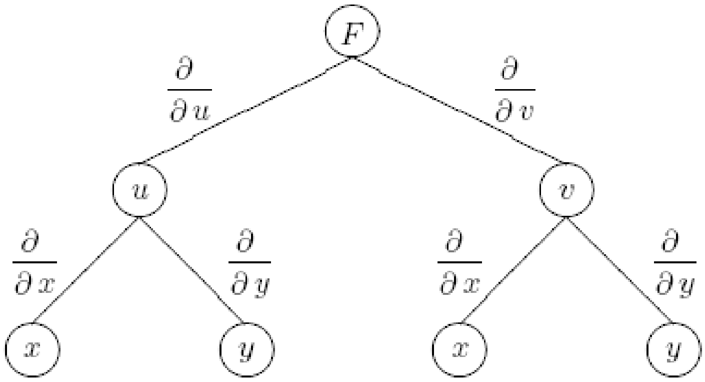
$$\boxed{df = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

Chain Rule:

Case 1. Suppose $F = F(u, v)$ and $u = u(x, y)$, $v = v(x, y)$. Then F is also a function of x and y and

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

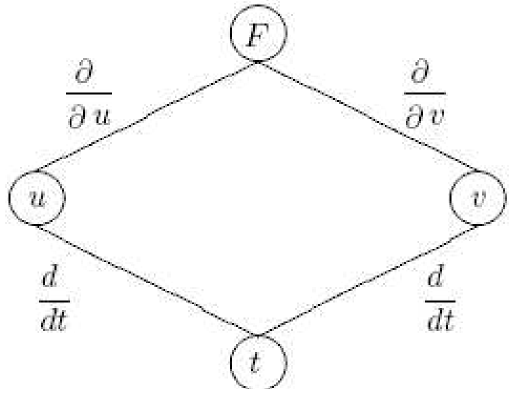
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}$$



Case 2. Suppose $F = F(u, v)$ and $u = u(t)$, $v = v(t)$. Then F is a function of t only and

$$\frac{dF}{dt} = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt}$$

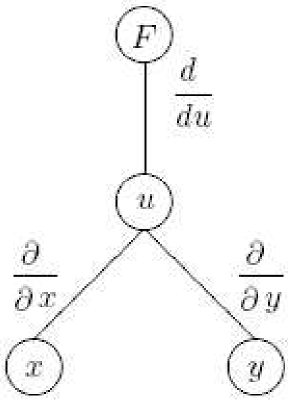
(Note the 'full' derivative here)



Case 3. $F = F(u)$, $u = u(x, y)$. Then F is a function of x and y and

$$\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}$$

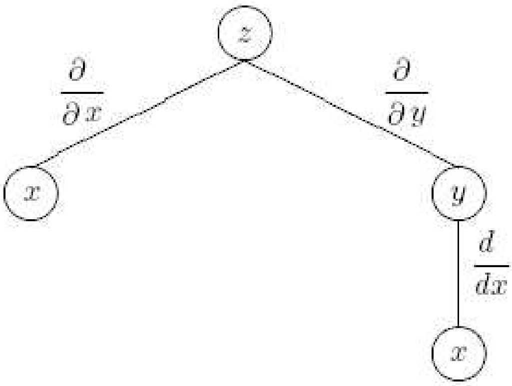
$$\frac{\partial F}{\partial y} = \frac{dF}{du}\frac{\partial u}{\partial y}$$



Implicit Differentiation:

Consider $z = F(x, y)$ where we assume the y is a function of x, *i.e.* $y = y(x)$. Then z is ultimately a function of x only and the chain rule tells us

$$\frac{dz}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$



Therefore,

$$\boxed{\frac{dy}{dx} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial y} = -\frac{F_x}{F_y}}$$

Higher Order Derivatives:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &\equiv \frac{\partial^2 f}{\partial x^2} \equiv f_{xx} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &\equiv \frac{\partial^2 f}{\partial y^2} \equiv f_{yy} \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &\equiv \frac{\partial^2 f}{\partial x \, \partial y} \equiv f_{yx} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &\equiv \frac{\partial^2 f}{\partial y \, \partial x} \equiv f_{xy}
\end{aligned}$$

PART 13: INTEGRATION

13.1 GENERAL RULES:

Power Rule:

$$\int f'_{(x)}\left[f_{(x)}\right]^{n}dx=\frac{\left[f_{(x)}\right]^{n+1}}{n+1}+C$$

$$a\int f'_{(x)}\left[f_{(x)}\right]^{n}dx=a\frac{\left[f_{(x)}\right]^{n+1}}{n+1}+C$$

By Parts:

$$\int udv=uv-\int vdu$$

Constants:

$$\int_{0}^{f(x)}kdy=kf(x)$$

13.2 RATIONAL FUNCTIONS:

$$\int (ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}+C \qquad (\text{for } n\neq -1)$$

$$\int \frac{c}{ax+b}dx=\frac{c}{a}\ln|ax+b|+C$$

$$\int x(ax+b)^n dx=\frac{a(n+1)x-b}{a^2(n+1)(n+2)}(ax+b)^{n+1}+C \qquad (\text{for } n\notin\{-1,-2\})$$

$$\int \frac{x}{ax+b}dx=\frac{x}{a}-\frac{b}{a^2}\ln|ax+b|+C$$

$$\int \frac{x}{(ax+b)^2}dx=\frac{b}{a^2(ax+b)}+\frac{1}{a^2}\ln|ax+b|+C$$

$$\int \frac{x}{(ax+b)^n}dx=\frac{a(1-n)x-b}{a^2(n-1)(n-2)(ax+b)^{n-1}}+C \qquad (\text{for } n\notin\{1,2\})$$

$$\int \frac{x^2}{ax+b}dx=\frac{b^2\ln(|ax+b|)}{a^3}+\frac{ax^2-2bx}{2a^2}+C$$

$$\int \frac{x^2}{(ax+b)^2}dx=\frac{1}{a^3}\left(ax-2b\ln|ax+b|-\frac{b^2}{ax+b}\right)+C$$

$$\int \frac{x^2}{(ax+b)^3}dx=\frac{1}{a^3}\left(\ln|ax+b|+\frac{2b}{ax+b}-\frac{b^2}{2(ax+b)^2}\right)+C$$

$$\int \frac{x^2}{(ax+b)^n}dx=\frac{1}{a^3}\left(-\frac{(ax+b)^{3-n}}{(n-3)}+\frac{2b(ax+b)^{2-n}}{(n-2)}-\frac{b^2(ax+b)^{1-n}}{(n-1)}\right)+C \qquad (\text{for } n\notin\{1,2,3\})$$

$$\int \frac{1}{x(ax+b)}dx=-\frac{1}{b}\ln\left|\frac{ax+b}{x}\right|+C$$

$$\int \frac{1}{x^2(ax+b)}dx=-\frac{1}{bx}+\frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right|+C$$

$$\int \frac{1}{x^2(ax+b)^2}dx=-a\left(\frac{1}{b^2(ax+b)}+\frac{1}{ab^2x}-\frac{2}{b^3}\ln\left|\frac{ax+b}{x}\right|\right)+C$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{1}{x^2-a^2}dx = \begin{cases} -\frac{1}{a}\operatorname{arctanh}\frac{x}{a} = \frac{1}{2a}\ln\frac{a-x}{a+x} + C & (\text{for } |x| < |a|) \\ -\frac{1}{a}\operatorname{arccoth}\frac{x}{a} = \frac{1}{2a}\ln\frac{x-a}{x+a} + C & (\text{for } |x| > |a|) \end{cases}$$

For $a \neq 0:$

$$\int \frac{1}{ax^2+bx+c}dx = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (\text{for } 4ac-b^2>0) \\ -\frac{2}{\sqrt{b^2-4ac}}\operatorname{arctanh}\frac{2ax+b}{\sqrt{b^2-4ac}} + C = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & (\text{for } 4ac-b^2<0) \\ -\frac{2}{2ax+b} + C & (\text{for } 4ac-b^2=0) \end{cases}$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c} + C$$

$$\int \frac{mx+n}{ax^2+bx+c}dx = \begin{cases} \frac{m}{2a}\ln\left|ax^2+bx+c\right| + \frac{2an-bm}{a\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C & (\text{for } 4ac-b^2>0) \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a\sqrt{b^2-4ac}}\operatorname{arctanh}\frac{2ax+b}{\sqrt{b^2-4ac}} + C & (\text{for } 4ac-b^2<0) \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a(2ax+b)} + C & (\text{for } 4ac-b^2=0) \end{cases}$$

$$\int \frac{1}{(ax^2+bx+c)^n}dx = \frac{2ax+b}{(n-1)(4ac-b^2)(ax^2+bx+c)^{n-1}} + \frac{(2n-3)2a}{(n-1)(4ac-b^2)}\int\frac{1}{(ax^2+bx+c)^{n-1}}dx + C$$

$$\int \frac{x}{(ax^2+bx+c)^n}dx = -\frac{bx+2c}{(n-1)(4ac-b^2)(ax^2+bx+c)^{n-1}} - \frac{b(2n-3)}{(n-1)(4ac-b^2)}\int\frac{1}{(ax^2+bx+c)^{n-1}}dx + C$$

$$\int \frac{1}{x(ax^2+bx+c)}dx = \frac{1}{2c}\ln\left|\frac{x^2}{ax^2+bx+c}\right| - \frac{b}{2c}\int\frac{1}{ax^2+bx+c}dx + C$$

$$\int \frac{dx}{x^{2^n}+1} = \sum_{k=1}^{2^{n-1}}\left\{\frac{1}{2^{n-1}}\left[\sin\left(\frac{(2k-1)\pi}{2^n}\right)\arctan\left[\left(x-\cos\left(\frac{(2k-1)\pi}{2^n}\right)\right)\csc\left(\frac{(2k-1)\pi}{2^n}\right)\right]\right] - \frac{1}{2^n}\left[\cos\left(\frac{(2k-1)\pi}{2^n}\right)\ln\left|x^2-2x\cos\left(\frac{(2k-1)\pi}{2^n}\right)+1\right|\right]\right\}$$

13.3 TRIGONOMETRIC FUNCTIONS (SINE):

$$\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$$

$$\int \sin^2 ax\ dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax + C = \frac{x}{2} - \frac{1}{2a}\sin ax\cos ax + C$$

$$\int x\sin^2 ax\ dx = \frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax + C$$

$$\int x^2\sin^2 ax\ dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x}{4a^2}\cos 2ax + C$$

$$\int \sin b_1x\sin b_2x\ dx = \frac{\sin((b_1-b_2)x)}{2(b_1-b_2)} - \frac{\sin((b_1+b_2)x)}{2(b_1+b_2)} + C \qquad (\text{for } |b_1| \neq |b_2|)$$

$$\int \sin^n ax\;dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\;dx \qquad (\text{for } n > 0)$$
$$\int \frac{dx}{\sin ax} = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right| + C$$
$$\int \frac{dx}{\sin^n ax} = \frac{\cos ax}{a(1-n)\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax} \qquad (\text{for } n > 1)$$
$$\int x \sin ax\;dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + C$$
$$\int x^n \sin ax\;dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\;dx \qquad (\text{for } n > 0)$$
$$\int_{\frac{-a}{2}}^{\frac{a}{2}} x^2 \sin^2 \frac{n\pi x}{a}\;dx = \frac{a^3(n^2\pi^2 - 6)}{24n^2\pi^2} \qquad (\text{for } n = 2, 4, 6...)$$
$$\int \frac{\sin ax}{x}dx = \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n+1}}{(2n+1)\cdot(2n+1)!} + C$$
$$\int \frac{\sin ax}{x^n}dx = -\frac{\sin ax}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}dx$$
$$\int \frac{dx}{1 \pm \sin ax} = \frac{1}{a}\tan\left(\frac{ax}{2} \mp \frac{\pi}{4}\right) + C$$
$$\int \frac{x\;dx}{1 + \sin ax} = \frac{x}{a}\tan\left(\frac{ax}{2} - \frac{\pi}{4}\right) + \frac{2}{a^2}\ln\left|\cos\left(\frac{ax}{2} - \frac{\pi}{4}\right)\right| + C$$
$$\int \frac{x\;dx}{1 - \sin ax} = \frac{x}{a}\cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right| + C$$
$$\int \frac{\sin ax\;dx}{1 \pm \sin ax} = \pm x + \frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + C$$

13.4 TRIGONOMETRIC FUNCTIONS (COSINE):

$$\int \cos ax\;dx = \frac{1}{a}\sin ax + C$$
$$\int \cos^2 ax\;dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax + C = \frac{x}{2} + \frac{1}{2a}\sin ax\cos ax + C$$
$$\int \cos^n ax\;dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\;dx \qquad (\text{for } n > 0)$$
$$\int x\cos ax\;dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a} + C$$
$$\int x^2\cos^2 ax\;dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x}{4a^2}\cos 2ax + C$$
$$\int x^n \cos ax\;dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\;dx$$
$$\int \frac{\cos ax}{x}dx = \ln|ax| + \sum_{k=1}^{\infty}(-1)^k\frac{(ax)^{2k}}{2k\cdot(2k)!} + C$$

$$\int \frac{\cos ax}{x^n} dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1}\int \frac{\sin ax}{x^{n-1}} dx \qquad (\text{for } n \neq 1)$$
$$\int \frac{dx}{\cos ax} = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$
$$\int \frac{dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax} \qquad (\text{for } n > 1)$$
$$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$$
$$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$$
$$\int \frac{x\ dx}{1+\cos ax} = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right| + C$$
$$\int \frac{x\ dx}{1-\cos ax} = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right| + C$$
$$\int \frac{\cos ax\ dx}{1+\cos ax} = x - \frac{1}{a}\tan\frac{ax}{2} + C$$
$$\int \frac{\cos ax\ dx}{1-\cos ax} = -x - \frac{1}{a}\cot\frac{ax}{2} + C$$
$$\int \cos a_1 x \cos a_2 x\ dx = \frac{\sin(a_1-a_2)x}{2(a_1-a_2)} + \frac{\sin(a_1+a_2)x}{2(a_1+a_2)} + C \qquad (\text{for } |a_1| \neq |a_2|)$$

13.5 TRIGONOMETRIC FUNCTIONS (TANGENT):

$$\int \tan ax\ dx = -\frac{1}{a}\ln|\cos ax| + C = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \tan^n ax\ dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\ dx \qquad (\text{for } n \neq 1)$$
$$\int \frac{dx}{q\tan ax + p} = \frac{1}{p^2+q^2}(px + \frac{q}{a}\ln|q\sin ax + p\cos ax|) + C \qquad (\text{for } p^2+q^2 \neq 0)$$
$$\int \frac{dx}{\tan ax} = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \frac{dx}{\tan ax + 1} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$
$$\int \frac{dx}{\tan ax - 1} = -\frac{x}{2} + \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$
$$\int \frac{\tan ax\ dx}{\tan ax + 1} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$
$$\int \frac{\tan ax\ dx}{\tan ax - 1} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$

13.6 TRIGONOMETRIC FUNCTIONS (SECANT):

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \sec^n ax\,dx = \frac{\sec^{n-1} ax \sin ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx \qquad (\text{for } n \neq 1)$$
$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$
$$\int \frac{dx}{\sec x + 1} = x - \tan\frac{x}{2} + C$$
$$\int \frac{dx}{\sec x - 1} = -x - \cot\frac{x}{2} + C$$

13.7 TRIGONOMETRIC FUNCTIONS (COTANGENT):

$$\int \cot ax\ dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \cot^n ax\ dx = -\frac{1}{a(n-1)}\cot^{n-1} ax - \int \cot^{n-2} ax\ dx \qquad (\text{for } n \neq 1)$$
$$\int \frac{dx}{1+\cot ax} = \int \frac{\tan ax\ dx}{\tan ax + 1}$$
$$\int \frac{dx}{1-\cot ax} = \int \frac{\tan ax\ dx}{\tan ax - 1}$$

13.8 TRIGONOMETRIC FUNCTIONS (SINE & COSINE):

$$\int \frac{dx}{\cos ax \pm \sin ax} = \frac{1}{a\sqrt{2}}\ln\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right| + C$$
$$\int \frac{dx}{(\cos ax \pm \sin ax)^2} = \frac{1}{2a}\tan\left(ax \mp \frac{\pi}{4}\right) + C$$
$$\int \frac{dx}{(\cos x + \sin x)^n} = \frac{1}{n-1}\left(\frac{\sin x - \cos x}{(\cos x + \sin x)^{n-1}} - 2(n-2)\int \frac{dx}{(\cos x + \sin x)^{n-2}}\right)$$
$$\int \frac{\cos ax\ dx}{\cos ax + \sin ax} = \frac{x}{2} + \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$
$$\int \frac{\cos ax\ dx}{\cos ax - \sin ax} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$
$$\int \frac{\sin ax\ dx}{\cos ax + \sin ax} = \frac{x}{2} - \frac{1}{2a}\ln|\sin ax + \cos ax| + C$$
$$\int \frac{\sin ax\ dx}{\cos ax - \sin ax} = -\frac{x}{2} - \frac{1}{2a}\ln|\sin ax - \cos ax| + C$$
$$\int \frac{\cos ax\ dx}{\sin ax(1+\cos ax)} = -\frac{1}{4a}\tan^2\frac{ax}{2} + \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$\int \frac{\cos ax\ dx}{\sin ax(1-\cos ax)} = -\frac{1}{4a}\cot^2\frac{ax}{2} - \frac{1}{2a}\ln\left|\tan\frac{ax}{2}\right| + C$$

$$\int \frac{\sin ax\ dx}{\cos ax(1+\sin ax)} = \frac{1}{4a}\cot^2\left(\frac{ax}{2}+\frac{\pi}{4}\right) + \frac{1}{2a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \frac{\sin ax\ dx}{\cos ax(1-\sin ax)} = \frac{1}{4a}\tan^2\left(\frac{ax}{2}+\frac{\pi}{4}\right) - \frac{1}{2a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| + C$$

$$\int \sin ax \cos ax\ dx = -\frac{1}{2a}\cos^2 ax + C$$

$$\int \sin a_1x \cos a_2x\ dx = -\frac{\cos((a_1-a_2)x)}{2(a_1-a_2)} - \frac{\cos((a_1+a_2)x)}{2(a_1+a_2)} + C \qquad (\text{for } |a_1| \neq |a_2|)$$

$$\int \sin^n ax \cos ax\ dx = \frac{1}{a(n+1)}\sin^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

$$\int \sin ax \cos^n ax\ dx = -\frac{1}{a(n+1)}\cos^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

$$\int \sin^n ax \cos^m ax\ dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m}\int \sin^{n-2} ax \cos^m ax\ dx \qquad (\text{f}$$

also:

$$\int \sin^n ax \cos^m ax\ dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m}\int \sin^n ax \cos^{m-2} ax\ dx \qquad (\text{for}$$

$$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a}\ln|\tan ax| + C$$

$$\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin ax\ dx}{\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax} + C \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^2 ax\ dx}{\cos ax} = -\frac{1}{a}\sin ax + \frac{1}{a}\ln\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| + C$$

$$\int \frac{\sin^2 ax\ dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^n ax\ dx}{\cos ax} = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax\ dx}{\cos ax} \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sin^n ax\ dx}{\cos^m ax} = \frac{\sin^{n+1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\sin^n ax\ dx}{\cos^{m-2} ax} \qquad (\text{for } m \neq 1)$$

also:

$$\int \frac{\sin^n ax\ dx}{\cos^m ax} = -\frac{\sin^{n-1} ax}{a(n-m)\cos^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} ax\ dx}{\cos^m ax} \qquad (\text{for } m \neq n)$$

also:

$$\int \frac{\sin^n ax\ dx}{\cos^m ax} = \frac{\sin^{n-1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\sin^{n-2} ax\ dx}{\cos^{m-2} ax} \qquad (\text{for } m \neq 1)$$

$$\int \frac{\cos ax\ dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} + C \qquad (\text{for } n \neq 1)$$

$$\int \frac{\cos^2 ax\ dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \ln\left|\tan\frac{ax}{2}\right|\right) + C$$

$$\int \frac{\cos^2 ax\ dx}{\sin^n ax} = -\frac{1}{n-1}\left(\frac{\cos ax}{a\sin^{n-1} ax)} + \int \frac{dx}{\sin^{n-2} ax}\right) \qquad (\text{for } n \neq 1)$$

$$\int \frac{\cos^n ax\ dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m-2}{m-1}\int \frac{\cos^n ax\ dx}{\sin^{m-2} ax} \qquad (\text{for } m \neq 1)$$

also:

$$\int \frac{\cos^n ax\ dx}{\sin^m ax} = \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} ax\ dx}{\sin^m ax} \qquad (\text{for } m \neq n)$$

also:

$$\int \frac{\cos^n ax\ dx}{\sin^m ax} = -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\ dx}{\sin^{m-2} ax} \qquad (\text{for } m \neq 1)$$

13.9 TRIGONOMETRIC FUNCTIONS (SINE & TANGENT):

$$\int \sin ax \tan ax\ dx = \frac{1}{a}(\ln|\sec ax + \tan ax| - \sin ax) + C$$

$$\int \frac{\tan^n ax\ dx}{\sin^2 ax} = \frac{1}{a(n-1)}\tan^{n-1}(ax) + C \qquad (\text{for } n \neq 1)$$

13.10 TRIGONOMETRIC FUNCTIONS (COSINE & TANGENT):

$$\int \frac{\tan^n ax\ dx}{\cos^2 ax} = \frac{1}{a(n+1)}\tan^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

13.11 TRIGONOMETRIC FUNCTIONS (SINE & COTANGENT):

$$\int \frac{\cot^n ax\ dx}{\sin^2 ax} = \frac{1}{a(n+1)}\cot^{n+1} ax + C \qquad (\text{for } n \neq -1)$$

13.12 TRIGONOMETRIC FUNCTIONS (COSINE & COTANGENT):

$$\int \frac{\cot^n ax\ dx}{\cos^2 ax} = \frac{1}{a(1-n)}\tan^{1-n} ax + C \qquad (\text{for } n \neq 1)$$

13.13 TRIGONOMETRIC FUNCTIONS (ARCSINE):

$$\int \arcsin x\ dx = x\arcsin x + \sqrt{1-x^2} + C$$

$$\int \arcsin\frac{x}{a}\ dx = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2} + C$$

$$\int x \arcsin\frac{x}{a}\, dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2} + C$$

$$\int x^2 \arcsin\frac{x}{a}\, dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{x^2 + 2a^2}{9}\sqrt{a^2 - x^2} + C$$

$$\int x^n \arcsin x\, dx = \frac{1}{n+1}\left(x^{n+1}\arcsin x + \frac{x^n\sqrt{1-x^2} - nx^{n-1}\arcsin x}{n-1} + n\int x^{n-2}\arcsin x\, dx\right)$$

$$\int \cos^n x \arcsin x\, dx = \left(x^{n^2+1}\arccos x + \frac{x^n\sqrt{1-x^4} - nx^{n^2-1}\arccos x}{n^2-1} + n\int x^{n^2-2}\arccos x\, dx\right)$$

13.14 TRIGONOMETRIC FUNCTIONS (ARCCOSINE):

$$\int \arccos x\, dx = x\arccos x - \sqrt{1-x^2} + C$$

$$\int \arccos\frac{x}{a}\, dx = x\arccos\frac{x}{a} - \sqrt{a^2 - x^2} + C$$

$$\int x\arccos\frac{x}{a}\, dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2 - x^2} + C$$

$$\int x^2\arccos\frac{x}{a}\, dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{x^2 + 2a^2}{9}\sqrt{a^2 - x^2} + C$$

13.15 TRIGONOMETRIC FUNCTIONS (ARCTANGENT):

$$\int \arctan x\, dx = x\arctan x - \frac{1}{2}\ln(1 + x^2) + C$$

$$\int \arctan(\frac{x}{a})dx = x\arctan(\frac{x}{a}) - \frac{a}{2}\ln(1 + \frac{x^2}{a^2}) + C$$

$$\int x\arctan(\frac{x}{a})dx = \frac{(a^2 + x^2)\arctan(\frac{x}{a}) - ax}{2} + C$$

$$\int x^2\arctan(\frac{x}{a})dx = \frac{x^3}{3}\arctan(\frac{x}{a}) - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2 + x^2) + C$$

$$\int x^n\arctan(\frac{x}{a})dx = \frac{x^{n+1}}{n+1}\arctan(\frac{x}{a}) - \frac{a}{n+1}\int\frac{x^{n+1}}{a^2 + x^2}\, dx, \quad n \neq -1$$

