Mid-Semester Examination

(MA6.102) Probability and Random Processes, Monsoon 2024

21 September, 2024

Max. Duration: 90 Minutes

Question 1 (2 Marks). For n events $A_1, A_2, \ldots, A_n$, show that

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

[*Hint: Analyze* $P\left(\bigcup_{i=1}^{n} A_i^c\right)$.]

Question 2 (3 Marks). If the discrete random variables X and Y are independent, then show that $Z = g(X)$ and $W = h(Y)$ are also independent, where $g, h : \mathbb{R} \to \mathbb{R}$.

Question 3 (5 Marks). Let $\Omega = \{1, 2, 3, \ldots\}$ be a sample space equipped with the σ-algebra of all subsets of Ω and a probability law P such that $P(\{\omega\}) = 2^{-\omega}$ for each $\omega = 1, 2, 3, \ldots$. Consider the random variables $X(\omega) = \omega$ and $Y(\omega) = (-1)^{\omega}$.

Find $\mathbb{E}[X \mid Y]$, i.e., express the random variable $\mathbb{E}[X \mid Y]$ as a function from Ω to $\mathbb{R}$.

Question 4 (5 Marks). Consider the closed [redacted] circle of radius r, i.e., $S = \{(x, y) : x^2 + y^2 \leq r^2\}$. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. Concretely, this means that the probability that the dart lands in any particular region A (that is entirely inside the circle of radius r) is equal to $\frac{\text{area}(A)}{\pi r^2}$.

Let X be the distance the dart lands from the center. Find the CDF F_X, PDF f_X, expected value $\mathbb{E}[X]$, and variance $\text{Var}(X)$.

Question 5 (5 Marks). A stick of length 1 is split at a point U that is uniformly distributed over $[0, 1]$. Determine the expected length of the substick that does not contain a given point $p \in [0, 1]$. Also, find the value of p that minimizes this expected length.

[*Hint: Express the quantity of interest as an expected value of a function of the random variable* U.]