Lecture 1

(1 August 2025)

Probability and Random Processes

Module 1 – Basics of Probability

Module 2 – Discrete Random Variables

Module 3 – Continuous Random Variables

Module 4 – Tail Bounds and Limit Theorems

Module 5 – Random Processes

Textbooks:

- 'Probability, Random Variables, and Stochastic Processes', Papoulis and Pillai,
- 'Introduction to Probability', Bertsekas and Tsitsiklis.
- 'Probability and Random Processes', Grimmett and Stirzaker.

Grading Plan (Tentative)

Assignments	—	15 %
Quiz 1	—	15 %
Quiz 2	—	15 %
In-class Quizzes	—	5 %
Mid-sem	—	20 %
End-sem	—	30 %

Module 1

- Different Approaches to Probability
- Probability Space
- Conditional Probability, Independence
- Total Probability Theorem, Bayes' Theorem
- Continuity of Probability
- Review of Counting

In day-to-day life, the outcomes in many situations are not deterministic. A random experiment is an experiment whose outcomes we cannot predict with certainty. We regularly use the word or the concept of 'probability' when there is certain non-deterministic or random nature in an outcome. For example,

- Weather forecast: 40% chance of rain

- Overbooking of Flights : The airlines allow overbooking of seats based on the probabilities of cancellations.

- Sports : probability of winning a cricket match

- Insurance Rates : Depends on probabilities of certain incidents

Probability theory is a mathematical framework that allows us to describe and analyze random experiments.

$\boxed{\text{Probability (roughly) means possibility}}$

So when things are random, what is the need or meaning of having a theory for this? Is this not a contradiction?

NO! This is not a contradiction. How is it that from initial ignorance we can later deduce knowledge? How can it be that although we state that we know nothing about a single toss of a coin yet we make definitive statements about the result of many tosses, results that are often closely realized in practice?

Hence there is a need for probability theory. It helps us to predict how likely or unlikely and event will occur.

Different Approaches to Probability

A. Classical Approach

Probability of an event E

$$P(E) = \frac{\text{No. of outcomes favourable to event } E}{\text{Total no. of possible outcomes}}$$

- Equitable distribution of ignorance

Example, We roll a pair of (unbiased) dice. What is the probability that $(\underline{4}\ 6)$ appears?

$E = \{\text{outcome is } (\underline{4}\ 6)\}$

$$P(E) = \frac{1}{36}.$$

This approach suffers from at least two problems.

(1) It cannot deal with outcomes that are not 'equally likely'.

In the example above what is the probability that sum of numbers is 9 ?

Total no. of possible sum values is 11 (2 3 ... 12).

$E = \{ \text{sum is } 9 \}$.

$$P(E) = \frac{1}{11} \; ?$$

This is not correct as the implicit assumption that all the sum values are equally likely is wrong.

(2) It cannot handle scenarios when the total no. of possible outcomes is infinite.

B. Relative Frequency Approach

Perform the experiment n times. Let n_E be the no. of times E occurs.

$$P(E) = \lim_{n \to \infty} \frac{n_E}{n}.$$

The issues with this approach are

(1) We cannot perform the experiment infinite no. of times.

(2) The ratio $\frac{n_E}{n}$ may not converge as $n \to \infty$.

[Despite the problems with the relative frequency approach of probability, the concept of relative frequency is essential in applying probability theory to the real world as we will see in the latter half of the course.]

C. Axiomatic Approach

This approach is based on conceptual/thought experiment.

Probability Space – $(\Omega, \mathcal{F}, P)$

Ω – Sample space

$\mathcal{F}$ – Event space or sigma algebra

P – Probability law

We shall review set theory before we proceed.

Set Theory

- A set is a well-defined collection of objects, which are called the elements of the set.

- A set with no elements is called the empty set, denoted by ϕ or $\{\}$.

- A set with a finite (resp. infinite) no. of elements is a finite (resp. infinite) set.

- A set S is a countably infinite set if there exists a bijective mapping between natural numbers N and S, i.e., if $S = \{x_1, x_2, \ldots\}$.

  E.g. Set of all integers $= \{\ldots, -1, 0, 1, \ldots\}$.

- A countable set is finite or countably infinite.
- A set S is uncountable if $\exists$ an injection $f: N \to S$ but no bijection exists.

<u>Exercise</u>. Prove that $\mathbb{Q} \cap [0,1]$ is a countably infinite set.

Exercise. Prove that $\{0,1\}^\infty$ is an uncountably infinite set.

[Use Cantor's diagonalization argument]

Subset notation:

$$A \subseteq B \Rightarrow (x \in A \Rightarrow x \in B)$$

Set difference:

$$A \setminus B = \{x \in A \text{ s.t. } x \notin B\}.$$

Universal set Ω contains all objects that could be of interest in a particular context.

Set Operations:

- Complement of a set S

$$S^c = \{x \in \Omega : x \notin S\}.$$

- Union of two sets A and B

$$A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}$$

- Intersection of two sets A and B

$$A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}$$

- Infinite union

If for every $n \in \mathbb{N}$, we are given S_n,

$$\bigcup_{n=1}^{\infty} S_n = \{x \in \Omega : x \in S_n \text{ for some } n \in \mathbb{N}\}.$$

- Infinite intersection

$$\bigcap_{n=1}^{\infty} S_n = \{x \in \Omega : x \in S_n \text{ for all } n \in \mathbb{N}\}$$

Examples.

(i) $S_n = (0, x - \frac{1}{n}]$, $x > 1$

$$\bigcup_{n=1}^{\infty} S_n = (0, x).$$

(ii) $S_n = (0, x + \frac{1}{n})$

$$\bigcap_{n=1}^{\infty} S_n = (0, x].$$

Properties.

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Exercise.

(i) For $n \in \mathbb{N}$ show that

$$\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c .$$

[Use mathematical induction]

(ii) Given sets $S_1 S_2 ----$ show that

$$\left(\bigcup_{i=1}^{\infty} S_i \right)^c = \bigcap_{i=1}^{\infty} S_i^c .$$

Note that (i) $\not\Rightarrow$ (ii). That is if we prove a statement T_n for all $n \in \mathbb{N}$ then this may not imply that T_∞ is true.

To see this consider the following.

Let $A_i = \{ i, i+1, i+2, \ldots \}$, $n \in \mathbb{N}$.

It is clear that $\bigcap_{i=1}^{n} A_i = A_n$ is non-empty.

T_n : $\bigcap_{i=1}^{n} A_i$ is non-empty.

Is T_∞ true?. That is is $\bigcap_{i=1}^{\infty} A_i$ non-empty?

We show that $\bigcap_{i=1}^{\infty} A_i$ is empty set.

We prove this via contradiction.

Suppose $\bigcap_{i=1}^{\infty} A_i$ is non-empty.

$\exists\, m \in \bigcap_{i=1}^{\infty} A_i \implies m \in A_i$, for all $i \in \mathbb{N}$

$$\implies m \in A_{m+1}.$$

However, $A_{m+1} = \{ m+1, m+2, \ldots \}$, i.e., $m \notin A_{m+1}$.

This is a contradiction.

So part (ii) of above exercise needs a needs a separate proof.

Hint. Use the definitions of infinite unions & intersections.