Lecture 4

(12 August 2025)

Independence

Three events A_1, A_2 & A_3 are (mutually) independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3).$$

In general, events $E_1, E_2, ---, E_n$ are mutually independent if

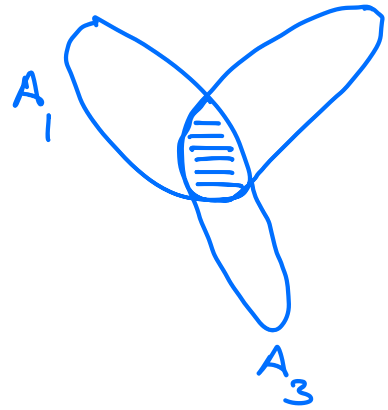
$$P\left(\bigcap_{i \in \mathcal{I}} A_i\right) = \prod_{i \in \mathcal{I}} P(A_i), \quad \mathcal{I} \subseteq [1:n].$$

$A_1, A_2, ---, A_n$ are pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j), \quad i \neq j.$$

Pairwise independence does not imply mutual independence.

Also $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ does not imply pairwise independence.



$P(A_1) = P(A_2) = P(A_3)$
$= 1/5$

$P(A_1 \cap A_2) = P(A_2 \cap A_3) = P(A_1 \cap A_3)$
$= p = P(A_1 \cap A_2 \cap A_3)$

$p = 1/25$: Pairwise independent but not mutually independent

$p = \frac{1}{125}$: $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ but not mutually independent.

A collection of events $A_1, A_2, \ldots, A_n$ is called a partition of Ω if $\bigcup_{i=1}^{n} A_i = \Omega$ and $A_i \cap A_j = \phi$ $i \neq j$. In other words $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive.

Total Probability Theorem

Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of Ω such that $P(A_i) > 0$ $\forall$ $i \in [1:n]$. Then, for any arbitrary event B,

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i)P(A_i) \quad \text{if } P(B_i) > 0 \;\; i \in [1:n].$$

Proof. $P(B) = P(B \cap \Omega)$

$$= P\left(B \cap \left(\bigcup_{i=1}^{n} A_i\right)\right)$$

$$= P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right)$$

$$= \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Bayes' Theorem

Let $\{A_1, A_2, ---, A_n\}$ be a partition of Ω such that $P(A_i) > 0$ $\forall i \in [1:n]$. Then for any arbitrary event B with $P(B) > 0$

$$P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{\sum_{j=1}^{n} P(B|A_j)\,P(A_j)}.$$

Proof. $$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(B|A_i)\,P(A_i)}{\sum_{i=1}^{n} P(B|A_i)\,P(A_i)}.$$

(by total probability theorem)

Example. A company producing electric relays has three manufacturing plants producing 50, 30 and 20 percent respectively.

Suppose that the probabilities that a relay manufactured by these plants is defective are 0.02, 0.05 and 0.01 respectively.

(i) If a relay is selected at random from the output of a company, what is the probability that it is defective?

$B = \{\text{random relay is defective}\}$

$A_i = \{\text{relay manufactured by plant } i\}$, $i \in [1:3]$.

$$P(B) = \sum_{i=1}^{3} P(B|A_i)\,P(A_i)$$

$$= 0.02 \times 0.5 + 0.05 \times 0.3 + 0.01 \times 0.2$$

$$= 0.027$$

(ii) If a relay selected at random is found to be defective, what is the probability that it was manufactured by plant 2?

$$P(A_2|B) = \frac{P(B|A_2)\,P(A_2)}{P(B)}$$

$$= \frac{0.05 \times 0.03}{0.027} = 0.556.$$

Multiplication Rule

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2 | A_1)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1 \cap A_2)\, P(A_3 | A_1 \cap A_2)$$

$$= P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2)$$

By induction we have

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} A_j\right) \quad \text{i.e.,}$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$= P(A_1)\, P(A_2 | A_1) \cdots P(A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Conditional Independence

The events A and B are conditionally independent given C with $P(C) > 0$ if

$$P(A \cap B | C) = P(A|C)\, P(B|C).$$

$$P(A|C)P(B|C) = P(A\cap B|C)$$

$$= \frac{P(A\cap B\cap C)}{P(C)}$$

$$= P(B|C)P(A|B\cap C)$$

$$\Rightarrow P(A|B\cap C) = P(A|C) \text{ if } P(B|C) > 0.$$

- The conditional independence of A and B given C neither implies nor is implied by the independence of A and B. The following examples illustrate this.

Example. Consider two independent fair coin tosses.

$A = \{1^{st} \text{ toss is a heads}\} = \{HT, HH\}$

$B = \{2^{nd} \text{ toss is a heads}\} = \{TH, HH\}$

$C = \{\text{the two tosses have different results}\}$

$= \{HT, TH\}$

$P(A\cap B) = P(A)P(B) = \frac{1}{4}$

$P(A\cap B|C) = 0, \quad P(A|C) = P(B|C) = \frac{1}{2}.$

$\therefore P(A\cap B|C) \neq P(A|C)P(B|C).$

Example. Consider two coins, a blue and a red one. We choose one of the two at random, each being chosen with probability $\frac{1}{2}$, and proceed with two independent tosses. The coins are biased: with the blue coin, the probability of heads in any given toss is 0,99, whereas for red coin it is 0,01.

Let B be the event that the blue coin was selected. Also let H_i be the event that the ith toss resulted in heads. Then

$$P(H_1 \cap H_2 \mid B) = P(H_1 \mid B)\, P(H_2 \mid B)$$

$$= 0{,}99 \times 0{,}99$$

(by the problem statement).

$$P(H_1 \cap H_2) \sim \tfrac{1}{2} \neq P(H_1)P(H_2) = \tfrac{1}{4}.$$

Exercise. If $P(C)=1$ is it true that

$$P(A \cap B) = P(A)P(B) \iff P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)\ ?$$

Review of Counting

Permutations: Given n distinct objects and let $k \le n$. We wish to count the number of different ways that we can pick k out of these n objects and arrange them in a sequence, i.e., no. of distinct k-object sequences

$$= n.(n-1)\cdots(n-k+1)$$

$$= {}^{n}P_{k} = \frac{n!}{(n-k)!}.$$

Combinations: Count the no. of k-element subsets of a given n-element set.

$$ {}^{n}C_{k} = \binom{n}{k} = {}^{n}P_{k} / k! = \frac{n!}{(n-k)!\,k!}.$$

Partitions: Consider n and $n_1, n_2, \ldots, n_r$ s.t. $n = n_1 + n_2 + \cdots + n_r$.

No. of partitions of n distinct elements into r disjoint subsets with ith subset containing exactly n_i elements

$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

$$= \frac{n!}{\cancel{(n-n_1)!}\, n_1!} \cdot \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\, n_2!} \cdots \frac{\cancel{(n-n_1-\cdots-n_{r-1})!}}{(n-n_1-\cdots-n_r)!\, n_r!}$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_r!}.$$