Lecture 6

(22 August 2025)

Theorem.

(a) If $x < y$ then $F_X(x) \leq F_X(y)$.

(b) $\lim_{x \to -\infty} F_X(x) = 0$ $\lim_{x \to \infty} F_X(x) = 1$.

(c) $F_X(x)$ is right continuous that is

$$\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x).$$

(d) $P(X > x) = 1 - F_X(x)$.

(e) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

(f) $P(X = x) = F_X(x) - \lim_{\varepsilon \to 0^+} F_X(x - \varepsilon)$.

Proof. (a) $F_X(x) = P(X \leq x)$

$$= P(X \leq y) \quad (\because x < y)$$

$$= F_X(y).$$

(b) Let $A_n = \{X \le n\}$

$A_1 \subseteq A_2 \subseteq \cdots$

$$\lim_{n\to\infty} F_X(n) = \lim_{n\to\infty} P(X \le n)$$

$$= P\left(\bigcup_{n=1}^{\infty} \{X \le n\}\right)$$

$$= P(\Omega) = 1.$$

$B_n = \{X \le -n\}$

$B_1 \supseteq B_2 \supseteq \cdots$

$$\lim_{n\to -\infty} F_X(n) = \lim_{n\to -\infty} P(X \le n)$$

$$= \lim_{n\to\infty} P(X \le -n)$$

$$= P\left(\bigcap_{n=1}^{\infty} \{X \le -n\}\right)$$

$$\overset{(a)}{=} P(\phi)$$

$$= 0.$$

[The proof of (a) is exactly along the same lines as a proof given in Lecture 1 in the discussion of mathematical induction.]

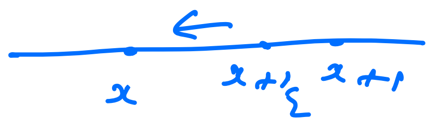
(c) $\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon)$

$$= \lim_{n\to\infty} F_X(x + 1/n)$$

$$= \lim_{n\to\infty} P(\underbrace{X \le x + 1/n}_{B_n})$$

$B_1 \supseteq B_2 \supseteq \cdots$

$$= P\left(\bigcap_{n=1}^{\infty} \{X \le x + \tfrac{1}{n}\}\right)$$



$$= P(X \le x) = F_X(x).$$

(d) $P(X > x) = 1 - P(X \le x) = 1 - F_X(x)$

(e) $P(x_1 < X \le x_2)$

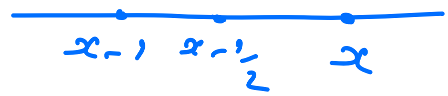
$$= P(\{X \le x_2\} \setminus \{X \le x_1\})$$

$$= P(X \le x_2) - P(X \le x_1)$$

$$= F_X(x_2) - F_X(x_1).$$

(f) $\lim_{\varepsilon \to 0^+} F_X(x-\varepsilon)$

$$= \lim_{n \to \infty} P(\underbrace{X \le x - \tfrac{1}{n}}_{A_n})$$



$$A_1 \subseteq A_2 \subseteq \cdots$$

$$= P\left(\bigcup_{n=1}^{\infty} \{X \le x - \tfrac{1}{n}\}\right)$$

$$= P(X < x).$$

$$P(X \le x) - P(X < x) = P(X = x).$$

- Actually the properties (a) (b) and (c) completely characterize cumulative distributive functions. That is F is the CDF of some random variable if and only if it satisfies (a) (b) and (c).

<u>Exercise</u>. Given a function F that satisfies (a) (b) and (c) construct a sample space Ω event space $\mathcal{F}$ probability law and random variable s.t. its CDF is equal to F.

Discrete Random Variable

A random variable X is called discrete if it takes values in some countable subset $\{x_1, x_2, \ldots\}$ of $\mathbb{R}$.

A discrete random variable has an associated probability mass function (pmf) $p_X : X \to [0,1]$ given by

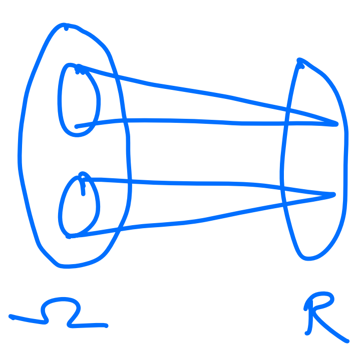
$$p_X(x) = P(X = x)$$
$$= P(\{\omega \in \Omega : X(\omega) = x\}).$$

$$F_X(x) = P(X \leq x)$$
$$= \sum_{i : x_i \leq x} p_X(x_i).$$

Lemma. Let X be a discrete random variable and it takes values $x_1, x_2, \ldots$. Then

$$\sum_{i \in \mathbb{N}} p_X(x_i) = 1.$$

Proof.



