MAT - Real Analysis I

Real Number System

① $\mathbb{R}$ - Real number

All real numbers are complex numbers

$\sqrt{4} = |2|$ (radical signs ÷ $\sqrt{\ }$)

② $\mathbb{Z}$ - Integers

$\{\ldots -2, -1, 0, 1, 2, 3 \ldots\}$

③ $\mathbb{N}$ - Natural numbers.

$\{1, 2, 3 \ldots\} = \mathbb{Z}^+$

④ $\mathbb{Q}$ - Set of rational number

$\{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$

eg÷ $1 \in \mathbb{Q} \rightarrow (1/1)$ $0 \in \mathbb{Q} \rightarrow (0/1)$

$$\boxed{\begin{array}{l} \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R} \\ \mathbb{Q} \cap \mathbb{Q}' = \phi \end{array}}$$

⑤ $\mathbb{P}$ - Set of prime numbers.

$\{2, 3, 5, 7, 11, 13 \ldots\}$.

$5 = 1 \times 5 \longrightarrow$ trivial factors.

$\left.\begin{array}{l} 10 = 2 \times 5 \\ \quad = 1 \times 10 \end{array}\right\} \longrightarrow$ non trivial factors.

twin prime $= (p, p+2)$

Field Axioms

~~Additive axioms~~

Def^n : A field is a set $\mathbb{F}$ together with two bin
operations.

$+ : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ (called addition)
$\times : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ (called multipication)
such that for all ($\forall$) $x, y, z \in \mathbb{F}$

$A = \{1, 2, 3\}$ ordered pairs
$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2;2), (2,3), (3,1), (3,2), (3,3)\}$

Here $(1,1) \in A \times A$
$(1,2) \in A \times A$

$+ : \underset{(1+2)}{A \times A} \longrightarrow A$ (3) ✓ (3 is in A)

$+ : \underset{(2+3)}{A \times A} \longrightarrow 5 \notin A$; $+ : A \times A \not\rightarrow A$ ✗

$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ ✓ (infinite numbers can be foun

Additive axioms

① A1 : $x + y = y + x$ (addition is commutative)

② A2 : $(x+y)+z = x+(y+z)$ (addition is associative)

③ A3 : There is an element $0 \in \mathbb{F}$ called additive identity such that $x + 0 = 0 + x = x \;\; \forall \; x \in \mathbb{F}$

④. A4 : For each $x \in \mathbb{F}$; there is an element $-x \in \mathbb{F}$; called additive inverse of x, such that $x + (-x) = 0$
(0 is additive identity element of F field)

Multiplicative axioms

① M1 : $x \times y = y \times x$ (multipication is commutative)

② M2 : $(x \times y) \times z = x \times (y \times z)$ (multipication is associative)

$(2,3) \in \mathbb{F} \times \mathbb{F}$	$(6,4) \in \mathbb{F} \times \mathbb{F}$	$(3,4) \in \mathbb{F} \times \mathbb{F}$
$2 \times 3 = 6 \in \mathbb{F}$	$6 \times 4 = 24$	$3 \times 4 = 12 \in \mathbb{F}$
$(2 \times 3) \times 4 = 24$		$(2,12) \in \mathbb{F} \times \mathbb{F}$
		$2 \times 12 = 24$

③ M3 : ~~∄~~ There is an element $e \in \mathbb{F}$; called multiplicative identity, such that $x \times e = x = e \times x$

④ M4 : For each $x \in \mathbb{F} \setminus \{0\}$

There is an element $x^{-1} \in \mathbb{F}$ multiplicative inverse of x such that $x \times x^{-1} = e$

⑤ D1 : $x \times (y + z) = (x \times y) + (x \times z)$
Then $(\mathbb{F}, +, \times)$ called field.

~~Ever If ... a set is ...~~

- If a set is finite, the cardinality is also a finite.

2009 3/08

Field Axioms

Order axioms

Def^n: An ordered field is field $\mathbb{F}$ on which an order relation '$<$' is defined such that,

O_1 Trichotomy

'$\forall$' $x, y \in \mathbb{F}$ exactly one of the following hold

$x < y$, $x = y$, $y < x$

O_2. Transitivity

For all $x, y \in \mathbb{F}$

and $x < y \wedge y < z$ then

$x < z$

AND $\wedge$

OR $\vee$

XOR $\Delta / \oplus$

O_3. For all $x, y, z \in \mathbb{F}$ $x < y \Rightarrow x+z < y+z$

Furthermore, if $z > 0$, then $xz < yz$

$O_1, O_2, O_3 \rightarrow$ ordered set

axioms + O_1 + O_2 + O_3 $\rightarrow$ ordered field

$z_1 < z_2$

$|z_1| < |z_2|$

eg:- $\mathbb{R}$ is an ordered field

H.W: Check the ordered field axioms.

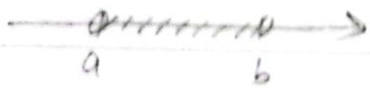
$\cdot\ \mathbb{C} = \{a+ib \mid a ; b \in \mathbb{R}\}$

H.W: Check the field axioms.

Completeness axioms

open interval $(0,1)$ — all real no.s between 0 & 1, $a < x < b$

$a < x < b$.



$[a,b]$: $a \leq x \leq b$

$(a,b]$: $a < x \leq b$

$[a,b)$: $a \leq x < b$

$\mathbb{Q}$ $\quad \mathbb{Q} : \{ p/q \mid p, q \in \mathbb{Z}, q \neq 0 \}$

$(0,1)$ ← infinite $\quad x \in (0,1) \quad 0 < x < 1, .5, 3$ upper bound

set of upper bounds → $[1, \infty)$ $\quad$ least upper bound

$[a,b] \quad a \leq x \leq b$ $\quad b \in [a,b]$ $\quad K \leq k$

upper boundary $\quad \sup[a,b] = b = \max[a,b]$ $\quad$ glb

~~largest~~ greatest lower bound = 0

only supreme exists .. bounded above

$(-\infty, \infty)$ ← unbounded set.

$\mathbb{N} = \{1, 2, 3 \dots\}$

Defn :-

Let A be a subset of an ordered field $\mathbb{F}$.

① An element $a \in \mathbb{F}$ is an upper bound for A if $x \leq a \;\forall\; x \in A$.

② An element $b \in \mathbb{F}$ is a lower bound for A if $b \leq x \;\forall\; x \in A$.

③ A is said to be bounded if it has both an upper bound and a lower bound.

④ An element $M \in \mathbb{F}$ is a least upper bound for A if, ~~the can~~

(1). M is an upper bound for A and

(2). $\forall$ upper bounds m for A we have $M \leq m$

M is called supremum of A $\quad \sup A = M$.

⑤. An element $K \in \mathbb{F}$ is a greatest lower bound for A if;

(1) K is an lower bound for A and.

(2) $\forall$ lower bounds k for A we have $K \geq k$

K is called infimum of A.

$\inf A = K$.

eg: (i) $A = (0,1)$ $\qquad$ $\text{Inf } A = 0$

$\text{Sup } A = 1$

(ii) $A = (-\infty, 2]$ $\qquad$ $\text{Sup } A = 2$

Inf A: Does not exist.

Note: ~~Inf ~ Sup~~

* If $\text{Sup } A \in A$ then;

  $\text{Sup } A$ = maximum of A.

* If $\text{Inf } A \in A$ then;

  $\text{Inf } A$ = minimum of A.

Defn

An ordered field $\mathbb{F}$ is said to be complete if every non-empty subset S of $\mathbb{F}$ which is bounded above has the least upper bound in $\mathbb{F}$.

