Differential Equations.

Name of the Course :- Differential Equations I
Course Code :- MAT 1204
No of Credits :- 2

Assessment Strategy :-

Homework Assignments :- 15% (3 or 4)
Mid Semester Exam :- 25% (1 hour)
End Semester Exam :- 60% (2 hours) [Q4]

References :-
Simmous, G.F (2017)
Differential Equations with applications and historical notes (George Simmons)

Chapter 0 :- Preliminaries
Chapter 01 :- First order ordinary Differential Equations.
Chapter 02 :- Higher order Ordinary Differential Equations.
Chapter 03 :- First order nonlinear Differential Equations.

No. .... Date ....

Chapter 0 :- Preliminaries.

Consider the function $y = f(x) = 2x^2 + x$.

x = Independent Variable

y = Dependent variable.

$\frac{dy}{dx} = 4x+1$:- The first derivative of y

:- The rate of Change of x with respect to (w.r.t) x.

$\frac{d^2y}{dx^2} = 4$:- The second derivative of y.

Next, Consider the function.

$z = x^2 + y^2$.

This is a function of two variables. x and y are independent variable. z is the dependent variable.

$\frac{\partial z}{\partial x} = 2x$:- First partial derivative of z with respect to x. [This is obtained by holding y Constant) {∂ = ഡോ}.

$\frac{\partial z}{\partial y} = 2y$:- First partial derivative of z w.r.t y. (This is obtained by holding x Constant)

$\frac{\partial^2 z}{\partial x^2} = 2$:- Second partial derivative of z w.r.t x.

$\frac{\partial^2 z}{\partial y^2} = 2$:- Second partial derivative of z w.r.t y

No. .... Date. ....

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 2\left(\frac{\partial y}{\partial x}\right) \neq 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(2x) = 0$$

eg:- $w = x^2 + y^2 + z^2$ ← Constant.

$$\frac{\partial w}{\partial x} = 2x.$$

Differential Equations :-

An Equations involving one dependent variable and its derivatives w.r.t one or more independent variables is Called a Differential Equations. (DE)

Eg:- i) $\frac{dy}{dx} = 4x+1$

ii) $y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{\left(\frac{dy}{dx}\right)}$ (k- Constant).

iii) $2\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 10y = e^{-3x}\sin x.$

iv) $k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}.$

v). $y = x\frac{dy}{dx} + k\left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right]$

vi) $\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2}\left(\frac{dx}{dt}\right)^4 = e^t$ t- independent variable, x- dependent variable.

vii) $\frac{\partial^3 u}{\partial t^3} = k\left(\frac{\partial^2 u}{\partial x^2}\right)$

* There are two Classes of DE'S :-

01. Ordinary Differential Equations (ODE'S)
02. Partial Differential Equations (PDE'S).

Definition (Def^n) :- ODE'S

A DE which involves derivatives w.r.t a Single independent variable (i.v) is known as an Ordinary differential Equations (ODE).

eg:- (i) $\frac{d^2y}{dx^2} = -ky$

(ii) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 8y = 100 \sin(20x)$

iii) $y + 5\frac{dy}{dx} = y^2$

Def^n :- PDE'S

A DE which involves two or more independent variables and partial Derivatives w.r.t then is Called a PDE'S

eg:- i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

ii) $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

iii) $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

iv) $\frac{\partial^3 u}{\partial t^3} = k\left(\frac{\partial^2 u}{\partial x^2}\right)^2$

The n^{th} derivative of a function is said to be the derivative of order n.

$\frac{dy}{dx}$ = Derivative of order 1

$\frac{d^2y}{dx^2}$ = Derivative of order 2.

$\frac{d^3y}{dx^3}$ = Derivative of order 3.

Def^n :- Order of a DE.

The order of the highest order derivative involved in a DE is called the order of the DE

eg:- (i) $\frac{dy}{dx} = \cos x + \sin x$:- A first order DE.

ii) $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}}$:- order 1

$\boxed{\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2}$

iii) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$:- order 2.

iv) $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$:- order 2.

(v) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = e^x$:- order 3.

Def^n :- Degree of a DE

The degree of a DE is the degree of the highest order derivative present in the equation (eq^n) after the DE has been made free from the radicals and fractions as far as the derivatives are concerned.

radicals = square roots $(\sqrt{x}, \sqrt{y})$ / $\sqrt{\frac{dy}{dx}}$ $\left(\frac{d^2y}{dx^2}\right)^{3/7}$ Power

fractions = $\frac{1}{2}, \frac{1}{3}, \left(\frac{dy}{dx}\right)^{3/5}$.

Eg:-

(i) $x\left(\frac{d^2y}{dx^2}\right)^1 + \frac{dy}{dx} + xy = 0$ Order 2 degree 1

ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ order 2 degree 1

iii) $y = \sqrt{x}\,\frac{dy}{dx} + \frac{1}{\left(\frac{dy}{dx}\right)}$ order - 1 degree - 2.

iv) $y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + 1$ order - 1 Degree - 2.

(v) $\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\left(\frac{d^2y}{dx^2}\right)^2 = \frac{w^2}{H^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]$ order - 2 degree - 2.

Linear and non-linear differential Equations :-

A DE in which the dependent variables and all its derivatives present occur in the 1st degree only and no products of dependent variables and/or derivatives occur is known as a linear DE. that is linear DE we don't see them as $y^2, y^3, y^4, y\frac{dy}{dx}, y^2\frac{dy}{dx}, \left(\frac{d^4y}{dx^4}\right)^{4/3}, \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$ A DE which is not linear is Called non-linear DE.

