Probability and Random Processes — Monsoon 2023

Assignment 3 Solutions

PRP TAs

October 12, 2023

Problem 1

Given, $P_K(k) = \begin{cases} \frac{1}{4} & \text{if } k = 1, 2, 3, 4, \\ 0 & \text{otherwise} \end{cases}$

$P_{N|K}(n|k) = \begin{cases} \frac{1}{k} & \text{if } n = 1, \ldots, k, \\ 0 & \text{otherwise} \end{cases}$

(a) Find the joint PMF of K and N.
Applying the chain rule, we have
$p_{N,K}(n,k) = p_K(k) p_{N|K}(n|K)$

substituting $p_K(k)$ and $p_{N|K}(n|k)$ we obtain

$p_{N,K}(n,k) = \begin{cases} \frac{1}{4k} & \text{if } k = 1, 2, 3, 4 \; and \; n = 1, 2, ..k \\ 0 & \text{otherwise} \end{cases}$

(b)

$$\begin{aligned} p_N(n) &= \sum_k p_{N,K}(n,k) \\ &= \sum_{k=n}^{4} \frac{1}{4k} \end{aligned} \tag{1}$$

$$p_N(n) = \begin{cases} \frac{25}{48} & \text{if } n = 1, \\ \frac{13}{48} & \text{if } n = 2, \\ \frac{7}{48} & \text{if } n = 3, \\ \frac{3}{48} & \text{if } n = 4 \end{cases} \tag{2}$$

(c) Conditional PMF of K given that N = 2 we have

$$p_{K|N}(K|2) = \frac{p_{N,K}(2,K}{p_N(2)} = \begin{cases} \frac{6}{13} & \text{if } k = 2, \\ \frac{4}{13} & \text{if } k = 3, \\ \frac{3}{13} & \text{if } k = 4, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(d) the conditional mean and variance of K, given that he bought at least two but no more than three books.
Let A be the event $2 \leq N \leq 3.We know that$

$$p_{k|A}(k) = \frac{Pr(K = k, A}{Pr(A)}$$
$$Pr(A) = p_N(2) + p_N(3) = \frac{5}{12}$$
$$Pr(K = k, A) = \begin{cases} \frac{1}{8} & \text{if } k = 2, \\ \frac{1}{6} & \text{if } k = 3, \\ \frac{1}{8} & \text{if } k - 4, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

And finally

$$p_{k|A}(k) = \begin{cases} \frac{3}{10} & \text{if } k = 2, \\ \frac{2}{5} & \text{if } k = 3, \\ \frac{3}{10} & \text{if } k = 4, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The conditional PMF of K given A is symmetric around k = 3.
The conditional variance of K given A by calculating you get as $3\frac{}{5.}$
(e)
Condition on the events N = 1, . . . , N = 4, and use the total expectation theorem.
Given, $E[C_i] = 30$, where $T = C_1 + C_2 + ... + C_N$.

$$\begin{aligned} \mathbb{E}[T] &= E[E[T|N]] \\ &= E[E[\sum_{i=1}^{N} C_i(N]] \\ &= E[N.30] = 30E[N] \\ &= 52.5 \end{aligned} \tag{6}$$

Problem 2

$$
\begin{aligned}
\mathbb{E}[N] &= \sum_{n=0}^{\infty} nP_N(n) \\
&= \sum_{n=1}^{\infty} nP(N = n) \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(N = n) && (7) \\
&= \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} P(N = n) && (8) \\
&= \sum_{i=1}^{\infty} P(N \geq i)
\end{aligned}
$$

To see the equivalence between eqn. (1) and (2), write down each term of the summation in the expression for expectation in a different row. Then, the summation can be visualised as a lower triangular matrix where each P(N = n) would take up a cell. Thus, n would denote the row no. and i denotes the col no.

