Practice Problem Set 2

(MA6.102) Probability and Random Processes, Monsoon 2023

Problem 1. Let $X = [X_1, X_2, X_3]$ be a random vector such that all $X_i's$ follow $\mathcal{N}(0, \sigma^2)$ independently of each other. Find the distribution of $Y = \|X\|^2$. Then find $\mathbb{E}[Y]$ and var(Y).

Problem 2. Let X and Y be two random variables (discrete or continuous, doesn't matter). Show that,

$$\text{var}[X] = \mathbb{E}[\text{var}[X|Y]] + \text{var}[\mathbb{E}[X|Y]]$$

Problem 3. Derive the MGF of the sum of K independent binomial random variables with parameters p_k and N_k for $k = 1, \ldots, K$. Use the derived MGF to determine the mean and variance of the sum.

Problem 4. For any finite collection of discrete random variables $X_1, X_2, \cdots, X_n$ with finite expectations and any discrete random variable Y, prove that

$$\mathbb{E}[\sum_{i=1}^{n} X_i \mid Y = y] = \sum_{i=1}^{n} \mathbb{E}[X_i \mid Y = y]$$

Problem 5. Let $X_1, X_2, \cdots, X_n$ be independent random variables such that

$$Pr(X_i = 1 - p_i) = p_i \quad \text{and} \quad Pr(X_i = -p_i) = 1 - p_i.$$

Let $X = \sum_{i=1}^{n} X_i$. Prove

$$Pr(|X| \geq a) \leq 2e^{-2a^2/n}.$$

Hint: May use the following inequality

$$p_i e^{\lambda(1-p_i)} + (1 - p_i)e^{-\lambda p_i} \leq e^{\lambda^2/8}$$

Problem 6. Let $X_1, \cdots, X_n$ be independent and identically distributed random variables having distribution function F and density f. The quantity $M \triangleq [X_1 + X_n]/2$, defined to be the average of the smallest and largest values in $X_1, \cdots, X_n$, is called the *midrange* of the sequence. Show that its distribution function is

$$F_M(m) = n \int_{-\infty}^{m} [F(2m - x) - F(x)]^{n-1} f(x) dx$$

Problem 7. Bob is finished shopping for groceries in a departmental store. Ready to checkout, he finds that both checkout staff are in the midst of serving their last customer. The staff member on the left and the right takes an $\text{Expo}(\lambda_1)$ and $\text{Expo}(\lambda_2)$ time respectively to serve a customer. Let T_1 and T_2 be the time taken by the checkout staff on the left and right respectively to finish serving his or her current customer.

(a) If $\lambda_1 = \lambda_2$, is T_1/T_2 independent of $T_1 + T_2$?

(b) Find $P(T_1 < T_2)$ (do not assume $\lambda_1 = \lambda_2$ in this and the next part).

(c) Find the expected total amount of time that Bob spends in the departmental store (assuming that he leaves immediately after he is done being served).

Problem 8. A group of travellers find themselves lost in a cave. They come upon 3 tunnels $A,\ B,\ C$. Both tunnels A and B are closed loops that do not lead to an exit and in fact lead right back to the entrance of the 3 tunnels. Tunnel C is the tunnel which leads to the exit. If they go through tunnel A, then it takes 2 days to go through the tunnel. If they go through tunnel B, then it takes 1 day to go through the tunnel. If they go through tunnel C, then they immediately leave the cave. Suppose the travellers choose tunnels A, B and C with constant probability 0.3, 0.5, 0.2 every time.

(a) Suppose we record the travellers' choices as a sequence (e.g., $ABBA...C$). What is the probability that the pattern AAB appears in the sequence before any BAA appears? [Note: You should also count cases where AAB appears in the sequence and BAA does not.]

(b) What is the expected number of days that the travellers will be lost in the cave?

(c) What is the variance of days that the travellers will be lost in the cave? (Hint: To compute $Var(T)$ for a random variable T, compute $E[T^2]$ first and apply the definition $Var(T) = E[T^2] - E[T]^2$)

Problem 9. **(a)** An exponential distribution with parameter λ has the probability density:

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(i) Given some i.i.d. data $\{x_i\}_{i=1}^n \sim \text{Exp}(\lambda)$, derive the maximum likelihood estimate (MLE) $\hat{\lambda}_{MLE}$.