$$\sum_{i=1}^{\infty} p_X(x_i) = \sum_{i=1}^{\infty} P(X=x_i)$$

$$= P\left(\bigcup_{i=1}^{\infty} (X=x_i)\right) \quad \text{[by additivity]}$$

$$= P(\Omega)$$

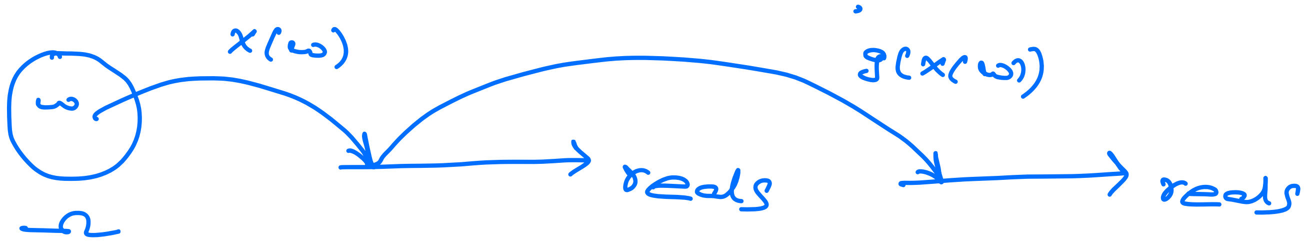
$$= 1.$$

Functions of Random Variable

Let $X:\Omega \to \mathbb{R}$ be a random variable.
Consider a real function $g:\mathbb{R}\to\mathbb{R}$,

$Y = g(X)$, i.e., $Y(\omega) = g(X(\omega))$, $\forall\, \omega \in \Omega$.

Is Y a random variable?



$$\{\omega : Y(\omega) \le y\} = \{\omega : g(X(\omega)) \le y\}$$

$$= \{\omega : X(\omega) \in B \text{ for some } B \in \mathcal{B}\}.$$

where $\mathcal{B}$ is is Borel σ-algebra - the smallest σ-field that contains the sets $(-\infty, x]$, $\forall x \in \mathbb{R}$.

$$\mathcal{B} = \{(-\infty, x], (-\infty, x), (x_1, x_2), [x_1, x_2], \ldots\}.$$

Since X is a random variable,

$\{\omega : X(\omega) \in B\} \in \mathcal{F}$ as B can be expressed as a countable union of sets of the form $(-\infty, x]$ and their complements.

$\therefore Y$ is also a random variable.

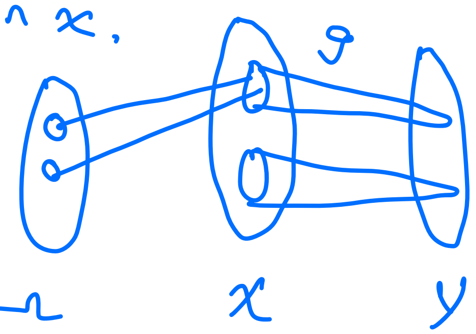
<u>Lemma</u>. Let X be a discrete random variable with PMF p_X and $Y = g(X)$. Then

$$p_Y(y) = \sum_{\substack{x \in \mathcal{X} : \\ g(x) = y}} p_X(x),$$

where X takes values in $\mathcal{X}$.

<u>Proof</u>. $p_Y(y)$

$= P(Y = y)$



$$= P(g(X) = y)$$

$$= P(\{\omega : g(X(\omega)) = y\})$$

$$= P(\{\omega : X(\omega) = x \text{ for some } x \text{ s.t. } g(x) = y\})$$

$$= P\Big(\bigcup_{\substack{x \in \mathcal{X}: \\ g(x) = y}} \{X(\omega) = x\}\Big)$$

$$= \sum_{\substack{x \in \mathcal{X}: \\ g(x) = y}} P(X = x) = \sum_{\substack{x \in \mathcal{X}: \\ g(x) = y}} p_X(x).$$

<u>Example</u>. Let $Y = |X|$ and

$$p_X(x) = \begin{cases} 1/9, & \text{if } x \in \{-4, -3, \dots, 0, 1, 2, 3, 4\} \\ 0, & \text{o.w.} \end{cases}$$

$$p_Y(y) = \begin{cases} p_X(y) + p_X(-y) = \frac{2}{9}, & \forall y \in [1:4] \\ 1/9, & y = 0. \end{cases}$$

Expectation

Suppose we have a collection of numbers $a_1, a_2, \ldots, a_n$ their average is a single number that describes the whole collection. Now consider a random variable X. We would like to define a similar notion.