Eg:- i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - x^2y = \cos x$.   y - dependent Variable
x - independent Variable

$\Rightarrow$ linear DE.

ii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y^2 = 0$. $\Rightarrow$ non-linear DE.

iii) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial y^2} = 0$ v - DV / x, y, z - IV

$\Rightarrow$ linear DE.

iv) $\frac{d^3y}{dx^3} + y\frac{dy}{dx} + (\sin y) = x^2$ $\Rightarrow$ non-lianear DE.

(v) $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}}$.

$y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$. $\Rightarrow$ non-linear DE

vi) $\frac{\partial^3 v}{\partial t^3} = k\left(\frac{\partial^2 v}{\partial x^2}\right)^2$ $\Rightarrow$ non-linear DE.

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Solution of a Differential Equations. (btw)

A Solution of a DE is ~~the~~ a relation between the dependent variable and independent variables not involving the derivatives such that this relation and the derivatives obtained from it satisfies the given DE.

Eg :- The function $y = 2e^{3x}$: $e^x : 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}, \dots$

Solution of the DE $\frac{dy}{dx} = 3y$. for all $x \in (-\alpha, \alpha)$ → Exponential fuctions

why,

Consider $y = \cancel{2x}\ 2e^{3x}$ $\left[\frac{dy}{dx}(e^x) = e^x\right]$

Differentially, we have $\cancel{\frac{dy}{dx} = 2e^x}$

$$\frac{d}{dx}\left[e^{f(x)}\right] = \frac{df}{dx} \times e^{f(x)} = f'(x)\, e^{f(x)}$$

$$\frac{dy}{dx} = 2(3e^{3x})$$

$$= 3(2e^{3x})$$

$$= 3y.$$

$$\frac{d}{dx} e^{ax} = a \cdot e^{ax}.$$

that is $y = 2e^{3x}$ is a ~~such~~ Solution of the DE $\frac{dy}{dx} = 3y$.

Formation of Differential Equations.

the problem of elimination gives as an idea as to what kind of Solution. a DE may have.

Eg :-

i) Consider $y = A\cos(px - \alpha)$ [This gives DE of Simple harmonic motion (SHM)]

Here A and α are arbitary Constant.

Let us eliminate A and α and from a DE.

$$y = A\cos(Px - \alpha)$$

$\frac{d}{dx}\cos f(x) = -f'(x)\sin f(x)$

Differentiating y w.r.t x

$\frac{d}{dx}\sin f(x) = f'(x)\cos f(x)$

$$\frac{dy}{dx} = A.P\left(-\sin(Px-\alpha)\right)$$

$$\frac{d^2y}{dx^2} = -AP\left[P.\cos(Px-\alpha)\right]$$

$$\frac{d^2y}{dx^2} = -P^2\underbrace{\left[A\cos(Px-\alpha)\right]}_{y}.$$

$$\frac{d^2y}{dx^2} = -P^2y.$$ [This DE does not include A and α].

$y = x^n = f(x)$

$\frac{d}{dx}f(x) = n.x^{n-1}$

Next eliminate P

$$y = A\cos(Px-\alpha)$$

$\cancel{\frac{d}{dx}[f(x)]^n = n f'(x)}$

we already have

$$\frac{d^2y}{dx^2} = -P^2y \quad \text{— ①}$$

Differentiating w.r.t x gives :-

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-P^2y\right)$$

$$\frac{d^3y}{dx^3} = -P^2\frac{dy}{dx} \quad \text{— ②}$$

$$\frac{②}{①} \Rightarrow \frac{d^3y}{dx^3}\Big/\frac{d^2y}{dx^2} = \frac{-P^2\frac{dy}{dx}}{-P^2y}.$$

$$\Rightarrow y\frac{d^3y}{dx^3} = \frac{dy}{dx}\frac{d^2y}{dx^2}.$$

H.w:-

find the DE of the family of Curves given $y = Ae^{2x} + Be^{-2x}$ (where A and B are orbirat
ely Constant).

$y = Ae^{2x} + Be^{-2x}$.

$\frac{dy}{dx} = A.2e^{2x} + B.(-2)e^{-2x}$.

$= 2Ae^{2x} - B2e^{-2x}$.

$= 2[Ae^{2x} - Be^{-2x}]$.

$\frac{d^2y}{dx^2} = 2\{A.2e^{2x} - B(-2)e^{-2x}\}$

$= 4\underbrace{[Ae^{2x} + Be^{-2x}]}_{y}$

$= 4y$

Chapter 01:-

An ODE of the first-order and first degree takes the form~~ation~~:-

$$\frac{dy}{dx} + f(x,y) = 0$$

this Cannot be slove for every function $f(x,y)$ But we shall discuss some possible cases. Some time it is Convenient to write the above DE in the form: $Mdx + Ndy = 0$ where $M = M(x,y)$ and $N = N(x,y)$ are functions of x and ~~f~~ y.

$$\frac{dy}{dx} + f(x,y) = 0$$

$$\frac{dy}{dx} = -f(x,y)$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = \frac{-M}{N}.$$

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No .... Date ....

$N\,dy = -M\,dx$.

$M\,dx + N\,dy = 0$.

$$\boxed{M\,dx + N\,dy = 0}$$

Separation of Variables :-

If the DE $M\,dx + N\,dy = 0$

Can be written in the form :

$$f_1(x)\,dx + f_2(y)\,dy = 0,$$

we say that the variables are separable.

In this case the DE is called a separable DE.

Eg :- ~~Slove~~ Solve $\frac{dy}{dx} = 2y$.

* Divided by zero is not possible.

