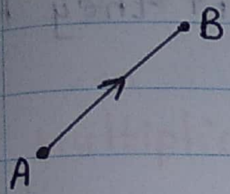
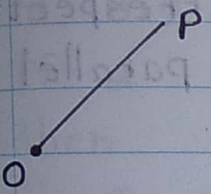
Vector

Vector quantities are those quantities that have magnitude and direction. It is generally reprosend by directed line segment.



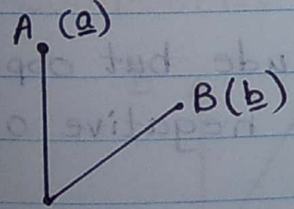
we represent a vector as $\overrightarrow{AB}$, where initial point of vector is deneted by A and the terminal point by B. magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}|$. But scalar has only magnitude.



Positrn vector

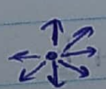
Let us denoted by origin as 'O'. such that this is a direct point. There is a point, say P at a distance from O have the position vector of point P is given by the vector $\overrightarrow{OP}$.

$\underline{a}$, $\vec{a}$, $\overrightarrow{AB}$, $\mathbf{a}$



$\underline{a}$ and $\underline{b}$ which represent the position vectors of two points A and B. Then we can write $\overrightarrow{AB} = (\underline{b} - \underline{a})$

Type of vectors

1. zero vectors 

It has zero magnitude. This means that vector has the same initial and terminal point. It is denoted by $\vec{0}$ or $\underline{0}$. The direction zero vector is indeterminate.

2. Unit vector $\frac{\underline{a}}{|\underline{a}|}$

It has a unit magnitude and unit vector in derection of a vector $\underline{a}$ is denoted by $\underline{a}$ and symbolieally as, $\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$

3. Co-initial vector

Two or more vectors are said to be co-initial if they have the same intial point.

4. Equal vector

Two vectors are said to be equal if they have the same magnitude and direction.

5. Collinear Vectors.

Two or more vectors are said to be collinear if they are paralled to the same line irrespective of their direction. They are also called parallel vectors.

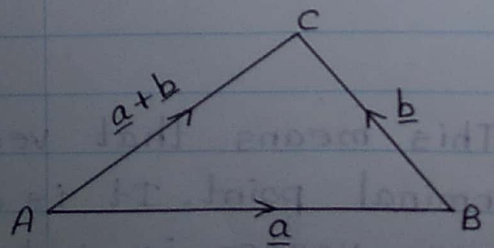
6. Coplanar vectors.

Those vectors is which lie on the same plane and they are all parallel to the sameplane.

7. Negative vectors $\vec{AB} \Rightarrow -\vec{AB}$

A vector which has same magnitude but opposite direction to another vector is called negative of that vectors.

Triangle Law



$\vec{AC} = \underline{a} + \underline{b}$

$\vec{AB} = \underline{a}$ $\vec{BC} = \underline{b}$

Considered a triangle ABC. Let the sum of two vectors $\underline{a}$ and $\underline{b}$ be represented by $\underline{c}$. The position vectors are represented by $\vec{AB}$, $\vec{BC}$ & $\vec{AC}$.

$$\vec{AC} = \vec{AB} + \vec{BC}$$

Properties of vector addition.

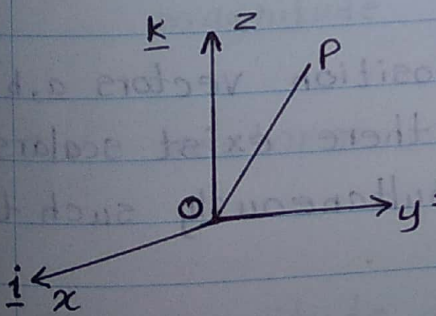
a) Commutative property $\underline{a}+\underline{b} = \underline{b}+\underline{a}$
b) Associative property $(\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$
c) zero is the additive identity $\underline{a}+\underline{0} = \underline{0}+\underline{a} = \underline{a}$
d) $\underline{a}+(-\underline{a}) = \underline{0}$

Multiplication of a vector by scalar

If $\underline{a}$ is a vector and m is a scalar. Then the product is $m\underline{a}$.
If $\underline{a}$ and $\underline{b}$ are vectors and m and n are scalars, then

I $m(\underline{a}) = (\underline{a})m = m\underline{a}$
II $m(n\underline{a}) = n(m\underline{a}) = mn\underline{a}$
III $m(\underline{a}+\underline{b}) = m\underline{a}+m\underline{b}$
IV $(m+n)\underline{a} = m\underline{a}+n\underline{a}$

Component from of vectors



We have consider the axis x, y, z and a point in the cordinate axis so, the position vector for such a point would be written as,

$$\overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$$

This is called components are x, y, z
Vector components are $x\underline{i}$, $y\underline{j}$, $z\underline{k}$

Consider the two vectors.

$\underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$ and $\underline{B} = p\underline{i} + q\underline{j} + r\underline{k}$

No :

I Sum is given by,

$$\underline{A} + \underline{B} = (a+p)\underline{i} + (b+q)\underline{j} + (c+r)\underline{k}$$

II Difference is given by,

$$\underline{A} - \underline{B} = (a-p)\underline{i} + (b-q)\underline{j} + (c-r)\underline{k}$$

III Multipication by a scalar m is given by,

$$m\underline{A} = ma\underline{i} + mb\underline{j} + mc\underline{k}$$

IV The vectors are equal of $a=p$, $b=q$, and $c=r$.

Test for collinearils

Three points A, B and C with position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are respectively are collinear, if and only if (iff) there exist scalar x, y, z not all zero simultaneously such that,

$$x\underline{a} + y\underline{b} + z\underline{c} = 0$$

where $x + y + z = 0$

Test for coplanar points.

For points A, B, C, D with position vectors $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ respectively are coplanar iff there exist scalars x, y, z, w as not all zero simultaneously such that,

$$x\underline{a} + y\underline{b} + z\underline{c} + w\underline{d} = \underline{0}$$

$$x + y + z + w = 0$$

Coplanar $\overset{\text{iff} \leftarrow \text{if and only if}}{\Longleftrightarrow}$ $x\underline{a} + y\underline{b} + z\underline{c} + w\underline{d} = \underline{0}$

$$x + y + z + w = 0$$

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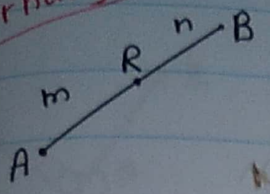
Section formula

Internally



Let $\underline{a}$ and $\underline{b}$ position vectors two points A and B. A point R vectors position vectors $\underline{r}$ divide AB such that,

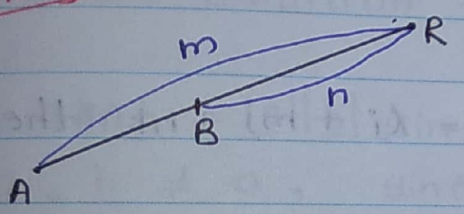
$\frac{AR}{RB} = \frac{m}{n}$ and this denotes that

AB is devided internally in the ratio $m:n$.

Then position vector of $\underline{r}$,

$$\underline{r} = \frac{n\underline{a} + m\underline{b}}{n+m}$$

Externally



$$\underline{r} = \frac{m\underline{b} - n\underline{a}}{m-n}$$

Magnitude of vectors.

a) For a vector $\underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$

magnitude $|\underline{A}| = \sqrt{a^2+b^2+c^2}$

b) For vectors $\underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$ and

$\underline{B} = l\underline{i} + m\underline{j} + n\underline{k}$

The magnitude is,

$$|\overrightarrow{AB}| = \sqrt{(l-a)^2 + (m-b)^2 + (n-c)^2}$$

Product of vectors

1) Scalar product

For a two vector $\underline{a}$ and $\underline{b}$ the dot product can be represented as $\underline{a}\cdot\underline{b}$.

It is defined, $\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta \quad (0 \leq \theta \leq \pi)$

we have the below possibilities:

a) if θ is a cule, then $\underline{a}.\underline{b} > 0$

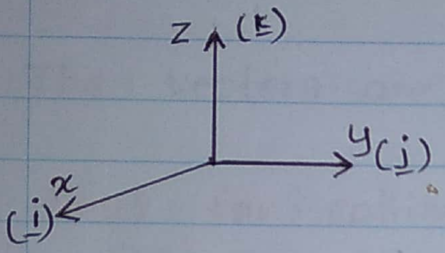
b) if θ is obtuse then $\underline{a}.\underline{b} < 0$

c) if θ is zero then $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|$

d) if θ is π then $\underline{a}.\underline{b} = -|\underline{a}|.|\underline{b}|$

e) if $\underline{a}.\underline{b} = 0$ then $\underline{a}$ and $\underline{b}$ are perpendiculars.

f) Consider the unit vector $\underline{i}$, $\underline{j}$ and $\underline{k}$



$$\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$$

$$\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$$

g) $\underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$ and $\underline{B} = l\underline{i} + m\underline{j} + n\underline{k}$ then,

$$\underline{A}.\underline{B} = al + bm + cn.$$

Properties of scalar product

I) $\underline{a}.\underline{a} = |\underline{a}||\underline{a}|\cos 0 = |\underline{a}|^2$

$\underline{a}.\underline{b} = \underline{b}.\underline{a}$ (commutative)

II $\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$ (Distributive)

III $(m\underline{a}).\underline{b} = \underline{a}.(m\underline{b}) = m(\underline{a}.\underline{b})$ (Associative)

Projection of vectors $\underline{a}$ on $\underline{b}$

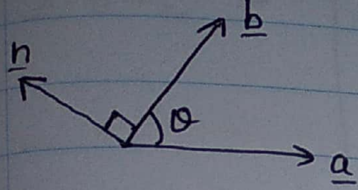
$$= \frac{\underline{a}.\underline{b}}{|\underline{b}|}$$

Maximum value of $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|$

Minimum value of $\underline{a}.\underline{b} = -|\underline{a}||\underline{b}|$

2) Vector product

It is called cross product.



$\underline{a}$ and $\underline{b}$ are two vectors

$\therefore \underline{a} \wedge \underline{b} = |\underline{a}||\underline{b}| \sin\theta \, \underline{n}$

where θ is the angle between them and n is the unit vector perpendicular to both $\underline{a}$ and $\underline{b}$. such that (s.t) $\underline{a}, \underline{b}, \underline{n}$ form a right handed screw system.

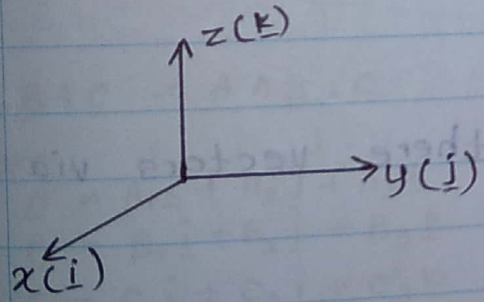
If vectors $\underline{a}$ and $\underline{b}$ are non zero and $\underline{a} \wedge \underline{b} = 0$ then it is the condition for them to be parallel vectors.

$$a \wedge b = |\underline{a}||\underline{b}| \sin\theta \, \underline{n} = 0$$

$$\underline{a}, \underline{b} \neq 0, \quad \sin\theta = 0$$

$$\theta = 2\pi$$

$$\underline{a} \parallel \underline{b}$$



I) $\underline{i} \wedge \underline{i} = \underline{j} \wedge \underline{j} = \underline{k} \wedge \underline{k} = 0$

II) $\underline{i} \wedge \underline{j} = \underline{k}$

$\underline{j} \wedge \underline{k} = \underline{i}$

$\underline{k} \wedge \underline{i} = \underline{j}$

III If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

then,

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i[a_2 b_3 - a_3 b_2] - \underline{j}[a_1 b_3 - a_3 \ldots + \underline{k}[a_1 b_2 - a_2 b_1]$$

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}[a_2 b_3 - a_3 b_2] - a_1[j b_3 - k b_2] \ldots b_1[j a_3 - k a_2]$$

$$= i[a_2 b_3 - a_3 b_2] - j[a_1 b_3 - b_1 a_3] + k[a_1 b_2 - b_1 a_2]$$

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Projection and component vector

Projection of $\underline{a}$ on $\underline{b}$ = $\frac{a \cdot b}{|\underline{b}|}$

Projection of $\underline{b}$ on $\underline{a}$ = $\frac{a \cdot b}{|\underline{a}|}$

work done by a force
$\underline{F} \cdot \underline{s}$ = d of product of force and displacement

suppose $F_1, F_2 \dots F_x$ are n forces acted on a parti
then during the displacement s of the particle, th
separate forces do quantities of work or Total
work done is

$$\sum_{i=1}^{n} f \cdot \underline{s}$$

Triple product

1. The scalar Triple product
It is a means of combining there vectors via cro
product and a dot product.
Given the vectors

$\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$

$\underline{B} = B_1\underline{i} + B_2\underline{j} + B_3\underline{k}$

$\underline{C} = C_1\underline{i} + C_2\underline{j} + C_3\underline{k}$

scalar tripal product
$\underline{A} \cdot (\underline{B} \wedge \underline{C})$

$$\underline{B} \wedge \underline{C} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \underline{i}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \underline{j}\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \underline{k}\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

$$\underline{A} \cdot \underline{B} \wedge \underline{C} = \left[A_1\underline{i} + A_2\underline{j} + A_3\underline{k}\right]$$

$$\underline{i}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \underline{j}\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \underline{k}\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

$$= A_1 \begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - A_2 \begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + A_3 \begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

$$\underline{A} \cdot \underline{B} \wedge \underline{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\underline{A} \cdot \underline{B} \wedge \underline{C} = \underline{A} \wedge \underline{B} \cdot \underline{C}$$
$$= \underline{C} \cdot \underline{A} \wedge \underline{B}$$
$$= \underline{C} \wedge \underline{A} \cdot \underline{B}$$
$$= \underline{B} \cdot \underline{C} \wedge \underline{A}$$

$$\underline{A} \cdot \underline{B} \wedge \underline{C} = -\underline{A} \cdot \underline{C} \wedge \underline{B}$$
$$= -\underline{A} \wedge \underline{C} \cdot \underline{B}$$
$$= -\underline{B} \cdot \underline{A} \wedge \underline{C}$$
$$= \underline{B} \cdot \underline{C} \wedge \underline{A}$$
$$= \underline{C} \wedge \underline{A} \cdot \underline{B}$$
$$= \underline{C} \cdot \underline{A} \wedge \underline{B}$$
$$= -\underline{C} \cdot \underline{B} \wedge \underline{A}$$

