MATHEMATICAL METHODS 01

Course name : Mathematical Methods I

Course code : MAT 1201

No. of Credits : 02

Assesmemt strategy

Homework assignments	-	15%
Mid sem. exam	-	25%
End sem. exam	-	60%
(4 Questions), 2hr.		100%

Course Outline

Chapter 1 : Algebra of the vectors
Chapter 2 : Vector applications
Chapter 3 : Calculus of vector value functions
Chapter 4 : Scalar and vector fields

References

Narayan and Mittal P. K (2003).
A textbook of vector analysis
New Delhi : S. Chand and company.

Chapter 1 : Algebra of vectors

Introduction to vectors

* In science, Mathematics, Engineering we distinguish between two types of quantities, namely, scalars and vectors

Scalars අදිශ

* A scalar is a quantity which has a magnitude but does not have a direction.

ex:- Mass, Volume, Density, Temprature, Blood pressure, Length, Voltage etc

Vectors දෛශික

* A vector is a quantity which has both magnitude and direction

Examples:- Displacement, Velocity, Accelaration, Fose, Electric fields, Momentum, Moment
ඝූර්ණය

Geometric Vectors ජ්‍යාමිතික

Directed line segments

* A line segement (a position of the line) having a direction ~~at~~ attached to it, is called a

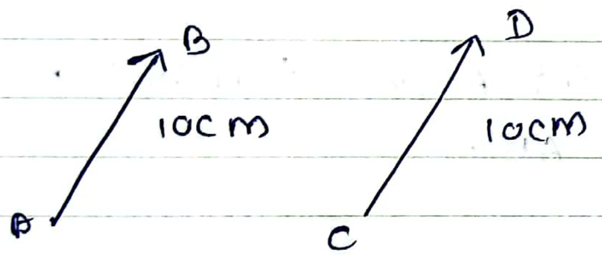
Intial point - ආරම්භක ලක්ෂ්‍යය
terminal point → අවසාන ලක්ෂ්‍යය

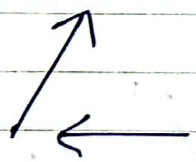
directed line segement

Line segement   Directed line segement

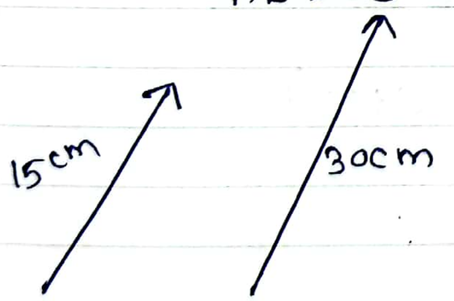
Equivalent line segment

* Two line segment are equivalent if they have the same length and the same length and same direction





AB = CD.



* Non equilavent

* Any two distinct points A and B in space determine directed line segment $\overrightarrow{AB}$ from A to B indicating its length (magnitude) and direction.

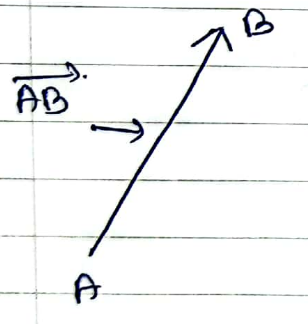
* It is useful to consider $\overrightarrow{AA}$ as a line segment. Geometrically $\overrightarrow{AA}$ is the point A and its length is 0. However, its direction is indeterminate.

note: $\boxed{\overrightarrow{AA} \equiv A}$

* A vector is commonly define as a family of ~~equa~~ equivalent line segments.

* There fore a vector should have a tail called its initial point and a tip called its terminal point.

* A vector with intial point A and terminal point B is denoted by $\overrightarrow{AB}$.



A $\equiv$ Initial point (tail)

B $\equiv$ Terminal point (tip)

Date No

Notation

A vector is denoted by a bold face symbol or a symbol with an arrow over symbol or by a symbol with an arrow over it

For example

$\overrightarrow{AB}$, $\vec{a}$, $\underline{a}$, $\mathbf{a}$

However, in handwriting we shall denote a vector by a letter over a bar.

In our work, we shall write $\vec{a}$ or $\underline{a}$

Magnitude of a vector

The magnitude (norm, modulous) (mod) of a vector $\underline{a} = \overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$ or $||\overrightarrow{AB}||$ or AB

$|\underline{a}| = ||\underline{a}|| = a \equiv$ the magnitude of $\underline{a}$

$|\overrightarrow{AB}| = ||\overrightarrow{AB}|| = AB \equiv$ the magnitude of $\overrightarrow{AB}$

$\boxed{\therefore\ |\underline{a}| \geq 0 \quad \text{or} \quad |\overrightarrow{AB}| \geq 0}$

ProMate

Equal vectors.

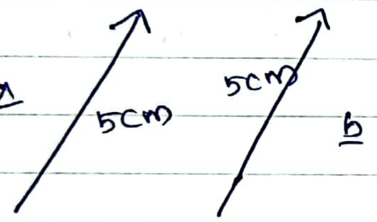
* Two vectors are said to be equal if they have equal magnitude and equal direction

$\underline{a} = \underline{b}$ if and only = iff

= $\Leftrightarrow$

Double implication.

* if and only if $|\underline{a}| = |\underline{b}|$ and $\underline{a}$ and $\underline{b}$ have the same direction



$\underline{a} = \underline{b}$

Zero vector

* A vector with magnitute 0 is called the zero vector

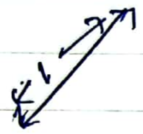
The zero vector is denoted by $\underline{0}$.

* The direction of the zero vector is indeterminate

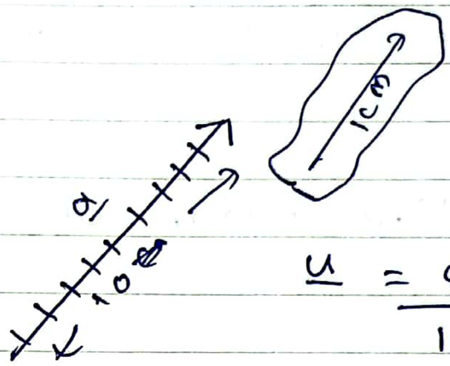
Date No

Unit vector

Any vector of unit magnitude is called the unit vector



$|u| = 1$



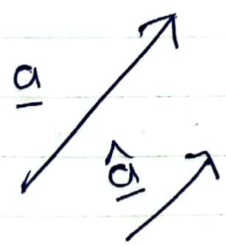
$$\underline{u} = \frac{\underline{a}}{10}$$

$$\hat{v} = \frac{\underline{v}}{|\underline{v}|} \qquad = \frac{\underline{a}}{|\underline{a}|}$$

$$\hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|}$$

Let $\underline{a}$ be a non zero vactor

i.e $|\underline{a}| \neq 0$ or $\underline{a} \neq \underline{0}$



A unit vector whose direction is that of $\underline{a}$ is donated $\hat{\underline{a}}$

$$\boxed{\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}}$$

$$\boxed{\hat{\underline{a}}\,|\underline{a}| = a\hat{\underline{a}} = \underline{a}}$$

ProMate

Date

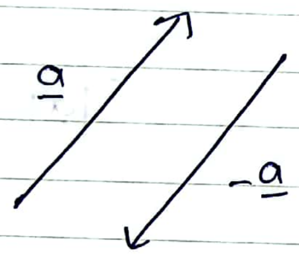
No

Negative vector

* The negative vector of a given vector $\underline{a}$ is a vector whose magnitude is $|\underline{a}|$ and whose direction is opposite to that ~~of~~ of $\underline{a}$

* The negative vector of $\underline{a}$ is denoted by $-\underline{a}$

$3 \rightarrow -3$



Position vector

* When a vector $\overrightarrow{OP}$ is used to specify the position of point P in a space, it is called the position vector of point P ~~for~~ w.r.t origin $\underline{O}$.

$\overrightarrow{OP}$ is called the position vector of P relative to ~~of~~ O

Free vector

* Vectors such as $\underline{a}$ describing quantity not related to fixed position in space are called free vectors.

Date 2023/03/02.

Multiplication of a vector by a scalar

$\underline{v} \xrightarrow{5} 5\underline{v}$ $\qquad |\underline{u}| = |5\underline{v}| = 5|\underline{v}|$

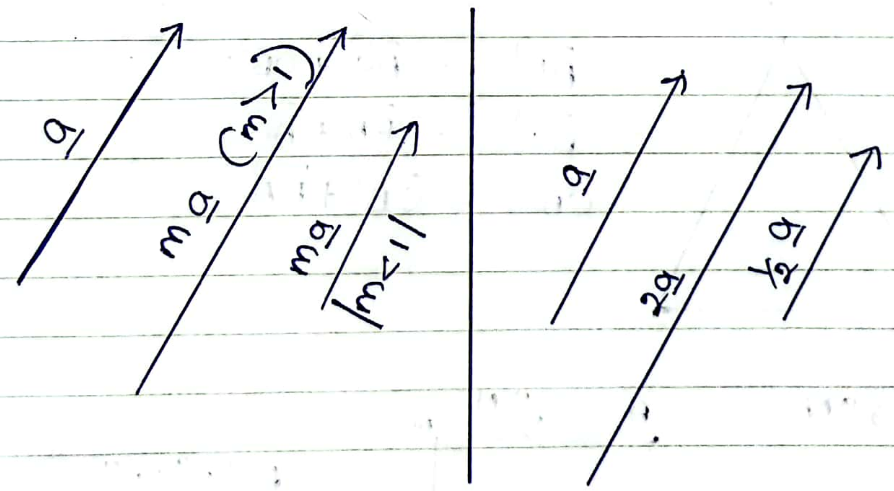
$\underline{v} \xrightarrow{-5} -5\underline{v}$ $\qquad |\underline{w}| = |-5\underline{v}| = -5|\underline{v}|$

$$|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}|$$

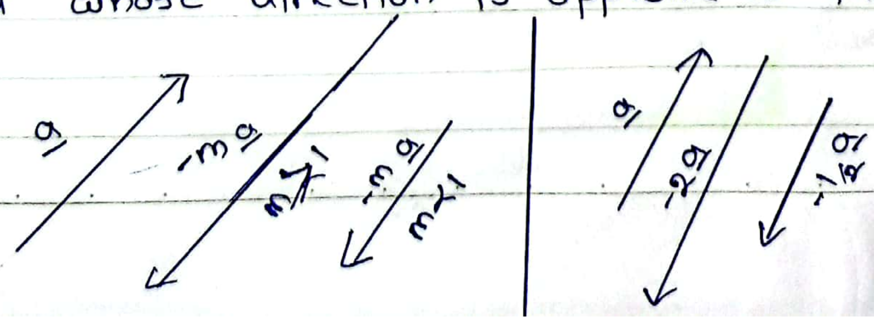
* Let m be a positive number (a positive scaler), and Let $\underline{a}$ be a non zero vector,

Then,

i) $m\underline{a}$ is a vector whose magnitude is $m|\underline{a}|$ and whose direction is that of vector $\underline{a}$.



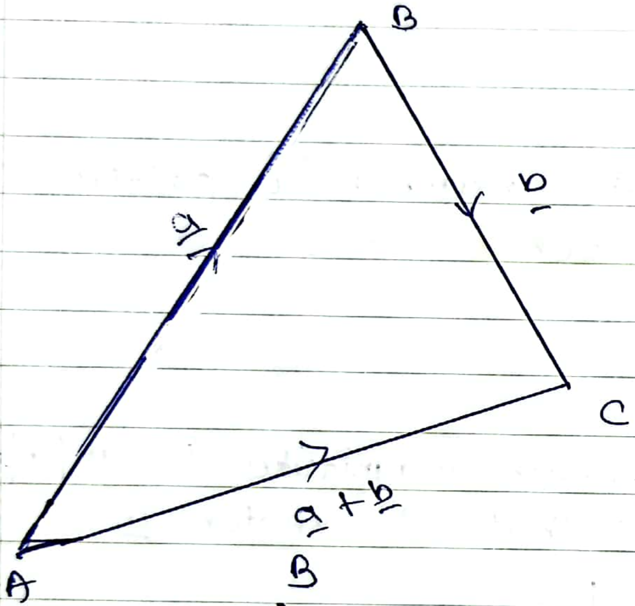
ii) $-m\underline{a}$ is a vector whose magnitude is $m|\underline{a}|$ and whose direction is opposite to that of $\underline{a}$.



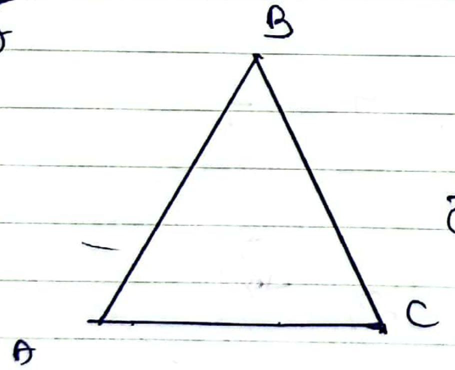
ProMate

Addition vectors

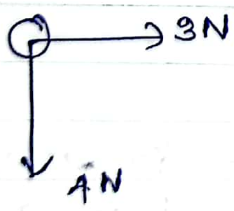
* Let, ABC be a triangle such that ~~AB~~ $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{BC} = \underline{b}$. Then the third side $\overrightarrow{AC}$ represents the vector sum of $\underline{a}$ and $\underline{b}$ which is denoted by $\underline{a}+\underline{b}$.

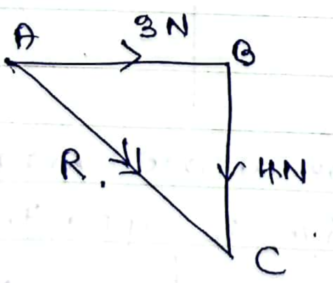


$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

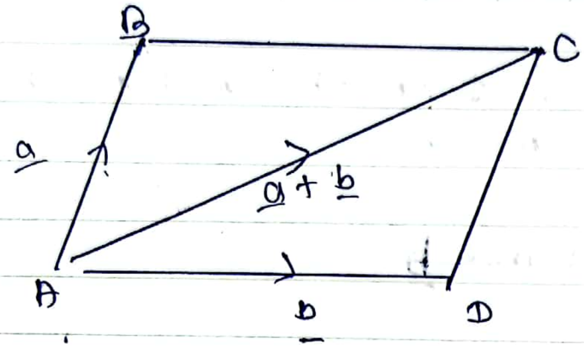


$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$$
$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$$
$$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$$





$$R = \sqrt{3^2+4^2}$$
$$= 5\,N$$



$\overrightarrow{BC} \parallel \overrightarrow{AD}$

$\overrightarrow{BC} = \overrightarrow{AD}.$

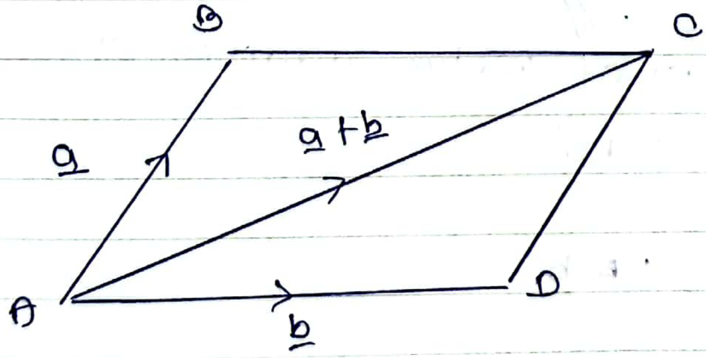
$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$$\boxed{\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}}$$

Parallelogram law

* Let, $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{AD} = \underline{b}$ be non parallel vectors that are ~~side~~ sides of a parallelogram ABDC with main diaganol $\overrightarrow{AC}$.