Solution,

$$\frac{dy}{dx} = 2y.$$

$$\frac{dy}{y} = 2\,dx \quad \text{(separation for integrative)}$$

This is valid for $y \neq 0$

But we can obseve that by substitution $y=0$ on the DE it is a solution of the DE.

$$\frac{dy}{dx} = 2y \left\langle \begin{array}{l} y=0 \\ \frac{dy}{dx} = \frac{d(0)}{dx} = 0 \\ \\ 2y = 2\times 0 = 0. \end{array} \right.$$

Integrating both solves gives,

$$\int \frac{dy}{y} = \int 2\,dx$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + \text{Constant} \quad (\text{if } n \neq -1)$$

$$\int x^3 \, dx = \frac{x^4}{4} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C \qquad \ln x = \log_e x \quad (x > 0).$$

$$\Rightarrow \int \frac{dy}{y} = \int 2 \, dx.$$

$\ln|y| = 2x + C_1$ (where C_1 is a Constant).

$$|y| = e^{(2x+C_1)}$$

$$|y| = e^{C_1} \cdot e^{2x}$$

$$\boxed{\begin{aligned} \ln|a| &= b \\ |a| &= e^b \end{aligned}}$$

$$y = \pm e^{C_1} \cdot e^{2x}.$$

$$y = C_2 e^{2x} \qquad [C_2 = \pm e^{C_1}].$$

now for any real constant C_2

we have $y = Ce^{2x}$. is the solution of DE.

observe that when $C = 0$ we have $y = 0$.

$y = Ce^{2x}$ is Called a one-parameter family of Solutions of $\frac{dy}{dx} = 2e^{2x}$.

solve: $\frac{dy}{dx} = e^{x-y}$.

$$\frac{dy}{dx} = \frac{e^x}{e^y}.$$

$$e^y dy = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

when "c" is Constant.

Consider the ln equation both side

$$\ln(e^y) = \ln(e^x + c)$$

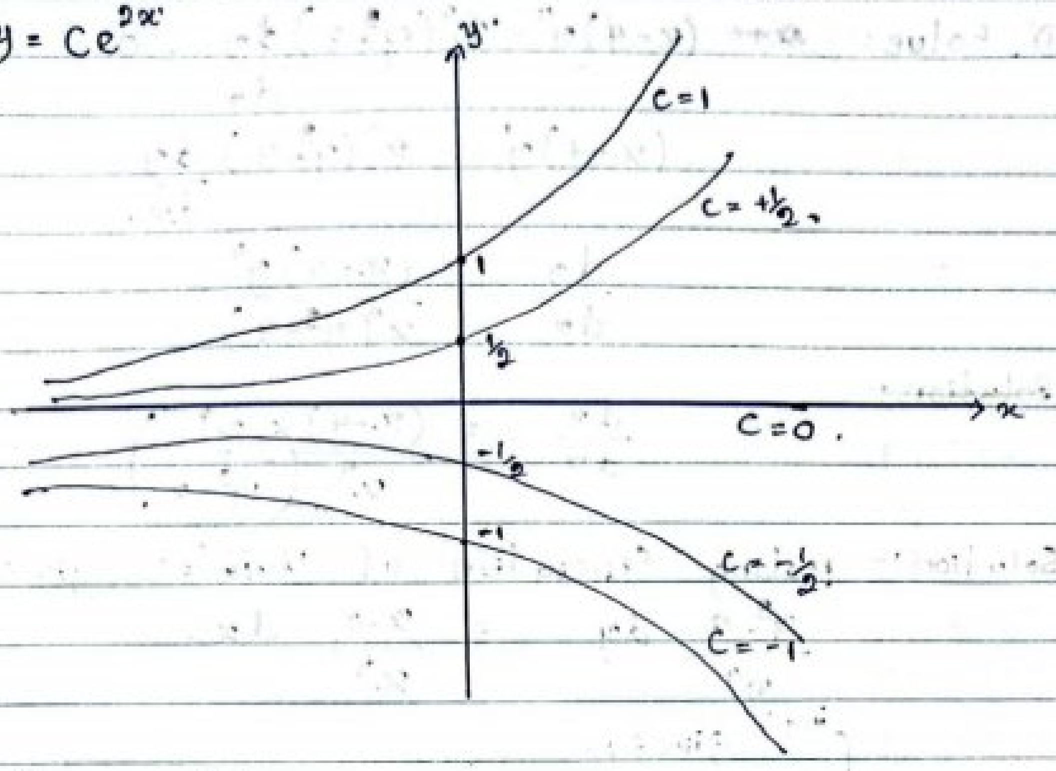
$$y \ln e = \ln(e^x + c)$$

$$y \times 1 = \ln(e^x + c)$$

$$y = \ln(e^x + c)$$

No. .... Date ....

$y = Ce^{2x}$



H.W:- $\frac{dy}{dx} = e^{x-y}$

$$\boxed{\int e^t\,dt = e^t + C}$$

Solution :-

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$

$$e^y dy = e^x dx$$

$\int^{ing}$ gives.

$$\int e^y dy = \int e^x dx$$

$$\boxed{\begin{aligned} e^y + C_1 &= e^x + C_2 \\ e^y &= e^x + \underbrace{C_2 - C_1}_{C} \end{aligned}}$$

$$e^y = e^x + C$$

where is an arbitrary constant.

$\Rightarrow \ln(e^y) = \ln(e^x + C)$

$y \ln_e e = \ln[e^x + C]$

$\underline{\underline{y = \ln[e^x + C]}}$

eg:-① Solve:- $(x-4)y^4 - x^3(y^2-3)\frac{dy}{dx} = 0.$

$$(x-4)y^4 = x^3(y^2-3)\frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

~~Solution:-~~

$$\frac{dy}{dx} = \frac{(x-4)}{x^3}\left(\frac{y^4}{y^2-3}\right)$$

Solution:- Making separation of varibles gives.

$$\frac{y^2-3}{y^4}\,dy = \frac{x-4}{x^3}\,dx.$$

$\int^{ing}$ gives:-

$$\int \frac{y^2-3}{y^4}\,dy = \int \frac{x-4}{x^3}\,dx.$$

$$\int \frac{y^2}{y^4}\,dy - 3\int \frac{1}{y^4}\,dy = \int \frac{x}{x^3}\,dx - 4\int \frac{1}{x^3}\,dx.$$

$$\cancel{\int \frac{1}{y}\,dy - 3\int y^{-4}\,dy = \int \frac{1}{x^2}\,dx - 4\int x^{-3}\,dx}$$

~~ln y~~

$$\int (y^{-2} - 3y^{-4})\,dy = \int (x^{-2} - 4x^{-3})\,dx.$$

$$\frac{1}{(-1)y} - \frac{3}{y^3(-3)} = \frac{1}{x(-1)} - \frac{4}{x^2(-2)} + C$$

$$\frac{-1}{y} + \frac{1}{y^3} = \frac{-1}{x} + \frac{2}{x^2} + C$$

where C is an arbitrary constant.