(ii) An estimator is unbiased if its expected value is equal to the true parameter it's estimating. Is this estimator biased?

(b) A gamma distribution with parameters α, β has a density function:

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma(t)$ is the gamma function (see https://en.wikipedia.org/wiki/Gamma_distribution).

Suppose we start with a prior distribution for some parameters θ, and observe some data $\{x_i\}_{i=1}^n$ with likelihood $P(\text{ data } \mid \theta)$. If the posterior for θ has the same form as the prior, then we say that the given prior is conjugate for the given likelihood.

(i) Show that the Gamma distribution (that is $\lambda \sim \text{Gamma}(\alpha, \beta)$) is a conjugate prior of the $\text{Exp}(\lambda)$ distribution. In other words, show that if the data points $x_i \sim \text{Exp}(\lambda)$ and $\lambda \sim \text{Gamma}(\alpha, \beta)$ then $P(\lambda|\text{data}) \sim \text{Gamma}\left(\alpha^*, \beta^*\right)$ for some values α^*, β^*.

(ii) Derive the maximum a posteriori estimator (MAP) $\hat{\lambda}_{MAP}$ as a function of α, β. What happens as the number of data points n gets large?

Problem 10. A Factory produces X_n devices on day n, where X_n are independent and identically distributed random variables with mean 5 and variance 9.

(a) Find an approximation to the probability that the total number devices produced in 100 days is less than 440.

(b) Find (approximately) the largest value of n such that $P(X_1 + X_2 ..... + X_n \geq 200 + 5n) \leq 0.05$

(c) Let N be the first day on which the total number of devices produced exceeds 1000. Calculate an approximation to the probability that $N \geq 220$.

Problem 11. A statistician wants to estimate the mean height h (in meters) of a population, based on n independent samples $X_1, ..., X_n$ chosen uniformly from the entire population. He uses the sample mean $M_n = (X_1 + ... + X_n)/n$ as the estimate of h, and a rough guess of 1.0 meters for the standard deviation of the samples X_i.

(a) How large should n be so that the standard deviation of M_n is at most 1 centimeter?

(b) How large should n be so that Chebyshev's inequality guarantees that the estimate is within 5 centimeters from h, with probability at least 0.99?

Problem 12. Suppose $X_1, X_2, ...$ are independent and identically distributed random variables. Suppose N is a positive integer valued random variable independent of all X_is. Then Prove that

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[N\right].E\left[X_1\right]$$

(Hint: use $E[E[X|Y]] = E[X]$.)

Problem 13. S_n represents stock price on the day n with S_0 given. S_n is given by

$$S_n = S_{n-1} + X_n, \qquad n \geq 1.$$

$X_1, X_2, ...$ are i.i.d. continuous RVs with mean 0 and variance 1. Suppose a stock's price today is 100. What can you say about the probability that the stock price will be between 95 and 105 on the 10^{th} day.

Problem 14. Suppose that you are given two independent Gaussian random variables, X_1 with mean 0 and variance σ_1^2 and X_2 with mean 0 and variance σ_2^2. Show that for any real number a_1, a_2, it holds that $Y = a_1 X_1 + a_2 X_2$ is also Gaussian. (Hint: Use MGF of Gaussian RV).

Problem 15. (a) Suppose $X(t)$ is a wide-sense stationary (WSS) process with autocorrelation function $R_X(\tau)$. Show that

$$\mathbb{E}[(X(t) - X(t+\tau))^2] = 2(R_X(0) - R_X(\tau)).$$

(b) Given a random process $X(t)$ with mean $\mu_X(t)$ and covariance $C_X(t_1, t_2)$, we construct a new process as the difference

$$\tilde{X}(t) = X(t) - \mu_X(t).$$

Find a necessary and sufficient condition for $\tilde{X}(t)$ to be a WSS process.

All the best for your examinations