Then, the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. is given by $\overrightarrow{AD}$



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$

$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{CD}.$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{BC}.$

$\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}.$

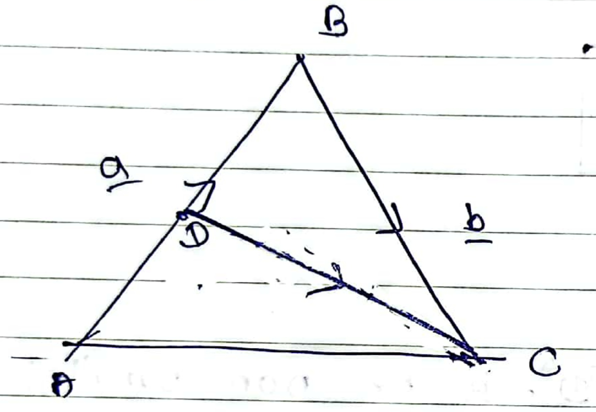
Date No

Example 1

In the triangle ABC, D is the mid point of AB. Let, $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{BC} = \underline{b}$

~~Write in terms of $\underline{a}$ and $\underline{b}$~~

write the vectors $\overrightarrow{CA}$ and $\overrightarrow{DC}$ in terms of $\underline{a}$ and $\underline{b}$



$$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$$

$$= -\underline{b} - \underline{a}$$

$$= -(\underline{b} + \underline{a}) \text{ //}$$

Using the triangle law of vector addition of Δ DBC, we have.

$$\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB} + \overrightarrow{BC}$$

$$= \frac{1}{2}(\underline{a}) + \underline{b}$$

$$= \frac{\underline{a}}{2} + \underline{b}$$

$$= \frac{\underline{a} + 2\underline{b}}{2} \text{ //}$$

ProMate

$\vec{AB} = \underline{a}$

$\vec{BC} = \underline{b}$

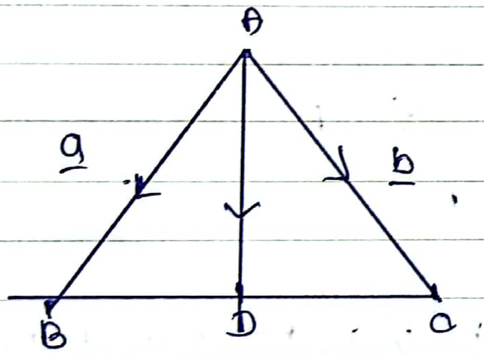
$\boxed{\underline{a} + \underline{b} = \underline{b} + \underline{a}}$ (True)

$2+3 = 3+2$

(i) Using the triangle law of vector addition of $\triangle ABC$ we have,

Hw

Show that $\vec{AB} + \vec{AC} = 2\vec{AD}$, where ABC is triangle D is the mid point of BC



~~AB~~

Let see $\vec{AB} = \underline{a}$, $\vec{AC}$ $\underline{b}$

Using the triangle law of vector addition of $ABD \triangle$

$\cancel{\vec{AB}}$ $\vec{AD} = \vec{AB} + \vec{BD}$ $\quad \vec{BC} = \vec{BA} + \vec{AC}$

$= -\underline{a} + \underline{b}$

$= \underline{a} + \frac{1}{2}(\underline{b} - \underline{a})$ — (1) $= \underline{b} - \underline{a}$

$ACD \triangle$

$\vec{AD} = \vec{AC} + \vec{CD}$

$= \underline{b} + \frac{1}{2}\vec{CB}$

$= \underline{b} + \frac{1}{2}(-\underline{b} + \underline{a})$ — (2)

① + ②

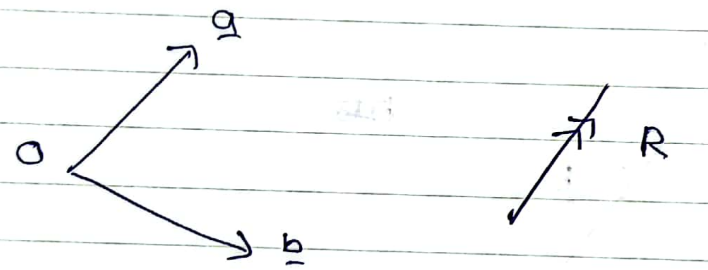
$$2\vec{AD} = \underline{a} + \underline{b} + \frac{1}{2}(-b + \underline{a} + \underline{b} - \underline{a})$$

$$2\vec{AD} = \underline{a} + \underline{b}$$

$$2\vec{AD} = \vec{AB} + \vec{AC}.$$

(2D)

Vectors in two dimensional space (2-space)

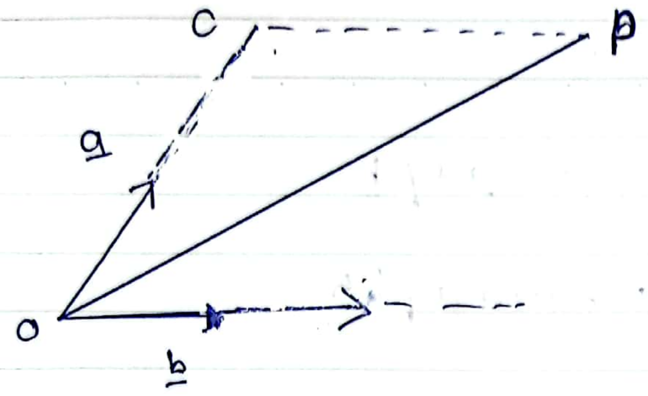


* Let $\underline{a}$ ($\underline{a} \neq 0$) and $\underline{b}$ ($\neq 0$) be non parallel vectors o be a fixed point

* Then there is exactly one plane containing O and the vectors $\underline{a}$ and $\underline{b}$

* Let P be any point, in this plane. Let C be a point ~~that~~ such that $\vec{OC}$ is parallel to $\underline{a}$

*



* $\overrightarrow{OC} \parallel \underline{a}$
$\overrightarrow{OC}$ is parallel to $\underline{a}$

* $\overrightarrow{CP} \parallel \underline{b}$
$\overrightarrow{CP}$ is parallel to $\underline{b}$

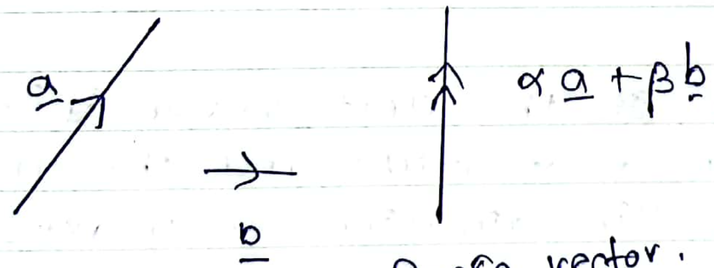
~~Then oc is~~

Let, $\overrightarrow{OC} = \alpha \underline{a}$
$\overrightarrow{CP} = \beta \underline{b}$

Using Δ law to vector addition,

$$\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$$

$$\boxed{\overrightarrow{OP} = \alpha \underline{a} + \beta \underline{b}}$$

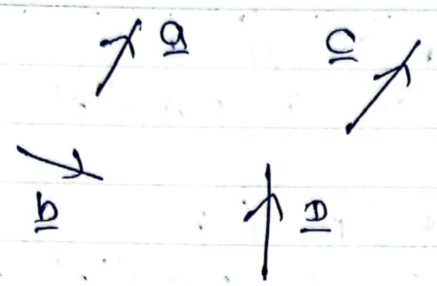


Linear combination of a and b

Basic vector.

* The vectors $\underline{a}$ and $\underline{b}$ are called base or basis vectors for the plane formed by $\underline{a}$ and $\underline{b}$

* $\alpha \underline{a} + \beta \underline{b}$ is called a linear combination of $\underline{a}$ and $\underline{b}$. It follows that any vector parallel to $\overrightarrow{OP}$ can be written as a ~~line~~ unique (exactly one) linear combination of $\underline{a}$ and $\underline{b}$



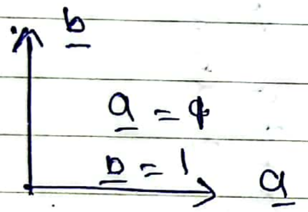
$\underline{c} = \alpha \underline{a} + \beta \underline{b}$

$\underline{D} = \alpha' \underline{a} + \beta' \underline{b}$

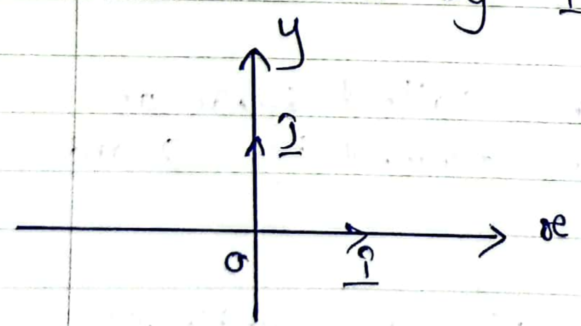
Cartesian Components

*Calculation are greatly simplified when the base vectors are perpendicular and unique vectors

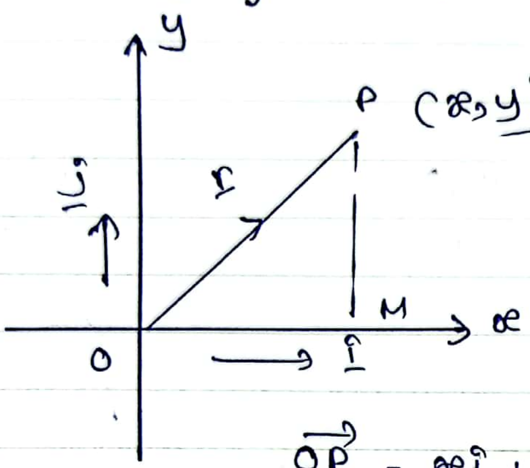
i.e $\underline{a} \perp \underline{b}$ and $|\underline{a}| = |\underline{b}| = 1$

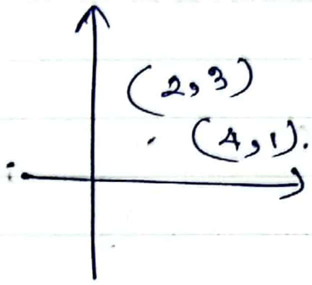


* Let ~~us define~~ the unit vetor the positive direction of x axis and the unit vector is the positive direction of y axis be denoted by $\underline{i}$ and $\underline{j}$, respective.



(i) Let P be any point in this plane (cartesian plane) (x y - plane) with ordinates (x, y)





$\vec{OP} = x\underline{i} + y\underline{j}$

$\vec{OM} = OM\,\underline{i}$

$= x\underline{i}$

$\vec{MP} = MP\,\underline{j}$

$= y\underline{j}$

$\vec{OP} = \vec{OM} + \vec{MP}$

$$\boxed{\vec{OP} = x\underline{i} + y\underline{j}}$$

2023 . 04. 20 .

Let P be any point in this plane with coordinates (x, y). The we can write the position vector point P by,

Also, $|\vec{OP}| = |\underline{r}| = r = \sqrt{x^2+y^2}$

$$\boxed{r = \sqrt{x^2+y^2}}$$

* Alternatively the vector $x\hat{i} + y\underline{j}$ can also be denoted by $\begin{pmatrix} x \\ y \end{pmatrix}$ which is called a column vector or column matrix with ~~the~~ this notation we can write

$\hat{i} = 1\underline{\hat{i}} + 0\underline{j}$

ProMate

Date
No

$\hat{i} = 1\underline{i} + 0\underline{j}$ $\begin{pmatrix}1\\0\end{pmatrix}$

$x\underline{i} + y\underline{j} = \begin{pmatrix}x\\y\end{pmatrix}$

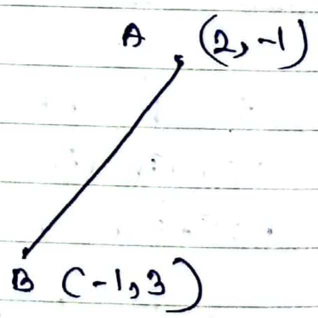
$\underline{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\underline{j} = \begin{pmatrix}0\\1\end{pmatrix}$

Somtimes

$x\underline{i} + y\underline{j} = (x, y)$ ← Raw matrix

$= \begin{pmatrix}x\\y\end{pmatrix}$

ex:- Find $\overrightarrow{AB}$ where A is the point $(2,-1)$ and B is the point $(-1,3)$



$\cancel{\overrightarrow{AB}} =$

$\overrightarrow{OA} = 2\underline{i} - \underline{j}$

$\overrightarrow{OB} = -1\underline{i} + 3\underline{j}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

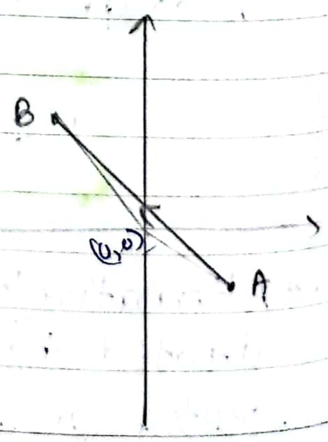
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= -2\underline{i} + \underline{j} + 3\underline{j} - 1\underline{i}$

$= -3\underline{i} + 4\underline{j}$



ProMate

Date No

** Vectors in three dimensional space (3 space).

- Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be any three non parallel and non-coplanar (there are not in same plane).

