Lecture 7

(26 August 2027)

Theorem. Consider a binomial RV Y with parameters n and p. As $n \to \infty$, while keeping $np = \lambda$ a constant, we have

$$\lim_{n\to\infty} P_Y(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Proof. $P_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$= \frac{n!}{(n-k)!\,k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \quad \frac{n!}{(n-k)!} \quad \frac{1}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \quad \underbrace{\frac{n(n-1)\ldots(n-k+1)}{n \quad n \quad - - \quad n}}_{\to 1} \quad \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{\to 1} \quad \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n}}_{\to e^{-\lambda}}$$

$$\therefore \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}.$$

Multiple Random Variables

Two random variables X and Y on the probability space $(\Omega, \mathcal{F}, P)$ are called jointly discrete if (X, Y) takes values in some countable subset of R^2.

$$X : \Omega \to R$$

$$Y : \Omega \to R$$

Let $\text{Range}(X) = \mathcal{X}$, $\text{Range}(Y) = \mathcal{Y}$.

The associated joint pmf is given by

$$p_{X,Y}(x, y) = P(\{\omega : X(\omega) = x, Y(\omega) = y\})$$

Example. Roll a pair of dice.

$$\Omega = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in [1:6]\}.$$

$$X(\omega) = \omega_1 + \omega_2, \quad Y(\omega) = \frac{\omega_1}{\omega_2}.$$

The pmfs of X and Y can be obtained from joint pmf using the formulas:

$$P_X(x) = \sum_{y \in \mathcal{Y}} p_{X,Y}(x, y), \quad p_Y(y) = \sum_{x \in \mathcal{X}} p_{X,Y}(x, y).$$

Reason: $p_X(x) = P(X=x)$

$$= P\left(X=x \cap \bigcup_{y \in \mathcal{Y}} \{Y=y\}\right)$$

$$= P\left(\bigcup_{y \in \mathcal{Y}} \{X=x\} \cap \{Y=y\}\right)$$

$$= P\left(\bigcup_{y \in \mathcal{Y}} \{X=x, Y=y\}\right)$$

$$= \sum_{y \in \mathcal{Y}} P(X=x, Y=y)$$

$$= \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y).$$

Functions of Multiple Random Variables

Consider two jointly discrete random variables X and Y.

Let $Z = g(X,Y)$, i.e.,

$Z(\omega) = g(X(\omega), Y(\omega))$.

Analogous to the way we argued $g(X)$ is a random variable, $Z = g(X,Y)$ is also a random variable.

$$P_Z(z) = \sum_{(x,y):\, g(x,y)=z} p_{XY}(x,y).$$

Exercise.

Prove that

$$E[g(X,Y)] = \sum_{x,y} g(x,y)\, p_{XY}(x,y).$$

Independence

Two discrete random variables X and Y are said to be independent if $p_{XY}(x,y) = p_X(x)\, p_Y(y) \;\forall\, x,y$, i.e., the events $\{X=x\}$ and $\{Y=y\}$ are independent for all x,y.

Example. Two random variables X and Y take values in $\{0,1\}$ and p_{XY} is their joint pmf. Suppose $p_{XY}(1,1) = p_X(1)\, p_Y(1)$.

Are X and Y independent?

$$p_{XY}(1,0) = p_X(1) - p_{XY}(1,1)$$

$$= p_X(1) - p_X(1)\, p_Y(1)$$

$$= p_X(1)\,(1-p_Y(1))$$

$$= p_X(1)\,p_Y(0).$$

Similarly $p_{XY}(x,y) = p_X(x)\,p_Y(y)\ \forall\, x, y.$

Yes X and Y are independent.

Exercise. Prove that the indicator random variables 1_A and 1_B are independent if and only if the events A and B are independent.

