Probability and Random Processes — Monsoon 2023

Assignment 1 Solutions

PRP TAs

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Problem 1

(a): Claim 1:

$$pow(A \bigcap B) \subseteq pow(A) \bigcap pow(B)$$

Proof 1:

$$\text{Let } x \in pow(A \bigcap B)$$

$$\Rightarrow \ x \subseteq \ A \bigcap B \text{ , By definition of } pow(A \bigcap B)$$

$$\Rightarrow \ x \subseteq \ A \,, x \subseteq \ B$$
$$\Rightarrow \ x \in pow(A),\, x \in pow(B)$$

$$\Rightarrow \ x \in pow(A) \bigcap pow(B)$$

Claim 2:

$$pow(A) \bigcap pow(B) \subseteq pow(A \bigcap B)$$

Proof 2:

$$\text{Let } x \in pow(A) \bigcap pow(B)$$

$$\Rightarrow \ x \in pow(A),\, x \in pow(B)$$
$$\Rightarrow \ x \subseteq \ A \,, x \subseteq \ B$$

$$\Rightarrow \ x \subseteq A \bigcap B$$

$$\Rightarrow \ x \in pow(A \bigcap B) \qquad \square$$

(b):

Let $x \in pow(A) \bigcup pow(B)$

There are 3 cases:

Case 1: $x \in pow(A)$, $x \notin pow(B)$
$\Rightarrow x \subseteq \ A$, $x \nsubseteq \ B$
$\Rightarrow x \subseteq \ A \setminus B$
$\Rightarrow x \subseteq \ A \bigcup B \ \ [\because A \setminus B \subseteq A \bigcup B]$

$\Rightarrow x \in pow(A \bigcup B)$

Case 2: $x \notin pow(A)$, $x \in pow(B)$
Similar as Case 1

Case 3: $x \in pow(A)$, $x \in pow(B)$
$\Rightarrow x \subseteq \ A$, $x \subseteq \ B$

$\Rightarrow x \subseteq \ A \bigcap B$

$\Rightarrow x \subseteq \ A \bigcup B \ \ [\because A \bigcap B \subseteq A \bigcup B]$

$\Rightarrow x \in pow(A \bigcup B)$ $\square$

Equality Claim

$pow(A) \bigcup pow(B) = pow(A \bigcup B)$ **if and only if** $A \subseteq \ B$ *or* $B \subseteq \ A$

$pow(A) \bigcup pow(B) \ \subseteq \ pow(A \bigcup B)$ **(This holds regardless, as proved above)**

Rephrased Claim:

$pow(A \bigcup B) \subseteq \ pow(A) \bigcup pow(B)$ **if and only if** $A \subseteq \ B$ *or* $B \subseteq \ A$

Proof:

Claim a): $A \subseteq \ B$ *or* $B \subseteq \ A \quad \Rightarrow \quad pow(A \bigcup B) \subseteq \ pow(A) \bigcup pow(B)$

Proof a): Case 1) $A \subseteq \ B \quad \Rightarrow \quad A \bigcup B = B$

Let $x \in pow(A \bigcup B) \quad \Rightarrow \quad x \subseteq A \bigcup B$ *or* $\quad x \subseteq B$

$x \subseteq B \quad \Rightarrow \quad x \in pow(B) \quad \Rightarrow \quad x \in pow(B) \bigcup pow(A)$

$pow(A \bigcup B) \subseteq \ pow(A) \bigcup pow(B)$

Claim b): $pow(A \bigcup B) \subseteq pow(A) \bigcup pow(B) \quad \Rightarrow \quad A \subseteq\ B \ or \ B \subseteq\ A$

Proof b): Assume the contrary i.e. $\neg(A \subseteq\ B \ or \ B \subseteq\ A)$
$\Rightarrow A \setminus B \neq \emptyset \quad and \quad B \setminus A \neq \emptyset$
$\therefore \exists x \in A \setminus B \quad and \quad \exists y \in B \setminus A$

Let's consider a set $S = \{x, y\}$, Now $x, y \in A \bigcup B$

$\Rightarrow S \subseteq\ (A \bigcup B) \quad \Rightarrow \quad S \in pow(A \bigcup B)$

$\because x \notin B \quad \Rightarrow \quad S \notin pow(B)$
$\because y \notin A \quad \Rightarrow \quad S \notin pow(A)$

