PRP A1 - Q1, 2

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1 Introduction

Question 1
The power set of the 8 disjoint set as shown in Figure 1 will be the smallest σ field that can be created. So. it will have 2^8 elements in it.

Question 2
To show that $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$, let's start by applying some basic principles from probability theory, specifically the inclusion-exclusion principle.

Step 1: Apply the Inclusion-Exclusion Principle The inclusion-exclusion principle states:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Given that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$, we can plug in these values:

$$P(A \cup B) = \frac{1}{2} + \frac{2}{3} - P(A \cap B)$$

Step 2: Apply the Boundaries for Probabilities Since $P(A \cup B)$ must be between 0 and 1, we have:

$$0 \leq \frac{1}{2} + \frac{2}{3} - P(A \cap B) \leq 1$$

Simplifying this, we find:

$$0 \leq \frac{7}{6} - P(A \cap B) \leq 1$$

This inequality splits into two inequalities: 1. $\frac{7}{6} - P(A \cap B) \geq 0 \Rightarrow P(A \cap B) \leq \frac{7}{6}$, which is trivial as probabilities cannot exceed 1. 2. $\frac{7}{6} - P(A \cap B) \leq 1 \Rightarrow P(A \cap B) \geq \frac{1}{6}$

Thus, we have:

$$\frac{1}{6} \leq P(A \cap B)$$

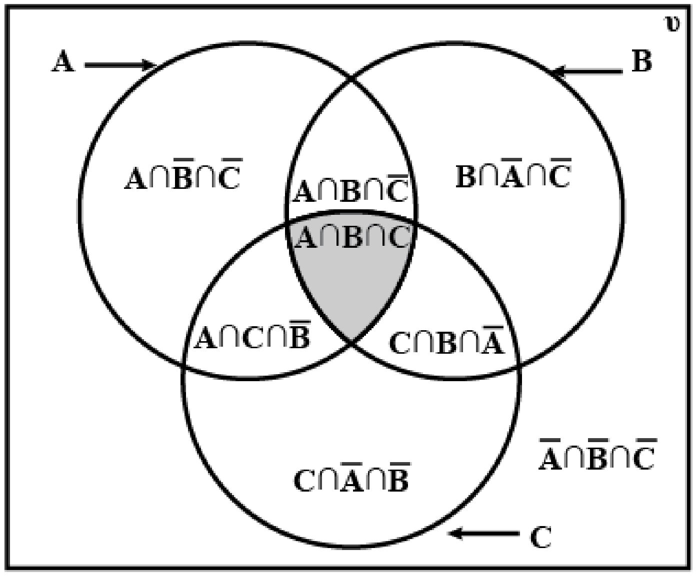

Figure 1: Three sets A, B, C

Step 3: Establish the Upper Bound Also, $P(A \cap B)$ cannot be larger than $P(A)$, so:

$$P(A \cap B) \leq P(A) = \frac{1}{2}$$

Conclusion Combining the results from Step 2 and Step 3, we have:

$$\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$$

Step 4: Provide Examples

1. Lower Bound Example: $P(A \cap B) = \frac{1}{6}$

Suppose that A and B are independent events. Then:

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

To achieve the lower bound, let's consider a scenario where: - $P(A) = \frac{1}{2}$ - $P(B) = \frac{2}{3}$ - Let $P(A \cap B) = \frac{1}{6}$

This scenario can occur if B is only partially within A. For example, if the probability space consists of 6 equally likely outcomes, where: - 3 outcomes are in A, and 4 outcomes are in B with 1 outcome common to both A and B.

Here, $P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{4}{6} = \frac{2}{3}$, and $P(A \cap B) = \frac{1}{6}$.

2. Upper Bound Example: $P(A \cap B) = \frac{1}{2}$

This occurs when $A \subseteq B$, meaning A is entirely contained within B.

For instance, let B be a subset of A, and $P(A) = \frac{1}{2}$. If $A = B$, then:

$$P(A \cap B) = P(A) = \frac{1}{2}$$

This demonstrates that the bounds $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$ can indeed be attained.

