Question 1. *Let* $(\Omega, \mathcal{F})$ *be a* σ*-field. If* $A, B \in \mathcal{F}$ *are atoms and* $A \neq B$*, then* $A \cap B = \varnothing$*.*

Proof. Recall: $A \in \mathcal{F}$ is an *atom* iff $A \neq \varnothing$ and for every $C \in \mathcal{F}$ with $C \subseteq A$, one has $C = \varnothing$ or $C = A$.

Let $A, B \in \mathcal{F}$ be atoms with $A \neq B$. Since $\mathcal{F}$ is a σ-field, $A \cap B \in \mathcal{F}$ and $A \cap B \subseteq A$. By the defining property of an atom applied to A, either

$$A \cap B = \varnothing \quad \text{or} \quad A \cap B = A. \tag{1}$$

Similarly, $A \cap B \subseteq B$, so by the atom property for B,

$$A \cap B = \varnothing \quad \text{or} \quad A \cap B = B. \tag{2}$$

If $A \cap B \neq \varnothing$, then (1) and (2) force $A \cap B = A$ and $A \cap B = B$, hence $A = B$, contradicting $A \neq B$. Therefore $A \cap B = \varnothing$.

Thus any two distinct atoms are disjoint. $\square$

Assignment 1, Q2 Solution

Problem

Let $E_1, E_2, ..., E_n$ be mutually exclusive and exhaustive events in a sample space Ω. Determine the smallest σ-field that contains all the events E_i for $i \in \{1, ..., n\}$.

Proof

The whole problem boils down to identifying that the sets E_i mutually exhaust the sample space, and are mutually exclusive.

We thus claim that the smallest σ-field is the one which consists of all possible unions of the events E_i. This can be denoted mathematically as given below:

$$\mathcal{F} = \left\{ \bigcup_{i \in S} E_i : S \subseteq \{1, 2, ..., n\} \right\}$$

Since any σ-field must be:

1. Containing the sample space Ω and the empty set ($\emptyset$)
2. Containing event E_i for $i = 1, ..., n$
3. Closed under complements of sets.
4. Closed under countable unions of sets in the σ-field.

Also observe that,

1. Since $\bigcup_{i=1}^{n} E_i = \Omega$ (mutually exhaustive), we have $\Omega \in \mathcal{F}$ when $S = \{1, 2, 3, ..., n\}$.
2. The events are mutually exclusive, thus when $S = \emptyset$, we get $\bigcup_{i \in \emptyset} E_i = \emptyset$.
3. When it comes to closure under complements, again due to exhaustivity and exclusivity, for any $A = \bigcup_{i \in S} E_i \in \mathcal{F}$, we have:

$$A^c = \left(\bigcup_{i \in S} E_i \right)^c = \bigcap_{i \in S} E_i^c = \bigcup_{j \in \{1, ..., n\} \setminus S} E_j$$

4. For closure under countable union, since the collection of sets $\mathcal{F}$ is finite, any countable union of sets is equivalent to a finite union of its distinct members. For any finite collection $\{A_k\}_{k=1}^{m} \subseteq \mathcal{F}$ where each $A_k = \bigcup_{i \in S_k} E_i$, the union is:

$$\bigcup_{k=1}^{m} A_k = \bigcup_{k=1}^{m} \left(\bigcup_{i \in S_k} E_i \right) = \bigcup_{i \in \bigcup_{k=1}^{m} S_k} E_i$$

Let $S_{\text{final}} = \bigcup_{k=1}^{m} S_k$. As $S_{\text{final}} \subseteq \{1, ..., n\}$, the resulting union is in $\mathcal{F}$. This demonstrates closure.

Having satisfied all axioms, $\mathcal{F}$ is a σ-field. Also, for any specific event E_j (where $j \in \{1, ..., n\}$), we can choose the singleton index set $S = \{j\}$ to show that $E_j \in \mathcal{F}$.

Verifying that $\mathcal{F}$ is the smallest such σ-field

Let $\mathcal{G}$ be any σ-field that contains all the events $E_1, E_2, ..., E_n$. We must show that $\mathcal{F} \subseteq \mathcal{G}$.

Let A be an arbitrary element of $\mathcal{F}$. By the definition of $\mathcal{F}$, A can be written as a union:

$$A = \bigcup_{i \in S} E_i$$

for some $S \subseteq \{1, 2, ..., n\}$.

By assumption, for every $i \in S$, we have $E_i \in \mathcal{G}$. Since $\mathcal{G}$ is a σ-field, it must be closed under countable unions, which implies it is also closed under finite unions.

As S is a finite set, the union $\bigcup_{i \in S} E_i$ must be an element of $\mathcal{G}$. Thus, $A \in \mathcal{G}$.

This implies $\mathcal{F} \subseteq \mathcal{G}$, proving that $\mathcal{F}$ is indeed the smallest σ-field containing all the events E_i.

Thus the σ-field generated by the partition $\{E_1, ..., E_n\}$ contains 2^n distinct elements, one for each subset of the index set $\{1, ..., n\}$.

