Chapter 5: Complex Numbers and Quadratic Equations

Introduction to Complex Numbers

* **Definition:** A number of the form *a + ib*, where *a* and *b* are real numbers, is defined as a complex number. Here, *i* represents the imaginary unit, where *i^2 = -1*.

* **Examples:** 2 + 3*i*, -7 + $\sqrt{2}$ *i*, $\sqrt{5}$ *i*, 4 + 1*i*, 5 (which is 5 + 0*i*), -7 (which is -7 + 0*i*)

* **Real and Imaginary Parts:**
* For a complex number *z = a + ib*:
* Re(*z*) = *a* (real part)
* Im(*z*) = *b* (imaginary part)

* Example: If *z* = 2 + 5*i*, then Re(*z*) = 2 and Im(*z*) = 5.

Algebra of Complex Numbers

1. Addition of Complex Numbers
* **(a + ib) + (c + id) = (a+c) + i(b+d)**
* Example: (2 + 3*i*) + (-3 + 2*i*) = (2 + -3) + *i*(3 + 2) = -1 + 5*i*

2. Subtraction of Complex Numbers
* **(a + ib) - (c + id) = (a-c) + i(b-d)**
* Example: (2 + 3*i*) - (-3 + 2*i*) = (2 + 3) + *i*(3 - 2) = 5 + *i*

3. Multiplication of Complex Numbers
* **(a + ib) * (c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)**
* Example: (2 + 3*i*) * (-3 + 2*i*) = 2*(-3 + 2*i*) + 3*i*(-3 + 2*i*)
= -6 + 4*i* - 9*i* + 6*i^2
= -6 - 5*i* - 6 (since *i^2* = -1)
= -12 - 5*i*

4. Division of Complex Numbers
* To divide complex number we multiply numerator and denominator by conjugate of the denominator.
* Example: (2 + 3*i*) / (-3 + 2*i*)
* Step 1: Multiply the numerator and the denominator by the conjugate of the denominator (-3 - 2*i*).
(2+3*i*) / (-3+2*i*) * (-3-2*i*)/(-3-2*i*) = (2+3i)(-3-2i) / (-3+2i)(-3-2i)
* Step 2: Expand the numerator and denominator:
* Numerator: (2 * -3) + (2 * -2*i*) + (3*i* * -3) + (3*i* * -2*i*) = -6 - 4*i* - 9*i* - 6*i^2 = -6 - 13*i* + 6 = -13*i*
* Denominator: $(-3)^2$ - $(2*i*)^2$ = 9 - 4*i^2 = 9 + 4 = 13
* Step 3: Combine the results: -13*i* / 13 = -*i*

5. Equality of Two Complex Numbers
* *a + ib = c + id* if and only if *a = c* and *b = d*

6. Complex Number Equal to Zero
* *a + ib = 0* if and only if *a = 0* and *b = 0*

Modulus, Conjugate, and Multiplicative Inverse

a) Modulus of a Complex Number
* If *z = a + ib*, then the modulus of *z*, denoted by |*z*|, is given by:

* |*z*| = √(*$a^2 + b^2$*)

b) Conjugate of a Complex Number
* The conjugate of *z = a + ib* is denoted by *z■* and is given by:
* *z■* = *a - ib*

c) Multiplicative Inverse of a Complex Number
* The multiplicative inverse of *a + ib* is:
* (a - ib) / (*$a^2 + b^2$*)

Polar Representation of a Complex Number

* A complex number *a + ib* can be represented in polar form as:
* *a + ib = r(cos θ + i sin θ)*
* where *r* = |*z*| = √(*$a^2 + b^2$*) is the modulus, and *θ* is the argument (or amplitude) of *z*.

Argument (θ) of a Complex Number
* The argument has many values, but the principal argument is the unique value of *θ* such that -π < *θ* ≤ π.

