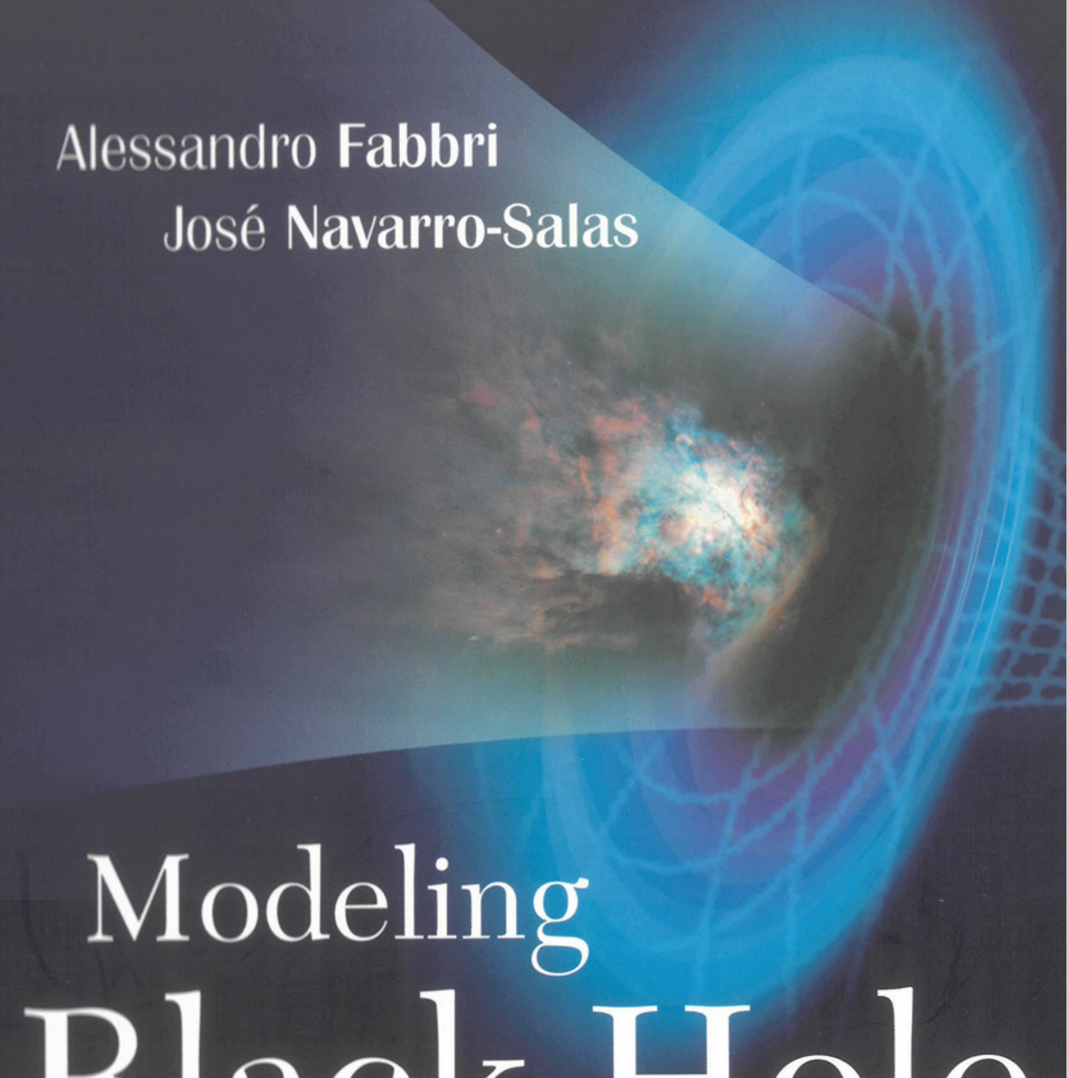

Alessandro **Fabbri**
José **Navarro-Salas**

Modeling Black Hole Evaporation

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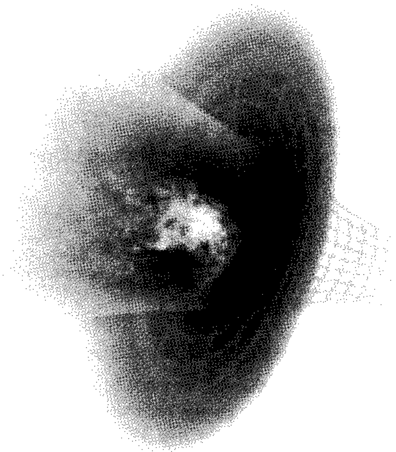

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MODELING BLACK HOLE EVAPORATION

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Printed in Singapore.

