Probability and Random Processes — Monsoon 2023

Assignment 2 Solutions

PRP TAs

September 6, 2023

Problem 1

$$
\begin{aligned}
P(A) &= 0.5 \\
P(B) &= 0.46 \\
P(A \cup B) &= 2 \cdot P(A \cap B) - (1)
\end{aligned}
$$

$$
\begin{aligned}
P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\
P(A \cap B) &= P(A) + P(B) - 2 \cdot P(A \cap B) - (from 1)
\end{aligned}
$$

$$
\begin{aligned}
3 \cdot P(A \cap B) &= P(A) + P(B) \\
3 \cdot P(A \cap B) &= 0.96 \\
P(A \cap B) &= 0.32 \\
\therefore P(A \cup B) &= 2 \cdot P(A \cap B) = 2 \cdot 0.32 = 0.64
\end{aligned}
$$

1.2

$$
\begin{aligned}
P(A \cup B \cup C) &= P(X \cup C) \\
&= P(X) + P(C) - P(X \cap C) \\
&= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cup B) \cap C) \\
&= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C)) \\
&= P(A) + P(B) + P(C) - P(A \cap B) - (P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))) \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)
\end{aligned}
$$

1.3

Given:

$$
\begin{aligned}
P(A) = P(B) = P(C) &= 0.5 - (eq\ 3) \\
P(A \cup B) &= 0.55 - (eq\ 4) \\
P(A \cup C) &= 0.7 - (eq\ 5) \\
P(B \cap C) &= 0.3 - (eq\ 6)
\end{aligned}
$$

$$P(A \cap B \cap C) = 2 \cdot P(A \cap B \cap C') - (eq\ 7)$$

$$
\begin{aligned}
P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\
&= 0.5 + 0.5 - 0.55 \\
&= 0.45 - (eq\ 8)
\end{aligned}
$$

$$
\begin{aligned}
P(A \cap C) &= P(A) + P(C) - P(A \cup C) \\
&= 0.5 + 0.5 - 0.7 \\
&= 0.3 - (eq\ 9)
\end{aligned}
$$

The sets C and C^c are disjoint, as $C \cup C^c = \Omega$. Hence, $(A \cap B) \cap C$ and $(A \cap B) \cap C^c$ are also disjoint.

$$
\begin{aligned}
A \cap B &= (A \cap B) \cap \Omega \\
A \cap B &= (A \cap B) \cap (C \cup C^c) \\
A \cap B &= (A \cap B \cap C) \cup (A \cap B \cap C^c) \\
P(A \cap B) &= P((A \cap B \cap C) \cup (A \cap B \cap C^c)) \\
&= P(A \cap B \cap C) + P(A \cap B \cap C^c) - (eq\ 10)
\end{aligned}
$$

$$
\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\
&\quad - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \\
&= 0.5 + 0.5 + 0.5 - 0.45 \\
&\quad - 0.3 - 0.3 + 0.3 \\
&= 0.75
\end{aligned}
$$

Therefore, $P(A \cap B \cap C) = 0.3$ and $P(A \cup B \cup C) = 0.75$.

Problem 2

Let G be the event of the child being a girl and B be that of the child being a boy.
Then we are given that, $P(G) = P(B) = \frac{1}{2}$.

(a)

We are given the condition that the first of the two children is a girl. This narrows down the sample space from GG, GB, BG, BB to GG, GB. The probability of both being girls, therefore, is,

$$\begin{aligned} P(GG|GG \cup GB) &= \frac{P(GG \cap (GG \cup GB))}{P(GG \cup GB)} \\ &= \frac{P(GG)}{P(GG \cup GB)} \\ &= \frac{1}{2} \end{aligned}$$

(b)

Now the condition is at least one child is a girl. Now the sample space is reduced to GG, GB, BG. The probability of both children being girls is therefore,

$$\begin{aligned} P(GG|GG \cup GB \cup BG) &= \frac{P(GG \cap (GG \cup GB \cup BG))}{P(GG \cup GB \cup BG)} \\ &= \frac{P(GG)}{P(GG \cup GB \cup BG)} \\ &= \frac{1}{3} \end{aligned}$$

(c)

In this section we need to introduce a new set of events which we shall call G_l, for when a girl is named Lilly and G when her name is anything other than Lilly. The following analysis follows,

$$P(\text{both girls} \mid \text{at least one daughter is named Lilly}) = \frac{\frac{1}{4}\frac{3}{4}}{\frac{1}{4}\frac{3}{4} + \frac{1}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{2}} = \frac{3}{7}$$

Problem 3

Let us assume the probability of getting a head on the biased coin $P(H) = p$ and the probability of getting tail be $P(T) = 1 - p$.
We want to use this biased coin so that two options are equally likely. So we need two mutually exclusive events with same probability. One possible choice is the set of events $\{TH, HT\}$ in the case when the coin is tossed twice.

