Real Analysis I

Course name :- Real Analysis I
Course Code :- MAT1205
Credits :- 2
Evaluation Criteria :-
Tutorial and Assignment :- 10%
Attendance : 05%
Mid Sem Exam :- 25%
End Sem Exam :- 60%
References :-
1. Real Analysis by Rudin
2. Elements of Real Analysis by Shanthi Narayan

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Chapter 01 :- Real number System:-

01. Real number $\Rightarrow \mathbb{R}$

$\mathbb{R} = \{\dots, e, \sqrt{2}, \pi\}$

02. Integers $\Rightarrow \mathbb{Z}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$

03. $\mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{Z}^+$

04. $\mathbb{Q}$ = Set of rational number.

$= \left\{\frac{p}{q} \; (p, q \in \mathbb{Z}), q \neq 0\right\}$

Eg:- $1 \in \mathbb{Q}$ $\quad -\frac{1}{2} \in \mathbb{Q}$ $\quad \frac{\sqrt{2}}{1} \notin \mathbb{Q}$ $\quad \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$

$\mathbb{Q} \cap \mathbb{Q}' = \phi$

05. Set of Prime number $\Rightarrow \mathbb{P}$ [It has 2 factors]

$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$

$5 = 1 \times 5 \rightarrow$ Trivial factors

$10 = 1 \times 10$ $\rightarrow$ non trivial factors

2×5

06. Field Axioms :- $\mathbb{F}$

Def^n :- A field is a set $\mathbb{F}$ together with two binary operations.

$+ : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ (called addition)

and $\overset{(\cdot)}{\times} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ (Called multiplication).

Such that for all ($\forall$) $x, y, z \in \mathbb{F}$

$A = \{1, 2, 3\}$

$A = \{1, 2, 3\}$ $\quad A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$B = \{5, 6\}$

$A \times B = \{(1,5), (1,6), (2,5), (2,6), (3,5), (3,6)\}$

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$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$(2, 71) \in \mathbb{R} \times \mathbb{R}$ $(2, 71) \in \mathbb{R} \times \mathbb{R}$

$2 + 71 = 73 \in \mathbb{R}$ $2 \times 71 = 142$

$(e, \pi) \in \mathbb{R} \times \mathbb{R}$

$e + \pi \in \mathbb{R}$

A1 :- $x+y = y+x$ (Addition is Commutative)

A2 :- $(x+y)+z = x+(y+z)$ (Addition is associative)

A3 :- There is a element $0 \in \mathbb{F}$ Called additive identity such that $x+0 = 0+x = x \; \forall x \in \mathbb{F}$

eg:- $(x, 0), (0, x)$ $+ : \mathbb{F} \times \mathbb{F}$

$x + 0 = x$

$0 + x = x \in \mathbb{F}$

$\therefore \boxed{x+0 = 0+x.}$

A4 :- For each $x \in \mathbb{F}$, there is an element $-x \in \mathbb{F}$ Called additive inverse of x, Such that

$x + (-x) = 0 = (-x)+(x)$

eg:- $(2, -2) \in \mathbb{F} \times \mathbb{F}$

$2 + (-2) = 0$

M1 :- $x \times y = y \times x$ (Multiplication is Commutative)

M2 :- $(x \times y) \times z = x \times (y \times z)$ (Multiplication is Associative)

eg:- $(2, 3) \in \mathbb{F} \times \mathbb{F}$, $(6, 4) \in \mathbb{F} \times \mathbb{F}$

$2 \times 3 = 6 \in \mathbb{F}$ $6 \times 4 = 24$

$(2 \times 3) \times 4 = 24$

$(3,4) \in \mathbb{F} \times \mathbb{F}$, $(2,12) \in \mathbb{F} \times \mathbb{F}$

$3 \times 4 = 12 \in \mathbb{F}$ $\quad$ $2 \times 12 = 24 \in \mathbb{F}$

$$\underbrace{\underbrace{(2 \times 3)}_{6} \times 4}_{24} = \underbrace{2 \times \underbrace{(3 \times 4)}_{12}}_{24}$$

M3 :- There is an element $e \in \mathbb{F}$, called multiplicative identity such that $x \times e = x = e \times x$

M4 = For each $x \in \mathbb{F} \setminus \{0\}$ there is an element $x^{-1} \in \mathbb{F}$ multiplicative inverse of x such that $x \times x^{-1} = e$.

D1 :- $x \times (y+z) = (x \times y) + (x \times z)$

Then $(\mathbb{F}, +, \times)$ called field.

Order Axioms:

Defⁿ :-

An ordered field is field $\mathbb{F}$ on which an order relation "$<$" is defined such that

O1. (Trichotomy)

"$\forall$" $x, y \in \mathbb{F}$ exactly one of the following holds.

$$x < y, \quad x = y, \quad y < x.$$

AND $\wedge$

O2. (Transitivity) OR $\vee$

$\forall$ $x, y, z \in \mathbb{F}$ XOR Δ $\oplus$

and $x < y \wedge y < z$ then $x < z$

O_3 For all $x, y, z \in \mathbb{F}$
$x < y \Rightarrow x + z < y + z$
Furthermore,
if $z > 0$, then
$xz < yz$

eg:- $\mathbb{R}$ is an ordered field. H/W:- Cheak ordered field axioms.

eg:- $\mathbb{C} = \{a + ib \,/\, a, b \in \mathbb{R}\}$ H/W Cheak ~~a~~ the field axioms.

eg:- $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$
$x = q \cdot 7 + r \quad (0 \leq r \leq 7)$

Completemess Axioms :-

Def^n :-

Let A be a Subset of an ordered field $\mathbb{F}$

1. An element $a \in \mathbb{F}$ is an upper bound for A
ඉහළ සීමාව
if $x \leq a \quad \forall x \in A$

$b \leq x \leq a$
lower bound, upper bound

2. An element ~~b~~ $b \in \mathbb{F}$ is a lower bound for A
if $b \leq x \quad \forall x \in A$.

3. A is Said to be bounded if it has both an upper bound and a lower bound.
ඉහළ සීමාවක්, හා පහළ සීමාවක් තිබේනම් A ට සීමිත යැයි කියනු ලැබේ:

4. An element $M \in \mathbb{F}$ is the least upper bound for A if
කුඩා ඉහළ සීමාව

(1) M is an upper bound for A and
(2) $\forall$ upper bounds m for A we have $M \leq m$.
M is Called Supremum of A $\Rightarrow$ Sup A = M.

5. An element ~~k~~ $K \in F$ is the Greatest lower bound for A if
විශාලතම පහළ සීමාව

$k \leq k$

(1) k is ~~to~~ an lower bound for A and
(2) ~~to~~ lower bounds k for A we have
$k \leq K$
K is called Infimum of $A \Rightarrow \text{Inf } A = K$

eg:- $A = (0, 1)$
$\text{Inf } A = 0$
$\text{Sup } A = 1$

eg:- $A = (-\alpha, 2]$
$\text{Sup } A = 2$
$\text{Inf } A$ = does not exist

Note:- * If $\text{Sup } A \in A$ then $\text{Sup } A$ = Maximum of A
* If $\text{Inf } A \in A$ then $\text{Inf } A$ = minimum of A

Def^n:-

An ordered field $\mathbb{F}$ is said to be Compleate if every non.empty subset S of $\mathbb{F}$ which is bounded above has the least upper bound in $\mathbb{F}$

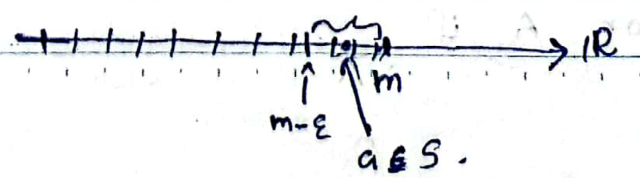
Def^n:-

An ordered field $\mathbb{F}$ is said to be Complete if every non empty subset A of $\mathbb{F}$ which is bounded below has the lower upper bound in $\mathbb{F}$

Th^m:-

Let S be a non empty subset of an ordered field $\mathbb{F}$ and $M \in \mathbb{F}$, Then $m = \text{Sup } S$ if and only if (iff)

i. m is an upper bound for S and. (1)
ii) for any $\varepsilon \in \mathbb{F}$ with $\varepsilon > 0$, then there exists an element ~~$S \in$~~ $a \in S$



Such that

~~S~~. $M - \varepsilon < a$.

Pf:- (Proof / నిరూపణ)

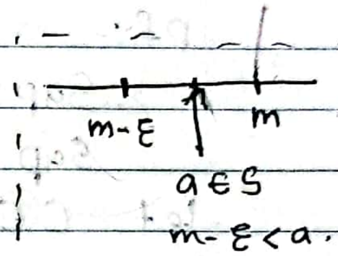
$S \neq \phi$, $S \subseteq \mathbb{F}$

$m = \sup S \iff$

Assume that $m = \sup S$, $S \subseteq \mathbb{F}$, $S \neq \phi$.

1). then by def^n m is un upper bound for S

11) If $\varepsilon \in \mathbb{F}$, $\varepsilon > 0$ for which $m - \varepsilon \geq a$ ~~$\forall$~~ $a \in S$



$\Rightarrow$ $m - \varepsilon$ is an upper bound for S

but $m \geq m - \varepsilon$ and m = least upper bound of S ($\sup S$)

$\therefore$ Contradiction

$\therefore$ for some $a \geq m - \varepsilon$.

$\Leftarrow$ assume that (i) and (ii) hold since S is bounded above

$\Rightarrow$ S has a supremum say k

by (1) m is un upper bound for S

$\Rightarrow k \leq m$. $\boxed{k = m}$

if $k < m$ then let $m - k = \varepsilon > 0$ by (ii) and ~~and~~ then there exist an element $b \in S$ such that $m - \varepsilon < b \leq k$.

$m - (m-k) < b \leq k$

$m - (m-k) < b \leq k$.

$k < b \leq k$

$k < k$.

there fore, $k = m$.

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Thm :-

Let A and B be non-empty subsets of $\mathbb{R}$ which are bounded above then, the set $S = \{a+b \mid a \in A, b \in B\}$ is bounded above and $\sup S = \sup A + \sup B$

$\sup(A+B) = \sup A + \sup B$.

Pf :-

$\because$ $\sup A$ exists

$\sup B$ exists

let $c \in S$. then $c = a+b$ such that $a \in A$ and $b \in B$.