$\Rightarrow S \notin pow(A) \bigcup pow(B)$

$\Rightarrow pow(A \bigcup B) \nsubseteq\ pow(A) \bigcup pow(B) \qquad \Box$

Problem 2

Case 1: $A = B$, Clearly, only one choice for C i.e. $A = B = C$, $\quad \therefore C \setminus A = \emptyset$

$$\Rightarrow \quad |C \setminus A| = 0 \quad \therefore \quad \sum_{\forall C: A \subseteq C \subseteq B} (-1)^{|C \setminus A|} = (-1)^0 = 1$$

Case 2: $A \subset B$

$$Let \quad S = \sum_{\forall C: A \subseteq C \subseteq B} (-1)^{|C \setminus A|}$$

Note that all the C's such that $C \setminus A$ has even cardinality, would contribute +1 to the summation while all the C's such that $C \setminus A$ has odd cardinality, would contribute -1 to the summation

$\therefore \quad S =$ # of even cardinality sets in $pow(B \setminus A)$ - # of odd cardinality sets in $pow(B \setminus A)$ (1)

[$\because C$ would always include elements of A, the possible choices for C would depend on the elements selected from $B \setminus A$ or another way to see it is: $C \setminus A \subseteq B \setminus A \quad (\because C \subseteq B) \Rightarrow$ all possible $C \setminus A$ constitute $pow(B \setminus A)$]

Claim a): For any non-empty set X, no. of even cardinality subsets of X are equal to the number of odd cardinality subsets of X

Proof a): Let $X = \{x_1, x_2, \cdots, x_n\}$

Construction of an even cardinality subset $Z \subseteq X$:

Each of $x_1, x_2, \cdots, x_{n-1}$ has 2 choices: either to be included or not

But, note that x_n does not have a choice. $\because$ whatever set Z constructed so far would have some cardinality $|Z|$. If $|Z|$ is even, $x_n \notin Z$, else, $x_n \in Z$.

$\therefore$ total # of subsets of X with even cardinality $= 2 * 2 * \cdots * 2 \quad (n-1)$ times

$= 2^{n-1}$ [By Multiplication Rule]

Also, total # of subsets of $X = 2^n$

$\Rightarrow$ total # of subsets of X with odd cardinality $= 2^n - 2^{n-1} = 2^{n-1} =$ total # of subsets of X with even cardinality

Hence, from (1) and Claim a), we get $S = 0 \quad when \quad A \subset B$ $\square$

Problem 3

It is given that A and B are independent events. This means that,

$$P(A \cap B) = P(A)P(B)$$

Now, consider the probability,

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$

Since, $P(A^c \cap B^c) = P(A^c)P(B^c)$, we can say that if A and B are independent, then A^c and B^c are also independent.

Now we prove that given the conditions, A^c and B are also independent. Similar to the previous argument,

$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B)(why?) \\ &= P(B)(1 - P(A)) \\ &= P(A^c)P(B) \end{aligned}$$

Therefore, if A and B are independent, A^c and B are also independent.

Problem 4

In the first sampling, m sea turtles are captured and tagged, so when they are returned, m sea turtles will be tagged and the remaining N - m will not be tagged. Now, when we take a sample of n turtles, we want to find the probability that k out of them will have been previously tagged and the rest will not be tagged.
We can solve this problem by going back to the basic approach to probability, i.e., the no. of desirable outcomes divided by the no. of total outcomes. This applies because the probability of each sample of m turtles being chosen is equal. If they weren't, we would have to take a weighted sum of probabilities.

$$\text{No. of desirable outcomes} = \binom{m}{k}\binom{N-m}{n-k}$$
$$\text{No. of total outcomes} = \binom{N}{n}$$

Therefore, the probability of exactly k previously tagged sea turtles being captured the secnd time is,

$$P(\text{k previously tagged}) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

Problem 5

(a) If the no. of people i.e. $k \geq 365$. Then, by Pigeonhole principle $P(\text{atleast one birthday match}) = 1$. Now, let's consider the case for $k < 365$.
We can write P(atleast one birthday match) as:

$$P(\text{atleast one birthday match}) = 1 - P(\text{No birthday match})$$

If we want no birthday matches, we have to choose k unique days out of 365 days in a year. This can be done $\binom{365}{k}$ ways and we can permute it among k people in $k!$ ways.
The total no. of ways of choosing k birthdays out of 365 days is 365^k. Since, every person can have a birthday on anyone of the 365 day. For k people, since their birthdays are independent, we multiply the no. of ways i.e. we multiply 365 k times.