02). Solve $(x+1)\frac{dy}{dx} = x(y^2+1)$.

$$\frac{dy}{dx} = \frac{x(y^2+1)}{x+1}$$

$$\frac{dy}{dx} = \left(\frac{x}{x+1}\right) y^2+1$$

Solution :- Making separation of variables gives.

$$\left(\frac{1}{y^2+1}\right)dy = \left(\frac{x}{x+1}\right)dx$$

$$\boxed{\int \frac{dt}{1+t^2} = \tan^{-1}(t) + C}$$

$\int$ -ing gives :-

$$\int \frac{1}{y^2+1}\,dy = \int \frac{x}{x+1}\,dx$$

$$\int \frac{x+1-1}{x+1}\,dx.$$

$$\int 1\,dx - \int \frac{1}{x+1}\,dx$$

$$x - \ln|x+1|$$

$$\tan^{-1} y = x - \ln|x+1| + C$$

$$y = \tan(x - \ln|x+1| + C)$$

where "C" is an arbitary Constant //

H.W :- Solve (i) $(xy^2+x)dx + (yx^2+y)dy = 0$

(ii) $\frac{dy}{dx} = e^{x-y}$ if $y(0) = \ln 2$. (ivp - initial value Problem)

(iii) $\frac{dy}{dx} = 2x^2(y^2+1)$ if $y(0) = 1$

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No. Date.

Homogeneous Equations:-

Consider the DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x,y)$$

In DE is said to be Homogeneous if there exists a function g such that $f(x,y) = g\left(\frac{y}{x}\right)$

Eg:- $f(x,y) = \dfrac{x+y}{2x+y} = \dfrac{\left(\frac{x+y}{x}\right)}{\left(\frac{2x+y}{x}\right)} = \dfrac{\left(1+\frac{y}{x}\right)}{\left(2+\frac{y}{x}\right)} = g(v)$ $\quad v = \left(\frac{y}{x}\right)$

Let, $v = y/x = y \times \frac{1}{x}$

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

~~$\frac{dv}{dx} = y\frac{d}{dx}$~~

$$\frac{dv}{dx} = y \times (-1)x^{-2} + \frac{1}{x}\frac{dy}{dx}$$

$$x\frac{dv}{dx} = \left(-\frac{y}{x}\right) + \frac{dy}{dx}$$

$$\rightarrow x\frac{dv}{dx} = -v + \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

No. .... Date ....

$$v = \frac{y}{x}.$$

$$\Rightarrow y = vx.$$

$$\frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$= v\frac{dx}{dx} + x\frac{dv}{dx}.$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}.} \quad \text{—— (1)}.$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = f(x,y) = g\left(\frac{y}{x}\right) = g(v) \quad \text{—— (2)}.$$

Now, (1), (2) gives.

$$v + x\frac{dv}{dx} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}. \quad (\text{Separable form}).$$

Eg:-

Solve :- $y + (x-2y)\dfrac{dy}{dx} = 0.$

Solution:- The given DE can be written as:-

$$\frac{dy}{dx} = \frac{-y}{x-2y}$$

$$\frac{dy}{dx} = \frac{-y/x}{\frac{x-2y}{x}} = \frac{-y/x}{1-2\left(y/x\right)} \leftarrow \text{This is homogeneous form}$$

(1)

Let $y = vx$.
Diff^{ing} gives.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then we have :-, from ①.

$$v + x\frac{dv}{dx} = \frac{-v}{1-2v} \quad (\text{as } y/x = v).$$

$$x\frac{dv}{dx} = \frac{-v}{1-2v} - v$$

$$x\frac{dv}{dx} = \frac{-v-v+2v^2}{1-2v} = \frac{2v^2-2v}{1-2v} = \frac{2v(v^2-v)}{1-2v}.$$

$$x\frac{dv}{dx} = \frac{2(v^2-v)}{1-2v}$$

$$\frac{1-2v}{2(v^2-v)}dv = \frac{dx}{x}. \quad (\text{separable form}).$$

$$\frac{2v-1}{2(v-v^2)}dv = \frac{dx}{x}.$$

$$\frac{d(v-v^2)}{dv} = (1-2v) = -(2v-1)$$

$\int^{ing}$ gives :-

$$\int \frac{2v-1}{2(v-v^2)}dv = \int \frac{dx}{x}.$$

$$-\frac{1}{2}\ln|v-v^2| = \ln x + c_1$$

$$-\ln|v-v^2| = 2(\ln|x| + c_1) = 2(\ln|x| + \ln|c_1|)$$
$$= 2\ln|xc| = \ln|xc|^2 = \ln|x^2c^2|$$

$$\frac{+1}{v - v^2} = x^2 C^2$$

$$v - v^2 = \frac{1}{x^2 C^2}$$

$$\frac{y}{x} - \left(\frac{y}{x}\right)^2 = \frac{1}{x^2 A}, \quad \text{where } A = c^2.$$

