PRP
Assignment - 1

2023102013
Nethavath Praveen

Problem - 1

Are any two distinct atoms in σ-field necessarily disjoint?

Proof:- in a σ-field $\mathcal{F}$ on Ω, a set $A \in \mathcal{F}$ is an atom if

* $A \neq \phi$
* For any $B \in \mathcal{F}$, if $B \subseteq A$, then $B = \phi$ or $B = A$

Let A and C be atoms in $\mathcal{F}$ with $A \neq C$
To show $A \cap C = \phi$.

Assume $A \cap C \neq \phi$
since $C \in \mathcal{F}$ and $A \in \mathcal{F}$, their intersection $A \cap C \in \mathcal{F}$ and also $A \cap C \subseteq A$

→ as A is an atom
Possible subsets of A are ϕ or A itself
since we assumed $A \cap C \neq \phi$ it cannot be ϕ
so, $A \cap C = A \Rightarrow \boxed{A \subseteq C}$ — (1)

→ as C is an atom
$A \cap C \in \mathcal{F}$ and $A \cap C \subseteq C$
possible subsets ϕ or C itself
Again $A \cap C \neq \phi$ so: $A \cap C = C$
$\boxed{C \subseteq A}$ — (2)

from — (1) and (2)
$A \subseteq C$ and $C \subseteq A$, hence $A = C$

This contradicts $A \neq C$

So ~~our~~ assumption $A \cap C \neq \phi$ is false

$\therefore A \cap C = \phi$

Which Means Any two distinct atoms in a σ-field are disjoint.

Problem - 2

$E_1, E_2, E_3 \ldots\ldots E_n$ are $E_i \cap E_j = \phi$ for $i \neq j$

$\bigcup_{i=1}^{n} E_i = \Omega$

Determine smallest σ-field that contains all the events E_i for $i \in [1:n]$.

Solution :-

Consider $\mathcal{F} = \left\{ \bigcup_{i \in I} E_i \mid I \subseteq \{1, 2, \ldots n\} \right\}$

all possible unions of E_i

$\mathcal{F}$ is a σ field $\rightarrow$ ϕ and Ω contained.

$\phi = \bigcup_{i \in \phi} E_i \in \mathcal{F}$ $\quad \Omega = \bigcup_{i=1}^{n} E_i \in \mathcal{F}$

for any $A = \bigcup_{i \in I} E_i \in \mathcal{F}$ $\quad A^c = \Omega \setminus \bigcup_{i \in I} E_i = \bigcup_{i \in I^c} E_i \in \mathcal{F}$

because the E_i are disjoint and exhaustive

$\rightarrow$ Closed under countable union A_k $\quad k \in 1, 2, 3 \ldots . n$

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{i \in I_k} E_i = \bigcup_{i \in \bigcup_k I_k} E_i \in \mathcal{F} \qquad \bigcup_k I_k = \{1, 2, \ldots, n\}$$

hence $\mathcal{F}$ is a σ-field

Let $\mathcal{G}$ be any σ-field with $E_i \in \mathcal{G}$ $\forall i$

so $\bigcup_{i \in I} E_i$ belongs in $\mathcal{G}$ so, $\mathcal{F} \subseteq \mathcal{G}$

so, $\mathcal{F}$ is smallest σ-field containing all E_i

$$\sigma(E_1, E_2 \ldots E_n) = \left\{ \bigcup_{i \in I} E_i \;\middle|\; I \subseteq \{1, 2, \ldots n\} \right\}$$

it has 2^n sets

Problem - 3 A and B are two events

$P(A) = \frac{3}{4}$, $P(B) = \frac{1}{3}$

(a) Show that $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$.

Solution: Upper bounds and lower bounds of $P(A \cap B)$

→ Upper bound : event that both A & B occur is a subset A and also subset of B
This means probability of intersection cannot be greater than individual event

~~As~~ $A \cap B \subseteq A \rightarrow P(A \cap B) \leq P(A)$
$A \cap B \subseteq B \rightarrow P(A \cap B) \leq P(B)$

∴ $P(A \cap B)$ is less than equal to smaller of two probs

$$P(A \cap B) \leq \min \{ P(A), P(B) \}$$

$P(A) = \frac{3}{4}$ $P(B) = \frac{1}{3}$ = min = $P(B)$

So upper bound is $\frac{1}{3}$ ⇒ $P(A \cap B) \leq \frac{1}{3}$

→ Lower bound :-
inclusion - exclusion - principle
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cap B) = P(A) + P(B) - P(A \cup B)$
remove max possible value
$P(A \cup B) \leq 1$
$P(A \cap B) \geq P(A) + P(B) - 1$
$P(A \cap B) \geq \frac{3}{4} + \frac{1}{3} - 1$
$P(A \cap B) \geq \frac{9+4-12}{12}$

$$\boxed{\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}}$$

so,

(5) Construct sample space Ω, events $A + B$ and probability P.

Let $\Omega = [0,1]$ and prob law P is length of that interval of Ω.

