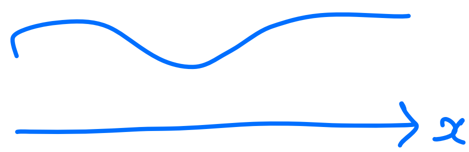
Lecture 3

(8 August 2025)

Continuity of Probability

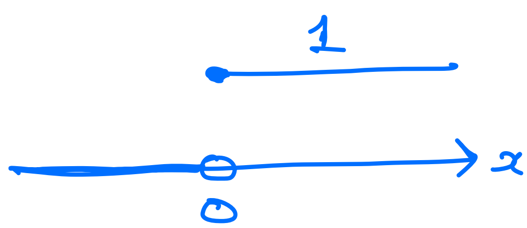
We first recall the notion of continuity of a real function.

$f(x)$



continuous

$u(x)$



discontinuous at $x=0$

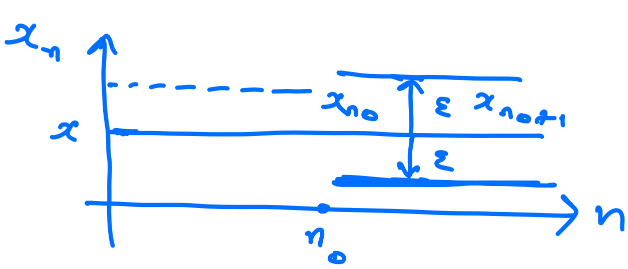
A function $f: \mathbb{R} \to \mathbb{R}$ is continuous if

$$x_n \to x \implies f(x_n) \to f(x) \text{ as } n \to \infty.$$

That is
$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

$x_n \to x$ as $n \to \infty$ if

for every $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ $\forall n \geq n_0$.



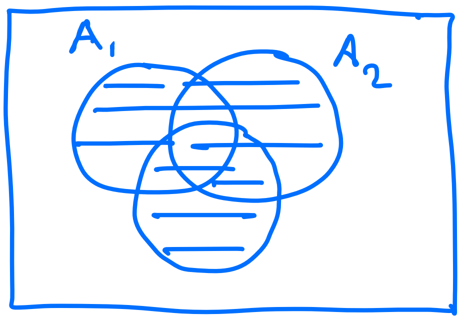
We have a similar notion of continuity for the set function probability law.

Theorem (Continuity of Probability).

For a sequence of events $A_1, A_2, \ldots$, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right).$$

Proof.



$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

Let $B_1 = A_1$,

$$B_i = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j\right).$$

Claim 1. $B_1, B_2, \ldots$ are disjoint, i.e., $B_i \cap B_{i'} = \phi$, $i \neq i'$.

Claim 2. $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$, $n \in \mathbb{N}$, and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i.$$

Proof of claim 1. WLOG let $i > i'$.

Let $x \in B_i \implies x \in A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j \right)$

$\implies x \notin A_{i'}$

$\implies x \notin B_{i'}$.

Let $x \in B_{i'} \implies x \in A_{i'} \setminus \left(\bigcup_{j=1}^{i'} A_j \right)$

$\implies x \in A_{i'}$

$\implies x \notin A_i \setminus \left(\bigcup_{j=1}^{i} A_j \right)$

(as $i' < i$)

$\implies x \notin B_i$.

Proof of claim 2. We prove the first equality by induction.

$A_1 = B_1$.

Assume $\bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{k} B_i$.

Let $C_k = \bigcup_{i=1}^{k} A_i$.

$$\bigcup_{i=1}^{k+1} A_i = C_{k+1} = C_k \cup A_{k+1}$$

$$= C_k \cup (A_{k+1} \setminus C_k)$$

$$= C_k \cup \left(A_{k+1} \setminus \bigcup_{j=1}^{k} A_i\right)$$

$$= C_k \cup B_{k+1}$$

$$= \bigcup_{i=1}^{k} B_k \cup B_{k+1}$$

$$= \bigcup_{i=1}^{k+1} B_i .$$

$\therefore \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ for all $n \in \mathbb{N}$.