13.16 TRIGONOMETRIC FUNCTIONS (ARCCOSECANT):

$$\int \operatorname{arccsc} x\, dx = x\operatorname{arccsc} x + \ln\left|x + x\sqrt{\frac{x^2-1}{x^2}}\right| + C$$

$$\int \operatorname{arccsc}\frac{x}{a}\, dx = x\operatorname{arccsc}\frac{x}{a} + a\ln(\frac{x}{a}(\sqrt{1 - \frac{a^2}{x^2}} + 1)) + C$$

$$\int x \operatorname{arccsc} \frac{x}{a}\ dx = \frac{x^2}{2} \operatorname{arccsc} \frac{x}{a} + \frac{ax}{2}\sqrt{1-\frac{a^2}{x^2}} + C$$

13.17 TRIGONOMETRIC FUNCTIONS (ARCSECANT):

$$\int \operatorname{arcsec} x\, dx = x \operatorname{arcsec} x - \ln\left|x + x\sqrt{\frac{x^2-1}{x^2}}\right| + C$$

$$\int \operatorname{arcsec} \frac{x}{a}\ dx = x \operatorname{arcsec} \frac{x}{a} + \frac{x}{a|x|} \ln\left|x \pm \sqrt{x^2-1}\right| + C$$

$$\int x \operatorname{arcsec} x\ dx = \frac{1}{2}\left(x^2 \operatorname{arcsec} x - \sqrt{x^2-1}\right) + C$$

$$\int x^n \operatorname{arcsec} x\, dx = \frac{1}{n+1}\left(x^{n+1} \operatorname{arcsec} x - \frac{1}{n}\left[x^{n-1}\sqrt{x^2-1} + [1-n]\left(x^{n-1} \operatorname{arcsec} x + (1-n)\int x^{n-2} \operatorname{arcsec} x\ dx\right)\right]\right)$$

13.18 TRIGONOMETRIC FUNCTIONS (ARCCOTANGENT):

$$\int \operatorname{arccot} x\, dx = x \operatorname{arccot} x + \frac{1}{2}\ln(1+x^2) + C$$

$$\int \operatorname{arccot} \frac{x}{a}\ dx = x \operatorname{arccot} \frac{x}{a} + \frac{a}{2}\ln(a^2+x^2) + C$$

$$\int x \operatorname{arccot} \frac{x}{a}\ dx = \frac{a^2+x^2}{2} \operatorname{arccot} \frac{x}{a} + \frac{ax}{2} + C$$

$$\int x^2 \operatorname{arccot} \frac{x}{a}\ dx = \frac{x^3}{3} \operatorname{arccot} \frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2+x^2) + C$$

$$\int x^n \operatorname{arccot} \frac{x}{a}\ dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1}\int \frac{x^{n+1}}{a^2+x^2}\ dx, \quad n \neq -1$$

13.19 EXPONETIAL FUNCTIONS

$$\int e^x\ \mathrm{d}x = e^x$$

$$\int e^{cx}\ \mathrm{d}x = \frac{1}{c}e^{cx}$$

$$\int a^{cx}\ \mathrm{d}x = \frac{1}{c \cdot \ln a}a^{cx} \text{ for } a > 0,\ a \neq 1$$

$$\int x e^{cx}\ \mathrm{d}x = \frac{e^{cx}}{c^2}(cx-1)$$

$$\int x^2 e^{cx}\ \mathrm{d}x = e^{cx}\left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right)$$

$$\int x^n e^{cx}\ \mathrm{d}x = \frac{1}{c}x^n e^{cx} - \frac{n}{c}\int x^{n-1}e^{cx}\mathrm{d}x$$

$$\int \frac{e^{cx}}{x}\; \mathrm{d}x = \ln|x| + \sum_{n=1}^\infty \frac{(cx)^{n}}{n\cdot n!}$$

$$\int \frac{e^{cx}}{x^n}\; \mathrm{d}x = \frac{1}{n-1}\left(-\frac{e^{cx}}{x^{n-1}} + c\int \frac{e^{cx} }{x^{n-1}}\,\mathrm{d}x\right) \qquad\text{(for } n\neq 1\text{)}$$

$$\int e^{cx}\ln x\; \mathrm{d}x = \frac{1}{c}e^{cx}\ln|x| - \operatorname{Ei}\,(cx)$$

$$\int e^{cx}\sin bx\; \mathrm{d}x = \frac{e^{cx}}{c^2+b^2}(c\sin bx - b\cos bx)$$

$$\int e^{cx}\cos bx\; \mathrm{d}x = \frac{e^{cx}}{c^2+b^2}(c\cos bx + b\sin bx)$$

$$\int e^{cx}\sin^n x\; \mathrm{d}x = \frac{e^{cx}\sin^{n-1} x}{c^2+n^2}(c\sin x - n\cos x) + \frac{n(n-1)}{c^2+n^2}\int e^{cx}\sin^{n-2} x\;\mathrm{d}x$$

$$\int e^{cx}\cos^n x\; \mathrm{d}x = \frac{e^{cx}\cos^{n-1} x}{c^2+n^2}(c\cos x + n\sin x) + \frac{n(n-1)}{c^2+n^2}\int e^{cx}\cos^{n-2} x\;\mathrm{d}x$$

$$\int x e^{cx^2 }\; \mathrm{d}x= \frac{1}{2c} \; e^{cx^2}$$

$$\int e^{-cx^2}\; \mathrm{d}x=\sqrt{\frac{\pi}{4c}}\operatorname{erf}(\sqrt{c}x)$$ (erf is the Error function)

$$\int xe^{-cx^2}\; \mathrm{d}x=-\frac{1}{2c}e^{-cx^2}$$

$$\int \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}\; \mathrm{d}x= \frac{1}{2}\left(1 + \operatorname{erf}\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

$$\int e^{x^2}\,\mathrm{d}x = e^{x^2}\left(\sum_{j=0}^{n-1}c_{2j}\,\frac{1}{x^{2j+1}} \right)+(2n-1)c_{2n-2} \int \frac{e^{x^2}}{x^{2n}}\;\mathrm{d}x \quad \text{valid for } n > 0,$$

where $c_{2j}=\frac{ 1 \cdot 3 \cdot 5 \cdots (2j-1)}{2^{j+1}}=\frac{(2j)\,!}{j!\, 2^{2j+1}}\ .$

$$\int \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_m dx = \sum_{n=0}^m\frac{(-1)^n(n+1)^{n-1}}{n!}\Gamma(n+1,-\ln x) + \sum_{n=m+1}^\infty(-1)^na_{mn}\Gamma(n+1,-\ln x) \qquad\text{(for } x> 0\text{)}$$

where $a_{mn}=\begin{cases}1 & \text{if } n = 0, \\ \frac{1}{n!} & \text{if } m=1, \\ \frac{1}{n}\sum_{j=1}^{n}ja_{m,n-j}a_{m-1,j-1} & \text{otherwise}\end{cases}$

$$\int\frac{1}{ae^{\lambda x}+b}\;\mathrm{d}x=\frac{x}{b}-\frac{1}{b\lambda}\ln\left(ae^{\lambda x}+b\right)$$ when $b\neq0$, $\lambda\neq0$ and $ae^{\lambda x}+b>0\,.$

$$\int_0^1 e^{x\cdot\ln a+(1-x)\cdot\ln b}\,\mathrm{d}x = \int_0^1 \left(\frac{a}{b}\right)^x \cdot b\,\mathrm{d}x = \int_0^1 a^x\cdot b^{1-x}\,\mathrm{d}x = \frac{a-b}{\ln a-\ln b}$$

for $a>0,\ b>0,\ a\neq b$, which is the logarithmic mean

$$\int_0^\infty e^{-ax}\,\mathrm{d}x = \frac{1}{a}$$

$$\int_0^\infty e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a>0)$$

$$\int_{-\infty}^\infty e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}} \quad (a>0)$$

$$\int_{-\infty}^\infty e^{-ax^2}e^{-2bx}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{a}} \quad (a>0)\int_{-\infty}^\infty xe^{-a(x-b)^2}\,\mathrm{d}x = b\sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^\infty x^2e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \quad (a>0)$$

$$\int_0^\infty x^n e^{-ax^2}\,\mathrm{d}x = \begin{cases} \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)/a^{\frac{n+1}{2}} & (n>-1, a>0) \\ \frac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\frac{\pi}{a}} & (n=2k, k\ \text{integer}, a>0) \\ \frac{k!}{2a^{k+1}} & (n=2k+1, k\ \text{integer}, a>0) \end{cases}$$

(!! is the double factorial)

$$\int_0^\infty x^n e^{-ax}\,\mathrm{d}x = \begin{cases} \frac{\Gamma(n+1)}{a^{n+1}} & (n>-1, a>0) \\ \frac{n!}{a^{n+1}} & (n=0,1,2,\ldots, a>0) \end{cases}$$

$$\int_0^\infty e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{a^2+b^2} \quad (a>0)$$

$$\int_0^\infty e^{-ax}\cos bx\,\mathrm{d}x = \frac{a}{a^2+b^2} \quad (a>0)$$

$$\int_0^\infty xe^{-ax}\sin bx\,\mathrm{d}x = \frac{2ab}{(a^2+b^2)^2} \quad (a>0)$$

$$\int_0^\infty xe^{-ax}\cos bx\,\mathrm{d}x = \frac{a^2-b^2}{(a^2+b^2)^2} \quad (a>0)$$

$$\int_0^{2\pi} e^{x\cos\theta}d\theta = 2\pi I_0(x)$$
(I_0 is the modified Bessel function of the first kind)

$$\int_0^{2\pi} e^{x\cos\theta+y\sin\theta}d\theta = 2\pi I_0\left(\sqrt{x^2+y^2}\right)$$

13.20 LOGARITHMIC FUNCTIONS

$$\int \ln(x)\,dx = x\ln(x) - x + C,$$

$$\int \ln ax\ dx = x\ln ax - x$$

$$\int \log_a x\,dx = x(\log_a x - \log_a e) + C$$

$$\int \ln(ax+b)\,dx = \frac{(ax+b)\ln(ax+b)-(ax)}{a}$$
$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x$$
$$\int (\ln x)^n\,dx = x\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}(\ln x)^k$$
$$\int \frac{dx}{\ln x} = \ln|\ln x| + \ln x + \sum_{k=2}^{\infty}\frac{(\ln x)^k}{k\cdot k!}$$
$$\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1}\int\frac{dx}{(\ln x)^{n-1}} \qquad (\text{for } n\neq 1)$$
$$\int x^m \ln x\,dx = x^{m+1}\left(\frac{\ln x}{m+1} - \frac{1}{(m+1)^2}\right) \qquad (\text{for } m\neq -1)$$
$$\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx \qquad (\text{for } m\neq -1)$$
$$\int \frac{(\ln x)^n\,dx}{x} = \frac{(\ln x)^{n+1}}{n+1} \qquad (\text{for } n\neq -1)$$
$$\int \frac{\ln x^n\,dx}{x} = \frac{(\ln x^n)^2}{2n} \qquad (\text{for } n\neq 0)$$
$$\int \frac{\ln x\,dx}{x^m} = -\frac{\ln x}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2x^{m-1}} \qquad (\text{for } m\neq 1)$$
$$\int \frac{(\ln x)^n\,dx}{x^m} = -\frac{(\ln x)^n}{(m-1)x^{m-1}} + \frac{n}{m-1}\int\frac{(\ln x)^{n-1}dx}{x^m} \qquad (\text{for } m\neq 1)$$
$$\int \frac{x^m\,dx}{(\ln x)^n} = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1}\int\frac{x^m dx}{(\ln x)^{n-1}} \qquad (\text{for } n\neq 1)$$
$$\int \frac{dx}{x\ln x} = \ln|\ln x|$$
$$\int \frac{dx}{x^n\ln x} = \ln|\ln x| + \sum_{k=1}^{\infty}(-1)^k\frac{(n-1)^k(\ln x)^k}{k\cdot k!}$$
$$\int \frac{dx}{x(\ln x)^n} = -\frac{1}{(n-1)(\ln x)^{n-1}} \qquad (\text{for } n\neq 1)$$
$$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$$
$$\int \frac{x}{x^2+a^2}\ln(x^2+a^2)\,dx = \frac{1}{4}\ln^2(x^2+a^2)$$
$$\int \sin(\ln x)\,dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x))$$
$$\int \cos(\ln x)\,dx = \frac{x}{2}(\sin(\ln x) + \cos(\ln x))$$
$$\int e^x\left(x\ln x - x - \frac{1}{x}\right)\,dx = e^x(x\ln x - x - \ln x)$$
$$\int \frac{1}{e^x}\left(\frac{1}{x} - \ln x\right)\,dx = \frac{\ln x}{e^x}$$

$$\int (\ln x)^x \, dx = (\ln x)^{x-1} + (\ln(\ln x))(\ln x)^x$$
$$\int e^x \left(\frac{1}{\ln x} - \frac{1}{x \ln^2 x} \right) \, dx = \frac{e^x}{\ln x}$$

13.21 HYPERBOLIC FUNCTIONS

$$\int \sinh ax \, dx = \frac{1}{a} \cosh ax + C$$
$$\int \cosh ax \, dx = \frac{1}{a} \sinh ax + C$$
$$\int \sinh^2 ax \, dx = \frac{1}{4a} \sinh 2ax - \frac{x}{2} + C$$
$$\int \cosh^2 ax \, dx = \frac{1}{4a} \sinh 2ax + \frac{x}{2} + C$$
$$\int \tanh^2 ax \, dx = x - \frac{\tanh ax}{a} + C$$
$$\int \sinh^n ax \, dx = \frac{1}{an} \sinh^{n-1} ax \cosh ax - \frac{n-1}{n} \int \sinh^{n-2} ax \, dx \qquad (\text{for } n > 0)$$
$$\int \sinh^n ax \, dx = \frac{1}{a(n+1)} \sinh^{n+1} ax \cosh ax - \frac{n+2}{n+1} \int \sinh^{n+2} ax \, dx \qquad (\text{for } n < 0,\, n \neq -1)$$
$$\int \cosh^n ax \, dx = \frac{1}{an} \sinh ax \cosh^{n-1} ax + \frac{n-1}{n} \int \cosh^{n-2} ax \, dx \qquad (\text{for } n > 0)$$
$$\int \cosh^n ax \, dx = -\frac{1}{a(n+1)} \sinh ax \cosh^{n+1} ax - \frac{n+2}{n+1} \int \cosh^{n+2} ax \, dx \qquad (\text{for } n < 0,\, n \neq -1)$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln \left| \tanh \frac{ax}{2} \right| + C$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln \left| \frac{\cosh ax - 1}{\sinh ax} \right| + C$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln \left| \frac{\sinh ax}{\cosh ax + 1} \right| + C$$
$$\int \frac{dx}{\sinh ax} = \frac{1}{a} \ln \left| \frac{\cosh ax - 1}{\cosh ax + 1} \right| + C$$
$$\int \frac{dx}{\cosh ax} = \frac{2}{a} \arctan e^{ax} + C$$
$$\int \frac{dx}{\sinh^n ax} = -\frac{\cosh ax}{a(n-1) \sinh^{n-1} ax} - \frac{n-2}{n-1} \int \frac{dx}{\sinh^{n-2} ax} \qquad (\text{for } n \neq 1)$$
$$\int \frac{dx}{\cosh^n ax} = \frac{\sinh ax}{a(n-1) \cosh^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} ax} \qquad (\text{for } n \neq 1)$$
$$\int \frac{\cosh^n ax}{\sinh^m ax} dx = \frac{\cosh^{n-1} ax}{a(n-m) \sinh^{m-1} ax} + \frac{n-1}{n-m} \int \frac{\cosh^{n-2} ax}{\sinh^m ax} dx \qquad (\text{for } m \neq n)$$

$$\int \frac{\cosh^n ax}{\sinh^m ax} dx = -\frac{\cosh^{n+1} ax}{a(m-1)\sinh^{m-1} ax} + \frac{n-m+2}{m-1} \int \frac{\cosh^n ax}{\sinh^{m-2} ax} dx \qquad (\text{for } m \neq 1)$$

$$\int \frac{\cosh^n ax}{\sinh^m ax} dx = -\frac{\cosh^{n-1} ax}{a(m-1)\sinh^{m-1} ax} + \frac{n-1}{m-1} \int \frac{\cosh^{n-2} ax}{\sinh^{m-2} ax} dx \qquad (\text{for } m \neq 1)$$

$$\int \frac{\sinh^m ax}{\cosh^n ax} dx = \frac{\sinh^{m-1} ax}{a(m-n)\cosh^{n-1} ax} + \frac{m-1}{n-m} \int \frac{\sinh^{m-2} ax}{\cosh^n ax} dx \qquad (\text{for } m \neq n)$$

$$\int \frac{\sinh^m ax}{\cosh^n ax} dx = \frac{\sinh^{m+1} ax}{a(n-1)\cosh^{n-1} ax} + \frac{m-n+2}{n-1} \int \frac{\sinh^m ax}{\cosh^{n-2} ax} dx \qquad (\text{for } n \neq 1)$$

$$\int \frac{\sinh^m ax}{\cosh^n ax} dx = -\frac{\sinh^{m-1} ax}{a(n-1)\cosh^{n-1} ax} + \frac{m-1}{n-1} \int \frac{\sinh^{m-2} ax}{\cosh^{n-2} ax} dx \qquad (\text{for } n \neq 1)$$

$$\int x \sinh ax \, dx = \frac{1}{a} x \cosh ax - \frac{1}{a^2} \sinh ax + C$$

$$\int x \cosh ax \, dx = \frac{1}{a} x \sinh ax - \frac{1}{a^2} \cosh ax + C$$

$$\int x^2 \cosh ax \, dx = -\frac{2x \cosh ax}{a^2} + \left(\frac{x^2}{a} + \frac{2}{a^3}\right) \sinh ax + C$$

$$\int \tanh ax \, dx = \frac{1}{a} \ln|\cosh ax| + C$$

$$\int \coth ax \, dx = \frac{1}{a} \ln|\sinh ax| + C$$

$$\int \tanh^n ax \, dx = -\frac{1}{a(n-1)} \tanh^{n-1} ax + \int \tanh^{n-2} ax \, dx \qquad (\text{for } n \neq 1)$$

$$\int \coth^n ax \, dx = -\frac{1}{a(n-1)} \coth^{n-1} ax + \int \coth^{n-2} ax \, dx \qquad (\text{for } n \neq 1)$$

$$\int \sinh ax \sinh bx \, dx = \frac{1}{a^2-b^2}(a \sinh bx \cosh ax - b \cosh bx \sinh ax) + C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \cosh ax \cosh bx \, dx = \frac{1}{a^2-b^2}(a \sinh ax \cosh bx - b \sinh bx \cosh ax) + C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \cosh ax \sinh bx \, dx = \frac{1}{a^2-b^2}(a \sinh ax \sinh bx - b \cosh ax \cosh bx) + C \qquad (\text{for } a^2 \neq b^2)$$

$$\int \sinh(ax+b)\sin(cx+d)\, dx = \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + C$$

$$\int \sinh(ax+b)\cos(cx+d)\, dx = \frac{a}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\sin(cx+d) + C$$

$$\int \cosh(ax+b)\sin(cx+d)\, dx = \frac{a}{a^2+c^2}\sinh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + C$$

$$\int \cosh(ax+b)\cos(cx+d)\, dx = \frac{a}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\sin(cx+d) + C$$

13.22 INVERSE HYPERBOLIC FUNCTIONS

$$\int \operatorname{arsinh}\frac{x}{a} \, dx = x \operatorname{arsinh}\frac{x}{a} - \sqrt{x^2+a^2} + C$$

$$\int \operatorname{arcosh} \frac{x}{a}\, dx = x \operatorname{arcosh} \frac{x}{a} - \sqrt{x^2 - a^2} + C$$

$$\int \operatorname{artanh} \frac{x}{a}\, dx = x \operatorname{artanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2| + C \qquad (\text{for } |x| < |a|)$$

$$\int \operatorname{arcoth} \frac{x}{a}\, dx = x \operatorname{arcoth} \frac{x}{a} + \frac{a}{2} \ln |x^2 - a^2| + C \qquad (\text{for } |x| > |a|)$$

$$\int \operatorname{arsech} \frac{x}{a}\, dx = x \operatorname{arsech} \frac{x}{a} - a \arctan \frac{x \sqrt{\frac{a-x}{a+x}}}{x-a} + C \qquad (\text{for } x \in (0,\, a))$$

$$\int \operatorname{arcsch} \frac{x}{a}\, dx = x \operatorname{arcsch} \frac{x}{a} + a \ln \frac{x + \sqrt{x^2 + a^2}}{a} + C \qquad (\text{for } x \in (0,\, a))$$

13.23 ABSOLUTE VALUE FUNCTIONS

$$\int |(ax+b)^n|\; dx = \frac{(ax+b)^{n+2}}{a(n+1)\,|ax+b|} + C \ [\, n \text{ is odd, and } n \neq -1\,]$$

$$\int |\sin ax|\; dx = \frac{-1}{a} |\sin ax| \cot ax + C$$

$$\int |\cos ax|\; dx = \frac{1}{a} |\cos ax| \tan ax + C$$

$$\int |\tan ax|\; dx = \frac{\tan(ax)[-\ln |\cos ax|]}{a\,|\tan ax|} + C$$

$$\int |\csc ax|\; dx = \frac{-\ln |\csc ax + \cot ax| \sin ax}{a\,|\sin ax|} + C$$

$$\int |\sec ax|\; dx = \frac{\ln |\sec ax + \tan ax| \cos ax}{a\,|\cos ax|} + C$$

$$\int |\cot ax|\; dx = \frac{\tan(ax)[\ln |\sin ax|]}{a\,|\tan ax|} + C$$

13.24 SUMMARY TABLE

$\int \frac{f'(u)}{f(u)}\,du = \ln\|f(u)\| + C, \qquad \int (f(u))^n\, f'(u)\,du = \frac{(f(u))^{n+1}}{n+1} + C$
$\int \sin u\,du = -\cos u + C, \quad \int \cos u\,du = \sin u + C, \quad \int \sec^2 u\,du = \tan u + C$
$\int \tan u\,du = \ln\|\sec u\| + C, \qquad \int \tanh u\,du = \ln(\cosh u) + C$
$\int \cot u\,du = \ln\|\sin u\| + C, \qquad \int \coth u\,du = \ln\|\sinh u\| + C$
$\int \sec u\,du = \ln\|\sec u + \tan u\| + C, \quad \int \operatorname{sech} u\,du = \tan^{-1}\|\sinh u\| + C$
$\int \operatorname{cosec} u\,du = \ln\|\operatorname{cosec} u - \cot u\| + C, \quad \int \operatorname{cosech} u\,du = \ln\|\tanh \frac{u}{2}\| + C$
$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left\|\frac{u+a}{u-a}\right\| + C, \qquad \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left\|\frac{u-a}{u+a}\right\| + C$
$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C = \ln\|u+\sqrt{u^2+a^2}\| + C$
$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C = \ln\|u+\sqrt{u^2-a^2}\| + C$
$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C = -\frac{1}{a}\ln\left\|\frac{a+\sqrt{a^2-u^2}}{u}\right\| + C$

13.25 SQUARE ROOT PROOFS

$\int \sqrt{a^2+x^2}\,dx$

Let $x = a\tan\theta \therefore dx = a\sec^2\theta d\theta \to \tan\theta = \frac{x}{a}$