Question 3: Bounds on $P(A \cap B)$ and Examples

Lower Bound

$P(A \cup B) \le P(\Omega) = 1$:

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &\ge P(A) + P(B) - 1. \end{aligned}$$

Substitute $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{3}$:

$$\begin{aligned} P(A \cap B) &\ge \tfrac{3}{4} + \tfrac{1}{3} - 1 \\ &= \tfrac{9}{12} + \tfrac{4}{12} - \tfrac{12}{12} \\ &= \boxed{\tfrac{1}{12}}. \end{aligned}$$

Upper Bound

$$P(A \cap B) \le \min\{P(A),\, P(B)\} = \boxed{\tfrac{1}{3}}.$$

Examples for Part (b)

Work on the finite probability space

$$\Omega = \{1, 2, \ldots, 12\}, \qquad P(\{\omega\}) = \tfrac{1}{12} \text{ for each } \omega \in \Omega.$$

Lower bound example $\big(P(A \cap B) = \frac{1}{12}\big)$**.**

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \qquad B = \{1, 10, 11, 12\}.$$

Upper bound example $\big(P(A \cap B) = \frac{1}{3}\big)$**.**

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \qquad B = \{1, 2, 3, 4\}.$$

Q.4

Number of events = 3 (three coins: HH, TT and HT).

$$P(\text{HH}) = P(\text{TT}) = P(\text{HT}) = \tfrac{1}{3} \quad \text{(all are uniform)}$$

Required probability: $P(\text{Tails on opposite face} \mid \text{coin toss gives Heads})$

Need to apply Bayes' Theorem and the Total Probability Theorem.

Note: Always state the formula or theorem being used in the answers.

Bayes' Theorem:

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

So,

$$P(\text{Tails on opposite side} \mid \text{Heads}) = \frac{P(\text{Heads AND Tails on opposite side})}{P(\text{Heads})}$$

Total Probability Theorem:

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i), \quad \text{if } P(A_i) > 0,\ i \in [1:n]$$

$$P(\text{Heads}) = P(\text{Heads} \mid \text{HH}) \cdot P(\text{HH}) + P(\text{Heads} \mid \text{HT}) \cdot P(\text{HT}) + P(\text{Heads} \mid \text{TT}) \cdot P(\text{TT})$$

$$= 1 \cdot \tfrac{1}{3} + \tfrac{1}{2} \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3}$$

$$= \boxed{\tfrac{1}{2}}$$

$P(\text{Heads AND Tails on opposite side}) \;\rightarrow\;$ Only coin HT contributes.

Essentially, this also follows from the Total Probability Theorem, where the other two coins give 0 probability for this case.

$$P(\text{Heads AND Tails on opp. side}) = \tfrac{1}{3} \cdot 0 + \tfrac{1}{3} \cdot 0 + \tfrac{1}{3} \cdot \tfrac{1}{2}$$

$$= P(\text{HT chosen}) \cdot P(\text{Heads on HT}) = \boxed{\tfrac{1}{6}}$$

$$\therefore\ P(\text{Tails on opposite side} \mid \text{Heads}) = \frac{\frac{1}{6}}{\frac{1}{2}} = \boxed{\tfrac{1}{3}}$$

Question 5. *Let $\mathcal{C} = \{A_1, A_2, \ldots, A_n\}$ be a collection of mutually independent events. For any chosen index $i \in \{1, 2, \ldots, n\}$, the collection $\mathcal{C}' = (\mathcal{C} \setminus \{A_i\}) \cup \{A_i^c\}$ is also a collection of mutually independent events.*

Proof. **Given:** The collection $\mathcal{C} = \{A_1, \ldots, A_n\}$ is mutually independent. Thus, for any subcollection $\{B_1, \ldots, B_k\} \subseteq \mathcal{C}$:

$$\mathbf{P}\left(\bigcap_{j=1}^{k} B_j\right) = \prod_{j=1}^{k} \mathbf{P}(B_j)$$

To Prove: The collection $\mathcal{C}' = \{A_1, \ldots, A_{i-1}, A_i^c, A_{i+1}, \ldots, A_n\}$ is mutually independent. For any subcollection $S \subseteq \mathcal{C}'$, we must show:

$$\mathbf{P}\left(\bigcap_{B \in S} B\right) = \prod_{B \in S} \mathbf{P}(B)$$

The proof follows by considering an arbitrary subcollection $S \subseteq \mathcal{C}'$.

Case 1: $A_i^c \notin S$

If $A_i^c \notin S$, then $S \subseteq \mathcal{C}$. Since $\mathcal{C}$ is mutually independent, the property holds for S by definition.

$$\mathbf{P}\left(\bigcap_{B \in S} B\right) = \prod_{B \in S} \mathbf{P}(B)$$

Case 2: $A_i^c \in S$

Let $S = S' \cup \{A_i^c\}$, where $S' \subseteq \mathcal{C} \setminus \{A_i\}$. Let $B_{\text{int}} = \bigcap_{B \in S'} B$. We must show $P(B_{\text{int}} \cap A_i^c) = P(B_{\text{int}})P(A_i^c)$.

