Probability and Random Processes — Monsoon 2023

Assignment 4 Solutions

PRP TAs

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Problem 1

Consider a random variable X with the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda \mathsf{e}^{-\lambda x}, & \text{if } x \geq 0, \\ (1-p)\lambda \mathsf{e}^{\lambda x}, & \text{if } x < 0, \end{cases}$$

where λ and p are scalars with $\lambda > 0$ and $p \in [0,1]$.
To find the expectation value we use the formula:

$$E[X] = \int_0^\infty x f_X(x)\, dx$$

t For given PDF we split it into two halves:

$$\begin{aligned} E[X] &= \int_0^\infty xp\lambda.e^{-\lambda x}.dx + \int_{-\infty}^0 x(1-p)\lambda.e^{\lambda x}.dx \\ &= p\int_0^\infty x\lambda.e^{-\lambda x}.dx + (1-p)\int_{-\infty}^0 x\lambda.e^{\lambda x}.dx \\ &= p(x\int_0^\infty \lambda.e^{-\lambda x}.dx - \int_0^\infty 1.\int \lambda e^{-\lambda x}.dx) + (1-p)(x.\int_{-\infty}^0 \lambda.e^{\lambda x}.dx - \int_{-\infty}^0 \int 1.\lambda e^{\lambda x}.dx) \end{aligned}$$

On simplification you get mean $E[X] = \dfrac{(2p-1)}{\lambda}$

To find the variance we need $E[X^2] = \displaystyle\int_{-\infty}^\infty x^2 f_X(x).dx$

$\implies p(\displaystyle\int_0^\infty x^2\lambda e^{-\lambda x}.dx) + (1-p)\int_{-\infty}^\infty x^2\lambda.e^{\lambda x}.dx$

By using integration by parts and after simplification you get $E[X^2] = \dfrac{2}{\lambda^2}$

Thus Variance of x if given by $Var(X) = E[X^2] - E[X]^2$

$\implies Var(X) = \dfrac{2}{\lambda^2} - (\dfrac{(2p-1)}{\lambda})^2$

$Var(X) = \dfrac{1+4p-4p^2}{\lambda^2}$

Problem 2

We are given a gaussian random variable with pdf,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(a)

We need to prove that the gaussian pdf given is valid, i.e.,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

Proof. Let $I := \int_{-\infty}^{\infty} f_X(x)\, dx$. Then, taking the transformation of variables $z = \frac{x-\mu}{\sigma}, \quad dz = \frac{dx}{\sigma}$.

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{z^2}{2}}\, dz \end{aligned}$$

Then, if we square both sides of the above equation, we get,

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\Big(\int_{-\infty}^{\infty} e^{\frac{z^2}{2}}\Big)^2 \\ &= \frac{1}{2\pi}\Big[\int_{-\infty}^{\infty} e^{\frac{z^2}{2}}\Big]\Big[\int_{-\infty}^{\infty} e^{\frac{w^2}{2}}\Big] && \text{(Change in variables)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z^2+w^2}{2}}\, dz\, dw \end{aligned}$$

Now, we transform the problem into polar coordinates by applying the transformation, $x = rcos\theta$, $y = rsin\theta$, $dxdy = rdrd\theta$.

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} re^{-\frac{r^2}{2}}\, dr\, d\theta \\ &= \frac{1}{2\pi}(-2\pi)e^{-\frac{r^2}{2}}\Big|_0^{\infty} \\ &= 1 \end{aligned}$$

Therefore, $I^2 = 1 \Rightarrow I = 1 \quad (\because I > 0)$.
Q.E.D.

$\square$

(b)

Now we prove that the mean of the gaussian r.v. is μ.

Proof. We use the definition of expectation as follows,

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx\end{aligned}$$

Applying the same transformation of equations as in (a),

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{\sqrt{2\pi}}\left(\sigma \cancelto{0}{\int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}}}\,dz + \mu\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz\right) \\ &= \mu\end{aligned}$$

Hence, $\mathbb{E}[x] = \mu$.
Q.E.D.

$\square$

(c)

Finally, we need to prove that $\mathrm{var}(X) = \sigma^2$.

Proof.

$$\begin{aligned}var(X) &= \mathbb{E}[(X-\mu)^2] \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx\end{aligned}$$

Applying the same transformation as in part (b), we get,

$$\begin{aligned}var(X) &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\,dz \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\left[\cancelto{0}{-ze^{-\frac{z^2}{2}}\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz\right] \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz \\ &= \sigma^2\end{aligned}$$

Therefore, $\mathrm{var}(X) = \sigma^2$.
Q.E.D.

$\square$

Problem 3

Let x be the point where the stick gets split. We know that $X \sim$U[0,1]. Thus $f_X(x) = 1$
If $x < p$, then point p is contained in the right substick which has length $1 - x$. Whereas if $x >= p$, the point p is contained in the left substick having length x.