Upperbound $P(A \cap B) = \frac{1}{3}$: $B \subseteq A$

$$A = \left[0, \frac{3}{4}\right], \quad B = \left[0, \frac{1}{3}\right]$$

Then $P(A) = \frac{3}{4}$ $\quad P(B) = \frac{1}{3}$

$A \cap B = B$ so, $P(A \cap B) = \frac{1}{3}$

Lowerbound $P(A \cap B) = \frac{1}{12}$: "disjoint"

$$A = \left[0, \frac{3}{4}\right] \quad B = \left[\frac{2}{3}, 1\right]$$

$P(A) = \frac{3}{4}$ $\quad P(B) = \frac{1}{3}$

Intersection $\left[\frac{2}{3}, \frac{3}{4}\right] \Rightarrow \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

Problem - 4

- coin 1 = both heads
- coin 2 = both tails
- coin 3 = head & tail

HH, TT, HT choose a coin uniformly an
Toss it, the result is H.

Let H be heads showing and
A be tails on opposite

$\Rightarrow$ A possible only in coin 3 (HT)

$P(H) \rightarrow$ HH $-$ $P(H) = 1$
HT $-$ $P(H) = 1/2$
TT $-$ $P(H) = 0$

$$P(H) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{2}$$

$$P(A \cap H) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Bayes $\quad P(A \mid H) = \dfrac{P(A \cap H)}{P(H)}$

$$P(A \mid H) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

1 out of 3 has tails in it's back

Problem - 5

$C = \{A_1, A_2, A_3 \dots A_n\}$ Collection of Mutually indep events

prove $(C \setminus A_i) \cup \{A_i^c\}$ $i \in [1:n]$ is also MIE

Proof : Let z be any subset of set $\{1,2,3 \dots n\}$ and size of z is k

Then $$P\left(\bigcap_{i=1}^{k} A_{z_i}\right) = \prod_{i=1}^{k} P(A_{z_i})$$

firstly, if A_i & A_j are independent then for $(i \neq j)$ A_i & A_j^c are also independent

$$A_i \cap A_j^c = A_i \setminus A_j$$

$$\begin{aligned} P(A_i \cap A_j^c) &= P(A_i) - P(A_i \cap A_j) \\ &= P(A_i) - P(A_i)\,P(A_j) \\ &= P(A_i)\left[1 - P(A_j)\right] \\ &= P(A_i)\,P(A_j^c) \end{aligned}$$

Therefore A_i and A_j^c are also independent

Now, Let z be a subset of set $\{1, \dots n\} \setminus \{i\}$

$$\therefore P\left(\bigcap_{j=1}^{k} A_{z_j}\right) \cap A_i^c = \left(\prod_{j=1}^{k} P(A_{z_j})\right) P(A_i^c)$$

A_i^c is independent of all A_{z_j} proved

$\therefore$ $(C \setminus A_i) \cup \{A_i^c\}$ is also a collection of Mutually independent events.

Problem - 6

E - event E actually occured
E' - " E did not occur
A - Alice asserts E occured
B - BOB asserts E occured

Given, $P(E) = \frac{1}{1000}$ $P(E') = 1 - P(E)$

$= 1 - \frac{1}{1000} = \frac{999}{1000}$

both A and B truthful is $\frac{9}{10}$

Compliment (lie) is $1 - \frac{9}{10} = \frac{1}{10}$

To find probability that E actually occured given A and B declared it occured.

Which Means $P(E \mid A \cap B)$

Baye's Theorem. $\boxed{P(E \mid A \cap B) = \frac{P(A \cap B \mid E) P(E)}{P(A \cap B)}}$

→ $P(A \cap B \mid E) \cdot P(E)$ => that Event happened and and A & B declared occured.

$P(A \cap B \mid E) = P(A \mid E) \cdot P(B \mid E)$

$= \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$

=> $P(A \cap B \mid E) \cdot P(E) = \frac{81}{100} \times \frac{1}{1000}$

$= \frac{81}{10^4}$

$P(A\cap B)$ probability of A & B that both delcare that Event E has occured

This can happen in 2 Mutually exclusive ways

Case ① E occured and both are truth this is equal to $P(A\cap B|E) \cdot P(E) = \frac{81}{10000}$

Case ② That E did not occur & both are false

$$P(E') = \frac{999}{1000}$$

$$P(A|E')\, P(B|E') = \left(\frac{1}{10}\right) \cdot \left(\frac{1}{10}\right) = \frac{1}{100}$$

$$P(A\cap B|E') \cdot P(E') = \frac{1}{100} \cdot \frac{999}{1000} = \frac{999}{100000}$$

$P(A\cap B) =$ Case ① + case ②

$$P(A\cap B) = \frac{81}{100000} + \frac{999}{100000} = \frac{1080}{100000}$$

$$P(E|A\cap B) = \frac{P(A\cap B|E) \cdot P(E)}{P(A\cap B)}$$

$$= \frac{\frac{81}{100000}}{\frac{1080}{100000}} = \frac{81}{1080}$$

$$P(E|A\cap B) = \frac{9}{120} = \frac{3}{40}$$

$\therefore$ The probability of E acutually occured, given that both A & B declared is $\frac{3}{40}$

Problem - 7

i - initial amount $i = 200$

N - To by a car amount $N = 2,000,000$

Prob of win $= 1/2$ loss $= \frac{1}{2}$

1₹   1₹

$$p = q = \frac{1}{2}$$

So probability of reaching N before reaching 0 starting with i

$$P_{win} = \frac{i}{N}$$

$$P_{ruin} = 1 - P_{win} \Rightarrow 1 - \frac{i}{N}$$

$$P_{ruin} = 1 - \frac{200}{2000000} = \frac{9999}{10,000}$$

Probability that man is ultamitely bankrupted is $\frac{9999}{10000}$

99.99 %. he will be bankrupted if he starts with i and Gambles with a fair coin.