③ $\sum_{i=1}^{n} P(A_i) \geq P(\bigcup_{i=1}^{n} A_i)$

where $P(\bigcup_{i=1}^{n} A_i) = 1$ since atleast one of the events A_i [$i \in \{1,2,\dots n\}$] is certain to occur

Thus, $\sum_{i=1}^{n} P(A_i) = np \geq 1$

$\Rightarrow p \geq \frac{1}{n}$

Since atmost two events can occur,

$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i,j\, i\neq j} P(A_i \cap A_j)$

$1 = np - {}^{n}C_2\, q$

$\Rightarrow \frac{n(n-1)}{2} q = np - 1$

Since $p \leq 1$, $np \leq n \Rightarrow np - 1 \leq n - 1$

$\frac{n(n-1)}{2} q \leq n - 1$

$\Rightarrow q \leq \frac{2}{n}$

④ (a) Given: $\lim_{n\to\infty} a_n = a$

To prove: $\lim_{n\to\infty} |a_n - a| = 0$

Proof: $\lim_{n\to\infty} a_n = a$

By definition of convergence of sequence,

$\forall \epsilon > 0, \exists n_0$ s.t. $\forall n > n_0$,

$$|a_n - a| < \epsilon$$

$$\Rightarrow ||a_n - a| - 0| < \epsilon$$

$$\Rightarrow \lim_{n \to \infty} |a_n - a| = 0$$

Given: $\lim_{n \to \infty} |a_n - a| = 0$

To prove: $\lim_{n \to \infty} a_n = a$

Proof: $\lim_{n \to \infty} |a_n - a| = 0$

$\forall \epsilon > 0, \exists n_0$ s.t. $\forall n > n_0$,

$$||a_n - a| - 0| < \epsilon$$

$$\Rightarrow |a_n - a| < \epsilon$$

$$\Rightarrow \lim_{n \to \infty} a_n = a$$

(b) Given: $\lim_{n \to \infty} P(A_n \Delta A) = 0$

To prove: $\lim_{n \to \infty} P(A_n) = P(A)$

Proof: $\forall \epsilon > 0, \exists n_0$ s.t. $\forall n > n_0$,

$$|P(A_n \Delta A) - 0| < \epsilon$$

$$\Rightarrow P(A_n \Delta A) < \epsilon$$

Now, $|P(A_n) - P(A)| < P(A \cup A_n) - P(A)$

$$= P(A_n \setminus A) < P(A_n \Delta A)$$

$\forall \epsilon > 0, \exists n_0$ s.t. $\forall n > n_0$,

$$|P(A_n) - P(A)| < P(A_n \Delta A) < \epsilon$$

$$\Rightarrow \lim_{n \to \infty} P(A_n) = P(A)$$

(# 5) $p(H) = p$

$p(T) = 1 - p$

Event $E_n \rightarrow$ Even number of heads in n tosses.

(a) To prove $\Rightarrow$ $P(E_n \mid n^{th} \text{ toss} = \text{Tails}) = p_{n-1}$

Proof: LHS: $\dfrac{P(E_n \cap n^{th} \text{ toss} = \text{Tails})}{P(n^{th} \text{ toss} = \text{Tails})}$

$$E_n = (\bar{E}_{n-1} \cap n^{th} \text{ toss} = \text{Head}) \cup (E_{n-1} \cap n^{th} \text{ toss} = \text{Tails})$$

$$\Rightarrow (E_n \cap n^{th} \text{ toss} = \text{Tails}) = \cancelto{\varphi}{(\bar{E}_{n-1} \cap n^{th} \text{ toss} = \text{Head} \cap n^{th} \text{ toss} = \text{Tails})} \cup (E_{n-1} \cap n^{th} \text{ toss} = \text{Tails})$$

$$\Rightarrow \frac{P(E_n \cap n^{th} \text{ toss} = \text{Tails})}{P(n^{th} \text{ toss} = \text{Tails})} = \frac{P(E_{n-1} \cap n^{th} \text{ toss} = \text{Tails})}{P(n^{th} \text{ toss} = \text{Tails})}$$

E_{n-1} & n^{th} toss are independent event

$$\Rightarrow \frac{P(E_{n-1}) \cdot \cancel{P(n^{th} \text{ Toss} = \text{Tails})}}{\cancel{P(n^{th} \text{ toss} = \text{Tails})}} = p_{n-1}$$

Hence, Proved.