Trick Method to Find θ

* **Step 1: Find the Reference Angle**
* **1.** If cos θ = 1 and sin θ = 0, then angle = 0
* **2.** If cos θ = 0 and sin θ = 1, then angle = π/2
* **3.** If sin θ = √3/2 and cos θ = 1/2, then angle = π/3
* **4.** If sin θ = 1/2 and cos θ = √3/2, then angle = π/6

* **Step 2: Determine the Quadrant and Calculate** θ
* **1.** If both sin θ and cos θ are positive, then θ = angle (first quadrant).
* **2.** If sin θ is positive and cos θ is negative, then θ = π - angle (second quadrant).
* **3.** If both sin θ and cos θ are negative, then θ = π + angle (third quadrant).
* **4.** If sin θ is negative and cos θ is positive, then θ = 2π - angle (fourth quadrant) or θ = -angle.
* **5.** If sin θ = 0 and cos θ = -1, then θ = π.
* Note: sin(-θ) = -sinθ and cos(-θ) = cosθ

Square Root of a Complex Number

* Formula: $(a + b)^2 = (a - b)^2 + 4ab$
* This is used in the form: $[x^2 + y^2]^2 = [x^2 - y^2]^2 + 4x^2y^2$

Powers of *i*

* *i* = √-1
* *i^2* = -1
* *i^3* = *i^2* *i* = -*i*
* *i■* = (*i^2*)^2 = $(-1)^2$ = 1
* In general, *i^{4k}* = 1, *i^{4k+1}* = *i*, *i^{4k+2}* = -1, *i^{4k+3}* = -*i* for any integer *k*.

Solutions of Quadratic Equations

* For a quadratic equation *$ax^2 + bx + c = 0$* where *a, b, c* are real coefficients:

* If *$b^2 - 4ac$* ≥ 0, the roots are given by *x* = $(-b \pm \sqrt{(b^2 - 4ac)}) / 2a$ (Real Roots)
* If *$b^2 - 4ac$* < 0, the roots are given by *x* = $(-b \pm \sqrt{(b^2 - 4ac)}) / 2a$ (Complex Roots)

Example Problems and Solutions

Ex 5.2 Q 2: Express *$Z = -\sqrt{3} + i$* in polar form and find modulus and argument.

* **Solution:**
* Given *$Z = -\sqrt{3} + i$*, here $a = -\sqrt{3}$, and $b = 1$
* **1. Find the modulus (r):**
* $r = |Z| = \sqrt{((-\sqrt{3})^2 + 1^2)} = \sqrt{(3+1)} = \sqrt{4} = 2$
* **2. Express in terms of r, cosθ and sinθ:**
* $-\sqrt{3} + i = 2(\cos\theta + i \sin\theta)$
* $2\cos\theta = -\sqrt{3}$ and $2\sin\theta = 1$
* $\cos\theta = -\sqrt{3}/2$ and $\sin\theta = 1/2$

* **3. Find the argument (θ) using the trick method:**
* Since $\cos \theta$ is negative and $\sin \theta$ is positive, θ lies in the second quadrant.
* The reference angle is $\pi/6$ (because $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$)
* Therefore, $\theta = \pi - \pi/6 = 5\pi/6$
* **4. Write the polar form:**
* $Z = 2(\cos(5\pi/6) + i \sin(5\pi/6))$
* **5. State the modulus and argument:**
* $|Z| = 2$ and the argument of Z is $5\pi/6$

Ex 5.2 Q 1-8 (Note on some solutions):
* Q1) *θ* = $4\pi/3$, principal argument *θ* = $4\pi/3 - 2\pi = -2\pi/3$
* Q5) *θ* = $5\pi/4$, principal argument *θ* = $5\pi/4 - 2\pi = -3\pi/4$

Miscellaneous and Supplementary Examples
* Note: Please refer to the textbook for detailed solutions to Ex 5.1, 5.2, 5.3, Misc examples, and exercises

Extra/HOT Questions

1. Find the square roots of the following complex numbers

i) 6 + 8*i*

* **Step 1:** Let the square root be *$x + iy$*. Then, $(x + iy)^2 = 6 + 8i$
* Expanding, we get $x^2 + 2ixy - y^2 = 6 + 8i$
* Equating the real and imaginary parts, we have:
* $x^2 - y^2 = 6$ (Equation 1)
* $2xy = 8 \Rightarrow xy = 4$ (Equation 2)
* **Step 2:** Use the formula $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$
* $(x^2 + y^2)^2 = 6^2 + 8^2 = 36 + 64 = 100$
* $x^2 + y^2 = 10$ (Equation 3)
* **Step 3:** Solve equations 1 and 3:
* Add Equations 1 and 3: $2x^2 = 16 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$
* Subtract Equation 1 from Equation 3: $2y^2 = 4 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$
* **Step 4:** Use Equation 2:
* If $x = 2\sqrt{2}$, then $y = 4/2\sqrt{2} = \sqrt{2}$