H.W :- Solve :- i) $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2} = \frac{1 + (y/x)^2}{2}$

ii) $(x^2 - y^2)\, dx + 2xy\, dy = 0.$

Home work :-

i) $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0.$

$$y[x^2+1]\, dy = -x(y^2+1)\, dx$$

$$\frac{y}{y^2+1}\, dy = \frac{-x}{x^2+1}\, dx$$

Solution :- Making separation of variables gives,

$$\left(\frac{y}{y^2+1}\right) dy = \left(\frac{-x}{x^2+1}\right) dx.$$

$\int$ ing gives :-

$$\int \frac{y}{y^2+1}\, dy = -\int \frac{x}{x^2+1}\, dx.$$

$$\frac{1}{2}\int \frac{2y}{y^2+1}\, dy = -\frac{1}{2}\int \frac{2x}{x^2+1}\, dx.$$

$$\frac{1}{\cancel{2}} \ln|y^2+1| = -\frac{1}{\cancel{2}} \ln|x^2+1| + C$$

$$\ln|y^2+1| = -\ln|x^2+1| + C$$

$$\ln|y^2+1| = -\ln|x^2+1| + \ln|C|$$

$$\ln|y^2+1| = \ln\left|\frac{C}{x^2+1}\right|$$

$$y^2+1 = \frac{C}{x^2+1}$$

$$y^2 = \frac{C}{x^2+1} - 1$$

$$y^2 = \frac{C - x^2 - 1}{x^2+1}$$

$$y^2 = \frac{x^2 + C_1}{x+1}$$

$$y = \cancel{x} \pm \sqrt{\frac{C_1}{x+1}}$$

C_1 is an arbi

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(i) $\frac{dy}{dx} = e^{x-y}$

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(i) $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

$\frac{dy}{dx} = \frac{1 + (y/x)^2}{2}$ — ①

Let $y = vx$.

Deff^ing gives.

$\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Then we have from ①.

$v + x\frac{dv}{dx} = \frac{1+v^2}{2}$ (as $y/x = v$)

$x\frac{dv}{dx} = \frac{1+v^2}{2} - v \Rightarrow \frac{1+v^2-2v}{2} \Rightarrow \frac{(1 \mp v)^2}{2}$

$x\frac{dv}{dx} = \frac{(1-v)^2}{2}$

$\frac{2}{(1-v)^2}dv = \frac{dx}{x}$ (separable form).

$\int$^ing gives :-

$2\int\frac{1}{(1-v)^2}dv = \int\frac{1}{x}dx$.

$2\int(1-v)^{-2}dv = \ln x + C$,

$2\frac{(1-v)^{-1}}{(-1)\times(-1)} = \ln x + C$

$\frac{2}{1-v} = \ln x$

ii) $(x^2-y^2)\,dx + 2xy\,dy = 0.$

$$2xy\,dy = (y^2-x^2)\,dx.$$

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2-1}{2\left(\frac{y}{x}\right)} \quad \text{—} \; ①$$

Let $y = vx.$

Deff^{ing} gives:-

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then we have from ①

$$v + x\frac{dv}{dx} = \frac{\left(\frac{y}{x}\right)^2-1}{2\left(\frac{y}{x}\right)}$$

$$v + x\frac{dv}{dx} = \frac{v^2-1}{2v} \quad \text{as} \left(\frac{y}{x} = v\right)$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} - v \Rightarrow \frac{-(v^2+1)}{2v}$$

$$\frac{-2v}{v^2+1}\,dv = \frac{dx}{x}.$$

$\int^{ing}$ gives:-

$$-\int \frac{2v}{v^2+1}\,dv = \int \frac{1}{x}\,dx.$$

$$-\ln|v^2+1| = \ln|x| + C \Rightarrow \ln|x| + \ln|C_1| \Rightarrow \ln|xC_1|$$

$$\frac{1}{v^2+1} = \cancel{\ln}\; xC_1$$

$$v^2+1 = \frac{1}{xC_1}$$

$$\left(\frac{y}{x}\right)^2 = \frac{1}{xC_1} - 1 \quad //$$

Differential Equations Reducible to Homogeneous form:-

- we Consider the DE,

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

where a', b', c', a, b and c are constants

observe that when $c=0$ and $c'=0$,

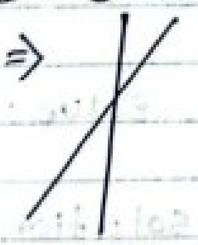
The DE reduces to $\frac{dy}{dx} = \frac{ax+by}{a'x+b'y} = \frac{a+b(y/x)}{a'+b'(y/x)}$,

which is a Homogeneous DE.

But when at least one of c and c' is non zero, the DE is not Homogeneous.

$$\boxed{\begin{array}{l} c=0 \text{ and } c'\neq 0 \\ c\neq 0 \text{ and } c'=0 \\ c\neq 0 \text{ and } c'\neq 0. \end{array}}$$

suppose that the lines $ax+by+c=0$ and $a'x+b'y+c'=0$ are not parallel $\Rightarrow$



to make the given DE Homogeneous, Let ~~$x = x+h$~~ ~~and~~ $x = X+h$ and $y = Y+k$, where X and Y are new variables and h and k constants yet to be chosen.

$$\left.\begin{array}{l} x = X+h \\ y = Y+k \end{array}\right\} \Rightarrow \begin{array}{l} dx = dX \\ dy = dY \end{array} \qquad \frac{dy}{dx} = \frac{dY}{dX}$$

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(h, k)

Now the given DE reduces to

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'} \quad \text{—} \ (2)$$

$$= \frac{aX+bY+ah+bk+c}{a'X+b'Y+a'h+b'k+c'}$$

In order to have eq (2) as a Homogeneous DE, Choose h and k such that the following equations are satisfied:

$$ah+bk+c = 0$$
$$a'h+b'k+c' = 0$$

Now, (2) $\Rightarrow$ $\dfrac{dY}{dX} = \dfrac{aX+bY}{a'X+b'Y}$

$$\frac{dY}{dX} = \frac{a+b(Y/X)}{a'+b'(Y/X)}$$

This can be solved using $Y = vX$

eg:- a. Solve: $(2x+y-3)\,dy = (x+2y-3)\,dx$

Solution:-

The DE can be written as

$$\frac{dy}{dx} = \cancel{2x+y-3} \ \frac{x+2y-3}{2x+y-3}$$

Let $x = X+h$ and $y = Y+k$, where h and k are constants to be determined.