$\forall a \in A$ $\quad a \leq \sup A \rightarrow$ ①

$\forall b \in B$ $\quad b \leq \sup B$ — ②.

① + ② $\Rightarrow$ $a+b \leq \sup A + \sup B$ $\quad \forall a \in A, b \in B$.

any element of $c \leq \sup A + \sup B \rightarrow$ ③ $\because$ any $c \in S$

$\Rightarrow$ $\sup A + \sup B$ is an upper bound of S therefore S is bounded above set

$\Rightarrow$ $\sup S$ exist. — ④

$\Rightarrow$ ③, ④ $\Rightarrow$

$\sup S \leq \sup A + \sup B$ — ⑤.

to show that,

$\sup S \geq \sup A + \sup B$

let $\varepsilon > 0$, $\varepsilon/2 > 0$

then there exists an element $x \in A$ such that

$$\sup A - \varepsilon/2 < x. \quad — ⑥$$

Similarlly $\Rightarrow$ // ^{y} $\varepsilon/2 > 0$.

Then there exist an element $y \in B$ such that
$\text{Sup } B - \varepsilon/2 < y$ — ⑦.

⑥ + ⑦ ∴ $\text{Sup } A - \varepsilon/2 + \text{Sup } B - \varepsilon/2 < x+y$.

$$\text{Sup } A + \text{Sup } B - \varepsilon < x+y \quad \text{— ⑧}.$$

Since $x+y \in S$, $x+y \leq \text{Sup } S \rightarrow$ ⑨.

$$\text{Sup } A + \text{Sup } B - \varepsilon \leq \text{Sup } S' \quad \text{— ⑩}.$$

ε be an positive Value $\varepsilon \rightarrow 0$

$$\text{Sup } A + \text{Sup } B \leq \text{Sup } S' \quad \text{— ⑪}.$$

there fore,

⑤ + ⑪ $\Rightarrow$ $\text{Sup } S = \text{Sup } A + \text{Sup } B$.

H.W:- Let A and B be non-empty sub sets of $\mathbb{R}$ which are bounded below. then the set $A+B = S = \{a+b \mid a \in A, b \in B\}$. is bounded below.

$\text{Inf } S = \text{Inf } A + \text{Inf } B$.

$\text{Inf } (A+B) = \text{Inf } A + \text{Inf } B$.

the archimedean properly of Real numbers.

Th^{m}:-

The set $\mathbb{N}$ of natural numbers is not bounded above.

i.e.

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Pf :-

Assume that $\mathbb{N}$ is bounded above, $\therefore$ by Completeness axiom Sup $\mathbb{N}$ exists.

Let $m = \text{Sup } \mathbb{N}$

let $\varepsilon > 0$, $\varepsilon = 1$

then there exist an element $k \in \mathbb{N}$ such that $m - \varepsilon < k \leq m$.

$m - 1 < k \leq m$:-

$m - 1 < k$

$m < k + 1 \leq m$

but $k + 1 \in \mathbb{N}$

$\Rightarrow$ $m < m$.

$\therefore$ Sup $\mathbb{N}$ Does not exist.

Corrollary :-

for every real number b there exist an integer $m < b$.

Corrollary :-

given any real number x there exist an integer k Such that $x - 1 \leq k < x$

Corrollary :-

If x and y are two positive real numbers. then there exist a natural number n Such that $nx > y$

Pf:- Assume that $nx \leq y$ $\forall n \in \mathbb{N}$.
then $n \leq y/x$ $\forall$ $n \in \mathbb{N}$
$\Rightarrow$ $\mathbb{N}$ is bounded above
as Set of natural numbers are unbounded.

Home work :-

$\text{Inf } S = \text{Inf } A + \text{Inf } B$

$\text{Inf } (A+B) = \text{Inf } A + \text{Inf } B$.

Pf :- Inf A exists
Inf B exists.
Let $D \in S$, Then $D = a+b$
Such that $a \in A$ and $b \in B$.
$\forall$ $a \in A$ $a \geq \text{Inf } A$ — ①
$\forall$ $b \in B$ $b \geq \text{Inf } B$ — ②.
①+② $\Rightarrow$ $a+b \geq \text{Inf } A + \text{Inf } B$ $\forall$ $a \in A$, $b \in B$.
any element of $D \geq \text{Inf } A + \text{Inf } B$ — ③ any $D \in S$.

$\Rightarrow$ Inf A + Inf B is an lower bound of S therefore, S is bounded ~~above~~ below set.
$\Rightarrow$ Inf S exist — ④.
③ ; ④ $\Rightarrow$ $\text{Inf } S \geq \text{Inf } A + \text{Inf } B$ — ⑤.

to show that,
$\text{Inf } S \leq \text{Inf } A + \text{Inf } B$
Let $\varepsilon > 0$, $\varepsilon/2 > 0$
then there, exist an element $x \in A$ Such that
$\text{Inf } A - \varepsilon/2 \geq x$. — ⑥
$//^{y}$ $\varepsilon/2 > 0$
Then there exist an element $y \in B$ Such that
$\text{Inf } B - \varepsilon/2 \geq y$ — ⑦

$⑤ + ⑦ \Rightarrow$. $\text{Inf } A - \frac{\varepsilon}{2} + \text{Inf } B - \frac{\varepsilon}{2} \geq x + y$.

$\text{Inf } A + \text{Inf } B - \varepsilon \geq x + y$ — ⑧

Since $x+y \in S$, $x+y \geq \text{Inf } S$ — ⑨

$\text{Inf } A + \text{Inf } B - \varepsilon \geq \text{Inf } S$ — ⑩

ε be an positive value $\varepsilon \to 0$.

$\text{Inf } A + \text{Inf } B \geq \text{Inf } S$ — ⑪.

There fore :-

⑤ + ⑪ $\Rightarrow$

$\text{Inf } S = \text{Inf } A + \text{Inf } B$

$\text{Inf}(A+B) = \text{Inf } A + \text{Inf } B$

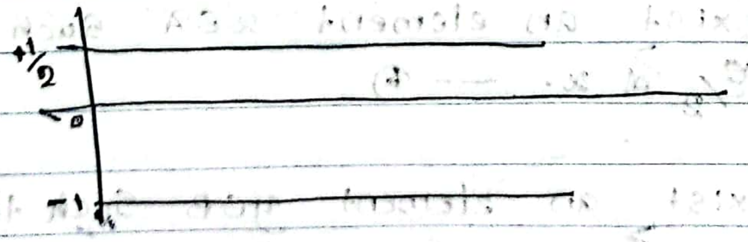
Eg:- Consider the set

$A = \left\{ \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{N} \right\}$.

(i). Show that A is bounded above & find the supremum. Is this Sup maximum of A?

(ii) Show that A is bounded below find the infimum. Is this Infimum a minimum of A?

$A = \left\{ -1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6} \dots \right\}$.



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i) Clearly, $\frac{1}{2}$ is an upper bound of A

$\therefore$ A is bounded above.

Let $M>0$ be an upper bound for A.

we must show that $\frac{1}{2} \leq M$.

Suppose by Contrarily,

$$M < \frac{1}{2}$$

Since M is upper bound of A. we have :

$$\frac{(-1)^n}{n} \leq M \ \forall \ n \in \mathbb{N}$$

In particularly, $n=2$,

$$\frac{1}{2} \leq M.$$

$$\Rightarrow \frac{1}{2} \leq M < \frac{1}{2}$$

$\therefore$ Controdiction.

$$\Rightarrow \frac{1}{2} \leq M$$

$\Rightarrow$ Sup A $= \frac{1}{2}$.

Since $\frac{1}{2} \in A$, Max A $= \frac{1}{2}$.

$\Rightarrow$ Sup A = Max A //.

ii) Clearly, -1 is an lower bound of A.

$\therefore$ A is bounded below.

Let $m<0$ be an ~~upper~~ lower bound for A.

We must Show that $-1 \geq m$

Suppose by Contrarily,

$$m > -1$$

Since m is lower bound of A,

we have $\frac{(-1)^n}{n} \geq m \ \forall \ m \in \mathbb{N}$.

In particular, $n = 1$

$-1 \geq m$,

$\Rightarrow -1 < m \leq -1$

Contradiction,

$\Rightarrow$ ~~$m > -1$~~ $\quad m \leq -1$

$\Rightarrow$ $\operatorname{Inf} A = -1$

$\therefore$ Since $-1 \in A$

$\min A = -1 = \operatorname{Inf} A$ //

Topology of the Real line

$$\longrightarrow \mathbb{R}$$

Def^n:-

Let $x \in \mathbb{R}$. The absolute value of x is difined by $|x|$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z}, \; q \neq 0 \}$ Set of rational number.

$\mathbb{Q}' =$ Set of irrational number

$= \{ \pi, e, \sqrt{2}, \ldots \}$

1) Show that $\sqrt{2}$ is irrational.

Sol^n

Suppose $\sqrt{2}$ is rational.

$\Rightarrow \sqrt{2} = p/q$. such that $p, q \in \mathbb{Z}$; $q \neq 0$.

$\boxed{(p, q) = 1}$ $\leftarrow$ relatively prime.

$$\boxed{2 \mid p \Rightarrow \frac{p}{2}}$$

$(12, 15) = 3 \rightarrow$ Greatest Common division.

$(14, 17) = 1 \rightarrow$ relatively Prime.

now $\sqrt{2} = p/q$

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2 \quad \text{—— (1)}$$

$\Rightarrow p^2$ - is an even number

p - must be even number.

$p = 2k \quad k \in \mathbb{Z}$ some integer.

(1) $\Rightarrow$ $(2k)^2 = 2q^2$

$4k^2 = 2q^2$

$2k^2 = q^2$

$\Rightarrow q^2$ is an even number.

$\Rightarrow q$ is an even number.

$2 \mid p$ and $2 \mid q$

but $(p, q) = 1$

but we found that $(p, q) \neq 1$

\# Contradiction

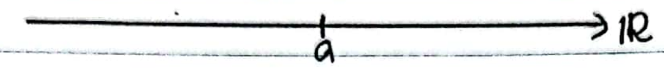
$\Rightarrow \sqrt{2}$ is irrational.

Home work:-

Show that followings are irrational number.

1. $\sqrt{3}$
2. $\sqrt{2} + \sqrt{3}$
3. $\sqrt{2} + 7$.