To my family
Sandro

To Lola, Sergi and María Dolores
Pepe

Preface

The aim of this book is to offer a comprehensible monograph on black hole quantum radiance and black hole evaporation. Most of the existing books on quantum field theory in curved spacetimes describe with detail the Hawking effect. However none of them extensively treats the issue of the backreaction of the evaporation process. Since this is a rather involved topic we shall take a modest, but pedagogical line to approach the subject.

The history of research on black holes tells us that important physical insights have been gained through the study of simplified models of gravitational collapse. The paradigm is the Oppenheimer and Snyder model (1939). Despite its simplicity, forced by the difficulties of the technical treatment of more realistic situations, it turns out to produce a very accurate picture of the gravitational collapse and its final outcome. All the main ingredients were there: different descriptions for external and infalling observers, divergent redshift at the horizon and existence of the internal singularity. The assumption of perfect spherical symmetry, the main criticism of black hole opponents, was not, in the end, a real drawback to invalidate the full picture offered by the model. In recent years, a model inspired by string theory, and proposed by Callan, Giddings, Harvey and Strominger (1992), also offers a simplified scenario which allows to study analytically the process of black hole formation and subsequent evaporation, including semiclassical backreaction effects. The results and techniques generated by this model renewed the interest in black hole evaporation.

Motivated by these two paradigmatic models, we take this line of thought and try to present a pedagogical view of the subject of black hole radiance and black hole evaporation. This is the reason for calling "modeling" the approach we take in this book.

The style and presentation of the different aspects involved have been chosen to make them accessible to a broad audience. We want to stress that we assume a basic knowledge of general relativity and also of quantum field theory, at the level of introductory graduate courses. Therefore, a wide spectrum of physicists, ranging from particle physicists to astrophysicists, and from beginner graduate students to senior researchers, can follow all chapters. Even those who are not very familiar with either general relativity or quantum field theory can find, we hope, this monograph accessible. With this respect we want to remark that this is indeed a book on quantum aspects of black holes and not on "quantum field theory in curved spacetimes". Since the latter subject plays a fundamental role to address the backreaction problem, we approach it following a simple, although somewhat unconventional, way. We try to escape from the standard technicalities of regularization schemes to derive conformal anomalies, effective actions, etc. Rather we try to present and rederive the fundamental results on a physically motivated basis.

The most delicate parts of the text were written while we worked together at the Department of Theoretical Physics and IFIC of the University of Valencia, and also at the Department of Physics of the University of Bologna, the Schrödinger Institute in Vienna and the Institut d'Astrophysique de Paris. We wish to thank our colleagues and collaborators in these institutions, and in particular Roberto Balbinot for a critical reading of the draft of the book. We also wish to thank the students who attended a Ph.D. course, delivered by both authors at the University of Valencia, that was based on preliminary notes of the present book. We found very useful the comments and questions posed by the students and our young collaborators Sara Farese and Gonzalo Olmo to improve the pedagogical style of the text. A special thanks to all our past and present collaborators, from whom we have benefited a lot. Finally we acknowledge financial support from the Spanish Ministerios de Educación y Ciencia y Tecnología, the Generalitat Valenciana, and the collaborative program CICYT-INFN.

Alessandro Fabbri and José Navarro-Salas

Contents

Chapter 1

Introduction

Black holes are among the most fascinating predictions of Einstein's theory of general relativity. They are an exotic, but natural outcome of the very basic feature of the gravitational interaction. The universality of gravity, nicely expressed by the equivalence principle, allows to produce an accumulative effect that can result in a very strong gravitational field. Indeed, it can be so strong that it cannot be counterbalanced by other forces and continues to grow up until extreme situations. Since gravity affects the spacetime geometry, the gravitational field produced by matter could become so strong as to substantially modify the ordinary causal structure of spacetime, and produce a region where even light is trapped and no particle can escape to infinity.

Quantum mechanics enters immediately into this game and its first consequence is to prevent, partly, this extreme situation. Chandrasekhar showed that the quantum degeneracy pressure of electrons can balance the gravitational force and avoid a complete implosion. However, this requires a limiting mass for the star of about 1.4 times the solar mass. Above this limit the full gravitational collapse can be prevented only by the degeneracy pressure of neutrons, but this requires again an upper mass limit (2–3 solar masses). Therefore for sufficiently massive stars, that do not throw away enough matter or radiation to reach the neutron star limit, complete collapse is inevitable and a black hole forms.