To show $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, let

$x \in \bigcup_{i=1}^{\infty} A_i \Longrightarrow x \in A_n$ for some $n \in \mathbb{N}$

$\Longrightarrow x \in \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$

$\Longrightarrow x \in B_m$ for some $m \in \mathbb{N}$

$$\Rightarrow x \in \bigcup_{i=1}^{\infty} B_i$$

Let $x \in \bigcup_{i=1}^{\infty} B_i \Rightarrow x \in B_n$ for some $n \in \mathbb{N}$

$$\Rightarrow x \in \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

$$\Rightarrow x \in A_m \text{ for some } m \in \mathbb{N}$$

$$\Rightarrow x \in \bigcup_{i=1}^{\infty} A_i$$

Consider

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) \quad [\text{by claim 2}]$$

$$= \sum_{i=1}^{\infty} P(B_i) \quad [\text{by additivity}]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \quad [\text{by claim 2}].$$

Corollary.

1) If $A_1, A_2, \ldots$ is a sequence of increasing nested events, i.e., $A_i \subseteq A_{i+1}$, $\forall i \in \mathbb{N}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} P(A_n).$$

2) If $B_1, B_2, \ldots$ is a sequence of decreasing nested events, i.e., $B_i \supseteq B_{i+1}$, $\forall i \in \mathbb{N}$, then

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n \to \infty} P(B_n).$$

Exercise. Prove the above corollary from the continuity of probability.

Union Bound for Infinite number of events

We first prove the union bound for finite no. of events: $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Longrightarrow P(A \cup B) \leq P(A) + P(B).$$

As induction hypothesis suppose

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i).$$

Consider $P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1})$

[by base case $k=2$]

$$\leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$$

[by induction hypothesis]

$$= \sum_{i=1}^{k+1} P(A_i)$$

Now consider

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \quad \text{[by continuity]}$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i)$$

[since if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent sequences and $x_n \leq y_n$ $\forall n\in\mathbb{N}$ then
$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.]

$$= \sum_{i=1}^{\infty} P(A_i).$$

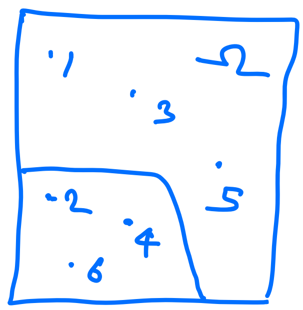
$$\therefore P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Conditional Probability

Roll a die, uniform probability on $\Omega = \{1, 2, \dots, 6\}$.

$P(\text{outcome is } 1 \mid \text{outcome is even})$

$$= 0$$



$$P(A \mid B) = 0 \quad \text{if} \quad A \cap B = \phi$$

$$\neq 0 \quad \text{if} \quad A \cap B \neq \phi$$

$P(\text{outcome is } 2 \mid \text{outcome is even})$

$$= 1/3$$

$P(\text{outcome} \in \{2, 3\} \mid \text{outcome is even})$

$$= 1/3 = \frac{P(A \cap B)}{P(B)}$$

$$P(A|B) \propto P(A \cap B)$$

(for a fixed B)

$$\Rightarrow P(A|B) = k\, P(A \cap B)$$

$$\Rightarrow 1 = P(B|B) = k\, P(B) \Rightarrow k = \frac{1}{P(B)}.$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

<u>Definition</u>. The conditional probability of an event A given an event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$

<u>Exercise</u>. Show that $P_B(A) \triangleq P(A|B)$, $P(B) > 0$ is a probability law w.r.t. Ω & $\mathcal{F}$, i.e., it satisfies all the three axioms, given that $(\Omega, \mathcal{F}, P)$ is a probability space.

<u>Example</u>. Roll a pair of fair dice.

(i) What is the probability that two faces have the same outcome?

$$P(E) = \frac{6}{36} = 1/6.$$

(ii) What is the probability of the same event given that the sum is not greater than 3.

$$B = \{ \text{sum} \le 3 \} = \{ (1\,1)\,(1\,2)\,(2\,3) \}.$$

$$P(E \mid B) = \frac{1}{3}.$$

Independence.

Two events A & B are called independent events if $P(A \cap B) = P(A)P(B)$.

Interpretation: $P(A \mid B) = P(A)$.

Example. Two fair dice are rolled.

$A = \{\text{sum is } 7\}$ $B = \{1^{st} \text{ roll is } 1\}$.

A & B are independent, $P(A \cap B) = P(A)\,P(B)$.

Intuition: For the sum to be 7 there is always exactly one possible outcome irrespective of whether given 1^{st} roll is 1 or 2 or --- 6.

$C = \{ \text{sum is } 8 \}$.

C & B are not independent, $P(B \cap C) \neq P(B)P(C)$.

Intuition: