Chapter 5: Complex Numbers and Quadratic Equations

Introduction to Complex Numbers

* The square roots of negative numbers (e.g., √-1, √-2) do not exist within the real number system. To address this, we introduce the concept of imaginary numbers.
* **Definition:** We define the imaginary unit as *i*, where $i^2 = -1$.
* **Complex Number Definition:** A complex number is a number of the form *a + ib*, where *a* and *b* are real numbers.
* Examples: 2 + 3*i*, -7 + √2 *i*, √5 *i*, 4 + 1*i*, 5 (which can be written as 5 + 0*i*), -7 (which can be written as -7 + 0*i*)
* **Real and Imaginary Parts:** For a complex number *z = a + ib*:
* Re(*z*) = *a* (the real part)
* Im(*z*) = *b* (the imaginary part)
* For example, if *z* = 2 + 5*i*, Re(*z*) = 2 and Im(*z*) = 5.

Algebra of Complex Numbers

Operations

1. **Addition:** (2 + 3*i*) + (-3 + 2*i*) = (2 - 3) + (*i*(3 + 2)) = -1 + 5*i*
2. **Subtraction:** (2 + 3*i*) - (-3 + 2*i*) = (2 + 3) + *i*(3 - 2) = 5 + *i*
3. **Multiplication:** (2 + 3*i*)(-3 + 2*i*) = 2(-3 + 2*i*) + 3*i*(-3 + 2*i*)
= $-6 + 4i - 9i + 6i^2 = -6 - 5i - 6 = -12 - 5i$ (since $i^2 = -1$)
4. **Division:** (2 + 3*i*) / (-3 + 2*i*)
* Multiply the numerator and the denominator by the conjugate of the denominator (-3 - 2*i*):
((2 + 3*i*)(-3 - 2*i*)) / ((-3 + 2*i*)(-3 - 2*i*))
= $(-6 - 4i - 9i - 6i^2) / ((-3)^2 - (2i)^2)$
= (-6 - 13*i* + 6) / (9 - (-1)*4)
= -13*i* / 13 = -*i*
5. **Equality:** *a + ib* = *c + id* if and only if *a = c* and *b = d*.
6. **Zero Complex Number:** *a + ib* = 0 if and only if *a* = 0 and *b* = 0.

Modulus, Conjugate, and Multiplicative Inverse

* **Modulus:** If *z = a + ib*, the modulus of *z* (denoted as |*z*|) is given by $|z| = \sqrt{a^2 + b^2}$.
* **Conjugate:** The conjugate of *z = a + ib* (denoted as *z■*) is *a - ib*.
* **Multiplicative Inverse:** The multiplicative inverse of *a + ib* is $(a - ib) / (a^2 + b^2)$. This can also be written as $a/(a^2+b^2) - i[b/(a^2+b^2)]$.

Polar Representation of a Complex Number

* **Form:** A complex number *a + ib* can be represented in polar form as *r*(cos θ + *i*sin θ), where:
* $r = |z| = \sqrt{a^2 + b^2}$ (the modulus)
* θ = arg *z* (the argument or amplitude of *z*)
* The principal argument is the value of θ in the range $-\pi < \theta \leq \pi$

Finding the Argument (θ) - Trick Method

1. **Find the Reference Angle:**
* If cos θ = 1 and sin θ = 0, then θ = 0.
* If cos θ = 0 and sin θ = 1, then θ = π/2.
* If sin θ = √3/2 and cos θ = 1/2, then θ = π/3.

* If sin θ = 1/2 and cos θ = √3/2, then θ = π/6.