Nevertheless, quantum mechanics seems to continue conspiring against the black hole, but in a different way. In a remarkable discovery Hawking showed how this takes place. Black holes are not as "black" as general relativity predicts, but rather radiate thermally all types of existing quanta, although for kinematical reasons light particles dominate the emission. This is a very small effect, at least initially, for black holes created from gravitational collapse. Hawking's discovery, which can be rederived from different perspectives, puts forward an intriguing and close relation between black holes and thermodynamics. This astonishing result is obtained using the approximation that the spacetime background is kept fixed at all times. This is an accurate approximation for the first period of the evaporation, where the effect is small, and, likely, a reasonable approximation until the black hole reaches Planck-size. Eventually, by extrapolation of the results, the black hole will disappear. The conspiracy of quantum mechanics is such that it turns, as a "boomerang" effect, against itself. Indeed, the resulting physical picture, as it was pointed out by Hawking himself almost thirty years ago, seems not to be compatible with the principles of quantum mechanics. The type of radiation emitted does not allow the recovery of the information about the star from which the black hole was created. Therefore, with the disappearance of the black hole this information will be lost forever. But this is forbidden by the basic principles of quantum mechanics itself.

However, there is no definitive picture of the full evaporation process and the reason comes out immediately. The fixed background approximation ignores the effects of the radiation on the spacetime geometry, in other words, the backreaction effects. They play an important role, even before reaching the Planck scale, when a still unknown quantum theory of gravity is expected to dominate the process. They need to be taken into account to have a detailed view of the evaporation process, and indeed they could serve as an inspiration to attack the deep problems or paradoxes of black hole evaporation mentioned above. The hope of many researchers is that, at the end, quantum mechanics will keep conspiring in such a way that a full quantum gravity approach will modify Hawking's original picture and allow information retrieval. Most remarkably, Hawking himself now seems to be converging towards this belief after almost thirty years of skepticism.

In general it is hard to try to go beyond the fixed background approximation to include the backreaction, even at the semiclassical level. This requires to solve the so called semiclassical Einstein equations

$$G_{\mu\nu} = 8\pi \langle \Psi | T_{\mu\nu}(g_{\alpha\beta}) | \Psi \rangle \ . \tag{1.1}$$

This is not at all a purely technical problem. First of all one needs to have an explicit expression for the expectation values $\langle \Psi | T_{\mu\nu}(g_{\alpha\beta}) | \Psi \rangle$ for a large family of metrics (including those that could be potentially the solution of the semiclassical equations). Moreover, these quantities also depend on the quantum state of the matter $|\Psi\rangle$, and this is a rather non-trivial issue.

To properly model the process of black hole evaporation one should bypass these difficulties somewhat. This is a remarkable open problem which should be addressed, in the appropriate limit, by any theory containing general relativity and quantum mechanics.

The organization of the material follows the above brief historical "tour". In Chapter 2 we start by briefly overviewing the Oppenheimer-Snyder model and the main features of the gravitational collapse. After this we introduce the basic ingredients of stationary (charged and rotating) black holes, using the simplest one, i.e., the Schwarzschild solution. To make the presentation accessible for a wide community of scientists we have based our arguments, as much as possible, on the equivalence principle to derive the basic results. The Kruskal coordinates, usually introduced for global purposes, are motivated here on a local basis in terms of locally inertial coordinates, where the intuition of readers more familiar with flat spacetime physics is more solid. Nevertheless the global aspects of the solutions are, of course, very important and we also pay special attention to this issue. In addition to the most popular notion of "event horizon" we also introduce the concept of "apparent horizon", which is of special relevance in the analysis of time dependent settings such as the evaporation process of Chapter 6. We also present the intriguing formal analogy between the "laws of black hole mechanics" and the laws of thermodynamics. We point out the loophole of the analogy, which will be filled in as a result of Hawking's discovery (to be presented in Chapter 3). Finally we also introduce black hole solutions in theories different from pure general relativity, motivated by the fact that they will be the inspiration of the model proposed by Callan, Giddings, Harvey and Strominger, the main theme of Chapter 6.