* If $x = -2\sqrt{2}$, then $y = 4/-2\sqrt{2} = -\sqrt{2}$
* **Answer:** Therefore, the square roots are $2\sqrt{2} + \sqrt{2}i$ and $-2\sqrt{2} - \sqrt{2}i$

ii) $3 - 4i$
* Following similar steps as above
* Let $(x + iy)^2 = 3 - 4i$
* $x^2 - y^2 = 3$ and $2xy = -4$
* $(x^2 + y^2)^2 = 3^2 + (-4)^2 = 25$
* $x^2 + y^2 = 5$
* $2x^2 = 8 \Rightarrow x = \pm 2$
* $2y^2 = 2 \Rightarrow y = \pm 1$
* Using $2xy = -4$
* If $x = 2$ then $y = -1$ and if $x = -2$ then $y = 1$
* **Answer:** $2 - i$ and $-2 + i$

iii) $2 + 3i$
* Let $(x + iy)^2 = 2 + 3i$
* $x^2 - y^2 = 2$ and $2xy = 3$
* $(x^2 + y^2)^2 = 2^2 + 3^2 = 13$
* $x^2 + y^2 = \sqrt{13}$
* $2x^2 = 2 + \sqrt{13} \Rightarrow x = \pm\sqrt{((2+\sqrt{13})/2)}$
* $2y^2 = -2+ \sqrt{13} \Rightarrow y = \pm\sqrt{((-2+\sqrt{13})/2)}$
* Since $2xy = 3$ which is positive x and y should have same sign
* **Answer:** $\pm(\sqrt{(2+\sqrt{13})/2} + i\sqrt{(-2+\sqrt{13})/2})$

iv) $7 - 30\sqrt{2}\, i$
* Let $(x + iy)^2 = 7 - 30\sqrt{2}\, i$
* $x^2 - y^2 = 7$ and $2xy = -30\sqrt{2}$
* $(x^2 + y^2)^2 = 7^2 + (-30\sqrt{2})^2 = 49 + 1800 = 1849$
* $x^2 + y^2 = \sqrt{1849} = 43$
* $2x^2 = 50 \Rightarrow x = \pm 5$
* $2y^2 = 36 \Rightarrow y = \pm 3\sqrt{2}$
* Using $2xy = -30\sqrt{2}$ we find if $x = 5$ then $y = -3\sqrt{2}$ and $x = -5$ then $y = 3\sqrt{2}$
* **Answer:** $5 - 3\sqrt{2}\, i$, $-5 + 3\sqrt{2}\, i$

v) $3+4i / 3-4i$
* $(3+4i) / (3-4i) = (3+4i) / (3-4i) * (3+4i) / (3+4i) = (9 + 24i - 16) / (9+16) = (-7+24i) / 25$
* Now let $(x+iy)^2 = (-7+24i)/25$
* $x^2 - y^2 = -7/25$ and $2xy = 24/25$
* $(x^2 + y^2)^2 = (-7/25)^2 + (24/25)^2 = 49/625 + 576/625 = 625/625 = 1$
* $x^2 + y^2 = 1$
* $2x^2 = 1-7/25 = 18/25 \Rightarrow x = \pm 3/5$
* $2y^2 = 1+7/25= 32/25 \Rightarrow y = \pm 4/5$
* Using $2xy = 24/25$ if $x = 3/5$ then $y = 4/5$ and if $x = -3/5$ then $y = -4/5$
* **Answer:** $3/5 + 4/5\, i$ and $-3/5 - 4/5\, i$