Then, the DE becomes,

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3}$$

Date.

$$\frac{dy}{dx} = \frac{x+2y+(h+2k-3)}{2x+y+(2h+k-3)}$$

To find h and k ~~set~~ solve the equations

$h+2k-3=0$ ——①

$2h+k-3=0$ ——②

$h+2-3=0$

$\boxed{h=1}$

①×2 − ② ⟹ $2h+4k\cancel{-6}^{-6}-2h-k\cancel{+3}^{+3}=0$

$3k=3$

$\boxed{k=1}$

having solved, we have, $h=1$ and $k=1$

$\therefore\ X = x-1$ and $Y = y-1$

and $\dfrac{dY}{dx} = \dfrac{X+2Y}{2X+Y} = \dfrac{1+2(Y/X)}{2+(Y/X)}$

to solve the HDE put $Y = vX$

Then $\dfrac{dY}{dX} = \dfrac{d}{dX}(vX) \Rightarrow v\dfrac{dX}{dX} + X\dfrac{dv}{dX}$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Then from ⊛ we have:

$$v + X\frac{dv}{dX} = \frac{1+2v}{2+v}$$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$= \frac{1+2v-v(2+v)}{2+v}$$

$$= \frac{1+\cancel{2v}-\cancel{2v}-v^2}{2+v}$$

$$X\frac{dv}{dX} = \frac{1-v^2}{2+v}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$\therefore$ Separationg Variables we have $\int \frac{dx}{x} = \ln|x| + c$

$$\frac{v+2}{(1-v^2)}\,dv = \frac{dx}{x}$$

$\int$ ing gives

$$\int \frac{v+2}{1-v^2}\,dv = \int \frac{dx}{x}$$

using Partial fractions

$$\frac{v+2}{1-v^2} = \frac{v+2}{(1+v)(1-v)}$$

$$= \frac{A}{1-v} + \frac{B}{1+v}$$

$$= \frac{A(1+v) + B(1-v)}{(1-v)(1+v)}$$

$$v+2 = A(1+v) + B(1-v)$$

$v = -1 \Rightarrow 1 = 2B$ $\qquad$ $v = 1 \Rightarrow 3 = 2A$

$B = \frac{1}{2}$ $\qquad$ $A = \frac{3}{2}$

$$\therefore \frac{v+2}{1-v^2} = \frac{3/2}{1-v} + \frac{1/2}{1+v}$$

$$\Rightarrow \int \frac{3/2}{1-v}\,dv + \int \frac{1/2}{1+v}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{3}{2}\int \frac{1}{1-v}\,dv + \frac{1}{2}\int \frac{1}{1+v}\,dv = \ln|x| + \ln c$$

$$-\frac{3}{2}\int \frac{-1}{1-v}\,dv + \frac{1}{2}\int \frac{1}{1+v}\,dv = \ln|x| + \ln c$$

$$-\frac{3}{2}\ln|1-v| + \frac{1}{2}\ln|1+v| = \ln|x| + \ln c'$$

$$\boxed{\ln \frac{a}{b} = \ln a - \ln b}$$

$\ln ab = \ln a + \ln b.$

$$\ln\left|\frac{(1+v)^{1/2}}{(1-v)^{3/2}}\right| = \ln|xc|$$

$$\frac{(1+v)^{1/2}}{(1-v)^{3/2}} = xc$$

$$\frac{1+v}{(1-v)^3} = x^2c^2 \Rightarrow 1+v = (1-v)^3x^2c^2$$

Substituting $v = Y/X = \frac{y-1}{x-1}$

we get,

$$1+\frac{y-1}{x-1} = \left(1-\frac{y-1}{x-1}\right)^3(x-1)^2c^2$$

$$= \left(\frac{x-\cancel{1}-y+\cancel{1}}{x-1}\right)^3(x-1)^2c^2$$

$$x-1+y-1 = \left(\frac{x-y}{x-1}\right)^3(x-1)^2c^2$$

$$x+y-2 = \frac{(x-y)^3}{x-1}c^2$$

$$x+y-2 = \frac{(x-y)^3c^2}{(x-1)}$$

$x+y-2 = A(x-3)^3$, where $A=c^2$, arbitary Constan

Home work:-

Solve :- (i) $(2x-y-1)\,dx + (2y-x-1)\,dy = 0$

$\frac{dy}{dx} =$

(ii) $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

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(i) $(2y-x-1)\,dy = -(2x-y-1)\,dx.$

$$\frac{dy}{dx} = \frac{-2x+y+1}{2y-x-1}$$

Let $x = X+h$ and $y = Y+k$, where h and k are constants to be determined.

Then, DE becomes,

$$\frac{dY}{dX} = \frac{-2(X+h)+Y+k+1}{-(X+h)+2(Y+k)-1}$$

$$\frac{dY}{dX} = \frac{-2X+Y-2h+k+1}{-X+2Y-h+2k-1}$$

to fine h and k solve the equations

$-2h+k+1=0$ —— ①

$-h+2k-1=0$ —— ②.