Defn

An ordered field $\mathbb{F}$ is said to be complete if every non-empty subset S of $\mathbb{F}$ which is bounded below has the greatest lower bound in $\mathbb{F}$.

Theorom

Let S be a non-empty subset of an ordered field $\mathbb{F}$ and $M \in \mathbb{F}$.

Then $M = \text{Sup } S$ if and only if (iff)

I. M is an upper bound for such

II. For any $\varepsilon \in \mathbb{F}$ with $\varepsilon > 0$ then.

Their exists an element $a \in S$.

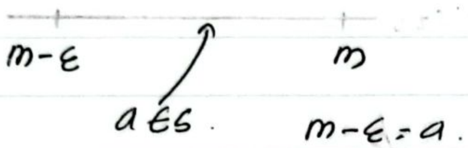
$$M - \varepsilon < a$$

$S \neq \phi$, $S \subseteq \mathbb{F}$

$m = \sup S$

Pf: Assume that $m = \sup S$
$S \subseteq \mathbb{F}$, $S \neq \phi$

then by def^n m is an upper bound for S.
If $\varepsilon \in \mathbb{F}$, $\varepsilon > 0$ for which $m - \varepsilon \geq a$ for all $(\forall)$ $a \in S$.



$m - \varepsilon$ is an upper bound for S.
But $m \geq m - \varepsilon$ and $m =$ least upper of S

$\therefore$ contradiction

$\therefore$ for some $a > m - \varepsilon$

assume that (i) and (ii) holds since S is bounded above.
S has a supremum say k by (i) m is an upper bound for S.

$\rightarrow k \leq m$

If $k < m$ then set $m - k = \varepsilon > 0$

an element $b \in S$ set
$m - \varepsilon < b \leq k$
$m - (m - k) \leq b \leq k$
$m - (m - k) < b \leq k$
$k < b \leq k$
$k < k$ #

$\rightarrow k = m$.

Theorem

Let A and B be non-empty subsets of $\mathbb{R}$ which are bounded above. then the set,

$S = \{ a+b \mid a \in A, b \in B \}$ is bounded above

$\sup S = \sup A + \sup B$

$\sup(A+B) = \sup A + \sup B$

Pf ÷ $\sup A$ → exists

$\sup B$ " exists.

Let $c \in S$ Then $c = a+b$

Set $a \in A$ and $b \in B$.

$\forall a \in A \quad a \leq \sup A \rightarrow (1)$.

$\forall b \in B \quad b \leq \sup B \rightarrow (2)$

(1) + (2)

$a+b \leq \sup A + \sup B \quad (\forall a \in A, b \in B)$

$c \leq \sup A + \sup B \rightarrow (3)$.

$\Rightarrow$ $\sup A$ $\sup B$ is an upper bound of S.

$\therefore$ S is bounded above set.

$\Rightarrow$ $\sup S$ exist $\rightarrow (4)$.

(3), (4) $\Rightarrow$

$\sup S \leq \sup A + \sup B \rightarrow (5)$

To show that

$\sup S \geq \sup A + \sup B$

Let $\varepsilon > 0$, $\varepsilon/2 > 0$

Then exist a element $x \in A$
set $\sup A - \varepsilon/2 < x \rightarrow (6)$

Similarly (11^{y})
$\varepsilon/2 > 0$

then exist an element $y \in B$, set $\sup B - \varepsilon/2 < y \rightarrow (7)$

(6) + (7).

$\sup A - \varepsilon/2 + \sup B - \varepsilon/2 < x + y$
$\sup A + \sup B - \varepsilon < x+y \rightarrow (8)$
$x+y \leq \sup S \rightarrow (1)$.

Since $x+y \in S$.
$\sup A + \sup B - \varepsilon \leq \sup S \rightarrow (10)$.

$\sup A + \sup B - \varepsilon \leq \sup S$
ε be an alone $\varepsilon \rightarrow 0$
$\sup A + \sup B \leq \sup S \rightarrow (11)$.

(5) + (11).
$\sup S = \sup A + \sup B$.

The Archimedean property of Real number.

Thm.

The set $\mathbb{N}$ of natural numbers is not bounded above.

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Pf: Assume that $\mathbb{N}$ is bounded above
$\therefore$ by completeness axion $\sup \mathbb{N}$ exists

Let $m = \sup \mathbb{N}$
Let $\varepsilon > 0$, $\varepsilon = 1$

Then exists an element $k \in \mathbb{N}$ set
$$m - \varepsilon < k \leq m.$$

$$m - 1 < k \leq m$$
$$m - 1 < k$$
$$m < k + 1 \leq m$$

but $k + 1 \in \mathbb{N}$.
$$m < m \quad \#.$$
$\therefore$ $\sup \mathbb{N}$ does not exists

Corrollary;
For every real number b there ~~is~~ exists an integer $m < b$.

Corrollary;
Given any real number x. There exists an integer k set $x - 1 \leq k \leq x$

Corollary;

If x and y are two positive real numbers th
There exists a natural number n set $nx > y$.

Pf: Assume that $nx \leq y$
$\forall\ n \in \mathbb{N}$

Then $n \leq y/x \quad \forall\ n \in \mathbb{N}$

$\Rightarrow$ $\mathbb{N}$ is bounded above.

as set of natural (N) numbers are unbounded.

eg: Consider the set.

$$A = \left\{ \frac{(-1)^n}{n} \mid n \in \mathbb{N} \right\}$$

(1) Show that A is bounded above. Find the supremum. Is the supremum maximum of A?

(II) Show that A is bounded below. Find the imflimum. Is this inflimum a minimum of A?

inflimum must be a element of A

Proof.

(i) Clearly $1/2$ is an upper bound of A.
$\therefore$ A is bounded above.

Let $M > 0$ be an another upper bound for A.
We must show that $1/2 \leq M$.

Suppose by contrarily;

$\cancel{M > 1/2}$ $M < 1/2$.

Since M is upper bound of A we have $\frac{(-1)^n}{n} \leq M$ for $\forall n \in \mathbb{N}$.

In particular $n = 2$,

$1/2 \leq M$

This implies that $\rightarrow$ $1/2 \leq M < 1/2$

$\therefore$ contradiction.

$\Longrightarrow$ $1/2 \leq M$.

$\Longrightarrow$ $\sup A = 1/2$

Since $1/2 \in A$, max of $A = 1/2$, $\therefore$ $\sup A = \max A$.

(ii) Clearly (-1) is a lower bound of A.

$\therefore$ A is bounded below.

Let $m < 0$ be any lower bound for A.

We must show that $m \leq (-1)$

Suppose by controrily;

$m > (-1)$

Since m is lower bound of A we have $\frac{(-1)^n}{n} \geq m$ $\forall n \in \mathbb{N}$

In particular $n = 1$;

$-1 \geq m$

$\Longrightarrow$ $-1 < m \leq -1$ #.

Contradiction.

$\Rightarrow m \geq (-1)$ and $m \leq (-1)$

$\text{Inf } A = (-1)$

$\therefore$ Since $-1 \in A$

$\min A = (-1) = \text{Inf } A$.

Topology of the Real line.

Def: Let x be any real number. The absolute value of x is defined by $|x|$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z}, q \neq 0 \}$

= set of rational numbers

$\mathbb{Q}' = \{ \pi, e, \sqrt{2} \ldots \}$

= set of irrational numbers.

$\pi \xrightarrow{\text{goes to}} \frac{22}{7}$.

①. Show that $\sqrt{2}$ is irrational.

