Practice Problem Set 1

(MA6.102) Probability and Random Processes, Monsoon 2023

Problem 1. Let X be the sum of Bernoulli random variables, $X = \sum_{i=1}^{n} X_i$. The X_i do not need to be independent. Show that $\mathbb{E}[X^2] = \sum_{i=1}^{n} P_{X_i}(1)\mathbb{E}[X|X_i = 1]$

Problem 2. Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive-or of the ith pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$ be the number of Y_i that equal 1.
(a) Show that each Y_i is 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$.
(b) Show that the Y_i are not mutually independent.
(c) Show that the Y_i satisfy the property that $\mathbb{E}[Y_iY_j] = \mathbb{E}[Y_i]\mathbb{E}[Y_j]$.
(d) Find Var$[Y]$.

Problem 3. We say that α is a median of a random variable X if $P(X \leq \alpha) \geq \frac{1}{2}$ and $P(X \geq \alpha) \geq \frac{1}{2}$. It is possible for the median to be non-unique, with all values in an interval satisfying the definition.
(a) Let X be a ternary random variable taking on the three values 0, 1, and 2 with probabilities p_0, p_1, and p_2, respectively. Find the median of X for each of the cases below.
(i) $p_0 = 0.2$, $p_1 = 0, 4$, $p_2 = 0.4$.
(ii) $p_0 = 0.2$, $p_1 = 0.2$, $p_2 = 0.6$.
(iii) $p_0 = 0.2$, $p_1 = 0.3$, $p_2 = 0.5$.
(b) Suppose X is a non-negative continuous random variable with PDF f_X such that $f_X(x) = 1$, for $0 \leq x \leq 0.5$ and $f_X(x) = 0$, for $0.5 < x \leq 1$. We know that $f_X(x)$ is positive for all $x > 1$, but is otherwise unknown. Find the median or interval of medians.

Problem 4. Suppose X and Y are two independent random variables such that $\mathbb{E}[X^4] = 2, \mathbb{E}[Y^2] = 1, \mathbb{E}[X^2] = 1$ and $\mathbb{E}[Y] = 0$. Compute Var(X^2Y).

Problem 5. Consider two random variables X and Y with joint PMF given as follows:

	$Y = 1$	$Y = 2$
$X = 1$	1/3	1/12
$X = 2$	1/6	0
$X = 4$	1/12	1/3

a) Find the marginal PMFs of X and Y.
b) Find $P(Y = 2|X = 1)$.
c) Are X and Y independent?
d) Let Z be a R.V. defined by $Z = X - 2Y$. Find the PMF of Z and $P(X = 2|Z = 0)$.

Problem 6. Alice plays the following game on a math show. There are 7 boxes and identical prizes are hidden inside 3 of the boxes. Alice is asked to choose a box where a prize might be. She chooses a box uniformly at random. From the unchosen boxes which do not have a prize, the host opens an arbitrary box and shows Alice that there is no prize in it. The host then allows Alice to change her choice if she so wishes. Alice chooses a box uniformly at random from the other 5 boxes (other than the one she chose first and the one opened by the host). Find the probability that Alice wins the prize.

Problem 7. Consider the random variables N, H, and T sampled as follows. First, N is sampled from the Poisson(10) distribution. This means that for each integer $n \geq 0$,

$$P[N = n] = e^{-10} 10^n / n!.$$

Then N independent fair coins are tossed. H is then the number of heads, and T the number of tails, obtained in this process.
a) Prove or disprove that H and T are independent.
b) What is the relation of Var(H) with Var(N)?

Problem 8. Let N be the number of customers that visit a certain store in a given day. Suppose that we know $\mathbb{E}[N]$ and Var(N). Let X_i be the amount that the ith customer spends on average. We assume X_i's are independent of each other and also independent of N. We further assume they have the same mean $\mathbb{E}[X]$ and same variance Var(X). Let Y be the store's total sales, i.e., $Y = \sum_{i=1}^{N} X_i$. Find $\mathbb{E}[Y]$ and Var(Y).

Problem 9. Alvin goes to the supermarket shortly, with probability $\frac{1}{3}$ at a time uniformly distributed between 0 and 2 hours from now; or with probability $\frac{2}{3}$, later in the day at a time uniformly distributed between 6 and 8 hours from now. What is the PDF of the time Alvin goes to the supermarket, viewed as a random variable X?

Problem 10. For a continuous random variable X, show that

$$\mathbb{E}[X] = \int_0^\infty P(X > x)\ dx - \int_0^\infty P(X < -x)\ dx.$$

Using this, prove that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\ dx.$$

Hint: Note that the function g can be expressed as the difference of two non-negative functions as $g(x) = g^+(x) - g^-(x)$, where $g^+(x) = \max\{g(x), 0\}$, and $g^-(x) = \max\{-g(x), 0\}$.

All the best for your examinations