To show that this is a valid choice, firstly we know that both are mutually exclusive events. Now, to compare their probabilities,

$$\begin{aligned} P(HT) &= P(H|T)P(T) \\ &= P(H)P(T) \\ &= p(1-p) \end{aligned}$$

$$\begin{aligned} P(TH) &= P(T|H)P(H) \\ &= P(T)P(H) \\ &= (1-p)p \end{aligned}$$

Since the probability of both these events is the same, this is a valid choice for two events which are equally likely. Also, we know that since these events are not exhaustive, we will perform the experiment until one of the two events occur. We can prove that when we keep on doing experiments, these events will definitely occur(assuming $0 < p < 1$) using geometric random variable with $\{HT, TH\}$ as the event. So, the probability of either HT or TH event occurring if we keep tossing becomes 1.

Problem 4

a) In order to prove $(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c$. Let's assume for the sake of contradiction $(\bigcup_{i=1}^{\infty} A_i)^c \neq \bigcap_{i=1}^{\infty} A_i^c$.

This implies $\exists$ some element $x \in (\bigcup_{i=1}^{\infty} A_i)^c$, but $x \notin \bigcap_{i=1}^{\infty} A_i^c$ or $x \in \bigcap_{i=1}^{\infty} A_i^c$ but $x \notin (\bigcup_{i=1}^{\infty} A_i)^c$

i) Let $x \in (\bigcup_{i=1}^{\infty} A_i)^c$, but $x \notin \bigcap_{i=1}^{\infty} A_i^c$

$$\begin{aligned} &\text{If } x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c \\ &\implies x \notin \bigcup_{i=1}^{\infty} A_i \\ &\implies x \notin A_i \text{ for all } i \in \{1 \ldots \infty\} \\ &\implies x \in A_i^c \text{ for all } i \in \{1 \ldots \infty\} \\ &\implies x \in \bigcap_{i=1}^{\infty} A_i^c \end{aligned}$$

A contradiction! We have equivalently also proved that $(\bigcup_{i=1}^{\infty} A_i)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$

ii) Let $x \in \bigcap_{i=1}^{\infty} A_i^c$, but $x \notin (\bigcup_{i=1}^{\infty} A_i)^c$

$$\begin{aligned}
&\text{If } x \in \bigcap_{i=1}^{\infty} A_i^c \\
&\implies x \in A_i \text{ for all } i \in \{1 \dots \infty\} \\
&\implies x \notin A_i^c \text{ for all } i \in \{1 \dots \infty\} \\
&\implies x \notin \bigcup_{i=1}^{\infty} A_i^c \\
&\implies x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c
\end{aligned}$$

A contradiction! We have equivalently also proved that $\bigcap_{i=1}^{\infty} A_i^c \subseteq (\bigcup_{i=1}^{\infty} A_i)^c$.

$$\therefore \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

b) If $A_1, A_2, \dots$ is an arbitrary infinite sequence of events. We know that

$$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} A_i)$$

Please refer to the lecture notes for the proof of this.

The above equation is valid for any arbitrary infinite sequence of events. Consider the events to be $A_1^c, A_2^c \dots, A_i^c, \dots$ all the way up to ∞ events. Applying the above equation we get:

$$\begin{aligned}
P(\bigcup_{i=1}^{\infty} A_i^c) &= \lim_{n\to\infty} P(\bigcup_{i=1}^{n} A_i^c) \\
1 - P(\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c) &= 1 - \lim_{n\to\infty} P(\left(\bigcup_{i=1}^{n} A_i^c\right)^c) && \text{(Using P(E) = 1-P(E}^c\text{))} \\
1 - P(\bigcap_{i=1}^{\infty} A_i) &= 1 - \lim_{n\to\infty} P(\bigcap_{i=1}^{n} A_i) && \text{(Using result from part (a))} \\
P(\bigcap_{i=1}^{\infty} A_i) &= \lim_{n\to\infty} P(\bigcap_{i=1}^{n} A_i)
\end{aligned}$$

Hence proved.

Problem 5

Let E_i be the event that the hiker reaches the destination by taking the trail i initially. There are n such events as $i \in [1:n]$. Note E_i's are disjoint.

Now,

$$\begin{aligned} P(\mathit{Reaching\ the\ destination}) &= P(E_1 \cup \cdots \cup E_n) \\ &= P(E_1) + \cdots + P(E_n) \quad \text{[By Additivity Axiom]} \end{aligned}$$

Now, let F_i be the event that hiker chooses ith trail initially and let S_i be the event that hiker chooses the correct subtrail among the possible subtrails for trail i. Again, there are n such possible F_i's and S_i's as $i \in [1:n]$

$$\begin{aligned} P(E_i) &= P(F_i)P(S_i|F_i) \quad , \forall i \in [1:n] \\ ` &= 1/n * (1/(i+1)) \quad [\because \text{Both choices are uniformly random}] \end{aligned}$$

$$\begin{aligned} \therefore P(E_1 \cup \cdots \cup E_n) &= \sum_{i=1}^{n} P(E_i) \\ &= 1/n \sum_{i=1}^{n} 1/(i+1) \end{aligned}$$