Proof by contradiction.

Soln. Suppose that $\sqrt{2}$ is rational.

$\sqrt{2} = p/q \qquad p, q \in \mathbb{Z}$ and $q \neq 0$

relatively prime $\rightarrow (p, q) = 1$. = g.c.d.

If g.c.d = 1 ; the 2 numbers are relatively prime.
(greatest common divisor)   ex : (14, 17) g.c.d = 1

Now $\sqrt{2} = \frac{p}{q}$

$2 = \frac{p^2}{q^2}$

$2q^2 = p^2$

$\Rightarrow$ p^2 is even number

$\Rightarrow$ p must be even number.

2 divides q
& Also
$2|q = \frac{q}{2}$

$p = 2k$ $k \in \mathbb{Z}$ (some integer).

① $\Rightarrow$ $(2k)^2 = 2q^2$

$4k^2 = 2q^2$

$2k^2 = q^2$

$\Rightarrow$ q^2 is an even number

$\Rightarrow$ q is an even number.

~~$2|q$~~.

$2|p$ and $2|q$

but $(p, q) = 1$.

but we found that $(p, q) \neq 1$

contradiction.

$\sqrt{2}$ is irrational.

H.W Show that followers are irrational numbers.

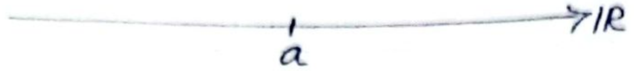
(i) $\sqrt{3}$

(ii) $\sqrt{2} + \sqrt{3}$

(iii) $\sqrt{2} + 7$

if $|a|<b$;
then $-b<a<b$

Open Sets and Closed sets.



$a \in \mathbb{R}$ and let $\varepsilon > 0$.

definitions

① An ε neighbourhood of 'a' is the set

$$N(a,\varepsilon) = \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \}$$

any point between this can be neighbourhood of a.

$-\varepsilon < x-a < \varepsilon$

$a-\varepsilon < x < \varepsilon + a$.

neighbourhood of a

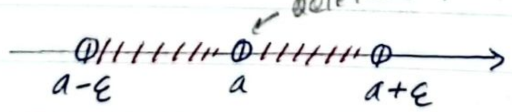
$N(a+\varepsilon, a-\varepsilon)$

② A deleted neighbourhood of 'a' is the set.

$$N^*(a,\varepsilon) = \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \} \setminus \{a\}$$

$$= (a-\varepsilon, a) \cup (a+\varepsilon, a)$$

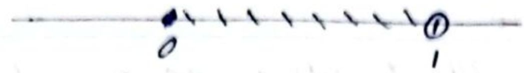
deleted.



③. A subset U of $\mathbb{R}$ is said to be open if for each $a \in U$ $\exists$ an $\varepsilon > 0$

$(a-\varepsilon, a+\varepsilon) \subset U$ $\qquad u = (0,1)$.

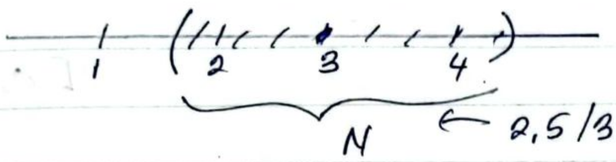
4) $u = [0, 1)$



0 cannot have neighbouhood which inside $[0,1)$

5) $\mathbb{N} \subseteq \mathbb{R}$

$\mathbb{N}$ is not a open set.
can not have a neighbourhood in a ~~Nn~~ bound.



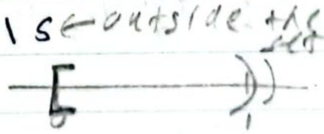
$\leftarrow$ 2.5 / 3.4 - not natural numbers.

$N \not\subseteq \mathbb{N}$

6. Let S be a subset of $\mathbb{R}$.

(i) $x \in S$ is called an interior point of S if exist an $\varepsilon > 0$. s.t $(x-\varepsilon, x+\varepsilon) \subset S$. The set of all interior points of a set S is denoted by S^o

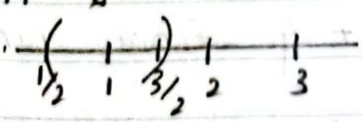
(ii). x is called boundary point of S if $\forall \varepsilon > 0$ the interval $(x-\varepsilon, x+\varepsilon)$ contains points of S as well as points of $\mathbb{R} \setminus S$ $\leftarrow$ outside the set
'if we take any ε'



(iii) $x \in S$ is called an isolated point of S if there existing an $\varepsilon > 0$ (positive) s.t $(x-\varepsilon, x+\varepsilon) \cap S = \{x\}$

$\mathbb{N}$

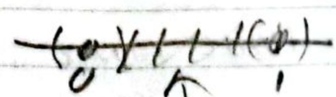
eg: ①



$\longrightarrow (1/2, 3/2) \cap \mathbb{N} = \{1\}$

1 is an isolated point.

②



no isolated point.

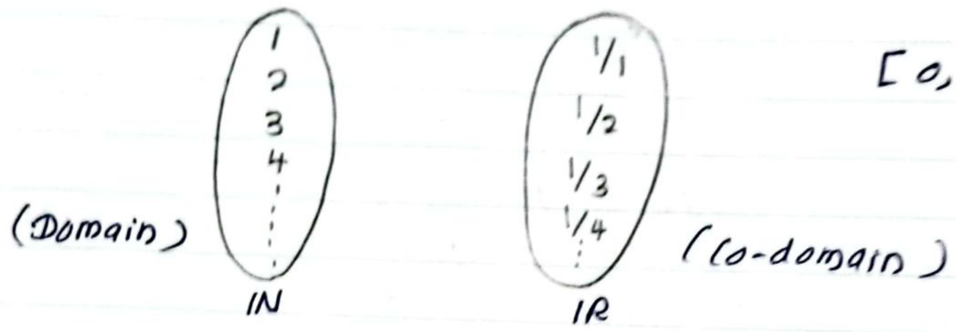
Sequence of Real Number.

Range of sequence

* The set of all distinct terms of sequence is called its range or range set.
* Range may be finite or infinite without ever being null set.

$$\{1, 1/2, 1/3, \ldots, 1/n, 1/(n+1), \ldots\}$$

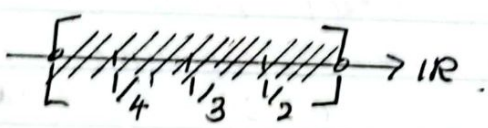
eg ÷ $\{1/n\}$



$[0,1] \subset \mathbb{R}$

$$\{1/1, 1/2, 1/3, \ldots, 1/n, 1/(n+1) \ldots\}$$

$$0 < a_n \leq 1 \quad \forall \; n \in \mathbb{N}.$$



Infinite range $= (0, 1]$.

eg ÷ $\{(-1)^n\} = a^n = \{-1, +1, -1, +1, +1\}$

Range of $\{a\} = \{-1, +1\}$ finite range

eg ÷ $\{3\} = \{3, 3, 3 \ldots\}$

Range of $\{3\} = \{3\}$ = finite range.

$\{1, 7, 3, -1, 7, 9, -2, \dots\}$.