2. **Determine the Quadrant and Adjust:**
* **Quadrant I (both sin θ and cos θ are positive):** θ = reference angle
* **Quadrant II (sin θ is positive, cos θ is negative):** θ = π - reference angle
* **Quadrant III (both sin θ and cos θ are negative):** θ = π + reference angle
* **Quadrant IV (sin θ is negative, cos θ is positive):** θ = 2π - reference angle or θ = -reference angle (since sin(-θ) = -sin(θ) and cos(-θ) = cos(θ))
* **If sin θ = 0 and cos θ = -1:** θ = π

Square Roots of Complex Numbers

* **Formula:** $(a+b)^2 = (a-b)^2 + 4ab$
* This can be used to solve problems where you need to find the square root of a complex number. This formula is derived from the identity $[x^2 + y^2]^2 = [x^2 - y^2]^2 + 4x^2y^2$.

Powers of *i*

* *i*^{1} = *i*
* *i*^{2} = -1
* *i*^{3} = -*i*
* *i*■ = 1
* *i*^(4*k*) = 1, *i*^(4*k*+1) = *i*, *i*^(4*k*+2) = -1, *i*^(4*k*+3) = -*i*, where *k* is an integer.

Solutions of Quadratic Equations

* **General Form:** *ax^2 + bx + c = 0*, where *a, b, c* are real coefficients and *a ≠ 0*.
* **Real Roots:** If *b^2 - 4ac ≥ 0*, the solutions are given by *x* = (-*b* ± √(*b^2 - 4ac*)) / (2*a*).
* **Complex Roots:** If *b^2 - 4ac < 0*, the solutions are given by *x* = (-*b* ± *i*√(|*b^2 - 4ac*|)) / (2*a*).

Properties of Modulus and Conjugate

Refer to text page 102 for the properties of modulus and conjugate. (Properties (i) to (v) are listed there).

Exercises

Example 5.1
* **Q3:** (Concept-based problem)
* **Q8:** (4 marks)
* **Q11, Q12, Q13, Q14:** (4 marks each)

Example 5.2
* **Q2:** Express *z* = -√3 + *i* in the polar form and write the modulus and argument.
* **Solution:**
1. Let -√3 + *i* = *r*(cos θ + *i* sin θ). Here *a* = -√3, *b* = 1.
2. *r* = √($(-\sqrt{3})^2 + 1^2$) = √(3+1) = √4 = 2.
3. -√3 + *i* = 2cos θ + *i* * 2sin θ.
4. 2cos θ = -√3, cos θ = -√3/2 and 2sin θ = 1, sin θ = 1/2.
5. Since cos θ is negative and sin θ is positive, θ is in the second quadrant.
6. The reference angle is π/6. Therefore, θ = π - π/6 = 5π/6.
7. Polar form of *z* = -√3 + *i* is 2(cos(5π/6) + *i*sin(5π/6)).

8. |z| = 2, arg(*z*) = 5π/6.
* **Q1 to Q8:**

* **Q1:** θ = 4π/3 or principal argument θ = 4π/3 - 2π = -2π/3.
* **Q5:** θ = 5π/4 or principal argument θ = 5π/4 - 2π = -3π/4.
* **Q7, Q8:** (Concept-based problems)

Example 5.3
* **Q1, Q8, Q9, Q10:** (1 mark each)

Miscellaneous Examples
* **Q12 to Q16:** (Concept-based problems)

Miscellaneous Exercise
* **Q4, Q5, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q20:**

Supplementary Material
* **Example 12:**

Example 5.4
* **Q1 to Q6:**

Extra/HOT Questions

1. **Find the square roots of the following complex numbers (4 marks each)**
* i. 6 + 8*i*
* Let √(6+8*i*) = x+i*y*. Then 6 + 8*i* = (x+i*y*)^2 = x^2-y^2+2*i*xy
* Equating real and imaginary parts, we have x^2-y^2=6 and 2xy=8 => xy=4 => y=4/x
* Substituting in first equation: x^2 - (16/x^2) = 6 => x^4 -6x^2 - 16=0
* Let a = x^2, a^2 -6a-16 =0 => (a-8)(a+2)=0. Then a=8 or a=-2, since a=x^2, x^2 =8 => x=±2√2.
* If x = 2√2, y = 4/2√2 = √2. If x = -2√2, y = -√2
* Thus, the square roots are 2√2 + √2 *i* and -2√2 - √2 *i*