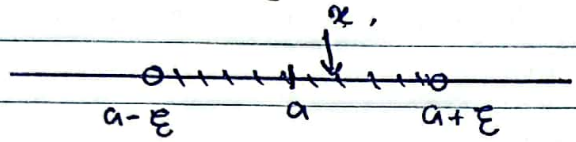
Open Sets and Closed Sets.



$a \in \mathbb{R}$ and Let $\varepsilon > 0$

(i) An ε-neighbourhood of "a" is the set.

$N(a, \varepsilon) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$.

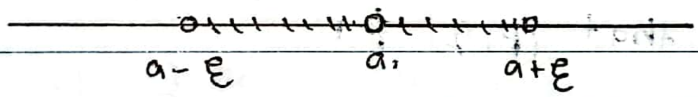


$-\varepsilon < x - a < \varepsilon$

$a - \varepsilon < x < a + \varepsilon$

(ii) A deleted neighbourhood (nbhd) of a is the set

$N^*(a, \varepsilon) = \{x \in \mathbb{R} \mid \cancel{a-\varepsilon, a} \; |x - a| < \varepsilon\} \setminus \{a\}$

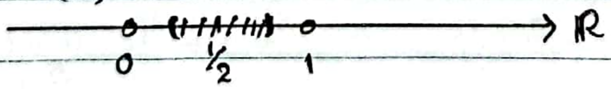


$= (a - \varepsilon, a) \cup (a, a + \varepsilon)$

Def^n:-

A Subset U of $\mathbb{R}$ is said to be open if for each $a \in U$ there exist ($\exists$) an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset U$

i) $U = (0, 1)$

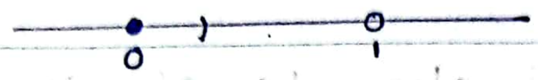


$\varepsilon = 1/4$

$\left(\frac{1}{2} - \frac{1}{4}, \frac{1}{2} + \frac{1}{4}\right) = \left(\frac{1}{4}, \frac{3}{4}\right)$

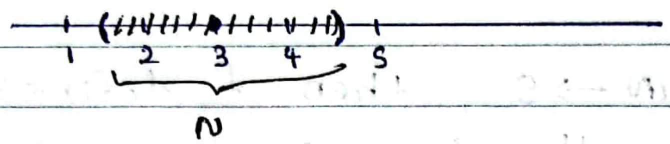
$R \setminus S \Leftrightarrow R - S.$

ii) $u = [0, 1)$



"0" can not have a nbhd which inside $[0, 1)$

iii) $\mathbb{N} \subset \mathbb{R}$ $\quad \mathbb{N} = \{1, 2, 3, \ldots\}$



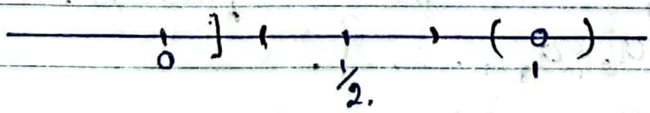
$N \not\subset \mathbb{N}$ (not a subset).

Def^n

Let S be a subset of $\mathbb{R}$

a. $x \in S$ is called an interior point of S if ~~ext~~ there exist an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset S$ the set of all interior points of a set S is denoted by S°

ii) x is called boundedrary point of S if $\forall \varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon)$ containts points of S as well as points of $\mathbb{R} \setminus S$.

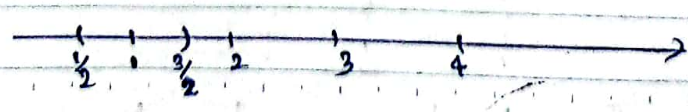


iii) $x \in S$ is called an isolated point of S if there exist an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap S = \{x\}$

eg:-

eg:- $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\left(\frac{1}{2}, \frac{3}{2}\right)$ nbhd of 1



$\left(\frac{1}{2}, \frac{3}{2}\right) \cap \mathbb{N} = \{1\}$

Sequences of Real number!

Def^n :-

A Sequence of elements of any non-empty Set S is defined through function $\mathbb{N}$ to S

that is, $f ; \mathbb{N} \longrightarrow S$

01 Note :- If $f ; \mathbb{N} \to S$, then f defines a sequence when the images are arranged in the Natural Order of natural number.

$1 \longrightarrow f(1) = a_1$

$2 \longrightarrow f(2) = a_2$

$3 \longrightarrow f(3) = a_3$

02: The expression $f(1), f(2), \ldots, f(n), f(n+1)$ is Called a sequance of elements of S

03. If $S = \mathbb{R}$, then it is Called Sequance of Real numbers.

04. We write $f(n)$ as a_n and use, $\langle a_n \rangle$ or $\{a_n\}$ for the sequence.

i.e. $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$.

eg :- $\left\{\frac{1}{n}\right\}$

$a_n = f(n) = \frac{1}{n}$

$f ; \mathbb{N} \longrightarrow \mathbb{R}$

$a_1 = f(1) = 1/1$

$a_2 = f(2) = 1/2$

$a_3 = f(3) = 1/3$

$\vdots \quad \vdots \quad \vdots$

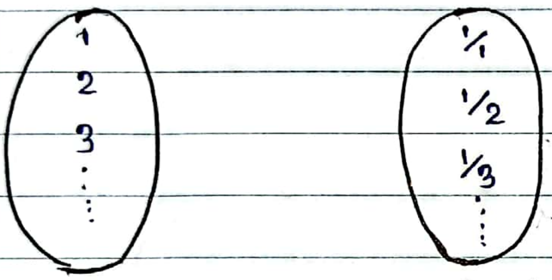
$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right\}$

Range of Sequence:-

* The set of all distinct terms of sequence is called its range or range set.

* Range may be finite or infinite, without ever being null set.

Eg:- $\left\{ \frac{1}{n} \right\}$

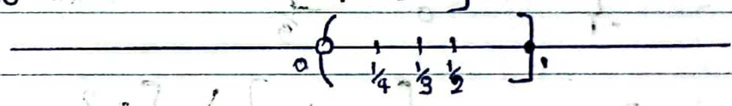


$\mathbb{N}$ $\mathbb{R}$

$\left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots \right\}$.

$0 < a_n \leq 1 \quad \forall n \in \mathbb{N}$,

$(0, 1]$



Infinite range $= (0, 1]$

eg:- $\left\{ (-1)^n \right\} = \left\{ a_n \right\}$

$= \left\{ -1, +1, -1, +1, \ldots \right\}$

Range of $\{a_n\} = \{-1, +1\}$

$=$ finite range.

eg:- $\{3\} = \{3, 3, 3, \ldots\}$.

Range of $\{3\} = \{3,\}$

$=$ finite range.

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Eg:- a_n = last digit of 7^n

$a_1 = 7$

$a_2 = 9$

$a_3 = 3$ $\qquad \{7, 9, 3, 1, 7, 9, 3, \ldots\}$

$a_4 = 1$

$a_5 = 7$ $\qquad$ Range of $\{a_n\}$ = finite

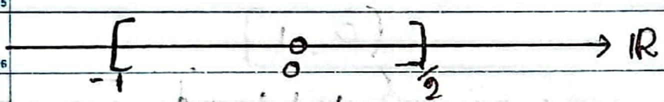
$a_6 = 9$ $\qquad = \{7, 9, 3, 1\}$

$a_7 = 3$

$a_8 = 1$

eg:- $\left\{\frac{(-1)^n}{n}\right\}$

$\left\{-\frac{1}{1}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4} \ldots\right\}$



Range $= \left[-1, \frac{1}{2}\right] \setminus \{0\}$

Bounded Sequence :-

* A Sequence is Said to be bounded if and only if its range Set is bounded.

* A sequence $\{a_n\}$ is said to be bounded above if and only if a real number M. Such that $a_n \leq M \; \forall n \in \mathbb{N}$

eg:- $\left\{\frac{1}{n}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

$a_n \leq 1 \quad \forall n \in \mathbb{N}$ bounded above.

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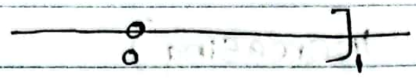
* A Sequance $\{a_n\}$ is said to be bounded below if and only if a real number m, such that $a_n \geq m$ $\forall n \in \mathbb{N}$

eg:- $\{\frac{1}{n}\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\}$



$a_n > 0$ $\forall n \in \mathbb{N}$ bounded below.

$0 < a_n \leq 1$ $\forall n \in \mathbb{N}$.

* A Sequance is said to be bounded if it is both bounded above and bounded below.

eg:- $\{(-1)^n\} = \{-1, +1, -1, +1, \dots\}$

$-1 \leq a_n \leq 1$ $\forall n \in \mathbb{N}$

* A Sequence is said to be unbounded if it is not bounded.

eg:- $\{2^n\} = \cancel{2, 4, 8, 8}\ \{2^1, 2^2, 2^3, \dots\}$

* bounded below.

$a_n \geq 2$ $\forall n \in \mathbb{N}$.

unbounded above.

Eg:- $\{-n^2\} = \{-1^2, -2^2, -3^2, -4^2, \dots\}$
$= \{\cancel{3}, \cancel{4}, \cancel{9}, \dots\}$

* bounded ~~below~~ above, but not bounded below.

$-1 \geq a_n$ $\forall n \in \mathbb{N}$

$\therefore$ un bounded.

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* Monotonic Sequances :-

* A Sequence is said to be monotonic if any of the following two conditions is satisfied

1. $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$
(Monotonically increasing)

2. $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$
(Monotonically decreasing).

* If both (1) and (2) satisfied simultaneously then it is called Constant Sequence.

eg:- $\{1, 1, 2, 2, 3, 3, \ldots\}$

$a_1 \leq a_2 < a_3 \leq a_4 < a_5$.

$\therefore$ Monotonic.

eg:- $\{1, 2, 3, \ldots\}$
$a_1 < a_2 < a_3 < \ldots$

* $a_n < a_{n+1} \quad \forall n \in \mathbb{N}$ strictly monotonic increasing

* $a_n > a_{n+1} \quad \forall n \in \mathbb{N}$ strictly monotonic decreasing.

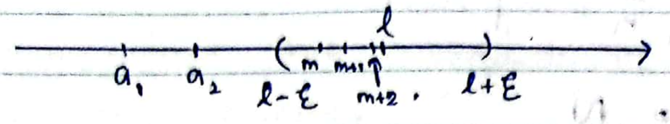
Eg:- $\{3\} = \{3, 3, 3, 3, \ldots\}$

$a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$.
$a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$

$a_n = a_{n+1} \quad \forall n \in \mathbb{N}$.