The whole Chapter 3 is devoted to the Hawking effect. We shall follow the original derivation of Hawking, together with the works of Parker and Wald, to explicitly show that the radiation emitted by a black hole at late times is exactly described by a thermal density matrix (modulated by a "grey-body" factor). We divide the derivation in two steps aiming to show in a clear way the skeleton of the argument and separate the technicalities from the main physical ideas. The simplest possible model, involving the matching of Minkowski and Schwarzschild spacetimes, contains all the ingredients that produce the Hawking radiation: the existence of an event horizon in a non-stationary spacetime. The intention is to introduce the Hawking radiation in an elementary way and make it more accessible to readers with a basic background on general relativity. The second step is to show that the late time thermal radiation obtained in this simplified spacetime, mimicking a gravitational collapse, is insensitive to the details of a realistic collapse. Apart from this, the derivation is quite standard. We based it on the properties of the late time Bogolubov transformations and the features of wave propagation in the Schwarzschild geometry. We finish the chapter presenting the physical implications of the Hawking effect. We shall discuss in detail the challenges that black hole evaporation poses to the interphase of quantum mechanics and general relativity. The possible breakdown of quantum predictability is the cornerstone of this more conceptual discussion. It will serve to warn the reader that not everything in this subject is well understood.

In Chapter 4 we approach the Hawking effect from a different perspective, aiming to get a better understanding. Since the basic feature is the existence of a horizon we simplify the Schwarzschild geometry working with its "near-horizon approximation" Rindler geometry, which is nothing else but a wedge of Minkowski space. We want to remark that it is quite common in the literature to discuss Rindler space and the Unruh effect before considering its curved spacetime "analog". However, we prefer to present things the other way around. The Unruh effect is even more shocking than the Hawking effect for readers that have learned "quantum field theory" with too much emphasis on "Poincaré symmetry" (see at this respect [Wald (1994)]). For this reason we rederive Hawking radiation from the near-horizon Rindler geometry and, only as a by-product, the Unruh effect. This result finds justification from the equivalence principle. All this discussion allows us to introduce into the game, in a natural way, one of the strongest spacetime symmetries in physics: the two-dimensional conformal symmetry. The theory of conformal fields is usually applied to analyse second-order phase transitions in condensed matter systems [Di Francesco *et al.* (1997)] and it is also widely used in the formulation of string theory [Polchinski (1998)]. We shall show how it also plays an important role in the Hawking effect. The thermal character of the emitted radiation can be nicely reobtained in terms of the conformal properties of the correlation functions of the effective matter theory around the horizon. This will offer a new perspective to grasp the deep physical meaning of the Hawking effect.

In Chapter 5 we approach the problem of determining an expression for the expectation value of the stress tensor for the matter fields in a curved background. This is the first obstacle one faces before attacking the backreaction problem. This is the fundamental problem of "quantum field theory in curved spacetime" and it is indeed very hard. No exact analytical expression is known even for the fixed Schwarzschild spacetime. Only for conformally flat geometries and for conformal fields one has a solution to the problem. We are far away from having such an expression in a generic four-dimensional spacetime. Reduction of the gravity-matter system under spherical symmetry leads to an effective two-dimensional theory that can serve as the starting point for the analysis. The advantage of doing so is that in two dimensions every metric is conformally flat, and therefore one expects to find an expression for the expectation value of the stress tensor. The near-horizon approximation, together with the ensuing conformal symmetry, allows to find an exact solution to this problem. This result is usually presented as a consequence of the fact that the trace anomaly determines univocally all the components of the stress tensor. An understanding of this, at least for a non-expert reader, will require a derivation of the trace anomaly itself. We want to avoid entering into the technicalities of regularization in curved spacetime and for this reason we present an unconventional derivation of the expectation values of the stress tensor, including the trace anomaly itself. Our derivation is based on the use of locally inertial coordinates and of the transformation law of the normal ordered stress tensor obtained in Chapter 4. In this way a non expert reader can easily follow the derivation. This approach also allows to determine an expression for the stress tensor when the spherically symmetric reduction is not restricted to the near-horizon geometry. We consider in detail the problem of selecting the appropriate quantum states relevant for quantum black hole physics. We stress that this is a highly non-trivial problem, especially when addressing the backreaction problem for the case of evaporating black holes. The last part of the chapter is devoted to this issue.

The first model which bypasses all these difficulties was proposed twelve years ago in [Callan *et al.* (1992)] and describes the near-horizon properties of near-extremal stringy black holes. Starting from it, the first analytic description of the process of black hole formation and evaporation was given in [Russo *et al.* (1992b)]. This is the central theme of Chapter 6. We also study the near-horizon dynamics of near-extremal Reissner–Nordström black holes. Also in this case the semiclassical equations are solvable, despite the fact that the selection of the appropriate quantum state to describe the evaporation is more involved. On the basis of these exact solutions we discuss the implications of this approach for what concerns the information loss paradox, but also its limitations.

Notation: throughout this book we shall follow the conventions for the metric and the curvature used in [Misner *et al.* (1973)]. The metric signature is $(- + ++)$ and the definition of the curvature tensors are: $R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \cdots$, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$. In most of the book we use geometrized units $G = 1 = c$. However, to emphasize the quantum aspects we maintain explicitly Planck's constant ($\hbar$) in the formulae.

Chapter 2

Classical Black Holes

2.1 Modeling the Gravitational Collapse

A physically realistic treatment of the process of gravitational collapse is certainly a hard one. However, as it historically happened, the assumption of spherical symmetry allows to simplify considerably the analysis keeping the main physical features of the process, in particular the eventual emergence of a region where gravity is so strong that even light cannot escape to infinity and remains trapped. To visualize and simplify further the process we identify two regions in the spacetime of a collapsing star. The first is the interior one described by a well-behaved metric, with all the complications involved in describing the matter making up the star. The second is the exterior region, which is in general a portion of a spherically symmetric vacuum solution to Einstein's field equations. Due to Birkhoff's theorem it is given by a portion of the Schwarzschild spacetime (obviously, in a realistic collapse the exterior region is much more complicated than that and only "at late times" it relaxes down to a vacuum geometry). The dynamics involved in the process will make the radius of the star decrease with time. This means that as time goes on bigger and bigger portions of the exterior vacuum solution will be needed for the description of the physical spacetime. Depending on the total mass M of the collapsing object we know that there are different possible outcomes. In particular we know that there is a mass threshold of order 1.4 times the solar mass, called the Chandrasekhar limit [Chandrasekhar (1931)], below which the collapse will stop and the final state of the collapsed object will be a white dwarf. If the initial mass is bigger but below a few solar masses the outcome is instead a neutron star.

One can get an estimate of the Chandrasekhar limit in the following simple way [Carter (1972); Hawking and Ellis (1973)].[1] A high density cold star at the end of the collapse, after having exhausted all its nuclear fuel, will be supported by fermion degeneracy pressure either of the electrons (white dwarf) or of neutrons (neutron star). Also, because electrons are much lighter than neutrons ($m_e \ll m_n$) electron degeneracy is the first to take place. Assuming one electron in a cube of size of its Compton wavelength λ_e we have that the average electron momentum $p_e \sim \frac{\hbar}{\lambda_e}$ is given, due to the uncertainty principle, by

$$p_e \sim \hbar n^{1/3} \ , \tag{2.1}$$

where n is the number density of electrons composing the star. For non-relativistic electrons the total energy of the star is given by

$$E \sim E_K - \frac{GM^2}{r} \ , \tag{2.2}$$

where r is the radius of the star and the kinetic term E_K is obtained by summing over the kinetic energies of the electrons

$$E_K \sim nr^3 E_{electron} \sim nr^3 \frac{p_e^2}{m_e} \sim \frac{\hbar^2 n^{5/3} r^3}{m_e} . \tag{2.3}$$

Observing that the mass of the star is due to protons and neutrons (with $m_p \sim m_n$)

$$M \sim n m_n r^3 \tag{2.4}$$

then we can eliminate n in E_K and get

$$E_K \sim \frac{\hbar^2}{m_e} \left(\frac{M}{m_n} \right)^{5/3} \frac{1}{r^2} \ . \tag{2.5}$$

The stable endpoint of the collapse is then the value of r minimizing E, namely

$$r_{min} \sim \frac{\hbar^2}{G m_e M^{1/3} m_n^{5/3}} \ . \tag{2.6}$$

It is important to stress that the condition for the validity of the non-relativistic approximation $p_e/m_e \ll c$ implies that $n \ll (m_e c/\hbar)^3$ which, in

[1] See also [Shapiro and Teukolsky (1983); Padmanabhan (2001)] for a detailed study.

turn, using Eq. (2.4) with $r = r_{min}$ is equivalent to

$$M \ll M_C = \frac{1}{m_n^2}\left(\frac{\hbar c}{G}\right)^{3/2} . \tag{2.7}$$

In the relativistic case $E_{electron} \sim p_e c$ and therefore

$$E_K \sim \hbar c \left(\frac{M}{m_n}\right)^{4/3} \frac{1}{r} . \tag{2.8}$$

Since E_K and the gravitational energy $E_G \sim -\frac{GM^2}{r}$ are both of order $1/r$ then equilibrium is possible only if

