Lecture 5

(19 August 2025)

Module 2 (Discrete Random Variables)

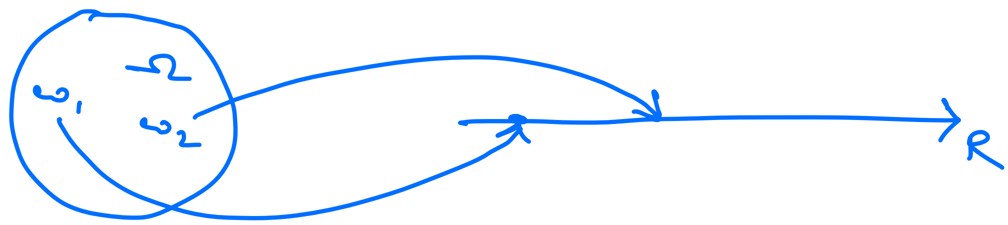
- The concept of a Random Variable
- Probability Distribution Function
- Types of Random Variables
- Expectation, Variance, Functions of RVs
- Multiple RVs, Conditioning, Independence

Random Variable

We may not be always interested in the actual outcome of a random experiment, but rather in some consequence of the random outcome.

A random variable is a function from sample space to real numbers.

$$X : \Omega \rightarrow \mathbb{R}$$



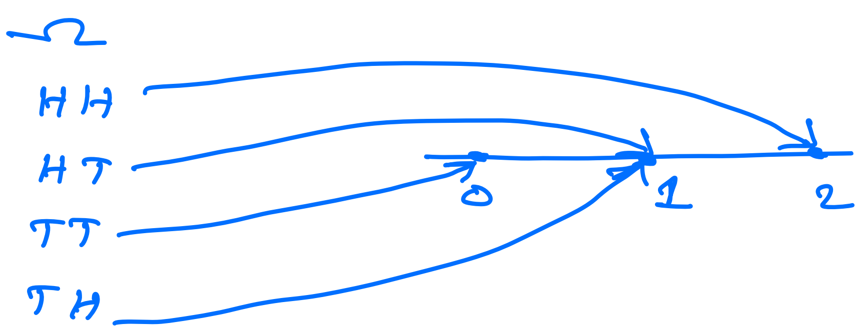
We would like to speak about events of the form $\{X \le x\} \triangleq \{\omega : X(\omega) \le x\}$, $x \in \mathbb{R}$.

<u>Definition</u>. A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega : X(\omega) \le x\} \in \mathcal{F}$, for all $x \in \mathbb{R}$, given Ω, $\mathcal{F}$.

<u>Examples</u>.

(i) $\Omega = \{HT, TH, HH, TT\}$

$X(\omega) =$ no. of heads



$-\infty < c < 0$, $\{X \le c\} = \phi$

$0 \le c < 1$, $\{X \le c\} = \{TT\}$

$1 \le c < 2$, $\{X \le c\} = \{TT, TH, HT\}$

$2 \le c < \infty$, $\{X \le c\} = \Omega$.

For Ω, $\mathcal{F} = 2^{\Omega}$, X above is a RV.

Notation: $\{\omega : X(\omega) \le c\} = X^{-1}((-\infty, x])$.

The above function is not a random variable w.r.t $\mathcal{F} = \{\phi, \Omega, \{HT, TH\}, \{HH, TT\}\}$ because $X^{-1}((-\infty, 1]) = \{HT, TH, TT\} \notin \mathcal{F}$.

Theorem. Given a sample space Ω and an event space $\mathcal{F}$. Let $X : \Omega \to R$ be a random variable. Then the following holds.

(i) $X^{-1}((-\infty, x)) = \{\omega : X(\omega) < x\} \in \mathcal{F}$.

(ii) $X^{-1}([x_1, x_2]) = \{\omega : x_1 \le X(\omega) \le x_2\} \in \mathcal{F}$.

(iii) $X^{-1}(\{x\}) = \{\omega : X(\omega) = x\} \in \mathcal{F}$.

(iv) $X^{-1}((x_1, x_2)) = \{\omega : x_1 < X(\omega) < x_2\} \in \mathcal{F}$.

Proof. (i) $A_n = X^{-1}((-\infty, x - \frac{1}{n}]) \in \mathcal{F}$.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} X^{-1}((-\infty, x - \tfrac{1}{n}])$$

$$= X^{-1}\left(\bigcup_{n=1}^{\infty} (-\infty, x - \tfrac{1}{n}]\right)$$

$$= X^{-1}((-\infty, x)) \in \mathcal{F}.$$

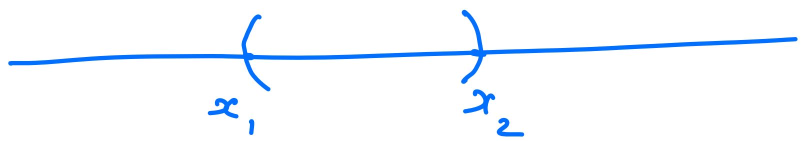
(ii) $$X^{-1}([x_1,\infty)) = \{\omega : X(\omega) \geq x_1\}$$
$$= \Omega \setminus \{\omega : X(\omega) < x_1\}$$
$$\in \mathcal{F}$$

$$X^{-1}((-\infty, x_2]) \in \mathcal{F}.$$

$$\Longrightarrow X^{-1}([x_1,\infty) \cap (-\infty, x_2]) = X^{-1}([x_1, x_2]) \in \mathcal{F}.$$

(iii) $x_1 = x_2 = x$ in (ii) gives $X^{-1}(\{x\}) \in \mathcal{F}$.