$\therefore$ Constant Sequance.

Def^n :-



$n \geq m$.

1. A Sequence $\{a_n\}$ is said to Converge to a real number l if, given $\varepsilon > 0$ if and only if a natural number N (depends on ε)

Such that :-

$$|a_n - l| < \varepsilon \quad \forall n \geq N$$

$$-\varepsilon < a_n - l < \varepsilon \quad \forall n \geq N$$

$$l - \varepsilon < a_n < \varepsilon + l. \quad \forall n \geq N$$

$$\lim_{n \to \infty} a_n = l$$

2. If the Sequence $\{a_n\}$ does not Converge to a real number we say that it diverges.

Eg:- Show that $\lim_{n \to \infty} \frac{1}{n} = 0.$

Solu^n :-

Let $\varepsilon > 0$ be given.

we must find $N \in \mathbb{N}$ Such that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \quad \forall n \geq N$$

Consider

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$$n > \frac{1}{\varepsilon}$$

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$\lceil 1.3 \rceil = 2$

$\lfloor 1.3 \rfloor = 1$

$n > \frac{1}{\varepsilon}$, $\left|\frac{1}{n} - 0\right| < \varepsilon$

$n \geq \left\lceil \frac{1}{\varepsilon} \right\rceil = N$

$\Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon \quad \forall n \geq N$

Such that $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$

Eg:- Show that, $\lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right) = 1$

$a_n = 1 - \frac{1}{2^n}$

$\lim_{n \to \infty} a_n = l.$

Let $\varepsilon > 0$ be given.

Consider

$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| = \frac{-1}{2^n}$

$= \frac{1}{2}^n < \varepsilon$

$= 2^n > \frac{1}{\varepsilon}$

$= \log 2^n > \log \frac{1}{\varepsilon}$

$= n \log 2 > \log \frac{1}{\varepsilon}$

$n > \frac{\log 1/\varepsilon}{\log 2}$

$n \geq \left\lceil \frac{\log \frac{1}{\varepsilon}}{\log 2} \right\rceil = N$

$\therefore \left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \varepsilon$

whenever $n \geq N$

where, $N = \left\lceil \frac{\log \frac{1}{\varepsilon}}{\log 2} \right\rceil$

$\frac{1}{2^n} < \varepsilon$

$\frac{1}{\varepsilon} < 2^n$

eg:- $(-1)^n$

Sol$^{\underline{n}}$:- Assume that this sequence, is converges to Some real number l.

$$\lim_{n\to\alpha} (-1)^n = l.$$

Let us take $\varepsilon = \frac{1}{2}$.

Then $\exists$ $N \in \mathbb{N}$

$|(-1)^n - l| < \frac{1}{2}$ $\forall n \geq N$

$\therefore$ $\forall n \geq \mathbb{N}$

$$2 = |(-1)^n - (-1)^{n+1}|.$$
$$= |(-1)^n - l + l - (-1)^{n+1}|$$

$$2 \leq |(-1)^n - l| + |l - (-1)^{n+1}|$$

$$2 < \frac{1}{2} + \frac{1}{2} = 1$$

$2 < 1$ #

There is a Contradiction.

$\therefore (-1)^n$ diverges.

* Th$^{\underline{m}}$:-

Let $\{a_n\}$ be sequence of real numbers.

If $\lim_{n\to\alpha} a_n = l_1$ and $\lim_{n\to\alpha} a_n = l_2$, then $l_1 = l_2$

(i.e. If a sequence is convergent then the limit of the sequence is unique).

Pf :-

let $\varepsilon > 0$ be given,

Since

$$\lim_{n \to \alpha} a_n = l_1$$

$\exists\ N_1 \in \mathbb{N}$ Such that

$$|a_n - l_1| < \frac{\varepsilon}{2} \quad \forall\ n \geq N_1 \quad \text{--- (1)}$$

$$\lim_{n \to \alpha} a_n = l_2$$

$\exists\ N_2 \in \mathbb{N}$ Such that

$$|a_n - l_2| < \varepsilon/2 \quad \forall\ n \geq N_2 . \quad \text{--- (2)}$$

let $N = \max\{N_1, N_2\}$.

then $\forall\ n \geq N$

$$\Rightarrow (1) \quad |a_n - l_1| < \varepsilon/2 \quad \forall n \geq N$$

$$\Rightarrow (2) \quad |a_n - l_2| < \varepsilon/2 \quad \forall n \geq N.$$

$$|l_1 - l_2| = |l_1 - a_n + a_n - l_2|$$

$$\leq |l_1 - a_n| + |a_n - l_2|$$

$$= |a_n - l_1| + |a_n - l_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall\ n \geq N$$

$$= \varepsilon$$

$\varepsilon > 0$

$\rightarrow \varepsilon \rightarrow 0$

$|l_1 - l_2| = 0$

$l_1 = l_2$.

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Th^m :-

every Convergent sequence of real numbers is bounded.

i.e. $\lim_{n\to\alpha} a_n = l$

$|a_n| \leq k \quad \forall$

Pf :- Let $\{a_n\}$ be a sequence of real numbers which converges to l. Let $\epsilon > 0$ with $\epsilon = 1$

Then $\exists$ $N \in \mathbb{N}$

$$|a_n - l| < 1 \quad \forall n \geq N$$

Consider

$$|a_n| = |a_n - l + l|$$
$$\leq |a_n - l| + |l| \quad \forall$$
$$< 1 + |l| \quad \forall n \geq N$$

Let $M = \max\{a_1, a_2, \ldots a_{N-1}, (1+|l|)\}$ $\{a_1, a_2, a_3, \ldots, a_{N-1}, a_N, a_{N+1} \ldots\}$

$\Rightarrow |a_n| \leq M \quad \forall n \in \mathbb{N}$ $\quad |a_n| < 1+|l| \quad \forall n \geq N$

so, $\{a_n\}$ is bounded.

Homework :-

Th^m :- (Sandwitch th^m :-)

Suppose that $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences such that

$$a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$$

If $\lim_{n\to\alpha} a_n = \lim_{n\to\alpha} c_n = l$ (say).

then $\lim_{n\to\alpha} b_n = l$.

$|x^n| = |x|^n \quad \forall x \in \mathbb{R}$

bounded $\not\Rightarrow$ Convergent.

$(-1)^n$

Eg :- $\lim_{n\to\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} = 0$

$0 \leq \left| \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} \right| \leq \frac{1}{n^2}$ $\quad \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} \leq \frac{1}{n^2}$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n^2} = 0$. $\quad \lim_{n\to\infty} \frac{1}{n^2} = 0$.

by ~~bound~~ Sandwitch the^m :

$\lim_{n\to\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} = 0$.

Th^m :-

A Sequence is monotonic and bounded then Sequence is Convergent.

Let $\{a_n\}$ be monotonic and bounded Sequence. without loss of generallity $\{a_n\}$ is increasing

$a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$

Since $\{a_n\}$ is bounded

$|a_n| \leq k \quad \forall n \in \mathbb{N}$

$-k \leq a_n \leq k \quad \forall n \in \mathbb{N}$

$\{a_n\}$ is bounded above

Let $\sup\{a_n\} = a$ (say).

we Claim that $\lim_{n\to\infty} a_n = a$.

Let $\varepsilon > 0$,

$\forall n \geq N$

$a_n \geq a_N$

$a_N \leq a_n < a, \forall n \geq N$ (increasing).

$\Rightarrow a_N \leq a_n < a + \varepsilon$ for $\varepsilon > 0$

$a - \varepsilon < a_N < a_n < a + \varepsilon \quad \forall n \geq N$

$\cancel{a_n}$

$a - \varepsilon < a_n < a + \varepsilon \quad \forall n \geq N$

$|a_n - a| < \varepsilon \quad \forall n \geq N$

$\Rightarrow \lim_{n \to \infty} a_n = a.$

Eg:- Let $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}}$

* ~~Monotonic~~ bounded.

by Mathematical Induction

$n = 1, \quad x_1 = \sqrt{2} \leq 2.$

Suppuse,

$n = k \quad x_k \leq 2 \quad \forall k.$

$x_k \leq 2.$

$x_k + 2 \leq 2 + 2.$

$x_k + 2 \leq 4$

$\sqrt{x_k + 2} \leq 2$

$x_{k+1} \leq 2$

by MI $x_n \leq 2 \quad \forall n \in \mathbb{N}$

$0 \leq x_n \leq 2.$

$\{x_n\}$ is bounded.

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$$x_{n+1} - x_n = \sqrt{2+x_n} - \sqrt{2+x_{n-1}}$$

$$= \frac{2+x_n - (2+x_{n-1})}{\left(\sqrt{2+x_n} + \sqrt{2+x_{n-1}}\right)}$$

$$= \frac{(x_n - x_{n-1})}{\sqrt{2+x_n} + \sqrt{2+x_{n-1}}}$$

for $n=1$

$x_1 = \sqrt{2}$

$x_2 = \sqrt{2+\sqrt{2}}$

$x_2 - x_1 = \sqrt{2+\sqrt{2}} - \sqrt{2} > 0$.

$x_2 > x_1$

for $n = k$ $\quad x_k > x_{k-1}$

$x_k - x_{k-1} > 0$

now

$n = k+1$

$x_{k+1} = \sqrt{2 + x_{k-1}}$

$$x_{k+1} - x_k = \frac{(x_k - x_{k-1})}{\sqrt{2+x_k} + \sqrt{2+x_{k-1}}} > 0$$

$x_{k+1} - x_k > 0$

$x_{k+1} > x_k$.

by MI $\quad x_{n+1} > x_n \quad \forall n \in \mathbb{N}$

$\{x_n\}$ is increasing

Since $\{x_n\}$ is monotonic and bounded $\Rightarrow \{x_n\}$ is convergent.