* ii. 3 - 4*i*
* Let √(3-4*i*) = x+i*y*. Then 3 - 4*i* = (x+i*y*)^2 = x^2-y^2+2*i*xy
* Equating real and imaginary parts, we have x^2-y^2=3 and 2xy=-4 => xy=-2 => y=-2/x
* Substituting in first equation: x^2 - (4/x^2) = 3 => x^4 -3x^2 - 4=0
* Let a = x^2, a^2 -3a-4 =0 => (a-4)(a+1)=0. Then a=4 or a=-1, since a=x^2, x^2 =4 => x=±2.
* If x = 2, y = -2/2 = -1. If x = -2, y = 1
* Thus, the square roots are 2 - *i* and -2 + *i*

* iii. 2 + 3*i*
* Let √(2+3*i*) = x+i*y*. Then 2 + 3*i* = (x+i*y*)^2 = x^2-y^2+2*i*xy
* Equating real and imaginary parts, we have x^2-y^2=2 and 2xy=3 => xy=3/2 => y=3/(2x)
* Substituting in first equation: x^2 - (9/4x^2) = 2 => 4x^4 -8x^2 - 9=0
* Let a = x^2, 4a^2 -8a-9 =0 => a = (8 ± √64+144) / 8 = (8±√208)/8 = (8± 4√13)/8 = 1±√13/2
* x^2 = 1+(√13/2) or x^2 = 1-(√13/2). Since x is real, x^2 = 1+(√13/2)
* x = ±√ [(2+√13)/2], y= (3/2)/±√ [(2+√13)/2] = ±3/(2√[(2+√13)/2])
* Then y = ± 3/√(2(2+√13))=±3/√(4+2√13) = ±3√((4-2√13)/ (16-52)) = ±3√((4-2√13)/ -36) = ±√((2√13-4)/16)=±√(2√13-4)/4
* The square roots are ±(√[(2+√13)/2]+ *i* (√[(2√13-4)/4])
* iv. 7 - 30√2 *i*

* Let $\sqrt{(7 - 30\sqrt{2}\,i)} = x+iy$. Then $7 - 30\sqrt{2}\,i = (x+iy)^2 = x^2-y^2+2ixy$
* Equating real and imaginary parts, we have $x^2-y^2=7$ and $2xy=-30\sqrt{2} \Rightarrow xy=-15\sqrt{2} \Rightarrow y=-15\sqrt{2}/x$
* Substituting in first equation: $x^2 - (450/x^2) = 7 \Rightarrow x^4 - 7x^2 - 450=0$
* Let $a = x^2$, $a^2 - 7a-450 =0 \Rightarrow (a-25)(a+18)=0$. Then $a=25$ or $a=-18$, since $a=x^2$, $x^2 =25 \Rightarrow x=\pm5$.
* If $x = 5$, $y = -15\sqrt{2}/5 = -3\sqrt{2}$. If $x = -5$, $y = 3\sqrt{2}$
* Thus, the square roots are $5 - 3\sqrt{2}\,i$ and $-5 + 3\sqrt{2}\,i$