$\uparrow \; \uparrow \; \uparrow \; \uparrow$

$a_1 \; a_2 \; a_3 \; a_5$

(This is sequence)

$\pi \neq 22/7$. $n \to \infty$ $1/n = 1/\infty \to 0$ $1/n \neq 0$.

eg :- a_n = last digit of 7^n

$a_1 = 7$ $a_5 = 7^4$

$a_2 = 9$ $a_6 = 9$

$a_3 = 3$ $a_7 = 3$

$a_4 = 1$ $a_8 = 1$

$\{7, 9, 3, 1, 7, 9, 3, 1 \dots\}$

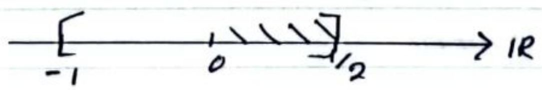
Range of $\{a_n\}$ = finite

(7, 9, 3, 1)

eg ÷ $\left\{\frac{(-1)^n}{n}\right\}$ $a_n = \frac{(-1)^n}{n}$.

$a_1 = (-1)$ $a_2 = 1/2$ $a_3 = -1/3$

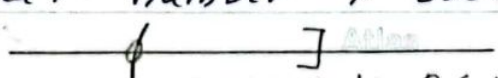
$\{-1, 1/2, -1/3, 1/4, -1/5, 1/6\}$.



Range $= [-1, 1/2] \setminus \{0\}$.

Bounded Sequnce.

* A sequence is said to be bounded if and only if its range set is bounded.
* A sequence $\{a_n\}$ is said to bounded above if $\exists$ a real number k such that $a_n \leq k \;\; \forall n \in \mathbb{N}$



$a_n \leq 1 \;\; \forall n \in \mathbb{N}$.

B seq $\{a_n\}$ is said to
~~$a_n > 0 \; \forall \; n \in \mathbb{N}$~~ bounded below if $\exists$ a real numb
k such that $a_n \geq k \;\; \forall \; n \in \mathbb{N}$.

$a_n > 0 \;\; \forall \;\; n \in \mathbb{N}$ bounded below. $0 < a_n \leq 1 \; \forall \; n \in \mathbb{N}$

eg: $\{1/n\} = \{1, 1/2, 1/3, \ldots\}$.

* A sequence is said to be bounded if it is both bounded above and bounded below.

eg: $\{(-1)^n\}$ $\qquad a_n = (-1)^n$

$-1 \leq a_n \leq 1 \;\; \forall \; n \in \mathbb{N}$.

* A sequence is said to be unbounded if it is not bounded.

eg: $\{2^n\} = \{2, 2^2, 2^3, \ldots\}$

bounded below.

$a_n \geq 2 \;\; \forall \; n \in \mathbb{N}$.

unbounded above.

eg: $\{-n^{+2}\} = \{1^2, -2^2, -3^2, -4^2 \ldots\}$.

bounded above $a_n \leq -1 \;\; \forall \; n \in \mathbb{N}$.

but not bounded

it is unbounded.

Monatonic Sequences.

A sequence is said to be monotonic if any of the following two conditions is satisfied.

(1) $a_{n+1} \geq a_n \;\; \forall \;\; n \in \mathbb{N}$ (monotonically increasing)

(2) $a_{n+1} \leq a_n \;\; \forall \;\; n \in \mathbb{N}$ (monotonically decreasing).

* If both (1) and (2) satistic simultaneously then it is called constant sequence.

$$eg \div \{1, +1, 2, 2, 3, 3 \ldots\}.$$

$$a_1 \leq a_2 < a_3 < a_4 \ldots$$

$$eg \div \{1, 2, 3, 4 \ldots\} \qquad a_1 < a_2 < a_3 < \ldots$$

* $a_n < a_{n+1} \ \forall \ n \in \mathbb{N}$ strictly monotonic increasing.

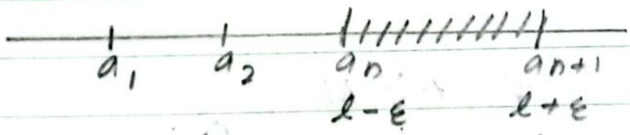
* $a_n > a_{n+1} \ \forall \ n \in \mathbb{N}$ strictly monotonically decreasing

$$eg \div \{3\} = \{3, 3, 3, 3 \ldots\}.$$

$$a_n \leq a_{n+1} \quad \forall \ n \in \mathbb{N}$$
$$a_n \geq a_{n+1} \quad \forall \ n \in \mathbb{N}.$$
$$a_n = a_{n+1} \quad \forall \ n \in \mathbb{N}.$$

def^n :-

$$a_1, a_2, a_3, \ldots, a_n, a_{n+1}.$$



$n \geq m$.

1. A sequence $\{a_n\}$ is said to converge to a real number l if, given $\varepsilon > 0$ there is a natural number N (dependent in ε). Such that.

$$|a_n - l| < \varepsilon \quad \forall \ n \geq N.$$

$$-\varepsilon < a_n - l < \varepsilon \quad \forall \ n \geq N.$$
$$l - \varepsilon < a_n < l + \varepsilon \quad \forall \ n \geq N.$$

$$\lim_{n\to\infty} a_n = l.$$

Q. ~~A seq~~ If the sequence $\{a_n\}$ does not converge to a real number we say that it diverges.

eg: Show that $\lim_{n\to\infty} 1/n = 0$.

sol^n: Let $\varepsilon > 0$ be given we must find $N \in \mathbb{N}$ such that

$$|1/n - 0| < \varepsilon.$$

- Consider

$$|1/n - 0| = 1/n$$
$$= 1/n < \varepsilon.$$

- $n > 1/\varepsilon$ $\quad |1/n - 0| < \varepsilon$

$$n \geq [1/\varepsilon] = N.$$

$\Rightarrow |1/n - 0| < \varepsilon \quad \forall \; n \geq N.$

such that $N = [1/\varepsilon]$

2023/05/24.

eg: Show that

$$\lim_{n\to\infty}\left(1 - 1/2^n\right) = 1.$$

$$a_n = 1 - 1/2^n$$

$$\lim_{n\to\infty} a_n = l.$$

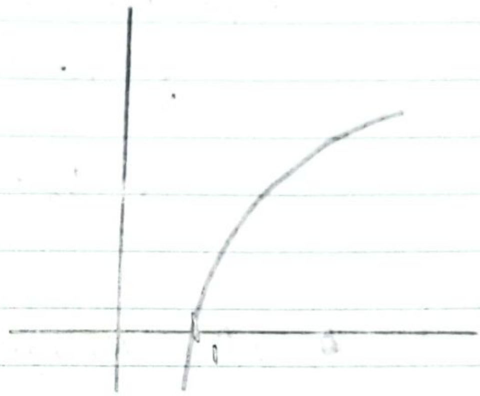
let $\varepsilon > 0$ be given consider;

$$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| = \left|\frac{-1}{2^n}\right| = \frac{1}{2^n} < \varepsilon$$

$$\frac{1}{2^n} < \varepsilon .$$

$$2^n > \frac{1}{\varepsilon} .$$

$$\log 2^n > \log \frac{1}{\varepsilon}$$



$$n \log 2 > \log \frac{1}{\varepsilon}$$

$$n > \frac{\log \frac{1}{\varepsilon}}{\log 2}$$

$$n \geq \left[\frac{\log \frac{1}{\varepsilon}}{\log 2}\right] = N .$$

$$\therefore \left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \varepsilon$$

whenever $n \geq N$.

when $N = \left[\frac{\log \frac{1}{\varepsilon}}{\log 2}\right]$

eg ÷ $(-1)^n$

Solution.

Assume that this sequence is converge to some real number l.

$$\lim_{n \to \infty} (-1) = l .$$

let us take
$\epsilon = 1/2$

then $\exists$ $N \in \mathbb{N}$.
$|(-1)^n - l| < 1/2$ $\forall$ $n \geq N$.

(i). $\forall$ $n \geq N$.