$= \int \sqrt{a^2+(a\tan\theta)^2} \times a\sec^2\theta d\theta$

$= \int \sqrt{a^2+a^2\tan^2\theta} \times a\sec^2\theta d\theta$

$= \int \sqrt{a^2+a^2(\sec^2\theta-1)} \times a\sec^2\theta d\theta$

$= \int \sqrt{a^2+a^2\sec^2\theta-a^2} \times a\sec^2\theta d\theta$

$= \int \sqrt{a^2\sec^2\theta} \times a\sec^2\theta d\theta$

$= \int a\sec\theta \times a\sec^2\theta d\theta$

$= \int a^2\sec^3\theta d\theta$

$= a^2\int \sec\theta \times \sec^2\theta d\theta$

$u = \sec\theta, dv = \sec^2\theta d\theta$

$du = \sec\theta\tan\theta d\theta, v = \tan\theta$

$\therefore a^2\int \sec^3\theta d\theta = \sec\theta \times \tan\theta - \int \tan\theta \times \sec\theta\tan\theta d\theta$

$a^2\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta$

$a^2\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int (\sec^2\theta-1)\sec\theta d\theta$

$a^2\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta d\theta$

$a^2\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$

$2a^2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta$

$\int \sec^3\theta d\theta = \frac{1}{2a^2}\left(\sec\theta\tan\theta + \int \sec\theta d\theta\right)$

$\int \sec^3\theta d\theta = \frac{1}{2a^2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right) + C$

$\therefore \int \sqrt{a^2+x^2}\,dx = \frac{1}{2a^2}\left(\frac{\sqrt{a^2+x^2}}{a} \times \frac{s}{a} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{s}{a}\right|\right) + C$

$$\int \sqrt{a^2 - x^2}\,dx$$

Let $x = a\sin\theta \therefore dx = a\cos\theta d\theta \rightarrow \sin\theta = \frac{x}{a}$

$$= \int \sqrt{a^2 - (a\sin\theta)^2} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2(1-\cos^2\theta)} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2 + a^2\cos^2\theta} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2\cos^2\theta} \times a\cos\theta d\theta$$

$$= \int a\cos\theta \times a\cos\theta d\theta$$

$$= \int a^2\cos^2\theta d\theta$$

$$= a^2 \int \cos^2\theta d\theta$$

$$= a^2 \int \frac{1+\cos(2\theta)}{2} d\theta$$

$$= \frac{a^2}{2} \int 1 + \cos(2\theta) d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{a^2}{2}\left[\arcsin\left(\frac{x}{a}\right) + \frac{x}{a} \times \frac{\sqrt{a^2 - x^2}}{a}\right] + C$$

$$\int \sqrt{x^2-a^2}\,dx$$

Let $x = a\sec\theta \therefore dx = a\sec\theta\tan\theta d\theta \rightarrow \sec\theta = \dfrac{x}{a}$

$$= \int \sqrt{(a\sec\theta)^2 - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2\sec^2\theta - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2(1+\tan^2\theta) - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2 + a^2\tan^2\theta - a^2} \times a\sec\theta\tan\theta d\theta$$

$$= \int \sqrt{a^2\tan^2\theta} \times a\sec\theta\tan\theta d\theta$$

$$= \int a\tan\theta \times a\sec\theta\tan\theta d\theta$$

$$= \int a^2\tan^2\theta\sec\theta d\theta$$

$$= a^2\int \tan^2\theta\sec\theta d\theta$$

$$= a^2\int (\sec^2\theta - 1)\sec\theta d\theta$$

$$= a^2\int \sec^3\theta - \sec\theta d\theta$$

$$= a^2\left(\int \sec^3\theta d\theta - \int \sec\theta d\theta\right)$$

$$= a^2\left(\left(\frac{1}{2a^2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right) - \left(\ln|\sec\theta + \tan\theta|\right)\right) + C$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) - a^2\left(\ln|\sec\theta + \tan\theta|\right) + C$$

$$= \frac{1}{2}\left(\frac{x}{a} \times \frac{\sqrt{x^2-a^2}}{a} + \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right|\right) - a^2\left(\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right|\right) + C$$

$$= \frac{1}{2}\frac{x \times \sqrt{x^2-a^2}}{a^2} + \frac{1}{2}\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| - \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

$$= \frac{1}{2}\frac{x \times \sqrt{x^2-a^2}}{a^2} + \left(\frac{1}{2} - a^2\right)\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

13.26 CARTESIAN APPLICATIONS

Area under the curve: $A = \int_a^b f_{(x)}dx$

Volume: $V = \int_a^b A$

Volume about x axis: $V_x = \pi\int_a^b [y]^2 dx = \pi\int_a^b \left[f_{(x)}\right]^2 dx$

Volume about y axis: $V_y = \pi \int_c^d [x]^2 dy$

Surface Area about x axis: $SA = 2\pi \int_a^b f_{(x)} \sqrt{1 + \left(f'_{(x)}\right)^2} dx$

Length wrt x-ordinates: $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Length wrt y-ordinates: $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$ (Where the function is continually increasing)

Length parametrically: $L = \int_{t1}^{t2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

PART 14: FUNCTIONS

14.1 COMPOSITE FUNCTIONS:

Odd ± Odd = Odd
Odd ± Even = Neither
Even ± Even = Even
Odd x Odd = Even
Odd / Odd = Odd
Even x Even = Even
Even / Even = Even
Even of Odd = Even
Even of Even = Even
Even of Neither = Neither
Odd of Odd = Odd
Odd of Even = Even
Odd of Neither = Neither

If f(x) is odd: $\int_{-a}^{a} f(x)dx = 0$

If f(x) is even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

14.2 MULTIVARIABLE FUNCTIONS:

Limit: $\lim_{(x,y)\to(0,0)}\left(f_{(x,y)}\right)=\lim_{(x,mx)\to(0,0)}\left(f_{(x,mx)}\right)=\lim_{(x)\to(0)}\left(f_{(x,mx)}\right)$

Discriminant: $D_{(x_0,y_0)} = z_{xx}z_{yy} - \left(z_{xy}\right)^2$

Critical Points: $z = f_{(x,y)}$

Solve for: $\left.\begin{aligned} z_x &= 0 \\ z_y &= 0 \end{aligned}\right\}$

- If the critical point (x_0,y_0) is a local maximum, then

  $D(x_0,y_0) >= 0$

  $f_{xx}(x_0,y_0) <= 0$ and $f_{yy}(x_0,y_0) <= 0$
- If $D(x_0,y_0) > 0$, and either

  $f_{xx}(x_0,y_0) < 0$ or $f_{yy}(x_0,y_0) < 0$

  then the critical point (x_0,y_0) is a local maximum.
- If the critical point (x_0,y_0) is a local minimum, then

  $D(x_0,y_0) >= 0$

  $f_{xx}(x_0,y_0) >= 0$ and $f_{yy}(x_0,y_0) >= 0$
- If $D(x_0,y_0) > 0$, and either

  $f_{xx}(x_0,y_0) > 0$ or $f_{yy}(x_0,y_0) > 0$

  then the critical point (x_0,y_0) is a local minimum.

- If the critical point (x_0,y_0) is a saddle point, then

  $D(x_0,y_0) <= 0$

- If

  $D(x_0,y_0) < 0$,

  then the critical point (x_0,y_0) is a saddle point.

14.3 FIRST ORDER, FIRST DEGREE, DIFFERENTIAL EQUATIONS:

Separable:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$
$$g(y)dy = f(x)dx$$
$$\int g(y)dy = \int f(x)dx$$

Linear:

$$\frac{dy}{dx} + P(x) \times y = Q(x)$$
$$I(x) = e^{\int P(x)dx}$$
$$y = \frac{1}{I(x)}\left(\int I(x) \times Q(x)dx\right)$$

Homogeneous:

$$f(\lambda x, \lambda y) = f(x, y)$$
$$\frac{dy}{dx} = f(x, y) = F\left(\frac{y}{x}\right)$$
Let $v(x) = \frac{y}{x}, \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$
$$\therefore v + x\frac{dv}{dx} = F(v)$$
$$x\frac{dv}{dx} = F(v) - v$$
$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$
$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

Exact:

$$\frac{dy}{dx} = f(x, y) \to M(x, y)dx + N(x, y)dy = 0$$

If: $M_y = N_x$

When: $F_X = M \;\&\; F_Y = N$

Therefore,

$$F = \int M(x,y)dx = \Phi(x,y) + g(y)$$

$$F_Y = \frac{\partial}{\partial y}\left(\Phi + g(y)\right) = \Phi_Y + g'(y) = N$$

$$\therefore g(y) = \ldots$$

So: $F(x,y) = \Phi(x,y) + g(y) = C$

14.4 SECOND ORDER

Where $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$

$ay'' + by' + cy = f(x)$

Homogeneous:

$$ay'' + by' + cy = 0$$

$$\Rightarrow am^2 + bm + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three possible outcomes:

1) m_1, m_2 where $m_1 \neq m_2$ $\quad \Rightarrow y_h = Ae^{m_1 x} + Be^{m_2 x}$
2) m_1, m_2 where $m_1 = m_2$ $\quad \Rightarrow y_h = (A + Bx)e^{m_1 x}$
3) $m_{1,2} = \alpha \pm \beta j$ $\quad \Rightarrow y_h = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$

Undetermined Coefficients

Where $f(x)$ is in the form of

1) A polynomial $\quad \Rightarrow y_p = A_n x^n + A_{n-1}x^{n-1} + A_1 x + A_0$
2) $\alpha \sin(kx)$ $\quad \Rightarrow y_p = A\sin(kx) + B\cos(kx)$
3) αe^{kt} $\quad \Rightarrow y_p = Ae^{kt}$

NB: Multiplication is OK: eg:

$$f(x) = 3x^3e^x$$

$$y_p = \left(Ae^x\right)\left(Bx^3 + Cx^2 + Dx + E\right)$$

$$y_p = \left(e^x\right)\left(Bx^3 + Cx^2 + Dx + E\right)$$

NB: If y_p is part of y_c, you multiply y_p by x

To determine the unknown variables, substitute back into the original equation with y_p, y'_p, y''_p and compare the coefficients.

Then,

$$y = y_h + y_{p1} + y_{p2} + y_{p3} + \ldots$$

Variation of Parameters

$$y_h = c_1 u_1(x) + c_2 u_2(x)$$
$$y_p = v_1(x) u_1(x) + v_2(x) u_2(x)$$
Where,
$$v_1' = \frac{-u_2(x) f(x)}{\Delta}, v_2' = \frac{u_1(x) f(x)}{\Delta}, \Delta = \begin{vmatrix} u_1 & u_2 \\ u_1', & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1'$$
$$\therefore v_1 = \int \frac{-u_2(x) f(x)}{u_1 u_2' - u_2 u_1'} dx, v_2 = \int \frac{u_1(x) f(x)}{u_1 u_2' - u_2 u_1'} dx$$

PART 15: MATRICIES

15.1 BASIC PRINICPLES:

Size = $i \times j$, i=row, j=column

$A = \lfloor a_{ij} \rfloor$

15.2 BASIC OPERTAIONS:

Addition: $A + B = \lfloor a_{ij} + b_{ij} \rfloor$

Subtraction: $A - B = \lfloor a_{ij} - b_{ij} \rfloor$

Scalar Multiple: $kA = \lfloor ka_{ij} \rfloor$

Transpose: $\lfloor A^T \rfloor_{ij} = A_{ji}$

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$

$(A + B + C + ...)^T = A^T + B^T + C^T + ...$

$(ABCD...)^T = ...D^T C^T B^T A^T$

Scalar Product: $a \bullet b = \begin{bmatrix} a_1 & a_2 & a_3 & ... \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ ... \end{bmatrix}$

Symmetry: $A^T = A$

Cramer's Rule:

$Ax = B$

$x_i = \dfrac{\det(A_i)}{\det(A)}$ where Ai = column i replaced by B

Least Squares Solution

In the form $A\underline{x} = \underline{b}$, $\underline{x} = \left(A^T A\right)^{-1} A^T \underline{b}$

For a linear approximation: $r_0 + r_1 x = b$

For a quadratic approximation: $r_0 + r_1 x + r_2 x^2 = b$

Etc.

15.3 SQUARE MATRIX:

Diagonal: $\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$

Lower Triangle Matrix: $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

Upper Triangle Matrix: $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

15.4 DETERMINATE:

2x2: det $(A) = ad - bc$
3x3: det $(A) = aei + bfg + cdh - afh - bdi - ceg$
nxn: det $(A) = a_{11}C_{11} + a_{12}C_{12} + a_{1n}C_{1n} = \sum_{j=1}^{n} a_{1j}C_{ij} = \sum_{j=1}^{n} a_{1j}Mi_{1j} \times (-1)^{(1+j)}$

Rules:

1. If A has a row or a column of zeros, $\det(A) = 0$.

$$e.g. \begin{vmatrix} 5 & 0 & 6 & -1 \\ 0 & 0 & 8 & -2 \\ 1 & 0 & -3 & 4 \\ 3 & 0 & 0 & 1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 0.$$

2. Multiply any one row of A by a scalar k to obtain A'. Then

$$\det(A') = k \det(A)$$

(makes sense when you consider taking a cofactor expansion along that row) $e.g.$
$\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 5$ and $\begin{vmatrix} 3 & 9 \\ -1 & 2 \end{vmatrix} = 15$, as expected

3. Interchange any two rows in A to obtain A'. Then

$$\det(A') = -\det(A)$$

$$e.g. \begin{vmatrix} 13 & 1 \\ 2 & -1 \end{vmatrix} = -15 \text{ and } \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 15, \text{ as expected.}$$

4. Add a multiple of one row in A to another to obtain A'. Then

$$\det(A') = \det(A)$$

e.g. let $A = \begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$, then $\det(A) = 14$. Consider $\begin{bmatrix} 1 & 3 \\ -3 & 5 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$ $\sim \begin{bmatrix} 1 & 3 \\ 0 & 14 \end{bmatrix} = A'$ and $\det(A') = 14$, as expected.

Note that Rules 2, 3 and 4 deal in particular with how elementary row operation affect the determinant of a matrix. Rule 4 is particularly useful as it allows us to induce additional zeros in a matrix to simplify the determinant calculation.

<u>Ex:</u> Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 6 \\ -3 & 5 & 1 \end{vmatrix}$.

<u>Soln:</u> $$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 6 \\ -3 & 5 & 1 \end{vmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + 3R_2 \end{matrix} = \begin{vmatrix} 0^+ & -5 & -9 \\ 1^- & 2 & 6 \\ 0^+ & 11 & 19 \end{vmatrix} = -\begin{vmatrix} -5 & -9 \\ 11 & 19 \end{vmatrix} = -4.$$

5. If one row in A is a scalar multiple of another, then $\det(A) = 0$. This makes sense, since we can then apply an e.r.o. to get a zero row before applying Rule 1. *e.g.*
$\begin{vmatrix} 3 & -1 & 4 & 7 \\ 2 & 2 & 3 & -1 \\ -3 & 1 & -4 & -7 \\ 1 & 6 & 2 & 1 \end{vmatrix} = 0$, since $R_3 = -R_1$.

6. $\det(A^\top) = \det(A)$. This rule basically allows us to apply Rules 1-5 to columns as well as rows.

<u>Ex:</u> $$\begin{vmatrix} 2 & 1 & 0 & -1 \\ -5 & 0 & 4 & 2 \\ 1 & -3 & 0 & 4 \\ 0 & 0 & -1 & -2 \end{vmatrix} R_3 \to R_3 + 3R_1 = \begin{vmatrix} 2^+ & 1^- & 0 & -1 \\ -5 & 0^+ & 4 & 2 \\ 7 & 0^- & 0 & 1 \\ 0 & 0^+ & 1 & -2 \end{vmatrix} = -\begin{vmatrix} -5 & 4 & 2 \\ 7 & 0 & 1 \\ 0 & -1 & -2 \end{vmatrix} C_3 \to C_3 - 2C_2$$

$$= -\begin{vmatrix} -5^+ & 4 & -6 \\ 7^- & 0 & 1 \\ 0^+ & -1^- & 0^+ \end{vmatrix} = -(-(-1))\begin{vmatrix} -5 & -6 \\ 7 & 1 \end{vmatrix} = -(-5+42) = -37.$$

7. $\det(kA) = k^n \det(A)$ (This is simply an expanded version of Rule 2.)

8. $\det(AB) = \det(A)\det(B)$ This is not an obvious rule, but a very important one which is used frequently in practice. *e.g.* let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$. Then $\det(A) = 2 - 12 = -10$, $\det(B) = -1 - 6 = -7$, $AB = \begin{bmatrix} 7 & 7 \\ -1 & 9 \end{bmatrix}$ and $\det(AB) = 63 - (-7) = 70 = \det(A)\det(B)$, as expected.

9. An upper triangular matrix is square with all entries below the main diagonal equal to zero. A lower triangular matrix is square with all entries above the main diagonal equal to zero. Finally, the determinant of a lower triangular or upper triangular matrix is the product of all the diagonal elements. For example,

$$\text{(i)} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{vmatrix} = (1)\begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} = (1)((4)(2)-(0)(2)) = (1)(4)(2) = 8.$$

$$\text{(ii)} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 \\ -1 & 1 & 2 & 4 \end{vmatrix} = (2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 2 & 4 \end{vmatrix} = (2)(1)\begin{vmatrix} 5 & 0 \\ 2 & 4 \end{vmatrix} = (2)(1)(5)(4) = 40.$$

In particular, note that the identity matrix is both upper and lower triangular, so $\det(I) = 1^n = 1$ for any order n.

10. A square matrix A is invertible if and only if $\det(A) \neq 0$. This is the most important rule of determinants.

From Rule 10, it follows that:

(i) $\det(A) = 0$ shows that A is singular, *i.e.* A^{-1} does not exist. Compare this with the case of a scalar a. If $|a| = 0$ (*i.e.* if $a = 0$), then $a^{-1} = \frac{1}{a}$ also does not exist.

$\det(A) \neq 0$ shows that A is non-singular, *i.e.* A^{-1} does exist.

(ii) Note that if A is non-singular, then $AA^{-1} = I$. Hence,

$$\begin{array}{llcl} i.e. & \det(AA^{-1}) & = & \det(I) \\ i.e. & \det(A)\det(A^{-1}) & = & 1 \\ i.e. & \det(A^{-1}) & = & \dfrac{1}{\det(A)} \end{array}$$

15.5 INVERSE

2x2:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3x3:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{aei-afh-bdi+bfg+cdh-ceg}\begin{bmatrix} ei-fh & ch-bi & bf-ce \\ fg-di & ai-cg & cd-af \\ dh-eg & bg-ah & ae-bd \end{bmatrix}$$

Minor: Mij = Determinate of Sub matrix which has deleted row i and column j

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$M21 = \left\| \begin{bmatrix} b & c \\ h & f \end{bmatrix} \right\|$$

Cofactor: $Cij = Mij \times (-1)^{(i+j)}$

Adjoint Method for Inverse:

$$adj(A) = C^T$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

<u>Ex:</u> Find the adjoint of $A = \begin{bmatrix} 1^+ & 2^- & 1^+ \\ -1^- & 0^+ & 2^- \\ 1^+ & 1^- & 0^+ \end{bmatrix}$.

<u>Soln:</u> $C_{11} = +\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2$, $C_{12} = -\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = 2$, $C_{13} = +\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -1$, $C_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1$, $C_{22} = +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$, $C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$, $C_{31} = +\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$, $C_{32} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$, $C_{33} = +\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2$.

Hence, $C = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ and $\text{adj}(A) = C^T = \begin{bmatrix} -2 & 1 & 4 \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix}$.

Left Inverse:

$$AC = I$$

$$C = \left(A^T A\right)^{-1} A^T$$

(when rows(A)>columns(A))

Right Inverse:

$$CA = I$$

$$C = A^T \left(AA^T\right)^{-1}$$

(when rows(A)<columns(A))

15.6 LINEAR TRANSFORMATION

Axioms for a linear transformation:

If $F(\underline{u} + \underline{v}) = F(\underline{u}) + F(\underline{v})$ [Preserves Addition]

And $F(\lambda \underline{u}) = \lambda F(\underline{u})$ [Preserves Scalar Multiplication]

Transition Matrix:

The matrix that represents the linear transformation

$T(\underline{v}) = c_1 T(v_1) + c_2 T(v_2) + ... + c_n T(v_n)$
$T(x) = Ax$
$A = [T(\underline{e}_1) \mid T(\underline{e}_2) \mid ... \mid T(\underline{e}_3)]$ (With m columns and n rows)
$(T: V \to W, \dim(V) = m, \dim(W) = n)$

Zero Transformation:
$T(\underline{v}) = 0, \forall \underline{v} \varepsilon V$
Identity Transformation:
$T(\underline{v}) = \underline{v}, \forall \underline{v} \varepsilon V$

15.7 COMMON TRANSITION MATRICIES

Rotation (Clockwise): $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Rotation (Anticlockwise): $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Scaling: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Shearing (parallel to x-axis): $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Shearing (parallel to y-axis): $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

15.8 EIGENVALUES AND EIGENVECTORS

Definitions: All solutions of $Ax = \lambda x$
Eigenvalues: All solutions of λ of det(A-λI)=0
Eigenvectors: General solution of [A-λI][X]=0 (ie: the nullspace)
Characteristic Polynomial: The function $p(\lambda) = \det(A - \lambda I)$

<u>Result 1:</u> *Let A be an $n \times n$ matrix and let $p(\lambda)$ be its characteristic polynomial. Suppose that λ is an eigenvalue of A with corresponding eigenvector $\boldsymbol{x}$. Then:*

(i) *The leading coefficient of $p(\lambda)$ is $(-1)^n$.*

(ii) *The constant term of $p(\lambda)$ is $\det(A)$.*

(iii) *λ^k is an eigenvalue of A^k with corresponding eigenvector $\boldsymbol{x}$.*

(iv) *If A is invertible, then $\lambda \neq 0$ and $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\boldsymbol{x}$.*

Algebraic Multiplicity: The number of times a root is repeated for a given eigenvalue.
$\sum$ of all algebraic multiplicity = degree of the characteristic polynomial.
Geometric Multiplicity: The number of linearly independent eigenvectors you get from a given eigenvalue.

Transformation:

$$T: V \to V$$
$$T(x) = \lambda x$$

The same process for an ordinary matrix is used.

Linearly Independence: The set of eigenvectors for distinct eigenvalues is linearly independent.