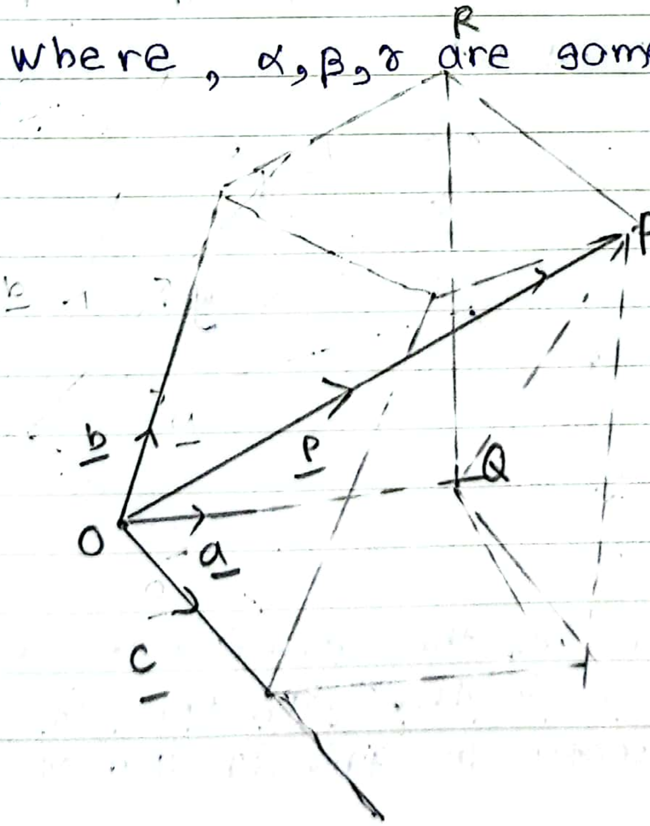
- ~~Let~~ If O is a fixed point in space and if P is any other point in space

- Then we can draw the closed polygon OQPR such that $\overrightarrow{OQ}$ is parallel to $\underline{a}$, $\overrightarrow{QR}$ is parallel to $\underline{b}$, $\overrightarrow{RP}$ is parallel to $\underline{c}$

That is,

$$\overrightarrow{OQ} = \alpha \underline{a} \qquad \overrightarrow{OQ} \parallel \underline{a}$$
$$\overrightarrow{QR} = \beta \underline{b} \qquad \overrightarrow{QR} \parallel \underline{b}$$
$$\overrightarrow{RP} = \gamma \underline{c} \qquad \overrightarrow{RP} \parallel \underline{c}$$

where, α, β, γ are some scalars.



$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$$
$$= \overrightarrow{OQ} + \overrightarrow{QR} + \overrightarrow{RP}$$
$$= \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c}$$

ProMate

Date No

Thus, the position vector of any point in space can uniquely (exactly one) be expressed in terms of $\underline{a}$ and $\underline{b}$ and $\underline{c}$.

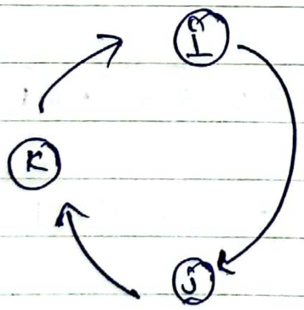
* $\alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}$ is called a linear combination of $\underline{a}$, $\underline{b}$ and $\underline{c}$

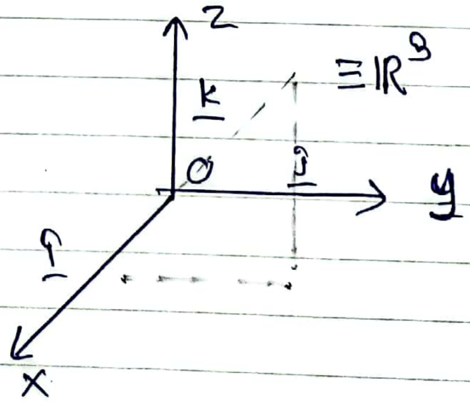
* $\underline{a}$, $\underline{b}$ and $\underline{c}$ are called base vectors.

Cartesian components:

* Starting with three mutually perpendicular directions leads to cartesian frame of reference. This consist of a fixed point O, the origin three mutually perpendicular axes OX, OY and OZ

* The axes are placed in such a way that they form a right handed set





Let

$\underline{i}$: the unit vector in the direction of OX
$\underline{j}$: the unit vector in the direction of OY
$\underline{k}$: the unit vector in the direction of OZ

Date No

* For a general point $P(x, y, z)$ the position vector $\vec{OP}$ is given by $\vec{OP} = x\underline{i} + y\underline{j} + z\underline{k}$

Let, $\underline{r} = \vec{OP}$

$$\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\underline{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, y, z).$$

$$\hat{\underline{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{\underline{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{OP} = r = \sqrt{x^2 + y^2 + k^2}}$$

Example 4.

A triangle ABC has its vertices at the poin $A(2, -1, 4)$, $B(3, -2, 5)$, $C(-1, 6, 2)$ find ~~ABC~~ $\vec{AB}$, $\vec{BC}$, $\vec{CA}$ and then find lengths of the sides of triangle.

$\vec{OA} = 2\hat{\underline{i}} - 1\hat{\underline{j}} + 4\underline{k}$ $\quad \vec{OB} = 3\hat{\underline{i}} - 2\hat{\underline{j}} + 5\underline{k}$ $\quad \vec{OC} = -1\hat{\underline{i}} + 6\hat{\underline{j}} + 2\underline{k}$

$$\vec{AB} = \vec{AO} + \vec{OB}$$
$$= -2\hat{\underline{i}} + 1\hat{\underline{j}} - 4\underline{k} + 3\hat{\underline{i}} - 2\hat{\underline{j}} + 5\underline{k}$$
$$= \hat{\underline{i}} - \hat{\underline{j}} + \underline{k}$$

129

1

Length of $AB = \sqrt{1^2 + (-1)^2 + 1^2}$
$= \sqrt{3}$ //

16
64
49
129

$$\vec{BC} = \vec{BO} + \vec{OC}$$
$$= -3\hat{\underline{i}} + 2\hat{\underline{j}} - 5\underline{k} - 1\hat{\underline{i}} + 6\hat{\underline{j}} + 2\underline{k}$$
$$= -4\hat{\underline{i}} + 8\hat{\underline{j}} - 3\underline{k}$$ //

Length of BC
$= \sqrt{16 + 64 + 49}$
$\sqrt{129}$ //

$\vec{CA} = \vec{CO} + \vec{OA}$

$= 1\hat{i} - 6\hat{j} - 2\hat{k} + 2\hat{i} - 1\hat{j} + 4\hat{k}$

$= 3\hat{i} - 7\hat{j} + 2\hat{k}$

Length of CA $= \sqrt{9 + 49 + 4}$

$2 \underline{|62}$
31

$= \sqrt{62}$ //

H.W

~~A ABC~~

Δ ABC has its vertices at the points A(−1, 3, 0) B(−3, 0, 7) C(−1, 2, 3) find the vectors ~~ABC~~ $\vec{AB}$, $\vec{AC}$, $\vec{CB}$. length of the sides of Δ

$\vec{OA} = -1\hat{i} + 3\hat{j} + 0\hat{k}$

$\vec{OB} = -3\hat{i} + 0\hat{j} + 7\hat{k}$

$\vec{OC} = -1\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{AB} = \vec{AO} + \vec{OB}$

$= \hat{i} - 3\hat{j} - 0\hat{k} - 3\hat{i} + 0\hat{j} + 7\hat{k}$

$= -2\hat{i} - 3\hat{j} + 7\hat{k}$

$\vec{AC} = \vec{AO} + \vec{OC}$

$= \cancel{\hat{i}} - 3\hat{j} - 0\hat{k} - \cancel{\hat{i}} + 2\hat{j} + 3\hat{k}$

$= -\hat{j} + 3\hat{k}$

$\vec{CB} = \vec{CO} + \vec{OB}$

$= \hat{i} - 2\hat{j} - 3\hat{k} + -3\hat{i} + 0\hat{j} + 7\hat{k}$

$= -2\hat{i} - 2\hat{j} + 4\hat{k}$

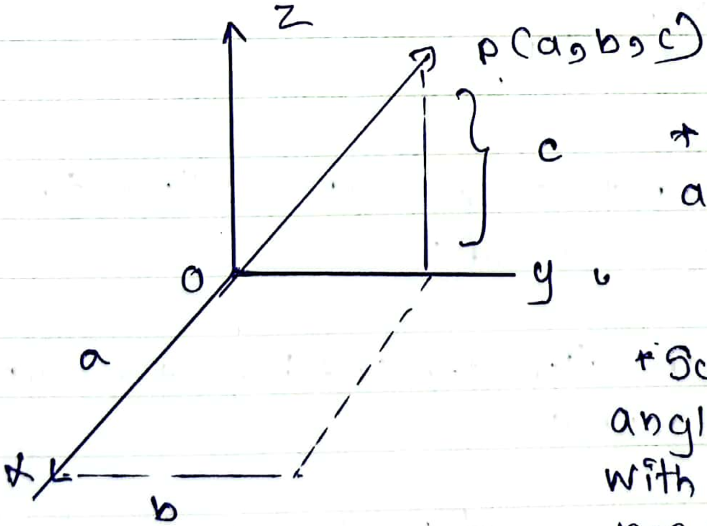
$$|\vec{AB}| = \sqrt{4+9+49}$$
$$= \sqrt{62}$$

$$|\vec{AC}| = \sqrt{1+9}$$
$$= \sqrt{10}$$

$$|\vec{CB}| = \sqrt{4+4+16}$$
$$= \sqrt{24} = 2\sqrt{6}$$

~~Direction~~ Direction Ratios and Direction Cosines of a vector.

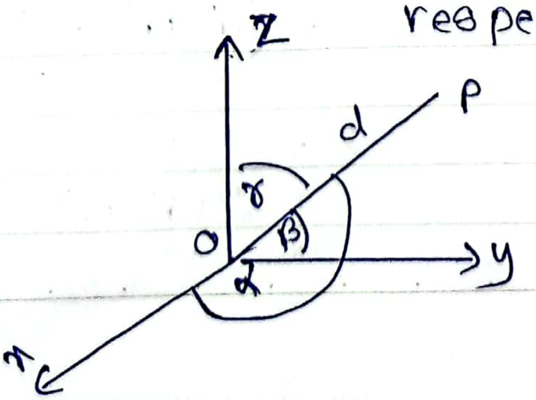
* Let P be the point (a, b, c) consider the position vector of $\vec{OP}$. The coordinates of P determines the direction of OP relatives to the ~~axes~~ axis

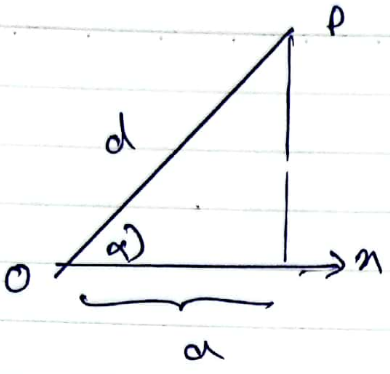


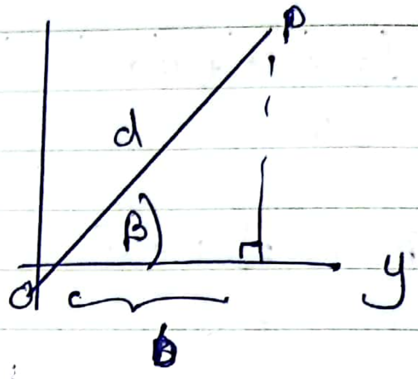
* The ratios a : b : c are called the direction ratios of $\vec{OP}$

* Suppose OP makes angles α, β and γ with OX, OY and OZ respectively

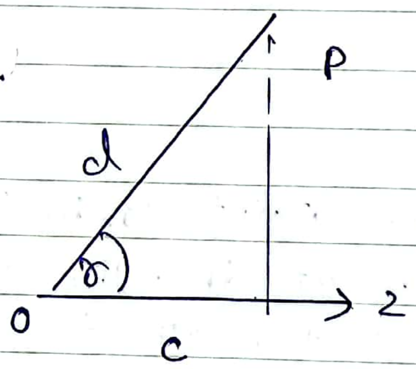
* Let OP = d







$\cos\alpha = \frac{a}{d}$ $\cos\beta = \frac{b}{d}.$



$\cos\gamma = \frac{c}{d}.$

$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of $\vec{OP}$.

* Direction cosines usually denoted by the symbols $l, m,$ and n. so that

$$\boxed{\begin{aligned} l &= \cos\alpha \\ m &= \cos\beta \\ n &= \cos\gamma \end{aligned}}$$

$$d = \sqrt{a^2+b^2+c^2}$$

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

Date No

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} \qquad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$l^2 + m^2 + n^2$$

$$= \frac{a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2}$$

$$= \frac{a^2+b^2+c^2}{a^2+b^2+c^2} = 1$$

$$\boxed{l^2 + m^2 + n^2 = 1}.$$

Example 6

* The direction cosines of the vector ~~at~~ $\overrightarrow{OP}$ where P is the point $(2, -3, 6)$

$\cancel{\overrightarrow{OP} =}$ $\cancel{\cos\alpha}$

$$d = \sqrt{4 + 9 + 36}$$ direction ratios.

$2 : -3 : 6$

$$= \sqrt{49} = 7$$

$$l = \cos\alpha = \frac{a}{d} = \frac{2}{7}$$

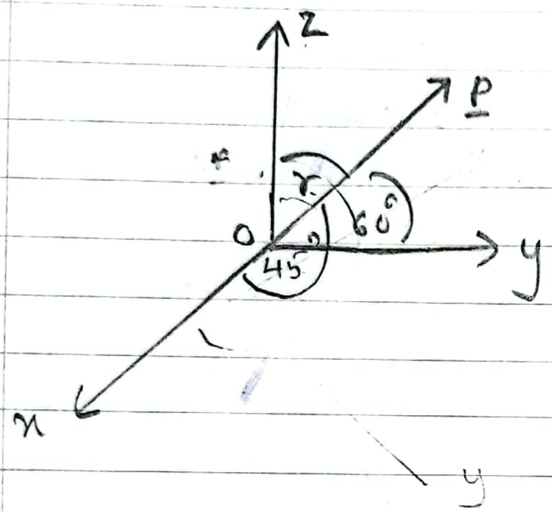
$$m = \cos\beta = \frac{-3}{7}$$

$$n = \cos\gamma = \frac{6}{7}$$

Date
No

Example 7

A vector $\underline{r}$ is ~~impling~~ inclining to ox at 45° degrees and to oy 60° ~~&~~, find its inclination to oz



$|\underline{r}| = 12$ units
express the $a\hat{i}$, $b\hat{j}$ & $c\hat{k}$

~~$\vec{OX}$~~ $= \cos\alpha = \dfrac{a}{d}$

Let r be inclined to oz at p;

The direction cosines of $\underline{r}$ are $\cos 45^\circ, \cos 60^\circ, \cos\gamma$

~~$\cos 45^\circ =$~~

$l = \cos 45^\circ$ $\quad m = \cos 60^\circ$ $\quad n = \cos\gamma$

$= \dfrac{1}{\sqrt{2}}$ $\quad m = \dfrac{1}{2}$ $\quad n = \cos\gamma$

$l^2 + m^2 + n^2 = 1$ $\quad \cos\gamma = \dfrac{1}{2}$

$\dfrac{1}{2} + \dfrac{1}{4} + n^2 = 1$

$n^2 = 1 - \dfrac{3}{4}$ $\quad n = \pm\dfrac{1}{2}$ $\quad r = \cos^{-1}\left(\dfrac{1}{2}\right)$

$= \dfrac{1}{4}$ $\quad \gamma = 60^\circ$ or (120°)

ProMate

Date No

$l = \frac{a}{d}$, $m = \frac{b}{d}$,

$\frac{1}{\sqrt{2}} = \frac{a}{12}$ $\frac{1}{2} = \frac{b}{12}$

$b = \frac{12}{2}$ //

$a = \frac{12}{\sqrt{2}}$ //

$a : b : c$

$c = \frac{n}{d}$

$\frac{12\sqrt{2}}{2} : \frac{12}{2} : \frac{12}{2}$

$\frac{c}{12} = \frac{1}{2}$

$\sqrt{2} : 1 : 1$

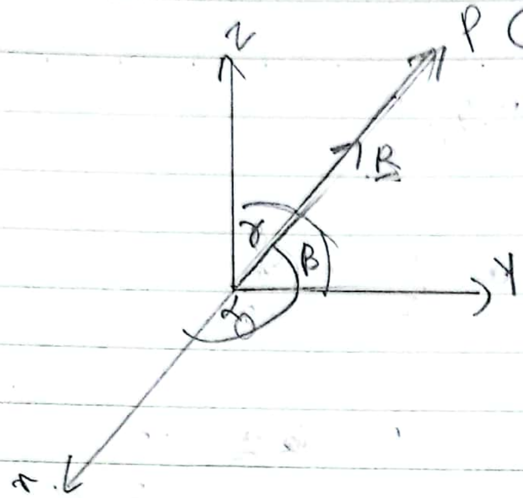
$c = \frac{12}{2}$ //

$P = 6\sqrt{2}\,\hat{i} + 6\,\hat{j} + 6\underline{k}$ //

H.W

* Find the co ordinates of P if OP is of length 5 units and is in the direction of $\overrightarrow{OR}$ where $R \equiv (2, -1, 4)$.

ProMate



$P(a_1, b_1, c_1)$

$|\overrightarrow{OP}| = 5$

$\overrightarrow{OR} = 2\underline{i} - \underline{j} + 4\underline{k}$

$|\overrightarrow{OR}| = \sqrt{2^2 + (-1)^2 + 4^2}$

$= \sqrt{4 + 1 + 16}$

$= \sqrt{21}$

$\cos\alpha = \frac{a}{d} = \frac{2}{\sqrt{21}}$

~~or~~ ⑥

$\cos\beta = \frac{b}{d} = \frac{-1}{\sqrt{21}}$

$l^2 + m^2 + n^2 = 1$

$\cos\gamma = \frac{c}{d} = \frac{4}{\sqrt{21}}$

$\cos\alpha = \frac{a_1}{5} = \frac{2}{\sqrt{21}}$

$\cos\beta = \frac{-1}{\sqrt{21}} = \frac{b_1}{5}$

$a_1 = \frac{10}{\sqrt{21}}$

$b_1 = \frac{-5}{\sqrt{21}}$

$\cos\gamma = \frac{c_1}{5} = \frac{4}{\sqrt{21}}$

$p = \left(\frac{10}{\sqrt{21}}\underline{i} - \frac{5}{\sqrt{21}}\underline{j} + \frac{20}{\sqrt{21}}\underline{k}\right)$

$c_1 = \frac{20}{\sqrt{21}}$

$\frac{4}{21} + \frac{1}{21} + \frac{16}{21}$

$\frac{21}{21} = 1$

Date 2023.04.27.

Linear Combination of vectors

* Let $\underline{v_1}, \underline{v_2}, \ldots, v_n$ be n vectors and let $c_1, c_2, \ldots, c_n$ be n scalars. The expression $c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_n\underline{v_n}$ is called a linear combination of vectors.

$3\underline{i} + 2\underline{j} + 3\underline{k}$.

$\Rightarrow \underline{i}, \underline{j}, \underline{k}$ linear combination.

$2\underline{i} + 5\underline{j}$

$\underline{i}, \underline{j}$ linear combination

* A vector $\underline{v}$ is called a linear combination of the vectors v_1, v_2 ~~and~~ $\ldots v_n$ if it can be express as a linear combination of the vectors ~~of~~ $\underline{v_1}, \underline{v_2}, \ldots, v_n$.

That is a there exist some n scalars such that your $\underline{v}$ is equal to $c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_n\underline{v_n}$

$$\underline{v} = c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_n\underline{v_n} = \sum_{k=1}^{n} c_k \underline{v_k}$$

Example :-

(i) $\underline{a} = 3\underline{i} + 2\underline{j} + 4\underline{k}$ is a linear combination of $\underline{i}, \underline{j}, \underline{k}$

(ii) $\underline{v} = 2\underline{a} + \underline{b} - 3\underline{c}$ is a linear combination of $\underline{a}, \underline{b}, \underline{c}$

ProMate

(iii) $\underline{w} = x\underline{i} + b\underline{j} + 2\underline{k}$ is a linear combination of $\underline{i}, \underline{j}, \underline{k}$

Spanning sets.

Let $\{ \underline{v_1}, \underline{v_2}, \ldots, \underline{v_n} \}$ be a set of n vectors, The set of all linear combinations.

* The set of all linear combination of the ~~vectors~~ in vectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ is called span by the vectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ and is denoted by span $\{v_1, v_2, v_3, \ldots, v_n\}$.

That is span of $\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \ldots, v_n\}$ =

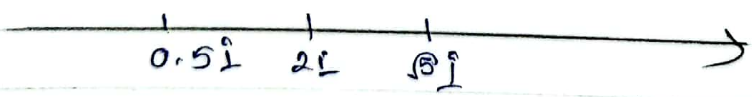
$\{c_1 v_1, c_2 v_2, c_3 v_3 + \ldots + c_n v_n\}$ are scalars.

example:

(1) Spanning set of $\underline{i}$

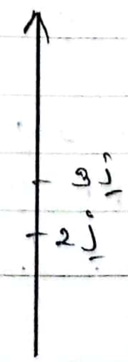
span$\{i\} = \{x\underline{i} \mid x$ is a scalar$\}$

; x-axis.



(2) span$\{\underline{j}\} = \{y\underline{j} \mid y$ is a scalar$\}$.

; y axis.

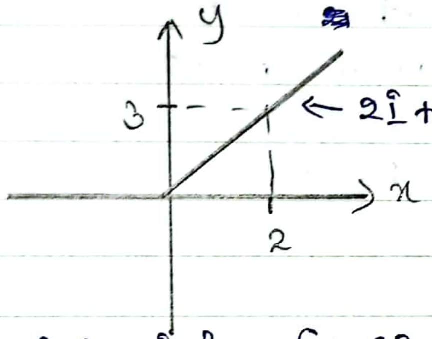


span$\{\underline{k}\}$ = $\{z\underline{k} \mid z$ is a scalar$\}$
; z axis.

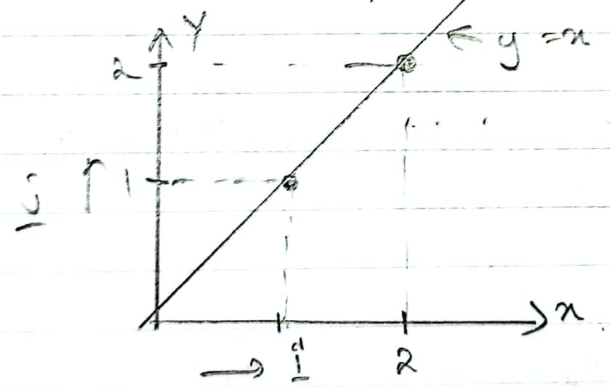
span$\{\underline{0}\}$ = $\{c\,\underline{0} \mid c$ is a scalar$\}$.

= $\{\underline{0}\}$.

span$\{\underline{i}, \underline{j}\}$ = $\{x\underline{i}+y\underline{j} \mid x, y$ is a scalar$\}$ $2\underline{i}+3\underline{j}$



span$\{\underline{i}+\underline{j}\}$ = $\{c(\underline{i}+\underline{j}) \mid c$ is a scalar$\}$.



$c(\underline{i}+\underline{j})$

$c\underline{i}+c\underline{j}$

span$\{\underline{i}, \underline{j}, \underline{k}\}$ = $\{x\underline{i}+y\underline{j}+z\underline{k} \mid x, y, z$ are scalar$\}$.

= xyz space.

Date

Linear Independence.

* Let $\underline{v_1}, \underline{v_2} \ldots \underline{v_n}$ be a set of n vectors If the only solution to the vector equation

* $x_1\underline{v_1} + x_2\underline{v_2} + \cdots + x_n\underline{v_n} = 0$ is a ~~preemfer~~ trivial solution, namely $x_1 = 0, x_2 = 0 \ldots x_n = 0$

The set is linearly independent or the vectors $\underline{v_1}, \underline{v_2}, \underline{v_3} \ldots \underline{v_n}$ are linearly independent

	x	y	$x+y$
Non trivial solution	1	-1	0
	-1	1	0
	x	$-x$	0
	2	-2	0
Trivial Solution →	0	0	0

$x+y=0$

There are infinity solutions, has this equation

Also,

$x+y+z=0.$

$x_1 + x_2 + x_3 + x_4 \cdots + x_n = 0$

$$\left.\begin{array}{ll} x_1 = 0 & x_3 = 0 \\ x_2 = 0 & x_4 = 0 \end{array}\right\} \text{trivial}$$

* If there are non-trivial solutions the set is linearly dependent or the vector $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ are linearly dependent

ProMate

CamScanner

non trivial - dependent
trivial - independent

Only has trivial solution. $x\underline{i} = 0$
$\neq$ nontrivial $\to$ $2\underline{i} = 0$
$\to 0\underline{I} = 0$

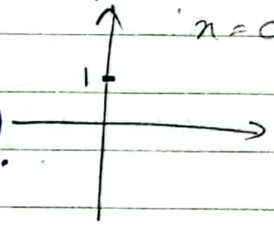
Example 10

(i) $V = \{\underline{i}\}$ is linearly independent
To see this, set $x\underline{i} = 0$ if $x = 0$ (since $\underline{i} \neq 0$).
$\therefore$ $\{\underline{i}\}$ is linearly independent

$x\underline{i} + y\underline{0} = 0$

(ii) $V = \{\underline{i}, 0\}$ is linearly dependent
To see this,
$x_1\underline{i} + x_2\underline{0} = 0$
$x_1\underline{i} = 0$ (for any x_2).
$x_1 = 0$

$x = 0$.



This equation has non trivial solution
For example. $x_1 = 0$ and $x_2 = 1$.

NOTE

Any set of vectors including the zero vector $\underline{0}$ is linearly dependent.
$\{\underline{V}_1, \underline{V}_2 \ldots \underline{0} \ldots \underline{V}_4, \underline{V}_5 \ldots \underline{V}_n\}$

Example

Show that the vectors
$\underline{V}_1 = 2\underline{i} - \underline{j}$
$\underline{V}_2 = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{V}_3 = 7\underline{i} - \underline{j} + 5\underline{k}$ are linearly independent.

Solution.

Date

(i) To show that vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly independent, we must show that the vector equation

$$x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3 = \underline{0}$$ has only the trivial solution

Setting, $x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3$ equal to zero gives,

$$x_1(2\underline{i} - \underline{j}) + x_2(\underline{i} + 2\underline{j} + 3\underline{k}) + x_3(7\underline{i} - \underline{j} + 5\underline{k}) = \underline{0}$$

$\underline{i}, \underline{j}, \underline{k}$ are linearly independent

$$(2x_1 + x_2 + 7x_3)\underline{i} + (-x_1 + 2x_2 - x_3)\underline{j} + (0x_1 + 3x_2 + 5x_3)\underline{k} = \underline{0}$$

Since,

$\underline{i}, \underline{j}, \underline{k}$ are linearly independent we must have

$$\left[\begin{aligned} 2x_1 + x_2 + 7x_3 &= 0 \quad \text{—(1)} \\ -x_1 + 2x_2 - x_3 &= 0 \quad \text{—(2)} \\ 3x_2 + 5x_3 &= 0 \quad \text{—(3)} \end{aligned}\right]$$

Linear system

(1) + (2) × 2

$$\cancel{2x_1} + x_2 + 7x_3 - \cancel{2x_1} + 4x_2 - 2x_3 = 0$$

$$5x_2 + 5x_3 = 0$$

$$x_2 = -x_3.$$

$$-3x_3 + 5x_3 = 0$$

$$x_3 = 0$$

$$x_2 = 0$$

$$2x_1 = -7x_3 - x_2$$

$$x_1 = 0$$

ProMate

CamScanner

Date No

Then, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

it means that the above linear system has only the trivial solution. $x_1 = 0, x_2 = 0, x_3 = 0$

* Therefore, the vectors $\underline{v_1}, \underline{v_2}, \underline{v_3}$ are linearly independent

Example

Proof that the vetors $v_1 = 2\underline{i} + \underline{j} + \underline{k}$, $v_2 = \underline{i} + 2\underline{j} + 3\underline{k}$, $v_3 = 7\underline{i} - 4\underline{j} - 7\underline{k}$ are linearly independent

$$x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = 0$$

$$x_1(2\underline{i} + \underline{j} + \underline{k}) + x_2(\underline{i} + 2\underline{j} + 3\underline{k}) + x_3(7\underline{i} - 4\underline{j} - 7\underline{k}) = 0$$

$$(2x_1 + x_2 + 7x_3)\underline{i} + (x_1 + 2x_2 - 4x_3)\underline{j} + (x_1 + 3x_2 - 7x_3)\underline{k} = 0$$

Since $\underline{i}, \underline{j}, \underline{k}$ are linearly independent we must have,

$$2x_1 + x_2 + 7x_3 = 0 \quad — ①$$

$$x_1 + 2x_2 - 4x_3 = 0 \quad — ③$$

$$x_1 + 3x_2 - 7x_3 = 0 \quad — ②$$

① + ②

$$3x_1 + 4x_2 = 0$$

$$x_1 = -\frac{4}{3}x_2$$

② - ③

$$\cancel{x_1} + 3x_2 - 7x_3 - \cancel{x_1} - 2x_2 + 4x_3 = 0$$

$$x_2 - 3x_3 = 0$$

$$x_2 = 3x_3$$

$$2\left(-\frac{4}{3}x_2\right) + x_2 + 7\left(\frac{x_2}{3}\right) = 0$$

$$-\frac{8}{3}x_2 + \frac{7}{3}x_2 + x_2 = 0$$

$$\frac{2}{3}x_2 = 0$$

$$x_2 = 0$$

$$x_1 = 0$$

$$x_3 = 0.$$

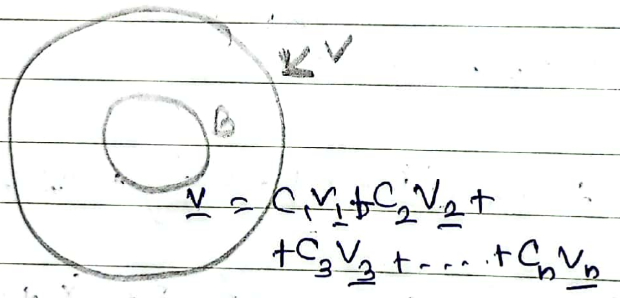
ProMate

Then $x_1 = 0, x_2 = 0, x_3 = 0$
it means that the above linear system has only the trivial solutions

2023.05.04

Basis.

+ A set $B = \{ \underline{v_1}, \underline{v_2}, \cdots, \underline{v_n} \}$ is a 'basis of a space V including B if B spans V and B is linearly independent;



(i) B spans V

That is any vector $\underline{v} \in V$ can be written as linear combination of
$\underline{v_1}, \underline{v_2}, \underline{v_3}, \cdots, \underline{v_n}$

$V = \text{span}\{B\} = \text{span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \cdots, \underline{v_n}\}$

(ii) $\underline{v_1}, \underline{v_2}, \cdots, \underline{v_n}$ are linearly indendent

ie $c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_n\underline{v_n} = 0.$

has only the trivial sol$^{\underline{n}}$;

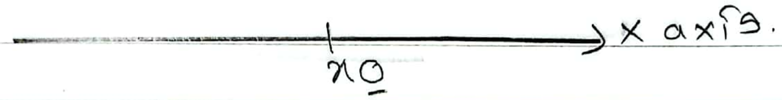
$c_1 = 0,\ c_2 = 0,\ \ldots,\ c_n = 0$

$v_1 = 1 \times \underline{v_1} + 0\,\underline{v_2} + \cdots + 0\,\underline{v_n}$

$v_2 = 0\,v_1 + 1 \times \underline{v_2} + 0\,\underline{v_3} + \cdots + 0\,\underline{v_n}$

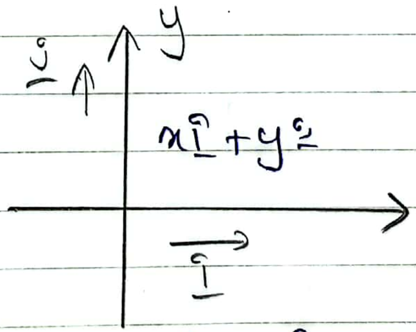
Ex:-

i. $B_1 = \{\underline{e}\}$ is a basis of $\mathbb{R}$ - 1 - space

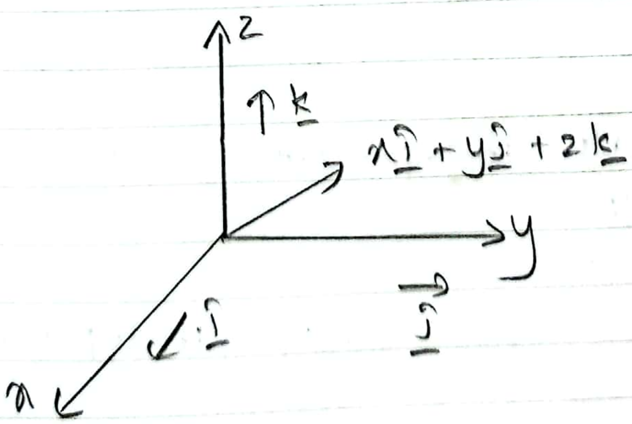


(ii) $B_2 = \{\underline{i}, \underline{j}\}$ is a basis $\mathbb{R}^2$ - 2 - space

$\mathbb{R}^2 = \text{span}\{\underline{i}, \underline{j}\}$



(iii) $B_3 = \{\underline{i}, \underline{j}, \underline{k}\}$ is a basis of $\mathbb{R}^3$ - 3 - space



Date

No

Note:

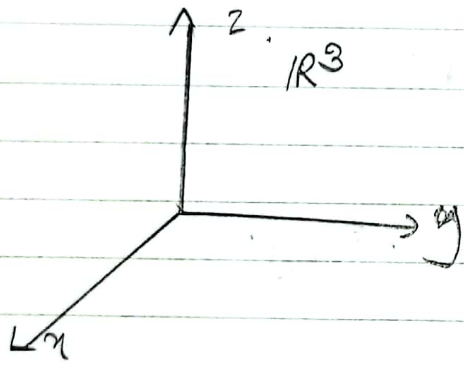
$\{\underline{i}\}$, $\{\underline{i}, \underline{j}\}$ and $\{\underline{i}, \underline{j}, \underline{k}\}$ are called the standard Bases for $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ respectively

ex:-

Let $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and

$\underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ show that $B = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$

is a basis of $\mathbb{R}^3$



$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$

Solution:

To show that B is a Basis of $\mathbb{R}^3$ we must Show that

(i) B is a linearly independent set

(ii) B spans $\mathbb{R}^3$.

To show that B is linearly independent set.

ProMate

Date No

$x_1\underline{v_1} + x_2\underline{v_2} + x_3\underline{v_3} = 0$

$$x_1\begin{pmatrix}1\\1\\1\end{pmatrix} + x_2\begin{pmatrix}1\\2\\2\end{pmatrix} + x_3\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}x_1 + x_2 + x_3\\ x_1 + 2x_2 + x_3\\ x_1 + 2x_2 + 0x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$(x_1 + x_2 + x_3)\underline{i} + (x_1 + 2x_2 + x_3)\underline{j} + (x_1 + 2x_2 + 0x_3)\underline{k} = 0$$

Since $\underline{i}$, $\underline{j}$, $\underline{k}$ are linearly independent we must have,

$x_1 + x_2 + x_3 = 0$ —— (1)

$x_1 + 2x_2 + 2x_3 = 0$ —— (2)

$x_1 + 2x_2 = 0$ —— (3)

by (3) $x_1 = -2x_2$

(2) $-2x_2 + 2x_2 + 2x_3 = 0$

$x_3 = 0$

(1) $-2x_2 + 2x_2 + x_3 = 0$

$x_2 = x_3$

$\therefore x_2 = 0$, $x_1 = -2x_2$

$x_1 = 0$

Only the trivial solutions

ProMate

Date

No

$\therefore$ $\underline{v_1}$, $\underline{v_2}$ and $\underline{v_3}$ are linearly independent

Next,

we shall show that B spans $\mathbb{R}^3$ for this, we must show that any vector $\underline{v} \in \mathbb{R}^3$ can be written as linear combination of $\underline{v_1}$, $\underline{v_2}$, $\underline{v_3}$

Let $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\underline{i} + y\underline{j} + z\underline{k}$;

So that, x, y; and z are scalars;

Suppose;

$$\underline{v} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_3 \\ c_1 + 2c_2 + c_3 \\ c_1 + 2c_2 + 0c_3 \end{pmatrix}.$$

(we have to show that there exist scalars satisfying the above system.

ProMate

since $\underline{i}, \underline{j}$ and $\underline{k}$ are linearly independent

$$c_1 + c_2 + c_3 = x \quad \text{--- (1)}$$
$$c_1 + 2c_2 + c_3 = y \quad \text{--- (2)}$$
$$c_1 + 2c_2 = z \quad \text{--- (3)}$$

$\Rightarrow$ (1) $-$ (2),

$$-c_2 = x - y$$
$$c_2 = y - x$$

(2) $-$ (3)

$$c_3 = y - z$$

by (1)

$$c_1 + y - x + y - z = x$$
$$c_1 = 2x + z - 2y$$

(That is)

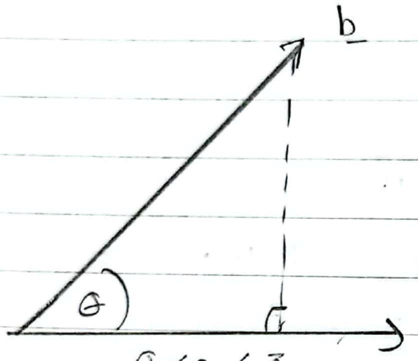
i.e. $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (z - 2y + 2x)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (y - x)\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + (y - z)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now we have shown that
$\underline{v_1}, \underline{v_2}, \underline{v_3}$ are ~~linear~~ linearly independent and they span $\mathbb{R}^3$, $\therefore$ B is a basis of $\mathbb{R}^3$.

Note:- $\mathbb{R}^3$ has several bases, for
ex:- The standard Basis
$B = \{\underline{i}, \underline{j}, \underline{k}\}$.

Scalar · Product

Def^n: The scalar product of two vectors $\underline{a}$ and $\underline{b}$, denoted by $\underline{a} \cdot \underline{b}$, is defined by $|\underline{a}|\,|\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$. so that $0 \le \theta \le \pi$



$0 \le \theta \le \pi$

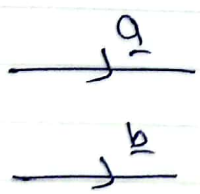
$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta}$$

$\underline{a} \cdot \underline{b}$ is read as $\underline{a}$ dot $\underline{b}$

$OA = |\underline{b}| \cos\theta$ is the projection (scalar) of vector $\underline{b}$ on $\underline{a}$.

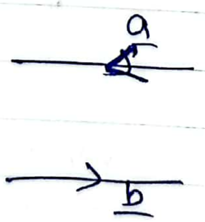
Propeties.

(i) If $\underline{a}$ and $\underline{b}$ are parallel then $\theta = 0$ or π.



$\theta = 0$

(like parallel)



$\theta = \pi$ (unlike parallel)

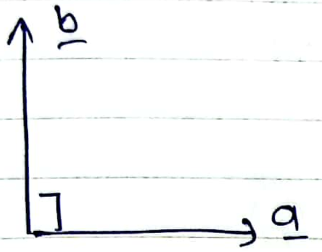
$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos 0$
$= |\underline{a}|\,|\underline{b}| \ge 0$

$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\pi$
$= -|\underline{a}|\,|\underline{b}| < 0$

Date No

(ii) Perpendicular vectors

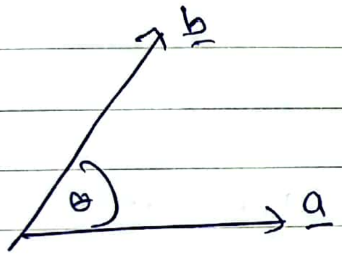
If $\underline{a}$ and $\underline{b}$ are perpendiculars then $\theta = \pi/2$



$$\underline{a} \cdot \underline{b} = |\underline{a}| \, |\underline{b}| \cos \pi/2$$

$$= 0 \in \mathbb{R}.$$

(iii) Angle between two vectors



Then,

$$\underline{a} \cdot \underline{b} = |\underline{a}| \, |\underline{b}| \cos\theta$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| \, |\underline{b}|}$$

$$= \left(\frac{\underline{a}}{|\underline{a}|}\right) \cdot \left(\frac{\underline{b}}{|\underline{b}|}\right)$$

$$= \hat{\underline{a}} \cdot \hat{\underline{b}}$$

$$\boxed{\theta = \cos^{-1}\left(\hat{\underline{a}} \cdot \hat{\underline{b}}\right)}$$

(iv) Scalar product holds commutatively

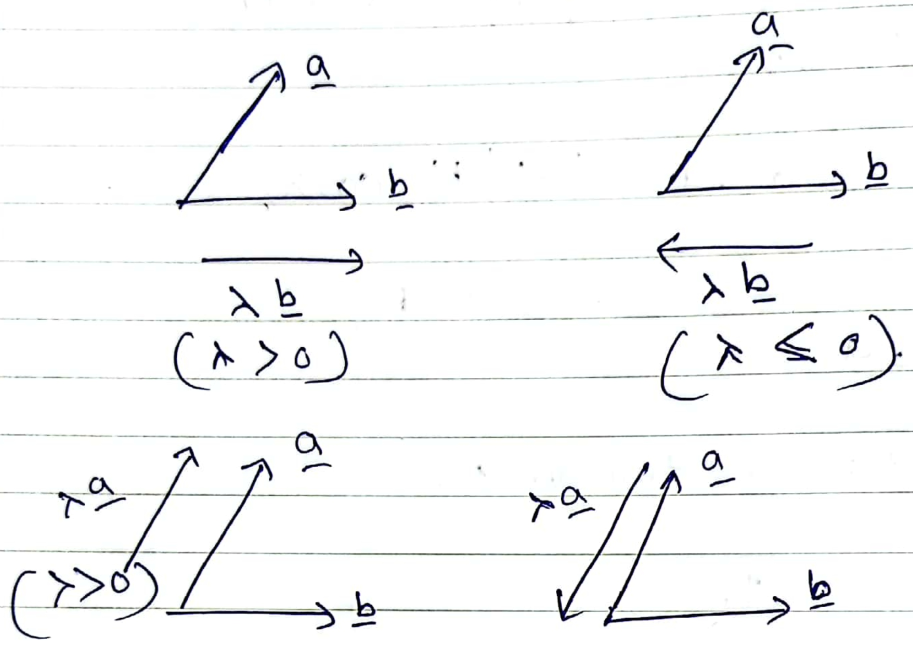
i.e. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.

(v) The Scalar product is distributive

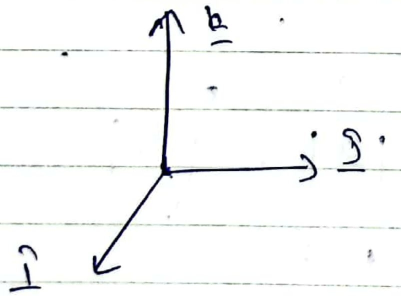
i.e. $\underline{a}(\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c}$

where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors.

6. ~~The~~ Let λ be a scalar and $\underline{a}$ and $\underline{b}$ be vectors.



(vii) Let $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$



$\underline{i} \cdot \underline{i} = |\underline{i}|\,|\underline{k}| \cos\theta$

$= 1 \times 1 \times 1$

$= 1$

$\underline{k} \cdot \underline{k} = |\underline{k}|\,|\underline{k}| \cos\theta$

$= 1 \times 1 \times 1$

$= 1$

$\underline{i} \cdot \underline{j} = |\underline{i}|\,|\underline{j}| \cos\frac{\pi}{2}$

$= 0$

$\underline{j} \cdot \underline{k} = 0 \qquad \underline{k} \cdot \underline{i} = 0.$

Date No

$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 \underline{b_2} + a_3 b_3$

ex :- $\underline{a} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

$\underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$

$\underline{a} \cdot \underline{b} = (2\underline{i} - 3\underline{j} + 4\underline{k})(\underline{1} + 3\underline{j} + 2\underline{k})$

$\hat{a} = \dfrac{a}{|a|} = \dfrac{(2\underline{i} + 3\underline{j} + 4\underline{k})}{\sqrt{29}}$

$\hat{b} = \dfrac{b}{|b|} = \dfrac{\underline{i} + 3\underline{j} + 2\underline{k}}{\sqrt{14}}$

$\hat{a} \cdot \underline{b} = \dfrac{1}{\sqrt{29}\sqrt{14}} = \dfrac{1}{\sqrt{406}}$

Let θ be the angle between $\underline{a}$ and $\underline{b}$

$\theta = \cos^{-1}(\hat{\underline{a}} \cdot \hat{\underline{b}})$

$= \cos^{-1}\left(\dfrac{1}{\sqrt{406}}\right)$

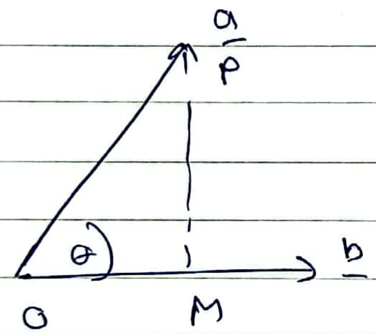
(viii) components of $\underline{a}$ ~~and~~ on $\underline{b}$

Let $\underline{a} = a_1 \underline{\hat{i}} + a_2 \underline{\hat{j}} + a_3 \underline{\hat{k}}$

$\underline{a} \cdot \underline{\hat{i}} = a_1$ (The component $\underline{a}$ on $\underline{\hat{i}}$)
$\underline{a} \cdot \underline{\hat{j}} = a_2$ (The component $\underline{a}$ on $\underline{\hat{j}}$)
$\underline{a} \cdot \underline{\hat{k}} = a_3$ (The component of $\underline{a}$ on $\underline{\hat{k}}$)

$\underline{a} = (\underline{a} \cdot \underline{\hat{i}})\, \underline{\hat{i}} + (\underline{a} \cdot \underline{\hat{j}})\, \underline{\hat{j}} + (\underline{a} \cdot \underline{\hat{k}})\, \underline{\hat{k}}$

Let θ be the angle between a and b.



$OM = OP \cos\theta$
$= |\underline{a}| \cos\theta$
= The componet of $\underline{a}$ on $\underline{b}$ (projection)

$$OM = \frac{|\underline{a}|\,|\underline{b}| \cos\theta}{|\underline{b}|} \quad \text{if } \underline{b} \neq \underline{0}$$

$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} = \underline{a} \cdot \underline{\hat{b}}.$$

Date No

ex:-

Let $\underline{a} = 2\hat{i} + 3\hat{j} - 4\underline{k}$ and $\underline{b} = \hat{i} + \hat{j} + 2\underline{k}$

find the componet of $\underline{a}$ ~~and~~ on $\underline{b}$

$|\underline{b}| \sqrt{1^2 + 1^2 + 2^2}$

$= \sqrt{6}$

$\hat{\underline{b}} = \dfrac{\underline{b}}{|\underline{b}|} = \dfrac{\hat{i} + \hat{j} + 2\underline{k}}{\sqrt{6}}$

There fore The component of a on $\underline{b}$

$= \underline{a} \cdot \hat{b}$

$= (2\hat{i} + 3\hat{j} - 4\underline{k}) \dfrac{(\hat{i} + \hat{j} + 2\underline{k})}{\sqrt{6}}$

$= \dfrac{(2 + 3 - 8)}{\sqrt{6}} = -3/\sqrt{6}$

ProMate

Therefore the Component of $\underline{a}$ on $\underline{b}$

$$= \underline{a} \cdot \hat{\underline{b}}$$

$$= (2\hat{i} + 3\hat{j} - 4\hat{k}) \frac{(\underline{i} + \underline{j} + 2\underline{k})}{\sqrt{6}}$$

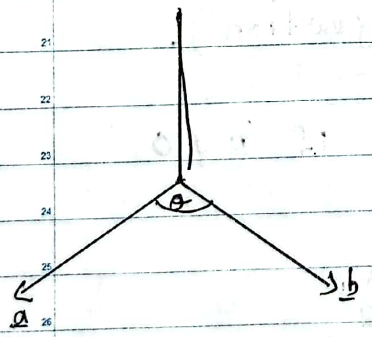
$$= \frac{(2+3-8)}{\sqrt{6}} = -3/\sqrt{6},$$

Cross vector product.

The Cross vector product of two vectors $\underline{a}$ and $\underline{b}$, denoted by $\underline{a} \times \underline{b}$ or $\underline{a} \wedge \underline{b}$, is the vector defined by $|a||b| \sin\theta\, \underline{n}$ where $\underline{n}$ is a unit vector given by the right hand rule. that is orthogonal to the plane spanned by $\underline{a}$ and $\underline{b}$ with $0 \leq \theta \leq \pi$.

That is,

$$\underline{a} \times \underline{b} = \underline{a} \wedge \underline{b} = |\underline{a}||\underline{b}| \sin\theta\, \underline{n}$$



Also, $\underline{a} \wedge \underline{0} = \underline{0}$

Properties of the Cross Product :-

01). $\underline{a} \wedge \underline{b} = -\underline{b} \wedge \underline{a}$

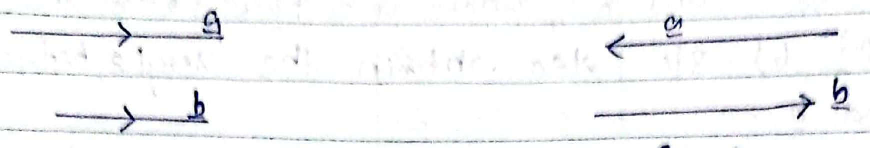
2) $\underline{a} \wedge \underline{a} = \underline{0}$

3) $|\underline{a} \wedge \underline{b}| = \left| |\underline{a}||\underline{b}| \sin\theta\, \underline{n} \right|$

$$= |\underline{a}||\underline{b}| \sin\theta,$$

4) Two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \underset{\sim}{0}$$



$\theta = 0$ $\qquad\qquad$ $\theta = \pi$.

05). $(\lambda \underset{\sim}{a} \wedge \underset{\sim}{b}) = (\underset{\sim}{a} \wedge \lambda \underset{\sim}{b}) = \lambda(\underset{\sim}{a} \wedge \underset{\sim}{b})$ where λ is a scalar.

06) $\underset{\sim}{i} \wedge \underset{\sim}{i} = \underset{\sim}{j} \wedge \underset{\sim}{j} = \underset{\sim}{k} \wedge \underset{\sim}{k} = \underset{\sim}{0}$

$\underset{\sim}{i} \wedge \underset{\sim}{j} = \underset{\sim}{k}$ $\qquad\qquad$ $\underset{\sim}{j} \wedge \underset{\sim}{k} = \underset{\sim}{i}$

$\underset{\sim}{j} \wedge \underset{\sim}{i} = -\underset{\sim}{k}$ $\qquad\qquad$ $\underset{\sim}{k} \wedge \underset{\sim}{i} = \underset{\sim}{j}$

07) Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \qquad \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$= \underset{\sim}{i}(a_2b_3 - a_3b_2) - \underset{\sim}{j}(a_1b_3 - b_1a_3] + \underset{\sim}{k}[a_1b_2 - a_2b_1].$$

08) Angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \, \underset{\sim}{n}$$

$$|\underset{\sim}{a} \wedge \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$$

$$\sin\theta = \frac{|\underset{\sim}{a} \wedge \underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{b}|} \qquad (\underset{\sim}{a}, \underset{\sim}{b} \neq 0)$$

$$\theta = \sin^{-1}\left[\frac{|\underset{\sim}{a} \wedge \underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right] \qquad \hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} \qquad \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \sin^{-1}(\hat{\underset{\sim}{a}} \wedge \hat{\underset{\sim}{b}})$$

No.

Date

Example:-

Determine a unit vector perpendicular to the plane of $\underline{a}$ and $\underline{b}$, where $\underline{a} = 4\underline{i} + 7\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ also obtain the angle between $\underline{a}$ and $\underline{b}$.

$$\boxed{\begin{aligned} \underline{c} &= \underline{a} \wedge \underline{b} \\ \hat{\underline{c}} &= \frac{\underline{c}}{|\underline{c}|} \end{aligned}}$$

now, $\underline{c} = \underline{a} \wedge \underline{b}$

$$= \begin{array}{c} \begin{array}{ccc} (+) & (-) & (+) \end{array} \\ \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 7 & -1 \\ 2 & -6 & -3 \end{vmatrix} \end{array}$$

$$= \underline{i}\left[7\times(-3) - (-1)\times(-6)\right] - \underline{j}\left[4\times(-3) - (-1)\times 2\right] + \underline{k}\left[4\times(-6) - 7\times 2\right]$$

$$= (-21 - 6)\underline{i} - \left[-12 + 2\right]\underline{j} + \left[-24 - 14\right]\underline{k}$$

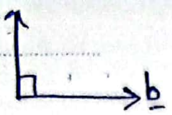
$$= -27\underline{i} + 10\underline{j} - 38\underline{k}$$

$$\hat{\underline{c}} = \frac{\underline{c}}{|\underline{c}|} = \frac{-27\underline{i} + 10\underline{j} - 38\underline{k}}{\left|\sqrt{27^2 + 10^2 + 38^2}\right|}$$

$$= \frac{-27\underline{i} + 10\underline{j} - 38\underline{k}}{\sqrt{2273}}$$

$$= \frac{-27}{\sqrt{2273}}\underline{i} + \frac{10}{\sqrt{2273}}\underline{j} - \frac{38}{\sqrt{2273}}\underline{k}.$$

No.



$\underline{a} \cdot \underline{b} = 0$

$\underline{a} \wedge \underline{b} = \underline{0}$

ex :- Determine whether $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = -6\underline{i} - 3\underline{j} + 3\underline{k}$ are parallel.

Consider $\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ -6 & -3 & 3 \end{vmatrix}$

$= \underline{i}(\underline{0}) - \underline{j}(0) + \underline{k}(0)$

$= \underline{0}$

$\therefore$ $\underline{a}$ and $\underline{b}$ are parallel.

Home work :-

If $\underline{a} \wedge \underline{b} = \underline{c} \wedge \underline{d}$ and $\underline{a} \wedge \underline{c} = \underline{b} \wedge \underline{d}$, the show that $\underline{a} - \underline{d}$ and $\underline{b} - \underline{c}$ are parallel

<u>more properties on $\underline{a} \wedge \underline{b}$</u>

$\underline{a} \wedge (\underline{b} + \underline{c}) = \underline{a} \wedge \underline{b} + \underline{a} \wedge \underline{c}$

$(\underline{a} + \underline{b}) \wedge \underline{c} = \underline{a} \wedge \underline{c} + \underline{b} \wedge \underline{c}$

$\underline{a} \cdot (\underline{a} \wedge \underline{b}) = 0$ (Scalar)

$\underline{b} \cdot (\underline{a} \wedge \underline{b}) = 0$

Homework :-

If $\underline{a} \wedge \underline{r} = \underline{b} + \lambda \underline{a}$ and $\underline{a} \cdot \underline{r} = 3$ where $\underline{a}$ is $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = -\underline{i} - 2\underline{j} + \underline{k}$, then find $\underline{r}$ and λ

$r = x\underline{i} + y\underline{j} + z\underline{k}$.

Scalar Triple product / Mixed triple product

Let $\underline{a}, \underline{b}$, and $\underline{c}$ be vectors. the quantity $\underline{a} \cdot (\underline{b} \wedge \underline{c})$ is the scalar dot product of $\underline{a}$ and $\underline{b} \wedge \underline{c}$

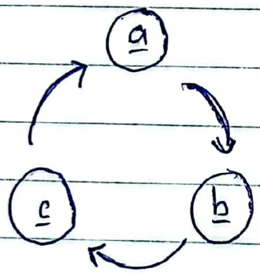
$\underline{a} \cdot (\underline{b} \wedge \underline{c})$ is called a scalar triple product of $\underline{a}, \underline{b}$ and $\underline{c}$.

Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$

$\underline{b} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}$

$\underline{c} = c_1 \underline{i} + c_2 \underline{j} + c_3 \underline{k}$

$$\underline{a} \cdot (\underline{b} \wedge \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



$\underline{a} \cdot (\underline{b} \wedge \underline{c}) = \underline{b} \cdot (\underline{a} \wedge \underline{c}) = \underline{c} \cdot (\underline{a} \wedge \underline{b})$

Note that $\underline{a} \cdot (\underline{b} \cdot \underline{c})$ and $\underline{a} \wedge (\underline{b} \cdot \underline{c})$ are meaningless.

we also denote $\underline{a} \cdot (\underline{b} \wedge \underline{c})$ by $[\underline{a}, \underline{b}\ \underline{c}]$.

Vector triple Product.

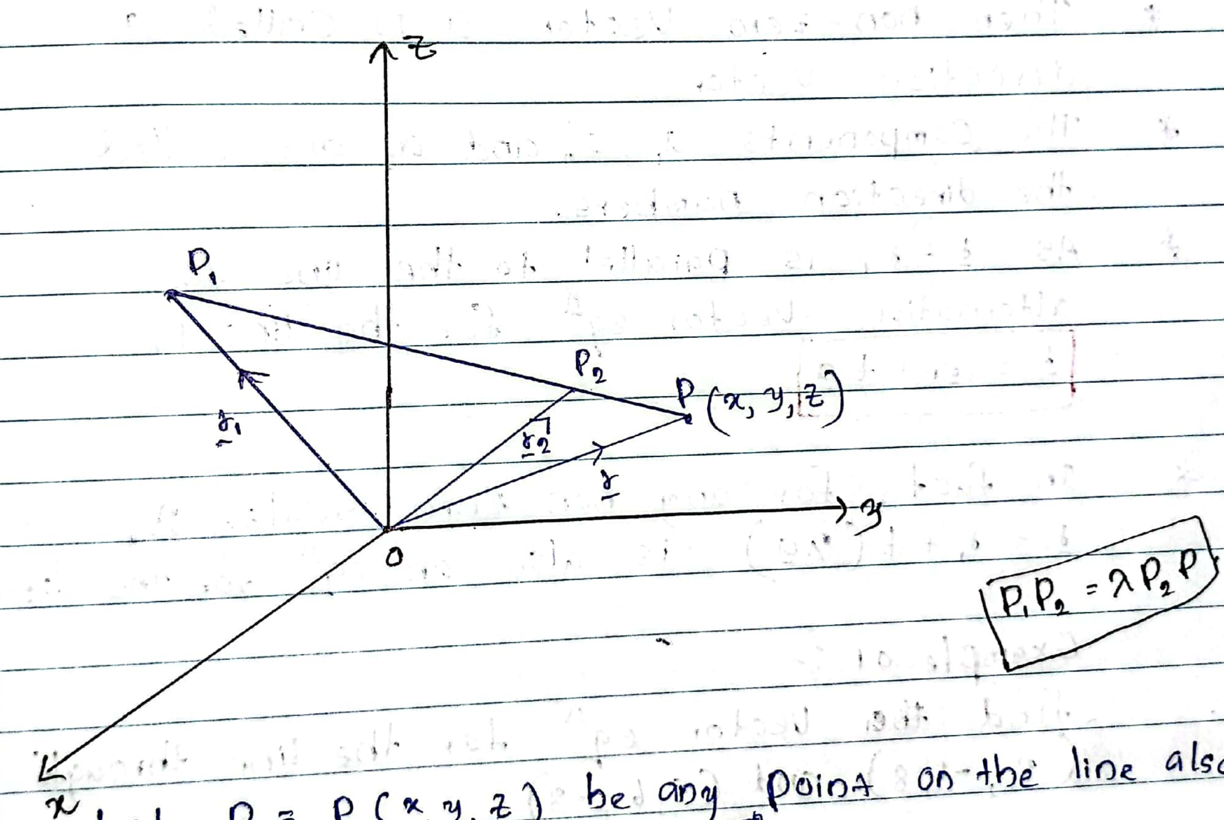
Let $\underline{a}, \underline{b}$ and $\underline{c}$ be vectors the quantity $\underline{a} \wedge (\underline{b} \wedge \underline{c})$ is a vector product of $\underline{a}$ and $\underline{b} \wedge \underline{c}$ is called a vector triple product of $\underline{a}, \underline{b}$ and $\underline{c}$, It can be shown that $\underset{\sim}{a} \wedge (\underset{\sim}{b} \wedge \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

Chapter 2 :-

Lines and planes in 3 - space.

Lines :- A vector Equation.