Theorem. If X and Y are independent discrete random variables then $E[XY] = E[X]\,E[Y].$

Proof. $$E[XY] = \sum_{x,y} xy\, p_{XY}(x,y)$$

$$= \sum_{x,y} xy\, p_X(x)\, p_Y(y)$$

$$= \sum_{x} x\, p_X(x) \sum_{y} y\, p_Y(y)$$

$$= E[X]\,E[Y].$$

__Exercise__. X and Y are independent implies that $g(X)$ and $h(Y)$ are independent.

__Independence__ of __several__ RVs

n random variables $\underline{X_1}, \underline{X_2}, \ldots, X_n$ are independent if

$$P_{\underline{X_1}, \underline{X_2}, \ldots, X_n}(\underline{x_1}, \underline{x_2}, \ldots, x_n) = \prod_{i=1}^{n} P_{X_i}(\underline{x_i}) \quad \forall\, \underline{x_1}, \underline{x_2}, \ldots, x_n.$$

__Exercise__. $\underline{X_1}, \underline{X_2}, \ldots, X_n$ are independent implies

$$P_{X_{\mathcal{I}}}(x_{\mathcal{I}}) = \prod_{i \in \mathcal{I}} P_{X}(\underline{x_i}) \quad \forall\, \mathcal{I} \subseteq [1:n].$$

$[x_{\mathcal{I}} = (x_i : i \in \mathcal{I})]$

Computation of Mean and Variance for Examples

$X \sim \text{Bernoulli}(p)$:

$$E[X] = p$$

$$E[X^2] = p \implies \text{var}(X) = p - p^2 = p(1-p)$$

$X \sim \text{Binomial}(n, p)$: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$E[X] = \sum_{k=0}^{n} k \, p_X(k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{(n-k)! \, k!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)! \, (k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k'=0}^{n-1} \frac{(n-1)!}{(n-k')! \, k'!} p^{k} (1-p)^{n-1-k'}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} = np.$$

$$E[X^2] = \sum_{k=0}^{n} k^2 p_X(k)$$

$$= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k'=0}^{n-1} (k'+1) \binom{n-1}{k'} p^{k'} (1-p)^{n-1-k'}$$

$$= np \left[1 + \sum_{k=0}^{n-1} k \binom{n-1}{k} p^k (1-p)^{n-1-k} \right]$$

$$= np \left[1 + (n-1)p \right].$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np(1-p).$$

$X \sim \text{Geometric}(p)$: $p_X(k) = (1-p)^{k-1} p$, $k = 1, 2, \ldots$

$$E[X] = \sum_{k=1}^{\infty} k\, p_X(k)$$

$$= \sum_{k=1}^{\infty} k (1-p)^{k-1} p$$

$$\left\{ \begin{array}{l} 1 + x + x^2 + \cdots = \dfrac{1}{1-x} \\ 1 + 2x + 3x^2 + \cdots = \dfrac{1}{(1-x)^2} \end{array} \right.$$

$$= p \sum_{k=1}^{\infty} k (1-p)^{k-1}$$

$$= p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p}.$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

$$\sum_{k=2}^{\infty} k(k-1) x^{k-2} = \frac{2}{(1-x)^3}$$

$$\sum_{k=2}^{\infty} (k^2-k) x^{k-2} = \frac{2}{(1-x)^3}$$

$$E[X^2] = \sum_{k=1}^{\infty} k^2 p_X(k)$$

$$= \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

$$\sum_{k=1}^{\infty} k^2 (1-p)^{k-1} = \sum_{k=1}^{\infty} (k^2-k)(1-p)^{k-1} + \sum_{k=1}^{\infty} k (1-p)^{k-1}$$

$$= \frac{2(1-p)}{p^3} + \frac{1}{p^2} = \frac{2-p}{p^3}$$

$$\Rightarrow E[X^2] = p \cdot \frac{2-p}{p^3} = \frac{2-p}{p^2}.$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

$X \sim \mathrm{Poisson}(\lambda)$: $p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$

$$E[X] = \sum_{k=0}^{\infty} k\, p_X(k)$$

$$= \sum_{k=0}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} = \lambda.$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 p_X(k)$$

$$= \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \lambda \sum_{k=1}^{\infty} k\, e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \sum_{k'=0}^{\infty} (k'+1)\, e^{-\lambda} \frac{\lambda^{k'}}{k'!}$$

$$= \lambda [\lambda + 1]$$

$$\Rightarrow \mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$= \lambda$$