$$\hbar c \left(\frac{M}{m_n}\right)^{4/3} = GM^2 , \tag{2.9}$$

which in turn implies

$$M = M_C . \tag{2.10}$$

The radius of the star can then be estimated taking into account that, in the relativistic regime, $m_e c \leq p_e$. Therefore $n \geq (m_e c/\hbar)^3$ and then

$$r_C \leq \frac{1}{m_e m_n}\left(\frac{\hbar^3}{Gc}\right)^{1/2} . \tag{2.11}$$

The typical mass density in this situation is $\rho \sim 10^7 - 10^{11}\mathrm{g/cm}^3$. The critical value for the mass so obtained is the so called Chandrasekhar limit [Chandrasekhar (1931)] and corresponds to $\sim$ 1.4 solar masses. Stars of mass bigger than M_C cannot be maintained in equilibrium by the degeneracy pressure of electrons. Note that the existence of this limit is a direct consequence of the shift in the compressibility index of the Fermi gas, from 5/4 to 4/3, when the velocities approach the relativistic regime.[2]

In the above extreme situation the electrons can produce neutrons, via inverse β decay

$$e^- + p^+ \rightarrow n + \nu_e \tag{2.12}$$

once the electrons have reached the energy $(m_n - m_p)c^2$. The conversion of electrons into neutrons makes the star unstable until it is converted into a neutron star. Then one would be tempted to use the same ideal Fermi gas approximation to find equilibrium configurations due to neutron degeneracy

[2]In the non-relativistic situation the equation of state of the (cold) degenerate Fermi gas is $PV^{\frac{5}{3}} = const.$, while in the relativistic regime one has $PV^{\frac{4}{3}} = const.$

pressure. This can be easily done, since it amounts to replace m_e with m_n in the previous formulae. The critical mass M_C stays the same, but the estimation for the radius has changed to

$$r_C \sim \frac{1}{m_n^2}\left(\frac{\hbar^3}{Gc}\right)^{1/2} = \frac{GM_C}{c^2} \quad . \tag{2.13}$$

On one hand this shows that neutron stars are much more compact than white dwarfs. However, since r_C is of the order of the Schwarzschild radius, general relativistic corrections can no more be neglected and must be included in the analysis.

More reasonable approximations for the matter inside the star have to be considered. This requires to solve Einstein's equations (from now on we shall use geometrized units $G = 1 = c$)

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.14}$$

for a matter stress tensor of the *perfect fluid* form[3]

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \ , \tag{2.15}$$

where u^μ is the four-velocity of a matter point, ρ the energy density and p its pressure. Spherical symmetry leads to the Oppenheimer–Tolman–Volkoff equation [Oppenheimer and Volkoff (1939); Tolman (1939)] for hydrostatic equilibrium[4]

$$\frac{dp}{dr} = -(p+\rho)\frac{m(r) + 4\pi r^3 p}{r(r-2m(r))} \ , \tag{2.16}$$

where

$$m(r) = 4\pi \int_0^r dr' \rho r'^2 \ . \tag{2.17}$$

This together with the equation of state $p = p(\rho)$ provides a system of differential equations which allows to study the equilibrium configurations of cold matter. [Oppenheimer and Volkoff (1939)] were able to find, using a crude model of neutron stars, an upper mass limit for equilibrium. A detailed series of investigations were initiated, after the second world war, by Wheeler and his group and they confirm that for very high densities the

[3]For an introductory account see [Schutz (1985)].

[4]In the Newtonian limit it reduces to the usual hydrostatic equation $\frac{dp}{dr} = -\frac{\rho m(r)}{r^2}$.

only possible equilibrium configurations are neutron stars,[5] with an upper bound for the mass to be of the order of 2–3 solar masses. This indicates that a dramatic scenario happens when the initial mass of the collapsing star is above the threshold for neutron star formation. In this case no known force is capable to counterbalance the gravitational force responsible for the collapse until complete implosion, i.e., the radius of the star will decrease without limit down to, ideally, zero size. In this case a black hole is said to be formed. In 1957, Wheeler summarized the situation as follows [Adams *et al.* (1958)]: "Of all the implications of general relativity for the structure and evolution of the universe, this question of the fate of great masses of matter is one of the most challenging ... Perhaps there is no final equilibrium state: this is the proposal of Oppenheimer and Snyder ..."

However, before the black hole idea was generally accepted, additional investigations were required. The main reason was that a black hole possesses "very uncomfortable and disturbing properties", as was first described by the model of Oppenheimer and Snyder.

