Assignment - 2

PRP

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Problem 1:

Smallest σ-field on $\Omega = \{-2, -1, 0, 1, 2\}$

(a) $X(\omega) = \omega^2$:

$X(\omega) = 0$ for $\omega = \{0\} = X(\omega) = 0^2 = 0$

$X(\omega) = 1$ for $\omega = \{-1, 1\} = X(1) = (1)^2, X(-1) = (-1)^2 = 1$

$X(\omega) = 4$ for $\omega = \{-2, 2\} = X(2) = 2^2, X(-2) = (-2)^2 = 4$

$X = \{0, 1, 4\}$

$X^{-1}(0) = \{0\}$, $X^{-1}(1) = \{-1, 1\}$, $X^{-1}(4) = \{-2, 2\}$

Pre image's are $\{\{0\}, \{-1, 1\}, \{-2, 2\}\}$

σ- field consists of all possible union set in C

σ- field $= \{\phi, \Omega, \{0\}, \{-1, 0, 1\}, \{-1, 1\}, \{-2, 2\}, \{-2, 0, 2\},$
$\{-2, -1, 1, 2\}\}$

$n = 3$ $\quad 2^3 = 8$ sets

(b) $X(\omega) = \omega + 1$

$X(-2) = -2 + 1 = -1$

$X(-1) = -1 + 1 = 0$

$X(0) = 0 + 1 = 1$

$X(1) = 1 + 1 = 2$

$X(2) = 2 + 1 = 3$

$\{\{-1\}, \{0\}, \{1\}, \{2\}, \{3\}\}$

$n = 5$

$2^n = 2^5 = 32$ sets

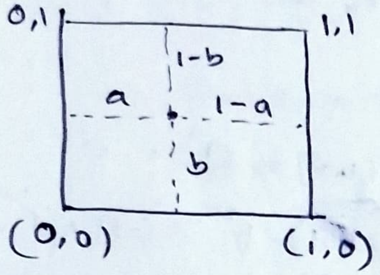
for smallest σ- field is the powerset of Ω (2^n)

Problem 2 :- $\Omega = [0,1] \times [0,1]$ unit - square

$A \subseteq \Omega$,

The RV X is the distance from a point $\omega = (a,b)$ in Unit square $(x=0, x=1, y=0, y=1)$

distance $X(\omega) = \min\{a, 1-a, b, 1-b\}$



Range

- min value : if point is on edge it's nearest edge is 0
- max value : point furthest from all edges is center $\omega = \{0.5, 0.5\}$

Range of X is interval $[0, 0.5]$

CDF $F_X(x) = P(X \leq x)$

- $x < 0$ distance cannot be -ve so min is 0 so, prob of X being less than -ve is 0

$$\therefore F_X(x) = P(X \leq x) = 0$$

- $x \geq 0.5$ Max value of X is 0.5 prob is 1

$$F_X(x) = P(X \leq x) = 1$$

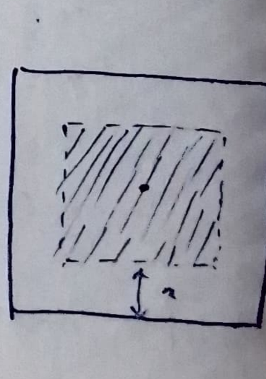
- $0 \leq x \leq 0.5$ complement rule . .

$$F_X(x) = P(X \leq x) = 1 - P(X > x)$$

even $X > x$ means distance from (a,b) to nearest edge greater than x. true when all edges is greater than x

so, $a > x$, $1-a > x \Rightarrow a < 1-x$

$b > x$, $1-b > x \Rightarrow b < 1-x$



inner square side $(1-x) - x = 1-2x$

$P(X > x)$ is are of this square

$P(X > x) = (\text{side})^2 = (1-2x)^2$

complement $F_X(x) = 1 - (1-2x)^2$

$$F_X(x) = (1-(1-4x+4x^2)) = 1-1+4x-4x^2$$
$$= 4x - 4x^2$$

$$\text{CDF} = F_X(x) = \begin{cases} 0 & x < 0 \\ 4x - 4x^2 & 0 \le x < 0.5 \\ 1 & x \ge 0.5 \end{cases}$$

Problem-3

if the given functions are valid CDFs by checking
Three properties :

1. Non-decreasing

   if $x_1 \leq x_2$ then $F_X(x_1) \leq F_X(x_2)$

2. External limits

   - $\lim\limits_{x \to -\infty} F_X(x) = 0$
   - $\lim\limits_{x \to \infty} F_X(x) = 1$

3. Right continuity

   $\lim\limits_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x)$

(a) $1-(1-F_X(x))^{\gamma}$, $\gamma \in \mathbb{N}$

function approaches 0 as $x \to -\infty$

1 as $x \to \infty$

- $\lim\limits_{x \to -\infty} G(x) = 1-(1-\lim\limits_{x \to -\infty} F_X(x))^{\gamma}$

  $= 1-(1-0)^{\gamma} = 1-1 = 0$

- $\lim\limits_{x \to +\infty} G(x) = 1-(1-\lim\limits_{x \to \infty} F_X(x))^{\gamma}$

  $= 1-(1-1)^{\gamma} = 1-0 = 1$

Function follows External limits.