* v. $3 + 4i$
* Let $\sqrt{(3+4i)} = x+iy$. Then $3 + 4i = (x+iy)^2 = x^2-y^2+2ixy$
* Equating real and imaginary parts, we have $x^2-y^2=3$ and $2xy=4 \Rightarrow xy=2 \Rightarrow y=2/x$
* Substituting in first equation: $x^2 - (4/x^2) = 3 \Rightarrow x^4 - 3x^2 - 4=0$
* Let $a = x^2$, $a^2 - 3a-4 =0 \Rightarrow (a-4)(a+1)=0$. Then $a=4$ or $a=-1$, since $a=x^2$, $x^2 =4 \Rightarrow x=\pm2$.
* If $x = 2$, $y = 2/2 = 1$. If $x = -2$, $y = -1$
* Thus, the square roots are $2+i$ and $-2 -i$
* $3-4i$
* Let $\sqrt{((3-4i)/5)} = x+iy$.
* Then $3/5 - 4/5\, i = x^2-y^2+2ixy$
* Equating real and imaginary parts, $x^2-y^2=3/5$ and $2xy=-4/5 \Rightarrow xy=-2/5 \Rightarrow y=-2/(5x)$
* Substituting: $x^2 - (4/25x^2) = 3/5 \Rightarrow 25x^4-15x^2-4=0$
* Let $a=x^2$, $25a^2 -15a-4 =0$. Then $a=(15\pm\sqrt{225+400})/50=(15\pm25)/50$. Then $a = 4/5$ or $a=-1/5$. x^2 cannot be negative so $x^2 =4/5$. Therefore $x = \pm 2/\sqrt{5}$
* If $x =2/\sqrt{5}$, $y = -2/5\cdot(\sqrt{5}/2) =-1/\sqrt{5}$. If $x = -2/\sqrt{5}$, $y = 1/\sqrt{5}$
* Therefore $\sqrt{(3/5 - 4/5\, i)} = 2/\sqrt{5} -i/\sqrt{5}$ or $-2/\sqrt{5} + i/\sqrt{5}$.
* The roots are thus $3/5 +4/5\, i$ and $-3/5 -4/5 i$

2. **Convert the following complex numbers to polar form:**
* i. $3\sqrt{3} + 3i$
* $r= \sqrt{(27+9)} = \sqrt{36}=6$. $3\sqrt{3} + 3i=6(\cos\theta+i\sin\theta)$
* $6\cos\theta= 3\sqrt{3} \Rightarrow \cos\theta=\sqrt{3}/2$, $6\sin\theta=3 \Rightarrow \sin\theta =1/2$.
* $\theta = \pi/6$. Then $6(\cos \pi/6 + i\sin\pi/6)$
* ii. $(1 - i) / (1 + i) = (1-i)/(1+i) \cdot (1-i)/(1-i) =(1-2i-1)/(1-(-1)) = -2i/2 = -i$
* $-i = 0 - i$. $r = \sqrt{(0^2+1^2)}=1$
* Then $0-i = \cos\theta+i\sin\theta$. $\cos\theta =0$, $\sin\theta=-1$. $\theta = -\pi/2$ or $3\pi/2$.
* So $1(\cos (-\pi/2) +i \sin(-\pi/2))$

* iii. $1 + i$
* $r = \sqrt{2}$. $1+i = \sqrt{2}(\cos\theta+i\sin\theta)$. $\sqrt{2}\cos\theta=1$, $\cos\theta =1/\sqrt{2}$. $\sqrt{2}\sin\theta=1$, $\sin\theta =1/\sqrt{2}$. $\theta=\pi/4$
* $\sqrt{2}(\cos \pi/4 + i\sin\pi/4)$
* iv. $-1 + \sqrt{3}\, i$
* $r = \sqrt{(1+3)}= \sqrt{4}=2$. $-1 + \sqrt{3}\, i = 2(\cos\theta +i\sin\theta)$
* $2\cos\theta=-1$, $\cos\theta=-1/2$. $2\sin\theta =\sqrt{3}$, $\sin\theta = \sqrt{3}/2$. $\theta = 2\pi/3$.
* Then $2(\cos 2\pi/3+i\sin2\pi/3)$.
* v. $-3 + 3i$
* $r=\sqrt{((9+9))} = 3\sqrt{2}$. $-3+3i = 3\sqrt{2}(\cos\theta+i\sin\theta)$
* $3\sqrt{2}\cos\theta=-3$. $\cos\theta = -1/\sqrt{2}$. $3\sqrt{2}\sin\theta=3$. $\sin\theta = 1/\sqrt{2}$. $\theta =3\pi/4$
* $3\sqrt{2}(\cos 3\pi/4 +i\sin3\pi/4)$
* vi. $-2 - 2i$
* $r=\sqrt{(4+4)}=\sqrt{8}=2\sqrt{2}$. $-2 -2i = 2\sqrt{2}(\cos\theta+i\sin\theta)$
* $2\sqrt{2}\cos\theta=-2$, $\cos\theta=-1/\sqrt{2}$. $2\sqrt{2}\sin\theta=-2$, $\sin\theta =-1/\sqrt{2}$. $\theta= 5\pi/4$ or $-3\pi/4$
* $2\sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4)$

3. If $a + ib = (x + i) / (x - i)$, where x is a real number, prove that $a^2 + b^2 = 1$ and $b/a = 2x$ /

($x^2 - 1$).