Digitalization: For a nxn matrix with n distinct eigenvalues; if and only if there are n Linearly Independent Eigenvectors:

$$D = P^{-1}AP$$

Where $P = [P_1 | P_2 | ... | P_n]$, P_n is an eigenvector.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

<u>**Result 7:**</u> *If $B = C^{-1}AC$, then*

(i) *the eigenvalues of B are the same as those of A.*

(ii) *each eigenvalue of A or B has the same algebraic and geometric multiplicity, regardless of which matrix they correspond to.*

(iii) *if $\boldsymbol{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $C^{-1}\boldsymbol{x}$ is an eigenvector of B corresponding to λ.*

Cayley-Hamilton Theorem: Every matrix satisfies its own polynomial:

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} ... + a_1\lambda + a_0 = 0$$
$$P(\lambda) = a_nA^n + a_{n-1}A^{n-1} ... + a_1A + a_0 = 0$$

Orthonormal Set: The orthonomal basis of a matrix A can be found with $P = [P_1 | P_2 | ... | P_n]$, the orthonormal set will be

$$B = \left\{ \frac{P_1}{\|P_1\|}, \frac{P_2}{\|P_2\|}, ... \frac{P_n}{\|P_n\|} \right\}$$

QR Factorisation:

$$A = [u_1 | u_2 | ... | u_n] = QR$$
$$\dim(A) = n \times k, k \le n$$

All columns are Linearly Independent

$Q = [v_1 | v_2 | ... | v_n]$ by the Gram-Schmidt Process

$$R = \begin{bmatrix} \|q_1\| & u_2^Tv_1 & u_3^Tv_1 & u_4^Tv_1 & ... & u_k^Tv_1 \\ 0 & \|q_2\| & u_3^Tv_2 & u_4^Tv_2 & ... & u_k^Tv_2 \\ 0 & 0 & \|q_3\| & u_4^Tv_3 & ... & u_k^Tv_3 \\ 0 & 0 & 0 & \|q_4\| & ... & u_k^Tv_4 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \|q_k\| \end{bmatrix}$$

$$R=\begin{bmatrix} \|q_1\| & u_2 \bullet v_1 & u_3 \bullet v_1 & u_4 \bullet v_1 & ... & u_k \bullet v_1 \\ 0 & \|q_2\| & u_3 \bullet v_2 & u_4 \bullet v_2 & ... & u_k \bullet v_2 \\ 0 & 0 & \|q_3\| & u_4 \bullet v_3 & ... & u_k \bullet v_3 \\ 0 & 0 & 0 & \|q_4\| & ... & u_k \bullet v_4 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \|q_k\| \end{bmatrix}$$

15.9 JORDAN FORMS

Generalised Diagonlisation:

$$P^{-1}AP = J$$

$$A = PJP^{-1}$$

Jordan Block:

$$J_B = \begin{bmatrix} \lambda & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & 1 & ... & 0 & 0 \\ 0 & 0 & \lambda & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda \end{bmatrix}$$

Jordan Form:

$$J = \begin{bmatrix} J_1 & 0 & ... & 0 \\ 0 & J_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & J_n \end{bmatrix}$$

Algebraic Multiplicity: The number of times λ appears on main diagonal

Geometric Multiplicity: The number of times λ appears on main diagonal without a 1 directly above it

Generalised Chain: $=\{u_m, u_{m-1}, ..., u_2, u_1\}$, where u_1 is an eigenvector

$$u_k = (A-\lambda I)u_{k+1}$$

$$u_{k+1} = [A-\lambda I \,|\, u_k]$$

$P = [P_1 \,|\, P_2 \,|\, ... \,|\, P_m \,|\, ...]$, for every eigenvector of A

Powers:

$$A^k = PJ^k P^{-1}$$

$$J^k = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_n \end{bmatrix}^k = \begin{bmatrix} J_1{}^k & 0 & \dots & 0 \\ 0 & J_2{}^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_n{}^k \end{bmatrix}$$

$$J_B{}^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \dots \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \dots \\ 0 & 0 & \lambda^k & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

15.10 COMPLEX MATRICIS:

Conjugate Transpose:

$$A^* = \overline{A^T}$$

$$A^{*^*} = A$$

$$(A+B)^* = A^* + B^*$$

$$(zA)^* = \bar{z}A^*$$

$$(AB)^* = B^* A^*$$

Hermitian Matrix: (Similar to Symmetric Matricis in the real case)
A square matrix such that A^*=A
Eigenvalues of A are purely real
Eigenvectors from distinct eigenvalues are orthogonal. This leads to a unitary digitalisation of the Hermitian matrix.
These are normal

Skew-Hermitian:

A square matrix such that A^*=-A
Eigenvalues of A are purely imaginary
Eigenvectors from distinct eigenvalues are orthogonal.
If A is Skew-Hermitian, iA is normal as: $(iA)^* = \bar{i}A^* = (-i)(-A) = iA$
These are normal

Unitary Matrix: (Similar to Orthogonal Matricis in the real case)
A square matrix such that A^*A=I
Columns of A form an orthonormal set of vectors
Rows of A from an orthonormal set of vectors

Normal Matrix:

Where $AA^* = A^*A$
These will have unitary diagonalisation
All Hermition and Skew-Hermitian matricis are normal ($A^*A = AA = AA^*$)

Diagonalisation:

For a nxn matrix with n distinct eigenvalues; if and only if there are n Linearly Independent Eigenvectors:

$$D = P^{-1}AP$$

Where $P = [P_1 \,|\, P_2 \,|\, ... \,|\, P_n]$, P_n is an eigenvector.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

If A is Hermitian, $D = P^{-1}AP = P^{*}AP$ as P are an orthonormal set of vectors.

Spectral Theorem:

For a nxn Normal matrix and eigenvectors form an orthonormal set

$$P = [P_1 \,|\, P_2 \,|\, ... \,|\, P_n]$$

$$A = \lambda_1 P_1 P_1^{*} + \lambda_2 P_2 P_2^{*} + ... + \lambda_n P_n P_n^{*}$$

Therefore, A can be represented as a sum of n matricis, all of rank 1.
Therefore, A can be approximated as a sum of the dominant eigenvalues

15.11 NUMERICAL COMPUTATIONS:

Rayleigh Quotient:

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top A \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$$

if (λ;*v*) is an eigenvalue/eigenvector pair of *A*, then

$$R(\boldsymbol{v}) = \frac{\boldsymbol{v}^\top A \boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \frac{\boldsymbol{v}^\top (\lambda \boldsymbol{v})}{\boldsymbol{v}^\top \boldsymbol{v}} = \lambda \frac{\boldsymbol{v}^\top \boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} = \lambda,$$

In particular, suppose that A is a symmetric matrix. Then we know that A can be orthogonally diagonalized, *i.e.* there exists an orthogonal matrix Q such that

$$Q^\top A Q = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (real) eigenvalues of A. Now, without loss of generality, suppose that λ_1 is the largest of all the eigenvalues, *i.e.* $\lambda_1 \geq \lambda_i$, for all $i = 2, 3, \ldots, n$.

Let $\boldsymbol{x}$ be any vector in $I\!R^n$ and let $\boldsymbol{s} = Q^\top \boldsymbol{x} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \in I\!R^n$, so that $\boldsymbol{x} = Q\boldsymbol{s}$. Then

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top A \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} = \frac{\boldsymbol{s}^\top Q^\top A Q \boldsymbol{s}}{\boldsymbol{s}^\top Q^\top Q \boldsymbol{s}} = \frac{\boldsymbol{s}^\top D \boldsymbol{s}}{\boldsymbol{s}^\top \boldsymbol{s}} = \frac{\lambda_1 s_1^2 + \lambda_2 s_2^2 + \ldots + \lambda_n s_n^2}{s_1^2 + s_2^2 + \ldots + s_n^2}.$$

Since λ_1 is the largest eigenvalue,

$$R(\boldsymbol{x}) \leq \frac{\lambda_1 s_1^2 + \lambda_1 s_2^2 + \ldots + \lambda_1 s_n^2}{s_1^2 + s_2^2 + \ldots + s_n^2}, \quad i.e.\ R(\boldsymbol{x}) \leq \lambda_1, \text{ for all } \boldsymbol{x} \in I\!R^n.$$

i.e. For a real symmetric matrix, the maximum value of Rayleigh's quotient is equal to the largest eigenvalue of A. This fact can sometimes be used to find the largest eigenvalue of a real symmetric matrix. (Note that the above argument is not valid if A is not symmetric.)

<u>Ex 1:</u> Find the largest eigenvalue of $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and the corresponding eigenvector.

<u>Soln:</u> Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then

$$\boldsymbol{x}^\top A \boldsymbol{x} = [x_1\ x_2\ x_3] \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [x_1\ x_2\ x_3] \begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + x_2 \\ 2x_3 \end{bmatrix} = x_1^2 - 4x_1x_2 + x_2^2 + 2x_3^2.$$

We have

$$\begin{aligned} R(\boldsymbol{x}) &= \frac{\boldsymbol{x}^\top A \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} \\ &= \frac{x_1^2 - 4x_1x_2 + x_2^2 + 2x_3^2}{x_1^2 + x_2^2 + x_3^2} \\ &= \frac{3x_1^2 + 3x_2^2 - (2x_1^2 + 4x_1x_2 + 2x_2^2) + 2x_3^2}{x_1^2 + x_2^2 + x_3^2} \\ &= \frac{3x_1^2 + 3x_2^2 + 3x_3^2 - 2(x_1 + x_2)^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2} \\ &= 3 - \frac{(2(x_1 + x_2)^2 + x_3^2)}{x_1^2 + x_2^2 + x_3^2} \\ &\leq 3, \end{aligned}$$

for all x_1, x_2 and x_3. In other words, the maximum eigenvalue is $\lambda = 3$. Note that $R(\boldsymbol{x}) = 3$ only if $x_3 = 0$ and $x_1 + x_2 = 0$, *i.e.* a corresponding eigenvector is $\boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Power method:

If A is a nxn matrix with Linearly Independent Eigenvectors, and distinct eigenvectors arranged such that: $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ and the set of eigenvectors are: $\{v_1, v_2, \ldots, v_n\}$

Any vector "w" can be written as:

$$w_0 = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

$$w_1 = A w_0 = c_1 A v_1 + c_2 A v_2 + \ldots + c_n A v_n = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n$$

$$w_s = Aw_{s-1} = c_1\lambda_1^{\ s}v_1 + c_2\lambda_2^{\ s}v_2 + ... + c_n\lambda_n^{\ s}v_n = \lambda_1\left(c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^s v_2 + ... + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^s v_n\right)$$

As $\left|\frac{\lambda_i}{\lambda_1}\right| < 1,\ \lim_{s\to\infty}\left(\left(\frac{\lambda_i}{\lambda_1}\right)^s\right) = 0$

$\therefore w_s \to c_1\lambda_1^{\ s}v_1$

Appling this with the Rayleigh Quotient:

$w_s = A\left(\frac{w_{s-1}}{|w_{s-1}|}\right), \lambda = R(w_s), w_0$ can be any vector usually $\begin{bmatrix} 1 \\ 0 \\ ... \end{bmatrix}$

PART 16: VECTORS

16.1 Basic Operations:

Addition: $\underline{a}+\underline{b}=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$

Subtraction: $\underline{a}-\underline{b}=\begin{bmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{bmatrix}$

Equality: $\underline{a}=\underline{b} \Leftrightarrow a_1=b_1, a_2=b_2, a_3=b_3$

$k\underline{a}+l\underline{b}=\lambda\underline{a}+\mu\underline{b} \Rightarrow k=\lambda, l=\mu$

Scalar Multiplication: $k\underline{a}=\begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$

Parallel: $\underline{a}=k\underline{b} \Leftrightarrow \underline{a}\|\underline{b}$

Magnitude: $|\underline{a}|=\sqrt{(\underline{a}_1)^2+(\underline{a}_2)^2+(\underline{a}_3)^2}$

Unit Vector: $\hat{\underline{a}}=\frac{\underline{a}}{|\underline{a}|}$

Zero Vector: A vector with no magnitude and no specific direction

Dot Product: $\underline{a}\bullet\underline{b}=|\underline{a}|\cdot|\underline{b}|\cdot\cos\theta$

$\underline{a}\bullet\underline{b}=\underline{a}_1\underline{b}_1+\underline{a}_2\underline{b}_2+\underline{a}_3\underline{b}_3$

Angle Between two Vectors: $\cos\theta=\frac{\underline{a}\bullet\underline{b}}{|\underline{a}|\cdot|\underline{b}|}$

$$\cos\theta=\frac{\underline{a}_1\underline{b}_1+\underline{a}_2\underline{b}_2+\underline{a}_3\underline{b}_3}{\left(\sqrt{\underline{a}_1{}^2+\underline{a}_2{}^2+\underline{a}_3{}^2}\right)\cdot\left(\sqrt{\underline{b}_1{}^2+\underline{b}_2{}^2+\underline{b}_3{}^2}\right)}$$

Angle of a vector in 3D: $\hat{\underline{a}}=\begin{bmatrix} \frac{a_1}{|\underline{a}|} \\ \frac{a_2}{|\underline{a}|} \\ \frac{a_3}{|\underline{a}|} \end{bmatrix}=\begin{bmatrix} \cos(\alpha) \\ \cos(\beta) \\ \cos(\gamma) \end{bmatrix}$

Perpendicular Test: $\underline{a}\bullet\underline{b}=0$

Scalar Projection: a onto b: $P = \underline{a} \bullet \hat{\underline{b}}$

Vector Projection: a onto b: $\underline{P} = \left(\underline{a} \bullet \hat{\underline{b}} \right) \hat{\underline{b}} = \frac{1}{|\underline{b}|^2} (\underline{a} \bullet \underline{b}) \underline{b}$

Cross Product:

$$\underline{a} \times \underline{b} = \langle \underline{a}_2 \underline{b}_3 - \underline{a}_3 \underline{b}_2, \underline{a}_3 \underline{b}_1 - \underline{a}_1 \underline{b}_3, \underline{a}_1 \underline{b}_2 - \underline{a}_2 \underline{b}_1 \rangle$$

$$\underline{a} \times \underline{b} = |\underline{a}| \cdot |\underline{b}| \cdot \sin\theta \cdot \underline{n}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}| \cdot |\underline{b}| \cdot \sin\theta$$

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a}) = \underline{c} \cdot (\underline{a} \times \underline{b})$$

$$\underline{a} \times \underline{b} = \det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \underline{i} \left(\det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \right) - \underline{j} \left(\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \right) + \underline{k} \left(\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right)$$

16.2 Lines

$\underline{r} = \underline{a} + \lambda \underline{b}$, where a is a point on the line, and b is a vector parallel to the line

$$x = a_1 + \lambda b_1$$

$$y = a_2 + \lambda b_2$$

$$z = a_3 + \lambda b_3$$

$$\lambda = \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

16.3 Planes

$$\underline{n} \bullet \overrightarrow{AR} = 0$$

$$\underline{n} \bullet \underline{r} = \underline{n} \bullet \underline{a}$$

$$\underline{n} \bullet \underline{r} = k$$

Where: $\underline{n} = \langle a, b, c \rangle$ & $\underline{r} = \langle x, y, z \rangle$: $ax + by + cz = k$

16.4 Closest Approach

Two Points: $d = \left| \overline{PQ} \right|$

Point and Line: $d = \left| \overline{PQ} \times \hat{\underline{a}} \right|$

Point and Plane: $d = \left| \overline{PQ} \bullet \hat{\underline{n}} \right|$

Two Skew Lines: $d = \left| \overline{PQ} \bullet \hat{\underline{n}} \right| = \left| \overline{PQ} \bullet (\underline{a} \times \underline{b}) \right|$

Solving for t: $\left[\underline{r}_b(t) - \underline{r}_a(t)\right] \bullet \left[\underline{v}_b - \underline{v}_a\right] = 0$
$\left[{}_a\underline{r}_b(t)\right] \bullet \left[{}_a\underline{v}_b\right] = 0$

16.5 Geometry

Area of a Triangle: $A = \dfrac{|AB \times AC|}{2}$

Area of a Parallelogram: $A = |AB \times AC|$

Area of a Parallelepiped: $A = |AD \bullet (AB \times AC)|$

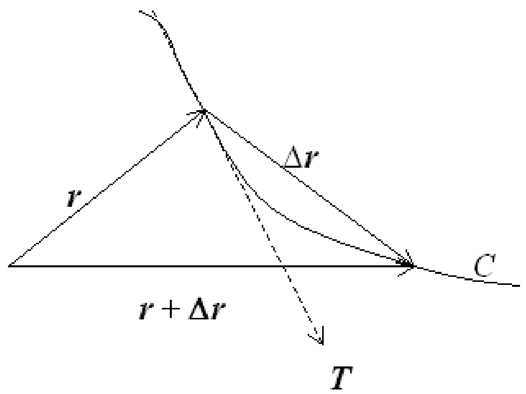
16.6 Space Curves

Where: $r(t) = x(t)i + y(t)j + z(t)k$

Velocity: $v(t) = r'(t) = x'(t)i + y'(t)j + z'(t)k$

Acceleration: $a(t) = v'(t) = r''(t) = x''(t)i + y''(t)j + z''(t)k$

Definition of "s":



The length of the curve from r to r+Δr

Unit Tangent: $T = \dfrac{dr}{ds} = \dfrac{r'(t)}{|r'(t)|}$

$|T| = 1$

Chain Rule: $\dfrac{dr}{dt} = \dfrac{dr}{ds} \times \dfrac{ds}{dt}$

As $\left|\frac{dr}{ds}\right|=1,\ \left|\frac{dr}{dt}\right|=\left|\frac{ds}{dt}\right|$=speed

Normal:

$$T \bullet T = 1$$
$$\frac{d}{ds}(T \bullet T) = 0$$
$$\frac{dT}{ds} \bullet T + T \bullet \frac{dT}{ds} = 0$$
$$2T \bullet \frac{dT}{ds} = 0$$
$$T \bullet \frac{dT}{ds} = 0$$

As T is tangent to the curve, $\frac{dT}{ds}$ is normal

$$N = \frac{\left(\frac{dT}{ds}\right)}{\left|\frac{dT}{ds}\right|}$$

Curvature:

$$\frac{dT}{ds} = \left|\frac{dT}{ds}\right| N = \kappa N$$
$$\therefore \kappa = \left|\frac{dT}{ds}\right| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|v(t) \times a(t)|}{|v(t)|^3}$$

Unit Binomial: $B = T \times N$

Tortion: $\tau = \left|\frac{dB}{ds}\right|$

16.7 Vector Space

16.8 ABBREVIATIONS

λ = a scalar value
μ = a scalar value
θ = the angle between the vectors
$\underline{a}$ = a vector
$\underline{b}$ = a vector
k = a scalar value
l = a scalar value
$\underline{n}$ = the normal vector
$\underline{r}$ = the resultant vector

PART 17: SERIES

17.1 MISCELLANEOUS

General Form: $S_n = a_1 + a_2 + a_2 + a_4 + ... + a_n = \sum_{n=1}^{n} a_n$

Infinite Form: $S_\infty = a_1 + a_2 + a_2 + a_4 + ... = \sum_{n=1}^{\infty} a_n$

Partial Sum of a Series: $S_i = a_1 + a_2 + a_2 + a_4 + ... + a_i = \sum_{n=1}^{i} a_n$

17.2 TEST FOR CONVERGENCE AND DIVERGENCE

Test For Convergence: $\lim_{n\to\infty}(S_n) = L$, if L exists, it is convergent

Test For Divergence: $\lim_{n\to\infty}(a_n) \neq 0$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} Divergent, |r| \geq 1 \\ Convergent, |r| < 1 \end{cases}$$

P Series

$$\sum_{n=1}^{\infty} \frac{1}{x^p} \begin{cases} Divergent, p \leq 1 \\ Convergent, p > 1 \end{cases}$$

The Sandwich Theorem

If there is a positive series so that $a_n \leq b_n \leq c_n$

If $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n) = L$, then, $\lim_{n\to\infty}(b_n) = L$

Hence, if a_n & c_n are convergent, b_n must also be convergent

The Integral Test

If $a_n = f_{(x)}$ if $f_{(x)}$ is continuous, positive and decreasing

If S_∞ or $\int_1^\infty f_{(x)}$ is true, then the other is true

Eg:

$$a_n = \frac{1}{n} = f_{(n)} = \frac{1}{x} = f_{(x)}$$

$$\therefore \int_1^\infty f_{(x)} dx = \int_1^\infty \frac{1}{x} dx = [\ln x]_1^\infty = D.N.E.$$

$\therefore a_n$ is divergent

The Direct Comparison Test

If we want to test a_n, and know the behaviour of b_n, where a_n is a series with only non-negative terms

If b_n is convergent and $a_n \le b_n$, then a_n is also convergent

The Limit Comparison Test

If there is a convergent series $\sum_{n=1}^{\infty} c_n$, then if $\lim_{n\to\infty}\left(\frac{a_n}{c_n}\right) < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges

If there is a divergent series $\sum_{n=1}^{\infty} d_n$, then if $\lim_{n\to\infty}\left(\frac{a_n}{d_n}\right) > 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

D'almbert's Ratio Comparison Test

FOR POSITIVE TERMS:

Converges: $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) < 1$

Diverges: $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) > 1$

Not enough information: $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = 1$

The n^th Root Test

For $\sum_{n=1}^{\infty} a_n$ where $a_n \ge 0$, then if $\lim_{n\to\infty}\sqrt[n]{a_n}$,

Converges: $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$

Diverges: $\lim_{n\to\infty}\sqrt[n]{a_n} > 1$

Not enough information: $\lim_{n\to\infty}\sqrt[n]{a_n} = 1$

Abel's Test:

If $\sum_{n=1}^{\infty} a_n$ is positive and decreasing, and $\sum_{n=1}^{\infty} c_n$ is a convergent series.

Then $\sum_{n=1}^{\infty} a_n \times c_n$ Converges

Negative Terms

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent

Alternating Series Test

This is the only test for an alternating series in the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \times b_n$

Let b_n be the sequence of positive numbers. If $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

Alternating Series Error

$|R_n| = |S - s_n| \le b_{n+1}$, where R_n is the error of the partial sum to the n^th term.

17.3 ARITHMETIC PROGRESSION:

Definition: $a, a+d, a+2d, a+3d, ...$

Nth Term: $= a + d(n-1)$

Sum Of The First N Terms: $\sum_{a=1}^{n} a = \frac{n}{2}(2a + d(n-1))$

17.4 GEOMETRIC PROGRESSION:

Definition: $a, ar, ar^2, ar^3, ...$

Nth Term: $= ar^{n-1}$

Sum Of The First N Terms: $S_n = \sum_{a=1}^{n} a = \frac{a(1-r^n)}{1-r}$

Sum To Infinity: $S_\infty = \lim_{n\to\infty}\left(\frac{a(1-r^n)}{1-r}\right) = \frac{a}{1-r}$ (given $|r| < 1$)

$P, A, Q, ...$

Geometric Mean: $\frac{A}{P} = r, \frac{Q}{A} = r$

$\therefore \frac{A}{P} = \frac{Q}{A} \Rightarrow A^2 = PQ \Rightarrow A = \sqrt{PQ}$

17.5 SUMMATION SERIES

Linear: 1+2+3+4+… $\sum_{a=1}^{n} a = \frac{n(n+1)}{2}$

Quadratic: $1^2+2^2+3^2+4^2+\ldots$ $\sum_{a=1}^{n} a^2 = \frac{n(n+1)(2n+1)}{6}$

Cubic: $1^3+2^3+3^3+4^3+\ldots$ $\sum_{a=1}^{n} a^3 = \left(\frac{n(n+1)}{2}\right)^2$

17.6 APPROXIMATION SERIES

Taylor Series

$$f_{(x)} = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}{}_{(x_0)}}{n!}(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + ...$$

where, $a_n = \frac{f^{(n)}{}_{(x_0)}}{n!}$

Maclaurun Series

Special case of the Taylor Series where $x_0 = 0$

Linear Approximation:

$$f_{(x)} \approx L_{(x)} = \sum_{n=0}^{1} a_n (x - x_0)^n = \sum_{n=0}^{1} \frac{f^{(n)}{}_{(x_0)}}{n!} (x - x_0)^n = a_0 + a_1 (x - x_0)$$

Quadratic Approximation:

$$f_{(x)} \approx Q_{(x)} = \sum_{n=0}^{2} a_n (x - x_0)^n = \sum_{n=0}^{2} \frac{f^{(n)}{}_{(x_0)}}{n!} (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2$$

Cubic Approximation:

$$f_{(x)} \approx C_{(x)} = \sum_{n=0}^{3} a_n (x - x_0)^n = \sum_{n=0}^{3} \frac{f^{(n)}{}_{(x_0)}}{n!} (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3$$

17.7 MONOTONE SERIES

Strictly Increasing: $a_{n+1} > a_n$ $\qquad \frac{a_{n+1}}{a_n} > 1$

Non-Decreasing: $a_{n+1} \geq a_n$

Strictly Decreasing: $a_{n+1} < a_n$ $\qquad \frac{a_{n+1}}{a_n} < 1$

Non-Increasing: $a_{n+1} \leq a_n$

Convergence: A monotone sequence is convergent if it is bounded, and hence the limit exists when $a_n \rightarrow \infty$

17.8 RIEMANN ZETA FUNCTION

Form:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

Euler's Table:

n=2	$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots = \frac{\pi^2}{6}$
n=4	$\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots = \frac{\pi^4}{90}$
n=6	$\zeta(6) = \sum_{k=1}^{\infty} \frac{1}{k^6} = 1 + \frac{1}{64} + \frac{1}{729} + \frac{1}{4096} + \ldots = \frac{\pi^6}{945}$
n=8	$\zeta(8) = \frac{\pi^8}{9450}$
n=10	$\zeta(10) = \frac{\pi^{10}}{93555}$

n=12	$\zeta(12)=\dfrac{691\pi^{12}}{638512875}$
n=14	$\zeta(14)=\dfrac{2\pi^{14}}{18243225}$
n=16	$\zeta(16)=\dfrac{3617\pi^{16}}{325641566250}$
n=18	$\zeta(18)=\dfrac{43867\pi^{18}}{38979295480125}$
n=20	$\zeta(20)=\dfrac{174611\pi^{20}}{1531329465290625}$
n=22	$\zeta(22)=\dfrac{155366\pi^{22}}{13447856940643125}$
n=24	$\zeta(24)=\dfrac{236364091\pi^{24}}{201919571963756521875}$
n=26	$\zeta(26)=\dfrac{1315862\pi^{26}}{11094481976030578125}$

Alternating Series:

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^1=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\arctan 1=\frac{\pi}{4}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^2=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^3=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^4=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots=\frac{\pi^4}{96}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^5=\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\cdots=\frac{5\pi^5}{1536}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\right)^6=\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\cdots=\frac{\pi^6}{960}$$

Proof for n=2:

Taylor Series Expansion: $\sin(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots$

$\sin(x)=x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\ldots$

Polynomial Expansion: $\sin(x)=x(x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)\ldots$

$\sin(x)=Ax\left(1-\dfrac{x^2}{\pi^2}\right)\left(1-\dfrac{x^2}{2^2\pi^2}\right)\left(1-\dfrac{x^2}{3^2\pi^2}\right)\ldots$

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)=1=A$$

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)..$$

Comparing the Coefficient of x^3:

$$-\frac{1}{3!}=-\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\frac{1}{4^2\pi^2}\ldots$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\ldots$$

17.9 SUMMATIONS OF POLYNOMIAL EXPRESSIONS

$$\sum_{i=m}^{n} 1 = n+1-m$$

$$\sum_{i=1}^{n} \frac{1}{i} = H_n$$

(Harmonic number)

$$\sum_{i=m}^{n} i = \frac{(n+1-m)(n+m)}{2}$$

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4} = \left[\sum_{i=1}^{n} i\right]^2$$

$$\sum_{i=0}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$$

$$\sum_{i=0}^{n} i^p = \frac{(n+1)^{p+1}}{p+1} + \sum_{k=1}^{p} \frac{B_k}{p-k+1}\binom{p}{k}(n+1)^{p-k+1}$$

where B_k denotes a Bernoulli number

$$\left(\sum_{i=m}^{n} i\right)^2 = \sum_{i=m}^{n} (i^3 - im(m-1))$$

$$\sum_{i=m}^{n} i^3 = \left(\sum_{i=m}^{n} i\right)^2 + \sum_{i=m}^{n} i$$

17.10 SUMMATIONS INVOLVING EXPONENTIAL TERMS

Where $x \neq 1$

$$\sum_{i=m}^{n-1} x^i = \frac{x^m - x^n}{1-x}$$

$(m < n)$

$$\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$$ (geometric series starting at 1)

$$\sum_{i=0}^{n-1} i x^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2}$$

$$\sum_{i=0}^{n-1} i 2^i = 2 + (n-2)2^n$$ (special case when $x = 2$)

$$\sum_{i=0}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}}$$ (special case when $x = 1/2$)

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\sum_{k=0}^{\infty} k\frac{z^k}{k!} = ze^z$$

$$\sum_{k=0}^{\infty} k^2\frac{z^k}{k!} = (z+z^2)e^z$$

$$\sum_{k=0}^{\infty} k^3\frac{z^k}{k!} = (z+3z^2+z^3)e^z$$

$$\sum_{k=0}^{\infty} k^4\frac{z^k}{k!} = (z+7z^2+6z^3+z^4)e^z$$

$$\sum_{k=0}^{\infty} k^n\frac{z^k}{k!} = z\frac{d}{dz}\sum_{k=0}^{\infty} k^{n-1}\frac{z^k}{k!} = e^z T_n(z)$$

where $T_n(z)$ is the Touchard polynomials.