① − ② × 2 ⇒ $-\cancel{2h}+k+1+\cancel{2h}-4k+2 = 0$

$-3k = -3$

$\boxed{k = 1}$

$-h+2-1=0$

$\boxed{h=1}$

having solved, we have, $h=1$ and $k=1$

∴ $X = x-1$ and $Y = y-1$

and $\frac{dY}{dX} = \frac{-2X+Y}{-X+2Y} = \frac{-2+(Y/X)}{-1+2(Y/X)}$

to solve the HDE put $Y = vX$

Then,

$$\frac{dY}{dX} = \frac{d}{dX}(vX)$$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

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Then, form we have,

$$\cancel{\frac{dy}{dx}} \quad v + x\frac{dv}{dx} = \frac{-2+v}{-1+2v}$$

$$\therefore x\frac{dv}{dx} = \frac{-2+v}{-1+2v} - v = \frac{-2+v-v(-1+2v)}{-1+2v}$$

$$x\frac{dv}{dx} = \frac{-2+v+v-2v^2}{-1+2v}$$

$$x\frac{dv}{dx} = \frac{-2+2v-2v^2}{-1+2v} = \frac{-2v^2+2v-2}{2v-1}$$

$\therefore$ Separating variables we have

$$\frac{2v-1}{-2v^2+2v-2} = \frac{dx}{x}$$

$\int^{ing}$ gives,

$$\int \frac{2v-1}{-2v^2+2v-2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\int \frac{-2(2v-1)}{-2v^2+2v-2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\int \frac{-4v+2}{-2v^2+2v-2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln|-2v^2+2v-2| = \ln|x| + \ln c$$

$$\ln\left|\frac{1}{(-2v^2+2v-2)^{1/2}}\right| = \ln|xc|$$

$$\frac{1}{(-2v^2+2v-2)^{1/2}} = xc$$

$$\frac{1}{-2v^2+2v-2} = x^2c^2 \Rightarrow 1 = (-2v^2+2v-2)x^2c^2.$$

Substituting $u = Y/X = \frac{y-1}{x-1}$

we get,

$$1 = \left[-2\left(\frac{y-1}{x-1}\right)^2 + 2\left(\frac{y-1}{x-1}\right) - 2\right](x-1)^2 C^2$$

$$1 = \left[\frac{-2(y-1)^2 + 2(y-1)(x-1) - 2(x-1)}{(x-1)^2}\right](x-1)^2 C^2.$$

(ii) $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

Let $x = X+h$ and $y = Y+k$, where h and k are Constants to be determined.

Then, The DE becomes.

$$\frac{dY}{dX} = \frac{Y+k-(X+h)+1}{Y+k+(X+h)+5}$$

$$\frac{dY}{dX} = \frac{Y-X+k-h+1.}{Y+X+k+h+5}$$

to fine h and k solve the equations

$k-h+1 = 0$ —— ①

$k+h+5 = 0$ —— ②

①+② $\Rightarrow$ $k-\cancel{h}+1+k+\cancel{h}+5 = 0$

$2k = -6$

$\boxed{k = -3}$

$-3-h+1 = 0$

$\boxed{-2 = h}$

having sloved, we have, $h = -2$ and $k = -3$

$\therefore X = x+2$ and $Y = y+3$

and $\dfrac{dY}{dX} = \dfrac{Y-X}{Y+X} = \dfrac{(Y/X)-1}{(Y/X)+1}$

to solve the HDE put $Y = vX$ Then;

$$\frac{dY}{dX} = \frac{d}{dX}(vX)$$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

Then form we have,

$$v + X\frac{dv}{dX} = \frac{v-1}{v+1}$$

$$X\frac{dv}{dX} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1}$$

$$X\frac{dv}{dX} = \frac{\cancel{v}-1-v^2-\cancel{v}}{v+1}$$

$$X\frac{dv}{dX} = \frac{-(v^2+1)}{v+1}$$

Separating variables we have,

$$\frac{-(v+1)}{v^2+1} = \frac{dX}{X}$$

$\int$ ing gives

$$-\int \frac{(v+1)}{v^2+1}\,dv = \int \frac{1}{X}\,dX.$$

$$-\frac{1}{2}\int \frac{2v+2}{v^2+1}\,dv = \int \frac{1}{X}\,dX.$$

* Suppose that $ax+by+c=0$ and $a'x+b'y+c'=0$ are parallel.

Then we have,

$$\frac{a}{a'} = \frac{b}{b'} = t \text{ (say)}$$

$a = a't$ and $b = b't$

Then, $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$

$$\frac{dy}{dx} = \frac{a'tx + b'ty + c}{a'x+b'y+c'}$$

$ax+by+c=0$

$y = \frac{-a}{b}x - c$

$y = \frac{-a'}{b'}x - c'$

$\frac{-a}{b} = \frac{-a'}{b'}$

$\frac{a}{a'} = \frac{b}{b'}$

$$\frac{dy}{dx} = \frac{(a'x+b'y)t+C}{(a'x+b'y)+C'}$$

Let $z = a'x+b'y$.

$$\frac{dz}{dx} = a' + b'\frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{b'}\left(\frac{dz}{dx} - a'\right)$$

Then we have :-

$$\frac{1}{b'}\left(\frac{dz}{dx} - a'\right) = \frac{zt+C}{z+C'}$$

$$\frac{dz}{dx} = b'\left(\frac{zt+C}{z+C'}\right)+a'$$

Eg:- Solve :- $\frac{dy}{dx} = \frac{y-x+1}{y-x+5}$

Note:- that the lines $y-x+1=0$ and $y-x+5=0$ are parallel.