$$P(\text{No birthday match}) = \frac{\binom{365}{k}}{365^k} \cdot k!$$

$$P(\text{At least one birthday match}) = 1 - \frac{\binom{365}{k} \cdot k!}{365^k}$$

Writing a simple program to calculate this expression, we get:
$P(\text{Atleast one birthday match}) \geq \frac{1}{2}$ for $k \geq 23$. Therefore, the minimum k is 23.

(b)

i) If $k \geq 365$, $P(\text{Atleast one birthday match}) = 1$.
If $k < 365$, we need to select k unique days out of 365 days in a year similar to the previous part. Let $D = \{d_1, d_2, \ldots d_k\}$ be that set. We can permute it in $k!$ ways for k persons. Now to calculate the total probability, we iterate over all such possible sets.

$$P(\text{No birthday match}) = \sum_{D=\{d_1,d_2,\ldots d_k\}} k! \cdot p_{d_1} \cdot p_{d_2} \cdot \ldots p_{d_k}$$

$$P(\text{At least one birthday match}) = 1 - k! \left(\sum_{D=\{d_1,d_2,\ldots d_k\}} p_{d_1} \cdot p_{d_2} \cdot \ldots p_{d_k} \right)$$

ii) An intuitive explanation for why $p_i = \frac{1}{365}$ minimizes the $P(\text{At least one birthday match})$ is if the probabilities are all equal then the birthdays are spread out as uniformly possible. Any change in probability increases the chances of a collision (a match in this case.) E.g. if $p_1 = 1$ and all other $p_i = 0$. Then $P(\text{At least one birthday match})$=1.

A more mathematical way of seeing this fact is as follows. If we want to minimize $P(\text{Atleast one birthday match})$, it is same maximizing the expression

$$\sum_{D=\{d_1,d_2,\ldots d_k\}} p_{d_1} \cdot p_{d_2} \cdot \ldots p_{d_k}$$

An intuitive way to say why the above expression is maximized for $p_i = \frac{1}{365}$, $\forall i$. Consider the simple case of two variables p_1 and p_2, where $p_1 = p_2 = \frac{1}{2} = p$. Now let's change p_1 and p_2, which is the same adding a number r to p_1 and subtracting r from p_2 as the sum of p_1 and p_2 is 1.

$$p_1 \cdot p_2 = (p+r)(p-r)$$
$$= p^2 - r^2 \leq p^2 = \left(\frac{1}{2}\right)^2$$

Therefore, the product $p_1 \cdot p_2$ is maximized for $p_1 = p_2 = \frac{1}{2}$. Now, let's convert it into a formal proof.

We will prove it using contraction. Assume the expression is maximum for some $p_i \neq p_j$. The above expression can be written as

$$\sum_{D=\{d_1,d_2,\ldots d_k\}} p_{d_1} \cdot p_{d_2} \cdot \ldots p_{d_k}$$

$$= p_i \cdot e_{k-1}(p_1, p_2, .., p_{365}) + p_j \cdot e_{k-1}(p_1, p_2, .., p_{365}) + p_i \cdot p_j \cdot e_{k-2}(p_1, p_2, .., p_{365}) + e_k(p_1, p_2, .., p_{365})$$

$$= e_{k-1}(p_1, p_2, .., p_{365}) \cdot (p_i + p_j) \ + \ p_i \cdot p_j \cdot e_{k-2}(p_1, p_2, .., p_{365}) \ + e_k(p_1, p_2, .., p_{365})$$

Where $(p_1, p_2, .., p_{365})$ is the tuple of probabilities $p_1, p_2, \ldots p_{365}$, but does not contain p_i and p_j. And e_k denotes the k^{th} elementary symmetric polynomial in the variables $x_1, \ldots, x_n$ is defined by

$$e_k\left(x_1, \ldots, x_n\right)=\sum_{1 \leq j_1<j_2<\cdots<j_k \leq n} x_{j_1} \ldots x_{j_k}.$$

This just says to add up all of the $\left(\begin{array}{l} n \\ k \end{array}\right)$ terms we can get by choosing and multiplying k of the variables. For example, $e_1\left(x_1, x_2, x_3\right)=x_1+x_2+x_3, e_2\left(x_1, x_2, x_3\right)=x_1 x_2+ x_1 x_3+x_2 x_3$, and $e_3\left(x_1, x_2, x_3\right)=x_1 x_2 x_3$.