$= |(-1)^n - (-1)^{n+1}|$
$= |(-1)^n - l + l - (-1)^{n+1}|$
$\leq |(-1)^n - l| + |l - (-1)^{n+1}|$

$< 1/2 + 1/2 = 1$
$2 \leq 1$ #

$\therefore$ 1^n diverges.

Theorom.

let $\{a_n\}$ be a sequence of real numbers. If $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} a_n = l_2$ then $l_1 = l_2$.

[i.e: If a sequence of convergent then the limit of the sequence is unique].

Pf: let $\epsilon > 0$ be given.

since.
$\lim_{n\to\infty} a_n = l_1$

$\exists$ $N_1 \in \mathbb{N}$ such that
$|a_n - l_1| < \epsilon/2$ $\forall \geq N_1$ $\rightarrow$ ①.

$\lim_{n\to\infty} a_n = l_2$.

$\exists\ N_2 \in \mathbb{N}$ such that

$|a_n - l_2| < \varepsilon/2 \quad \forall\ n \geq N_2 \qquad n \geq N_2 \to ②$.

let $N = \max\{N_1, N_2\}$.

then $\forall\ n \geq N$.

$\Rightarrow$ ① $|a_n - l_1| < \varepsilon/2 \quad \forall\ n \geq N$.

$\Rightarrow$ ②. $|a_n - l_2| < \varepsilon/2 \quad \forall\ n \geq N$.

$$|l_1 - l_2| = |l_1 - a_n + a_n - l_2|$$

$$\leq |l_1 - a_n| + |a_n - l_2|$$
$$= |a_n - l_1| + |a_n - l_2|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \forall\ n \geq N.$$
$$= \varepsilon.$$

$\varepsilon > 0$.

$\Rightarrow \varepsilon \to 0$.

$|l_1 - l_2| = 0$.

$l_1 = l_2$.

Theorem

Every convergent sequence of real numbers is bounded.

i.e $\lim_{n \to \infty} a_n = l$.

$|a_n| \leq K \quad \forall$.

P.f. :- let $\{a_n\}$ be a sequence of real numbers which convergent to l

let $\varepsilon > 0$, with $\varepsilon = 0$. 1

then $\exists \; N \in \mathbb{N}$.

$|a_n - l| < 1 \quad \forall \; n \geq N$

Consider

$$|a_n| = |a_n - l + l|$$
$$\leq |a_n - l| + |l| \quad \forall \; n \geq N.$$
$$< 1 + |l| \quad \forall \; n \geq N.$$

$\{a_1, a_2, \dots, a_{N-1}, a_N, a_{N+1} \dots\}$.

$|a_n| < 1 + |l| \quad \forall \; n \geq N$.

let $M = \max\{a_1, a_2, \dots, a_{N-1}, (1+|l|)\}$.

$\Rightarrow |a_n| \leq M \quad \forall \; n \in \mathbb{N}$

$\{a_n\}$ is bounded.

No : Date: / /

Theorom : Sandwitch theorom.

Suppose that $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences such that

$$a_n \leq b_n \leq c_n \quad \forall \ n \in \mathbb{N}.$$

If $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} c_n = \lim$ 6 (ay)

then $\lim\limits_{n \to \infty} b_n = l$.

eg: $\lim\limits_{n \to \infty} \dfrac{\cos(n\pi/2)}{n^2} = 0.$

$$0 \leq \left| \frac{\cos(n\pi/2)}{n^2} \right| \leq 1/n^2$$

$$\left| \frac{\cos(n\pi/2)}{n^2} \right| \leq 1/n^2$$

$$\lim_{n \to \infty} 1/n^2 = 0.$$

$\lim\limits_{n \to \infty} 0 = 0$, $\lim\limits_{n \to \infty} 1/n = 0.$

by sandwich theorom.

$$\lim_{n \to \infty} \frac{\cos(n\pi/2)}{n^2} = 0.$$

Theorom : Sequence is monotonic and bounded then the sequence is convergent.

Let $\{a_n\}$ be monotonic and bounded sequence

(without lose of g) $\{a_n\}$ is increasing

$$a_{n+1} \geq a_n \quad \forall \ n \in \mathbb{N}.$$

Since $\{a_n\}$ is bounded.

$$|a_n| \leq k \quad \forall \ n \in \mathbb{N}.$$

$$-k \leq a_n \leq k \quad \forall \ n \in \mathbb{N}.$$

$\{a_n\}$ is bounded above.

let $\sup \{a_n\} = a$.

we claim that $\lim_{n \to \infty} a_n = a$.

let $\varepsilon > 0 \quad \forall \ n \geq N$.

$$a_n \geq a_N.$$

$$a_N \leq a_n \leq a \quad \forall \ n \geq N \quad n \in \mathbb{N}.$$

(increasing)

$\Rightarrow a_N \leq a_n \leq a + \varepsilon$ for $\varepsilon > 0$

$$a - \varepsilon < a_N \leq a_n < a + \varepsilon \quad \forall \ n \geq N.$$

$$a - \varepsilon < a_n < a + \varepsilon \quad \forall \ n \geq N.$$

$$|a_n - a| < \varepsilon \quad \forall \ n \geq N.$$

$\Rightarrow \lim_{n \to \infty} a_n = a$ //

eg: Let $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}}$

Bounded.

by M_2

$n = 1$, $x_2 = \sqrt{2} \leq 2$

Suppose.

$m = k$ $\quad x_k \leq 2 \quad \forall k$.

$x_2 \leq 2$

$x_k + 2 \leq 2 + 2$

$x_{k+2} \leq 4$

$\sqrt{x_{k+2}} \leq 2 \longrightarrow x_{k+1} \leq 2$

by Mathematical induction $x_n \leq 2 \quad \forall n \in \mathbb{N}$.

$0 \leq x_n \leq 2$

$\{x_n\}$ is bounded.

$$x_{n+1} - x_n = \sqrt{2 + x_n} - \sqrt{2 + x_{n-1}}$$

$$= \frac{2 + x_n - (2 + x_{n-1})}{\sqrt{2 + x_n} + \sqrt{2 + x_{n-1}}}$$

$$= \frac{(x_n - x_{n-1})}{\sqrt{2 + x_n} + \sqrt{2 + x_{n-1}}}$$

$x_1 = \sqrt{2}$

$x_2 = \sqrt{2 + \sqrt{2}}$

$x_2 - x_1 = \sqrt{2 + \sqrt{2}} - \sqrt{2} > 0$

$x_2 > x_1$

for $n = k$, $\quad x_k > x_{k-1}$

$x_k - x_{k-1} > 0$.

Now $n = k+1$

$$x_{k+1} = \sqrt{2 + x_{k-1}}$$

$$x_{k+1} - x_k = \frac{x_k - x_{k-1}}{\sqrt{2+x_k} + \sqrt{2+x_{k-1}}} > 0$$

$$x_{k+1} - x_k > 0$$

$$x_{k+1} > x_k$$

by MI $\quad x_{n+1} > x_n \quad \forall \; n \in \mathbb{N}$.