Let $\lim\limits_{n \to \infty} x_n = l$.

$$\lim_{n \to \infty} x_{n-1} = l.$$

$$x_n = \sqrt{2 + x_{n-1}}$$

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \sqrt{2 + x_{n-1}}$$

$$l = \sqrt{2 + l}$$

$$l^2 = 2 + l.$$

$$l^2 - l - 2 = 0$$

$$(l-2)(l+1) = 0$$

$l = 2$ or $\boxed{l = -1}$ $\times \to$ because $0 \leq x_n < 2$.

$$\therefore \lim_{n \to \infty} x_n = 2$$

Home work :- $x_1 = \sqrt{3}$, $x_n = \sqrt{3 + x_{n-1}}$

* ~~bounded~~

by ~~mathematical induction,~~

~~$n = 1$, $x_1 = \sqrt{3} < 3$~~

~~Suppose,~~

~~$n = k$, $x_k < 3$ $\forall k$~~

~~$x_k < 3$~~

~~$x_k + 3 < 3 + 3$~~

~~$x_k + 3 < 6$~~

~~$\sqrt{x_k + 3} < \sqrt{6}$~~

~~$x_{k+1} < \sqrt{6}$~~

~~by MI $x_n < \sqrt{6}$ $\forall n \in \mathbb{N}$~~

~~$0 \leq x_n < \sqrt{6}$~~

~~$\{x_n\}$ is bounded~~

$x_{n+1} - x_n = \sqrt{3+x_n} - \sqrt{3+x_{n-1}}$

$= \sqrt{3+x_n} - \sqrt{3+x_{n-1}}$

$= \dfrac{3+x_n - (3+x_{n-1})}{\left(\sqrt{3+x_n} + \sqrt{3+x_{n-1}}\right)}$

$= \dfrac{(x_n - x_{n-1})}{\sqrt{3+x_n} + \sqrt{3+x_{n-1}}}$

for $n=1$

$x_1 = \sqrt{3}$

$x_2 = \sqrt{3+\sqrt{3}}$

$x_2 > x_1$

for $n = k$, $x_k > x_{k-1}$

$x_k - x_{k-1} > 0$

now,

$n = k+1$

$x_{k+1} = \sqrt{3+x_k}$

$x_{k+1} - x_k = \dfrac{(x_k - x_{k-1})}{\sqrt{3+x_k} + \sqrt{3+x_{k-1}}} > 0$

$x_{k+1} > x_k$

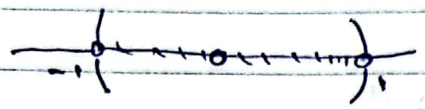
$\{x_n\}$ is increasing

Since $\{x_n\}$ is monotonic & bounded ∴ $\{x_n\}$ is convergent.

Eg:- Show that if $x \in \mathbb{R}$ and $|x| < 1$, then $\lim_{x \to a} x^n = 0$.

Sol^n :- If $x = 0$, then there is nothing to prove.

Assume that $x \neq 0$, $-1 < x < 1$



Since $|x| < 1$

we have $\frac{1}{|x|} > 1$

$\Rightarrow$ $\exists$ real number $a > 0$, such that $\frac{1}{|x|} = 1 + a$. —①

Let $\varepsilon > 0$ be given,

we want to find $N \in \mathbb{N}$ such that

$|x^n - 0| < \varepsilon \ \forall n \geq N$

$|x^n - 0| = |x^n|$

$= |x|^n$

$= \frac{1}{(1+a)^n}$ by ① —②.

$(1+a)^n = {}^nC_0 \, 1^n a^0 + {}^nC_1 \, 1^{n-1} a^1 + {}^nC_2 \, 1^{n-2} a^2 + \cdots + {}^nC_n \, 1^0 a^n$.

$= 1 + na + \frac{n(n-1)}{2} a^2 + \frac{n(n-1)(n-2)}{\lfloor 3} a^3 + \cdots + a^n > na$.

$(1+a)^n > na$.

$\frac{1}{(1+a)^n} < \frac{1}{na}$.

$|x^n - 0| < \frac{1}{na}$ by ②.

$< \varepsilon$ where.

$\frac{1}{na} < \varepsilon$

$$n > \frac{1}{a\varepsilon}$$

$$|x^n - 0| < \varepsilon \quad \forall n > \frac{1}{a\varepsilon}$$

Let $\left[\frac{1}{a\varepsilon}\right] = N$

$\Rightarrow |x^n - 0| < \varepsilon \quad \forall n \geq N.$

$\Rightarrow \lim\limits_{n \to \alpha} x^n = 0.$

Limits and Continuity of Real Valued functions :-

Limits :-

* **Left Limit :-**

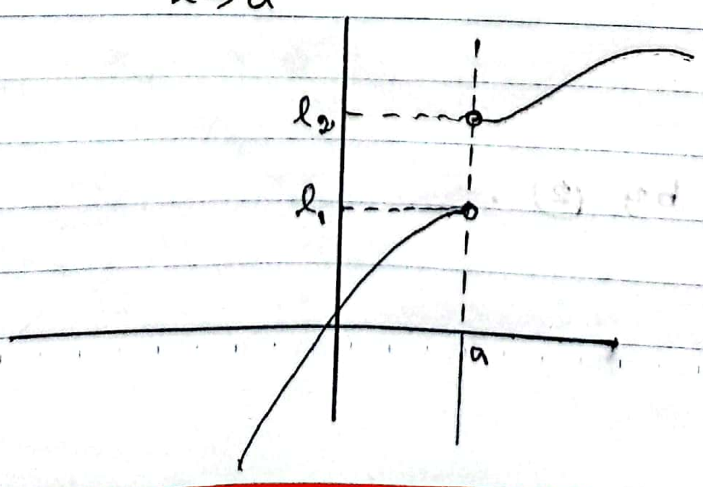
Let $f(x)$ be a real valued function. If x tends to "a" from left side then $f(x)$ tends to l_1.

i.e. $\lim\limits_{x \to a^-} f(x) = l_1$

* **Right Limit :-**

Let $f(x)$ be a real valued function. If x tends to "a" from right side then $f(x)$ tends to l_2.

i.e. $\lim\limits_{x \to a^+} f(x) = l_2.$



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$x + iy$ $\mathbb{R} \subset \mathbb{C}$ $\frac{3}{2} - 2$ $\frac{3-4}{2}$ $\frac{1}{2}$.

If $l_1 = l_2 = l$

Then $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

$\therefore \lim\limits_{x \to a} f(x)$ exists $\lim\limits_{x \to a} f(x) = l$.

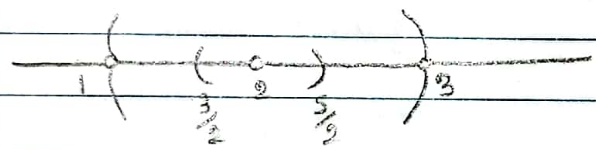
Def^n :- $\varepsilon - \delta$ def^n

Suppose a and l are real numbers, and $f(x)$ be a real valued function whose domain $D (\subseteq \mathbb{R})$ includes, all points in some open interval about a (except posibility the point a itself). ~~Then~~

~~$f : D \to \mathbb{R}$~~

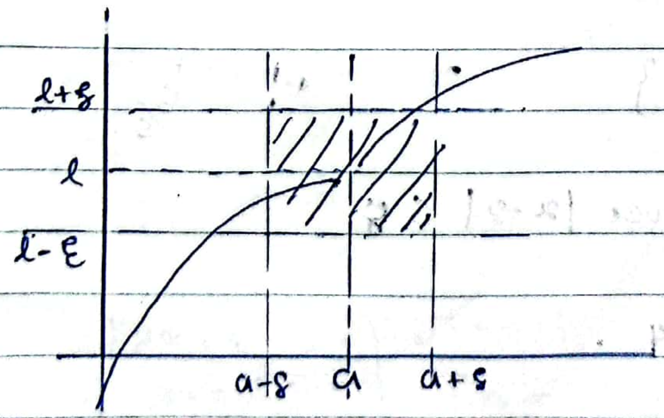
$A = (1, 3) \setminus \{2\}$

$2 \notin \left(\frac{3}{2}, \frac{5}{2}\right) \subseteq A$



Then l is called the limit of the f^n $f(x)$ at a if given any $\varepsilon > 0$,
$\exists\ \delta > 0$ (δ depend on a and ε)
Such that,

$|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$



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T.I = Triangular Inequality.

i.e. $\lim\limits_{x \to a} f(x) = l$.

Eg:- Show that $\lim\limits_{x \to 2} x^2 = 4$

$f(x) = x^2$
$l = 4$
$a = 2$.

$|x^2 - 4| < \varepsilon$ where $|x-2| < \delta$

now:- $|x^2-4| = |(x-2)(x+2)|$

$= |x-2||x+2|$

Consider all x which satisfies inequality $|x-2| < 1$, Then for all such x

$x - 2 < \pm 1$

$-1 < x-2 < 1$

$1 < x < 3$.

$|x+2| \leq |x| + 2$ (by T.I)

$< 3+2$

$|x| < 3$

$= 5$

$|x+2| < 5$

$|x^2-4| = |x-2||x+2|$

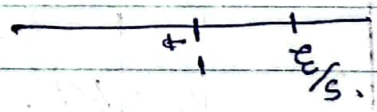
$< 5|x-2|$

$< \varepsilon$ whenever $|x-2| < \varepsilon$

$|x-2| < \varepsilon/5$.

$|x^2-4| < \varepsilon$

let $\delta = \min\{1, \varepsilon/5\}$



∴ $|x^2-4| < \varepsilon$ whenever $|x-2| < \delta$

∴ $\lim\limits_{x \to 2} x^2 = 4$.

Eg:- Show that,

$$\lim_{x \to -1} \frac{2x+3}{x+2} = 1$$

Show that,

$$\left| \frac{2x+3}{x+2} - 1 \right| < \varepsilon \quad \text{where } |x-(-1)| < \delta$$

$l = 1 \qquad a = -1.$

$$f(x) = \frac{2x+3}{x+2}$$

$|f(x) - 1|$

Now $\left| \frac{2x+3}{x+2} - 1 \right|$. ~~where $|x-(-1)| < \delta$~~

$$|f(x)-1| = \left| \frac{x+1}{x+2} \right|$$

$$= \frac{|x+1|}{|x+2|} \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad |ab| = |a||b|$$

$$= \frac{|x-(-1)|}{|x+2|}$$

Consider all x which satisfies inequality

$$|x+1| < \frac{1}{3} \leftarrow \text{උපකල්පනයකි}$$

$$-\frac{1}{3} < x+1 < \frac{1}{3}$$

$$-\frac{1}{3} - 1 < x < \frac{1}{3} - 1$$

$$-\frac{4}{3} < x < -\frac{2}{3}.$$

$$|x+2| > \frac{2}{3} \qquad |x+2| = |x-(-2)| > \left| -\frac{4}{3} - (-2) \right| = \frac{2}{3}.$$

$$\frac{1}{|x+2|} < \frac{3}{2}$$

Now

$$\left| \frac{2x+3}{x+2} - 1 \right| = \frac{|x+1|}{|x+2|} < \frac{3}{2}|x+1|$$

$$\left| \frac{2x+3}{x+2} - 1 \right| < \frac{3}{2}|x+1|$$

$< \varepsilon$ ~~whenever~~,

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whenever $\frac{3}{2}|x+1| < \varepsilon$

$$|x+1| < \frac{2\varepsilon}{3}$$

$$|x-(-1)| < \frac{2\varepsilon}{3}$$

let $\delta = \min\left\{\frac{2\varepsilon}{3}, \frac{1}{3}\right\}$.