2.1.1 *The Oppenheimer–Snyder model*

Motivated by the earlier striking result of the existence of an upper mass limit for neutron stars, [Oppendeimer and Snyder (1939)] studied the process of gravitational collapse ending with a black hole. They had the intuition that the internal pressure has no essential role since, in any case, no internal force is able to stop the gravitational contraction. For this reason, and for mathematical simplicity, they constructed a model neglecting the internal pressure and assumed a uniform density, see Fig. 2.1. This leads to model the interior of the star as a homogeneous and isotropic ball of dust described by the energy-momentum tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu \ . \tag{2.18}$$

Homogeneity means that ρ can only be a function of time τ in comoving and synchronous coordinates, i.e.,

$$\begin{aligned} u^\mu &= \delta^\mu_\tau \\ ds^2 &= -d\tau^2 + g_{ij}dx^i dx^j \ . \end{aligned} \tag{2.19}$$

[5]This result was obtained at the end of the fifties and is reported in [Harrison *et al.* (1965)].

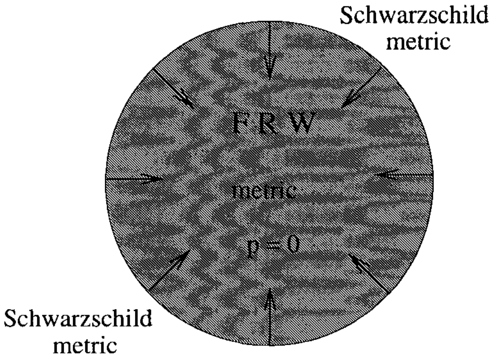

Fig. 2.1 A collapsing star in the Oppenheimer–Snyder model.

The solution to the field equations inside a spherical ball of dust is known and corresponds to a portion of a closed Friedmann–Robertson–Walker (FRW) spacetime of the form[6]

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega^2_{(3)} \ , \tag{2.20}$$

where $d\Omega^2_{(3)}$ is the metric on the unit three-sphere

$$d\Omega^2_{(3)} = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{2.21}$$

The closed geometry (three-sphere), instead of the open or flat universes, arises because we make the physical assumption that the star is at rest ($R = R_0$) at the initial time $\tau = 0$. Flat or open three-geometries would have meant that the star collapsed from infinite radius.

Let us now briefly recall how to determine the function $R(\tau)$. The conservation laws

$$\nabla^\mu T_{\mu\nu} = 0 \tag{2.22}$$

give the equation

$$\frac{d}{d\tau}\rho R^3 = 0 \ . \tag{2.23}$$

Therefore

$$\rho = \frac{A}{R^3} \ , \tag{2.24}$$

[6]See, for instance, [Misner *et al.* (1973); Stephani (1982)].

where A is a constant that, as we will see, is related to the initial radius R_0. The only relevant Einstein equation to be determined is the $\tau\tau$ component, namely $R_{\tau\tau} = 8\pi T_{\tau\tau}$ giving

$$\frac{1}{R^2}\left(\frac{dR}{d\tau}\right)^2 = -\frac{1}{R^2} + \frac{8}{3}\pi\rho = -\frac{1}{R^2} + \frac{8\pi A}{3R^3} . \tag{2.25}$$

We can then write

$$\left(\frac{dR}{d\tau}\right)^2 = -1 + \frac{R_0}{R} , \tag{2.26}$$

where we have defined

$$R_0 = \frac{8}{3}\pi A . \tag{2.27}$$

R_0 can be interpreted to be the value of $R(\tau)$ at the initial time $\tau = 0$, when the star starts to contract ($R(\tau) \leq R_0$). Actually, this is the maximum radius the ball can have, after which ($\tau > 0$) the collapse proceeds until $R = 0$ in a *finite time* $\Delta\tau$. This can be seen by integrating Eq. (2.26) close to $R \to 0$, leading to

$$(\tau - \Delta\tau) \sim \frac{R^{3/2}}{\sqrt{R_0}} . \tag{2.28}$$

The exact form of $R(\tau)$ can be given in parametric form by introducing a cycloidal coordinate η such that

$$\tau = \frac{R_0}{2}(\eta + \sin\eta) . \tag{2.29}$$

In terms of η the solution for R is

$$R = \frac{R_0}{2}(1 + \cos\eta) . \tag{2.30}$$

It is easy to see that for $\eta = \pi$ we reach $R = 0$. Therefore, we can determine $\Delta\tau$ immediately

$$\Delta\tau = \frac{\pi R_0}{2} . \tag{2.31}$$

At the point $R = 0$ the star has reached *infinite density* and moreover *geodesics* followed by the dust particles *cannot be extended beyond* $\tau = \Delta\tau$. For this reason they are called *incomplete* and this signals the presence of spacetime singularities [Geroch (1968); Wald (1984)]. In fact, all these features conspire to make the spacetime to develop there a *curvature singularity*,[7] as it can be inspected directly by calculating the curvature tensor.