$\therefore \{x_n\}$ is increasing //

Since $\{x_n\}$ is monotonic and bounded
$\Rightarrow \{x_n\}$ is convergent.

let $\lim\limits_{n \to \infty} x_n = l$.

$$\lim_{n\to\infty} x_{n-1} = l.$$

$$x_n = \sqrt{2 + x_{n-1}}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{2 - x_{n-1}}$$

$$l = \sqrt{2 + l}$$

$$l^2 = 2 + l.$$

$$l^2 - l - 2 = 0$$

$$(l-2)(l+1) = 0$$

$$l = 2 \quad \text{or} \quad l = (-1)$$

$$\lim_{n\to\infty} x_n = 2 \; //.$$

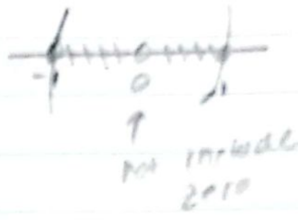
2023
05/31

Show that if $x \in \mathbb{R}$ and $|x| < 1$ then $\lim_{n \to \infty} x^n = 0$

Soln: If $x = 0$, then there is nothing to prove

Assume that $x \neq 0$, $-1 < x < 1$

$|x| < 1$



Since $|x| < 1$ we have $\frac{1}{|x|} > 1$

$\Rightarrow \exists$ real number $a > 0$ ← definitely a positive real number

$\frac{1}{|x|} = 1 + a \rightarrow ①$

Let $\varepsilon > 0$ be given.

we want to find $N \in \mathbb{N}$

$|x^n - 0| < \varepsilon \quad \forall n \geq N$

Consider $|x^n|$

$$|x^n - 0| = |x^n|$$
$$= |x|^n$$
$$= \frac{1}{(1+a)^n} \qquad (\text{by } ①)$$

$$\left[(1+a)^n = {}^nC_0 1^n a^0 + {}^nC_1 1^{n-1} a^1 + {}^nC_2 1^{n-2} a^2 + \cdots + {}^nC_n 1^0 a^n \right]$$

$$= \underbrace{1 + na + \frac{n(n-1)}{2!} a^2 + \frac{n(n-1)(n-2)}{3!} + \cdots + a^n}_{> na}$$

$$(1+a)^n > na$$

$$\frac{1}{(1+a)^n} > na.$$

$$|x^n - 0| < 1/_{na} \quad \text{by } ③.$$

$$|x^n - 0| < \varepsilon \quad \text{where } \frac{1}{na} < \varepsilon$$

$$n < \frac{1}{\varepsilon a}.$$

$$\therefore |x^n - 0| < \varepsilon \quad \forall\ n > 1/_{a\varepsilon}$$

Let $\left[1/_{a\varepsilon}\right] = N$

$$\Rightarrow |x^n - 0| < \varepsilon \quad \forall n \geq N$$

$$\Rightarrow \lim_{n \to \infty} x^n = 0$$

Limits and Continuity of Real valued functions

Limits :-

Left limit

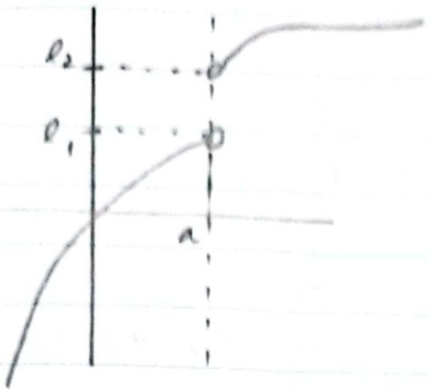
Let $f(x)$ be a real valued function. if x tends to 'a' from left side then $f(x)$ tends to l_1.

That is $\lim\limits_{x \to a^-} f(x) = l_1$

Right limit

Let $f(x)$ be a real valued function. If x tends to 'a' from right side then $f(x)$ tends to l_2

i.e : $\lim\limits_{x \to a^+} f(x) = l_2$



If $l_1 = l_2$

then $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$\therefore \lim_{x \to a} f(x)$ exists $\lim_{x \to a} f(x) = l$

f very complex valued fu^n is not a real valued fu^n

Def^n :- **$\varepsilon - \delta$ definition**

Suppose a and l are real numbers and $f(x)$ be a real valued fu^n. whose domain $D \subseteq \mathbb{R}$ includes all points in some open intervals about a (except possible the point a itself).

$$D \subseteq \mathbb{R} \to \mathbb{R}$$

Then l is called the limit of the fu^n $f(x)$ at a if given any $\varepsilon > 0$, $\exists$ $\delta > 0$ (δ depend on a and δ) such that,

$|f(x) - l| < \varepsilon$ whenever $|x-a| < \delta$

i.e : $\lim_{x \to a} f(x) = l$.

eg :- Show that $\lim_{x \to 2} x^2 = 4$

$f(x) = x^2$
$l = 4$
$a = 2$

we have to show that

sol^n :- ~~consider~~ $|x^2 - 4| < \varepsilon$

when $|x-2| < \delta$

$$\boxed{|ab| = |a|\,|b|}$$

Now

$$|x^2-4| = |(x+2)(x-2)|$$
$$= |x+2|\,|x-2|$$

Consider all x which satisfies inequality

$\Rightarrow$ $|x-2| < 1$, Then for all such x

$$-1 < x-2 < 1$$
$$1 < x < 3$$

$$|x+2| \leq |x| + 2 \quad \text{(by traingular inequality)}$$
$$< 3 + 2$$
$$= 5$$

$$|x+2| < 5$$
$$|x^2-4| = |x-2|\,|x+2|$$
$$< 5|x-2|$$
$$< \varepsilon \quad \text{whenever } 5|x-2| < \varepsilon$$
$$|x-2| < \varepsilon/5$$

let $\delta = \min\{1, \varepsilon/5\}$

$\therefore$ $|x^2-4| < \varepsilon$ whenever $|x-2| < \delta$

$\therefore$ $\lim_{x \to 2} x^2 = 4$

$2x+3-x-2 \quad x+1$

② . Show that $\lim_{x \to (-1)} \frac{2x+3}{x+2} = 1$.

$f(x) = \left(\frac{2x+3}{x+2}\right)$

$l = 1$

$a = (-1)$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Soln:

Consider

$$\left|\frac{(2x+3)}{(x+2)} - 1\right| < \varepsilon.$$

$$\left|\frac{(x+1)}{(x+2)}\right| < \varepsilon$$

$|x-(-1)|$

Now $\left|\frac{(2x+3)}{(x+2)} - 1\right| = \left|\frac{(x+1)}{(x+2)}\right| = \frac{|x+1|}{|x+2|} = \frac{|x-(-1)|}{|x+2|}$

Consider all x which satisfies the inequality

$|x-(-1)| < 1/3$ → This is a small interval between (-1)

$-1/3 < x+1 < 1/3$

$-4/3 < x < -2/3$

To find the best neighbourhood

$$|x+2| = |x-(-2)|$$
$$> \left|\frac{-4}{3} - (-2)\right| = 2/3$$

$\therefore |x+2| > 2/3$

$$\frac{1}{|x+2|} < 3/2$$

Now $\left|\frac{2x+3}{x+2} - 1\right| < 3/2 \, |x+1|$

$$\left|\frac{2x+3}{x+2} - 1\right| = \frac{|x+1|}{|x+2|}$$
$$< 3/2 \, |x+1|$$

Now $\left|\frac{2x+3}{x+2}-1\right| < \frac{3}{2}|x+1|$

$< \varepsilon$ whenever

$\frac{3}{2}|x+1| < \varepsilon$

$|x+1| < \frac{2\varepsilon}{3}$

$|x-(-1)| < \frac{2\varepsilon}{3}$

Let $\delta = \min\left\{\frac{2\varepsilon}{3}, \frac{1}{3}\right\}$

$\therefore |f(x)-1| < \varepsilon$ whenever $|x-(-1)| < \delta$
where $\delta = \min\left\{\frac{2}{3}\varepsilon, \frac{1}{3}\right\}$

$\therefore \lim_{x \to (-1)} f(x) = 1.$

<u>H.W.</u>

① $\lim_{x \to 3} (x^2+2x) = 15$

Consider

$|x^2+2x-15| = 0$

2003/06/07

① Show that $\lim_{x \to 0} f(x)$ does not exist where

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

This fn is not defined at zero. But limit may exist

soln. Suppose $\lim_{x \to 0} f(x) = l$. $l \in \mathbb{R}$

For given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x) - l| < \varepsilon$ where $|x - 0| < \delta$.