Problem 3

Let $X_i, \forall i \in [n]$ be indicator random variables which take value 1 when $\pi(i) = i$ and 0 otherwise. Then $X = \sum\limits_{i=1}^{n} X_i$, Thus,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=1}^{n} \mathbb{E}[X_i] \text{ [By linearity of expectation]} \\
&= \sum_{i=1}^{n} n.1/n = 1 \quad [\because \mathbb{E}[X_i] = \frac{1}{n!}(n-1)! = 1/n] \\
Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \mathbb{E}[(\sum_{i=1}^{n} X_i)^2] - 1 \\
&= \mathbb{E}[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j] - 1 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}[X_i X_j] - 1 \text{ [By linearity of expectation]} \\
&= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \mathbb{E}[X_i X_j] - 1 \\
&= 1 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \mathbb{E}[X_i X_j] - 1 \quad [\because \mathbb{E}[X_i^2] = \mathbb{E}[X_i] = 1/n] \\
&= \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \mathbb{E}[X_i X_j] = \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{1}{n}\frac{1}{n-1} = 1 \quad [\because \mathbb{E}[X_i X_j] = \frac{1}{n!}(n-2)! = \frac{1}{n}\frac{1}{n-1}]
\end{aligned}$$

The last step follows from the fact that $X_i X_j, i \neq j$, is also an indicator R.V. which takes value 1 with probability $\frac{1}{n!}(n-2)! = \frac{1}{n}\frac{1}{n-1}$ as there are $(n-2)!$ ways to permute remaining $[n] \setminus \{i, j\}$ numbers

Problem 4

Let p be the probability of A winning a single match. Since each team is equally likely to win each match $\implies p = \frac{1}{2}$.

The following possible scenarios are possible for a match:

- One team wins all the 4 matches straight up.
- One team wins 4 matches and other teams wins 1 match. The team which won 4 matches must won the last match, otherwise if the losing team wins the last match. That implies that the winning team won 4 games straight which is same as the previous case, therefore the game should have ended there only as per the rules. A contradiction.

- One team wins 4 matches and other teams wins 2 matches. (Team winning the game must have won the last match)
- One team wins 4 matches and other teams wins 3 matches. (Team winning the game must have won the last matches)

We will calculate the probabilities wrt. Team A winning. The probabilities of Team B winning can be obtained by switching p and $1-p$.
$P(\text{A wins 4 matches straight}) = p^4$

$P(\text{A wins 4 matches and loses one}) = {}^4C_3\ p^3(1-p)\cdot p$

$P(\text{A wins 4 matches and loses 2}) = {}^5C_3\ p^3(1-p)^2\cdot p$

$P(\text{A wins 4 matches and loses 3}) = {}^6C_3\ p^3(1-p)^3\cdot p$

Similarly,

$P(\text{B wins 4 matches straight}) = (1-p)^4$

$P(\text{B wins 4 matches and loses one}) = {}^4C_3\ p(1-p)^3\cdot(1-p)$

$P(\text{B wins 4 matches and loses 2}) = {}^5C_3\ p^2(1-p)^3\cdot(1-p)$

$P(\text{B wins 4 games and loses 3}) = {}^6C_3\ p^3(1-p)^3\cdot(1-p)$

Therefore, $P(\text{Game ending in 4 matches}) = p^4 + (1-p)^4$

$P(\text{Game ending in 5 matches}) = {}^4C_3\ p^4(1-p) + {}^4C_3\ p(1-p)^4$

$P(\text{Game ending in 6 matches}) = {}^5C_3\ p^4(1-p)^2 + {}^5C_3\ p^2(1-p)^4$

$P(\text{Game ending in 7 matches}) = {}^6C_3\ p^4(1-p)^3 + {}^6C_3\ p^3(1-p)^4$

For the sake of calculation we will assume $p = \frac{1}{2}$ from here onwards. You can solve for a general p as well. Let X be a RV which represents the no. of matches played. Clearly $X \in \{4, 5, 6, 7\}$.