(#6)

Argue $p_0 = 1$

0 toss $\rightarrow$ 0 Heads certain $\Rightarrow$ Even heads certain $\Rightarrow$ $p_0 = 1$

To show : $p_n = p(1 - p_{n-1}) + (1-p)\, p_{n-1}$

As in last part $\rightarrow$ $E_n = (\bar{E}_{n-1} \cap n^{th} = H) \cup (E_{n-1} \cap n^{th} = T)$

$$P(E_n) = P\left[(\bar{E}_{n-1} \cap n^{th} = H) \cup (E_{n-1} \cap n^{th} = T)\right]$$

$$= P(\bar{E}_{n-1} \cap n^{th} = H) + P(E_{n-1} \cap n^{th} = T) - \cancelto{0}{P(E_{n-1} \cap \bar{E}_{n-1} \cap n^{th} = H \cap n^{th} = T)}$$

$$= P(\bar{E}_{n-1})\, P(n^{th} = H) + P(E_{n-1})\, P(n^{th} = T)$$

$$= (1 - P(E_{n-1}))\, p + P(E_{n-1})\, (1-p)$$

$$= (1 - p_{n-1})\, p + (1-p)\, p_{n-1} = RHS$$

Hence, shown.

c) Solve the Recurrance relation

$$p_n = p(1-p_{n-1}) + (1-p)p_{n-1}$$

$$= p_{n-1}(1-2p) + p$$

$$p_n = [(p_{n-2})(1-2p) + p][1-2p] + p$$

$$= (p_{n-2})(1-2p)^2 + \underline{p(1-2p)} + p$$

$$= (p_{n-3})(1-2p)^3 + p(1-2p)^2 + p(1-2p) + p$$

$$\Rightarrow p_n = p\sum_{i=1}^{n}(1-2p)^{i-1} + p_0 \cdot (1-2p)^n$$

$$= p\sum_{i=1}^{n}(1-2p)^{i-1} + (1-2p)^n$$

$$= (1-2p)^n + p\left[\frac{1-(1-2p)^n}{2p}\right] = \frac{1}{2}\left[(1-2p)^n + 1\right]$$

#6 To prove :

$$\sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i) - \sum_{i\in[1:n]:\, i\ne k} P(A_i \cap A_k)$$

① To show:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j)$$

LHS RHS

$n=2$ case : $P(A_1 \cup A_2) = P(A_1) - P(A_1\cap A_2) + P(A_2)$ $n=2$ works. , $n=1$ case LHS $P(A_1)$ = RHS $P(A_1)$

Assume: $n=k$ is also valid for this inequality

To show: $n=k+1$ is valid

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$

let $B = \bigcup_{i=1}^{k} A_i$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P(B \cup A_{k+1}) \ge P(B) + P(A_{k+1}) - P(B \cap A_{k+1})$$

$$P\left(\bigcup_{i=1}^{K+1} A_i\right) \geq P(B) + P(A_{K+1}) - P\left(\left(\bigcup_{i=1}^{K} A_i\right) \cap A_{K+1}\right)$$

$$\geq P(B) + P(A_{K+1}) - P\left[\bigcup_{i=1}^{K} (A_i \cap A_{K+1})\right]$$

$$P(B) \geq \sum_{i=1}^{K} P(A_i) - \sum_{1 \leq i < j \leq K} P(A_i \cap A_j)$$

$$P\left(\bigcup_{i=1}^{K+1} A_i\right) \geq \sum_{i=1}^{K} P(A_i) - \sum_{1 \leq i < j \leq K} P(A_i \cap A_j) + P(A_{K+1})$$

$$- P\left[\bigcup_{i=1}^{K} (A_i \cap A_{K+1})\right]$$

$$\geq \underline{\sum_{i=1}^{K+1} P(A_i)} - \sum_{1 \leq i < j \leq K} P(A_i \cap A_j) - \underline{P\left[\bigcup_{i=1}^{K} (A_i \cap A_{K+1})\right]}$$

By Probability law : $\underline{P\left(\bigcup_{i=1}^{K} (A_i \cap A_{K+1})\right) \leq \sum_{i=1}^{K} P(A_i \cap A_{K+1})}$ (equality when disjoint sets)

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{k+1} A_i\right) \geq \sum_{i=1}^{k+1} P(A_i) - \sum_{1 \le i < j \le k+1} P(A_i \cap A_j)$$

Left side inequality proved using PMI.