Assume $\varepsilon = 1$

$\Rightarrow |f(x) - l| < 1$ whenever $|x - 0| < \delta$

Let $x = -\delta/2$

$|x| < \delta$

$\delta/2 < \delta$

$|x| = |-\delta/2| = \delta/2$

according to this x

$|f(x) - l| < 1$

$|-1 - l| =$

$|1 + l| =$

$\therefore$ $|1 + l| < 1$

$\therefore$ $-1 < 1 + l < 1$

$-2 < l < 0$

expansion.

$|a| < 1$

$-1 < a < 1$

$\Rightarrow$ $2 > l > 0$

-2 0 2

no any value satisfy both.

But there is no real number that simultaneously satisfy the inequality $0 < l < 2$ and $-2 < l < 0$.

#.

② Show that $\lim_{x\to 0} x\sin 1/x = 0$

Sol^n Let $\epsilon > 0$ be given.
We need to find a $\delta > 0$, such that $|x\sin 1/x - 0| <$
$\forall\ |x-0| < \delta$

$\forall x \in \mathbb{R}$
$-1 \leq \sin\theta \leq 1$
$|\sin\theta| \leq 1$

Now consider $|x\sin 1/x - 0| = |x\sin 1/x|$
$= |x|\,|\sin 1/x|$

$\leq |x|$
$< \epsilon$ where $|x| < \epsilon$

$|x\sin 1/x - 0| < \epsilon$ whereas $|x-0| < \delta$
where $\delta = \epsilon$

Hence the proof.

③ $f: \underset{\text{domain}}{\mathbb{R}} \longrightarrow \underset{\text{codomain}}{\{0,1\}}$ given by.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

ex:-
$f(2) = 1$
$f(1/\sqrt{2}) = 0$

Show that. If $a \in \mathbb{R}$ then $\lim_{x\to a} f(x)$ does not exists. ← (use contradiction method)

Do later

NW

Theorem :- Uniqueness of limit.

Let f be a function which is defined on some open interval I containing a (except possibility at a) If ~~limit~~ $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} f(x) = l_2$,

then $l_1 = l_2$

$I = (x, y) \subseteq \mathbb{R}$

Suppose $l_1 \neq l_2$

let $\varepsilon = \dfrac{|l_1 - l_2|}{3} > 0$ Then $\exists$ $\delta_1 > 0$ and $\delta_2 > 0$

such that $|f(x) - l_1| < \varepsilon/2$ whenever $x \in I$ $|x-a| < \delta_1$

$|f(x) - l_2| < \varepsilon/2$ whenever $x \in I$ $|x-a| < \delta_2$

to satisfy both

Let $\delta = \min\{\delta_1, \delta_2\}$

Then whenever $|x-a| < \delta$

$$0 < |l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \leq |l_1 - f(x)| + |f(x) - l_2|$$

$$\begin{aligned} 0 < |l_1 - l_2| &\leq |l_1 - f(x)| + |f(x) - l_2| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \\ &= \frac{|l_1 - l_2|}{3} \end{aligned}$$

whenever $|x-a| < \delta$

$0 < |l_1 - l_2| < \dfrac{|l_1 - l_2|}{3}$ (contradiction) #

our assumption is wrong

$\therefore$ $l_1 = l_2$

Continuous Function.

Defn

ε-δ defn

Let $D \subseteq \mathbb{R}$ and $f(D) \to \mathbb{R}$
The function f is said to be continous at $a \in D$ such that $|f(x) - f(a)| < \varepsilon$ whenever $\forall x \in D$ and $|x-a| < \delta$

① Show that $f(x) = x^2$ is continous anywhere on,

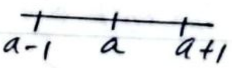
Soln :

Let $\varepsilon > 0$ be given and $a \in \mathbb{R}$ we need to find ~~prove~~ that $\delta > 0$.
such that
$$|f(x) - f(a)| < \varepsilon \quad \text{whenever } |x-a| < \delta$$

$\cancel{|x^2 < a^2 \delta|}$

$$|f(x) - f(a)| = |x^2 - a^2|$$
$$= |(x-a)(x+a)|$$
$$= |x-a||x+a|$$

Consider $|x-a| < 1$



$$-1 < x-a < 1$$
$$a-1 < x < a+1$$
definitely $\Rightarrow |x| < |a+1|$

Consider $|x+a| \leq |x| + |a|$
$$< |a+1| + |a|$$
$$\leq |a| + 1 + |a|$$
$$\cancel{\delta} = 2|a| + 1$$

$\Rightarrow$ ~~$|x+a| < 2|a+1|$~~

$\Rightarrow \quad |x+a| < 2|a|+1$

① $\Rightarrow$

$$|x^2-a^2| = |x+a||x-a|$$

$$< (1+2|a|)(|x-a|)$$

$$< \varepsilon \quad \text{where } (1+2|a|)(|x-a|) < \varepsilon$$

$$|x-a| < \frac{\varepsilon}{(1+2|a|)}$$

Let $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$

$\therefore \quad |x^2-a^2| < \varepsilon \quad$ whenever $|x-a| < \delta$

$\Rightarrow f(x) = x^2$ is continous $\forall\, a \in \mathbb{R}$.

① Show that $f(x) = \begin{cases} x\sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

is continous at $x = 0$.

Sol^n. Let $\varepsilon > 0$ be given we have to find $\delta > 0$

Consider

$$|f(x)-f(0)| = |x\sin 1/x - 0|$$

$$= |x\sin 1/x|$$

$$= |x||\sin 1/x|$$

$$\leq |x| \qquad \because |\sin 1/x| \leq 1$$

$$< \varepsilon \quad \text{whenever } |x| < \varepsilon$$

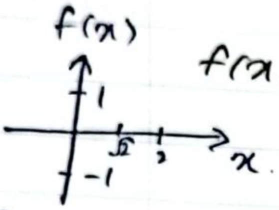
$$\delta = \varepsilon$$

$|f(x) - f(0)| < \varepsilon$ where $|x - 0| < \delta$ such that $\delta = \varepsilon$.

$\therefore$ $f(x)$ continuous at $x = 0$.

eg :- Show that the function

$F : \mathbb{R} \longrightarrow \{-1, 1\}$ given by.



$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$f(2) = 1$
$f(\delta) = (-1)$

is not continuous at every real number.

sol^n :-

Assume that f is continous at some $a \in \mathbb{R}$

Then given $\varepsilon = 1$
$\exists$ $\delta > 0$ such that
$|f(x) - f(a)| < \varepsilon$ whenever
$|x - a| < \delta$

let $\varepsilon = 1$

Then $|f(x) - f(a)| < 1$ whenever $|x - a| < \delta$

Since $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$ $\qquad \mathbb{Q}' = \mathbb{R} \setminus \mathbb{Q}$
the interval
$|x - a| < \delta$ contains both rationals and irrationals.