$\therefore |f(x)-1| < \varepsilon$ where $|x-(-1)| < \delta$.

where,

$$\delta = \min\left\{\frac{2}{3}\varepsilon, \frac{1}{3}\right\}.$$

$$\therefore \lim_{x \to -1} f(x) = 1$$

Homework :-

$\lim_{x \to 3} (x^2 + 2x) = 15$ $\qquad (x^2+2x) < \varepsilon$ where $(x-3) < \delta$

$l = 15 \qquad a = 3$

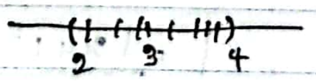
$$f(x) = x^2+2x$$

$\cancel{|f(x)-l|} = |f(x)-15| = |x^2+2x-15|$

$$= |(x+5)(x-3)|$$
$$= |x+5|\,|x-3|$$

Consider all x which satisfies ineqqualitty.

$$|x-3| < 1$$
$$-1 < x-3 < 1$$
$$2 < x < 4$$



$$|x+5| \leq |x| + 5$$
$$< 4+5$$
$$= 9$$
$$|x+5| < 9$$

$$|x^2+2x| = |x+5|\,|x-3|$$
$$< 9|x-3|$$
$$< \varepsilon \qquad \text{whenever } |x-3| < \varepsilon$$
$$|x-3| < \varepsilon/9$$

$$|x^2+2x| < \varepsilon$$

let $\delta = \min\{1, \varepsilon/9\}$

$\therefore |x^2+2x| < \varepsilon$ whenever $|x-3| < \varepsilon$

$$\therefore \lim_{x\to 3} (x^2+2x) = 15$$

[illegible] limit [illegible]

eg:- Show that $\lim_{x\to 0} f(x)$ does not exist, where, [illegible]

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Solution:- Suppose $\lim_{x\to 0} f(x) = l$, $l \in \mathbb{R}$

For given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(x) - l| < \varepsilon$ where $|x-0| < \delta$

Assume $\varepsilon = 1$

$\Rightarrow |f(x) - l| < 1$ whenever $|x-0| < \delta$

$|x| < \delta$

let $x = \dfrac{-\delta}{2}$

$$|x| = \left|-\frac{\delta}{2}\right| = \frac{\delta}{2}$$

$$\frac{\delta}{2} < \delta$$

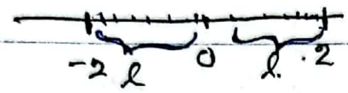
$$1 > |f(x) - l|$$
$$= |-1 - l| \qquad x = -\frac{\delta}{2} \text{ [illegible]}$$
$$= |1 + l|$$

$|1+l| < 1$

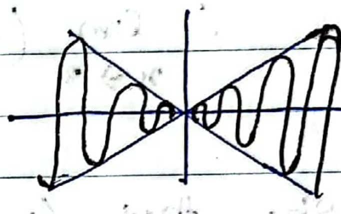
$-1 < 1+l < 1$

$-2 < l < 0$

$\times(-) \Rightarrow \quad 0 < l < 2$



But there is no real number that simultaneously satisfy the inequalities $0 < l < 2$ and $-2 < l < 0$ #



Eg:- Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

Solution :- Let $\varepsilon > 0$ be given, we need to find a $\delta > 0$ such that $\left|x \sin \frac{1}{x} - 0\right| < \varepsilon \;\; \forall \; |x-0| < \delta$

Now consider,

$\left| x \sin \frac{1}{x} - 0 \right| = \left| x \sin \frac{1}{x} \right|$

$\boxed{\begin{array}{l} -1 \leq \sin\theta \leq 1 \\ \therefore \sin\theta \leq 1 \end{array}}$

$= |x| \left|\sin \frac{1}{x}\right|$

$\left|x \sin \frac{1}{x}\right| \leq |x|$

$< \varepsilon$ whenever $|x| < \varepsilon$

$\left|x \sin \frac{1}{x} - 0\right| < \varepsilon$ whenever $|x - 0| < \delta$

where $\varepsilon = \delta$

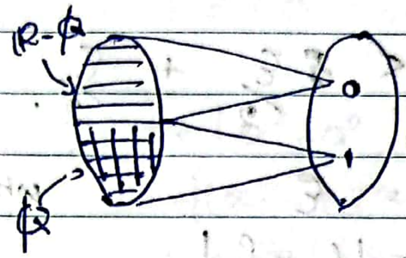
hence the proof.

* eg:- $f: \mathbb{R} \to \{0, 1\}$ given by,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$f(2) = 1$

$f\left(\frac{1}{\sqrt{3}}\right) = 0$



ඕනෑම Real number එකක
ළඟ සීමාව නොපවතී. ∴
ප්‍රත්‍යාදර්ශයක් ලෙස
ශ්‍රිතයේ සන්තතිකතාව බවට
පැහැදි උදාහරණය කරයි .

Show that if $a \in \mathbb{R}$, then $\lim_{x \to a} f(x)$ does not exists

(I will prove later)

Th^m :- [uniqueness of limit]

Let f be a function which is defined on some open interval I containing a (except possibly at a) If, $\lim_{x \to a} f(x) = l_1$ and

$\lim_{x \to a} f(x) = l_2$, Then $l_1 = l_2$.

If $l_1 \neq l_2$

Let $\varepsilon = \frac{|l_1 - l_2|}{3} > 0$. Then $\exists\ \delta_1 > 0$ and $\delta_2 > 0$

Such that,

$|f(x) - l_1| < \varepsilon/2$ whenever $x \in I$ $|x-a| < \delta_1$

$|f(x) - l_2| < \varepsilon/2$ whenever $x \in I$ $|x-a| < \delta_2$.

Let us take $\delta = \min\{\delta_1, \delta_2\}$.

Then whenever $|x-a| < \delta$

$0 < |l_1 - l_2|$ (because $l_1 \neq l_2$).

$\therefore 0 < |l_1 - l_2| = |l_1 - f(x) + f(x) - l_2|$

by triangular inequality,

$$\leq |l_1 - f(x)| + |f(x) - l_2|$$

$$0 < |l_1 - l_2| \leq |f(x) - l_1| + |f(x) - l_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

$$= \varepsilon \quad \text{whenever } |x-a| < \delta$$

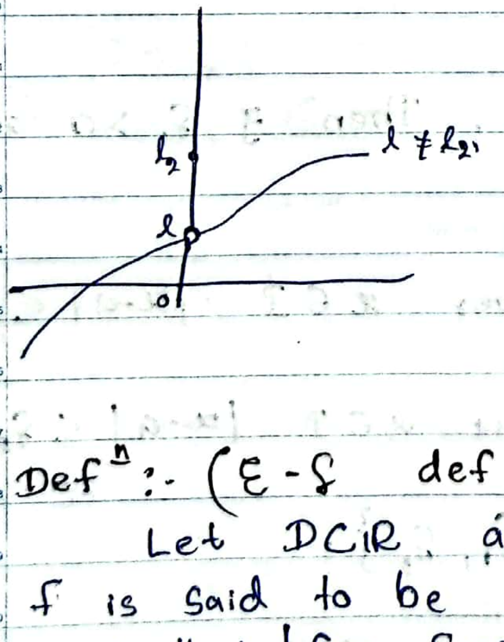
$$= \frac{|l_1 - l_2|}{3}$$

$$0 < |l_1 - l_2| < \frac{|l_1 - l_2|}{3} \quad \#$$

$\therefore$ Contradiction,

$\therefore l_1 = l_2.$

Continuous function :-



$f(0) \neq l$

$\lim_{x \to 0^-} f(x) = l$

$\therefore \lim_{x \to 0} f(x) = l$

$\lim_{x \to 0^+} f(x) = l$

Def^n :- (ε - δ def^n :-)

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ the function f is said to be continuous at $a \in D$ such that $|f(x) - f(a)| < \varepsilon$ whenever $\forall x \in D$ and $|x - a| < \delta$

Q1) Show that $f(x) = x^2$ is Continuous on $\mathbb{R}$

Solⁿ :-

Let $\varepsilon > 0$ be given and $a \in \mathbb{R}$

we need to ~~prove~~ find that $\delta > 0$

Such that $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$

$$|f(x) - f(a)| = |x^2 - a^2|$$
$$= |(x-a)(x+a)|$$
$$= |x-a||x+a| \quad \text{--- (1)}$$