Let X be a discrete random variable that takes values in $\mathcal{X}$. The expectation or expected value or mean of X is defined as

$$E[X] = \sum_{x \in \mathcal{X}} x \, P_X(x).$$

Interpretation. Consider a discrete RV that takes values $x_1, x_2, \ldots, x_m$. This random variable is a result of a random experiment. Suppose we repeat this experiment a very large number of times N, and that the trials are independent. Let x_i occurs N_i number of times for $i \in [1:m]$. We consider the average of all the observed values:

$$\frac{\sum_{i=1}^{m} N_i x_i}{N} = \sum_{i=1}^{m} \left(\frac{N_i}{N}\right) \cdot x_i$$

$$\approx \sum_{i=1}^{m} x_i \, p_X(x_i).$$

<u>Example</u>. $p_X(1) = p = 1 - p_X(0)$.

$$E[X] = p \cdot 1 + (1-p) \cdot 0 = p.$$

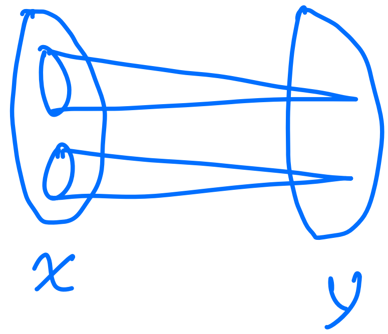
<u>Expectation of a function of RV</u>

Let X be a discrete random variable and $g: \mathbb{R} \to \mathbb{R}$ then $Y = g(X)$ is a RV. To calculate its expectation, it may appear that we first need to find its PMF p_Y and compute $\sum_y y\, p_Y(y)$. There is an easier way to do this without finding p_Y.

Law of the Unconscious Statistician

$$E[g(X)] = \sum_{x \in \mathcal{X}} g(x) p_X(x).$$

$Y = g(X)$



$$E[Y] = \sum_{y \in \mathcal{Y}} y\, p_Y(y)$$

$$= \sum_{y \in \mathcal{Y}} y \sum_{\substack{x \in \mathcal{X}: \\ g(x) = y}} p_X(x)$$

$$= \sum_{y \in \mathcal{Y}} \sum_{\substack{x \in \mathcal{X}: \\ g(x) = y}} y\, p_X(x)$$

$$= \sum_{y \in \mathcal{Y}} \sum_{\substack{x \in \mathcal{X}: \\ g(x) = y}} g(x)\, p_X(x)$$

$$= \sum_{x \in \mathcal{X}} g(x)\, p_X(x).$$

Example. Let $Y = |X|$ and

$$p_X(x) = \begin{cases} 1/9, & \text{if } x \in \{-4, -3, \ldots, 0, 1, 2, 3, 4\} \\ 0, & \text{o.w.} \end{cases}$$

$$E[Y] = E[|X|] = \frac{2}{9} \times (1+2+3+4) + 0$$
$$= \frac{20}{9}.$$

Moments: $E[X^n] = \sum_x x^n p_X(x)$.

Variance.

$$\text{Var}(X) = E[(X - E[X])^2],$$

measures the amount by which X tends to deviate from mean.

Let $\mu = E[X]$.

$$E[(X-\mu)^2] = \sum_x (x-\mu)^2 p_X(x)$$
$$= \sum_x (x^2 + \mu^2 - 2x\mu) p_X(x)$$
$$= \sum_x x^2 p_X(x) + \mu^2 - 2\mu \sum_x x p_X(x)$$
$$= \sum_x x^2 p_X(x) + \mu^2 - 2\mu^2$$
$$= \sum_x x^2 p_X(x) - \mu^2$$
$$= E[X^2] - E[X]^2.$$

Examples of Discrete RVs

Bernoulli Random Variable

Consider a coin toss, which comes up a head with probability p, and a tail with probability $1-p$.

$$X(H) = 1, \quad X(T) = 0.$$

$$p_X(1) = p, \quad p_X(0) = 1-p.$$

Exercise. $X \sim Be(p)$. Show that

(i) $E[X] = p$.

(ii) $Var(X) = p(1-p)$.

Binomial Random Variable

A coin is tossed n times independently. Let X be the total no. of heads in the n-toss sequence. $X \in [0:n]$.

$$P(\{H\}) = p = 1 - P(\{T\}).$$

$$p_X(k) = \binom{n}{k} p^k (1-p)^k, \quad k \in [0:n].$$

Exercise. $X \sim \text{Binom}(n, p)$. Show that

$E[X] = np$, $\quad Var(X) = np(1-p)$.

Geometric Random Variable

Toss a coin independently until we get a heads.

$$P(\{H\}) = p = 1 - P(\{T\}).$$

X = No. of coin tosses required to get a heads

$$P_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, 3, \ldots$$

Exercise. Let $X \sim \text{Geometric}(p)$. Show that

$E[X] = 1/p$, $\quad Var(X) = \dfrac{1-p}{p^2}$.

Poisson Random Variable

A Poisson random variable takes values $0, 1, 2, \ldots$ with PMF

$$P_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots, \text{ for } \lambda > 0.$$

In practice, a Poisson random variable can be viewed as a limiting case of a binomial random variable.

<u>Exercise</u>: Let $X \sim \text{Poisson}(\lambda)$. Show that

$$E[X] = \text{Var}[X] = \lambda.$$