17.11 SUMMATIONS INVOLVING TRIGONOMETRIC TERMS

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sin z$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sinh z$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z$$

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh z$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2^{2k}-1)2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \tan z, |z| < \frac{\pi}{2}$$

$$\sum_{k=1}^{\infty} \frac{(2^{2k}-1)2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \tanh z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k} z^{2k-1}}{(2k)!} = \cot z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{2^{2k} B_{2k} z^{2k-1}}{(2k)!} = \coth z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k-1} (2^{2k}-2) B_{2k} z^{2k-1}}{(2k)!} = \csc z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{-(2^{2k}-2) B_{2k} z^{2k-1}}{(2k)!} = \operatorname{csch} z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} z^{2k}}{(2k)!} = \sec z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{E_{2k} z^{2k}}{(2k)!} = \operatorname{sech} z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{(2k)! z^{2k+1}}{2^{2k} (k!)^2 (2k+1)} = \arcsin z, |z| \le 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k)! z^{2k+1}}{2^{2k} (k!)^2 (2k+1)} = \operatorname{arsinh} z, |z| \le 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \arctan z, |z| < 1$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} = \operatorname{artanh} z, |z| < 1$$

$$\ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k)! z^{2k}}{2^{2k+1} k (k!)^2} = \ln\left(1 + \sqrt{1+z^2}\right), |z| \le 1$$

$$\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k} = \frac{\pi - \theta}{2}, 0 < \theta < 2\pi$$

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} = -\frac{1}{2} \ln(2 - 2\cos\theta), \theta \in \mathbb{R}$$

$$\sum_{k=0}^{\infty} \frac{\sin[(2k+1)\theta]}{2k+1} = \frac{\pi}{4}, 0 < \theta < \pi$$

$$B_n(x) = -\frac{n!}{2^{n-1}\pi^n} \sum_{k=1}^{\infty} \frac{1}{k^n} \cos\left(2\pi k x - \frac{\pi n}{2}\right), 0 < x < 1$$

$$\sum_{k=0}^{n} \sin(\theta + k\alpha) = \frac{\sin\frac{(n+1)\alpha}{2} \sin(\theta + \frac{n\alpha}{2})}{\sin\frac{\alpha}{2}}$$

$$\sum_{k=1}^{n-1} \sin\frac{\pi k}{n} = \cot\frac{\pi}{2n}$$

$$\sum_{k=0}^{n-1} \csc^2\left(\theta + \frac{\pi k}{n}\right) = n^2 \csc^2(n\theta) \sum_{k=1}^{n-1} \csc^2\frac{\pi k}{n} = \frac{n^2-1}{3}$$

$$\sum_{k=1}^{n-1} \csc^4 \frac{\pi k}{n} = \frac{n^4 + 10n^2 - 11}{45}$$

17.12 INFINITE SUMMATIONS TO PI

$$\pi = \frac{1}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{((2n)!)^3(42n+5)}{(n!)^6 16^{3n+1}}$$

$$\pi = \frac{4}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(21460n+1123)}{(n!)^4 441^{2n+1} 2^{10n+1}}$$

$$\pi = \frac{4}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(6n+1)\left(\frac{1}{2}\right)_n^3}{4^n (n!)^3}$$

$$\pi = \frac{72}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(260n+23)}{(n!)^4 4^{4n} 18^{2n}}$$

$$\pi = \frac{3528}{Z} \qquad Z = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)!(21460n+1123)}{(n!)^4 4^{4n} 882^{2n}}$$

17.13 LIMITS INVOLVING TRIGONOMETRIC TERMS

$$\lim_{x\to a} \sin x = \sin a$$

$$\lim_{x\to a} \cos x = \cos a$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to n^{\pm}} \tan\left(\pi x + \frac{\pi}{2}\right) = \mp\infty \qquad \text{for any integer } n$$

ABBREVIATIONS

a = the first term
d = A.P. difference
r = G.P. ratio

17.14 POWER SERIES EXPANSION

Exponential:

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$\sum_{k=0}^{\infty} k\frac{z^k}{k!} = ze^z$$

$$\sum_{k=0}^{\infty} k^2 \frac{z^k}{k!} = (z+z^2)e^z$$
$$\sum_{k=0}^{\infty} k^3 \frac{z^k}{k!} = (z+3z^2+z^3)e^z$$
$$\sum_{k=0}^{\infty} k^4 \frac{z^k}{k!} = (z+7z^2+6z^3+z^4)e^z$$
$$\sum_{k=0}^{\infty} k^n \frac{z^k}{k!} = z\frac{d}{dz}\sum_{k=0}^{\infty} k^{n-1}\frac{z^k}{k!} = e^z T_n(z)$$
$$\ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k)!z^{2k}}{2^{2k+1}k(k!)^2} = \ln\left(1+\sqrt{1+z^2}\right), |z| \le 1$$

Trigonometric:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \\
\tan x &= \sum_{n=0}^{\infty} \frac{U_{2n+1} x^{2n+1}}{(2n+1)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1) B_{2n} x^{2n-1}}{(2n)!} \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}. \\
\csc x &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(2^{2n-1}-1) B_{2n} x^{2n-1}}{(2n)!} \\
&= x^{-1} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots, \qquad \text{for } 0 < |x| < \pi. \\
\sec x &= \sum_{n=0}^{\infty} \frac{U_{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} \\
&= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}.
\end{aligned}$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$
$$= x^{-1} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \cdots, \qquad \text{for } 0 < |x| < \pi.$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sinh z$$

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh z$$

$$\sum_{k=1}^{\infty} \frac{(2^{2k}-1)2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \tanh z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}z^{2k-1}}{(2k)!} = \coth z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{-(2^{2k}-2)B_{2k}z^{2k-1}}{(2k)!} = \operatorname{csch} z, |z| < \pi$$

$$\sum_{k=0}^{\infty} \frac{E_{2k}z^{2k}}{(2k)!} = \operatorname{sech} z, |z| < \frac{\pi}{2}$$

$$\sum_{k=0}^{\infty} \frac{(2k)!z^{2k+1}}{2^{2k}(k!)^2(2k+1)} = \arcsin z, |z| \le 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k)!z^{2k+1}}{2^{2k}(k!)^2(2k+1)} = \operatorname{arsinh} z, |z| \le 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \arctan z, |z| < 1$$

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} = \operatorname{artanh} z, |z| < 1$$

$$\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k} = \frac{\pi-\theta}{2}, 0 < \theta < 2\pi$$

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k} = -\frac{1}{2}\ln(2-2\cos\theta), \theta \in \mathbb{R}$$

$$\sum_{k=0}^{\infty} \frac{\sin[(2k+1)\theta]}{2k+1} = \frac{\pi}{4}, 0 < \theta < \pi$$

$$B_n(x) = -\frac{n!}{2^{n-1}\pi^n} \sum_{k=1}^{\infty} \frac{1}{k^n} \cos\left(2\pi kx - \frac{\pi n}{2}\right), 0 < x < 1$$

$$\sum_{k=0}^{n} \sin(\theta + k\alpha) = \frac{\sin\frac{(n+1)\alpha}{2} \sin(\theta + \frac{n\alpha}{2})}{\sin\frac{\alpha}{2}}$$

$$\sum_{k=1}^{n-1} \sin\frac{\pi k}{n} = \cot\frac{\pi}{2n}$$

$$\sum_{k=0}^{n-1} \csc^2\left(\theta + \frac{\pi k}{n}\right) = n^2 \csc^2(n\theta)$$

$$\sum_{k=1}^{n-1} \csc^2\frac{\pi k}{n} = \frac{n^2-1}{3}$$

$$\sum_{k=1}^{n-1} \csc^4\frac{\pi k}{n} = \frac{n^4+10n^2-11}{45}$$

$$\sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2}{(k+1)(2k+1)!} z^{2k+2} = (\arcsin z)^2, |z| \le 1$$

Exponential and Logarithm Series:

$$\ln(1+x) = x\left(\frac{1}{1} - x\left(\frac{1}{2} - x\left(\frac{1}{3} - x\left(\frac{1}{4} - x\left(\frac{1}{5} - \cdots\right)\right)\right)\right)\right) \quad \text{for} \quad |x| < 1.$$

$$\ln(x) = \ln\left(\frac{1+y}{1-y}\right) = 2y\left(\frac{1}{1} + \frac{1}{3}y^2 + \frac{1}{5}y^4 + \frac{1}{7}y^6 + \frac{1}{9}y^8 + \cdots\right)$$

$$= 2y\left(\frac{1}{1} + y^2\left(\frac{1}{3} + y^2\left(\frac{1}{5} + y^2\left(\frac{1}{7} + y^2\left(\frac{1}{9} + \cdots\right)\right)\right)\right)\right), \ y = \frac{x-1}{x+1}$$

$$\ln(1+x) = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \cfrac{x}{1 - 0x + \cfrac{1^2x}{2 - 1x + \cfrac{2^2x}{3 - 2x + \cfrac{3^2x}{4 - 3x + \cfrac{4^2x}{5 - 4x + \ddots}}}}}$$

$$\ln\left(1+\frac{2x}{y}\right)=\cfrac{2x}{y+\cfrac{x}{1+\cfrac{x}{3y+\cfrac{2x}{1+\cfrac{2x}{5y+\cfrac{3x}{1+\ddots}}}}}}=\cfrac{2x}{y+x-\cfrac{(1x)^2}{3(y+x)-\cfrac{(2x)^2}{5(y+x)-\cfrac{(3x)^2}{7(y+x)}}}}$$

Fourier Series:

$$f_W(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(kx)+b_k\sin(kx)$$

$$f_W(x)=\frac{a_0}{2}+a_1\cos(x)+a_2\cos(2x)+\ldots+a_n\cos(nx)+b_1\sin(x)+b_2\sin(2x)+\ldots+b_n\sin(nx)$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(kx)dx \quad k=0,1,2,\ldots,n$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(kx)dx \quad k=1,2,\ldots,n$$

17.15 Bernoulli Expansion:

Fundamentally:

$$1^k+2^k+3^k+\ldots+n^k=\begin{cases}\text{A polynomial in n(n+1)} & \text{k odd}\\ \text{(2n+1) x A polynomial in n(n+1)} & \text{k even}\end{cases}$$

Expansions:

$$1+2+3+...+n=\frac{1}{2}n(n+1)$$

$$1+2+3+...+n=\frac{1}{2}n^2+\frac{1}{2}n$$

$$1+2+3+...+n=\frac{1}{2}\left(\binom{2}{0}B_0n^2+\binom{2}{1}B_1n\right)$$

$$1^2+2^2+3^2+...+n^2=(2n+1)\frac{1}{6}n(n+1)$$

$$1^2+2^2+3^2+...+n^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$$

$$1^2+2^2+3^2+...+n^2=\frac{1}{3}\left(\binom{3}{0}B_0n^3+\binom{3}{1}B_1n^2+\binom{3}{2}B_2n\right)$$

$$1^3+2^3+3^3+...+n^3=(1+2+3+...+n)^2$$

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}(n(n+1))^2$$

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$$

$$1^3+2^3+3^3+...+n^3=\frac{1}{4}\left(\binom{4}{0}B_0n^4+\binom{4}{1}B_1n^3+\binom{4}{2}B_2n^2+\binom{4}{3}B_3n\right)$$

$$1^4+2^4+3^4+...+n^4=(2n+1)\frac{1}{30}n(n+1)(3n(n+1)-1)$$

$$1^4+2^4+3^4+...+n^4=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

$$1^4+2^4+3^4+...+n^4=\frac{1}{5}\left(\binom{5}{0}B_0n^5+\binom{5}{1}B_1n^4+\binom{5}{2}B_2n^3+\binom{5}{3}B_3n^2+\binom{5}{4}B_4n\right)$$

$$1^k+2^k+3^k+...+n^k=\frac{1}{k+1}\left(\binom{k+1}{0}B_0n^{k+1}+\binom{k+1}{1}B_1n^{k+1-1}+\binom{k+1}{2}B_2n^{k+1-2}+...+\binom{k+1}{k-1}B_{k-1}n^2+\binom{k+1}{k}B_kn\right)$$

List of Bernoulli Numbers:

n	B(n)
0	1
1	$-\frac{1}{2}$
2	$\frac{1}{6}$
3	0

4	$-\frac{1}{30}$
5	0
6	$\frac{1}{42}$
7	0
8	$-\frac{1}{30}$
9	0
10	$\frac{5}{66}$
11	0
12	$-\frac{691}{2730}$
13	0
14	$\frac{7}{6}$
15	0
16	$-\frac{3617}{510}$
17	0
18	$\frac{43867}{798}$
19	0
20	$-\frac{174611}{330}$

PART 18: ELECTRICAL

18.1 FUNDAMENTAL THEORY

Charge: $q = 6.24 \times 10^{18}$ Coulombs

Current: $I = \dfrac{dq}{dt}$

Resistance: $R = \dfrac{\rho \ell}{A}$

Ohm's Law: $V = IR$

Power: $P = VI = I^2 R = \dfrac{V^2}{R}$

Conservation of Power: $\sum P_{CONSUMED} = \sum P_{DELIVERED}$

Electrical Energy: $W = P \times t = I^2 \times R \times t = \int_0^t P dt$

Kirchoff's Voltage Law: The sum of the volt drops around a close loop is equal to zero.

$$\sum V = 0$$

Kirchoff's Current Law: The sum of the currents entering any junction is equal to the sum of the currents leaving that junction.

$$\sum I_{IN} = \sum I_{OUT}$$

Average Current:

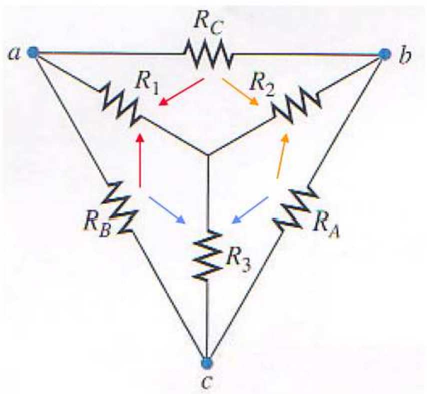
$$I_{AVE} = \frac{1}{T} \int_0^T I(t) dt$$

$$I_{AVE} = \frac{1}{T} \times Area \text{ (under I(t))}$$

RMS Current:

$$\sqrt{\frac{1}{T} \int_0^T (I(t))^2 dt}$$

Δ to Y Conversion:



$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

$$R_A = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_1}$$
$$R_B = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_2}$$
$$R_C = \frac{R_1R_2 + R_2R_3 + R_1R_3}{R_3}$$

18.2 COMPONENTS

Resistance in Series: $R_T = R_1 + R_2 + R_3 + \ldots$

Resistance in Parallel: $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$

Inductive Impedance: $X_L = j\omega L = j2\pi fL$

Capacitor Impedance: $X_C = -j\frac{1}{\omega C} = -j\frac{1}{2\pi fC}$

Capacitance in Series: $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$

Capacitance in Parallel: $C_T = C_1 + C_2 + C_3 + \ldots$

Voltage, Current & Power Summary:

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} i\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

18.3 THEVENIN'S THEOREM

Thevenin's Theorem:

V_{TH} = Open Circuit Voltage between a & b

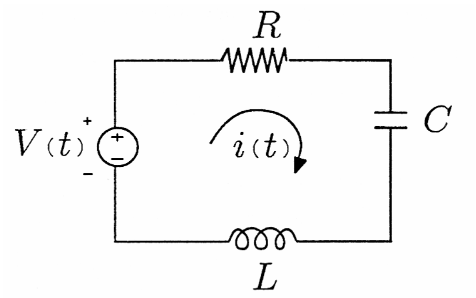
R_{TH} = Short Circuit any voltage source and Open Circuit any current source and calculate R_{TH} as the resistance from a & b. With dependant sources, SC terminals a & b and calculate the current in the wire (I_{SC}). $R_{TH} = \frac{V_{TH}}{I_{SC}}$

Maximum Power Transfer Theorem: $P_{MAX} = \frac{(V_{TH})^2}{4R_{TH}}$, where $R_L = R_{TH}$

18.4 FIRST ORDER RC CIRCUIT

18.5 FIRST ORDER RL CIRCUIT

18.6 SECOND ORDER RLC SERIES CIRCUIT



Calculation using KVL:

$$-V_S + V_R + V_L + V_C = 0$$

$$V_R + V_L + V_C = V_S$$

$$Ri + L\frac{di}{di} + V_C = V_S$$

Circuit current:

$$i = i_C = C\frac{dV_C}{dt}$$

$$\therefore \frac{di}{dt} = C\frac{d^2V_C}{dt^2}$$

$$\therefore RC\frac{dV_C}{dt} + LC\frac{d^2V_C}{dt^2} + V_C = V_S$$

$$LC\frac{d^2V_C}{dt^2} + RC\frac{dV_C}{dt} + V_C = V_S$$

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt} + \frac{1}{LC}V_C = \frac{V_S}{LC}$$

Important Variables

Standard Format: $s^2 + 2\alpha s + \omega_0{}^2 = 0$

Damping Factor: $\alpha = \frac{1}{2}\left(\frac{R}{L}\right)$

Natural Frequency: $s = \frac{dV_c}{dt}$

Undamped Natural Frequency: $\omega_0 = \sqrt{\left(\frac{1}{LC}\right)}$

Damping Frequency: $\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$

Mode Delta: $\Delta = \alpha^2 - \omega_0{}^2$

V_C: $V_C(t) = TRANSIENT + FINAL$

Solving:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2} = -\alpha \pm \sqrt{\Delta}$$

Mode 1:

If: $\Delta > 0$, then :

$$s = -\alpha \pm \sqrt{\Delta}$$

$$V_C(t) = TRANSIENT + FINAL$$

$$TRANSIENT = Ae^{s_1 t} + Be^{s_2 t}$$

$$FINAL = V_C(\propto) = V_S$$

$$V_C(t) = Ae^{s_1 t} + Be^{s_2 t} + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$

$$\therefore A + B + V_S = V_0 \rightarrow A + B = V_0 - V_S$$

$$\frac{dV_c}{dt} = As_1 e^{s_1 t} + Bs_2 e^{s_2 t}$$

$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = As_1 + Bs_2$$

$$\therefore \left.\begin{array}{l} V_0 - V_S = A + B \\ \frac{I_0}{C} = As_1 + Bs_2 \end{array}\right\}$$

Mode 2:

If: $\Delta = 0$, then :

$$s = -\alpha$$

$$V_C(t) = TRANSIENT + FINAL$$

$$TRANSIENT = (A + Bt)e^{st} = (A + Bt)e^{-\alpha t}$$

$$FINAL = V_C(\propto) = V_S$$

$$V_C(t) = (A + Bt)e^{-\alpha t} + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$
$$\therefore A + V_S = V_0 \rightarrow A = V_0 - V_S$$
$$\frac{dV_c}{dt} = (A + Bt)se^{st} + Be^{st}$$
$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = As + B$$
$$\therefore \left.\begin{array}{r} V_0 - V_S = A \\ \frac{I_0}{C} = As + B \end{array}\right\}$$

Mode 3:

If: $\Delta < 0$, and letting $\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$, then :

$$s = -\alpha \pm j\omega_d$$
$$V_C(t) = TRANSIENT + FINAL$$
$$TRANSIENT = (A\cos(\omega_d t) + B\sin(\omega_d t))e^{-\alpha t}$$
$$FINAL = V_C(\infty) = V_S$$
$$V_C(t) = (A\cos(\omega_d t) + B\sin(\omega_d t))e^{-\alpha t} + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$
$$\therefore A + V_S = V_0 \rightarrow A = V_0 - V_S$$
$$\frac{dV_c}{dt} = (-A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t))e^{-\alpha t} - \alpha(A\cos(\omega_d t) + B\sin(\omega_d t))e^{-\alpha t}$$
$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = B\omega_d - \alpha A$$
$$\therefore \left.\begin{array}{r} V_0 - V_S = A \\ \frac{I_0}{C} = B\omega_d - \alpha A \end{array}\right\}$$

Mode 4:

If: $R = 0$, then :

$$\alpha = 0, \omega_d = \omega_0$$
$$s = \pm j\omega_d = \pm j\omega_0$$
$$V_C(t) = TRANSIENT + FINAL$$
$$TRANSIENT = A\cos(\omega_d t) + B\sin(\omega_d t)$$
$$FINAL = V_C(\infty) = V_S$$
$$V_C(t) = A\cos(\omega_d t) + B\sin(\omega_d t) + V_S$$