Non- decreasing property

$x_1 \leq x_2 \Rightarrow F_x(x_1) \leq F_x(x_2)$

$y = F_x(x)$ we consider

$g(y) = 1 - (1-y)^r \quad y \in [0,1]$

$(1-y)$ → as y increase $(1-y)$ is non-incra

$(1-y)^r$ as r is +ve it is non-incrasing

$-(1-y)^r$ it is non- decreasing

adding 1 a constant does not change

so

$g(y) = 1-(1-y)^r$ is non-decreasing

since $G(x)$ is non-decreasing ofunct of $F_x(x)$. and $F_x(x)$ is a non-decreasing function of x, then their composition $G(x)$ must be non decrasing

if $x_1 < x_2$

$$F_x(x_1) < F_x(x_2)$$

$$-F_x(x_1) > F_x(x_2)$$

$$1 - F_x(x_1) > 1 - F_x(x_2)$$

$$(1-F_x(x_1))^r > (1-F_x(x_2))^r$$

$$-(1-F_x(x_1))^r < (1-F_x(x_2))^r$$

$$1-(1-(F_x(x_1))^r < \{1-(1-F_x(x_2))^r$$

Right continuity

$$\lim_{\varepsilon \to 0^+} F_y(x+\varepsilon) = F_y(x)$$

$$= \lim_{\varepsilon \to 0^+} \left(1 - (1 - F_x(x+\varepsilon))^r\right)$$

$$= 1 - \lim_{\varepsilon \to 0^+} (1 - F_x(x+\varepsilon))^r$$

$$= 1 - \left(1 - \lim_{\varepsilon \to 0^+} F_x(x+\varepsilon)\right)^r$$

$$= 1 - (1 - F_x(x))^r$$

$$= F_y(x)$$

$\therefore$ the given function satisfies all three properties it is a valid CDF.

(b) $F_x(x) + (1 - F_x(x)) \log(1 - F_x(x))$

$$\lim_{x \to \infty} F_y(x) = \lim_{x \to \infty} F_x(x) + (1 - F_x(x)) \log(1 - F_x(x))$$

$$= \lim_{x \to \infty} F_x(x) + \lim_{x \to \infty} (1 - F_x(x)) \log(1 - F_x(x))$$

$$= 1 + \lim_{x \to \infty} \frac{\log(1 - F_x(x))}{\frac{1}{1 - F_x(x)}}$$

$$= 1 + \lim_{F_x(x) \to 1} \frac{\log(1 - F_x(x))}{\frac{1}{1 - F_x(x)}} = \frac{\log(0) \text{ undefined}}{\infty}$$

So, L-Hopital

$$= 1 + \lim_{F_x(x) \to 1} \frac{\frac{-1}{1 - F_x(x)}}{\frac{-1}{(1 - F_x(x))^2}} = 1 + \lim_{F_x(x) \to 1} 1 - F_x(x) = 1$$

$\therefore \lim_{x\to\infty} F_Y(x) = 1$ —①

Now, $\lim_{x\to-\infty} F_Y(x)$

$\lim_{x\to-\infty} F_X(x) + (1-F_X(x))\log(1-F_X(x))$

$= \lim_{x\to-\infty} F_X(x) + \lim_{x\to-\infty} (1-F_X(x))\log(1-F_X(x))$

$= 0 + \lim_{F_X(x)\to 0} (1-F_X(x))\log(1-F_X(x))$

$= 0 + (1-0)\cdot\log(1)$

$= 0$ $\quad \lim_{x\to-\infty} F_Y(x) = 0$ —②

from ① and ② External limits are Valid

For $x > y$ $\quad F_X(x) > F_X(y)$

also $0 < F_X < 1$

$1 - F_X(x) < 1 - F_X(y)$

$0 < 1-F_X(x) < 1$

$(1-F_X(x))\log(1-F_X(x)) > (1-F_X(y))\log(1-F_X(y))$

in $x\log x$ $(x \in 0,1)$ as $x\uparrow$, $\log x \uparrow$ theirfor $x\log x$ increase and remains negative

$\therefore (1-F_X(x))\log(1-F_X(x)) > (1-F_X(y))\log(1-F_X(y))$

$F_X(x) > F_X(y)$

$\therefore F_X(x) + (1-F_X(x))\log(1-F_X(x)) > F_X(y) + (1-F_X(y))\log(1-F_X(y))$

