Lecture 2

(5 August 2025)

Axiomatic Approach

probability space

$$(\Omega, \mathcal{F}, P)$$

- Ω → Sample space
- $\mathcal{F}$ → Event space
- P → Probability law

Sample space Ω

sample space Ω is the set of all possible outcomes of a random experiment.

The random experiment should produce exactly one out of all the possible outcomes in Ω, That is the elements the sample space should be mutually exclusive and collectively exhaustive.

Examples

(i) Finite sample space

Roll a pair of dice

$$\Omega = \{ (i,j) : i,j \in [1:6] \}$$

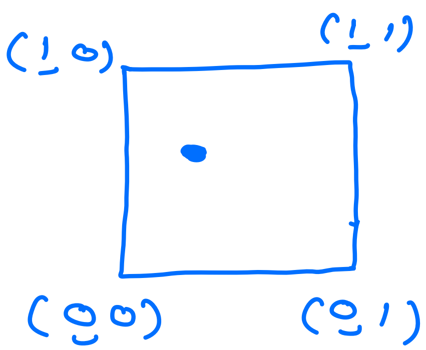
(ii) Countably infinite sample space

Toss a coin until you see a heads

$$\Omega = \{ H, TH, TTH, \dots \}$$

(iii) Uncountably infinite sample space

Consider throwing a dart on a 1×1 square target.



$$\Omega = \{ (x,y) : 0 \leq x \leq 1, \; 0 \leq y \leq 1 \}.$$

Non-Examples

Consider the random experiment of rolling a die, which of the following are potential sample spaces?

(i) $\{\underline{1}, \underline{2}, \underline{3}, \underline{3 \text{ or } 4}, \underline{5}, 6\}$

(ii) $\{\underline{1}\ \underline{3}\ \underline{4}\ \underline{5}\ 6\}$

(i) is not a sample space because it is not mutually exclusive.

(ii) is not a sample space because it is not collectively exhaustive.

$\Omega = \{\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{5}\ 6\}$ is a sample space

$\Omega = \{1 \text{ or } 2, \underline{3}\ \underline{4}\ \underline{5}\ 6\}$ is also a valid sample space.

Event space

An event is a subset of the sample space. The collection of all events is called an event space. An event space should be a σ-Field or σ-algebra.

A collection of sets $\mathcal{F}$ is said to be a σ-Field if it satisfies the following.

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(closed under complements)

(iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(closed under countable unions)

Proposition.

(i) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

(closed under countable intersections)

(ii) $A_1, A_2, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$.

(iii) $A, B \in \mathcal{F} \Rightarrow A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$.

Examples.

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unions

(i) $\mathcal{F} = \{\Omega, \phi\}$ (Smallest σ-field)

(ii) $\mathcal{F} = \{\Omega, E, E^c, \phi\}$

(Smallest σ-field that contains event E)

Example. What is the smallest σ-field that contains two events A and B?

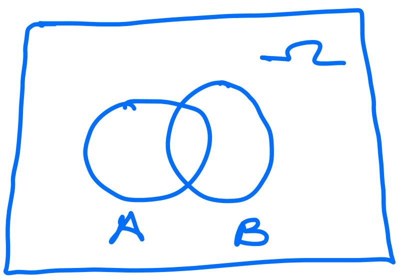
Let $\sigma(\mathcal{C})$ denote the smallest σ-field that contains the elements of collection $\mathcal{C}$.

We have

$$\sigma(A, B, \Omega, \phi)$$

$$= \{\Omega, \phi, A, B, A^c, B^c, A\cup B, A^c\cup B^c, A\cup B^c, A^c\cup B, A^c\cap B^c, A\cap B, A^c\cap B, A\cap B^c, A\triangle B, (A\triangle B)^c\}$$

An easy way to see this is the following.



$A\cap B$	$A\cap B^c$
$A^c\cap B$	$A^c\cap B^c$

$$\sigma(A, B, \Omega, \phi)$$

$$= \sigma(\underbrace{A\cap B, A\cap B^c, A^c\cap B, A^c\cap B^c}_{\text{collection of disjoint events}})$$

$$= \{\bigcup_{i\in I} E_i : I \subseteq [1:4]\},$$

where $E_1 = A \cap B,$

$E_2 = A \cap B^c,$

$E_3 = A^c \cap B,$

$E_4 = A^c \cap B^c.$

Thus the smallest σ-field can have at most 16 elements (no. of possible unions of the sets among $A \cap B$, $A^c \cap B$, $A \cap B^c$, $A^c \cap B^c$). It is at most 16 because not all of them are always distinct, e.g., $A \cap B$ can be ϕ.

Exercise: Find the smallest σ-field containing A & B in the following cases.

(i) $\Omega = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{2, 3\}$.

(ii) $\Omega = \{1, 2, 3, 4\}$, $A = \{2, 3\}$, $B = \{4\}$.

Probability Law

A probability law or a probability measure is a set function $P : \mathcal{F} \rightarrow R$ that satisfies the following axioms.

(1) (Non-negativity) $P(E) \geq 0$ for all $E \in \mathcal{F}$.

(2) (Normalization) $P(\Omega) = 1$.

(3) (Additivity) If $A_1, A_2, \ldots$ are disjoint events (i.e., mutually exclusive), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Examples.

(i) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$.

$P(\{1\}) = 0.2$ $\quad$ $P(\{4\}) = 0.3$

$P(\{2\}) = 0.04$ $\quad$ $P(\{5\}) = 0.15$

$P(\{3\}) = 0.06$ $\quad$ $P(\{6\}) = 0.25$

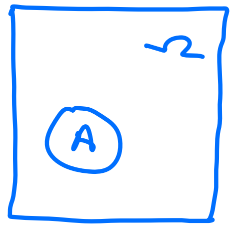
$$P(A) = \sum_{i \in A} P(\{i\}), \quad A \subseteq [1:6].$$

This defines a probability i.e., it satisfies all the axioms.

(ii) $\Omega = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$.

Let $\mathcal{F}$ contain all subsets of Ω.

$$P(A) = \frac{\text{area of } A}{\text{area of } \Omega} = \text{area of } A.$$



Properties of Probability Law

(a) $P(A) + P(A^c) = 1$, $P(A) \le 1$, $P(\phi) = 0$.

(b) If $A \subseteq B$, then $P(A) \le P(B)$.

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. (a) $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$.

$P(\phi) = 0$.

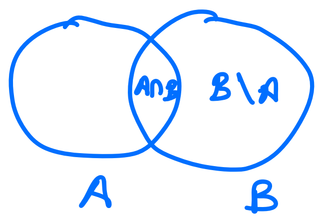
(b) $B = A \cup (B \setminus A) \implies P(B) = P(A) + P(B \setminus A) \ge P(A)$.

$A \subseteq \Omega \implies P(A) \le P(\Omega) = 1$.

(c) $P(A \cup B) = P(A \cup (B \setminus A))$

$= P(A) + P(B \setminus A)$

$= P(A) + P(B) - P(A \cap B)$.



More generally if $A_1, A_2, \ldots$ are events then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1 \le i \le n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k)$$

$$- \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

This is called the inclusion-exclusion principle, proof follows by induction.