2. Convert the following complex numbers in the polar form

i) $3\sqrt{3} + 3i$
* $r = \sqrt{((3\sqrt{3})^2 + 3^2)} = \sqrt{(27 + 9)} = \sqrt{36} = 6$
* $\cos\theta = 3\sqrt{3} / 6 = \sqrt{3} / 2$, $\sin\theta = 3 / 6 = 1 / 2$
* $\theta = \pi/6$

* **Answer:** $6(\cos \pi/6 + i \sin \pi/6)$

ii) $(1 - i) / (1 + i)$
* $(1 - i) / (1 + i) * (1 - i) / (1 - i) = (1 - 2i - 1) / (1 + 1) = -2i / 2 = -i$
* $r = \sqrt{(-1)^2} = 1$
* $\cos \theta = 0$, $\sin \theta = -1$
* $\theta = -\pi/2$
* **Answer:** $\cos(-\pi/2) + i \sin(-\pi/2)$

iii) $1 + i$
* $r = \sqrt{1^2 + 1^2} = \sqrt{2}$
* $\cos \theta = 1 / \sqrt{2}$, $\sin \theta = 1 / \sqrt{2}$
* $\theta = \pi/4$
* **Answer:** $\sqrt{2} (\cos \pi/4 + i \sin \pi/4)$

iv) $-1 + \sqrt{3}\, i$
* $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$
* $\cos \theta = -1/2$, $\sin \theta = \sqrt{3} / 2$
* $\theta = 2\pi/3$
* **Answer:** $2(\cos 2\pi/3 + i \sin 2\pi/3)$

v) $-3 + 3i$
* $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$
* $\cos \theta = -3/3\sqrt{2} = -1/\sqrt{2}$, $\sin \theta = 3/3\sqrt{2} = 1/\sqrt{2}$
* $\theta = 3\pi/4$
* **Answer:** $3\sqrt{2} (\cos 3\pi/4 + i \sin 3\pi/4)$

vi) $-2 - i$
* $r = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$
* $\cos \theta = -2 / \sqrt{5}$, $\sin \theta = -1 / \sqrt{5}$
* $\theta = 5\pi/4$ or $-3\pi/4$
* **Answer:** $\sqrt{5}(\cos 5\pi/4 + i \sin 5\pi/4)$ or $\sqrt{5} (\cos(-3\pi/4) + i \sin(-3\pi/4))$

3. If $a + ib = (x + i)^2 / (2x^2 + 1)$, where x is real, prove that $a^2 + b^2 = 1$ and $b/a = 2x/(x^2 - 1)$
* $a + ib = (x^2 + 2ix - 1) / (2x^2 + 1) = (x^2 - 1 + 2ix) / (2x^2 + 1)$
* $a = (x^2 - 1) / (2x^2 + 1)$ and $b = 2x / (2x^2 + 1)$
* $a^2 + b^2 = ((x^2 - 1)^2 + (2x)^2) / (2x^2 + 1)^2 = (x^{\blacksquare} - 2x^2 + 1 + 4x^2) / (2x^2 + 1)^2$
* $= (x^{\blacksquare} + 2x^2 + 1) / (2x^2 + 1)^2 = (x^2 + 1)^2 / (2x^2 + 1)^2$
* Also, we need to show $a^2 + b^2 = 1$, however it seems the question may have an error.
* However, $b/a = (2x / (2x^2 + 1)) / ((x^2 - 1) / (2x^2 + 1)) = 2x / (x^2 - 1)$

4. Find the real and imaginary part of $i + i^2 + i^3 + i^{\blacksquare}$
* $i + i^2 + i^3 + i^{\blacksquare} = i - 1 - i + 1 = 0$
* Real Part = 0 and Imaginary Part = 0

5. Compute : $i + i^2 + i^3 + i^{\blacksquare}$
* $i + i^2 + i^3 + i^{\blacksquare} = i - 1 - i + 1 = 0$

6. Solve the following quadratic equations
i) $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$
* $x = ((\sqrt{3} + 1) \pm \sqrt{ ((\sqrt{3} + 1)^2 - 4 \sqrt{3}) }) / 2$
* $= (\sqrt{3} + 1 \pm \sqrt{ (3 + 2\sqrt{3} + 1 - 4\sqrt{3}) }) / 2$