(2) 2^{nd} Inequality

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i \in [1:n] : i \neq j} P(A_i \cap A_j)$$

	LHS		RHS
n=1 case	$P(A_i)$	$=$	$P(A_i)$

n=2 case $\quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leftarrow$ [you can prove this easily]

Assume $n=k$ case

For $n=k+1$ case

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\underbrace{\bigcup_{i=1}^{k} A_i}_{B} \cup A_{k+1}\right) = P(B \cup A_{k+1})$$

$$\leq P(B) + P(A_{k+1}) - P(B \cap A_{k+1})$$

$$\leq P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$\leq \sum_{i=1}^{k} P(A_i) - \sum_{\substack{i \in [1,k] : i \neq r \\ r \in [1,k]}} P(A_i \cap A_r) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right) \leq \sum_{i=1}^{k} P(A_i \cap A_{k+1})$$

$$\leq \sum_{i=1}^{k+1} P(A_i) - \sum_{i \in [1,k+1]} P(A_i \cap A_r)$$

, Hence proved

Q7:

Solution 1.

You are given two envelopes with different amounts of money, `a` and `b` (Assume b>a WLOG)

Event Definitions:

- **`W`:** The event that you end up with the envelope containing the larger amount `b`.
- **`A`:** The event that `X < a`.
  - **`A1`:** `A` occurs, and you initially chose the envelope containing `b`.
  - **`A2`:** `A` occurs, and you initially chose the envelope containing `a`.
- **`B`:** The event that `a < X < b`.
  - **`B1`:** `B` occurs, and you initially chose the envelope containing `b`.
  - **`B2`:** `B` occurs, and you initially chose the envelope containing `a`.
- **`C`:** The event that `X > b`.
  - **`C1`:** `C` occurs, and you initially chose the envelope containing `b`.
  - **`C2`:** `C` occurs, and you initially chose the envelope containing `a`.

Calculating `P(W)` for Each Case:

1. **Event `A`: `X < a`**
   - If `A1` occurs (you initially chose `b`), you don't switch and keep `b`, so you win: `P(W | A1) = 1`.
   - If `A2` occurs (you initially chose `a`), you don't switch and keep `a`, so you lose: `P(W | A2) = 0`.
   - Therefore, the probability of winning given `A` is:

```
P(W | A) = 1/2 * (P(W | A1) + P(W | A2)) = 1/2 * (1 +
0) = 1/2
```

2. **Event `B`: `a < X < b`**
    - If `B1` occurs (you initially chose `b`), you don't switch and keep `b`, so you win: `P(W | B1) = 1`.
    - If `B2` occurs (you initially chose `a`), you switch to `b`, so you win: `P(W | B2) = 1`.
    - Therefore, the probability of winning given `B` is:

    ```
    P(W | B) = 1/2 * (P(W | B1) + P(W | B2)) = 1/2 * (1 +
    1) = 1
    ```

3. **Event `C`: `X > b`**
    - If `C1` occurs (you initially chose `b`), you don't switch and keep `b`, so you win: `P(W | C1) = 1`.
    - If `C2` occurs (you initially chose `a`), you switch to `b`, so you win: `P(W | C2) = 0`.
    - Therefore, the probability of winning given `C` is:

    ```
    P(W | C) = 1/2 * (P(W | C1) + P(W | C2)) = 1/2 * (1 +
    0) = 1/2
    ```

Total Probability Calculation:

Using the Law of Total Probability:

$P(W) = P(A) * P(W | A) + P(B) * P(W | B) + P(C) * P(W | C)$
$P(W) = P(A) * (1/2) + P(B) * 1 + P(C) * (1/2)$
$P(W) = 1/2 * (P(A) + P(C)) + P(B)$
$P(W) = 1/2 + 1/2 * P(B)$

Since `P(B) > 0` (since `a < X < b` is possible), `P(W)` is greater than `1/2`, confirming that the strategy increases your chances of ending up with the larger amount `b`.

Conclusion:

Your friend is correct—the strategy increases the probability of selecting the envelope with the larger amount `b` to greater than 50%.

Q8

Solution:

To solve this, we use Bayes' Theorem. Let's define the events:

- A_1: You play against a type I player.
- A_2: You play against a type II player.
- A_3: You play against a type III player.
- B: You win the game.

We want to find the probability that you played against a type II player given that you win the game, i.e., $P(A_2 \mid B)$.