Any two distinct points in 3-space determine exactly one line between them.
Suppose that we want to find an equation of the line between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$



Let $P \equiv P(x, y, z)$ be any point on the line also,
Let $\vec{OP_1} = \underset{\sim}{r}_1$, $\vec{OP_2} = \underset{\sim}{r}_2$ and $\vec{OP} = \underset{\sim}{r}$

Then $\vec{P_1P_2} = \underset{\sim}{r}_2 - \underset{\sim}{r}_1$ and $P_2P = \underset{\sim}{r} - \underset{\sim}{r}_2$
As $\vec{P_1P_2}$ is parallel to $\vec{P_2P}$ we have

$$\underset{\sim}{V} - \underset{\sim}{U}_2 = t(\underset{\sim}{U}_2 - \underset{\sim}{V}_1) \quad \text{—} \ (1)$$

where t is a scalar

No. .... Date. ....

$$\therefore \underline{r}_2 - \underline{r}_1 = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$= a \text{ (say)} =: a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$$

Then ① $\Rightarrow$ $\underline{r} - \underline{r}_2 = t(\underline{a})$

$\boxed{\underline{r} = \underline{r}_2 + t\,\underline{a}}$ This is a vector eq^n to the line through P_1 and P_2.

* t is called a parameter.
* Then non-zero vector $\underline{a}$ is called a direction vector
* The components a_1, a_2 and a_3 are called the direction numbers.
* As $\underline{r} - \underline{r}_1$ is parallel to the line an alternative vector eq^n for the line is

$\boxed{\underline{r} = \underline{r}_1 + t\,\underline{a}}$

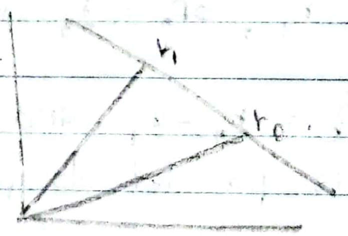
* In fact, for any non zero scalar λ, $\underline{r} = \underline{r}_1 + t(\lambda\underline{a})$ is also an eq^n for the line.

Example 01 :-

Find ~~the~~ vector eq^n for the line through $(2, -1, 8)$ and $(5, 6, -3)$

$\underline{r}_1 = 2\underline{i} - \underline{j} + 8\underline{k}$

$\underline{r}_2 = 5\underline{i} + 6\underline{j} - 3\underline{k}$



No. ....   Date. ....

Let $\underline{a} = \underline{r}_1 - \underline{r}_2$

$= (2\underline{i} - \underline{j} + 8\underline{k}) - (5\underline{i} + 6\underline{j} - 3\underline{k})$.

$= -3\underline{i} - 7\underline{j} + 11\underline{k}$ // the given line.

$\therefore \underline{r} = \underline{r}_1 + t\,\underline{a}$

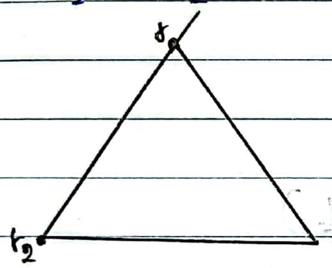
$= (2\underline{i} - \underline{j} + 8\underline{k}) + t(-3\underline{i} - 7\underline{j} + 11\underline{k})$

where t is a Scalar.

Parametric Equations of a Straight line:

Consider the equation,

$\underline{r} = r_2 + t(\underline{a})$



Straight line,

Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$\underline{r}_2 = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$

$\underline{r} = x_2\underline{i} + y_2\underline{j} + z_2\underline{k} + t[a_1\underline{i} + a_2\underline{j} + a_3\underline{k}]$

where, $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ for Same Scalar a_1, a_2, a_3.

$\therefore \underline{r} = (x_2 + t a_1)\underline{i} + (y_2 + t a_2)\underline{j} + (z_2 + t a_3)\underline{k} = x\underline{i} + y\underline{j} + z\underline{k}$

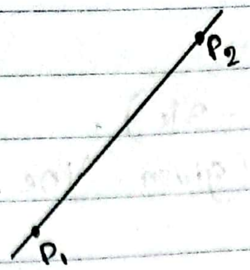
Since $\underline{i}, \underline{j}, \underline{k}$ and linearly independent vectors we have,

$x = x_2 + a_1 t$

$y = y_2 + a_2 t$

$z = z_2 + a_3 t$

those equation are Called parametric equations for the line through P_1 and P_2.



vector equation

$$\underline{r} = r_2 + t\underline{a}$$

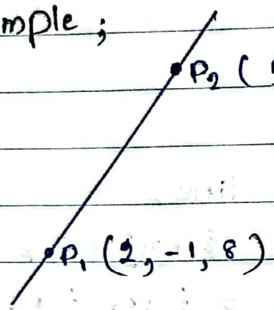
Parametric equation.

$$x = x_2 + ta_1$$
$$y = y_2 + ta_2$$
$$z = z_2 + ta_3.$$

Example:-

Find the parametric equation for the line in

Example;



$$\underline{a} = (2\underline{i} - \underline{j} + 8\underline{k}) - (5\underline{i} + 6\underline{j} - 3\underline{k})$$
$$\underline{a} = -3\underline{i} - 7\underline{j} + 11\underline{k}$$
$$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$$

$\therefore a_1 = -3, \quad a_2 = -7$ and $a_3 = 11$

The parametric equation of the line are,

$$x = 5 - 3t$$
$$y = 6 - 7t$$
$$z = -3 + 11t$$

Example:-

Find a vector $\underline{a}$ that is parallel to the line whose Parametric equation are $x = 4 + 9t$, $y = 14 + 5t$, $z = 1 - 3t$.

$$x = 4 + 9t \Rightarrow a_1 = 9$$
$$y = -14 + 5t \Rightarrow a_2 = 5$$
$$z = 1 - 3t \Rightarrow a_3 = -3.$$

$$a = 9\underline{i} + 5\underline{j} - 3\underline{k}.$$

This is a direction vector of the line and hence it is parallel to the line.

Symmetric Equation of the line :-

Consider parametric equations of the line through P_1 and P_2.

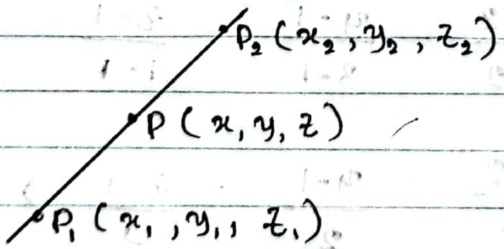
$$x = x_2 + a_1 t$$
$$y = y_2 + a_2 t$$
$$z = z_2 + a_3 t$$

Assume that a_1, a_2 and a_3 are non-zero.

$$\frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3} = t$$

$$\Rightarrow \boxed{\frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}}$$

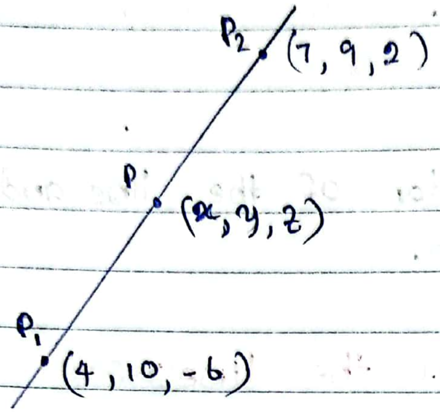
These equation are Called Symmetric equations for the line through P_1 and P_2.



$$\boxed{\frac{x - x_2}{x_1 - x_2} = \frac{y - y_2}{y_1 - y_2} = \frac{z - z_2}{z_1 - z_2}}$$

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

Eg:- Find the symmetrical equations for the line through $(4, 10, -6)$ and $(7, 9, 2)$.



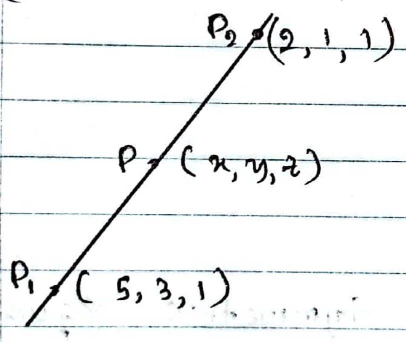
$$\frac{x-4}{7-4} = \frac{y-10}{9-10} = \frac{z-(-6)}{2-(-6)}$$

$$\frac{x-4}{3} = \frac{y-10}{-1} = \frac{z+6}{8}$$

$$\frac{x-7}{4-7} = \frac{y-9}{10-9} = \frac{z-2}{-6-(2)}$$

$$\frac{x-7}{-3} = \frac{y-9}{1} = \frac{z-2}{-8}$$

02) Find the Symmetric equation for the line through $(5, 3, 1)$ and $(2, 1, 1)$



$$\frac{x-5}{2-5} = \frac{y-3}{1-3} = \frac{z-1}{1-1}$$

$$\frac{x-5}{-3} = \frac{y-3}{-2} = \frac{z-1}{0} \Rightarrow z = 1$$

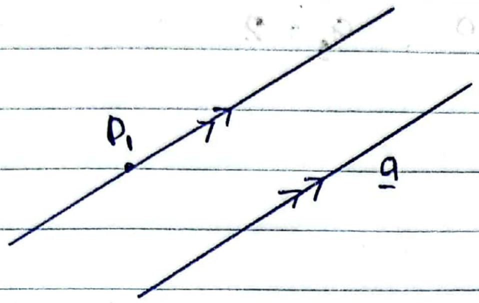
$$\frac{x-2}{5-2} = \frac{y-1}{3-1} = \frac{z-1}{1-1}$$

$$\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-1}{0} \Rightarrow z = 1$$

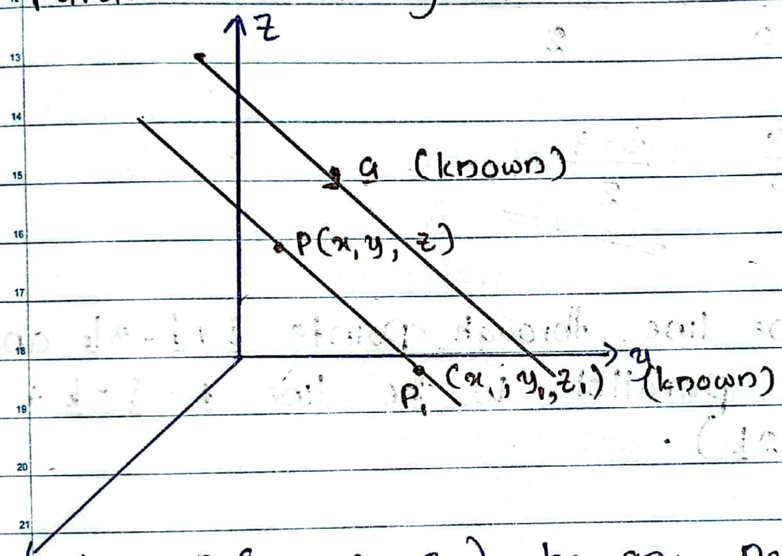
No. .... Date. ....

Note:-

A line is space is also uniquely determinined by a fixed point $P_1(x_1, y_1, z_1)$ and a non-zero direction vector $\underline{a}$.



through the point P_1, there passes exactly are line parallel to the given vector $\underline{a}$



Let $P(x_1, y_1, z_1)$ be any point on the line,

$$\vec{P_1P} = \vec{OP} - \vec{OP_1}$$

$$\vec{P_1P} = \underline{r} - \underline{r}_1$$

is parallel to the vector $\underline{a}$

$$\therefore \quad \underline{r} - \underline{r}_1 = t\underline{a}$$

$$\underline{r} = \underline{r}_1 + t\underline{a}$$

Eg:- write vector, parameter and symmetric equation for the line through $(4, 6, -3)$ and parallel to $\underline{a} = 5\underline{i} - 10\underline{j} + 2\underline{k}$.

Vector equation,

$\underline{r} = r_1 + t\,\underline{a}$

$\underline{r} = 4\underline{i} + 6\underline{j} - 3\underline{k} + t\,[5\underline{i} - 10\underline{j} + 2\underline{k}]$

where, t is a scalar parameter.

$\Rightarrow \quad a_1 = 5, \quad a_2 = -10, \quad a_3 = 2$

$x = 4 + 5t$

$y = 6 - 10t$

$z = -3 + 2t$

Symmetric equations;

$$\frac{x-4}{5} = \frac{y-6}{-10} = \frac{z-(-3)}{2}$$

$$\frac{x-4}{5} = \frac{y-6}{-10} = \frac{z+3}{2}$$

Eg:- Show that the line through points $\underline{i} + \underline{j} - 3\underline{k}$ and $4\underline{i} + 7\underline{j} + \underline{k}$ is parallel to the line $r = \underline{i} - \underline{k} + \alpha\left(\frac{3}{2}\underline{i} + 3\underline{j} + 2\underline{k}\right)$.

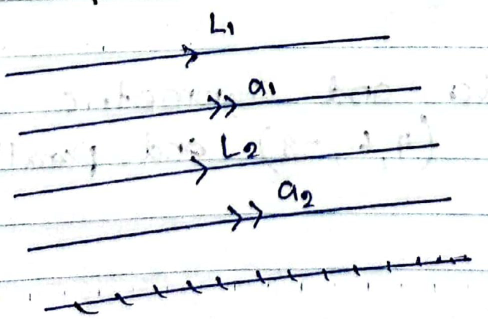
$\underline{i} + \underline{j} - 3\underline{k}$

$4\underline{i} + 7\underline{j} + \underline{k}$

$r = r_1 + t(a)$

$\underline{a} = \frac{3}{2}\underline{i} + 3\underline{j} + 2\underline{k}$

a is parallel to the line.



$L_1 \parallel L_2 = \text{if} = a_1 \parallel a_2$

No. ....

Date. ....

the line through $\underline{i}+\underline{j}-3\underline{k}$ and $4\underline{i}+7\underline{j}+\underline{k}$ is parallel to the direction vector.

$\underline{a} = (1-4)\underline{i} + (1-7)\underline{j} + (-3-1)\underline{k}$.

$\underline{a} = -3\underline{i} - 6\underline{j} - 4\underline{k}$

$\underline{a} = -2\left(\frac{3}{2}\underline{i} + 3\underline{j} + 2\underline{k}\right)$

Also the line $\underline{r} = \underline{i} - \underline{k} + \lambda\left(\frac{3}{2}\underline{i} + 3\underline{j} + 2\underline{k}\right)$ is parallel to the vector $\underline{b} = \frac{3}{2}\underline{i} + 3\underline{j} + 2\underline{k}$

that is $\underline{a} = -2\underline{b}$ and hence $\underline{a}$ is Parallel to $\underline{b}$ therefore; given two lines are parallel.

Homework:-

Find the Cordinates of the point where the line through $A(3,4,1)$ and $B(5,1,6)$ cross of the xy-plane.

Find the Point of intersection of the lines $\underline{r} = \underline{a} - 2\underline{b} + t(\underline{b} + 2\underline{a})$ and $\underline{r} = 2\underline{a} - \underline{b} + a(\underline{a} + 2\underline{b})$ where $\underline{a}$ and $\underline{b}$ are linearly independent Vectors.

Pairs of lines.