If $x \in \mathbb{Q}$;

$f(a) = 1$

$|1 - f(a)| < 1$

$\Rightarrow -1 < 1 - f(a) < 1$

$1 > f(a) - 1 > -1$

$2 > f(a) > 0$

$\Rightarrow \quad 0 < f(a) < 2$

$\therefore f(a) = 1$

If $x \in \mathbb{R} \setminus \mathbb{Q}$;

$f(x) = -1$

$|-1 - f(a)| < 1$

$-1 < -1 - f(a) < 1$

$1 > f(a) + 1 > -1$

$0 > f(a) > -2$

So #

$\therefore$ the function ~~f(x)~~ f is discontinous at every $x \in \mathbb{R}$.

eg:- $f(x) = 1/x$ Show that f is continous at $x = 1$.

$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

Soln. Let $\varepsilon > 0$ be given

We need to find $\delta > 0$ such that $|f(x) - f(1)| < \varepsilon$ whenever $|x - 1| < \delta$.

Consider

$|f(x) - f(1)| = |1/x - 1|$ whenever $|x-1| < \delta$

Let us fix $1/2$ such that $|x-1| < 1/2$.

$$-1/2 < x-1 < 1/2$$
$$1/2 < x < 3/2$$

$1/2 < x$
$1/x < 2$
$|1/x| < 2$

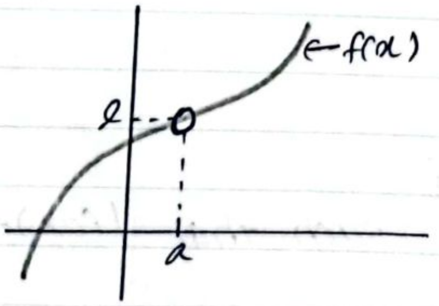
① $\Rightarrow$

$$|f(x) - f(1)| = |1/x - 1|$$
$$= \left|\frac{1-x}{x}\right|$$
$$= \left|\frac{x-1}{x}\right|$$
$$= \frac{1}{|x|}\,|x-1| < 2|x-1|$$
$$< \epsilon \quad \text{whenever}$$
$$2|x-1| < \epsilon$$
$$|x-1| < \epsilon/2$$

Let us choose
$$\delta = \min\{1/2, \epsilon/2\}$$

$\Rightarrow |f(x) - f(1)| < \epsilon$ wherever $|x-1| < \delta$

Differentiability



$\lim_{x \to a} f(x) = l$.

$\epsilon > 0 \quad \exists \delta > 0$.
$|f(x) - \delta| < \epsilon$ whenever
$|x-a| < \delta$

$f(a) \neq l$.

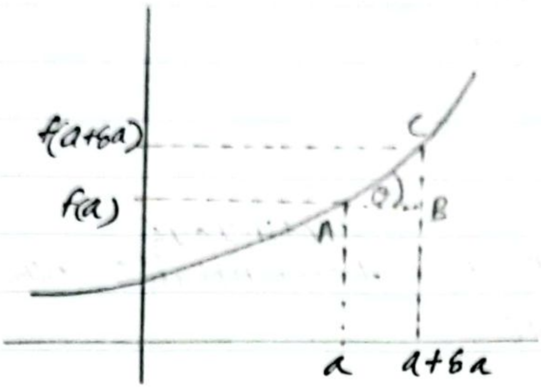
Here a limit exists at l. but $f(a) \neq l$.

At sharp points we can't draw tangents. There should be a curvature

- If limit exists, it is not continuous at always.
- If a function is continuous, a limit is exists

$\varepsilon > 0,\ \exists\, \delta > 0$

$|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$



$f(a) > 0$

$$\tan\theta = \frac{BC}{AB}$$

$$= \frac{f(a+\delta a) - f(a)}{(a+\delta a) - a}$$

$$\tan\theta = \frac{f(a+\delta a) - f(a)}{\delta a}$$

$$\lim_{\delta a \to 0} \frac{f(a+\delta a) - f(a)}{\delta a} = f'(a)$$

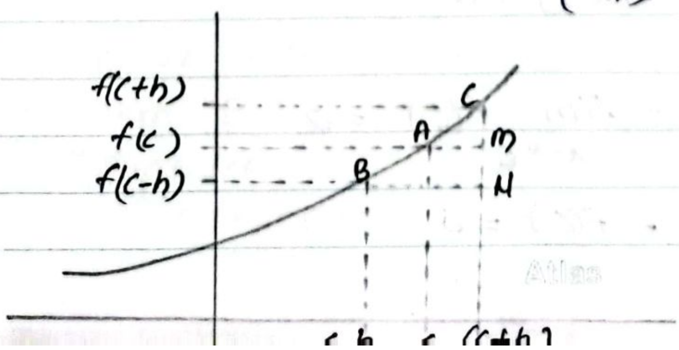
$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

Right hand deravative.

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = Rf'(c)$$

Left hand deravative.

$$Lf'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{(-h)}$$



$$Rf'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{c+h-c}.$$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$Lf'(c) = \lim_{h \to 0} \frac{f(c) - f(c-h)}{c-(c-h)}$$

$$= \lim_{h \to 0} \frac{f(c-h) - f(c)}{(-h)}$$

Theorem

Every differentiable ~~continous~~ fu^n is continuous ~~differentiable~~ but onverse is not true.

Proof

Let f be a differentiable at any point c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x-c} \quad \text{exists.}$$

Now consider $f(x) - f(c) = \frac{f(x) - f(c)}{(x-c)}(x-c)$

$$\lim_{x \to c}\left(f(x) - f(c)\right) = \lim_{x \to c} \frac{f(x) - f(c)}{(x-c)}(x-c)$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{(x-c)} \lim_{x \to c}(x-c)$$

$$= f'(c) \times \lim_{x \to c}(x-c)$$

$$= f'(c) \times 0$$

$$= 0.$$

$$\Rightarrow \lim_{x \to c}\left[f(x) - f(c)\right] = 0$$

$$\lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0$$

$$\lim_{x \to c} f(x) - f(c) = 0$$

$$\lim_{x \to c} f(x) = f(c)$$

$\therefore$ f is continuous at $x = c$.

Every continous fun is differentiable)?

- No
- counter example

$$f(x) = |x|$$

$x = 0$ $\quad f(0) = 0$

$$\frac{|x|}{x} = \begin{cases} 1 & \forall \ x > 0 \\ -1 & \forall \ x < 0 \end{cases}$$

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - 0}{x}$$

$$= \frac{|x|}{x}$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1 = Rf'(0)$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = (-1) = Lf'(1)$$

$$Lf'(0) \neq Rf'(0)$$

$\therefore$ f is not differentiable at $x = 0$.

for given $\varepsilon > 0$.

Consider.

$$|f(x) - f(0)| = ||x| - 0|$$

$$= ||x|| \quad \text{where } |x| < \varepsilon.$$

$$= |x|$$

$|f(x)-f(0)| < \varepsilon$ whenever $|x-0| < \delta$,
where $\delta = \varepsilon$

$\therefore$ f is continous at $x=0$.

Eg ÷ Show that the fu^n.

$$f(x) = \begin{cases} x\sin 1/x & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

is continous at $x=0$ but not differentiable at $x=0$.

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{(-h)}$$

$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-h\sin(-1/h) - 0}{(-h)}$$

$$= -\lim_{h\to 0} \frac{h\sin(1/h)}{h}$$

$$= -\lim_{h\to 0} \sin(1/h)$$

This is doesnot exists.

$\therefore$ $Lf'(0)$ DNE.

$\Rightarrow$ f is not differentiable at $x=0$

eg: Let $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$, $x \neq 0$

$f(0) = 0$.

Show that f is continous at $x=0$, and also show that f is differentiable at $x=0$.