* = $(\sqrt{3} + 1 \pm \sqrt{(4 - 2\sqrt{3})}) / 2$
* = $(\sqrt{3} + 1 \pm \sqrt{((\sqrt{3} - 1)^2)})/2$
* = $(\sqrt{3} + 1 \pm (\sqrt{3} - 1)) / 2$
* *x* = $(\sqrt{3} + 1 + \sqrt{3} - 1)/2$ or *x* = $(\sqrt{3} + 1 - \sqrt{3} + 1) / 2$
* *x* = $2\sqrt{3} / 2$ or $x = 2 /2$
* **Answer:** *x* = $\sqrt{3}$ or *x* = 1

ii) $2x^2 + 5 = 0$
* $2x^2 = -5$
* $x^2 = -5/2$
* $x = \pm\sqrt{(-5/2)} = \pm$ *i* $\sqrt{5}/\sqrt{2}$
* **Answer:** $x = \pm$*i* $\sqrt{10}/2$

7. Find the complex conjugate and multiplicative inverse of

i) (2 - 5*i*)²
* $(2-5i)^2 = 4 -20i -25 = -21 - 20i$
* Conjugate = $-21+20i$
* Multiplicative inverse = $(-21 + 20i) / (-21^2 + -20^2) = (-21+20i)/ (441 + 400) = (-21+20i) /841 = -21/841 + 20/841 i$

ii) (2 + 3*i*) / (3 - 7*i*)
* $(2+3i) / (3-7i) * (3+7i)/(3+7i) = (6 + 14i + 9i - 21) / (9 + 49) = (-15 + 23i) / 58$
* Conjugate = $(-15 - 23i) / 58$
* Multiplicative inverse = (3 - 7*i*) / (2 + 3*i*) = $(3-7i) / (2+3i) * (2-3i)/(2-3i) = (6 - 9i - 14i -21) / 13= (-15-23i) /13 = (-15-23i) / 58$
* Multiplicative inverse of $(-15+23i)/58 = (-15-23i) / 58$

8. If |Z| = 2 and arg Z = π/4 then Z= _______
* Z = r (cosθ + *i* sinθ)
* Z = 2 (cos π/4 + *i* sin π/4)
* Z = 2 ($1/\sqrt{2}$ + *i* $1/\sqrt{2}$)
* **Answer:** Z = $\sqrt{2} + \sqrt{2}$ *i*

Answers to Extra/HOT Questions

* **1.** i) $2\sqrt{2} + \sqrt{2}$*i*, $-2\sqrt{2} - \sqrt{2}$*i* ii) 2 – *i*, -2 + *i* iii) $\sqrt{((2+\sqrt{13})/2)} + i\sqrt{((-2+\sqrt{13})/2)}$, $-\sqrt{((2+\sqrt{13})/2)} - i\sqrt{((-2+\sqrt{13})/2)}$ iv) $5 - 3\sqrt{2}$ *i*, $-5 + 3\sqrt{2}$ *i* v) 3/5 + 4/5 *i*, -3/5 - 4/5 *i*
* **2.** i) 6(cos π/6 + *i* sin π/6) ii) cos (–π/2) + *i* sin(-π/2) iii) $\sqrt{2}$ (cos π/4 + *i* sin π/4) iv) 2(cos 2π/3 + *i* sin 2π/3) v) $3\sqrt{2}$ (cos 3π/4 + *i* sin 3π/4) vi) $\sqrt{5}$(cos 5π/4 + *i* sin 5π/4) or $\sqrt{5}$ (cos(-3π/4) + *i* sin(-3π/4))
* **3.** $a^2 + b^2 = (x^2+1)^2/(2x^2+1)^2$, $b/a = 2x/(x^2-1)$
* **4.** 0, 0
* **5.** 0
* **6.** i) $\sqrt{3}$, 1 ii) *i*$\sqrt{10}/2$, -*i*$\sqrt{10}/2$
* **7.** i) -21 + 20*i*, (-21 - 20*i*) / 841 ii) -15/58 - 23/58*i*, (-15-23i) /58
* **8.** $\sqrt{2} + \sqrt{2}$ *i*