Finding A & B:

$$V_C(0^+) = V_C(0^-) = V_0$$

$$\therefore A + V_S = V_0 \rightarrow A = V_0 - V_S$$

$$\frac{dV_c}{dt} = -A\omega_d \sin(\omega_d t) + B\omega_d \cos(\omega_d t)$$

$$\frac{dV_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{i_L(0^-)}{C} = \frac{I_0}{C} = B\omega_d$$

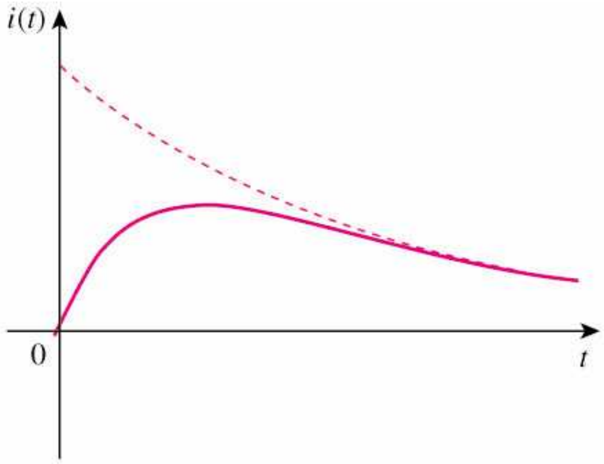
$$\therefore \left.\begin{matrix} V_0 - V_S = A \\ \frac{I_0}{C} = B\omega_d \end{matrix}\right\}$$

Current through Inductor:

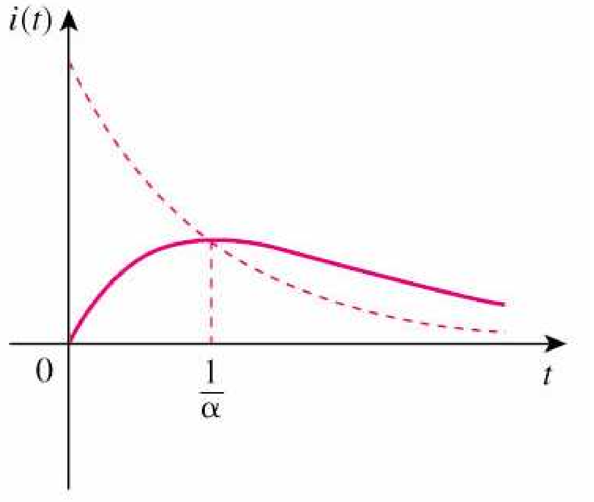
$$i_L = i_C = C\frac{dV_C}{dt}$$

Plotting Modes:

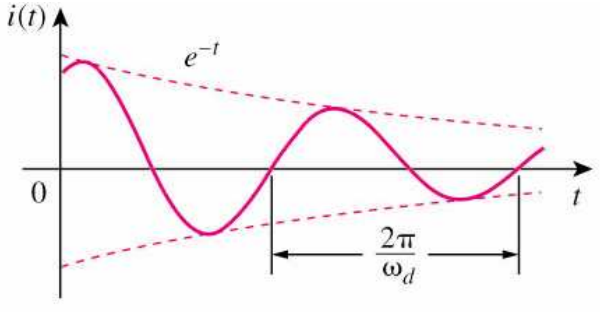
Mode 1: Over Damped



Mode 2: Critically Damped

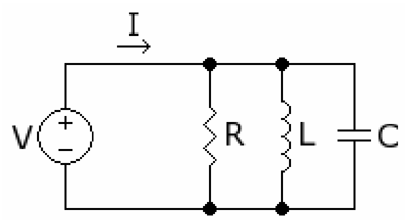


Mode 3: Sinusoidal Damped



Mode 4: Not Damped
(Oscillates indefinitely)

18.7 SECOND ORDER RLC PARALLEL CIRCUIT



Calculation using KCL:

$$i_S = i_R + i_L + i_C$$

$$i_S = \frac{V}{R} + i_L + C\frac{dV}{dt}$$

Node Voltage:

$$V_L = L\frac{di_L}{dt} = V$$

$$\frac{dV}{dt} = L\frac{d^2 i_L}{dt^2}$$

$$\therefore i_S = \frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt^2}$$

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = i_S$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{1}{LC}i_S$$

Important Variables

Standard Format: $s^2 + 2\alpha s + \omega_0{}^2 = 0$

Damping Factor: $\alpha = \frac{1}{2}\left(\frac{1}{RC}\right)$

Undamped Natural Frequency: $\omega_0 = \sqrt{\left(\frac{1}{LC}\right)}$

Damping Frequency: $\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$

Mode Delta: $\Delta = \alpha^2 - \omega_0{}^2$

Solving:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2} = -\alpha \pm \sqrt{\Delta}$$

18.8 LAPLANCE TRANSFORMATIONS

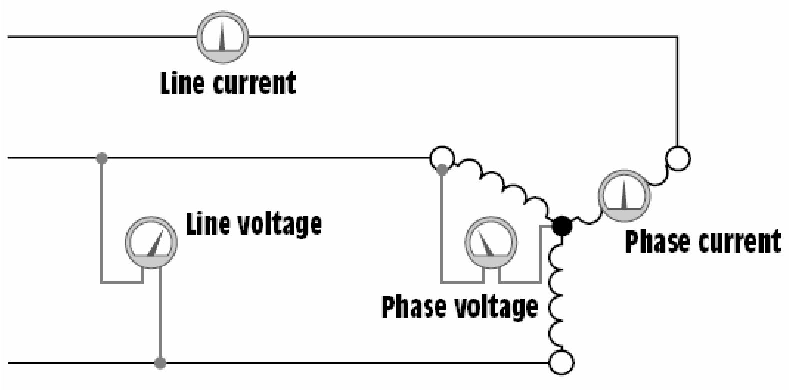
Identities:

1. $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}, \quad s > a$
9. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
10. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
11. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
12. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
13. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
14. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
15. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$

Properties:

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s} F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds} F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_1(t)$	$F_1(s)F_2(s)$

18.9 THREE PHASE – Y



Line Voltage: $V_{LINE} = V_{PHASE} \times \sqrt{3}$

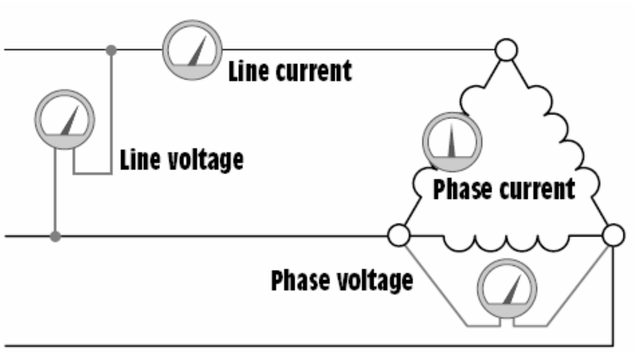
Phase Voltage: $V_{PHASE} = \frac{V_{LINE}}{\sqrt{3}}$

Line Current: $I_{LINE} = I_{PHASE}$

Phase Current: $I_{PHASE} = I_{LINE}$

Power:
$$S = \sqrt{3} \times V_{LINE} \times I_{LINE}$$
$$S = 3 \times V_{PHASE} \times I_{PHASE}$$

18.10 THREE PHASE – DELTA



Line Voltage: $V_{LINE} = V_{PHASE}$

Phase Voltage: $V_{PHASE} = V_{LINE}$

Line Current: $I_{LINE} = I_{PHASE} \times \sqrt{3}$

Phase Current: $I_{PHASE} = \frac{I_{LINE}}{\sqrt{3}}$

Power:
$$S = \sqrt{3} \times V_{LINE} \times I_{LINE}$$
$$S = 3 \times V_{PHASE} \times I_{PHASE}$$

18.11 POWER

Instantaneous: $P(t) = V(t) \times I(t)$

Average:

$$= \frac{1}{T}\int_0^T P(t)dt = \frac{1}{2}V_{MAX} I_{MAX} \cos(\theta_V - \theta_I) = V_{RMS} I_{RMS} \cos(\theta_V - \theta_I)$$

Maximum Power: $P_{MAX} = \frac{|V_{TH}|^2}{8R_{TH}}$ where $Z_L = \overline{Z_{TH}}$

Total Power: $= I_{RMS}{}^2 R$

Complex Power:

$$S = V_{RMS}\overline{I_{RMS}}$$

$$S = I_{RMS}{}^2 Z$$

$$S = P + jQ$$

where P = Average or Active Power (W) [positive = load, negative = generator]
where Q = Reactive Power (VAr) [positive = inductive, negative = capacitive]

18.12 Electromagnetics

Definitions:

Magnetic Flux	Φ	Strength of magnetic field	Wb
Reluctance	$\Re$	Relative difficulty for flux to establish	A-t/Wb
Permeability	μ	Relative ease for flux to establish	H/m
Magnetomotive Force	$\Im$	Ability of coil to produce flux	A-t
Flux density	B	Flux per unit area	Wb/m^2 or T
Magnetic Field Intensity	H	MMF per unit length	A-t/m

Permeability of free space: $\mu_0 = 4\pi \times 10^{-7} Hm^{-1}$

Magnetic Field Intensity: $H = \frac{\Im}{\ell} = \frac{NI}{\ell}$

Reluctance: $\Re = \frac{1}{\mu A}$

Ohm's Law: $\Phi = \frac{\Im}{\Re}$ OR $\Im = NI$

Magnetic Force on a conductor: $F = BI\ell$

Electromagnetic Induction: $EMF = -N\frac{\Phi_2 - \Phi_1}{t}$

$EMF = Bv\ell$

Magnetic Flux: $\Phi = BA$

Electric Field: $E = \frac{F}{q} = \frac{V}{d}$

Magnetic force on a particle: $F = qvB$

PART 19: GRAPH THEORY

19.1 Fundamental Explanations

List of vertices:

$$V = \{v_1, v_2, v_3, ...\}$$

List of edges:

$$E = \{e_1, e_2, e_3, ...\}$$

Subgaphs:

Any subgraph H such that
$V(H) \subset V(G) \& E(H) \subset E(G)$

Any subgraph H where $V(H) = V(G)$, there are no cycles and all verticies are connected.

Degree of vertex:

Number of edges leaving a vertex

$$\sum_{v \in V(G)} d(v) = 2|E(G)|$$

Distance: $d(u,v)$ =Shortest path between u & v

Diameter: $diam(G) = \max_{u \& v \in V(G)} \{d(u,v)\}$

Total Edges in a simple bipartite graph:

$$|E(G)| = \frac{|V(X)||V(Y)|}{2}$$

$$\sum_{x \in X} d(x) = \sum_{y \in Y} d(y)$$

Total Edges in K-regular graph:

$$|E(G)| = \frac{k(k-1)}{2}$$

19.2 Factorisation:

1 Factorisation:

A spanning union of 1 Factors and only exists if there are an even number of vertices.

1 Factors of a $K_{n,n}$ bipartite graph:

$$F_1 = [11',22',33',...]$$

$$F_2 = [12',23',34',...]$$

$$F_3 = [13',24',35'...]$$

$$F_n = ...$$

where all numbers are MOD(n)

1 Factors of a K_{2n} graph:

$$F_0 = \{(1,\infty),(2,0),(3,2n-2),...,(n,n+1)\}$$

$$F_i = \{(i,\infty),(i+1,2n-2+1),...,(i+n-1,i+n\}$$

$$F_{2n-2} = ...$$

Where all numbers are MOD(2n-1)

19.3 Vertex Colouring

Chromatic Number:

$\chi(G) \geq 3$ if there are triangles or an odd cycle

$\chi(G) \geq 2$ if is an even cycle

$\chi(G) \geq n$ if is K_n is a subgraph of G

Union/Intersection:

If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_m$, then

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{P(K_m,\lambda)}$$

Edge Contraction:

$$P(G,\lambda) = P(G-e,\lambda) - P(G.e,\lambda)$$

Common Chromatic Polynomials:

$$P(T_n,\lambda) = \lambda(\lambda-1)^{n-1}$$

$$P(C_n,\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$$

$$P(K_n,\lambda) = \lambda(\lambda-1)(\lambda-2)...(\lambda-n+1)$$

- Where the highest power is the number of vertices
- Where the lowest power is the number of components
- Where the the coefficient of the 2nd highest power is the number of edges.

19.4 Edge Colouring:

Common Chromatic Polynomials:

$$\chi'(G) \geq \Delta(G)$$

$$\chi'(K_{n,n}) = n$$

$$\chi'(C_{2n}) = 2$$

$$\chi'(C_{2n+1}) = 3$$

$$\chi'(K_{2n}) = 2n-1$$

$$\chi'(K_{2n+1}) = 2n+1$$

PART 99: CONVERSIONS

99.1 LENGTH:

Name of unit	Symbol	Definition	Relation to SI units
ångström	Å	≡ 1×10^{-10} m	= 0.1 nm
astronomical unit	AU	≈ Distance from Earth to Sun	≈ 149 597 871 464 m
barleycorn (H)		≡ ⅓ in (see note above about rounding)	= 8.46×10^{-3} m
bohr, atomic unit of length	a_0	≡ Bohr radius of hydrogen	≈ 5.291 772 0859 × 10^{-11} ± 3.6×10^{-20} m
cable length (Imperial)		≡ 608 ft	= 185.3184 m
cable length (International)		≡ 1/10 nmi	= 185.2 m
cable length (U.S.)		≡ 720 ft	= 219.456 m
chain (Gunter's; Surveyor's)	ch	≡ 66 ft ≡ 4 rods	= 20.1168 m
cubit (H)		≡ Distance from fingers to elbow ≈ 18in	≈ 0.5 m
ell (H)	ell	≡ 45 in	= 1.143 m
fathom	fm	≡ 6 ft	= 1.8288 m
fermi	fm	≡ 1×10^{-15} m	= 1×10^{-15} m
finger		≡ 7/8 in	= 0.022 225 m
finger (cloth)		≡ 4½ in	= 0.1143 m
foot (Benoît) (H)	ft (Ben)		≈ 0.304 799 735 m
foot (Clarke's; Cape) (H)	ft (Cla)		≈ 0.304 797 2654 m
foot (Indian) (H)	ft Ind		≈ 0.304 799 514 m
foot (International)	ft	≡ ⅓ yd = 12 inches	= 0.3048 m
foot (Sear's) (H)	ft (Sear)		≈ 0.304 799 47 m
foot (U.S. Survey)	ft (US)	≡ 1 200/3 937 m	≈ 0.304 800 610 m
french; charriere	F	≡ ⅓ mm	= 3.3×10^{-4} m
furlong	fur	≡ 10 chains = 660 ft = 220 yd	= 201.168 m
hand		≡ 4 in	= 0.1016 m
inch	in	≡ 1/36 yd = 1/12 ft	= 0.0254 m
league (land)	lea	≡ 3 US Statute miles	= 4 828.032 m
light-day		≡ 24 light-hours	= 2.590 206 837

			12×10^{13} m
light-hour		≡ 60 light-minutes	= 1.079 252 8488 × 10^{12} m
light-minute		≡ 60 light-seconds	= 1.798 754 748 × 10^{10} m
light-second		≡ Distance light travels in one second in vacuum	= 299 792 458 m
light-year	l.y.	≡ Distance light travels in vacuum in 365.25 days	= 9.460 730 472 5808 × 10^{15} m
line	ln	≡ 1/12 in	= 0.002 116 m
link (Gunter's; Surveyor's)	lnk	≡ 1/100 ch	= 0.201 168 m
link (Ramsden's; Engineer's)	lnk	≡ 1 ft	= 0.3048 m
metre (SI base unit)	m	≡ Distance light travels in 1/299 792 458 of a second in vacuum.	= 1 m
mickey		≡ 1/200 in	= 1.27 × 10^{-4} m
micron	µ		≡ 1 × 10^{-6} m
mil; thou	mil	≡ 1 × 10^{-3} in	= 2.54 × 10^{-5} m
mil (Sweden and Norway)	mil	≡ 10 km	= 10 000 m
mile	mi	≡ 1 760 yd = 5 280 ft = 80 chains	= 1 609.344 m
mile (geographical) (H)		≡ 6 082 ft	= 1 853.7936 m
mile (telegraph) (H)	mi	≡ 6 087 ft	= 1 855.3176 m
mile (U.S. Survey)	mi	≡ 5 280 *ft* (US Survey feet)	= 5 280 × 1 200/3 937 m ≈ 1 609.347 219 m
nail (cloth)		≡ 2¼ in	= 0.057 15 m
nautical league	NL; nl	≡ 3 nmi	= 5 556 m
nautical mile (Admiralty)	NM (Adm); nmi (Adm)	≡ 6 080 ft	≡ 1 853.184 m
nautical mile (international)	NM; nmi	≡ 1 852 m	= 1 852 m
pace		≡ 2.5 ft	= 0.762 m
palm		≡ 3 in	= 0.0762 m
parsec	pc	Distance of star with ***par***allax shift of one arc ***sec***ond from a base of one astronomical unit	≈ 3.085 677 82 × 10^{16} ± 6 × 10^{6} m
pica		≡ 12 points	Dependent on point measures.

point (American, English)	pt	≡ 1/72.272 in	≈ 0.000 351 450 m
point (Didot; European)	pt	≡ 1/12 × 1/72 of pied du roi; After 1878: ≡ 5/133 cm	≈ 0.000 375 97 m; After 1878: ≈ 0.000 375 939 85 m
point (PostScript) [11]	pt	≡ 1/72 in	= 0.000 352 7 m
point (TeX)	pt	≡ 1/72.27 in	= 0.000 351 4598 m
quarter		≡ ¼ yd	= 0.2286 m
rod; pole; perch (H)	rd	≡ 16½ ft	= 5.0292 m
rope (H)	rope	≡ 20 ft	= 6.096 m
span (H)		≡ 9 in	= 0.2286 m
spat			≡ 1 × 10^{12} m
stick (H)		≡ 2 in	= 0.0508 m
stigma; bicron (picometre)	pm		≡ 1 × 10^{-12} m
twip	twp	≡ 1/1 440 in	= 1.7638 × 10^{-5} m
x unit; siegbahn	xu		≈ 1.0021 × 10^{-13} m
yard (International)	yd	≡ 0.9144 m ≡ 3 ft ≡ 36 in	≡ 0.9144 m

99.2 AREA:

Name of unit	Symbol	Definition	Relation to SI units
acre (international)	ac	≡ 1 ch × 10 ch = 4 840 sq yd	= 4 046.856 4224 m^2
acre (U. S. survey)	ac	≡ 10 sq ch = 4 840 sq yd	= 4 046.873 m^2 [15]
are	a	≡ 100 m^2	= 100 m^2
barn	b	≡ 10^{-28} m^2	= 10^{-28} m^2
barony		≡ 4 000 ac	= 1.618 742 568 96 × 10^7 m^2
board	bd	≡ 1 in × 1 ft	= 7.741 92 × 10^{-3} m^2
boiler horsepower equivalent direct radiation	bhp EDR	≡ (1 ft^2) (1 bhp) / (240 BTU_{IT}/h)	≈ 12.958 174 m^2
circular inch	circ in	≡ π/4 sq in	≈ 5.067 075 × 10^{-4} m^2
circular mil; circular thou	circ mil	≡ π/4 mil^2	≈ 5.067 075 × 10^{-10} m^2
cord		≡ 192 bd	= 1.486 448 64 m^2
dunam		≡ 1 000 m^2	= 1 000 m^2
Guntha		≡ 33 ft x 33 ft[*citation*	≈ 101.17 m^2

		needed]	
hectare	ha	≡ 10 000 m^2	= 10 000 m^2
hide		≈ 120 ac (variable)	≈ 5 × 10^5 m^2
rood	ro	≡ ¼ ac	= 1 011.714 1056 m^2
shed		≡ 10^{-52} m^2	= 10^{-52} m^2
square (roofing)		≡ 10 ft × 10 ft	= 9.290 304 m^2
square chain (international)	sq ch	≡ 66 ft × 66 ft = 1/10 ac	= 404.685 642 24 m^2
square chain (U.S. Survey)	sq ch	≡ 66 ft(US) × 66 ft(US) = 1/10 ac	= 404.687 3 m^2
square foot	sq ft	≡ 1 ft × 1 ft	= 9.290 304 × 10^{-2} m^2
square foot (U.S. Survey)	sq ft	≡ 1 ft (US) × 1 ft (US)	≈ 9.290 341 161 327 49 × 10^{-2} m^2
square inch	sq in	≡ 1 in × 1 in	= 6.4516 × 10^{-4} m^2
square kilometre	km^2	≡ 1 km × 1 km	= 10^6 m^2
square link	sq lnk	≡ 1 lnk × 1 lnk	= 4.046 856 4224 × 10^{-2} m^2
square metre (SI unit)	m^2	≡ 1 m × 1 m	= 1 m^2
square mil; square thou	sq mil	≡ 1 mil × 1 mil	= 6.4516 × 10^{-10} m^2
square mile; section	sq mi	≡ 1 mi × 1 mi	= 2.589 988 110 336 × 10^6 m^2
square mile (U.S. Survey)	sq mi	≡ 1 mi (US) × 1 mi (US)	≈ 2.589 998 × 10^6 m^2
square rod/pole/perch	sq rd	≡ 1 rd × 1 rd	= 25.292 852 64 m^2
square yard	sq yd	≡ 1 yd × 1 yd	= 0.836 127 36 m^2
stremma		≡ 1 000 m^2	= 1 000 m^2
township		≡ 36 sq mi (US)	≈ 9.323 994 × 10^7 m^2
yardland		≈ 30 ac	≈ 1.2 × 10^5 m^2