$F_Y(x) > F_Y(y)$ for $x > y$

$\therefore F_Y$ is non-decreasing function

Now,

$$\lim_{\varepsilon \to 0^+} F_Y(x)$$

$$= \lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) + (1 - F_X(x+\varepsilon)) \log(1 - F_X(x+\varepsilon))$$

$$= F_X(x) + \lim_{\varepsilon \to 0^+} (1 - F_X(x+\varepsilon)) \log(1 - F_X(x+\varepsilon))$$

$$= F_X(x) + \lim_{\varepsilon \to 0^+} (1 - F_X(x+\varepsilon)) \lim_{\varepsilon \to 0^+} \log(1 - F_X(x+\varepsilon))$$

$$= F_X(x) + (1 - F_X(x)) \log(1 - F_X(x))$$

$$= F_Y(x)$$

→ $F_Y(x)$ is Right Continuous.

→ The Given function satisfies all three properties so is a CDF

Problem 4.

Show $E[N] = \sum_{i=1}^{\infty} P(N \geq i)$

Proof: for a non-negative integer RV. N

$$E[N] = \sum_{i=1}^{\infty} P(N \geq i)$$

defination $E[N] = \sum_{k=0}^{\infty} k \cdot P(N=k)$

Expanding RHS

$$\sum_{i=1}^{\infty} P(N \geq i) = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P(N=k)$$

Swap summations

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{k} P(N=k)$$

as $P(N=k)$ does not depend on inner Summation index i so treat as constant

$$= \sum_{k=1}^{\infty} P(N=k) \sum_{i=1}^{k} 1$$

$$\Rightarrow \sum_{i=1}^{k} 1 = \underbrace{1+1+\cdots}_{k \text{ times}} = k$$

$$= \sum_{k=1}^{\infty} k \cdot P(N=k)$$

$k=1$ same as $k=0$

for $k=0$ $P(N=0) = 0$

$$= \sum_{k=1}^{\infty} k \cdot P(N=k) = E[N]$$

$$\boxed{\therefore E[N] = \sum_{i=1}^{\infty} P(N \geq i)}$$

Problem 5

Given an example of a non-constant RV X such that $E\left[\frac{1}{X}\right] = \frac{1}{E[X]}$

Sol

X is non constant RV

Let's defin X

$$X = \begin{cases} 2 & \rightarrow \text{prob } 4/9 \\ \frac{1}{2} & \rightarrow \text{prob } 4/9 \\ -1 & \rightarrow \text{prob } 1/9 \end{cases}$$

X is non-constant

$\rightarrow$ $E[X] = \sum_i x_i p_i$

$$E[X] = 2 \cdot \frac{4}{9} + \frac{1}{2} \cdot \frac{4}{9} + (-1) \cdot \frac{1}{9}$$

$$E[X] = \frac{8}{9} + \frac{2}{9} - \frac{1}{9} = \frac{8+2-1}{9} = \frac{9}{9} = 1$$

$$E[X] = 1 \Rightarrow \frac{1}{E[X]} = 1 \quad \text{— (1)}$$

$\rightarrow$ $E\left[\frac{1}{X}\right] = \frac{1}{2} \cdot \frac{4}{9} + 2 \cdot \frac{4}{9} + (-1) \cdot \frac{1}{9}$

$$= \frac{2}{9} + \frac{8}{9} - \frac{1}{9} = \frac{2+8-9}{9} = \frac{9}{9} = 1$$

$$E\left[\frac{1}{X}\right] = 1 \quad \text{— (2)}$$

from (1) and (2) $\cancel{E[X] = E\left[\frac{1}{X}\right]}$ $E\left[\frac{1}{X}\right] = \frac{1}{E[X]} = 1$

Problem 6

$$P_X(k-1)\,P_X(k+1) \le P_X(k)^2$$

and Give an example Such $P_X(k)^2 = P_X(k-1)\,P_X(k+1)$

Sol :-

Given X is Binomial or possion

for $X \sim \text{Binomial}(n, p)$

PMF $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

the ratio $\dfrac{P_X(k)}{P_X(k-1)} = ?$

$$\frac{P_X(k)}{P_X(k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \frac{\dfrac{n!}{k!\,(n-k)!}}{\dfrac{n!}{(k-1)!\,(n-k+1)!}} \times \frac{p}{1-p}$$

$$\frac{P_X(k)}{P_X(k-1)} = \frac{n-k+1}{k} \cdot \frac{p}{1-p}$$

here ratio is function of k. as $k \uparrow$ the $(n-k+1) \downarrow$ and denom $k \uparrow$. Therefor $\frac{n-k+1}{k} \downarrow$