Ex :

$\underline{A} = 2\underline{i} + 3\underline{j} - \underline{k}$

$\underline{B} = -\underline{i} + \underline{j}$

$\underline{C} = 2\underline{i} + 2\underline{j}$ Find $\underline{A} \cdot \underline{B} \wedge \underline{C} = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - 3\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$2((1\times 0) - (2\times 0)) - 3[(-1\times 0) - (2\times 0)] - 1[(-1\times 2) - (2\times 1)]$$

$$= 0 - 0 - 1[(-2) - (2)] = 4$$

$\underline{A} \cdot \underline{B} \wedge \underline{C} = \underline{A} \wedge \underline{B} \cdot \underline{C}$

where

$\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$

$\underline{B} = B_1\underline{i} + B_2\underline{j} + B_3\underline{k}$

$\underline{C} = C_1\underline{i} + C_2\underline{j} + C_3\underline{k}$

$$\underline{A} \cdot \underline{B} \wedge \underline{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = (-1)\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = (-1)(-1)\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\underline{C} \cdot \underline{A} \wedge \underline{B} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \underline{C} \cdot \underline{A} \wedge \underline{B} = \underline{A} \wedge \underline{B} \cdot \underline{C}$$

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2. The vector triple product. H.W

$$\underline{A} \wedge (\underline{B} \wedge \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$

$\underline{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k}$

$\underline{B} = B_1 \underline{i} + B_2 \underline{j} + B_3 \underline{k}$

$\underline{C} = C_1 \underline{i} + C_2 \underline{j} + C_3 \underline{k}$

$$\boxed{\begin{array}{l} \underline{i} \wedge \underline{i} = \underline{j} \wedge \underline{j} = \underline{k} \wedge \underline{k} = 0 \\ \underline{i} \wedge \underline{j} = \underline{k} \\ \underline{j} \wedge \underline{k} = \underline{i} \\ \underline{k} \wedge \underline{i} = \underline{j} \end{array}}$$

L.H.S. $= \underline{A} \wedge (\underline{B} \wedge \underline{C})$

$$\underline{B} \wedge \underline{C} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \underline{i}(B_2C_3 - B_3C_2) - \underline{j}(B_1C_3 - B_3C_1) + \underline{k}(B_1C_2 -$$

$$\underline{A} \wedge (\underline{B} \wedge \underline{C}) = (A_1\underline{i} + A_2\underline{j} + A_3\underline{k}) \wedge [\underline{i}(B_2C_3 - B_3C_2) - \underline{j}(B_1C_3 - B_3C_1) + \underline{k}(B_1C_2 - B_2C_1)]$$

$$= -A_1\underline{k}(B_1C_3 - B_3C_1) + A_1\underline{j}(B_1C_2 - B_2C_1) + \underline{k}A_2(B_2C_3 - B_3C_2)$$
$$+ \underline{i}A_2(B_1C_2 - B_2C_1) + A_3\underline{j}(B_2C_3 - B_3C_2) - \underline{i}A_3(B_1C_3 - B_3$$

$$= \underline{i}[A_2B_1C_2 - A_2B_2C_1 - A_3B_1C_3 + A_3B_3C_1] +$$
$$\underline{j}[-A_1B_1C_2 + A_1B_2C_1 + A_3B_2C_3 - A_3B_3C_2] +$$
$$\underline{k}[-A_1B_1C_3 + A_1B_3C_1 + A_2B_2C_3 - A_2B_3C_2]$$

R.H.S. $= (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B}) \cdot \underline{C}$

$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$

$\underline{A} \cdot \underline{C} = (A_1\underline{i} + A_2\underline{j} + A_3\underline{k}) \cdot (C_1\underline{i} + C_2\underline{j} + C_3\underline{k})$

$= A_1C_1 + A_2C_2 + A_3C_3$

$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{i} \cdot \underline{k} = 0$

$\underline{A} \cdot \underline{B} = (A_1\underline{i} + A_2\underline{j} + A_3\underline{k}) \cdot (B_1\underline{i} + B_2\underline{j} + B_3\underline{k})$

$= A_1B_1 + A_2B_2 + A_3B_3$

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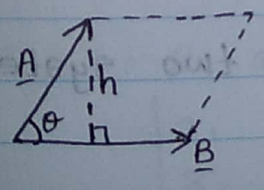
R.H.S. $= (\underline{A}\cdot\underline{C})\,\underline{B} - (\underline{A}\cdot\underline{B})\,\underline{C}$

$= [A_1C_1 + A_2C_2 + A_3C_3](B_1\underline{i} + B_2\underline{j} + B_3\underline{k}) - (A_1B_1 + A_2B_2 + A_3B_3)(C_1\underline{i} + C_2\underline{j} + C_3\underline{k})$

$=$

Area and Volume using cross product

The area of a parallelogram



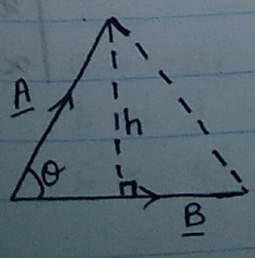
Area of a parllelogram $= |\underline{B}|\,h$

$= |\underline{A}\sin\theta|\,|\underline{B}|$

$= |BA\sin\theta|$

$= |\underline{A}\wedge\underline{B}|$

The area of Triangle



area of trangle $= \frac{1}{2}h|\underline{B}|$

$= \frac{1}{2}|AB\sin\theta|$

$= \frac{1}{2}|\underline{A}\wedge\underline{B}|$ //

Prove that

I $(\underline{A} \wedge \underline{B}) \wedge (\underline{C} \wedge \underline{D}) = \underline{C}(\underline{A} \cdot \underline{B} \wedge \underline{D}) - \underline{D}(\underline{A} \cdot \underline{B} \wedge \underline{C})$

II $(\underline{A} \wedge \underline{B}) \wedge (\underline{C} \wedge \underline{D}) = \underline{B}(\underline{A} \cdot \underline{C} \wedge \underline{D}) - \underline{A}(\underline{B} \cdot \underline{C} \wedge \underline{D})$

$$(\underline{A} \wedge \underline{B}) \wedge (\underline{C} \wedge \underline{D}) = \underline{C}(\underline{A} \cdot \underline{B} \wedge \underline{D}) - \underline{D}(\underline{A} \cdot \underline{B} \wedge \underline{C})$$

$$\underline{u} = \underline{A} \wedge \underline{B}$$

$$\begin{aligned}
(\underline{A} \wedge \underline{B}) \wedge (\underline{C} \wedge \underline{D}) &= \underline{u} \wedge (\underline{C} \wedge \underline{D}) \\
&= (\underline{u} \cdot \underline{D})\underline{C} - (\underline{u} \cdot \underline{C})\underline{D} \\
&= (\underline{A} \wedge \underline{B} \cdot \underline{D})\underline{C} - (\underline{A} \wedge \underline{B} \cdot \underline{C})\underline{D} \\
&= (\underline{A} \cdot \underline{B} \wedge \underline{D})\underline{C} - (\underline{A} \cdot \underline{B} \wedge \underline{C})\underline{D}
\end{aligned}$$

$$(\underline{A} \wedge \underline{B}) \wedge (\underline{C} \wedge \underline{D}) = \underline{B}(\underline{A} \cdot \underline{C} \wedge \underline{D}) - \underline{A}(\underline{B} \cdot \underline{C} \wedge \underline{D})$$

$$\underline{V} = \underline{C} \wedge \underline{D}$$

$$\begin{aligned}
(\underline{A} \wedge \underline{B}) \wedge (\underline{C} \wedge \underline{D}) &= (\underline{A} \wedge \underline{B}) \wedge \underline{V} \\
&= (\underline{A} \cdot \underline{V})\underline{B} - (\underline{B} \cdot \underline{V})\underline{A} \\
&= (\underline{A} \cdot \underline{C} \wedge \underline{D})\underline{B} - (\underline{B} \cdot \underline{C} \wedge \underline{D})\underline{A}
\end{aligned}$$

eg: If $\underline{a}, \underline{b}, \underline{c}$ and $\underline{p}, \underline{q}, \underline{r}$ are any two systems of three vectors, and if

$$\underline{p} = x_1\underline{a} + y_1\underline{b} + z_1\underline{c}$$
$$\underline{q} = x_2\underline{a} + y_2\underline{b} + z_2\underline{c}$$
$$\underline{r} = x_3\underline{a} + y_3\underline{b} + z_3\underline{c}$$

then $[\underline{p}, \underline{q}, \underline{r}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$[\underline{p}, \underline{q}, \underline{r}] = [\underline{p} \wedge \underline{q} \cdot \underline{r}]$$

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot \underline{b} \wedge \underline{c}$$
$$= \underline{a} \wedge \underline{b} \cdot \underline{c}$$

$$\underline{p} \wedge \underline{q} = [x_1 \underline{a} + y_1 \underline{b} + z_1 \underline{b} + z_1 \underline{c}] \wedge [x_2 \underline{a} + y_2 \underline{b} + z_2 \underline{c}]$$
$$= x_1 y_2\, \underline{a} \wedge \underline{b} + x_1 z_2\, \underline{a} \wedge \underline{c} + y_1 x_2\, \underline{b} \wedge \underline{a} + y_1 z_2\, \underline{b} \wedge \underline{c} + z_1 x_2\, \underline{c} \wedge \underline{a} + z_1 y_2\, \underline{c} \wedge \underline{b}$$
$$= (x_1 y_2 - y_1 x_2)\, \underline{a} \wedge \underline{b} + (y_1 z_2 - z_1 y_2)\, \underline{b} \wedge \underline{c} + (z_1 x_2 - x z_2)\, \underline{c} \wedge \underline{a}$$
$$= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \underline{a} \wedge \underline{b} + \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \underline{b} \wedge \underline{c} + \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} \underline{c} \wedge \underline{a}$$

$$\underline{p} \wedge \underline{q} \cdot \underline{r} = \left[\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \underline{a} \wedge \underline{b} + \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \underline{b} \wedge \underline{c} + \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} \underline{c} \wedge \underline{a} \right] \cdot (x_3 \underline{a} + y_3 \underline{b} + z_3 \underline{c})$$

$a \cdot \underline{b} \wedge \underline{c}$
$\underline{b} \wedge \underline{c} \cdot \underline{a}$
$\underline{b} \cdot \underline{c} \wedge \underline{a}$
$\underline{c} \wedge \underline{a} \cdot \underline{b}$
$\underline{c} \cdot \underline{a} \wedge \underline{b}$

$$= x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \underline{a} \cdot \underline{b} \wedge \underline{c} + y_3 \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} \underline{b} \cdot \underline{c} \wedge \underline{a} + z_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \underline{c} \cdot \underline{a} \wedge \underline{b}$$

$$= \left[x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} + y_3 \begin{vmatrix} z_1 & z_2 \\ x_1 & x_2 \end{vmatrix} + z_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right] \underline{a} \cdot \underline{b} \wedge \underline{c}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} (a, b, c)$$

Note

$\underline{a}, \underline{b}, \underline{c}$ are non-coplanar and

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \neq 0$$

then the three vectors $\underline{p}, \underline{q}, \underline{r}$ also non-coplanar.

,c]

1) If $\underline{a} = -3\underline{i} - \underline{j} + 5\underline{k}$
$\underline{b} = \underline{i} - 2\underline{j} + \underline{k}$
$\underline{c} = 4\underline{j} - 5\underline{k}$ Find $\underline{a} \cdot (\underline{b} \wedge \underline{c})$

$i \wedge j \cdot k$
$k \cdot k$
1

$i \cdot j \wedge k$
$i \cdot i = 1$

$$\underline{a} \cdot (\underline{b} \wedge \underline{c}) = \begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix} (i, j, k)$$
$$1$$

$$= -3\begin{vmatrix} -2 & 1 \\ 4 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 0 & -5 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix}$$

$$= -3(10-4) + 1(-5-0) + 5(4-0)$$

$$= -18 - 5 + 20$$

$$= -3$$

2) If $2\underline{i} - \underline{j} + 3\underline{k}$, $3\underline{i} + 2\underline{j} + \underline{k}$ and $\underline{i} + m\underline{j} + 4\underline{k}$ are coplanar find the value m ?

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$$

$$2\begin{vmatrix} 2 & 1 \\ m & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & m \end{vmatrix} = 0$$

$$2(8-m) + 1(12-1) + 3(3m-2) = 0$$

$$16 - 2m + 11 + 9m - 6 = 0$$

$$7m = -21$$

$$m = -3$$

10/25

Vector joining two parts.

If $P_1 = (x_1, y_1, z_1)$

$P_2 = (x_2, y_2, z_2)$ are any two pts.

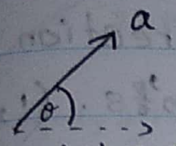
Then $\overrightarrow{P_1P_2} = \overrightarrow{P_1O} + \overrightarrow{OP_2}$

$$= -(x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) + x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$$

$$= (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

Projection of $\underline{a}$ alones $\underline{b}$ is $\frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$ and the projection vector of $\underline{a}$ along $\underline{b}$ is,

$$\left(\frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}\right) \cdot \underline{b}$$



$|a| \cos\theta$

$$= |\underline{a}| \cos\theta = \frac{|\underline{a}||\underline{b}|}{|\underline{b}|} \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$$

$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} = \frac{|\underline{a}|\cancel{|\underline{b}|}\cos\theta}{\cancel{|\underline{b}|}} = |\underline{a}| \cos\theta$$

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ are two vectors and λ is any scalar, then,

$$\underline{a} + \underline{b} = (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} + (a_3 + b_3)\underline{k}$$

$$\lambda\underline{a} = \lambda(a_1\underline{i} + a_2\underline{j} + a_3\underline{k})$$

$$= (\lambda a_1)\underline{i} + (\lambda a_2)\underline{j} + (\lambda a_3)\underline{k}$$

$$\underline{a} \cdot \underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \cdot (b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}(a_2 b_3 - a_3 b_2) - \underline{j}(a_1 b_3 - a_3 b_1) + \underline{k}(a_1 b_2 - a_2 b_1)$$

$$\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2$$

$$|a|^2 =$$

angle between two vectors.

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

1) Find the unit vector in the direction sum of the vectors. $\underline{a} = 2\underline{i} - \underline{j} + 2\underline{k}$ and $\underline{b} = -\underline{i} + \underline{j} + 3\underline{k}$

$$\underline{c} = \underline{a} + \underline{b} = (2\underline{i} - \underline{j} + 2\underline{k}) + (-\underline{i} + \underline{j} + 3\underline{k}) = \underline{i} + 5\underline{k}$$

$$|\underline{c}| = |\underline{a} + \underline{b}| = \sqrt{1^2 + 5^2} = \sqrt{26} =$$

Unit Vector of $\underline{c}$,

$$= \frac{\underline{c}}{|\underline{c}|} = \frac{\underline{i} + 5\underline{k}}{\sqrt{26}}$$

2) Find a vector of magnitude 11 in the direction opposite to that of $\overrightarrow{PQ}$, where P and Q are the p'ts. (1, 3, 2) and (-1, 0, 8) respectively.

$$\overrightarrow{QP} = \underline{p} - \underline{q}$$

$$= \underline{i} + 3\underline{j} + 2\underline{k} - (-\underline{i} + 0\underline{j} + 8\underline{k})$$

$$= 2\underline{i} + 3\underline{j} - 6\underline{k}$$

Unit vector of $\overrightarrow{QP} = \dfrac{2\underline{i} + 3\underline{j} - 6\underline{k}}{\sqrt{2^2 + 3^2 + 6^2}}$

$$= \frac{2\underline{i} + 3\underline{j} - 6\underline{k}}{7}$$

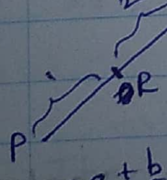
Vector of magnitude 11 in the $\overrightarrow{QP}$ direction

$$= \frac{11}{7}(2\underline{i} + 3\underline{j} - 6\underline{k})$$

3) Find the position vector of a point R which divides the line joining the two p'ts P and Q with position vectors $\overrightarrow{OP} = 2\underline{a} + \underline{b}$ and $\overrightarrow{OQ} = \underline{a} - 2\underline{b}$ respectively, in the ratio 1 : 2,

I internally II externally.

I



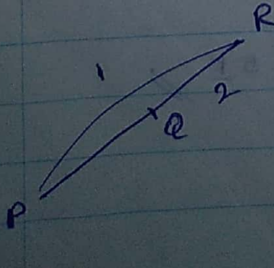
$q = \underline{a} - 2\underline{b}$

$p = 2\underline{a} + \underline{b}$

$$\underline{r} = \frac{2(2\underline{a} + \underline{b}) + 1(\underline{a} - 2\underline{b})}{2 + 1}$$

$$= \frac{5\underline{a}}{3}$$

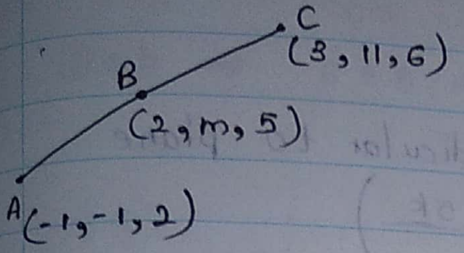
II



$$\underline{r} = \frac{1(\underline{a} - 2\underline{b}) - 2(2\underline{a} + \underline{b})}{1 - 2}$$

$$\underline{r} = \frac{-3\underline{a} - 4\underline{b}}{-1} = 3\underline{a} + 4\underline{b}$$

4) If the points $(-1,-1,2)$, $(2,m,5)$ and $(3,11,6)$ are collinear find the value of m.



$\cancel{\overrightarrow{AB} = (-\underline{i}, -\underline{j}, 2\underline{k}) + (2, m, 5)}$

$\overrightarrow{AB} = 3\underline{i} + (m+1)\underline{j} + 3\underline{k}$

$\overrightarrow{AC} = 4\underline{i} + 12\underline{j} + 4\underline{k}$

$\overrightarrow{AB} = \lambda \overrightarrow{AC}$

$3\underline{i} + (m+1)\underline{j} + 3\underline{k} = \lambda(4\underline{i} + 12\underline{j} + 4\underline{k})$

$(3-4\lambda)\underline{i} + (m+1-12\lambda)\underline{j} + (3-4\lambda)\underline{k} = 0$

$3 - 4\lambda = 0 \qquad 1 + m - 12\lambda = 0 \qquad 3 - 4\lambda = 0$

$\lambda = \frac{3}{4}$ // $\qquad 1 + m - 12 \times \frac{3}{4} = 0$

$m = 8$ //

5) If $\underline{a} = 2\underline{i} - \underline{j} + \underline{k}$, $\underline{b} = \underline{i} + \underline{j} - 2\underline{k}$ and $\underline{c} = \underline{i} + 3\underline{j} - \underline{k}$, find λ such that (ලබනු) (s.t) $\underline{a}$ is perpendicular $\lambda\underline{b} + \underline{c}$.

$\underline{a} \cdot (\lambda\underline{b} + \underline{c}) = 0$

$(2\underline{i} - \underline{j} + \underline{k}) \cdot [\lambda(\underline{i} + \underline{j} - 2\underline{k}) + \underline{i} + 3\underline{j} - \underline{k}] = 0$

$(2\underline{i} - \underline{j} + \underline{k}) \cdot [(\lambda+1)\underline{i} + (\lambda+3)\underline{j} - (2\lambda+1)\underline{k}] = 0$

$2(\lambda+1) - 1(\lambda+3) - (2\lambda+1) = 0$

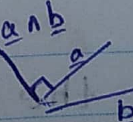
$-\lambda - 2 = 0$

$\lambda = -2$ //

6) Find all vectors of magnitude $10\sqrt{3}$ that are prependicular to the plane of $\underline{i} + 2\underline{j} + \underline{k}$ and $-\underline{i} + 3\underline{j} + 4\underline{k}$.

$\underline{a} = \underline{i} + 2\underline{j} + \underline{k}$

$\underline{b} = -\underline{i} + 3\underline{j} + 4\underline{k}$



$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix} = \underline{i}(8-3) - \underline{j}(4+1) + \underline{k}(3+2)$$

$$= 5\underline{i} - 5\underline{j} + 5\underline{k}$$

Unit vector of direction $(a \wedge b)$

$$\underline{n} = \frac{\underline{a} \wedge \underline{b}}{|\underline{a} \wedge \underline{b}|} = \frac{5\underline{i} - 5\underline{j} + 5\underline{k}}{5\sqrt{3}}$$

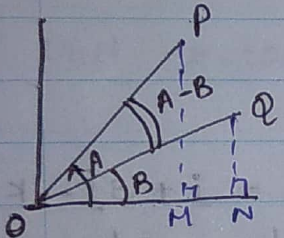
magnitude $10\sqrt{3}$ that are perpendicular to plane

$$= \cancel{10}^{2}\sqrt{3}\left(\frac{5\underline{i} - 5\underline{j} + 5\underline{k}}{\cancel{5}\sqrt{3}}\right)$$

$$= 2(5\underline{i} - 5\underline{j} + 5\underline{k})$$

7) Using vectors prove that,

$\cos(A-B) = \cos A . \cos B + \sin A . \sin B$



Let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ unit vectors making angle between A and B.

Then $Q\hat{O}P = A - B$

$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

$\overrightarrow{OP} = \cos A\,\underline{i} + \sin A\,\underline{j}$

//, $\overrightarrow{OQ} = \cos B\,\underline{i} + \sin B\,\underline{j}$

$\overrightarrow{OP} . \overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}| \cos(A-B)$

$= \cos(A-B)$ — ①

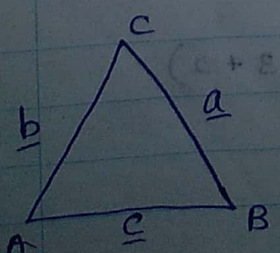
$\overrightarrow{OP} . \overrightarrow{OQ} = (\cos A\,\underline{i} + \sin A\,\underline{j}) . (\cos B\,\underline{i} + \sin B\,\underline{j})$

$= \cos A . \cos B + \sin A . \sin B$ — ②

①, ② $\cos A . \cos B + \sin A . \sin B = \cos(A-B)$

8) Prove that in a Δ ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ where

a, b, c represent the magnitude of the sides opposite to vertices A, B, C respectively.



Let the three side of tringular BC, CA, AB be respesent by $\underline{a}$, $\underline{b}$, $\underline{c}$ respectively.

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

$\underline{c} = -\underline{b} - \underline{a} \longrightarrow \underline{a} + \underline{b} = -\underline{c}$ — ①

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① $\wedge \underline{a} \Rightarrow \underline{a}\wedge\underline{a} + \underline{b}\wedge\underline{a} = -\underline{c}\wedge\underline{a}$

$\underline{b}\wedge\underline{a} = \underline{a}\wedge\underline{c}$ — ②

① $\wedge \underline{b} \Rightarrow \underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{b} = -\underline{c}\wedge\underline{b}$

$\underline{a}\wedge\underline{b} = \underline{b}\wedge\underline{c}$ — ③

② & ③

$\underline{a}\wedge\underline{b} = \underline{b}\wedge\underline{c} = \underline{c}\wedge\underline{a}$

$|\underline{a}\wedge\underline{b}| = |\underline{b}\wedge\underline{c}| = |\underline{c}\wedge\underline{a}|$

$|\underline{a}||\underline{b}|\sin(\pi - C) = |\underline{b}||\underline{c}|\sin(\pi - A) = |\underline{c}||\underline{a}|\sin(\pi - B)$

$ab \sin C = bc \sin A = ca \sin B$

$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ //

1. The ~~magnitude~~ of the vector $6i + 2j + 3k$ is
   a) 5 b) 7 c) 12 d) 1

2. The position vector of the point which devide the join points with position vector $\underline{a}+\underline{b}$ and $2\underline{a}-\underline{b}$ in the ratio 1 : 2 is

   $\frac{2(\underline{a}+\underline{b}) + (2\underline{a}-\underline{b})}{3}$

   a) $\frac{3\underline{a} + 2\underline{b}}{3}$ b) $\underline{a}$

   c) $\frac{5\underline{a} - \underline{b}}{3}$ d) $\frac{4\underline{a} + \underline{b}}{3}$

3. The angle between vectors $\underline{i} - \underline{j}$ & $\underline{j} - \underline{k}$ is, $(\underline{i}-\underline{j})\cdot(\underline{j}-\underline{k})$
   a) $\pi/3$ b) $2\pi/3$ c) $-\pi/3$ d) $5\pi/6$

   $\sqrt{2}\cdot\sqrt{2}\cos\theta$

   $2\cos\theta = -1/2$

4. The value of λ for with the two vectors $2\underline{i} - \underline{j} + 2\underline{k}$ & $3\underline{i} + \lambda\underline{j} + \underline{k}$ are prependicular is,

   $(2\underline{i} - \underline{j} + 2\underline{k})\cdot(3\underline{i} + \lambda\underline{j} + \underline{k}) = 0$

   a) 2 b) 4 c) 6 d) 8

   $6 - \lambda + 2 = 0$

   $\lambda = 8$

1. If $|a| = 8$, $|b| = 3$ and $|a \wedge b| = 12$ then value of $a \cdot b$

a) $6\sqrt{3}$ b) $8\sqrt{3}$ c) $12\sqrt{3}$ d) None of there.

$|a \wedge b| = 12$

$|a \wedge b| = |a||b| \sin\theta$

$12 = 8 \cdot 3 \sin\theta$

$\sin\theta = \frac{12}{24} = \frac{1}{2}$

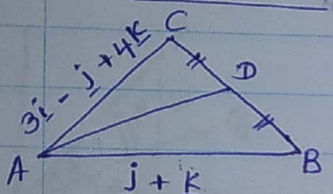
$a \cdot b = |a||b| \cos\theta$

$= 8 \times 3 \cdot \frac{\sqrt{3}}{2}$

$= 12\sqrt{3}$

2. The 2 vectors $j + k$ and $3i - j + 4k$ represents the two sides AB and AC, respectively of a Δ ABC. The length of the midian through A is

a) $\frac{\sqrt{34}}{2}$ b) $\frac{\sqrt{48}}{2}$ c) $\sqrt{18}$ D) None of there.



$$\vec{AD} = \frac{(j + k) + (3i - j + 4k)}{2} = \frac{3i + 5k}{2}$$

$$\vec{AD} = \sqrt{\frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{34}{4}} = \frac{\sqrt{34}}{2}$$

3. The projection of vector $a = 2i - j + k$ along $b = i + 2j + 2k$ is,

a) $\frac{2}{3}$ b) $\frac{1}{3}$ c) 2 d) $\sqrt{6}$

$$\frac{a \cdot b}{|b|} = \frac{(2i - j + k)(i + 2j + 2k)}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

4. The unit vector perpendicular to the vectors $i - j$ and $i + j$ forming a right handed system is,

a) k b) $-k$ c) $\frac{i - j}{\sqrt{2}}$ d) $\frac{i + j}{\sqrt{2}}$

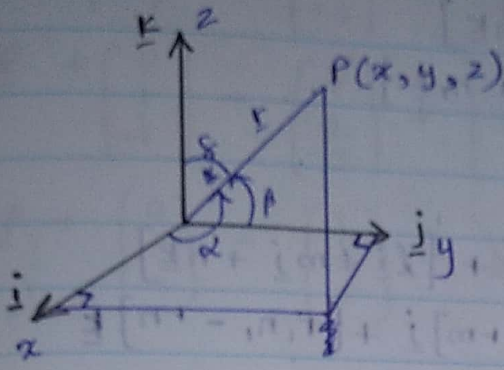
$$\frac{(i - j) \wedge (i + j)}{|(i - j) \wedge (i + j)|} =$$

5. If $|a| = 3$ and $-1 \leq k \leq 2$, then $|ka|$ lies in the interval

a) $\{0, 6\}$ b) $\{-3, 6\}$ c) $\{3, 6\}$ d) $\{1, 2\}$

$|k a| = \{0, 6\}$

Direction cosines vectors.



suppose that vector $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

makes angles α, β and γ with $\underline{i}, \underline{j}, \underline{k}$ respectively.

$$\cos\alpha = x/r$$
$$\cos\beta = y/r$$
$$\cos\gamma = z/r$$

The unit vector of the direction of $\underline{r}$ can be written as.

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$= r\cos\alpha\,\underline{i} + r\cos\beta\,\underline{j} + r\cos\gamma\,\underline{k}$$
$$= r[\cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}]$$

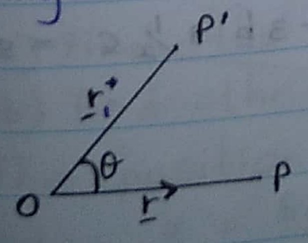
$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of the vector $\underline{r}$.

Further $\underline{r} = r[\cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}]$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

The direction consines of a vector are usually denoted by l, m, n — $l = \cos\alpha$ / $m = \cos\beta$ / $n = \cos\gamma$

Thus $l^2 + m^2 + n^2 = 1$

Angle between two vectors.



Position vectors of the points P and P' relative to O are $\underline{r}$ and $\underline{r}$ and direction cosines of OP and OP' are (l, m, n) and (l_1, m_1, n_1).

Find |PP'|.
If θ is the angle between $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ then usms tringular cosine rule for a show that.

$$\cos\theta = ll_1 + mm_1 + nn_1$$

Further $\underline{r} = r\{\cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k}]$

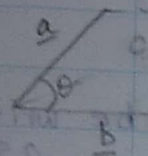
$\vec{OP} = \underline{r} = r[l\underline{i} + m\underline{j} + n\underline{k}]$

$\vec{OP'} = \underline{r}_1 = r_1[l_1\underline{i} + m_1\underline{j} + n_1\underline{k}]$

$\vec{PP'} = \underline{r}_1 - \underline{r}$

$= r_1[l_1\underline{i} + m_1\underline{j} + n_1\underline{k}] - r[l\underline{i} + m\underline{j} + n\underline{k}]$

$= [r_1l_1 - rl]\underline{i} + [r_1m_1 - rm]\underline{j} + [r_1n_1 - rn]\underline{k}$



$c^2 = a^2 + b^2 - 2ab\cos\theta$

$|\vec{PP'}|^2 = |OP'|^2 + |OP|^2 - 2|OP'||OP|\cos\theta$

$[r_1l_1 - rl]^2 + [r_1m_1 - rm]^2 + [r_1n_1 - rn]^2 = (r_1l_1)^2 + (r_1m_1)^2 + (r_1n_1)^2 + (rl)^2 + (rm)^2 + (rn)^2 - 2\sqrt{(r_1l_1)^2 + (r_1m_1)^2 + (r_1n_1)^2}\cdot\sqrt{(rl)^2 + (rm)^2 \ldots}\cos\theta$

$r_1^2l_1^2 + r^2l^2 - 2rr_1ll_1 + r_1^2m_1^2 + r^2m^2 - 2rr_1mm_1 + r_1^2n_1^2 + r^2n^2 - 2rr_1nn_1 = r_1^2[l_1^2 + m_1^2 + n_1^2] + r^2[l^2 + m^2 + n^2] - 2\sqrt{r_1^2[l_1^2 + m_1^2 + n_1^2]}\cdot\sqrt{r^2(l^2 + m^2 + \ldots}\cos\theta$

$\underbrace{r_1^2[l_1^2 + m_1^2 + n_1^2]}_{1} + r^2[l^2 + m^2 + n^2] - 2[rr_1ll_1 + rr_1mm_1 + rr_1nn_1] = r_1^2 + r^2 - 2r_1r\cos\theta$

$r_1^2 + r^2 - 2rr_1[ll_1 + mm_1 + nn_1] = r_1^2 + r^2 - 2r_1r\cos\theta$

$ll_1 + mm_1 + nn_1 = \cos\theta$

vector eq's

1. Make a the subject of the eqⁿ $6\underline{a} - 3\underline{b} + \frac{1}{2}\underline{c} = \underline{d}$

$6\underline{a} - 3\underline{b} + \frac{\underline{c}}{2} = \underline{d}$

$6\underline{a} = \underline{d} + 3\underline{b} - \frac{\underline{c}}{2}$

$\underline{a} = \frac{1}{6}\underline{d} + \frac{1}{2}\underline{b} - \frac{\underline{c}}{12}$

2. solve the vector eqⁿ

$\underline{p} + [6\underline{i} + \underline{j} - \underline{k}] = [4\underline{i} + 2\underline{j} - 3\underline{k}] + 3[-\underline{i} + \underline{k}]$

$\underline{p} = \underline{i} + 2\underline{j} - [6\underline{i} + \underline{j} - \underline{k}]$

$\underline{p} = -5\underline{i} + \underline{j} + \underline{k}$

$$\underline{p} + \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

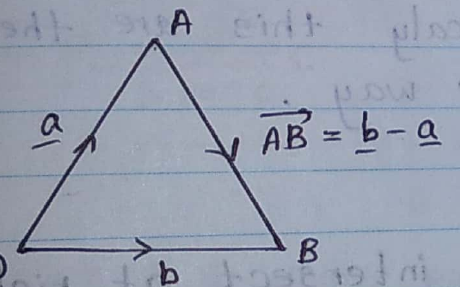
$$\underline{p} + \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$$

$$\underline{p} + \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \underline{p} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{p} = -5\underline{i} + \underline{j} + \underline{k}$$

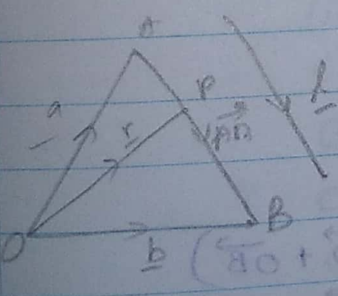
Vector eq's the straight line.

suppose $\underline{a} = \overrightarrow{OA}$ is the vector joining the origin O to the point A and $\underline{b} = \overrightarrow{OB}$ the vector joining O to B.



Thus $\overrightarrow{AB} = \underline{b} - \underline{a}$ is the vector from A to B.

$$\overrightarrow{AB} = \underline{b} - \underline{a}$$
$$\underline{b} = \underline{a} + \overrightarrow{AB}$$



$$\overrightarrow{AP} = \lambda \overrightarrow{AB}$$
$$\underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a})$$
$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$$

$$\overrightarrow{AP} = \mu \underline{l}$$
$$(\underline{r} - \underline{a}) = \mu \underline{l}$$
$$\underline{r} = \underline{a} + \mu \underline{l}$$

a) ~~find~~ the ~~vector $\overrightarrow{AB}$~~

b) Find the ~~vector~~

The position vector of the points. A and B are giveen by
$\underline{a} = 6\underline{i} + 2\underline{j} - \underline{k}$
$\underline{b} = -3\underline{i} + 2\underline{j} + \underline{k}$

a) Find the vector $\overrightarrow{AB}$

b) Find the vector $\overrightarrow{BA}$

c) Find the vector $\underline{p} = \underline{b} + 2(\overrightarrow{BA})$

d) Describe in terms of geometry the vector, $\underline{r} = \underline{b} + s(\overrightarrow{BA})$ where s is any real number.

a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (-3\underline{i} + 2\underline{j} + \underline{k}) - (6\underline{i} + 2\underline{j} - \underline{k})$

$= -9\underline{i} + 2\underline{k}$ //

b) $\overrightarrow{BA} = \underline{a} - \underline{b} = (6\underline{i} + 2\underline{j} - \underline{k}) - (-3\underline{i} + 2\underline{j} + \underline{k})$

$= 9\underline{i} - 2\underline{k}$ //

c) $\underline{P} = \underline{b} + 2(\overrightarrow{BA})$

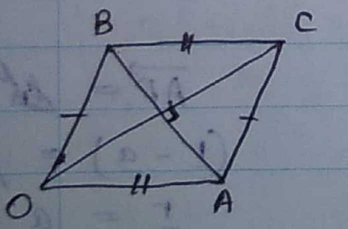
$\underline{P} = (-3\underline{i} + 2\underline{j} + \underline{k}) + 2(9\underline{i} - 2\underline{k}) = 15\underline{i} + 2\underline{j} - 3\underline{k}$ //

d) $\underline{r} = \underline{b} + s(\overrightarrow{BA}) \qquad s > 0,\ s = 0,\ s < 0$

S is a various or this position various. There ~~are~~ fo[r] which is the vector eq's of the staight line. S can take negative value we get the vector eq's of th[e] line joining A to B. Geometricaly, this are the same line ~~what~~ but travel~~ing~~ in opposite way.

H.W.

1) The diagonal of a rhombus intersect at right angle[s]



$\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$

$|\overrightarrow{OA}| = |\overrightarrow{BC}| = |\overrightarrow{OB}| = |\overrightarrow{AC}|$

$\overrightarrow{OA} = \overrightarrow{BC}, \quad \overrightarrow{OB} = \overrightarrow{AC}\ (\because //)$

~~2) The angle subtended~~ $\overrightarrow{OC} \cdot \overrightarrow{AB} = (\overrightarrow{OA} + \overrightarrow{AC}) \cdot (\overrightarrow{AO} + \overrightarrow{OB})$

$= (\overrightarrow{OA} + \overrightarrow{AC}) \cdot (-\overrightarrow{OA} + \overrightarrow{AC})$

$= -\overrightarrow{OA} \cdot \overrightarrow{OA} + \overrightarrow{OA} \cdot \overrightarrow{AC} - \overrightarrow{AC} \cdot \overrightarrow{OA} + \overrightarrow{AC} \cdot \overrightarrow{AC}$

$= \overrightarrow{AC} \cdot \overrightarrow{AC} - \overrightarrow{OA} \cdot \overrightarrow{OA}$

$= |\overrightarrow{AC}|^2 - |\overrightarrow{OA}|^2$

$= 0$ //

2) The angle substended by a Diameter at the circumfe[rence] of a semi-circle is a right angle.

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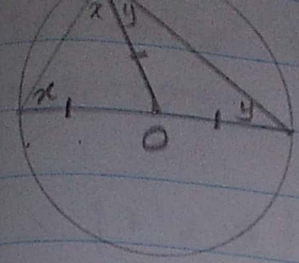


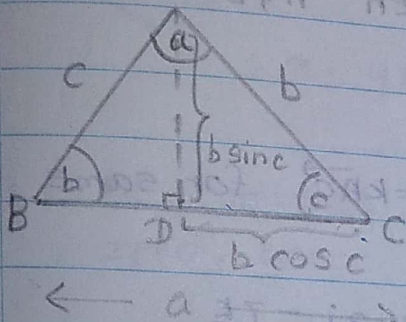
$OA = OB = OC$

$OAB\ \Delta$; $O\hat{A}B = A\hat{B}O$

$OBC\ \Delta$; $O\hat{B}C = O\hat{C}B$

$$x + y + x + y = 180^\circ$$
$$2x + 2y = 180^\circ$$
$$2(x+y) = 180^\circ$$
$$x + y = 90^\circ \Rightarrow A\hat{B}C = 90^\circ$$

3) show that $c^2 = a^2 + b^2 - 2ab\cos C$



$ACD\ \Delta$; $\sin C = \frac{AD}{b}$

$$\boxed{AD = b\sin C}$$

$$CD = b\cos C$$

$$\therefore\ BD = a - b\cos C$$

$ABD\ \Delta$; Using pythagora's theorem,

$$(AB)^2 = (AD)^2 + (BD)^2$$
$$c^2 = (b\sin C)^2 + (a - b\cos C)^2$$
$$c^2 = b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C$$
$$c^2 = a^2 + b^2(\sin^2 C + \cos^2 C) - 2ab\cos C$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

ex: 11/09

1. The position vectors of points A and B relative to the origin O are $4\underline{i} + 4\underline{j} - 7\underline{k}$ and $5\underline{i} - 2\underline{j} + 6\underline{k}$ respectively, find the direction cosines of $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$ and determine the angle between $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

$\overrightarrow{OA} = 4\underline{i} + 4\underline{j} - 7\underline{k}$

$\overrightarrow{OB} = 5\underline{i} - 2\underline{j} + 6\underline{k}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -4\underline{i} - 4\underline{j} + 7\underline{k} + 5\underline{i} - 2\underline{j} + 6\underline{k}$

$= \underline{i} - 6\underline{j} + 13\underline{k}$

$|\overrightarrow{OA}| = \sqrt{4^2 + 4^2 + (-7)^2} = \sqrt{81} = 9$

$|\overrightarrow{OB}| = \sqrt{5^2 + (-2)^2 + 6^2} = \sqrt{65}$

$|\overrightarrow{AB}| = \sqrt{1^2 + (-6)^2 + 13^2}$

$= \sqrt{206}$

$$\frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} = \frac{4\underline{i}+4\underline{j}-7\underline{k}}{9}$$

Direction cosine of $\overrightarrow{OA} = \left(-\frac{4}{9}, \frac{4}{9}, -\frac{7}{9}\right)$

$\overrightarrow{OB} = \left(\frac{5}{\sqrt{165}}, \frac{-2}{\sqrt{165}}, \frac{6}{\sqrt{165}}\right)$

$\overrightarrow{AB} = \left(\frac{1}{\sqrt{206}}, \frac{-6}{\sqrt{206}}, \frac{13}{\sqrt{206}}\right)$

2. Given points $A(5,-1,3)$ and B with position vectors $\overrightarrow{OB} = \underline{i}+2\underline{j}-4\underline{k}$, find the vector $\overrightarrow{AB}$. It is given furt that chascordinate $(13, y, 17)$. Given A, B and C are collinear findly

A, B and C are collinear $\Leftrightarrow$ $\overrightarrow{AB} = k\overrightarrow{AC}$ for same

$\overrightarrow{OA} = 5\underline{i} - \underline{j} + 3\underline{k}$ $\qquad$ $\overrightarrow{AB} = -4\underline{i} + 3\underline{j} - 7\underline{k}$

$\overrightarrow{OB} = \underline{i} + 2\underline{j} - 4\underline{k}$ $\qquad$ $\overrightarrow{OC} = 13\underline{i} + y\underline{j} + 17\underline{k}$

$\overrightarrow{AC} = 8\underline{i} + (y+1)\underline{j} + 14\underline{k}$

$\overrightarrow{AB} = k\overrightarrow{AC}$

$(-4\underline{i}+3\underline{j}-7\underline{k}) = k(8\underline{i}+(y+1)\underline{j}+14\underline{k})$

$\underline{i}(-4-8k)+\underline{j}(3-(y+1)k)+\underline{k}(-7-14k) = 0$

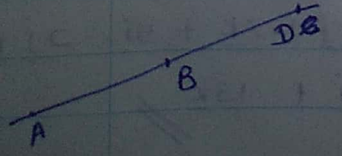
$\Rightarrow$ $-4-8k = 0 \Rightarrow k = -\frac{1}{2}$

$3-(y+1)k = 0 \longrightarrow (y+1)\times -\frac{1}{2} = 3$

$-7-14k = 0 \Rightarrow k = -\frac{1}{2}$ $\qquad$ $y+1 = -6$

$y = -7$

3. If D has cordinate $(-3,5,-1)$ show that $A(5,-1,3)$ $B(1,2,-4)$ and D lie on the same line.



$\overrightarrow{AD} = -8\underline{i} + 6\underline{j} - 4\underline{k}$ $\qquad$ $\overrightarrow{AD} = \lambda\overrightarrow{AB}$

$\overrightarrow{AB} = -4\underline{i} + 3\underline{j} - 7\underline{k}$ $\qquad$ $(-8,6,-4) = \lambda(-4,3,-7$

$\lambda = 2$

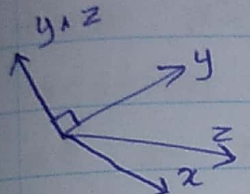
$\overrightarrow{AB}$ ...

4. Determine whether $x = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $y = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $z = \begin{pmatrix}1\\2\\1\end{pmatrix}$ are coplanar vectors.

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1(1-2) - 2(1-1) + 3(2-1) = 2 \neq 0$$

$\therefore$ x, y, z are non-coplanar.

we have to calculate scalar triple product.



$$y \wedge z = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = i(1-2) - j(1-1) + k(2-1)$$

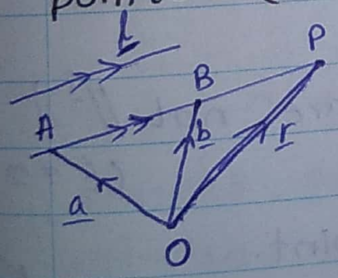
$$= -\underline{i} + \underline{k}$$

$$x \cdot y \wedge z = (\underline{i} + 2\underline{j} + 3\underline{k}) \cdot (-\underline{i} + \underline{k})$$

$$= 2 \neq 0$$

we can see, scalar triple product is not equal to zero, hence vector x, y and z are not coplanar.

5. write down the eq^n of the line that passes through the points $(2,-1,3)$ and $(1,4,-3)$.



$$\overrightarrow{AP} = \lambda \overrightarrow{AB}$$

$$\underline{r} - \underline{a} = \lambda(\underline{b} - \underline{a})$$

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$$

$$\overrightarrow{AP} = \mu \underline{l}$$

$$\underline{r} - \underline{a} = \mu \underline{l}$$

$$\underline{r} = \underline{a} + \mu \underline{l}$$

$$\overrightarrow{OA} = \underline{a} = (2,-1,3)$$

$$\overrightarrow{OB} = \underline{b} = (1,4,-3)$$

$$\underline{r} = (2,-1,3) + \lambda[(1,4,-3) - (2,-1,3)]$$

$$(x,y,z) = (2,-1,3) + \lambda(-1,5,-6)$$

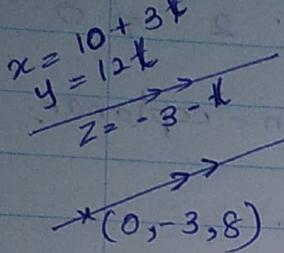
$$\Rightarrow x = 2 - \lambda \Rightarrow \lambda = 2 - x$$

$$y = -1 + 5\lambda \Rightarrow \lambda = \frac{y+1}{5}$$

$$z = 3 - 6\lambda \Rightarrow \lambda = \frac{3-z}{6}$$

$$\frac{2-x}{1} = \frac{y+1}{5} = \frac{3-z}{6}$$

6. Determine if the line that passes through the point $(0,-3,8)$ and is parallel to the line given by $x = 10+3t$, $y = 12t$ and $z = -3-t$ passes through the x-z plane. If does given the co-tinales of the point.



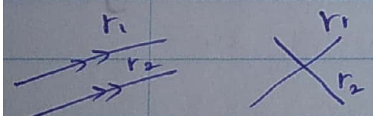
$$\frac{x-10}{3} = \cancel{t}\lambda \qquad \frac{y-0}{12} = \cancel{t}\lambda \qquad \frac{z+3}{-1} = \cancel{t}\lambda$$

$$\Rightarrow \frac{x-10}{3} = \frac{y+0}{12} = \frac{z+3}{-1}$$

$$\underline{r} = \underline{a} + \mu\underline{\lambda}$$
$$= (0,-3,8) + \mu(3,12,-1)$$

7. Let $\underline{a}, \underline{b}$ and $\underline{c}$ be three non-coplanar vectors, show that the lines

$$\underline{r} = 8\underline{a} - 9\underline{b} + 10\underline{c} + \lambda(3\underline{a} - 16\underline{b} + 7\underline{c})$$
$$\underline{r} = 15\underline{a} + 29\underline{b} + 5\underline{c} + \mu(3\underline{a} + 8\underline{b} - 5\underline{c})$$ are non-coplanar



$$\underline{r} = \underline{a} + \lambda\underline{l}$$
$$\underline{r} = \underline{b} + \mu\underline{m}$$ (Both lines are not $//^{l}$) ($\underline{l} \neq \underline{m}$)

Assume that the lines have a common point. ①

$$8\underline{a} - 9\underline{b} + 10\underline{c} + \lambda(3\underline{a} - 16\underline{b} + 7\underline{c}) = 15\underline{a} + 29\underline{b} + 5\underline{c} + \mu(3\underline{a} + 8\underline{b} - 5\underline{c})$$
$$(-7 + 3\lambda - 3\mu)\underline{a} + (-38 - 16\lambda - 8\mu)\underline{b} + (5 + 7\lambda + 5\mu)\underline{c} = 0$$

$$\Rightarrow -7 + 3\lambda - 3\mu = 0 \qquad -38 - 16\lambda - 8\mu = 0 \qquad 5 + 7\lambda + 5\mu = 0$$

$$3(\lambda - \mu) = 7$$
$$\lambda - \mu = \frac{7}{3}$$
$$\boxed{\lambda = \frac{7}{3} + \mu}$$

$$7\left(\frac{7}{3} + \mu\right) + 5\mu = -5$$
$$\frac{49}{3} + 7\mu + 5\mu = -5$$
$$12\mu = -5 - \frac{49}{3}$$
$$12\mu = \frac{-64}{3}$$
$$\mu = \frac{-64}{36}$$

$$\lambda = \frac{7}{3} - \frac{64}{36}$$
$$\lambda = \frac{20}{36} = \frac{5}{9}$$

① ; $8\underline{a} - 9\underline{b} + 10\underline{c} + \frac{5}{9}(3\underline{a} - 16\underline{b} + 7\underline{c}) = 15\underline{a} + 29\underline{b} + 5\underline{c} - \frac{16}{9}(3\underline{a} + 8\underline{b} - 5\underline{c})$

$8\underline{a} - 9\underline{b} + 10\underline{c} + \frac{5}{3}\underline{a} - \frac{80}{9}\underline{b} + \frac{35}{9}\underline{c} = 15\underline{a} + 29\underline{b} + 5\underline{c} - \frac{16\underline{a}}{3} - \frac{128\underline{b}}{9} + \frac{80}{9}\underline{c}$

$\frac{29}{3}\underline{a} - \frac{161}{9}\underline{b} + \frac{125}{9}\underline{c} \neq \frac{29}{3}\underline{a} + \frac{133}{9}\underline{b} + \frac{125}{9}\underline{c}$

$\therefore$

8. Air traffic control is tracking two planes invicinity of their airport. At a given moment. one plane is at a location 45 km east and 120 km north of the airport at an altitude of 7.5 km. The second plane is located 63 km east and 96 km south of the airport at an altitude (height) of 6.0 km. The 1st plane is flying directly toward the airport while the 2nd plane is continuins at a constant altitude with a headins defined by the vector $\overrightarrow{h_2} = (3, 4, 0)$ to land eventually at another airport to the northwest of our air traffic controllers Do the paths of these two aircraft cross?

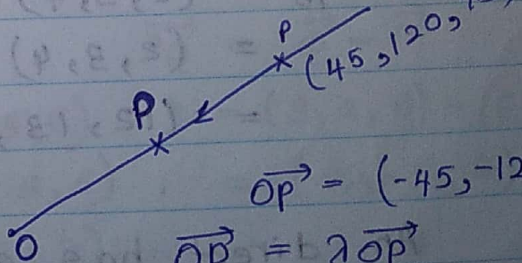
The 1st thing we need to do is to determine the position vectors of the two planes. Define our airport as the orision of cordinals and define x = east

y = north

z = upwest

Positiogn vector of plan ①

$\underline{P} = (45, 120, 7.5)$



$\overrightarrow{OP} = (-45, -120, 7$

$\overrightarrow{OP'} = \lambda\overrightarrow{OP}$

Position vector of plan ②

$\underline{q} = (63, -96, 6)$

Plane 1 is heading directly toward the airport. The vecto from the position of plane 1 to the orisin is given by

$\overline{V} = \underline{O} - \underline{P} = (0 - 45, 0 - 120, 0 - 7.5) = (-45, -120, -7.$

The q^n of the line representing plane 1 motion then becomes,

$$\underline{r}_1 = (45, 120, 7.5) + k_1(-45, -120, -7.5) \quad \text{— (1)}$$

Plane 2 continues at a constant altitude with heading $\vec{h_2} = (3, 4, 0)$.

The q^n of the line representing plane 2 motion then becomes.

$$\underline{r}_2 = (-63, 96, 6.6) + k_2(3, 4, 0) \quad \text{— (2)}$$

11/21

1) In physics, the motion of an object traveling at a constant speed is described by the eq: $\vec{s}_t = \vec{s}_i + \vec{v}_t$ where s_i is the initial position, s is the position at some later time t, and r is the object. write the vector eq^n which returns the set of position vectors $\vec{s}$ for an object having an initial position $\vec{s}_i = (2, 3, 4)$ and a velocity of $\vec{v} = (1, 1, 2)$ and determine the objects location at $t = 10s$

$$\vec{s}_t = \vec{s}_i + \vec{v}_t$$

$$= (2, 3, 4) + (1, 1, -2)t$$

At $t = 10s$

$$\vec{s}_{10} = (2, 3, 4) + (1, 1, -2)10$$

$$= (2, 3, 4) + (10, 10, -20)$$

$$= (12, 13, -16)$$

An direct has a position of $\vec{s}_i - (3, 3, 6)$ at $t = 0$ and a velocity of $\vec{v} = (10, 7, 3)$. use the vector eq^n $\vec{s} = \vec{s}_i +$ to determine the distance traveled by the object between $t = 3s$ and $t = 5s$. Distance measured in meters

$\vec{s} = \vec{s_1} + \vec{s_\lambda}$

$= (3, 3, 6) + (10, 7, 3)\lambda$

$s_3 = (3, 3, 6) + (10, 7, 3)3\lambda$

$s_5 = (3, 3, 6) + (10, 7, 3)5\lambda$

$s_5 - s_3 = (10, 7, 3)2$

$= (20, 14, 6)$ //

$|s_5 - s_3| = \sqrt{(20)^2 + (14)^2 + 6^2}$

$= \sqrt{400 + 196 + 36}$

$= \sqrt{632}$ //

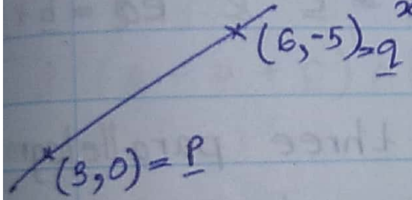
2) write the eq^n of $y = -\frac{5}{3}x + 5$ as a vector eq^n

$\underline{r} = \underline{a} + \lambda \underline{b}$ ~~$x = 3, y = 0$~~

start by choosing two pts on the line,

$x = 3, \quad y = 0$

$x = 6, \quad y = -\frac{5}{3} \times 6 + 5 = -5$



$\vec{PQ} = \vec{q} - \vec{P}$

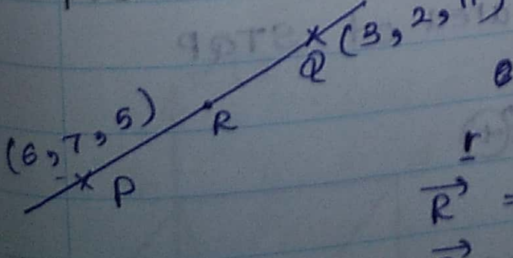
$= (6, -5) - (3, 0) = (3, -5)$

Finally vector eq^n of the line,

$\underline{r} = \underline{a} + \lambda \underline{b}$

$= (3, 0) + \lambda(3, -5) = (3 + 3\lambda, -5\lambda)$ //

3) Determine the eq^n for the line define by the points $P \equiv (6, 7, 5)$ and $Q \equiv (3, 2, 11)$. Then find the position vector for a point R, half-way between these two pts.



$\underline{r} = \underline{a} + \lambda \underline{b}$

$\underline{r} = (6, 7, 5) + \lambda(-3, -5, 6)$

$\underline{r} = (6, 7, 5) + \frac{1}{2}(-3, -5, 6)$

$\vec{R} = (6 - \frac{3}{2}, 7 - \frac{5}{2}, 5 + 3)$

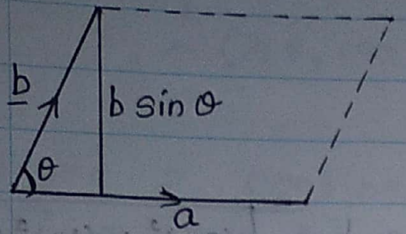
$\vec{R} = (\frac{9}{2}, \frac{9}{2}, 8)$ //

$\vec{PQ} = \vec{Q} - \vec{P}$

$\underline{b} = (3, 2, 11) - (6, 7, 5)$

$= (-3, -5, 6)$ //

Geometrical interpretation of vector product.



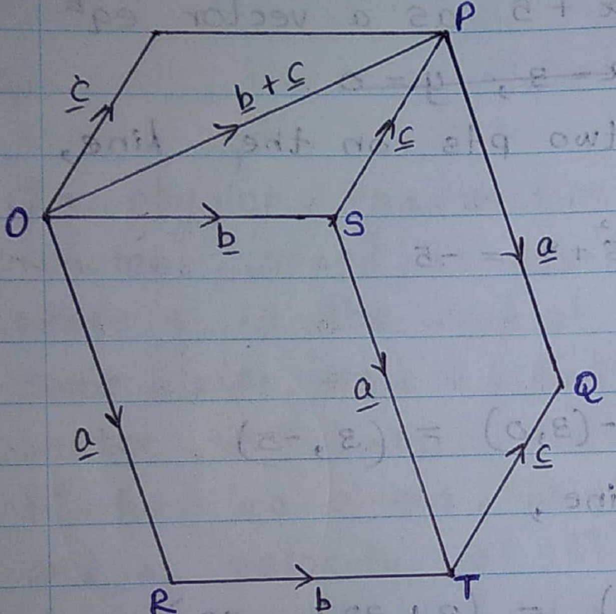
$\underline{A} = \underline{a} \wedge \underline{b} = |\underline{a}||\underline{b}| \sin\theta \, \underline{n}$

$|\underline{A}| = |\underline{a} \wedge \underline{b}| = |\underline{a}||\underline{b}| \sin\theta$ = area of the Parallelogram

Distributive law

I $\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$

II $(\underline{a} + \underline{b}) \wedge \underline{c} = \underline{a} \wedge \underline{c} + \underline{b} \wedge \underline{c}$



$\underline{a} = \overrightarrow{OR}$, $\underline{b} = \overrightarrow{OS}$, $\underline{c} = \overrightarrow{TQ}$

The triangles OSP and RTQ are made up of the same vectors. $\overrightarrow{RT} = \underline{b}$, $\overrightarrow{TQ} = \underline{c}$ & $\overrightarrow{RQ} = \underline{b} + \underline{c}$

shows the three parallelograms OStR, SPQT and OPQR in the same plane. Their areas are equal.

$|\underline{a} \wedge \underline{b}|$, $|\underline{a} \wedge \underline{c}|$ and $|(\underline{b} + \underline{c}) \wedge \underline{a}|$ respectively

$|\underline{a} \wedge (\underline{b} + \underline{c})| = |(\underline{b} + \underline{c}) \wedge \underline{a}|$

$|\underline{a} \wedge (\underline{b} + \underline{c})|$ = area of OPQR

= area of polygon OSPQTR

= area of OStR + area of STQP

$|\underline{a} \wedge (\underline{b} + \underline{c})| = |\underline{a} \wedge \underline{b}| + |\underline{a} \wedge \underline{c}|$ — (*)

Next, consider the directions.

The direction of each of the three vector, $|\underline{a} \wedge \underline{b}|$ and $|\underline{a} \wedge \underline{c}|$ is the same and let $\underline{n}$ be the unit vector in that direction.

From (*) sines $|\underline{a} \wedge (\underline{b} + \underline{c})| \underline{n} = |\underline{a} \wedge \underline{b}| \underline{n} + (\underline{a} \wedge \underline{c}) \underline{n}$

Hence, $\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$

1) Prove that

$$(\underline{a}+\underline{b})\wedge(\underline{a}-\underline{b}) = -2\underline{a}\wedge\underline{b}$$

$$L.H.S. = (\underline{a}+\underline{b})\wedge\underline{a} - (\underline{a}+\underline{b})\wedge\underline{b}$$

$$= \underbrace{\underline{a}\wedge\underline{a}}_{0} + \underline{b}\wedge\underline{a} - \underline{a}\wedge\underline{b} - \underbrace{\underline{b}\wedge\underline{b}}_{0}$$

$$= \underline{b}\wedge\underline{a} - \underline{a}\wedge\underline{b} = -\underline{a}\wedge\underline{b} - \underline{a}\wedge\underline{b} = -2\underline{a}\wedge\underline{b}$$

2) Simplify

I $(\underline{a}+\underline{b}+\underline{c})\wedge(\underline{b}-\underline{c})$

II $x\wedge y$ where $x = 2\underline{u}-\underline{v}$

$y = \underline{u}+2\underline{v}$ and $\underline{u}$ & $\underline{v}$ being andy two vectors.

I $$(\underline{a}+\underline{b}+\underline{c})\wedge(\underline{b}-\underline{c}) = (\underline{a}+\underline{b}+\underline{c})\wedge\underline{b} - (\underline{a}+\underline{b}+\underline{c})\wedge\underline{c}$$

$$= \underline{a}\wedge\underline{b} + \underline{c}\wedge\underline{b} - \underline{a}\wedge\underline{c} - \underline{b}\wedge\underline{c}$$

$$= \underline{a}\wedge\underline{b} - \underline{a}\wedge\underline{c} + 2\,\underline{c}\wedge\underline{b}$$

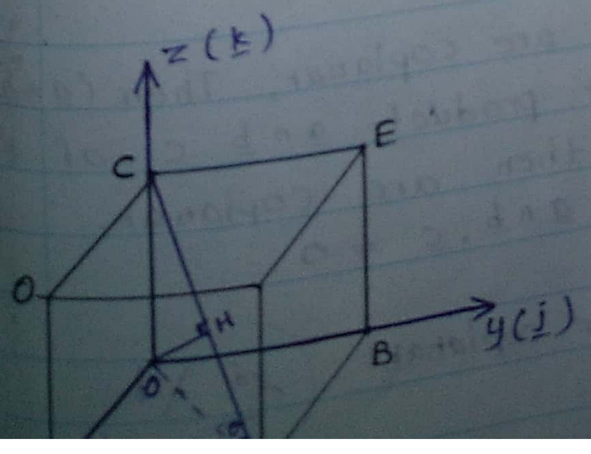
II $$x\wedge y = (2\underline{u}-\underline{v})\wedge(\underline{u}+2\underline{v})$$

$$= (2\underline{u}-\underline{v})\wedge\underline{u} + (2\underline{u}-\underline{v})\wedge 2\underline{v}$$

$$= 2\underline{u}\wedge\underline{u} - \underline{v}\wedge\underline{u} + 4\underline{u}\wedge\underline{v} - 2\underline{v}\wedge\underline{v}$$

$$= -\underline{v}\wedge\underline{u} + 4\underline{u}\wedge\underline{v} = 5\underline{u}\wedge\underline{v}$$

3) Find the perpendicular distance of a corner of a unit cube from a diagonal which ~~from a diagonal~~ does not pass through it.



$OH = OF\sin\theta$ ... (1)

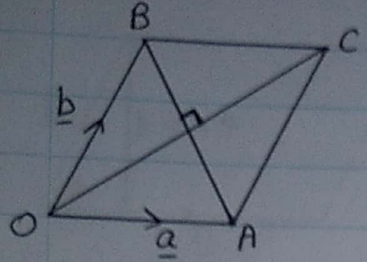
Now $|\vec{OF}\wedge\vec{FC}| = |\vec{OF}||\vec{FC}|\sin\theta$

$$\sin\theta = \frac{|\vec{OF}\wedge\vec{FC}|}{|\vec{OF}||\vec{FC}|}$$

From (1) $\Rightarrow$ $$OH = \frac{OF\,|\vec{OF}\wedge\vec{FC}|}{|\vec{OF}||\vec{FC}|}$$

$$= \frac{|\vec{OF}\wedge\vec{FC}|}{|\vec{FC}|}$$

4) The diasonal of a rhombus intersect at right angles (90°)



$\vec{OC} \cdot \vec{AB} = 0$

suppose that $\vec{OA} = \underline{a}$ and $\vec{OB} = \underline{b}$

$\vec{OC} = \vec{OA} + \vec{AC}$
$= \underline{a} + \underline{b}$

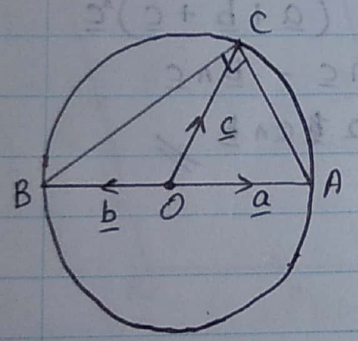
$\vec{AB} = \vec{AO} + \vec{OB}$
$= -\underline{a} + \underline{b}$

Since OACB is a rhombus then $|\underline{a}| = |\underline{b}|$

Now prove that

$\vec{OC} \cdot \vec{AB} = (\underline{a}+\underline{b}) \cdot (\underline{b}-\underline{a})$
$= \underline{a}\cdot\underline{b} - \underline{a}\cdot\underline{a} + \underline{b}\cdot\underline{b} - \underline{b}\cdot\underline{a}$
$= \underline{b}\cdot\underline{b} - \underline{a}\cdot\underline{a} = |\underline{b}|^2 - |\underline{a}|^2 = 0 \quad (\because |\underline{a}| = |$

5) The angle subtended by a diameter at the circumf-ence of a semi-circle is a right angle.



Taking O as the orism, & the posi vector of A, B, C be $\underline{a}, \underline{b}, \underline{c}$ recpective
The $|\underline{c}| = |\underline{b}| = |\underline{a}|$ = the radius.

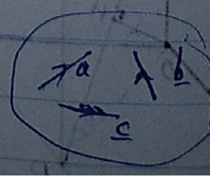
To prove BC is perpendicular to CA consider the dot product $\vec{CA}$ and $\vec{CB}$.

$\vec{CA} \cdot \vec{CB} = (\underline{a} - \underline{c}) \cdot (\underline{b} - \underline{c})$
$= \underline{a}\cdot\underline{b} - \underline{a}\cdot\underline{c} - \underline{b}\cdot\underline{c} + \underline{c}\cdot\underline{c}$

6) If there non-zero $\underline{a}, \underline{b}, \underline{c}$ are coplanar, Then $(\underline{a} \wedge \underline{b})\cdot\underline{c}$ & conversely, if the scalar product $\underline{a} \wedge \underline{b} \cdot \underline{c}$ of three non-zero vectors is zero, then are coplanar.

$\underline{a}, \underline{b}, \underline{c}$ are coplanar $\Leftrightarrow$ $\underline{a} \wedge \underline{b} \cdot \underline{c} = 0$

Proof Assume that the vectors are coplanar.



The $\underline{c}$ can be expressed as a linear combination $\Rightarrow \underline{c} = \lambda \underline{a} + \mu \underline{b}$

$\underline{a} \wedge \underline{b} \cdot \underline{c} = \underline{a} \wedge \underline{b} \cdot (\lambda \underline{a} + \mu \underline{b}) = \lambda\, \underline{a} \wedge \underline{b} \cdot \underline{a} + \mu\, \underline{a} \wedge \underline{b} \cdot \underline{b} = 0.$

Let assume that $\underline{a} \wedge \underline{b} \cdot \underline{c} = 0$

$\underline{a} \wedge \underline{b} \cdot \underline{c} = 0 \Rightarrow$ (implies) that the value of the parallelpiped formed by the vector vanishes.

Thus three vector are coplanar.

Some properties of the scalar triple product.

1. Scalar triple product $\{a, b, c\}$, where the vectors are given in terms of three non coplanar vectors $\underline{l}, \underline{m}, \underline{n}$

Let $\underline{a} = a_1 \underline{l} + a_2 \underline{m} + a_3 \underline{n}$

$\underline{b} = b_1 \underline{l} + b_2 \underline{m} + b_3 \underline{n}$

$\underline{c} = c_1 \underline{l} + c_2 \underline{m} + c_3 \underline{n}$ show that, $\underline{a} \wedge \underline{b} \cdot \underline{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} (l, m, n)$

2. If $\underline{a}, \underline{b}, \underline{c}$ are three vectors then,

$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

Proof let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$

$\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$

$\underline{c} = c_1 \underline{i} + c_2 \underline{j} + c_3 \underline{k}$

$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \wedge \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= [a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}] \wedge [\underline{i}(b_2 c_3 - b_3 c_2) - \underline{j}(b_1 c_3 - b_3 c_1) + \underline{k}(b_1 c_2 - b_2 c_1)]$

$= a_1 \underline{i}$

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Vector triple product.

(a, b, c) is defined as the cross product of vec[tor] a with the cross product of vectors b and c.

i.e. $a \wedge (b \wedge c)$

Here $a \wedge (b \wedge c)$ is coplanar with the vectors b & c and perpendicular to a.

Hence we can write.
$a \wedge (b \wedge c)$ as linear combination of vector b and [c]
That is,

$$a \wedge (b \wedge c) = x b + y c$$

Vector triple product formula,

$$a \wedge (b \wedge c) = (a \cdot c) b - (a \cdot b) c \quad \text{and}$$

$$(a \wedge b) \wedge c = (a \cdot c) b - (b \cdot c) a$$

$$\Rightarrow \quad a \wedge (b \wedge c) \neq (a \wedge b) \wedge c$$

Vector triple product proof :

$$(a \wedge b) \wedge c = (a \cdot c) b - (b \cdot c) a$$

$(a \wedge b) \wedge c$ as a linear combination of a & b.

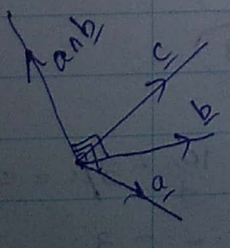
$$(a \wedge b) \wedge c = x a + y b \quad — ①$$

$$c \cdot ① \Rightarrow \quad c \cdot \underbrace{(a \wedge b) \wedge c} = c \cdot [x \cancel{y} a + y b]$$

$$c \wedge (a \wedge b) \cdot c = x c a + y c \cdot b$$

$$\Downarrow$$

$$0 = x c \cdot a + y c \cdot b$$



$$x c a = - y \cdot c \cdot b$$

$$\frac{x}{c \cdot b} = \frac{y}{c \cdot a} = \lambda$$

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From (1);

$$(\underline{a} \wedge \underline{b}) \wedge \underline{c} = \lambda(\underline{c}\cdot\underline{b})\,\underline{a} - \lambda(\underline{c}\cdot\underline{a})\,\underline{b}.$$

It is valid for every value of $\underline{a}$, $\underline{b}$ and $\underline{c}$ because it is an identity.

$\underline{a} = \underline{i}$, $\underline{b} = \underline{j}$, $\underline{c} = \underline{i}$

$$(\underline{i} \wedge \underline{j}) \wedge \underline{i} = \lambda(\underline{i}\cdot\underline{j})\,\underline{i} - \lambda(\underline{i}\cdot\underline{i})\,\underline{j}$$

$$\underline{k} \wedge \underline{i} = 0 - \lambda \underline{j}$$

$$\underline{j} = -\lambda \underline{j}$$

$$\lambda = -1$$

$$(\underline{a}\wedge\underline{b})\wedge\underline{c} = -(\underline{c}\cdot\underline{b})\underline{a} + (\underline{c}\cdot\underline{a})\underline{b} = (\underline{c}\cdot\underline{a})\underline{b} - (\underline{c}\cdot\underline{b})\underline{a}$$

Properties

1. $$\underline{a}\wedge\underline{b}\cdot\underline{c}\wedge\underline{d} = \begin{vmatrix} \underline{a}\cdot\underline{c} & \underline{a}\cdot\underline{d} \\ \underline{b}\cdot\underline{c} & \underline{b}\cdot\underline{d} \end{vmatrix}$$

$\underline{g} = \underline{a}\wedge\underline{b}$

$$\begin{aligned}
\underline{a}\wedge\underline{b}\cdot\underline{c}\wedge\underline{d} &\Rightarrow \underline{g}\cdot\underline{c}\wedge\underline{d} \\
&\Rightarrow \underline{g}\wedge\underline{c}\cdot\underline{d} \\
&= (\underline{a}\wedge\underline{b})\wedge\underline{c}\cdot\underline{d} \\
&= [(\underline{a}\cdot\underline{c})\underline{b} - (\underline{b}\cdot\underline{c})\underline{a}]\cdot\underline{d} \\
&= (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d}) - (\underline{b}\cdot\underline{c})(\underline{a}\cdot\underline{d}) \\
&= \begin{vmatrix} \underline{a}\cdot\underline{c} & \underline{a}\cdot\underline{d} \\ \underline{b}\cdot\underline{c} & \underline{b}\cdot\underline{d} \end{vmatrix}
\end{aligned}$$

2. $$\underline{a}\wedge\underline{b}\cdot(\underline{b}\wedge\underline{c})\wedge(\underline{c}\wedge\underline{a}) = [\underline{a}\wedge\underline{b}\cdot\underline{c}]^2$$

$$\underline{a}\wedge\underline{b}\cdot[(\underline{b}\wedge\underline{c})\wedge\underbrace{(\underline{c}\wedge\underline{a})}_{\underline{g}}]$$

$$\begin{aligned}
&\underline{b}\cdot\underline{c}\wedge\underline{a} \\
&= \underline{b}\wedge\underline{c}\cdot\underline{a} \\
&= \underline{a}\cdot\underline{b}\wedge\underline{c} \\
&= \underline{a}\wedge\underline{b}\cdot\underline{c}
\end{aligned}$$

$\Rightarrow \underline{a}\wedge\underline{b}\cdot[(\underline{b}\wedge\underline{c})\wedge\underline{g}]$

$\Rightarrow \underline{a}\wedge\underline{b}\,[(\underline{b}\cdot\underline{g})\underline{c} - (\underline{c}\cdot\underline{g})\underline{b}]$

$\Rightarrow \underline{a}\wedge\underline{b}\,[(\underline{b}\cdot\underline{c}\wedge\underline{a})\underline{c} - (\underline{c}\cdot\underline{c}\wedge\underline{a})\underline{b}]$

$\Rightarrow \underline{a}\wedge\underline{b}\,[(\underline{b}\cdot\underline{c}\wedge\underline{a})\underline{c}] = [\underline{b}\cdot\underline{c}\wedge\underline{a}][\underline{a}\wedge\underline{b}\cdot\underline{c}]$

$\Rightarrow \underline{a}\wedge\underline{b}\,[(\underline{b}\cdot\underline{c}\wedge\underline{a})\underline{c}]$

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$$\underline{b} \wedge \underline{d} = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \underline{i}(3-2) - \underline{j}(2-3) + \underline{k}(4-9) = \underline{i} + \underline{j} - 5\underline{k}$$

$$\gamma = \frac{(\underline{a} \cdot \underline{b} \wedge \underline{d})}{(\underline{a} \cdot \underline{b} \wedge \underline{c})} = \frac{(1,2,3)(1,1,-5)}{-18} = \frac{-12}{-18} = \frac{2}{3}$$

$$\underline{d} = \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c}$$
$$= -\tfrac{1}{3}\underline{a} - \tfrac{2}{3}\underline{b} + \tfrac{2}{3}\underline{c} = -\tfrac{1}{3}(\underline{a} + 2\underline{b} - 2\underline{c})$$

Product of two scalar triple product

Prove that,

$$[\underline{p}, \underline{q}, \underline{r}][\underline{p}', \underline{q}', \underline{r}'] = \begin{vmatrix} \underline{p}\cdot\underline{p}' & \underline{p}\cdot\underline{q}' & \underline{p}\cdot\underline{r}' \\ \underline{q}\cdot\underline{p}' & \underline{q}\cdot\underline{q}' & \underline{q}\cdot\underline{r}' \\ \underline{r}\cdot\underline{p}' & \underline{r}\cdot\underline{q}' & \underline{r}\cdot\underline{r}' \end{vmatrix}$$

$$[\underline{p}, \underline{q}, \underline{r}][\underline{p}', \underline{q}', \underline{r}'] = [\underline{p}\cdot\underline{q}\wedge\underline{r}][\underline{p}'\cdot\underline{q}'\wedge\underline{r}']$$

$$(\underline{a}\wedge\underline{b})\wedge(\underline{c}\wedge\underline{d}) = [\underline{a},\underline{b},\underline{d}]\cdot\underline{c} - [\underline{a},\underline{b},\underline{c}]\underline{d}$$
$$= [\underline{a},\underline{c},\underline{d}]\underline{b} - (\underline{b},\underline{c},\underline{d})\underline{a}$$

$$[\underline{a},\underline{b},\underline{d}]\underline{c} - (\underline{a},\underline{b},\underline{c})\underline{d} - [\underline{a},\underline{c},\underline{d}]\underline{b} + [\underline{b},\underline{c},\underline{d}]\underline{a} = 0$$
$$(\underline{b},\underline{c},\underline{d})\underline{a} - (\underline{a},\underline{c},\underline{d})\underline{b} + (\underline{a},\underline{b},\underline{d})\underline{c} - (\underline{a},\underline{b},\underline{c})\underline{d} = 0$$

Considering the four vectors $\underline{p}, \underline{q}, \underline{r}$ and $\underline{s}$. we obtain linear relation.

$$[\underline{q},\underline{r},\underline{s}]\underline{p} - [\underline{p},\underline{r},\underline{s}]\underline{q} + (\underline{p},\underline{q},\underline{s}]\underline{r} - [\underline{p},\underline{q},\underline{r}]\underline{s} = 0$$

Let consider $\underline{s} = \underline{p}'\wedge\underline{q}'$

$$[\underline{q},\underline{r},\underline{p}'\wedge\underline{q}']\underline{p} - [\underline{p},\underline{r},\underline{p}'\wedge\underline{q}')\underline{q} + [\underline{p},\underline{q},\underline{p}'\wedge\underline{q}']\underline{r} - (\underline{p},\underline{q},\underline{r}].\underline{p}'\wedge\underline{q}' =$$

$\cancel{*}\cdot\underline{r}$;

$$[\underline{q},\underline{r},\underline{p}'\wedge\underline{q}']\underline{p}\cdot\underline{r}' - [\underline{p},\underline{r},\underline{p}'\wedge\underline{q}')\underline{q}\cdot\underline{r}' + [\underline{p},\underline{q},\underline{p}'\wedge\underline{q}']\underline{r}\cdot\underline{r}' - (\underline{p},\underline{q},\underline{r}]\underline{p}'\wedge\underline{q}' = 0$$

Date: / /

$[p, q, r][p', q', r'] = [q, r, p'\wedge q']\, p\cdot r' - [p, r, p'\wedge q']\, q\cdot r' + [p, q, p'\wedge q']\, r\cdot r'$

$$[q, r, p'\wedge q']\, p\cdot r' = [q\cdot r\wedge(p'\wedge q')](p\cdot r')$$
$$= [q\cdot(r\cdot q')p' - (r\cdot p')q'](p\cdot r')$$
$$= [(q\cdot p')(r\cdot q') - (r\cdot p')(q\cdot q')](p\cdot r')$$

$$[p, r, p'\wedge q'](q\cdot r') = [p\cdot r\wedge(p'\wedge q')](q\cdot r')$$
$$= [p\cdot(r\cdot q')p' - (r\cdot p')q'](q\cdot r')$$
$$= [(p\cdot p')(r\cdot q') - (r\cdot p')(q\cdot q')][q\cdot r']$$

$$[p, q, p'\wedge q'](r\cdot r') = [p\cdot q\wedge(p'\wedge q')](r\cdot r')$$
$$= [p\cdot(q\cdot q')p' - (q\cdot p')q'](r\cdot r')$$
$$= [(p\cdot p')(q\cdot q') - (q\cdot p')(q\cdot q')](r\cdot r')$$

$$[p, q, r][p', q', r'] = [(q\cdot p')(r\cdot q') - (r\cdot p')(q\cdot q')](p\cdot r')$$
$$- [(p\cdot p')(r\cdot q') - (r\cdot p')(q\cdot q')](q\cdot r') +$$
$$[(p\cdot p')\cdot(q\cdot q') - (q\cdot p')(q\cdot q')](r\cdot r')$$

$$= \begin{vmatrix} p\cdot p' & p\cdot q' & p\cdot r' \\ q\cdot q' & q\cdot q' & q\cdot r' \\ r\cdot q' & r\cdot q' & r\cdot r' \end{vmatrix}$$

$$\cancel{= (p\cdot p')[(q\cdot q')(r\cdot r'}$$

Th^n

I) If a, b, c are three non coplanar vectors then prove that $b\wedge c$, $c\wedge a$, $a\wedge b$ are also non-coplanar and express a, b, c in terms of $b\wedge c$, $c\wedge a$, $a\wedge b$.

II) If a, b, c are three non coplanar vectors the express $b\wedge c$, $c\wedge a$, $a\wedge b$ in terms of a, b, c.

Since $\underline{a}, \underline{b}, \underline{c}$ are non coplanar $\underline{a}\cdot\underline{b}\wedge\underline{c} \neq 0$,

$(\underline{b}\wedge\underline{c})\cdot(\underline{c}\wedge\underline{a})\wedge(\underline{a}\wedge\underline{b}) \neq 0$

~~$\underline{b}\wedge\underline{c}\cdot[(\underline{c}\wedge\underline{a})\wedge(\underline{a}\wedge\underline{b})] =$~~

$$(\underline{b}\wedge\underline{c})\cdot[(\underline{c}\wedge\underline{a})\wedge(\underline{a}\wedge\underline{b})] = (\underline{b}\wedge\underline{c})[\underline{c}\cdot(\underline{a}\wedge\underline{b})\underline{a} - (\underline{c}\cdot\underline{a})(\underline{a}\wedge\underline{b})]$$

$$= (\underline{b}\wedge\underline{c}\cdot\underline{a})(\underline{c}\cdot(\underline{a}\wedge\underline{b}) - (\underline{c}\cdot\underline{a})\underline{b}\wedge\underline{c}(\underline{a}\wedge\underline{b})$$

$$= |\underline{b}\wedge\underline{c}\cdot\underline{a}|^2 - (\underline{c}\cdot\underline{a})[\underline{b}\cdot\underline{c}\wedge(\underline{a}\wedge\underline{b})]$$

The vector $\underline{b}\wedge\underline{c}$, $\underline{c}\wedge\underline{a}$, $\underline{a}\wedge\underline{b}$ being non coplanar every vector and in particule $\underline{a}, \underline{b}, \underline{c}$ can be expressible as linear combination of the some.

Let $\underline{a} = \lambda.\,\underline{b}\wedge\underline{c} + m.\,\underline{c}\wedge\underline{a} + n.\,\underline{a}\wedge\underline{b}$ —— (1)

The setting scalar product with $\underline{a}, \underline{b}, \underline{c}$ we have,

$\times\cdot\underline{a}$;

$$\underline{a}\cdot\underline{a} = \lambda.\,\underline{b}\wedge\underline{c}\cdot\underline{a} + m\,\underbrace{\underline{c}\wedge\underline{a}\cdot\underline{a}}_{0} + n(\underbrace{\underline{a}\wedge\underline{b}\cdot\underline{a}}_{0})$$

$$|\underline{a}|^2 = \lambda\,\underline{b}\wedge\underline{c}\cdot\underline{a}$$

$$\lambda = \frac{\underline{b}\wedge\underline{c}\cdot\underline{a}}{|\underline{a}|^2}$$

$\times\cdot\underline{b}$; $\underline{a}\cdot\underline{b} = m\,\underline{c}\wedge\underline{a}\cdot\underline{b}$

$$m = \frac{\underline{a}\cdot\underline{b}}{\underline{c}\wedge\underline{a}\cdot\underline{b}}$$

$\times\cdot\underline{c}$; $\underline{a}\cdot\underline{c} = n\,\underline{a}\wedge\underline{b}\cdot\underline{c}$

$$n = \frac{\underline{a}\cdot\underline{c}}{\underline{a}\wedge\underline{b}\cdot\underline{c}}$$

$$\underline{a}\wedge\underline{b}\cdot\underline{c} = \underline{a}\cdot\underline{b}\wedge\underline{c} = \underline{b}\wedge\underline{c}\cdot\underline{a} = \underline{b}\cdot\underline{c}\wedge\underline{a} = \underline{c}\wedge\underline{a}\cdot\underline{b}$$

$$\underline{a} = \frac{|\underline{a}|^2}{\underline{b}\wedge\underline{c}\cdot\underline{a}}\underline{b}\wedge\underline{c} + \frac{\underline{a}\cdot\underline{b}}{\underline{c}\wedge\underline{a}\cdot\underline{b}}\underline{c}\wedge\underline{a} + \frac{\underline{a}\cdot\underline{b}}{\underline{a}\wedge\underline{b}\cdot\underline{c}}\underline{a}\wedge\underline{b}$$

$$= \frac{1}{[\underline{b}, \underline{c}, \underline{a}]}[|\underline{a}|^2\underline{b}\wedge\underline{c} + \underline{a}\cdot\underline{b}.\,\underline{c}\wedge\underline{a} + \underline{a}\cdot\underline{c}\,\underline{a}$$

$//^{ly}$ $\underline{b}$ & $\underline{c}$ can be expressed terms of $\underline{b}\wedge\underline{c}$, $\underline{c}\wedge\underline{a}$, & $\underline{a}\wedge\underline{b}$.

12/12

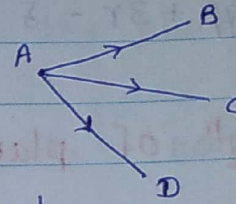
Th^m

A necessary and sufficient condition for four points A, B, C, D with position vectors $\underline{a}, \underline{b}, \underline{c}$ & $\underline{d}$ to be coplanar is that there exist four scalars $\alpha, \beta, \gamma, \delta$ not all zero such that,

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} + \delta \underline{d} = \underline{0}$$
$$\alpha + \beta + \gamma + \delta = 0$$

necessary condition

Assume that A, B, C, D are coplanar. Then the vector $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.



Thus there exists scalar λ and μ such that,

$$\overrightarrow{AD} = \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$ (expressed as linear combinations)

$$\underline{d} - \underline{a} = \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$$
$$\underline{d} - \underline{a} - \lambda(\underline{b} - \underline{a}) - \mu(\underline{c} - \underline{a}) = \underline{0}$$
$$(\lambda + \mu - 1)\underline{a} - \lambda \underline{b} - \mu \underline{c} + \underline{d} = 0$$
$$\alpha = \lambda + \mu - 1 \quad \beta = -\lambda \quad \gamma = -\mu \quad \delta = 1$$

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} + \delta \underline{d} = \underline{0}$$
$$\alpha + \beta + \gamma + \delta = \lambda + \mu - 1 - \lambda - \mu + 1 = 0 \,//$$

Sufficient $Cond^n$

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} + \delta \underline{d} = 0$$
$$\alpha + \beta + \gamma + \delta = 0$$ (assume that $\alpha, \beta, \gamma, \delta \neq 0$)

$$\alpha \underline{a} + \beta \underline{b} = -(\gamma \underline{c} + \delta \underline{d})$$
$$\alpha + \beta = -(\gamma + \delta)$$

$$\frac{\alpha \underline{a} + \beta \underline{b}}{\alpha + \beta} = \frac{\gamma \underline{c} + \delta \underline{d}}{\gamma + \delta}$$

No:

11

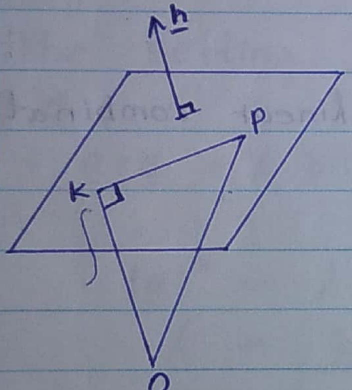
1. Show that the points A, B, C, D with position vectors $-6\underline{a}+3\underline{b}+2\underline{c}$, $3\underline{a}-2\underline{b}+4\underline{c}$, $5\underline{a}+7\underline{b}+3\underline{c}$ and $-13\underline{a}+17\underline{b}-\underline{c}$ are coplanar.

$$\alpha[-6\underline{a}+3\underline{b}+2\underline{c}]+\beta[3\underline{a}-2\underline{b}+4\underline{c}]+\gamma[5\underline{a}+7\underline{b}+3\underline{c}]+\delta[-13\underline{a}+17\underline{b}-\underline{c}]=0$$

$$\underline{a}[-6\alpha+3\beta+5\gamma-13\delta]+\underline{b}[3\alpha-2\beta+7\gamma+17\delta]+\underline{c}[2\alpha+4\beta+3\gamma-\delta]=0$$

$$\Rightarrow -6\alpha+3\beta+5\gamma-13\delta=0$$
$$3\alpha-2\beta+7\gamma+17\delta=0$$
$$2\alpha+4\beta+3\gamma-\delta=0$$

Vector eq^n of plane.



Let $\underline{n}$ be the unit vector normal to given plane and p the length of the perpendicular from the origin O, to the plane.

$$\overrightarrow{OK}=P.\underline{n}$$

Ok perpendicular to the plane, we have $\overrightarrow{Ok}=P\underline{n}$

Let P be any point on the plane with position vector $\underline{r}$ then $\overrightarrow{OP}=\underline{r}$.

$$\overrightarrow{KP}=\overrightarrow{KO}+\overrightarrow{OP}$$
$$=-P\underline{n}+\underline{r} = \underline{r}-P\underline{n}$$

kP lies on the plane

Ok $\perp$ kp

$$\overrightarrow{Ok}.\overrightarrow{kp}=0$$
$$P\underline{n}.(\underline{r}-P\underline{n})=0$$

$P\neq 0$;

$$\underline{n}.\underline{r}-P\underline{n}.\underline{n}=0$$
$$\underline{n}.\underline{r}-P=0$$
$$\underline{r}.\underline{n}=P$$

1. Find the distance from the origin to the plane. $2x+3y-4z=5$

$$(x\underline{i}+y\underline{j}+z\underline{k}).(2\underline{i}+3\underline{j}-4\underline{k}) = 5$$

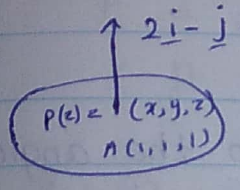
$$(x,y,z).(2,3,-4) = 5$$

$$\frac{(x,y,z).(2,3,-4)}{\sqrt{2^2+3^2+(-4)^2}} = \frac{5}{\sqrt{2^2+3^2+(-4)^2}}$$

$$\frac{(x,y,z)(2,3,-4)}{\sqrt{29}} = \frac{5}{\sqrt{29}}$$

Distance from the origin to the plane is $\frac{5}{\sqrt{29}}$ //

2. Find the plane through $A(1,1,1)$ normal to $2\underline{i}-\underline{j}+5\underline{k}$.



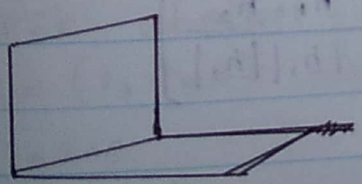
$$\overrightarrow{AP} \perp (2\underline{i}-\underline{j}+5\underline{k})$$

$$\overrightarrow{AP}.(2\underline{i}-\underline{j}+5\underline{k}) = 0$$

$$\overrightarrow{AP} = \underline{r}-\underline{a}$$
$$= (x,y,z)-(1,1,1)$$
$$= (x-1, y-1, z-1)$$

$$(x-1, y-1, z-1)(2\underline{i}-\underline{j}+5\underline{k}) = 0$$
$$2x-2-y+1+5z-5 = 0$$
$$2x-y+5z = 6$$

Vector $eq^{\underline{n}}$ of the plane passing a given point and coarial with two given planes.



given two planes

$$\underline{r}.\underline{n}_1 = \varepsilon_1$$
$$\underline{r}.\underline{n}_2 = \varepsilon_2$$ and which passes

through the point A with position vector $\underline{a}$

The $q^{\underline{n}}$

$$\underline{r}.\underline{n}_1 - q_1 + k(\underline{r}.\underline{n}_2 - q_2) = 0$$

$$\Rightarrow \underline{r}.(\underline{n}_1 + k\underline{n}_2) = q_1 + kq_2$$

represents a plane through the line of intersection of the two given planes for all values of k.

Scalar & vector field

what is scalar field

$\sin xyz$, $\cos x$, $x^2+y^2+z^2$, $ax+by+cz+d$

where a, b, c, d all are constant.

Definition :

two diamentional

Let D be a subset of $\mathbb{R}^2 / \mathbb{R}^3$. a vector field on

Three diamentional

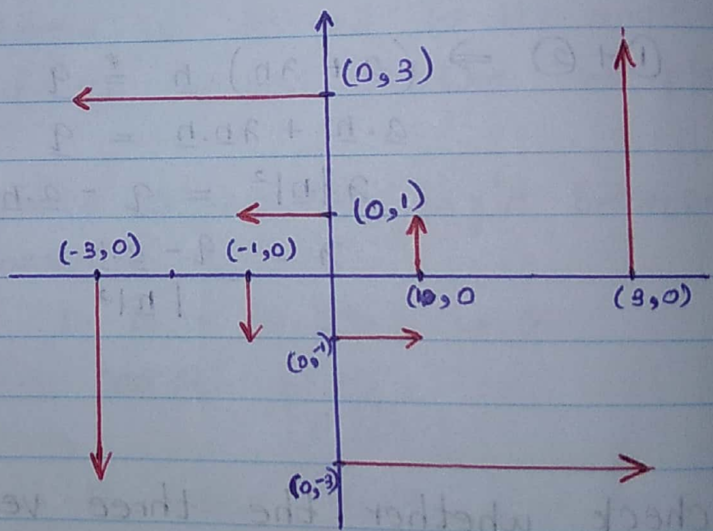
$\mathbb{R} / \mathbb{R}^3$ is a function.

$\vec{F}$ = assign to each point (x, y) , (x, y, z)

$\vec{F}.(x, y)$ / $\vec{F}.(x, y, z)$.

Let's see how can we sketch.

x, y	$\vec{F}(x, y)$
(1, 0)	$-0i+j$/<0,1>
(0, 1)	<-1, 0>
(-1, 0)	<0, -1>
(0, -1)	<1, 0>
(3, 0)	<0, 3>
(0, 3)	<-3, 0>
(-3, 0)	<0, -3>
(0, -3)	<3, 0>



Gradient

It is a vector field.

$\nabla \rightarrow$ gradient

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

for any scalar function $f(x,y,z) = x^2+y^2+z^2$ find it's gradient (∇f) ?

$$f(x,y,z) = x^2+y^2+z^2$$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$ //

If a scalar field $q = 3(x^2y - y^2x)$ calculate gradient at the point $(1,-2,-1)$.

$$\text{gradient } q = \vec{\nabla q} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](3(x^2y - y^2x))$$

$$= 3[i(2xy - y^2) + j(x^2 - 2xy)]$$

$$= 3i(2xy - y^2) + 3j(x^2 - 2xy)$$

$$= 3i(2\times1\times-2 - 4) + 3j(1 - 2\times1\times-2)$$

$$= -24i + 15j$$ //

Divergence and curl operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \qquad \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$\nabla \cdot \vec{F} \leftarrow$ Divergence of the vector field.

Ex: for any vector function,

$$V = xy\hat{i} + yz\hat{j} + zx\hat{k}$$

$$\text{divergence } \nabla\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (xy\hat{i} + yz\hat{j} + zx$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$(\because i.i = j.j = k.k = 1)$$

$$= x + y + z$$ //

Find the divergent of vector
$\vec{F} = x^2y\hat{i} - (z^3 - 3x)\hat{j} + 4y^2\hat{k}$

~~diverge~~ $\nabla \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(x^2y\,i - (z^3 - 3x)\,j + 4y^2k\right)$

$= \frac{\partial}{\partial x}x^2y - \frac{\partial}{\partial y}(z^3 - 3x) + \frac{\partial}{\partial z}(4y^2)$

$= 2xy + 0 + 0 = 2xy$ //

Curl $(\nabla \times \vec{F})$

$\vec{F} = P.(x, y, z), \ Q(x, y, z), \ R(x, y, z)$
P, Q, R ordinary scalar function.

$\vec{V} = xy\hat{i} + yz\hat{j} + xz\hat{k}$
Find the curl.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz)\right]i - \left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(xy)\right]j + \left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right]k$$

$= (0 - y)i - (z - 0)j + (0 - \cancel{y}x)k$
$= -y\underline{i} - z\underline{j} - x\underline{k} = -(y\underline{i} + z\underline{j} + x\underline{k})$ //

Determine the divergent of a vector field,
$\vec{F} = \langle x^2y, \ x/y, \ xyz \rangle$

$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{\partial}{\partial z}(xyz)$
$= 2xy - \frac{x}{y^2} + xy = 3xy - \frac{x}{y^2}$ //

Determine the curl of a vector field,
$\langle x^2y, -y/x, xyz \rangle$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -y/x & xyz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}\left(\frac{-y}{x}\right)\right] i - \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(x^2y)\right] j + \left[\frac{\partial}{\partial x}\left(\frac{-y}{x}\right) - \frac{\partial}{\partial y}(x^2y)\right] k$$

$$= (xz - 0)i - (yz - 0)j + \left(\frac{y}{x^2} - x^2\right)k$$

$$= xz\underline{i} - yz\underline{j} + \left(\frac{y}{x^2} - x^2\right)\underline{k} \quad //$$

Intersting properties of Diverges and curl.

1. $\nabla \times (\nabla F) = 0 \rightarrow$ The curl of its gradient is the zero vector.

2. $\nabla \times \vec{F} = 0 \rightarrow$ The curl of any conservative vector field is zero.

what is directional derivative ?

ex: find the directional derivation (d.d) of A, at $(1, -2, -1)$ in the direction
$A = x^2yz + 4xz^2$
$2\underline{i} - \underline{j} - 2\underline{k}$.

Direction derivative $= \vec{\nabla} \vec{A}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= i(2xyz + 4z^2) + j(x^2z + 0) + k(x^2y + 8xz)$$

$$= i(2\times1\times(-2)\times(-1) + 4(-1)^2) + j(1\times(-2)) + k(1\times(-2) + 8\times1\times(-1))$$

$$= i(4+4) + j(-1) + k(-10)$$

$$= 8\underline{i} - \underline{j} - 10\underline{k} \quad //$$