The location of two lines in space may be Such that,

(a) the lines are Parallel

(b) the lines are not parallel and intersect

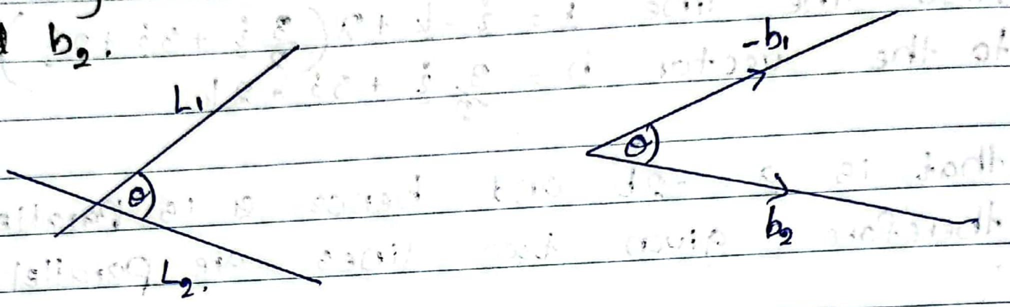
(c) the lines are not parallel and not intersect.

Such lines are Called Skew lines

Angle between a pair of lines:-

Let $r_1 = a_1 + \lambda b_1$ and $r_2 = a_2 + \mu b_2$ the two lines,

the angle between any two lines is defined as the angle between the direction vectors b_1 and b_2.



Let the angle between b_1 and b_2 be θ, As

$b_1 . b_2 = |b_1||b_2| \cos\theta$

$$\frac{b_1 . b_2}{|b_1||b_2|} = \cos\theta$$

This applies to a pair of skew lines as well as a pair of intersecting lines.

It follows that if the lines are perpendicular, then;

$$\cos\theta = 0 \;;\; b_1 . b_2 = 0 .$$

Homework:-

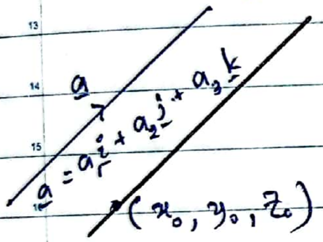
Find the angle between the lines $r_1 = i - 2j + 3k + \lambda(2i - 3j + 6k)$ and $r_2 = 2i - 7j + 10k + \mu(i + 2j + 2k)$.

$$\cos\theta = \frac{(2\hat{i}+3\hat{j}+6\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{|2\hat{i}+3\hat{j}+6\hat{k}|\cdot|\hat{i}+2\hat{j}+2\hat{k}|}$$

$$= \frac{2+6+12}{\sqrt{4+9+36}\times\sqrt{1+4+4}}$$

$$= \frac{20}{\sqrt{49}\times\sqrt{9}}$$

$$= \frac{20}{21}$$

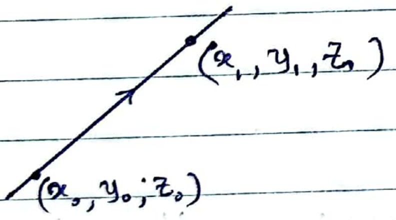


There are 3 forms:-

Vector form: $\underline{r} = \underline{r}_0 + \lambda \underline{a}$

Parametric form: $x = x_0 + \lambda a_1,\ y = y_0 + \lambda a_2,\ z = z_0 + \lambda a_3$

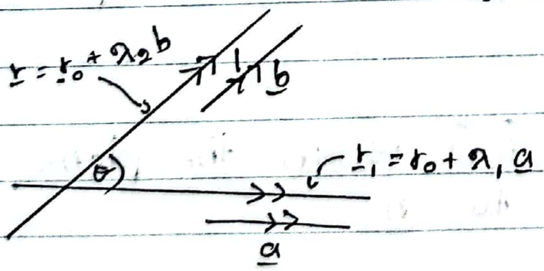
Symmetric form: $\dfrac{x-x_0}{a_1} = \dfrac{y-y_0}{a_2} = \dfrac{z-z_0}{a_3}$.



$a_1 = x_1 - x_0$

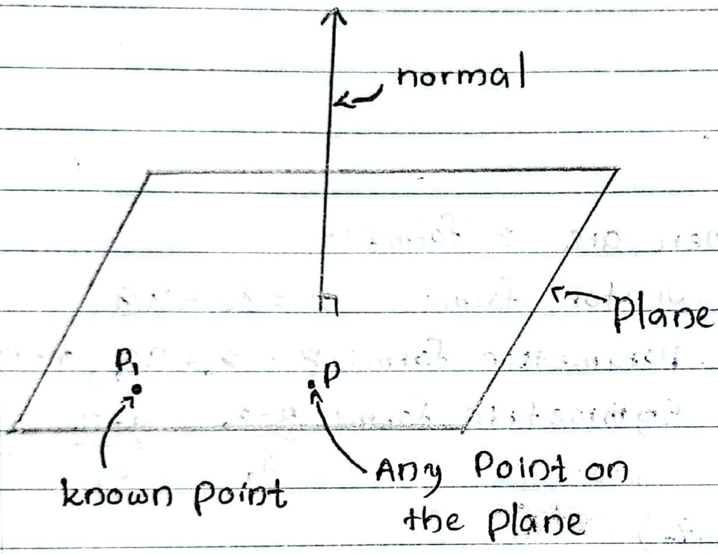
$a_2 = y_1 - y_0$

$a_3 = z_1 - z_0$.



Planes : vector eq^n / Cartesian eq^n

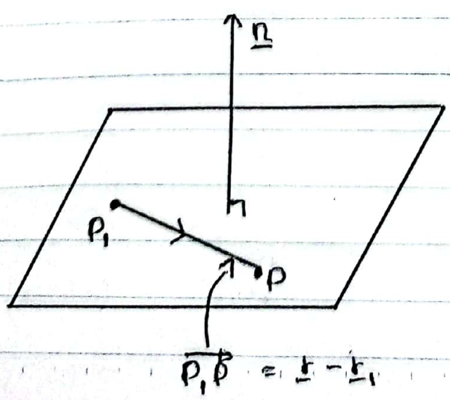
Through a given point $P_1(x_1, y_1, z_1)$ their pass and infinite number of planes. However, if a point P_1 and a vector $\underline{n}$ is specified, there is exactly one plane containing P_1 with $\underline{n}$ normal or perpendicular to the plane.



Let $\underline{r}_1 = \overrightarrow{OP_1}$

Let $P(x, y, z)$ be any point on the plane with $\underline{r} = \overrightarrow{OP}$

Then, Clearly $\underline{r} - \underline{r}_1 = \overrightarrow{P_1P}$ is in the plane and is perpendicular to $\underline{n}$



$\therefore \underline{n} \cdot (\underline{r} - \underline{r}_1) = 0$

This is a vector eq^n of the plane.

Cartesian Equation:-

Let $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$

Then $\underline{n} \cdot (\underline{r} - \underline{r}_1) = 0$ gives,

$(a\underline{i} + b\underline{j} + z\underline{k}) \cdot \left[(x - x_1)\underline{i} + (y - y_1)\underline{j} + (z - z_1)\underline{k}\right] = 0.$

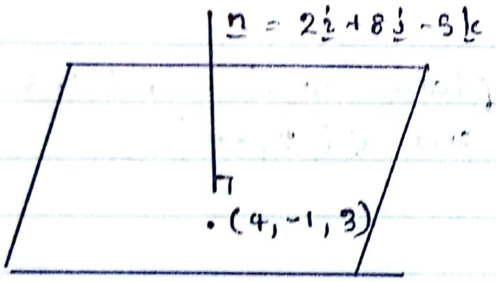
$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ← This can be reduced to the form:-

$ax + by + cz = d,$

$d = ax_1 + by_1 + cz_1$

This is a Cartesian eq^n of the plane and of the plane. It is sometimes called the point-normal form of the eq^n of a plane.

eg:- Find an eq^n of the ~~point~~ plane with normal vector $\underline{n} = 2\underline{i} + 8\underline{j} - 5\underline{k}$ containing point $(4, -1, 3)$



vector eq^n is given by,

$\underline{n} \cdot (\underline{r} - \underline{r}_1) = 0.$

$(2\underline{i} + 8\underline{j} - 5\underline{k}) \cdot [\underline{r} - (4\underline{i} - \underline{j} + 3\underline{k})] = 0$

Date.

No.

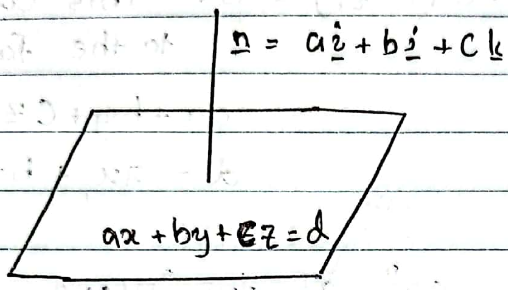
The Cartesian eq^n is,

$$2(x-4)+8[y-(-1)]-5(z-3) = 0.$$

$$2x+8y-5z = -15$$

Example:-

Find a vector normal to the plane $3x-4y+10z=8$

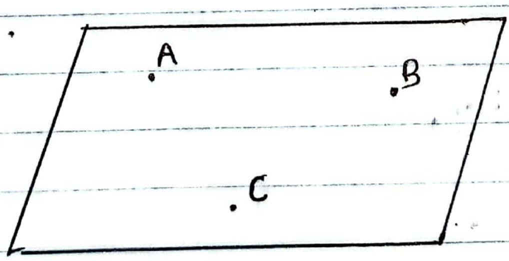


The normal to the plane is, ~~n = 3i + 4j~~

$$n = 3i - 4j + 10k$$

Example:-

Find an eq^n of the plane that contains the points $A(1,0,-1)$, $B(3,1,4)$ and $C(2,-2,0)$



$$u = \vec{AB} = 2i + j + 5k$$

$$v = \vec{AC} = i - 2j + k$$

$$\underline{n} = \underline{u} \wedge \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 5 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \underline{i}\,[1-(-10)] - \underline{j}\,[2-5] + \underline{k}\,[-4-1]$$

$$= 11\underline{i} + 3\underline{j} - 5\underline{k}$$

This is perpendicular to the plane.

Take any point $P(x, y, z)$ in the plane,

Then $\overrightarrow{CP} = \underline{w} = (x-2)\underline{i} + (y+2)\underline{j} + (z-0)\underline{k}$

$$= (x-2)\underline{i} + (y+2)\underline{j} + z\underline{k}.$$

$\underline{w}$ lies in the plane

$\therefore \underline{n} \cdot \underline{w} = 0$

$$(11\underline{i} + 3\underline{j} - 5\underline{k}) \cdot \left[(x-2)\underline{i} + (y+2)\underline{j} + z\underline{k}\right] = 0$$

The Cartesian eq^n is $\Rightarrow$ $11x + 3y - 5z = 16$.

Example :-

Find the eq^n of the plane passing through $(1, 3, -6)$ that is perpendicular to the line

$$\frac{x-1}{4} = \frac{y}{5} = \frac{z+5}{6}$$

The line is parallel to the vector $\underline{n} = 4\underline{i} + 5\underline{j} + 6\underline{k}$.

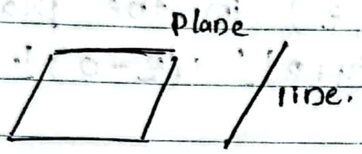
$\therefore$ the plane is perpendicular to $\underline{n}$

No. Date

$\therefore$ The eq^n of the plane is
$4(x-1) + 5(y-3) + 6(z+6) = 0$

$\Rightarrow$ ~~4x~~ $4x + 5y + 6z = -17$

Example:-
Shows that the plane whose vector eq^n is $\underline{r}\cdot(\underline{i} + 2\underline{j} - \underline{k}) = 3$ Contains the line whose vector eq^n is $\underline{r} = \underline{i} + \underline{j} + \lambda(2\underline{i} + \underline{j} + 4\underline{k})$



The line is Contained in the plane of any two points on the line are on the plane.

Taking $\lambda = 0$ and $\lambda = 1$, we find that $\underline{i} + \underline{j}$ and $3\underline{i} + 2\underline{j} + 4\underline{k}$ are two points on the ~~plane~~. line.

If $\underline{r} = \underline{i} + \underline{j}$, then $\underline{r}\cdot(\underline{i} + 2\underline{j} - \underline{k}) = (\underline{i} + \underline{j})\cdot(\underline{i} + 2\underline{j} - \underline{k}) = 3$

$\underline{r}\cdot(\underline{i} + 2\underline{j} - \underline{k}) = (\underline{i} + \underline{j})\cdot(\underline{i} + 2\underline{j} - \underline{k}) = 3.$

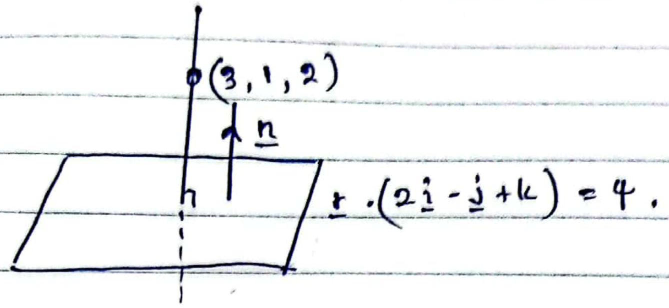
If $\underline{r} = 3\underline{i} + 2\underline{j} + 4\underline{k}$, then
$\underline{r}\cdot(\underline{i} + 2\underline{j} - \underline{k}) = (3\underline{i} + 2\underline{j} + 4\underline{k})\cdot(\underline{i} + \cancel{\underline{j}}\, 2\underline{j} - \underline{k}) = 3.$

$\therefore$ $\underline{r} = 3\underline{i} + 2\underline{j} + 4\underline{k}$ is a point on the line.

$\therefore$ The line is Contained in the plane.

Example:-

Find the vector eq^n of the line passing through the point $(3,1,2)$ and perpendicular to the plane. $\underline{r}\,.\,(2\underline{i} - \underline{j} + \underline{k}) = 4$.



Since the eq^n of the plane is $\underline{r}\,.\,(2\underline{i} - \underline{j} + \underline{k}) = 4$

$\underline{n} = 2\underline{i} - \underline{j} + \underline{k}$ is perpendicular to the plane.

Since the line perpendicular to the plane it is parallel to $\underline{n} = 2\underline{i} - \underline{j} + \underline{k}$

$\therefore$ the vector eq^n of the line is

$$\underline{r} = 3\underline{i} + \underline{j} + 2\underline{k} + t(2\underline{i} - \underline{j} + \underline{k})$$

where, t is a scalar parameter.

At the point of the plane and line

$$[3\underline{i} + \underline{j} + 2\underline{k} + t(2\underline{i} - \underline{j} + \underline{k})]\cdot(2\underline{i} - \underline{j} + \underline{k}) = 4$$

$$[(3+2t)\underline{i} + (1-t)\underline{j} + (2+t)\underline{k}]\,.\,(2\underline{i} - \underline{j} + \underline{k}) = 4$$

$$\Rightarrow 2(3+2t) - (1-t) + (2+t) = 4$$

$$\therefore \underline{r} = 3\underline{i} + \underline{j} + 2\underline{k} - \frac{1}{2}(2\underline{i} - \underline{j} + \underline{k}), \text{ at } t = -1/2$$

$$= 2\underline{i} + 3/2\,\underline{j} + 3/2\,\underline{k}$$

So the point of intersection of the line and the plane is $(2, 3/2, 3/2)$