From the law of total probability, $P(B_{\text{int}}) = P(B_{\text{int}} \cap A_i) + P(B_{\text{int}} \cap A_i^c)$, which implies:

$$P(B_{\text{int}} \cap A_i^c) = P(B_{\text{int}}) - P(B_{\text{int}} \cap A_i)$$

Since all events in S' and A_i are in the mutually independent collection $\mathcal{C}$, we have:

$$P(B_{\text{int}}) = \prod_{B \in S'} \mathbf{P}(B)$$

$$P(B_{\text{int}} \cap A_i) = \left(\prod_{B \in S'} \mathbf{P}(B)\right) \mathbf{P}(A_i)$$

Substituting these into the equation:

$$\begin{aligned}\mathbf{P}(B_{\text{int}} \cap A_i^c) &= \left(\prod_{B \in S'} \mathbf{P}(B)\right) - \left(\prod_{B \in S'} \mathbf{P}(B)\right) \mathbf{P}(A_i) \\ &= \left(\prod_{B \in S'} \mathbf{P}(B)\right) (1 - \mathbf{P}(A_i)) \\ &= \left(\prod_{B \in S'} \mathbf{P}(B)\right) \mathbf{P}(A_i^c)\end{aligned}$$

This is the required condition for the subcollection S.

Since the condition holds for all possible subcollections, $\mathcal{C}'$ is mutually independent. $\square$

Q6

We are interested in

$$P(E \mid A \cap B),$$

where

$$P(E) = \frac{1}{1000}, \qquad P(\overline{E}) = \frac{999}{1000},$$

and each witness is truthful with probability 0.9 and lies with probability 0.1. Let A denote the event that Alice asserts E occurred, and B the event that Bob asserts the same.

For a single witness,

$$P(A \mid E) = 0.9, \qquad P(A \mid \overline{E}) = 0.1,$$

and similarly for Bob.

Since Alice and Bob are independent witnesses and there is no collusion, their answers are *conditionally independent* given whether E occurred. Hence,

$$P(A \cap B \mid E) = P(A \mid E)\, P(B \mid E) = 0.9 \cdot 0.9 = \frac{81}{100},$$

$$P(A \cap B \mid \overline{E}) = P(A \mid \overline{E})\, P(B \mid \overline{E}) = 0.1 \cdot 0.1 = \frac{1}{100}.$$

Applying Bayes' theorem,

$$P(E \mid A \cap B) = \frac{P(E)\, P(A \cap B \mid E)}{P(E)\, P(A \cap B \mid E) + P(\overline{E})\, P(A \cap B \mid \overline{E})}.$$

Substituting values,

$$\begin{aligned} P(E \mid A \cap B) &= \frac{\frac{1}{1000} \cdot \frac{81}{100}}{\frac{1}{1000} \cdot \frac{81}{100} + \frac{999}{1000} \cdot \frac{1}{100}} \\ &= \frac{\frac{81}{100000}}{\frac{81}{100000} + \frac{999}{100000}} = \frac{81}{1080}. \end{aligned}$$

Reducing the fraction,

$$P(E \mid A \cap B) = \frac{3}{40}.$$

$$\boxed{P(E \mid A \cap B) = \frac{3}{40}}$$

Question 7

Writing down all the permutations will get very messy very quickly. A clever idea is to identify the recursive structure of the problem and exploit it.

Denote the probability of going bankrupt from a current balance of 'k' as P_k. From the Total Probability Theorem, we can write:

$$P(Bankruptcy) = P(\text{Heads})P(Bankruptcy \mid Heads) + P(\text{Tails})P(Bankruptcy \mid Tails)$$

$$P_k = P(\text{Heads})P_{k+1} + P(\text{Tails})P_{k-1}$$

With a fair coin, where $P(\text{Heads}) = P(\text{Tails}) = \frac{1}{2}$, the equation becomes:

$$P_k = \frac{P_{k+1} + P_{k-1}}{2}$$

P_k is the arithmetic mean of P_{k+1} and P_{k-1}. Let's analyze the boundary conditions.

- If the current balance is 0, the player is already bankrupt. So, the probability of going bankrupt is 1.

  $$P_0 = 1$$

- If the player reaches the target of 2,000,000, they have won and will not go bankrupt. So, the probability of going bankrupt is 0.

  $$P_{2,000,000} = 0$$

We are looking for P_{200}. Since the probabilities form an AP, the solution is of the form:

$$P_k = P_0 + \frac{(P_{2,000,000} - P_0)k}{2,000,000}$$

$$P_k = 1 - \frac{k}{2,000,000}$$

For a starting balance of $k = 200$:

$$P_{200} = 1 - \frac{200}{2,000,000} = 1 - 0.0001 = 0.9999$$

To build more intuition, think about how the probability of going bankrupt changes with a change in the starting position. Also, what happens if the coin is biased?

Interested students can also check out the concept of Markov Processes.