$P(X=4) = p^4 + (1-p)^4 = 2\cdot\left(\frac{1}{2}\right)^4 = \frac{1}{8}$

$P(X=5) = {}^4C_3\ p^4(1-p) + {}^4C_3\ p(1-p)^4 = 2 * {}^4C_3\cdot\left(\frac{1}{2}\right)^5 = \frac{1}{4}$

$P(X=6) = {}^5C_3\ p^4(1-p)^2 + {}^5C_3\ p^2(1-p)^4 = 2\cdot{}^5C_3\left(\frac{1}{2}\right)^6 = \frac{5}{16}$

$P(X=7) = {}^6C_3\ p^4(1-p)^3 + {}^6C_3\ p^3(1-p)^4 = 2\cdot{}^6C_3\left(\frac{1}{2}\right)^7 = \frac{5}{16}$

Summing these values up, we get 1. Thus $P(X)$ is a valid probability distribution. Let's now calculate $E[X]$ and $E[X^2]$.

$$
\begin{aligned}
E[X] &= \sum_{n=4}^{n=7} n \cdot P(X = n) \\
&= 4 \cdot \frac{1}{8} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{5}{16} + 7 \cdot \frac{5}{16} \\
&= \frac{8}{16} + \cdot \frac{20}{16} + \frac{30}{16} + \frac{35}{16} \\
&= \frac{93}{16} \\
&= 5.8125
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \sum_{n=4}^{n=7} n^2 \cdot P(X = n) \\
&= 16 \cdot \frac{1}{8} + 25 \cdot \frac{1}{4} + 36 \cdot \frac{5}{16} + 49 \cdot \frac{5}{16} \\
&= \frac{32}{16} + \cdot \frac{100}{16} + \frac{180}{16} + \frac{245}{16} \\
&= \frac{557}{16} \\
&= 34.8125
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= 34.8125 - (5.8125)^2 \\
&= 1.027
\end{aligned}
$$

Problem 5

We know that the condition for a given discrete function to be a valid pmf is that

1. It is non-negative.
2. Sum of its value over the domain(here real numbers) should be 1, i.e., $\sum_{x \in \mathbb{R}} p_X(x) = 1$

The function given here is $p_X(x) = 2^{-(|x|+1)}$ for non-zero integers. We can see that since it is in the form of 2^a, it will always be positive. So, this means the given function satisfies the 1st condition.

To check the 2nd condition which is the total probability,

$$\begin{aligned}
\sum_{x\in\mathbb{Z}-\{0\}} 2^{-(|x|+1)} &= \sum_{x\in\mathbb{Z}^-} 2^{-(|x|+1)} + \sum_{x\in\mathbb{Z}^+} 2^{-(|x|+1)} \\
&= \sum_{x\in\mathbb{Z}^-} 2^{-(-x+1)} + \sum_{x\in\mathbb{Z}^+} 2^{-(x+1)} \\
&= \sum_{x\in\mathbb{Z}^+} 2^{-(x+1)} + \sum_{x\in\mathbb{Z}^+} 2^{-(x+1)} \\
&= \sum_{x\in\mathbb{Z}^+} 2^{-(x)} \\
&= 1 \qquad \text{(Using the Sum of GP)}
\end{aligned}$$

Hence, it satisfies both the conditions and is thus a valid PMF.
Now, to calculate $E[X] = \sum_{x\in\mathbb{Z}-\{0\}}(x)p_X(x)$

$$\begin{aligned}
\sum_{x\in\mathbb{Z}-\{0\}} (x)2^{-(|x|+1)} &= \sum_{x\in\mathbb{Z}^-} (x)2^{-(|x|+1)} + \sum_{x\in\mathbb{Z}^+} (x)2^{-(|x|+1)} \\
&= \sum_{x\in\mathbb{Z}^-} (x)2^{-(-x+1)} + \sum_{x\in\mathbb{Z}^+} (x)2^{-(x+1)} \\
&= \sum_{x\in\mathbb{Z}^+} (-x)2^{-(x+1)} + \sum_{x\in\mathbb{Z}^+} (x)2^{-(x+1)} \\
&= 0
\end{aligned}$$

Therefore, the expectation of the random variable whose PMF is the given function is 0.