The term to be maximized reduces to maximizing $p_i \cdot p_j$ since p_i+p_j term is fixed as all the other probabilities are fixed (same or different, it doesn't matter) and the probability sum adds up to 1. By AG-GM inequality, $p_i \cdot p_j$ is maximized when $p_i=p_j$. Since, this is true for any pair i, j. Therefore $P(\text{At least one birthday})$ is minimized for a uniform distribution i.e. $p_i=\frac{1}{365}$.

Problem 6

Regardless of the number of seats (as long as the number of seats are greater than 2) the probability of the last person getting their seat is $\frac{1}{2}$. This can be proved by mathematical induction.

Consider the base case where there are only 2 seats. The first person selects one of the two seats randomly with equal probability. Hence, the second person gets their seat with probability of $\frac{1}{2}$.

Let $P(n)$ denote the probability of the last person getting their seat when there are n seats. We know $P(2)=\frac{1}{2}$. We will prove $P(n)=\frac{1}{2}$ by considering $P(m)=\frac{1}{2}$, for $2 \leq m<n$.

When the first person boards the train, there are 3 possibilities when $n \geq 3$:

1) He sits on his assigned seat (with a probability of $\frac{1}{n}$) - In this case, the last person definitely gets the correct seat because everyone else sat in correct seat.

2. He sits on the last person's seat (with a probability of $\frac{1}{n}$) - In this case, the last person definitely gets the wrong seat.

3. He sits on neither his seat nor the last person's seat - This happens with a probability of $\frac{n-2}{n}$. Say, the person sat in the seat of the k th person $(2 \leq k<n)$. Now 2nd passenger to the $(k-1)^{\text{th}}$ passenger occupy their correct seats. When the k th person comes, they have to randomly pick between the first persons seat, the seat of the $(k+1)$ th person, and every person after. Here, the problem is reduced to a variant of the same problem but with fewer seats. There are $n-k$ people after the k th person, therefore there are $n-k+1$ seats for the k th person to choose from.

By total probability theorem, we can write the probability conditioning on whether the first passenger occupies first seat, second seat until the last seat. So, the expression for $P(n)$ is

$$P(n)=\frac{1}{n}(1)\ +\ \frac{1}{n}(0)\ +\sum_{k=2}^{n-1} \frac{1}{n} P(n-k+1)$$

By induction hypothesis $P(n-k+1) = 1/2$ (as $2 \leq (n-k+1) < n$ when $2 \leq k < n-1$)

$$P(n) = \frac{1}{n}(1) \ + \ \frac{1}{n}(0) \ + \ \frac{n-2}{n}\left(\frac{1}{2}\right)$$

$$P(n) = \frac{1}{n} + \frac{n-2}{2n}$$

$$P(n) = \frac{2+(n-2)}{2n}$$

$$P(n) = \frac{1}{2}$$

Hence, for a train with 50 passengers. The P(Last passenger gets his assigned seat) $= \frac{1}{2}$.
Another explanation for this problem is as follows:

The first person in the queue (assume he does not sit in his own seat by random chance) displaces one person still in the queue by sitting in their seat . Passengers continue to board in their own seats until the displaced person comes to sit down. His own seat is taken so he in turn displaces someone else who is in the queue. There is always one displaced person in the queue as this cycle continues.

At some point this cycle must terminate i.e. one of the displaced person has to make a choice between 2 possible outcomes:

a) The displaced person randomly chooses the first person's seat at which point the cycle of displacement ends and the boarding can then continue to the end person with everyone else including the last man getting his own seat.

b) The last person's seat is selected by a displaced person then there is no more displacement until the last man tries to sit down and will find his seat already occupied.

The Probability of each event happening is $\frac{1}{2}$ as a displaced person has an equal probability of occupying first or the last person's seat. We can think of the first passenger as a displaced person since he randomly occupies a seat.

Problem 7

Please refer to this video for reference. The codes have been uploaded as a separate file.

Problem 8

Please refer to video Markov Chains and random walks and this video for Pagerank Algorithm