(iv)



$$(x_1, x_2) = (-\infty, x_2) \cap (x_1, \infty)$$

$$X^{-1}((-\infty, x_2)) \in \mathcal{F}.$$

$$X^{-1}((x_1,\infty)) = \Omega \setminus \{\omega : X(\omega) \leq x_1\}$$
$$\in \mathcal{F}.$$

$$\Longrightarrow X^{-1}((x_1, x_2)) = X^{-1}((-\infty, x_2)) \cap X^{-1}((x_1, \infty))$$
$$\in \mathcal{F}.$$

This also brings us to the consideration of Borel σ-Field or Borel σ-algebra:

The smallest σ-algebra on reals containing sets of the form $(-\infty, x]$, $x \in R$.

$$B = \{(-\infty, x),\ (-\infty, x],\ (x, \infty),\ [x, \infty),\ \{x\},\ (x_1, x_2),\ \ldots\}$$

The distribution function (or cumulative distribution function) of a random variable X is the function $F_X : R \to [0, 1]$ given by

$$F_X(x) = P(\{\omega : X(\omega) \leq x\})$$
$$= P(X \leq x).$$

Examples.

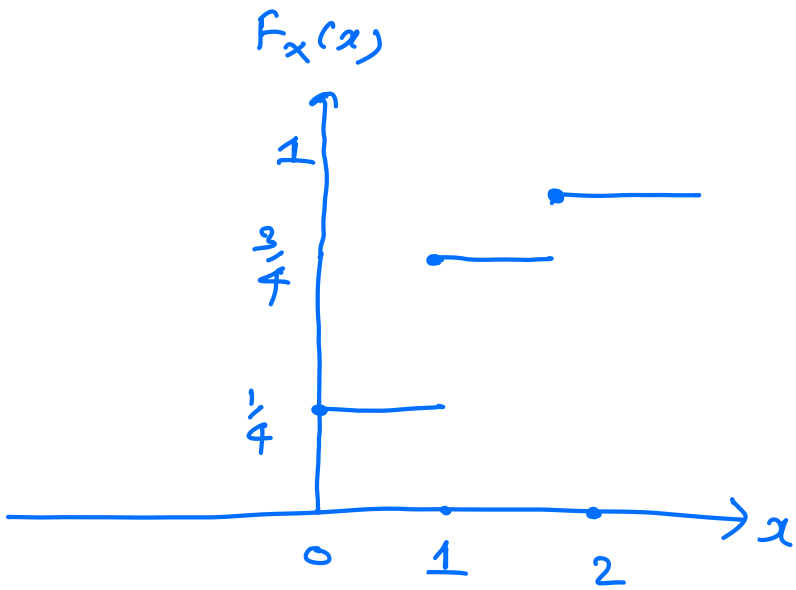
1) $\Omega = \{HH, TT, HT, TH\}$, $\mathcal{F} = 2^{\Omega}$,

$$P(\{\omega\}) = \frac{1}{4}.$$

$X(\omega) =$ no. of heads in ω.

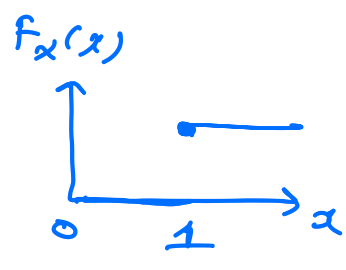
$$\{X \le x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \le x < 1 \\ \{TT, HT, TH\}, & 1 \le x < 2 \\ \Omega, & x \ge 2 \end{cases}$$

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$



2) Constant random variable

$X(\omega) = c,$ for all $\omega \in \Omega$.



$$F_X(x) = \begin{cases} P(\phi), & x < c \\ P(\Omega), & x \ge c \end{cases} = \begin{cases} 0, & x < c \\ 1, & x \ge c \end{cases}.$$

3) Bernoulli random variable

$\Omega = \{H, T\}$, $\mathcal{F} = \{H, T, \phi, \Omega\}$

$P(\{H\}) = p = 1 - P(\{T\})$.

$X(H) = 1$, $X(T) = 0$.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1-p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

4) Indicator random variable

Given Ω, $\mathcal{F}$ and $A \in \mathcal{F}$.

Indicator random variable of an event A is defined as

$$I_A : \Omega \to R :$$

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A. \end{cases}$$

<u>Exercise</u>. suppose $B_1, B_2, \ldots, B_n$ form a partition of Ω, we have $I_A = \sum_{i=1}^{n} I_{A \cap B_i}$, i.e.,

$$I_A(\omega) = \sum_{i=1}^{n} I_{A \cap B_i}(\omega), \quad \forall \omega \in \Omega.$$

The distribution function of X tells us about the values taken by X rather than about the sample space and the collection of events. For the time being we can forget all about probability spaces and concentrate on random variables and their distribution functions.

Theorem.

(a) If $x < y$ then $F_X(x) \le F_X(y)$.

(b) $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$.

(c) $F_X(x)$ is right continuous, that is

$$\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x).$$

(d) $P(X > x) = 1 - F_X(x)$.

(e) $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

(f) $P(X = x) = F_X(x) - \lim_{\varepsilon \to 0^+} F_X(x-\varepsilon)$.