Consider $|x-a| < 1$

$$-1 < x - a < 1$$
$$a - 1 < x < a + 1$$
$$|x| < |a+1|$$

Consider $|x+a| \leq |x| + |a|$

$$\leq |a+1| + |a|$$
$$\leq |a| + 1 + |a| \qquad |a+1| \leq |a| + 1$$
$$= 2|a| + 1$$

$$\Rightarrow |x+a| < 2|a| + 1$$

(1) $\Rightarrow$ $|x^2 - a^2| = |x+a| |x-a|$

$$< (1 + 2|a|) |x-a|$$

$< \varepsilon$ whenever $(1 + 2|a|)|x-a| < \varepsilon$

$$|x-a| < \frac{\varepsilon}{1 + 2|a|}$$

Let $\delta = \min \left\{ 1, \dfrac{\varepsilon}{1 + 2|a|} \right\}$

$|x^2 - a^2| < \varepsilon$ whenever, $|x-a| < \delta$

$\Rightarrow f(x) = x^2$ is Continuous $\forall a \in \mathbb{R}$

Eg:- Show that

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$ $\quad f(0) = 0$,

Solⁿ:-

Let $\varepsilon > 0$ be given we have to find $\delta > 0$

Consider,

$$|f(x) - f(0)| = \left| x \sin \frac{1}{x} - 0 \right|$$

$$= \left| x \sin \frac{1}{x} \right|$$

$$= |x| \left| \sin \frac{1}{x} \right|$$

$$\leq |x| \quad \because \left| \sin \frac{1}{x} \right| \leq 1$$

$$< \varepsilon \quad \text{whenever } |x| < \varepsilon$$

$\therefore$

$|f(x) - f(0)| < \varepsilon$ where $|x - 0| < \delta$

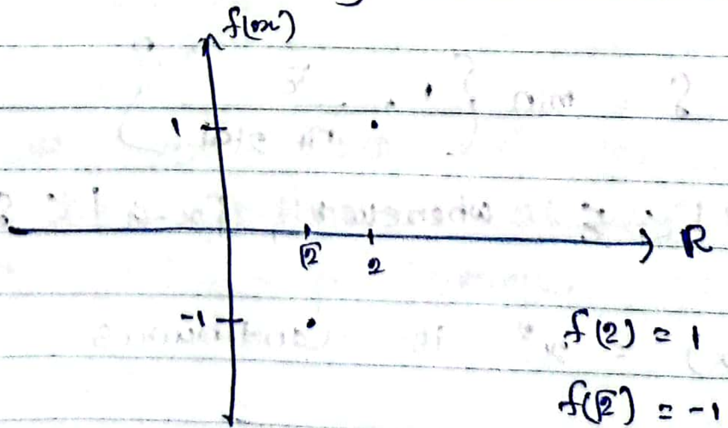
such that $\delta = \varepsilon$

Eg:- Show that the f^n $f : \underset{\text{Domain}}{\mathbb{R}} \to \underset{\text{Codomain}}{\{-1, 1\}}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is discontinuous at every real number.

Solu:-



$f(2) = 1$

$f(\sqrt{2}) = -1 \quad \sqrt{2} \notin \mathbb{Q}$

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Assume that f is Continuous at some $a \in \mathbb{R}$

Then, given $\varepsilon > 0$

$\exists \; \delta > 0$ Such that

$f(x) - f(a) < \varepsilon$ whenever

$|x - a| < \delta$

Let $\varepsilon = 1$

then $|f(x) - f(a)| < 1$, whenever $|x - a| < \delta$

Since $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$ $\mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$

the interval

$|x - a| < \delta$ Contains both rationals and

irrationals.

if $x \in \mathbb{Q}$,

$f(x) = 1$

$|1 - f(a)| < 1$

$-1 < 1 - f(a) < 1$

$1 > f(a) - 1 > -1$

$2 > f(a) > 0$

$\Rightarrow$ $0 < f(a) < 2.$

$\therefore f(a) = 1$

if $x \in \mathbb{R} \setminus \mathbb{Q}$

$f(x) = -1$

$|-1 - f(a)| < 1$

$-1 < -1 - f(a) < 1$

$1 > f(a) + 1 > -1$

$0 > f(a) > -2$

So $\#$

$\therefore$ the f^n f is discontinuous at every $x \in \mathbb{R}$

Eg:- $f(x) = \frac{1}{x}$, Show that f is Continuous at $x=1$

~~f(x)~~ $f : \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$

Sol^n:- Let $\varepsilon > 0$ be given we need to find $\delta > 0$ Such that $|f(x) - f(1)| < \varepsilon$ whenever $|x-1| < \delta$.

Consider,

$|f(x) - f(1)| = \left|\frac{1}{x} - 1\right|$ — ① whenever, $|x-1| < \delta$

Let us fix $\frac{1}{2}$ Such that $|x-1| < \frac{1}{2}$

$$-\frac{1}{2} < x - 1 < \frac{1}{2}$$

$$\frac{1}{2} < x < \frac{3}{2} \qquad \frac{1}{2} < x, \quad \frac{1}{x} < 2, \quad \left|\frac{1}{x}\right| < 2$$

$$① \Rightarrow |f(x) - f(1)| = \left|\frac{1}{x} - 1\right|$$

$$= \left|\frac{1-x}{x}\right|$$

$$= \left|\frac{x-1}{x}\right|$$

$$= \frac{1}{|x|}|x-1| < 2|x-1|$$

$< \varepsilon$ whenever,

$2|x-1| < \varepsilon$

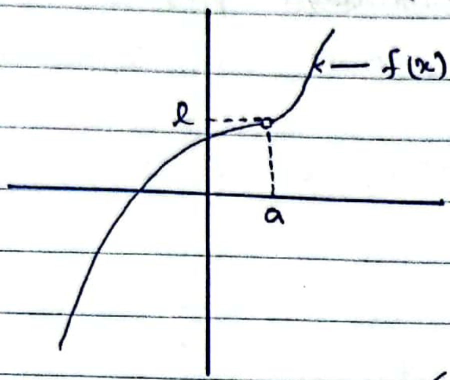
$|x-1| < \varepsilon/2$

Let us choose.

$$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}.$$

$\Rightarrow |f(x) - f(1)| < \varepsilon$, whenever $|x-1| < \delta$

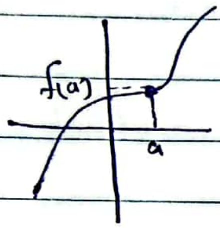
Differentiability :-



$$\lim_{x \to a} f(x) = l$$

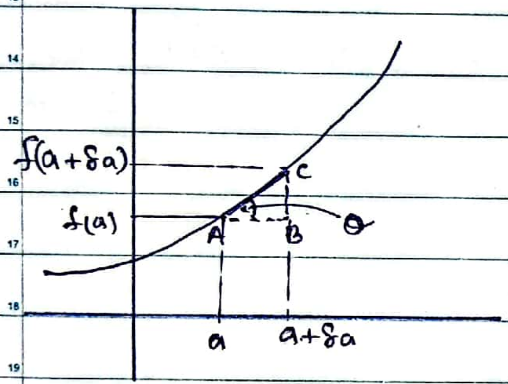
$\epsilon > 0 \ \exists \ \delta > 0$

$|f(x) - l| < \epsilon$ whenever $|x-a| < \delta$



$\epsilon > 0 \ \exists \ \delta > 0$

$|f(x) - f(a)| < \epsilon$ whenever $|x-a| < \delta$



$\delta a > 0$

$$\tan\theta = \frac{BC}{AB}$$

$$= \frac{f(a+\delta a) - f(a)}{(a+\delta a) - a}$$

$$\tan\theta = \frac{f(a+\delta a) - f(a)}{\delta a}$$

$$\lim_{\delta a \to 0} \frac{f(a+\delta a) - f(a)}{\delta a} = f'(a)$$

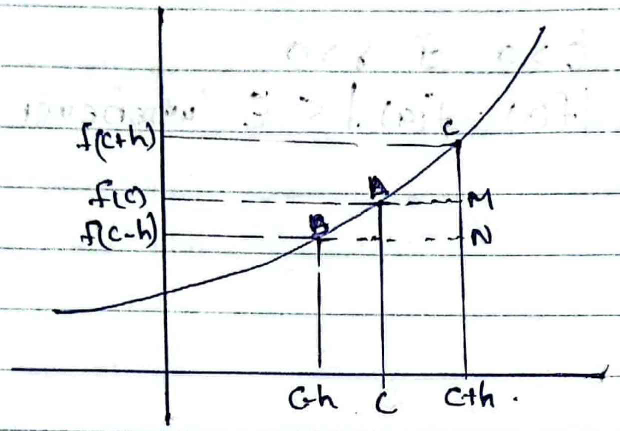
$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

Righ hand Derivetive

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = Rf'(c)$$

Left hand Derivative

$$f'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$



$$R f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{c+h-c}$$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

$$L f'(c) = \lim_{h \to 0} \frac{f(c) - f(c-h)}{c - (c-h)}$$

$$= \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}.$$

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* th^{m}.

Every ~~Continuous~~ differentiabe f^n is ~~differentiable~~ Continuous. but
onverse is not true.

Let f be a differetiable at any point c

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$

Now, $$f(x) - f(c) = \frac{f(x) - f(c)}{(x-c)} \cdot (x - c)$$

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x) - f(c)}{(x-c)} (x-c)$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{(x-c)} \cdot \lim_{x \to c} (x-c)$$

$$= f'(c) \times \lim_{x \to c} (x-c)$$

$$= f'(c) \times 0$$

$$= 0.$$

$$\Rightarrow \lim_{x \to c} (f(x) - f(c)) = 0$$

$$\lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0$$

$$\lim_{x \to c} f(x) - f(c) = 0$$

$$\lim_{x \to c} f(x) = f(c)$$

$\therefore f$ is Continuous at $x = c$ //.

No :- Counter Example.

$f(x) = |x|$

$x = 0 \quad f(0) = 0$

$$\frac{|x|}{x} = \begin{cases} 1 & \forall x > 0 \\ -1 & \forall x < 0 \end{cases}$$

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - 0}{x}$$

$$= \frac{|x|}{x}.$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1 \quad Rf'(0)$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1 \quad Lf'(1)$$

$Lf'(0) \neq Rf'(0)$

$\therefore f$ is not differentiable. at $x = 0$.

for given $\varepsilon > 0$

Consider

$|f(x) - f(0)| = ||x| - 0|$

$= ||x||$ whenever

$= |x|$ $\quad |x| < \varepsilon$

$< \varepsilon$

$|f(x) - f(0)| < \varepsilon$ whenever $|x - 0| < \delta$

where $\delta = \varepsilon$

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$\therefore f$ is Continuous at $x=0$.

Eg:- Show that the f^n

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is Continuous at $x=0$ but not differentiable at $x = 0$.

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h \sin(-1/h) - 0}{-h}$$

$$= -\lim_{h \to 0} \frac{\cancel{h} \sin 1/h}{\cancel{h}}$$

$$= -\lim_{h \to 0} \sin 1/h$$

$\therefore$ Does not exist.

$\therefore Lf'(0)$ Does not exist

$\Rightarrow f$ is not differentiable $x=0$.

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Eg:- Let $f(x) = \dfrac{x\,e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ $x \neq 0$.

$f(0) = 0$.

