Assignment 1

(MA6.102) Probability and Random Processes, Monsoon 2025

Release date: 7 August 2025, Due date: 16 August 2025

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Consider a sample space Ω and a σ-field $\mathcal{F}$ on Ω. An event $A \in \mathcal{F}$ is called an atom if

- $A \neq \emptyset$,
- For any $B \in \mathcal{F}$, if $B \subseteq A$, then $B = \emptyset$ or $B = A$.

For example, the atoms in the σ-field $\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$ on $\{1,2,3,4\}$ are $\{1\}$, $\{2\}$, and $\{3,4\}$.

Are any two distinct atoms in a σ-field necessarily disjoint?

Problem 2. Let $E_1, E_2, \ldots, E_n$ be mutually exclusive and exhaustive events in a sample space Ω. Determine the smallest σ-field that contains all the events E_i, for $i \in [1:n]$.

Problem 3. Let A and B be two events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$.

(a) Show that $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$.

(b) Construct a sample space Ω, events $A, B \subseteq \Omega$, and a probability law P such that the upper and lower bounds in part (a) are both achieved with equalities (in two separate constructions).

Problem 4. You are given three coins:

- One coin has heads on both faces.
- One coin has tails on both faces.
- One coin has heads on one face and tails on the other.

A coin is selected uniformly at random and tossed, resulting in heads. What is the probability that opposite face of the coin is tails, given this outcome?

Problem 5. Let $\mathcal{C} = \{A_1, A_2, \ldots, A_n\}$ be a collection of mutually independent events. Prove that, for any $i \in [1:n]$, $(\mathcal{C} \setminus A_i) \cup \{A_i^c\}$ is also a collection of mutually independent events.

Problem 6. A rare event E is under investigation in a court. The reliability of two independent witnesses called Alice and Bob is known to the court: each of Alice and Bob is truthful with probability $\frac{9}{10}$, and there is no collusion between the two of them. Let A and B be the events that Alice and Bob, respectively, assert that E occurred, and let $P(E) = \frac{1}{1000}$. What is the probability that E actually occurred given that both Alice and Bob declare that E occurred?

Problem 7. A man is saving up to buy a new car at a cost of Rs. 2000000. He starts with Rs. 200, and tries to win the remaining amount by the following gamble with his bank manager. He tosses a fair coin repeatedly; if it comes up heads then the manager pays him one rupee, but if it comes up tails then he pays the manager one rupee. He plays this game repeatedly until one of two events occurs: either he runs out of money and is bankrupted or he wins enough to buy the car. What is the probability that he is ultimately bankrupted?