* $a+ib = (x+i)/(x-i) = (x+i)^2/(x^2 + 1) = (x^2-1+2ix)/(x^2+1)$.
* Then $a=(x^2-1)/(x^2+1)$ and $b = 2x/(x^2+1)$.
* $a^2+b^2 = (x^2-1)^2/(x^2+1)^2 + 4x^2/(x^2+1)^2 = (x^4 -2x^2+1+4x^2)/(x^2+1)^2$
* $= (x^4 + 2x^2+1)/(x^2+1)^2= (x^2+1)^2/(x^2+1)^2=1$.
* $b/a = (2x/(x^2+1))/((x^2-1)/(x^2+1)) = 2x/(x^2-1)$.

4. **Find the real and imaginary part of** $(i + 1) / (i - 1)$
* $(i+1)/(i-1) = (i+1)/(i-1) * (i+1)/(i+1) = (i^2 +2i +1)/(i^2 -1)= (2i)/-2 = -i$
* Then real part =0, imaginary part =-1

5. **Compute:** $i + i^2 + i^3 + i$■
* $i + (-1) + (-i) + 1 = 0$.

6. **Solve the following quadratic equations:**
* i. $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$
* Using quadratic formula, $x = [(\sqrt{3} + 1) +- \sqrt{ ((\sqrt{3}+1)^2-4\sqrt{3})}]/2 = (\sqrt{3}+1+-\sqrt{((3+1+2\sqrt{3})-4\sqrt{3})})/2$
* $=(\sqrt{3}+1+-\sqrt{((4-2\sqrt{3}))})/2 = (\sqrt{3}+1+-\sqrt{((\sqrt{3}-1)^2)})/2 = (\sqrt{3}+1+- (\sqrt{3}-1))/2$
* Then $x=(\sqrt{3}+1+\sqrt{3}-1)/2 =\sqrt{3}$ or $x=(\sqrt{3}+1-\sqrt{3}+1)/2 =1$
* Therefore, $x = \sqrt{3}$ or $x = 1$

* ii. $2x^2 + 5 = 0$
* $2x^2=-5$
* $x^2 = -5/2$
* $x = \pm \sqrt{(-5/2)} = \pm i \sqrt{(5/2)}$

7. **Find the complex conjugate and multiplicative inverse of**

* i) $(2 - 5i)^2 = 4 - 20i +25i^2 = 4 - 20i -25= -21 - 20i$
* Conjugate = $-21+20i$
* Multiplicative inverse = $-21+20i/(21^2+20^2)= -21+20i/841 = -21/841 + i\ 20/841$
* ii) $(2 + 3i) / (3 – 7i)$
* Multiply by the conjugate of the denominator
* $= (2+3i)/(3-7i) * (3+7i)/(3+7i) = (6 +14i + 9i -21)/(9+49) = (-15 + 23i)/58 = -15/58 + i23/58$
* Conjugate = $-15/58 -23/58\, i$
* Multiplicative inverse = $3-7i/2+3i = (3-7i)(2-3i)/(4+9) = (6-9i-14i-21)/13 = (-15-23i)/13 = -15/13 -23/13i$

8. **If $|z| = 2$ and $\arg z = \pi/4$, then z = _______.**

* $z = r(\cos\theta + i \sin\theta)$
* $z = 2 (\cos \pi/4 + i \sin \pi/4) = 2(1/\sqrt{2} + i(1/\sqrt{2}))= \sqrt{2} + i\sqrt{2}$