Show that f is Continuous at $x=0$ also Show that f is differentiable at $x=0$.

$$f(x) = \begin{cases} x\,\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$h > 0$

$\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} f(h)$

$= \lim\limits_{h\to 0} h\,\dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$

$= \lim\limits_{h\to 0} h \times \lim\limits_{h\to 0} \dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$

$= \lim\limits_{h\to 0} h \times \lim\limits_{h\to 0} \dfrac{e^{1/h}(1 - e^{-2/h})}{e^{1/h}(1 + e^{-2/h})}$

$= \lim\limits_{h\to 0} h \times \lim\limits_{h\to 0} \dfrac{(1 - \cancel{e^{-2/h}}_{0})}{(1 + \cancel{e^{-2/h}}_{0})}$

$= \lim\limits_{h\to 0} h \times 1$

$= 0$

$\boxed{e^{-1/h} = \dfrac{1}{e^{1/h}} \qquad \dfrac{1}{h} \to \alpha}$

$\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(-h) = \lim\limits_{h\to 0} \dfrac{(-h)(e^{-1/h} - e^{1/h})}{(e^{-1/h} + e^{1/h})}$

$= \lim\limits_{h\to 0} (-h) \times \lim\limits_{h\to 0} \dfrac{(\cancel{e^{-2/h}}_{0} - 1)}{(\cancel{e^{-2/h}}_{0} + 1)}$

$= -\lim\limits_{h\to 0} (h) \times (-1)$

$= 0$

$\therefore \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0)$

f is Continuous at $x=0$.

Differentiability at $x=0$,

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\frac{(e^{1/h}-e^{-1/h})}{(e^{1/h}+e^{-1/h})}}{\cancel{h}}$$

$$Rf'(0) = \lim_{h\to 0} \left(\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right)$$

$$= 1$$

$$\text{11}^{ly} \quad Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= -\lim_{h\to 0} \frac{f(-h)}{h}$$

$$= -\lim_{h\to 0} \frac{(-\cancel{h})\left(\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}-e^{1/h}}\right)}{\cancel{h}}$$

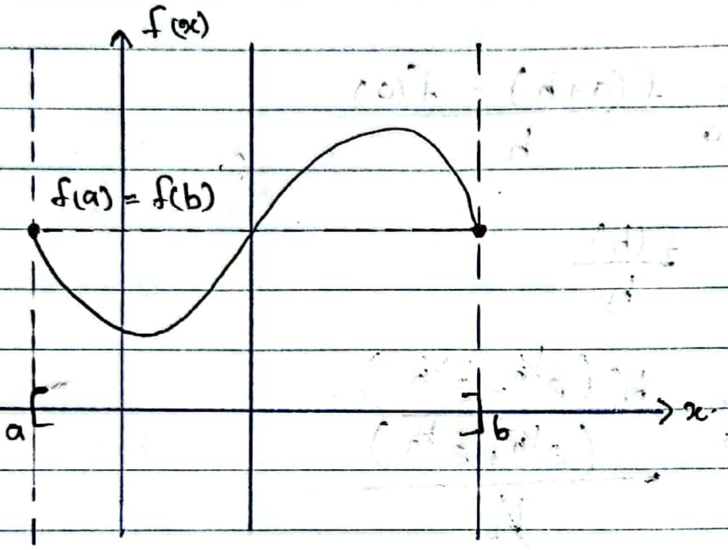
$$= \lim_{h\to 0} \left(\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}-e^{1/h}}\right)$$

$$= -1$$

$\therefore f$ is not differentiable at 0

Show that f is Continuous at 0 but not differentiable at 0

***. Rolle's thⁿ.



If a function f with domain $[a,b]$ is such that it is,

(i) Continuous in the Closed interval $[a,b]$
(ii) derivable in the open interval (a,b)
(iii) $f(a) = f(b)$ then $\exists$ exists atleast a $c \in (a,b)$ such that $f'(c) = 0$.

Eg:- Verify the Roll's thⁿ :-
for $f(x) = 2 + (x-1)^{2/3}$ in $[0,2]$.
$1 \in [0,2]$

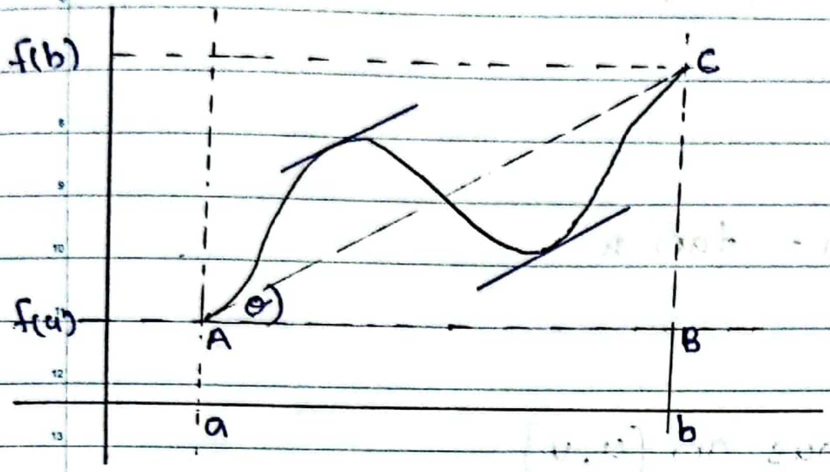
$f(1) = 2$
$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$

$\therefore f'(1) =$ Does not exists.

$\therefore$ f is not differentiable on $[0,2)$

$\therefore$ Rolle's Does not exist & Satisfies here

Th^n :- Lagrange's Mean Value th^n [1st MVT]



$\tan\theta = \frac{BC}{AB}$

If a function f with domain $[a,b]$ is Such that

(i) Continuous on $[a,b]$

(ii) differentiable $c \in (a,b)$

Such that $f'(c) = \frac{f(b) - f(a)}{b-a}$.

Eg:- find the value of c using Lagrange's MVT when $f(x) = 2x^2 + 3x + 4$ in $[1,2]$ $f'(x) = 4x + 3$

$f(1) = 9$

$f(2) = 18$

Since f is Continuous on $[1,2]$ and differentiable on $(1,2)$ $\exists\, c \in (1,2)$ Such that,

$f'(c) = \frac{f(2) - f(1)}{2-1} = \frac{18-9}{1} = 9$

$\therefore f'(c) = 9$

$4c + 3 = 9$

$4c = 9$

$c = 3/2.$

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**Eg: Show that $\dfrac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \dfrac{v-u}{1+u^2}$

if $0 < u < v < \pi/2$ and deduce that

$\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1} 4/3 < \dfrac{\pi}{4} + \dfrac{1}{6}$

$0 < u < v$

$f : [u, v] \to \mathbb{R}$

Such that $f(x) = \tan^{-1} x$

$f'(x) = \dfrac{1}{1+x^2}$

~~(i) f is Continuous on $[u,v]$~~

(ii) f is differentiable on (u,v).

then by Langrange's MVT $\exists c \in (u,v)$ $\quad c \in (u,v)$

Such that, $f'(c) = \dfrac{f(v) - f(u)}{v-u}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} v - \tan^{-1} u}{1-u} \quad \text{—} \ (1)$$

$u - v > 0$

$u < c < v$

$c > u > 0 \qquad\qquad c < v$

$c^2 > u^2 \qquad\qquad c^2 < v^2$

$1+c^2 > 1+u^2 \qquad\qquad 1+c^2 < 1+v^2$

$\dfrac{1}{1+c^2} < \dfrac{1}{1+u^2}$ — (2) $\qquad \dfrac{1}{1+c^2} < \dfrac{1}{1+v^2}$ — (3)

(2) and (3), $\dfrac{1}{1+v^2} < \dfrac{1}{1+c^2} < \dfrac{1}{1+u^2}$

$$\frac{1}{1+v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v-u} < \frac{1}{1+u^2}.$$

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$$

$v-u = \frac{4}{3} - 1$

$= \frac{1}{3}$

Let $u = 1$, $v = 4/3$

$1+v^2 = 1 + (4/3)^2 = \frac{25}{9}$

$[u,v] = \left[1, \frac{4}{3}\right]$

$1+u^2 = 2$.

④ $\Rightarrow \frac{1/3}{25/9} < \tan^{-1} 4/3 - \tan^{-1} 1 < \frac{1/3}{2}$

$$\frac{3}{25} < \tan^{-1} 4/3 - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \pi/4 + \frac{1}{6}.$$

Th^n :- Cauchy's MVT

If two function f and g are,

(i) Continuous in the closed interval $[a,b]$

(ii) differentiable on (a,b)

(iii) $g'(x) \neq 0 \quad \forall \; x \in (a,b)$

$\exists c \in (a,b)$ Such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(c)}$

Indeterminate form:-

$$\frac{\alpha}{\alpha}, \frac{0}{0}, 0^0, \alpha^\alpha, 0^\alpha$$

$$\lim_{x \to 0} \frac{x(x-1)}{x^2+x} = \frac{0}{0}.$$

Indeterminate form.

L'Hospital Rule :-

Let f, g be two functions such that,

(i) $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0} g(x) = 0$.

and.

(ii) $\lim_{x \to a} \left[\frac{f'(x)}{g'(x)}\right] = l.$

then, $\lim_{x \to a} \frac{f(x)}{g(x)} = l.$

Extended Version :-

If $\lim_{x \to a} f'(x) = 0$

$\lim_{x \to a} g'(x) = 0$, $\lim_{x \to a} \frac{f(x)}{g(x)} = l.$

and $\lim_{x \to a} \frac{f''(x)}{g''(x)} = l.$

Evaluate :-

$$\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{x \to 1} (1-x) \frac{\sin(\pi x/2)}{\cos(\pi x/2)}$$

$$= \lim_{x \to 1} \sin\left(\frac{\pi x}{2}\right) \times \lim_{x \to 1} \frac{(1-x)}{\cos(\pi x/2)}$$

$$= \lim_{x \to 1} \frac{(1-x)}{\cos(\pi x/2)} \quad \left(\frac{0}{0}\right)$$

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use L' Hospital rule.

Let $f(x) = 1 - x$

$f'(x) = -1$

$g(x) = \cos \frac{\pi x}{2}$, $g'(x) = \frac{-\pi}{2} \sin\left(\frac{\pi x}{2}\right)$.

$$\frac{f'(x)}{g'(x)} = \frac{-1}{-\frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right)} = \frac{2}{\pi} \frac{1}{\sin\left(\frac{\pi x}{2}\right)}$$

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{2}{\pi \sin\left(\frac{\pi x}{2}\right)} = \frac{2}{\pi}.$$