this means that the ratio for k will be greater than or equal to ratio for k+1

$$\frac{P_X(k)}{P_X(k-1)} \ge \frac{P_X(k+1)}{P_X(k)}$$

$$\therefore \left(P_X(k)\right)^2 \ge P_X(k-1)\,P_X(k+1)$$

For $X \sim$ poisson (λ)

$$PMF = P_X(k) = \frac{e^{-\lambda} \lambda^{k}}{k!}$$

Consecutive ratio

$$\frac{P_X(k)}{P_X(k-1)} = \frac{e^{-\lambda}\lambda^{k}/k!}{e^{-\lambda}\lambda^{k-1}/(k-1)!} = \frac{\lambda^{k}}{\lambda^{k-1}} \times \frac{(k-1)!}{k!} = \frac{\lambda}{k}$$

Now, As k increases the ratio $\frac{\lambda}{k}$ decrease

$$\therefore \frac{P_X(k)}{P_X(k-1)} \geq \frac{P_X(k+1)}{P_X(k)}$$

$$= (P_X(k))^2 \geq P_X(k-1)\, P_X(k+1)$$

→ Example

$$P_X(k)^2 = P_X(k-1)\, P_X(k+1)$$

$$\frac{P_X(k)}{P_X(k-1)} = c \Rightarrow P_X(k) = c \cdot P_X(k-1)$$

— geometric progression

Let X be geometric RV

PMF $\quad P_X(k) = (1-p)p^{k} \quad \forall\ k = 0, 1, 2, \ldots$

$$= P_X(k)^2 = ((1-p)p^{k})^2 = (1-p)^2 p^{2k} \quad \text{—①}$$

$$= P_X(k-1)\, P_X(k+1) = ((1-p)p^{k-1}) \cdot ((1-p)p^{k+1})$$

$$= (1-p)^2 p^{(k-1)+(k+)}$$

$$= (1-p)^2 p^{2k} \quad \text{—②}$$

both ① and ② are equal

$\therefore$ That equality holds

Problem 7

Given $X \sim \text{poisson}(\lambda)$

$\theta = e^{-3\lambda}$ and bias of $g(x) = E[g(x)] - \theta$

and if $E[g(x)] - \theta = 0$ it is unbiased

ⓐ $g(X) = e^{-3X}$

An estimator $g(x)$ is unbiased if it's expected value is equal

$E[g(x)] = \theta$ $\qquad \theta = e^{-3\lambda}$

$$E[g(x)] = \sum_{k=0}^{\infty} g(k) \cdot P(X=k)$$

for poisson distribution $\quad$ PMF $P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

(a) $g(x) = e^{-3x}$ $\quad E[e^{-3x}] = \sum_{k=0}^{\infty} e^{-3k} \cdot P(X=k)$

$$= \quad E[e^{-3x}] = \sum_{k=0}^{\infty} e^{-3k} \cdot \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

$$E[e^{-3x}] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{-3} \cdot \lambda)^k}{k!}$$

Apply Taylor Series $\quad e^z = \sum_{k=0}^{\infty} \dfrac{z^k}{k!}$ $\quad$ here $z = \lambda e^{-3}$

$$\sum_{k=0}^{\infty} \frac{(\lambda e^{-3})^k}{k!} = e^{\lambda e^{-3}}$$

$$E[e^{-3x}] = e^{-\lambda} \cdot e^{\lambda e^{-3}} = e^{-\lambda + \lambda e^{-3}} = e^{-\lambda(1-e^{-3})}$$

$$e^{-\lambda(1-e^{-3})} \neq e^{-3\lambda}$$

Since expected value of estimator is not equal to θ

The estimator $g(x) = e^{-3x}$ is Biased

(b) $g(x) = (-2)^x$

Expected Value of estimator function

$$E[(-2)^x] = \sum_{k=0}^{\infty} (-2)^k \cdot P(x=k)$$

Substitute PMF of poisson's

$$E[(-2)^x] = \sum_{k=0}^{\infty} (-2)^k \cdot \frac{e^{-\lambda} \lambda^k}{k!}$$

Poll $e^{-\lambda}$ out

$$E[(-2)^x] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(-2\lambda)^k}{k!}$$

This summation is Taylor Series for e^z, $z = -2\lambda$

$$\sum_{k=0}^{\infty} \frac{(-2\lambda)^k}{k!} = e^{-2\lambda}$$

$$E[(-2)^x] = e^{-\lambda} \cdot e^{-2\lambda}$$

$$= e^{-\lambda - 2\lambda} = e^{-3\lambda}$$

$$\therefore \quad E[g(x)] = e^{-3\lambda} = \theta$$

$\therefore$ as $E[g(x)] = e^{-3\lambda} = \theta$ The given estimator $g(x) = (-2)^x$ is UnBiased