Step 1: Calculate the Prior Probabilities

The probability of playing against each type of player:

- $P(A_1) = 1/2$ (since half the players are type I)
- $P(A_2) = 1/4$ (since a quarter of the players are type II)
- $P(A_3) = 1/4$ (since a quarter of the players are type III)

Step 2: Calculate the Conditional Probabilities

The probability of winning given that you play against each type:

- $P(B \mid A_1) = 0.3$
- $P(B \mid A_2) = 0.4$

- $P(B | A_3) = 0.5$

Step 3: Apply Bayes' Theorem

Bayes' Theorem is given by:

$P(A_2 | B) = [P(B | A_2) * P(A_2)] / P(B)$

Where P(B) is the total probability of winning, calculated as:

$P(B) = P(B | A_1) * P(A_1) + P(B | A_2) * P(A_2) + P(B | A_3) * P(A_3)$

Substitute the values:

$P(B) = (0.3 * 1/2) + (0.4 * 1/4) + (0.5 * 1/4)$

$P(B) = 0.15 + 0.1 + 0.125 = 0.375$

Step 4: Calculate the Posterior Probability

Now, calculate $P(A_2 | B)$:

$P(A_2 | B) = (0.4 * 1/4) / 0.375 = 0.1 / 0.375 \approx 0.267$

Final Answer:

The probability that you played against a type II player given that you won the game is approximately 0.267.

Problem 9

(a) If A is independent of B and A is independent of C, then A is independent of $B \cup C$

This statement is False

Example :- Let us toss two coins together

Event A: the first coin is head.

Event B: the second coin is head

Event C: the no. of head is exactly one

$$A = \{H, T\}, \{H, H\}$$
$$B = \{T, H\}, \{H, H\}$$
$$C = \{T, H\}, \{H, T\}$$

$P(A) = 1/2 \qquad P(B) = 1/2 \qquad P(C) = 1/2$

$P(A \cap B) = 1/4$

$P(A) . P(B) = 1/2 . 1/2 = 1/4$

$\therefore$ A and B are independent

$P(A \cap C) = 1/4$

$P(A) . P(C) = 1/2 . 1/2 = 1/4$

$\therefore$ A & C are independent

$B \cup C = \{T, H\} \{H, H\} \{H, T\}$

$P(B \cup C) = 3/4$

$P(A \cap (B \cup C)) = 1/2$

$P(A) \cdot P(B \cup C) = 1/4 \cdot 3/4$

$P(A \cap (B \cup C)) \neq P(A)\, P(B \cup C)$

therefore A is not independent of $(B \cup C)$

(b) LHS.

$$P(A \mid A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)}$$

$$= \frac{P(A \cap B) + P(A \setminus B)}{P(B) + P(A \setminus B)}$$

Let $P(A \cap B) = x$ $\quad P(A \setminus B) = z$

$P(B) = y$

LHS

$$P(A \mid A \cup B) = \frac{x + z}{y + z}$$

RHS

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{x}{y}$$

To prove

$$P(A \mid A \cup B) \geq P(A \mid B)$$

$$\frac{x+z}{y+z} \geq \frac{x}{y}$$

Since probabilities are always positive we can cross multiply

$$\cancel{xy} + zy \geq \cancel{xy} + zx$$

$$y \geq x$$

$$P(B) \geq P(A \cap B) \Rightarrow \text{this always holds true}$$

$\therefore \quad P(A \mid A \cup B) \geq P(A \mid B)$ is always true.

Problem 10

$C = \{A_1 A_2 \ldots A_n\}$ is a collection of mutually independent event

$\therefore$ Let Z be any subset of set $\{1, \ldots, n\}$.

$\therefore$ Let size of Z be k.

$$\therefore \quad P\left(\bigcap_{i=1}^{k} A_{z_i}\right) = \prod_{i=1}^{k} P(A_{z_i})$$

Let first prove :
if A_i & A_j are independent
then A_i & A_j^c are also independent
given $(i \neq j)$

$P(\cancel{A}$ $A_i \cap A_j^c = A_i \setminus A_j$.

$$P(A_i \cap A_j^c) = P(A_i) - P(A_i \cap A_j)$$
$$= P(A_i) - P(A_i) P(A_j)$$
$$= P(A_i) [1 - P(A_j)]$$
$$= P(A_i) \cancel{A_j} P(A_j^c)$$

therefore A_i & A_j^c are also independent

We need to prove
$(C \setminus A_i) \cup \{A_i^c\}$ is also a collection of ~~inde~~ mutually independent event
Let Z is be a ~~set~~ subset of set $\{1, \dots n\} \setminus \{i\}$

$\therefore$

$$P\left(\left(\bigcap_{j=1}^{k} A_{z_j}\right) \cap A_i^c\right) \cancel{=}$$

$$= \left(\prod_{j=1}^{k} P(A_{z_j})\right) . P(A_i^c)$$

A_i^c is independent of all A_{z_j} [proved above]

$\therefore$ $(C \setminus A_i) \cup \{A_i^c\}$ is also a collection of mutually independent events