99.3 VOLUME:

Name of unit	Symbol	Definition	Relation to SI units
acre-foot	ac ft	≡ 1 ac x 1 ft = 43 560 ft^3	= 1 233.481 837 547 52 m^3
acre-inch		≡ 1 ac × 1 in	= 102.790 153 128 96 m^3
barrel (Imperial)	bl (Imp)	≡ 36 gal (Imp)	= 0.163 659 24 m^3
barrel (petroleum)	bl; bbl	≡ 42 gal (US)	= 0.158 987 294 928 m^3
barrel (U.S. dry)	bl (US)	≡ 105 qt (US) = 105/32 bu (US lvl)	= 0.115 628 198 985 075 m^3
barrel (U.S. fluid)	fl bl (US)	≡ 31½ gal (US)	= 0.119 240 471 196 m^3
board-foot	fbm	≡ 144 cu in	= 2.359 737 216 × 10^{-3}

			m^3
bucket (Imperial)	bkt	≡ 4 gal (Imp)	= 0.018 184 36 m^3
bushel (Imperial)	bu (Imp)	≡ 8 gal (Imp)	= 0.036 368 72 m^3
bushel (U.S. dry heaped)	bu (US)	≡ 1 ¼ bu (US lvl)	= 0.044 048 837 7086 m^3
bushel (U.S. dry level)	bu (US lvl)	≡ 2 150.42 cu in	= 0.035 239 070 166 88 m^3
butt, pipe		≡ 126 gal (wine)	= 0.476 961 884 784 m^3
coomb		≡ 4 bu (Imp)	= 0.145 474 88 m^3
cord (firewood)		≡ 8 ft × 4 ft × 4 ft	= 3.624 556 363 776 m^3
cord-foot		≡ 16 cu ft	= 0.453 069 545 472 m^3
cubic fathom	cu fm	≡ 1 fm × 1 fm × 1 fm	= 6.116 438 863 872 m^3
cubic foot	cu ft	≡ 1 ft × 1 ft × 1 ft	= 0.028 316 846 592 m^3
cubic inch	cu in	≡ 1 in × 1 in × 1 in	= 16.387 064 × 10^{-6} m^3
cubic metre (SI unit)	m^3	≡ 1 m × 1 m × 1 m	= 1 m^3
cubic mile	cu mi	≡ 1 mi × 1 mi × 1 mi	= 4 168 181 825.440 579 584 m^3
cubic yard	cu yd	≡ 27 cu ft	= 0.764 554 857 984 m^3
cup (breakfast)		≡ 10 fl oz (Imp)	= 284.130 625 × 10^{-6} m^3
cup (Canadian)	c (CA)	≡ 8 fl oz (Imp)	= 227.3045 × 10^{-6} m^3
cup (metric)	c	≡ 250.0 × 10^{-6} m^3	= 250.0 × 10^{-6} m^3
cup (U.S. customary)	c (US)	≡ 8 US fl oz ≡ 1/16 gal (US)	= 236.588 2365 × 10^{-6} m^3
cup (U.S. food nutrition labeling)	c (US)	≡ 240 mL[16]	= 2.4×10^{-4} m^3
dash (Imperial)		≡ 1/384 gi (Imp) = ½ pinch (Imp)	= 369.961 751 302 08 3 × 10^{-9} m^3
dash (U.S.)		≡ 1/96 US fl oz = ½ US pinch	= 308.057 599 609 375 × 10^{-9} m^3
dessertspoon (Imperial)		≡ 1/12 gi (Imp)	= 11.838 776 0416 × 10^{-6} m^3
drop (Imperial)	gtt	≡ 1/288 fl oz (Imp)	= 98.656 467 013 8 × 10^{-9} m^3
drop (Imperial) (alt)	gtt	≡ 1/1 824 gi (Imp)	≈ 77.886 684 × 10^{-9} m^3
drop (medical)		≡ 1/12 ml	= 83.03 × 10^{-9} m^3
drop (metric)		≡ 1/20 mL	= 50.0 × 10^{-9} m^3
drop (U.S.)	gtt	≡ 1/360 US fl oz	= 82.148 693 22916 × 10^{-9} m^3
drop (U.S.) (alt)	gtt	≡ 1/456 US fl oz	≈ 64.854 231 × 10^{-9} m^3
fifth		≡ 1/5 US gal	= 757.082 3568 × 10^{-6} m^3
firkin		≡ 9 gal (US)	= 0.034 068 706 056 m^3
fluid drachm (Imperial)	fl dr	≡ ⅛ fl oz (Imp)	= 3.551 632 8125 × 10^{-6} m^3

fluid dram (U.S.); U.S. fluidram	fl dr	≡ ⅛ US fl oz	= 3.696 691 195 3125 × 10^{-6} m^3
fluid ounce (Imperial)	fl oz (Imp)	≡ 1/160 gal (Imp)	= 28.413 0625 × 10^{-6} m^3
fluid ounce (U.S. customary)	US fl oz	≡ 1/128 gal (US)	= 29.573 529 5625 × 10^{-6} m^3
fluid ounce (U.S. food nutrition labeling)	US fl oz	≡ 30 mL[16]	= 3×10^{-5} m^3
fluid scruple (Imperial)	fl s	≡ 1/24 fl oz (Imp)	= 1.183 877 60416 × 10^{-6} m^3
gallon (beer)	beer gal	≡ 282 cu in	= 4.621 152 048 × 10^{-3} m^3
gallon (Imperial)	gal (Imp)	≡ 4.546 09 L	= 4.546 09 × 10^{-3} m^3
gallon (U.S. dry)	gal (US)	≡ ⅛ bu (US lvl)	= 4.404 883 770 86 × 10^{-3} m^3
gallon (U.S. fluid; Wine)	gal (US)	≡ 231 cu in	= 3.785 411 784 × 10^{-3} m^3
gill (Imperial); Noggin	gi (Imp); nog	≡ 5 fl oz (Imp)	= 142.065 3125 × 10^{-6} m^3
gill (U.S.)	gi (US)	≡ 4 US fl oz	= 118.294 118 25 × 10^{-6} m^3
hogshead (Imperial)	hhd (Imp)	≡ 2 bl (Imp)	= 0.327 318 48 m^3
hogshead (U.S.)	hhd (US)	≡ 2 fl bl (US)	= 0.238 480 942 392 m^3
jigger (bartending)		≡ 1½ US fl oz	≈ 44.36 × 10^{-6} m^3
kilderkin		≡ 18 gal (Imp)	= 0.081 829 62 m^3
lambda	λ	≡ 1 mm^3	= 1 × 10^{-9} m^3
last		≡ 80 bu (Imp)	= 2.909 4976 m^3
litre	L	≡ 1 dm^3 [17]	= 0.001 m^3
load		≡ 50 cu ft	= 1.415 842 3296 m^3
minim (Imperial)	min	≡ 1/480 fl oz (Imp) = 1/60 fl dr (Imp)	= 59.193 880 208 3 × 10^{-9} m^3
minim (U.S.)	min	≡ 1/480 US fl oz = 1/60 US fl dr	= 61.611 519 921 875 × 10^{-9} m^3
peck (Imperial)	pk	≡ 2 gal (Imp)	= 9.092 18 × 10^{-3} m^3
peck (U.S. dry)	pk	≡ ¼ US lvl bu	= 8.809 767 541 72 × 10^{-3} m^3
perch	per	≡ 16½ ft × 1½ ft × 1 ft	= 0.700 841 953 152 m^3
pinch (Imperial)		≡ 1/192 gi (Imp) = ⅛ tsp (Imp)	= 739.923 502 60416 × 10^{-9} m^3
pinch (U.S.)		≡ 1/48 US fl oz = ⅛ US tsp	= 616.115 199 218 75 × 10^{-9} m^3
pint (Imperial)	pt (Imp)	≡ ⅛ gal (Imp)	= 568.261 25 × 10^{-6} m^3
pint (U.S. dry)	pt (US	≡ 1/64 bu (US lvl) ≡ ⅛	= 550.610 471

	dry)	gal (US dry)	3575 × 10^{-6} m^3
pint (U.S. fluid)	pt (US fl)	≡ ⅛ gal (US)	= 473.176 473 × 10^{-6} m^3
pony		≡ 3/4 US fl oz	= 22.180 147 171 875 × 10^{-6} m^3
pottle; quartern		≡ ½ gal (Imp) = 80 fl oz (Imp)	= 2.273 045 × 10^{-3} m^3
quart (Imperial)	qt (Imp)	≡ ¼ gal (Imp)	= 1.136 5225 × 10^{-3} m^3
quart (U.S. dry)	qt (US)	≡ 1/32 bu (US lvl) = ¼ gal (US dry)	= 1.101 220 942 715 × 10^{-3} m^3
quart (U.S. fluid)	qt (US)	≡ ¼ gal (US fl)	= 946.352 946 × 10^{-6} m^3
quarter; pail		≡ 8 bu (Imp)	= 0.290 949 76 m^3
register ton		≡ 100 cu ft	= 2.831 684 6592 m^3
sack (Imperial); bag		≡ 3 bu (Imp)	= 0.109 106 16 m^3[*citation needed*]
sack (U.S.)		≡ 3 bu (US lvl)	= 0.105 717 210 500 64 m^3
seam		≡ 8 bu (US lvl)	= 0.281 912 561 335 04 m^3[*citation needed*]
shot		≡ 1 US fl oz	≈ 29.57 × 10^{-6} m^3
strike (Imperial)		≡ 2 bu (Imp)	= 0.072 737 44 m^3
strike (U.S.)		≡ 2 bu (US lvl)	= 0.070 478 140 333 76 m^3
tablespoon (Canadian)	tbsp	≡ ½ fl oz (Imp)	= 14.206 531 25 × 10^{-6} m^3
tablespoon (Imperial)	tbsp	≡ 5/8 fl oz (Imp)	= 17.758 164 0625 × 10^{-6} m^3
tablespoon (metric)			≡ 15.0 × 10^{-6} m^3
tablespoon (U.S. customary)	tbsp	≡ ½ US fl oz	= 14.786 764 7825 × 10^{-6} m^3
tablespoon (U.S. food nutrition labeling)	tbsp	≡ 15 mL[16]	= 1.5×10^{-5} m^3
teaspoon (Canadian)	tsp	≡ 1/6 fl oz (Imp)	= 4.735 510 416 × 10^{-6} m^3
teaspoon (Imperial)	tsp	≡ 1/24 gi (Imp)	= 5.919 388 02083 × 10^{-6} m^3
teaspoon (metric)		≡ 5.0 × 10^{-6} m^3	= 5.0 × 10^{-6} m^3
teaspoon (U.S. customary)	tsp	≡ 1/6 US fl oz	= 4.928 921 595 × 10^{-6} m^3
teaspoon (U.S. food nutrition labeling)	tsp	≡ 5 mL[16]	= 5×10^{-6} m^3
timber foot		≡ 1 cu ft	= 0.028 316 846 592 m^3
ton (displacement)		≡ 35 cu ft	= 0.991 089 630 72 m^3
ton (freight)		≡ 40 cu ft	= 1.132 673 863 68 m^3

ton (water)		≡ 28 bu (Imp)	= 1.018 324 16 m^3
tun		≡ 252 gal (wine)	= 0.953 923 769 568 m^3
wey (U.S.)		≡ 40 bu (US lvl)	= 1.409 562 806 6752 m^3

99.4 PLANE ANGLE:

Name of unit	Symbol	Definition	Relation to SI units
angular mil	µ	≡ 2π/6400 rad	≈ 0.981 748 × 10^{-3} rad
arcminute	'	≡ 1°/60	≈ 0.290 888 × 10^{-3} rad
arcsecond	"	≡ 1°/3600	≈ 4.848 137 × 10^{-6} rad
centesimal minute of arc	'	≡ 1 grad/100	≈ 0.157 080 × 10^{-3} rad
centesimal second of arc	"	≡ 1 grad/(10 000)	≈ 1.570 796 × 10^{-6} rad
degree (of arc)	°	≡ π/180 rad = 1/360 of a revolution	≈ 17.453 293 × 10^{-3} rad
grad; gradian; gon	grad	≡ 2π/400 rad = 0.9°	≈ 15.707 963 × 10^{-3} rad
octant		≡ 45°	≈ 0.785 398 rad
quadrant		≡ 90°	≈ 1.570 796 rad
radian (SI unit)	rad	The angle subtended at the center of a circle by an arc whose length is equal to the circle's radius. One full revolution encompasses 2π radians.	= 1 rad
sextant		≡ 60°	≈ 1.047 198 rad
sign		≡ 30°	≈ 0.523 599 rad

99.5 SOLID ANGLE:

Name of unit	Symbol	Definition	Relation to SI units
steradian (SI unit)	sr	The solid angle subtended at the center of a sphere of radius r by a portion of the surface of the sphere having an area r^2. A sphere encompasses 4π sr.[14]	= 1 sr

99.6 MASS:

Name of unit	Symbol	Definition	Relation to SI units

atomic mass unit, unified	u; AMU		≈ $1.660\ 538\ 73 \times 10^{-27} \pm 1.3 \times 10^{-36}$ kg
atomic unit of mass, electron rest mass	m_e		≈ $9.109\ 382\ 15 \times 10^{-31} \pm 45 \times 10^{-39}$ kg [18]
bag (coffee)		≡ 60 kg	= 60 kg
bag (Portland cement)		≡ 94 lb av	= 42.637 682 78 kg
barge		≡ 22½ sh tn	= 20 411.656 65 kg
carat	kt	≡ 3 1/6 gr	≈ 205.196 548 333 mg
carat (metric)	ct	≡ 200 mg	= 200 mg
clove		≡ 8 lb av	= 3.628 738 96 kg
crith			≈ 89.9349 mg
dalton	Da		≈ $1.660\ 902\ 10 \times 10^{-27} \pm 1.3 \times 10^{-36}$ kg
dram (apothecary; troy)	dr t	≡ 60 gr	= 3.887 9346 g
dram (avoirdupois)	dr av	≡ 27 11/32 gr	= 1.771 845 195 3125 g
electronvolt	eV	≡ 1 eV (energy unit) / c^2	= 1.7826×10^{-36} kg
gamma	γ	≡ 1 μg	= 1 μg
grain	gr	≡ 64.798 91 mg	= 64.798 91 mg
hundredweight (long)	long cwt or cwt	≡ 112 lb av	= 50.802 345 44 kg
hundredweight (short); cental	sh cwt	≡ 100 lb av	= 45.359 237 kg
hyl (CGS unit)		≡ 1 gee × 1 g × 1 s^2/m	= 9.806 65 g
hyl (MKS unit)		≡ 1 gee × 1 kg × 1 s^2/m	= 9.806 65 kg
kilogram, grave	kg; G		(SI base unit)[8]
kip	kip	≡ 1 000 lb av	= 453.592 37 kg
mark		≡ 8 oz t	= 248.827 8144 g
mite		≡ 1/20 gr	= 3.239 9455 mg
mite (metric)		≡ 1/20 g	= 50 mg
ounce (apothecary; troy)	oz t	≡ 1/12 lb t	= 31.103 4768 g
ounce (avoirdupois)	oz av	≡ 1/16 lb	= 28.349 523 125 g
ounce (U.S. food nutrition labeling)	oz	≡ 28 g[16]	= 28 g
pennyweight	dwt; pwt	≡ 1/20 oz t	= 1.555 173 84 g
point		≡ 1/100 ct	= 2 mg
pound (avoirdupois)	lb av	≡ 7 000 grains	= 0.453 592 37 kg
pound (metric)		≡ 500 g	= 500 g
pound (troy)	lb t	≡ 5 760 grains	= 0.373 241 7216 kg

quarter (Imperial)		≡ 1/4 long cwt = 2 st = 28 lb av	= 12.700 586 36 kg
quarter (informal)		≡ ¼ short tn	= 226.796 185 kg
quarter, long (informal)		≡ ¼ long tn	= 254.011 7272 kg
quintal (metric)	q	≡ 100 kg	= 100 kg
scruple (apothecary)	s ap	≡ 20 gr	= 1.295 9782 g
sheet		≡ 1/700 lb av	= 647.9891 mg
slug; geepound	slug	≡ 1 gee × 1 lb av × 1 s^2/ft	≈ 14.593 903 kg
stone	st	≡ 14 lb av	= 6.350 293 18 kg
ton, assay (long)	AT	≡ 1 mg × 1 long tn ÷ 1 oz t	≈ 32.666 667 g
ton, assay (short)	AT	≡ 1 mg × 1 sh tn ÷ 1 oz t	≈ 29.166 667 g
ton, long	long tn or ton	≡ 2 240 lb	= 1 016.046 9088 kg
ton, short	sh tn	≡ 2 000 lb	= 907.184 74 kg
tonne (mts unit)	t	≡ 1 000 kg	= 1 000 kg
wey		≡ 252 lb = 18 st	= 114.305 277 24 kg (variants exist)
Zentner	Ztr.	Definitions vary; see [19] and.[14]	

99.7 DENSITY:

Name of unit	Symbol	Definition	Relation to SI units
gram per millilitre	g/mL	≡ g/mL	= 1,000 kg/m^3
kilogram per cubic metre (SI unit)	kg/m^3	≡ kg/m^3	= 1 kg/m^3
kilogram per litre	kg/L	≡ kg/L	= 1,000 kg/m^3
ounce (avoirdupois) per cubic foot	oz/ft^3	≡ oz/ft^3	≈ 1.001153961 kg/m^3
ounce (avoirdupois) per cubic inch	oz/in^3	≡ oz/in^3	≈ 1.729994044×10^3 kg/m^3
ounce (avoirdupois) per gallon (Imperial)	oz/gal	≡ oz/gal	≈ 6.236023291 kg/m^3
ounce (avoirdupois) per gallon (U.S. fluid)	oz/gal	≡ oz/gal	≈ 7.489151707 kg/m^3
pound (avoirdupois) per cubic foot	lb/ft^3	≡ lb/ft^3	≈ 16.01846337 kg/m^3
pound (avoirdupois) per cubic inch	lb/in^3	≡ lb/in^3	≈ 2.767990471×10^4 kg/m^3
pound (avoirdupois) per gallon (Imperial)	lb/gal	≡ lb/gal	≈ 99.77637266 kg/m^3
pound (avoirdupois) per gallon (U.S. fluid)	lb/gal	≡ lb/gal	≈ 119.8264273 kg/m^3
slug per cubic foot	slug/ft^3	≡ slug/ft^3	≈ 515.3788184 kg/m^3

99.8 TIME:

Name of unit	Symbol	Definition	Relation to SI units
atomic unit of time	au	≡ $a_0/(\alpha \cdot c)$	≈ 2.418 884 254 × 10^{-17} s
Callippic cycle		≡ 441 mo (hollow) + 499 mo (full) = 76 a of 365.25 d	= 2.398 3776 × 10^9 s
century		≡ 100 a (see below for definition of year length)	= 100 × year
day	d	= 24 h	= 86400 s
day (sidereal)	d	≡ Time needed for the Earth to rotate once around its axis, determined from successive transits of a very distant astronomical object across an observer's meridian (International Celestial Reference Frame)	≈ 86 164.1 s
decade		≡ 10 a (see below for definition of year length)	= 10 × year
fortnight		≡ 2 wk	= 1 209 600 s
helek		≡ 1/1 080 h	= 3.3 s
Hipparchic cycle		≡ 4 Callippic cycles - 1 d	= 9.593 424 × 10^9 s
hour	h	≡ 60 min	= 3 600 s
jiffy		≡ 1/60 s	= .016 s
jiffy (alternate)		≡ 1/100 s	= 10 ms
ke (quarter of an hour)		≡ ¼ h = 1/96 d	= 60 × 60 / 4 s = 900 s = 60 / 4 min = 15 min
ke (traditional)		≡ 1/100 d	= 24 × 60 × 60 / 100 s = 864 s = 24 * 60 / 100 min = 14.4 min
lustre; lustrum		≡ 5 a of 365 d	= 1.5768 × 10^8 s
Metonic cycle; enneadecaeteris		≡ 110 mo (hollow) + 125 mo (full) = 6940 d ≈ 19 a	= 5.996 16 × 10^8 s
millennium		≡ 1 000 a (see below for definition of year length)	= 1000 × year
milliday	md	≡ 1/1 000 d	= 24 × 60 × 60 / 1 000 s = 86.4 s
minute	min	≡ 60 s	= 60 s
moment		≡ 90 s	= 90 s
month (full)	mo	≡ 30 d[20]	= 2 592 000 s

month (Greg. av.)	mo	Average Gregorian month = 365.2425/12 d = 30.436875 d	≈ 2.6297×10^{6} s
month (hollow)	mo	≡ 29 d[20]	= 2 505 600 s
month (synodic)	mo	Cycle time of moon phases ≈ 29.530589 days (Average)	≈ 2.551×10^{6} s
octaeteris		= 48 mo (full) + 48 mo (hollow) + 3 mo (full)[21][22] = 8 a of 365.25 d = 2922 d	= $2.524\ 608 \times 10^{8}$ s
Planck time		≡ $(G\hbar/c^5)^{½}$	≈ $1.351\ 211\ 868 \times 10^{-43}$ s
second	s	time of 9 192 631 770 periods of the radiation corresponding to the transition between the 2 hyperfine levels of the ground state of the caesium 133 atom at 0 K[8] (but other seconds are sometimes used in astronomy)	(SI base unit)
shake		≡ 10^{-8} s	= 10 ns
sigma		≡ 10^{-6} s	= 1 μs
Sothic cycle		≡ 1 461 a of 365 d	= $4.607\ 4096 \times 10^{10}$ s
svedberg	S	≡ 10^{-13} s	= 100 fs
week	wk	≡ 7 d	= 604 800 s
year (Gregorian)	a, y, *or* yr	= 365.2425 d average, calculated from common years (365 d) plus leap years (366 d) on most years divisible by 4. See leap year for details.	= 31 556 952 s
year (Julian)	a, y, *or* yr	= 365.25 d average, calculated from common years (365 d) plus one leap year (366 d) every four years	= 31 557 600 s
year (sidereal)	a, y, *or* yr	≡ time taken for Sun to return to the same position with respect to the stars of the celestial sphere	≈ 365.256 363 d ≈ 31 558 149.7632 s
year (tropical)	a, y, *or* yr	≡ Length of time it takes for the Sun to return to the same position in the cycle of seasons	≈ 365.242 190 d ≈ 31 556 925 s

99.9 FREQUENCY:

Name of unit	Symbol	Definition	Relation to SI units
hertz (SI unit)	Hz	≡ Number of cycles per second	= 1 Hz = 1/s
revolutions per minute	rpm	≡ One unit rpm equals one rotation completed around a fixed axis in one minute of time.	≈ 0.104719755 rad/s

99.10 SPEED OR VELOCITY:

Name of unit	Symbol	Definition	Relation to SI units
foot per hour	fph	≡ 1 ft/h	≈ $8.466\,667 \times 10^{-5}$ m/s
foot per minute	fpm	≡ 1 ft/min	= 5.08×10^{-3} m/s
foot per second	fps	≡ 1 ft/s	= 3.048×10^{-1} m/s
furlong per fortnight		≡ furlong/fortnight	≈ $1.663\,095 \times 10^{-4}$ m/s
inch per minute	ipm	≡ 1 in/min	≈ $4.23\,333 \times 10^{-4}$ m/s
inch per second	ips	≡ 1 in/s	= 2.54×10^{-2} m/s
kilometre per hour	km/h	≡ 1 km/h	≈ $2.777\,778 \times 10^{-1}$ m/s
knot	kn	≡ 1 NM/h = 1.852 km/h	≈ 0.514 444 m/s
knot (Admiralty)	kn	≡ 1 NM (Adm)/h = 1.853 184 km/h[*citation needed*]	= 0.514 773 m/s
mach number	*M*	The ratio of the speed of an object moving through a fluid to the speed of sound in the same medium; typically used as a measure of aircraft speed.	Unitless. Actual speed of sound varies depending on atmospheric conditions. See "speed of sound" below for one specific condition.
metre per second (SI unit)	m/s	≡ 1 m/s	= 1 m/s
mile per hour	mph	≡ 1 mi/h	= 0.447 04 m/s
mile per minute	mpm	≡ 1 mi/min	= 26.8224 m/s
mile per second	mps	≡ 1 mi/s	= 1 609.344 m/s
speed of light in vacuum	*c*	≡ 299 792 458 m/s	= 299 792 458 m/s
speed of sound in air	*s*		≈ 344 m/s at 20 °C, 60% relative humidity [23]

99.11 FLOW (VOLUME):

Name of unit	Symbol	Definition	Relation to SI units
cubic foot per minute	CFM	≡ 1 ft^3/min	= 4.719474432×10^{-4} m^3/s
cubic foot per second	ft^3/s	≡ 1 ft^3/s	= 0.028316846592 m^3/s
cubic inch per minute	in^3/min	≡ 1 in^3/min	= 2.7311773×10^{-7} m^3/s
cubic inch per second	in^3/s	≡ 1 in^3/s	= 1.6387064×10^{-5} m^3/s
cubic metre per second (SI unit)	m^3/s	≡ 1 m^3/s	= 1 m^3/s
gallon (U.S. fluid) per day	GPD	≡ 1 gal/d	= $4.381263638 \times 10^{-8}$ m^3/s

gallon (U.S. fluid) per hour	GPH	≡ 1 gal/h	= 1.051503273 × 10^{-6} m^3/s
gallon (U.S. fluid) per minute	GPM	≡ 1 gal/min	= 6.30901964×10^{-5} m^3/s
litre per minute	LPM	≡ 1 L/min	= 1.6 × 10^{-5} m^3/s

99.12 ACCELERATION:

Name of unit	Symbol	Definition	Relation to SI units
foot per hour per second	fph/s	≡ 1 ft/(h·s)	≈ 8.466 667 × 10^{-5} m/s^2
foot per minute per second	fpm/s	≡ 1 ft/(min·s)	= 5.08 × 10^{-3} m/s^2
foot per second squared	fps^2	≡ 1 ft/s^2	= 3.048 × 10^{-1} m/s^2
gal; galileo	Gal	≡ 1 cm/s^2	= 10^{-2} m/s^2
inch per minute per second	ipm/s	≡ 1 in/(min·s)	≈ 4.233 333 × 10^{-4} m/s^2
inch per second squared	ips^2	≡ 1 in/s^2	= 2.54 × 10^{-2} m/s^2
knot per second	kn/s	≡ 1 kn/s	≈ 5.144 444 × 10^{-1} m/s^2
metre per second squared (SI unit)	m/s^2	≡ 1 m/s^2	= 1 m/s^2
mile per hour per second	mph/s	≡ 1 mi/(h·s)	= 4.4704 × 10^{-1} m/s^2
mile per minute per second	mpm/s	≡ 1 mi/(min·s)	= 26.8224 m/s^2
mile per second squared	mps^2	≡ 1 mi/s^2	= 1.609 344 × 10^3 m/s^2
standard gravity	*g*	≡ 9.806 65 m/s^2	= 9.806 65 m/s^2