Now that we have determined the spacetime metric associated to the region interior to the spherical star, let us have a look at what happens outside. There we have $T_{\mu\nu} = 0$ and therefore we must select a vacuum solution to the field equations. This is not difficult, since Birkhoff's theorem [Hawking and Ellis (1973)] ensures that the only spherically symmetric vacuum solution to the equations of general relativity is given by the Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\Omega^2 \ . \tag{2.32}$$

Here M is the mass of the star and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit two-sphere. We still have to relate, and this is very important, the interior and the exterior metrics along the star surface. For the interior metric the surface corresponds to a fixed coordinate $\chi = \chi_0$, for all τ, since fixed values of χ, θ and ϕ are timelike geodesics. In contrast, for the exterior metric the surface of the star is described by a (timelike geodesic) curve $r = r(t)$. Continuity of the metric implies that the induced metric must be the same on both sides of the star surface. This means that

$$-d\tau^2 + R^2(\tau)\sin^2\chi_0 d\Omega^2 = -\frac{(1 - \frac{2M}{r})^2 + (\frac{dr}{dt})^2}{(1 - \frac{2M}{r})} dt^2 + r(t)^2 d\Omega^2 \ . \tag{2.33}$$

In particular, by comparing the radius of the two-spheres we have that

$$r(t) = R(\tau)\sin\chi_0 \ , \tag{2.34}$$

which implies, evaluated at the initial point, that

$$r_0 = R_0 \sin\chi_0 \ , \tag{2.35}$$

where r_0 is the initial radius of the star. The relation between the mass of the star M and the internal parameters R_0 and χ_0 can be determined easily in the following way. By considering a dust particle on the surface of the star, due to the absence of any pressure then it must follow a geodesic both of the interior cosmological-type metric and of the exterior Schwarzschild spacetime. The interior trajectory has already been worked out in Eqs. (2.29) and (2.30). In the exterior Schwarzschild spacetime we similarly need the equation for a radial (timelike) geodesic starting at the maximum

[7] In the interior "cosmological" region this corresponds to the "big crunch" singularity.

radius r_0 at proper time $\tau = 0$ and then falling towards $r = 0$. The conserved quantity for geodesics $\tilde{E} \equiv -u_0 = -g_{0\mu}u^\mu$, together with the relation $u^\mu u_\mu = -1$, imply the following equations

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r} , \tag{2.36}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - (1 - 2M/r) , \tag{2.37}$$

where the numerical constant $\tilde{E}$, energy per unit mass, takes the value $\tilde{E}^2 = 1 - 2M/r_0$. We observe that Eq. (2.37) is similar to Eq. (2.26), which corresponds to the interior geometry. Therefore the solution $r = r(\tau)$ can be given in a similar parametric form [Misner *et al.* (1973)]

$$\begin{aligned} r &= \frac{r_0}{2}(1 + \cos\eta) , \\ \tau &= \left(\frac{r_0^3}{8M}\right)^{1/2} (\eta + \sin\eta) . \end{aligned} \tag{2.38}$$

By comparing the two trajectories one finally gets

$$M = \frac{R_0}{2} \sin^3 \chi_0 . \tag{2.39}$$

It is easy to check, from the above relations, that the full equality (2.33) is guaranteed. Moreover, it can be shown that not only the metric is continuous along the star's surface, but also the extrinsic curvature (which is proportional to the first derivatives of the metric) is continuous too. This means, as it should for physical consistency, that the matching is smooth [Misner *et al.* (1973)].

We have therefore succeeded in finding the full solution describing the spacetime of a collapsing sphere of dust. From the point of view of an ideal observer located on the star's surface the collapse starts at proper time $\tau = 0$, when the star has radius r_0, and then, as we have already remarked, after a finite amount of proper time $\Delta\tau$ the star and the observer will be destroyed at the spacetime singularity $r = 0$. It is very interesting to inspect what is the description of the collapse from the point of view of an asymptotic observer located very far from the star's surface. Straightforward integration of the geodesic equations gives

$$t = \int_{r_0}^{r} \frac{\sqrt{1 - 2M/r_0}}{\sqrt{2M/r - 2M/r_0}} \frac{dr}{(1 - 2M/r)} . \tag{2.40}$$