Let $l(x)$ be a function denoting the length of the susbtick given the splitting point x.

$$l(x) = \begin{cases} 1 - x & \text{if } x < p \\ x & \text{if } x \geq p \end{cases}$$

$$\begin{aligned}
E_X[l(x)] &= \int_0^1 f_X(x) \cdot l(x)\ dx \\
&= \int_0^p f_X(x) \cdot (1 - x)\ dx + \int_p^1 f_X(x) \cdot x\ dx \\
&= \int_0^p (1 - x)\ dx + \int_p^1 x\ dx \\
&= \left[x - \frac{x^2}{2}\right]_0^p + \left[\frac{x^2}{2}\right]_p^1 \\
&= [p - \frac{p^2}{2} - 0 + 0] + [\frac{1}{2} - \frac{p^2}{2}] \\
&= p - p^2 + \frac{1}{2}
\end{aligned}$$

To find the value of p which maximizes the expected length of substick containing p i.e. $L = E_X[l(x)]$. We differentiate the expression for L wrt. x and equate that to 0.

$$\frac{dL}{dp} = 1 - 2p = 0. \implies p = \frac{1}{2}$$

Since the equation for L is a downward parabola, the derivative is zero only at a maxima.

Problem 4

Firstly, we try to find $E[X]$ in terms of $P(X > x)$.

$$\begin{aligned}
E[X] &= \int_0^\infty x.f_X(x)\, dx \\
&= \int_0^\infty \left(\int_0^x dy\right) f_X(x)\, dx \\
&= \int_0^\infty \left(\int_y^\infty f_X(x)\, dx\right)\ dy \\
&= \int_0^\infty P(X > y)\, dy \\
&= \int_0^\infty P(X > x)\, dx
\end{aligned}$$

Now, use this to calculate $E[X^n]$.

$$\begin{aligned} E[X^n] &= \int_0^\infty P(X^n > x)\,dx \\ &= \int_0^\infty P(X > x^{\frac{1}{n}})\,dx \end{aligned}$$

Changing the variable from x to t such that $t = x^{\frac{1}{n}}$ and thus, $dx = n.t^{n-1}dt$.

$$\begin{aligned} E[X^n] &= \int_0^\infty n.t^{n-1}.P(X > t)\,dt \\ &= \int_0^\infty n.x^{n-1}.P(X > x)\,dx \end{aligned}$$

Hence, proved.

Problem 5

(a)

$$\begin{aligned} F_Y(y) &= P(Y \leq y) && (1) \\ &= P(F_X(X) \leq y) && (2) \\ &= P(X \leq F_X^{-1}(y)) && (3) \\ &= F_X(F_X^{-1}(y)) = y && (4) \end{aligned}$$

Note that the range of Y is $[0, 1]$ by definition of CDF. Equivalence between eqn. (2) and (3) holds due to the fact that F_X is strictly increasing in $(0, 1)$ and inverse of a monotonic function is well defined.

(b)

$$\begin{aligned} F_Z(z) &= P(Z \leq z) && (1) \\ &= P(-logF_X(x) \leq z) && (2) \\ &= P(logF_X(x) \geq -z) && (3) \\ &= P(e^{logF_X(x)} \geq e^{-z}) && (4) \\ &= P(F_X(X) \geq e^{-z}) && (5) \\ &= P(X \geq F_X^{-1}(e^{-z})) && (6) \\ &= 1 - P(X < F_X^{-1}(e^{-z})) && (7) \\ &= 1 - F_X(F_X^{-1}(e^{-z})) && (8) \\ &= 1 - e^{-z} && (9) \\ \Rightarrow f_Z(z) &= \frac{dF_Z}{dz}(z) = e^{-z} && (10) \end{aligned}$$

Note that equivalence between eqn. (3) and (4) holds as e^t is an increasing function in $(-\infty, \infty)$ and thus, there is a bijective mapping between events in (3) and (4).

Problem 6

$$
\begin{aligned}
P_Y(y) &= P(Y = y) \\
&= P(\lfloor X \rfloor = y) \\
&= P(y \leq X < y + 1) \\
&= \int_y^{y+1} f_X(t)dt \\
&= \int_y^{y+1} \lambda e^{-\lambda t} dt \\
&= \lambda \left[\frac{e^{-\lambda t}}{-\lambda} \right]_y^{y+1} \\
&= - \left[e^{-\lambda t} \right]_y^{y+1} \\
&= -[e^{-\lambda (y+1)} - e^{-\lambda y}] \\
&= e^{-\lambda y}(1 - e^{-\lambda})
\end{aligned}
$$

$$
\begin{aligned}
F_R(r) &= P(R \leq r) \\
&= P(X - \lfloor X \rfloor \leq r) \\
&= P(\bigcup_{i=0}^{\infty} \{\omega \in \Omega : X(\omega) \in [i, i + r]\}) \\
&= \sum_{i=0}^{\infty} P(\{\omega \in \Omega : X(\omega) \in [i, i + r]\}) \\
&= \sum_{i=0}^{\infty} \int_i^{i+r} f_X(t)dt \\
\Rightarrow f_R(r) &= \frac{dF_R}{dr}(r) = \sum_{i=0}^{\infty} \frac{d}{dr} \int_i^{i+r} f_X(t)dt = \sum_{i=0}^{\infty} f_X(i + r) \\
&= \sum_{i=0}^{\infty} \lambda e^{-\lambda(i+r)} \\
&= \lambda e^{-\lambda r} \sum_{i=0}^{\infty} e^{-\lambda i} \\
&= \frac{\lambda e^{-\lambda r}}{1 - e^{-\lambda}}
\end{aligned}
$$