99.13 FORCE:

Name of unit	Symbol	Definition	Relation to SI units
atomic unit of force		≡ $m_e \cdot \alpha^2 \cdot c^2 / a_0$	≈ 8.238 722 06 × 10^{-8} N [24]
dyne (cgs unit)	dyn	≡ g·cm/s^2	= 10^{-5} N
kilogram-force; kilopond; grave-force	kgf; kp; Gf	≡ *g* × 1 kg	= 9.806 65 N
kip; kip-force	kip; kipf; klbf	≡ *g* × 1 000 lb	= 4.448 221 615 2605 × 10^3 N
milligrave-force, gravet-force	mGf; gf	≡ *g* × 1 g	= 9.806 65 mN
newton (SI unit)	N	A force capable of giving a mass of one kg an acceleration of one meter per second, per second.[25]	= 1 N = 1 kg·m/s^2
ounce-force	ozf	≡ *g* × 1 oz	= 0.278 013 850 953 7812 N
pound	lb	≡ slug·ft/s^2	= 4.448 230 531 N
pound-force	lbf	≡ *g* × 1 lb	= 4.448 221 615 2605 N

poundal	pdl	≡ 1 lb·ft/s^2	= 0.138 254 954 376 N
sthene (mts unit)	sn	≡ 1 t·m/s^2	= 1 × 10^3 N
ton-force	tnf	≡ g × 1 sh tn	= 8.896 443 230 521 × 10^3 N

99.14 PRESSURE OR MECHANICAL STRESS:

Name of unit	Symbol	Definition	Relation to SI units
atmosphere (standard)	atm		≡ 101 325 Pa [26]
atmosphere (technical)	at	≡ 1 kgf/cm^2	= 9.806 65 × 10^4 Pa [26]
bar	bar		≡ 10^5 Pa
barye (cgs unit)		≡ 1 dyn/cm^2	= 0.1 Pa
centimetre of mercury	cmHg	≡ 13 595.1 kg/m^3 × 1 cm × g	≈ 1.333 22 × 10^3 Pa [26]
centimetre of water (4 °C)	cmH$_2$O	≈ 999.972 kg/m^3 × 1 cm × g	≈ 98.0638 Pa [26]
foot of mercury (conventional)	ftHg	≡ 13 595.1 kg/m^3 × 1 ft × g	≈ 40.636 66 × 10^3 Pa [26]
foot of water (39.2 °F)	ftH$_2$O	≈ 999.972 kg/m^3 × 1 ft × g	≈ 2.988 98 × 10^3 Pa [26]
inch of mercury (conventional)	inHg	≡ 13 595.1 kg/m^3 × 1 in × g	≈ 3.386 389 × 10^3 Pa [26]
inch of water (39.2 °F)	inH$_2$O	≈ 999.972 kg/m^3 × 1 in × g	≈ 249.082 Pa [26]
kilogram-force per square millimetre	kgf/mm^2	≡ 1 kgf/mm^2	= 9.806 65 × 10^6 Pa [26]
kip per square inch	ksi	≡ 1 kipf/sq in	≈ 6.894 757 × 10^6 Pa [26]
micron (micrometre) of mercury	μmHg	≡ 13 595.1 kg/m^3 × 1 μm × g ≈ 0.001 torr	≈ 0.133 3224 Pa [26]
millimetre of mercury	mmHg	≡ 13 595.1 kg/m^3 × 1 mm × g ≈ 1 torr	≈ 133.3224 Pa [26]
millimetre of water (3.98 °C)	mmH$_2$O	≈ 999.972 kg/m^3 × 1 mm × g = 0.999 972 kgf/m^2	= 9.806 38 Pa
pascal (SI unit)	Pa	≡ N/m^2 = kg/(m·s^2)	= 1 Pa [27]
pièze (mts unit)	pz	≡ 1 000 kg/m·s^2	= 1 × 10^3 Pa = 1 kPa
pound per square foot	psf	≡ 1 lbf/ft^2	≈ 47.880 25 Pa [26]
pound per square inch	psi	≡ 1 lbf/in^2	≈ 6.894 757 × 10^3 Pa [26]
poundal per square foot	pdl/sq ft	≡ 1 pdl/sq ft	≈ 1.488 164 Pa [26]

short ton per square foot		≡ 1 sh tn × g / 1 sq ft	≈ 95.760 518 × 10^3 Pa
torr	torr	≡ 101 325/760 Pa	≈ 133.3224 Pa [26]

99.15 TORQUE OR MOMENT OF FORCE:

Name of unit	Symbol	Definition	Relation to SI units
foot-pound force	ft lbf	≡ g × 1 lb × 1 ft	= 1.355 817 948 331 4004 N·m
foot-poundal	ft pdl	≡ 1 lb·ft^2/s^2	= 4.214 011 009 380 48 × 10^{-2} N·m
inch-pound force	in lbf	≡ g × 1 lb × 1 in	= 0.112 984 829 027 6167 N·m
metre kilogram	m kg	≡ N × m / g	≈ 0.101 971 621 N·m
Newton metre (SI unit)	N·m	≡ N × m = kg·m^2/s^2	= 1 N·m

99.16 ENERGY, WORK, OR AMOUNT OF HEAT:

Name of unit	Symbol	Definition	Relation to SI units
barrel of oil equivalent	bboe	≈ 5.8 × 10^6 $BTU_{59\ °F}$	≈ 6.12 × 10^9 J
British thermal unit (ISO)	BTU_{ISO}	≡ 1.0545 × 10^3 J	= 1.0545 × 10^3 J
British thermal unit (International Table)	BTU_{IT}		= 1.055 055 852 62 × 10^3 J
British thermal unit (mean)	BTU_{mean}		≈ 1.055 87 × 10^3 J
British thermal unit (thermochemical)	BTU_{th}		≈ 1.054 350 × 10^3 J
British thermal unit (39 °F)	$BTU_{39\ °F}$		≈ 1.059 67 × 10^3 J
British thermal unit (59 °F)	$BTU_{59\ °F}$	≡ 1.054 804 × 10^3 J	= 1.054 804 × 10^3 J
British thermal unit (60 °F)	$BTU_{60\ °F}$		≈ 1.054 68 × 10^3 J
British thermal unit (63 °F)	$BTU_{63\ °F}$		≈ 1.0546 × 10^3 J
calorie (International Table)	cal_{IT}	≡ 4.1868 J	= 4.1868 J
calorie (mean)	cal_{mean}		≈ 4.190 02 J
calorie (thermochemical)	cal_{th}	≡ 4.184 J	= 4.184 J
calorie (3.98 °C)	$cal_{3.98\ °C}$		≈ 4.2045 J
calorie (15 °C)	$cal_{15\ °C}$	≡ 4.1855 J	= 4.1855 J
calorie (20 °C)	$cal_{20\ °C}$		≈ 4.1819 J
Celsius heat unit (International Table)	CHU_{IT}	≡ 1 BTU_{IT} × 1 K/°R	= 1.899 100 534 716 × 10^3 J

cubic centimetre of atmosphere; standard cubic centimetre	cc atm; scc	≡ 1 atm × 1 cm^3	= 0.101 325 J
cubic foot of atmosphere; standard cubic foot	cu ft atm; scf	≡ 1 atm × 1 ft^3	= 2.869 204 480 9344 × 10^3 J
cubic foot of natural gas		≡ 1 000 BTU_{IT}	= 1.055 055 852 62 × 10^6 J
cubic yard of atmosphere; standard cubic yard	cu yd atm; scy	≡ 1 atm × 1 yd^3	= 77.468 520 985 2288 × 10^3 J
electronvolt	eV	≡ e × 1 V	≈ 1.602 177 33 × 10^{-19} ± 4.9 × 10^{-26} J
erg (cgs unit)	erg	≡ 1 g·cm^2/s^2	= 10^{-7} J
foot-pound force	ft lbf	≡ g × 1 lb × 1 ft	= 1.355 817 948 331 4004 J
foot-poundal	ft pdl	≡ 1 lb·ft^2/s^2	= 4.214 011 009 380 48 × 10^{-2} J
gallon-atmosphere (imperial)	imp gal atm	≡ 1 atm × 1 gal (imp)	= 460.632 569 25 J
gallon-atmosphere (US)	US gal atm	≡ 1 atm × 1 gal (US)	= 383.556 849 0138 J
hartree, atomic unit of energy	E_h	≡ $m_e \cdot \alpha^2 \cdot c^2$ (= 2 Ry)	≈ 4.359 744 × 10^{-18} J
horsepower-hour	hp·h	≡ 1 hp × 1 h	= 2.684 519 537 696 172 792 × 10^6 J
inch-pound force	in lbf	≡ g × 1 lb × 1 in	= 0.112 984 829 027 6167 J
joule (SI unit)	J	The work done when a force of one newton moves the point of its application a distance of one meter in the direction of the force.[25]	= 1 J = 1 m·N = 1 kg·m^2/s^2
kilocalorie; large calorie	kcal; Cal	≡ 1 000 cal_{IT}	= 4.1868 × 10^3 J
kilowatt-hour; Board of Trade Unit	kW·h; B.O.T.U.	≡ 1 kW × 1 h	= 3.6 × 10^6 J
litre-atmosphere	l atm; sl	≡ 1 atm × 1 L	= 101.325 J
quad		≡ 10^{15} BTU_{IT}	= 1.055 055 852 62 × 10^{18} J
rydberg	Ry	≡ $R_\infty \cdot h \cdot c$	≈ 2.179 872 × 10^{-18} J
therm (E.C.)		≡ 100 000 BTU_{IT}	= 105.505 585 262 × 10^6 J

therm (U.S.)		≡ 100 000 $BTU_{59\,°F}$	= 105.4804 × 10^6 J
thermie	th	≡ 1 $Mcal_{IT}$	= 4.1868 × 10^6 J
ton of coal equivalent	TCE	≡ 7 $Gcal_{th}$	= 29.3076 × 10^9 J
ton of oil equivalent	TOE	≡ 10 $Gcal_{th}$	= 41.868 × 10^9 J
ton of TNT	tTNT	≡ 1 $Gcal_{th}$	= 4.184 × 10^9 J

99.17 POWER OR HEAT FLOW RATE:

Name of unit	Symbol	Definition	Relation to SI units
atmosphere-cubic centimetre per minute	atm ccm	≡ 1 atm × 1 cm^3/min	= 1.688 75 × 10^{-3} W
atmosphere-cubic centimetre per second	atm ccs	≡ 1 atm × 1 cm^3/s	= 0.101 325 W
atmosphere-cubic foot per hour	atm cfh	≡ 1 atm × 1 cu ft/h	= 0.797 001 244 704 W
atmosphere-cubic foot per minute	atm·cfm	≡ 1 atm × 1 cu ft/min	= 47.820 074 682 24 W
atmosphere-cubic foot per second	atm cfs	≡ 1 atm × 1 cu ft/s	= 2.869 204 480 9344 × 10^3 W
BTU (International Table) per hour	BTU_{IT}/h	≡ 1 BTU_{IT}/h	≈ 0.293 071 W
BTU (International Table) per minute	BTU_{IT}/min	≡ 1 BTU_{IT}/min	≈ 17.584 264 W
BTU (International Table) per second	BTU_{IT}/s	≡ 1 BTU_{IT}/s	= 1.055 055 852 62 × 10^3 W
calorie (International Table) per second	cal_{IT}/s	≡ 1 cal_{IT}/s	= 4.1868 W
foot-pound-force per hour	ft lbf/h	≡ 1 ft lbf/h	≈ 3.766 161 × 10^{-4} W
foot-pound-force per minute	ft lbf/min	≡ 1 ft lbf/min	= 2.259 696 580 552 334 × 10^{-2} W
foot-pound-force per second	ft lbf/s	≡ 1 ft lbf/s	= 1.355 817 948 331 4004 W
horsepower (boiler)	bhp	≈ 34.5 lb/h × 970.3 BTU_{IT}/lb	≈ 9.810 657 × 10^3 W
horsepower (European electrical)	hp	≡ 75 kp·m/s	= 736 W
horsepower (Imperial electrical)	hp	≡ 746 W	= 746 W
horsepower (Imperial mechanical)	hp	≡ 550 ft lbf/s	= 745.699 871 582 270 22 W
horsepower (metric)	hp	≡ 75 m kgf/s	= 735.498 75 W
litre-atmosphere per minute	L·atm/min	≡ 1 atm × 1 L/min	= 1.688 75 W

litre-atmosphere per second	L·atm/s	≡ 1 atm × 1 L/s	= 101.325 W
lusec	lusec	≡ 1 L·µmHg/s[14]	≈ 1.333×10^{-4} W
poncelet	p	≡ 100 m kgf/s	= 980.665 W
square foot equivalent direct radiation	sq ft EDR	≡ 240 BTU_{IT}/h	≈ 70.337 057 W
ton of air conditioning		≡ 1 t ice melted / 24 h	≈ 3 504 W
ton of refrigeration (Imperial)		≡ 1 BTU_{IT} × 1 lng tn/lb ÷ 10 min/s	≈ $3.938\,875 \times 10^{3}$ W
ton of refrigeration (IT)		≡ 1 BTU_{IT} × 1 sh tn/lb ÷ 10 min/s	≈ $3.516\,853 \times 10^{3}$ W
watt (SI unit)	W	The power which in one second of time gives rise to one joule of energy.[25]	= 1 W = 1 J/s = 1 N·m/s = 1 kg·m^2/s^3

99.18 ACTION:

Name of unit	Symbol	Definition	Relation to SI units
atomic unit of action	au	≡ $\hbar = h/2\pi$	≈ $1.054\,571\,68 \times 10^{-34}$ J·s[28]

99.19 DYNAMIC VISCOSITY:

Name of unit	Symbol	Definition	Relation to SI units
pascal second (SI unit)	Pa·s	≡ N·s/m^2 , kg/(m·s)	= 1 Pa·s
poise (cgs unit)	P	≡ 10^{-1} Pa·s	= 0.1 Pa·s
pound per foot hour	lb/(ft·h)	≡ 1 lb/(ft·h)	≈ $4.133\,789 \times 10^{-4}$ Pa·s
pound per foot second	lb/(ft·s)	≡ 1 lb/(ft·s)	≈ 1.488164 Pa·s
pound-force second per square foot	lbf·s/ft^2	≡ 1 lbf·s/ft^2	≈ 47.88026 Pa·s
pound-force second per square inch	lbf·s/in^2	≡ 1 lbf·s/in^2	≈ 6,894.757 Pa·s

99.20 KINEMATIC VISCOSITY:

Name of unit	Symbol	Definition	Relation to SI units
square foot per second	ft^2/s	≡ 1 ft^2/s	= 0.09290304 m^2/s
square metre per second (SI unit)	m^2/s	≡ 1 m^2/s	= 1 m^2/s
stokes (cgs unit)	St	≡ 10^{-4} m^2/s	= 10^{-4} m^2/s

99.21 ELECTRIC CURRENT:

Name of unit	Symbol	Definition	Relation to SI units
ampere (SI base unit)	A	≡ The constant current needed to produce a force of 2 × 10^{-7} newton per metre between two straight parallel	= 1 A

		conductors of infinite length and negligible circular cross-section placed one metre apart in a vacuum.[8]	
electromagnetic unit; abampere (cgs unit)	abamp	≡ 10 A	= 10 A
esu per second; statampere (cgs unit)	esu/s	≡ (0.1 A·m/s) / c	≈ 3.335641×10^{-10} A

99.22 ELECTRIC CHARGE:

Name of unit	Symbol	Definition	Relation to SI units
abcoulomb; electromagnetic unit (cgs unit)	abC; emu	≡ 10 C	= 10 C
atomic unit of charge	au	≡ e	≈ 1.602 176 462 × 10^{-19} C
coulomb (SI unit)	C	≡ The amount of electricity carried in one second of time by one ampere of current.[25]	= 1 C = 1 A·s
faraday	F	≡ 1 mol × $N_A \cdot e$	≈ 96 485.3383 C
statcoulomb; franklin; electrostatic unit (cgs unit)	statC; Fr; esu	≡ (0.1 A·m) / c	≈ 3.335 641 × 10^{-10} C

99.23 ELECTRIC DIPOLE:

Name of unit	Symbol	Definition	Relation to SI units
atomic unit of electric dipole moment	ea_0		≈ 8.478 352 81 × 10^{-30} C·m

99.24 ELECTROMOTIVE FORCE, ELECTRIC POTENTIAL DIFFERENCE:

Name of unit	Symbol	Definition	Relation to SI units
abvolt (cgs unit)	abV	≡ 1 × 10^{-8} V	= 1 × 10^{-8} V
statvolt (cgs unit)	statV	≡ c· (1 μJ/A·m)	= 299.792 458 V
volt (SI unit)	V	The difference in electric potential across two points along a conducting wire carrying one ampere of constant current when the power dissipated between the points equals one watt.	= 1 V = 1 W/A = 1 kg·m^2/(A·s^3)

99.25 ELECTRICAL RESISTANCE:

Name of unit	Symbol	Definition	Relation to SI units
ohm (SI unit)	Ω	The resistance between two points in a conductor when one volt of electric potential difference, applied to these points, produces one ampere of current in the conductor.	= 1 Ω = 1 V/A = 1 $kg \cdot m^2/(A^2 \cdot s^3)$

99.26 CAPACITANCE:

Name of unit	Symbol	Definition	Relation to SI units
farad (SI unit)	F	The capacitance between two parallel plates that results in one volt of potential difference when charged by one coulomb of electricity.	= 1 F = 1 C/V = 1 $A^2 \cdot s^4/(kg \cdot m^2)$

99.27 MAGNETIC FLUX:

Name of unit	Symbol	Definition	Relation to SI units
maxwell (CGS unit)	Mx	$\equiv 10^{-8}$ Wb	$= 1 \times 10^{-8}$ Wb
weber (SI unit)	Wb	Magnetic flux which, linking a circuit of one turn, would produce in it an electromotive force of 1 volt if it were reduced to zero at a uniform rate in 1 second.	= 1 Wb = 1 V·s = 1 $kg \cdot m^2/(A \cdot s^2)$

99.28 MAGNETIC FLUX DENSITY:

Name of unit	Symbol	Definition	Relation to SI units
gauss (CGS unit)	G	$\equiv Mx/cm^2 = 10^{-4}$ T	$= 1 \times 10^{-4}$ T
tesla (SI unit)	T	$\equiv Wb/m^2$	= 1 T = 1 Wb/m^2 = 1 $kg/(A \cdot s^2)$

99.29 INDUCTANCE:

Name of unit	Symbol	Definition	Relation to SI units
henry (SI unit)	H	The inductance of a closed circuit that produces one volt of electromotive force when the current in the circuit varies at a uniform rate of one ampere per second.	= 1 H = 1 Wb/A = 1 $kg \cdot m^2/(A \cdot s)^2$

99.30 TEMPERATURE:

Name of	Symbol	Definition	Conversion to

unit			kelvin
degree Celsius	°C	°C = K − 273.15. A unit of °C is the same size as a unit of K; however, their numerical values differ as the zero point of Celsius is set at 273.15 K (the ice point).	[K] = [°C] + 273.15
degree Delisle	°De		[K] = 373.15 − [°De] × 2/3
degree Fahrenheit	°F	0 °F ≡ freezing pt. of H_2O+NaCl, 180°F between freezing and boiling pt of H_2O @ 1atm	[K] = ([°F] + 459.67) × 5/9
degree Newton	°N		[K] = [°N] × 100/33 + 273.15
degree Rankine	°R; °Ra	0 °R ≡ absolute zero	[K] = [°R] × 5/9
degree Réaumur	°Ré		[K] = [°Ré] × 5/4 + 273.15
degree Rømer	°Rø		[K] = ([°Rø] − 7.5) × 40/21 + 273.15
kelvin (SI base unit)	K	≡ 1/273.16 of the thermodynamic temperature of the triple point of water.	1 K

99.31 INFORMATION ENTROPY:

Name of unit	Symbol	Definition	Relation to SI units	Relation to bits
SI unit	J/K	≡ J/K	= 1 J/K	
nat; nip; nepit	nat	≡ k_B	= 1.380 650 5(23) × 10^{-23} J/K	
bit; shannon	bit; b; Sh	≡ ln(2) × k_B	= 9.569 940 (16) × 10^{-24} J/K	= 1 bit
ban; hartley	ban; Hart	≡ ln(10) × k_B	= 3.179 065 3(53) × 10^{-23} J/K	
nibble		≡ 4 bits	= 3.827 976 0(64) × 10^{-23} J/K	= 2^2 bit
byte	B	≡ 8 bits	= 7.655 952 (13) × 10^{-23} J/K	= 2^3 bit
kilobyte (decimal)	kB	≡ 1 000 B	= 7.655 952 (13) × 10^{-20} J/K	
kilobyte (kibibyte)	KB; KiB	≡ 1 024 B	= 7.839 695 (13) × 10^{-20} J/K	= 2^{10} bit

99.32 LUMINOUS INTENSITY:

Name of unit	Symbol	Definition	Relation to SI units
candela (SI base unit); candle	cd	The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 × 10^{12} hertz and that has a radiant	= 1 cd

		intensity in that direction of 1/683 watt per steradian.	
candlepower (new)	cp	≡ cd The use of *candlepower* as a unit is discouraged due to its ambiguity.	= 1 cd
candlepower (old, pre-1948)	cp	Varies and is poorly reproducible. Approximately 0.981 cd.	≈ 0.981 cd

99.33 LUMINANCE:

Name of unit	Symbol	Definition	Relation to SI units
candela per square foot	cd/ft^2	≡ cd/ft^2	≈ 10.763910417 cd/m^2
candela per square inch	cd/in^2	≡ cd/in^2	≈ 1,550.0031 cd/m^2
candela per square metre (SI unit);	cd/m^2	≡ cd/m^2	= 1 cd/m^2
footlambert	fL	≡ $(1/\pi)\ cd/ft^2$	≈ 3.4262590996 cd/m^2
lambert	L	≡ $(10^4/\pi)\ cd/m^2$	≈ 3,183.0988618 cd/m^2
stilb (CGS unit)	sb	≡ $10^4\ cd/m^2$	≈ $1 \times 10^4\ cd/m^2$

99.34 LUMINOUS FLUX:

Name of unit	Symbol	Definition	Relation to SI units
lumen (SI unit)	lm	≡ cd·sr	= 1 lm = 1 cd·sr

99.35 ILLUMINANCE:

Name of unit	Symbol	Definition	Relation to SI units
footcandle; lumen per square foot	fc	≡ lm/ft^2	= 10.763910417 lx
lumen per square inch	lm/in^2	≡ lm/in^2	≈ 1,550.0031 lx
lux (SI unit)	lx	≡ lm/m^2	= 1 lx = 1 lm/m^2
phot (CGS unit)	ph	≡ lm/cm^2	= 1×10^4 lx

99.36 RADIATION - SOURCE ACTIVITY:

Name of unit	Symbol	Definition	Relation to SI units
becquerel (SI unit)	Bq	≡ Number of disintegrations per second	= 1 Bq = 1/s
curie	Ci	≡ 3.7×10^{10} Bq	= 3.7×10^{10} Bq
rutherford (H)	rd	≡ 1 MBq	= 1×10^6 Bq

99.37 RADIATION – EXPOSURE:

Name of unit	Symbol	Definition	Relation to SI units
roentgen	R	1 R ≡ 2.58×10^{-4} C/kg	= 2.58×10^{-4} C/kg

99.38 RADIATION - ABSORBED DOSE:

Name of unit	Symbol	Definition	Relation to SI units
gray (SI unit)	Gy	≡ 1 J/kg = 1 m^2/s^2	= 1 Gy
rad	rad	≡ 0.01 Gy	= 0.01 Gy

99.39 RADIATION - EQUIVALENT DOSE:

Name of unit	Symbol	Definition	Relation to SI units
Röntgen equivalent man	rem	≡ 0.01 Sv	= 0.01 Sv
sievert (SI unit)	Sv	≡ 1 J/kg	= 1 Sv