Problem 6

We prove that the geometric random variable is memoryless by proving the following mathematical statement.

$$P(X > m + l | X > m) = P(X > l)$$

To evaluate the above equation, we use the fact that the cdf of the geometric r.v. is given by,

$$P(X < k) = 1 - (1 - p)^k$$

and therefore,

$$P(X > k) = (1 - p)^k$$

Now consider the LHS of the memorylessness equation,

$$
\begin{aligned}
P(X > m+l | X > m) &= \frac{P(X > m+l \cap X > m)}{P(X > m)} \\
&= \frac{P(P > m+l)}{P(X > m)} \\
&= \frac{(1-p)^{m+l}}{(1-p)^m} \qquad = (1-p)^l = P(X > l)
\end{aligned}
$$

Therefore, the geometric r.v. is memoryless.
Q.E.D.

Problem 7

9:30 AM to 10:30 AM is one hour, so in this poisson random function $\lambda = 20$, and we know that

$$P(X = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

Thus, P$(15 \leq X \leq 20) = P(X = 15) + P(X = 16) + P(X = 17) + P(X = 18) + P(X = 19) + P(X = 20)$

By solving this you get approximately 0.45423.

Problem 8

a) Assume $E[X|X = x_1] = f(x_1)$ thus, naturally $E[X|X] = f(X)$. Then,

$$
\begin{aligned}
f(x_1) =& E[X|X = x_1] \\
=& \sum_x x.p_{X|X}(x|x_1) \\
=& x_1 \\
\therefore f(X) =& X
\end{aligned}
$$

b) Assume $E[Xg(Y)|Y = y_1] = f(y_1)$ thus, naturally $E[Xg(Y)|Y] = f(Y)$. Then,

$$
\begin{aligned}
f(Y = y_1) =& E[X.g(Y)|Y = y_1] \\
=& \sum_x \sum_y x.g(y).p_{X,Y|Y}(x, y|y_1) \\
=& g(y_1). \sum_x x.p_{X|Y}(x|y_1) \\
=& g(y_1).E[X|Y = y_1] \\
\therefore f(Y) =& g(Y).E[X|Y]
\end{aligned}
$$

Now, assume X to be a single valued rv independent of Y such that $P_X(x \neq 1) = 0$. Then, $E[X.g(Y)|Y] = E[g(Y)|Y]$ and $g(Y).E[X|Y] = g(Y)$. Therefore, $E[g(Y)|Y] = g(Y)$.

c) Assume $E[E[X|Y,Z]|Y = y_1] = f(y_1)$ thus, naturally $E[E[X|Y,Z]|Y] = f(Y)$. Then,

$$
\begin{aligned}
f(Y = y_1) &= \sum_z \sum_y E[X|Y,Z].p_{Y,Z|Y}(y, z|y_1) \\
&= \sum_z E[X|Y = y_1, Z = z].p_{Z|Y}(z|y_1)
\end{aligned}
$$

Now, $E[X|Y = y_1, Z = z] = \sum_x x.p_{X|Y,Z}(x|y_1, z)$. Using this

$$
\begin{aligned}
f(Y = y_1) &= \sum_x \sum_z x.p_{X|Y,Z}(x|y_1, z).p_{Z|Y}(z|y_1) \\
&= \sum_x \sum_z x.p_{X,Z|Y}(x, z|y_1)\, dz \\
&= \sum_x x.p_{X|Y}(x|y_1) \\
&= E[X|Y = y_1] \\
\therefore f(Y) &= E[X|Y]
\end{aligned}
$$