Let $y-x = z$

$$\frac{dz}{dx} = \frac{dy}{dx} - 1$$

$$\frac{dy}{dx} = \frac{dz}{dx} + 1$$

Then we have,

$$\frac{dz}{dx} + 1 = \frac{z+1}{z+5} \Rightarrow \frac{z+1}{z+5} - 1$$

$$\frac{dz}{dx} = \frac{\cancel{z}+1-[\cancel{z}+5]}{z+5}$$

$$\frac{dz}{dx} = \frac{-4}{z+5}.$$

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Separating the variables gives;

$$(z+5)\,dz = -4\,dx.$$

$\int^{ing}$ gives,

$$\int (z+5)\,dz = \int -4\,dx.$$

$$\frac{z^2}{2} + 5z = -4x + C.$$

when C is an arbitatry constant.

* Linear Differential Equation

The linear DE of the first order takes the form:-

$$\frac{dy}{dx} + Py = Q \quad — (*)$$

where P and Q are functions of x alone.

$$P = P(x) \text{ and } Q = Q(x)$$

- A simple example is

$$\frac{dy}{dx} + \frac{1}{x}y = x^2 \quad — ①$$

Here $P = P(x) = \frac{1}{x}$ and $Q = Q(x) = x^2$.

multiplying ① by x gives:-

$$x\frac{dy}{dx} + y = x^3 \qquad \frac{d(yx)}{dx} = y\frac{dx}{dx} + x\frac{dy}{dx}.$$

$$\frac{d(yx)}{dx} = x^3. \qquad \frac{d}{dx}(yx) = y + x\frac{dy}{dx}.$$

$$d(yx) = x^3\,dx.$$

$$\int d(yx) = \int x^3\,dx.$$

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$$yx = \frac{x^4}{4} + C$$

$$y = \frac{x^4}{4} + \frac{C}{x}.$$

where C is an arbitrary Constant provided $x \neq 0$

Here x is Called, the intergrating factor.

Now Let us try to fine an intergrating factor R in general sense such that.

$$\frac{d}{dx}(yR) = R\left(\frac{dy}{dx} + Py\right)$$

$$\frac{d}{dx}(yR) = R\frac{dy}{dx} + RPy \quad \text{——} ①$$

Now multiplying ④ by R gives.

$$R\frac{dy}{dx} + RPy = RQ$$

$$\frac{d}{dx}(yR) = RQ$$

$\int^{ing}$ gives,

$$Ry = \int RQ\,dx + C$$

$$y = \frac{1}{R}\left[\int RQ\,dx + C''\right]$$

① Apply into product Rule,

$$R\cancel{\frac{dy}{dx}} + y\frac{dR}{dx} = R\cancel{\frac{dy}{dx}} + RPy.$$

$$\frac{dR}{dx} = RP \quad \text{if } y \neq 0 \quad \text{——} ②$$

$\int^{ing}$ ② w.r.t x gives.

$$\boxed{R = e^{\int P\,dx}.}$$

$$\frac{dR}{dx} = RP$$

$$\frac{dR}{R} = P\,dx.$$

$$\int \frac{dR}{R} = \int P\,dx.$$

$$\ln|R| = \int P\,dx.$$

$$R = e^{\int P\,dx}.$$

The factor $R = e^{\int P dx}$ is Called the intergrating factor of the DE. [Integrating factor => I.F].

$$\boxed{\begin{gathered}\frac{dy}{dx} + Py = Q \\ I.F = R = e^{\int P dx}\end{gathered}}$$

* To Solve the DE, multiplying the DE by R

Step 01 :-

$$\frac{dy}{dx} + \frac{1}{x} y = x^2$$

$$P = P(x) = \frac{1}{x}$$

$$I.F = R = e^{\int P dx} = e^{\int \frac{1}{x} dx}$$

$$= e^{\ln |x|}$$

$$= x$$

$$e^{\ln z} = a$$

$$\ln (e^{\ln z}) = a$$

$$\ln (e^{\ln z}) = \ln a$$

$$\log_e e^{\ln z} = \ln a$$

$$\ln z \log_e e = \ln a$$

$$\ln z = \ln a$$

$$z = a$$

$$\boxed{e^{\ln z} = z}$$

eg:- Solve $\frac{dy}{dx} + 2xy = 2e^{-x^2}$

Solution:-

$$P = 2x.$$

$$I.F = R = e^{\int P dx} = e^{\int 2x dx}$$

$$= e^{x^2}$$

$$\int 2x \, dx = \frac{\cancel{2}x^2}{\cancel{2}} = x^2$$

Now multiplying the DE by $R = e^{x^2}$ gives.

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2e^{x^2}e^{-x^2}$$

$$e^{x^2 + (-x^2)} = e^0 = 1$$

$$\frac{d}{dx}(ye^{x^2}) = 2$$

$\int^{ing}$ gives,

$$ye^{x^2} = 2x + C$$

$$y = e^{-x^2}(2x + C)$$

where C is an arbitary constant.

Eg:- Solve $(x+1)\dfrac{dy}{dx} - xy = 1 - x$

The given DE can be written as:-

$$\frac{dy}{dx} - \frac{x}{x+1}y = \frac{1-x}{x+1}$$

$$P = \frac{-x}{x+1} = \frac{-(x+1-1)}{x+1} = -1 + \frac{1}{x+1}$$

$$\text{I.F} = R = e^{\int P\,dx} = e^{\int\left(-1+\frac{1}{x+1}\right)dx}$$

$$-\int 1\,dx + \int \frac{1}{x+1}dx$$

$$-x + \ln|x+1|$$

$$= e^{-x+\ln|x+1|}$$

$$= e^{-x}.e^{\ln|x+1|}$$

$$= e^{-x}(x+1)$$

multiplying the DE by the I.F gives:-

$$(x+1)e^{-x}\frac{dy}{dx} - (x+1)e^{-x}\frac{x}{x+1}y = (1-x)e^{-x}$$

$$\frac{d}{dx}(y \times \text{I.F}) \quad \frac{d}{dx}\left[ye^{-x}(x+1)\right] = (1-x)e^{-x}$$

$\int^{ing}$ gives.

$$ye^{-x}(x+1) = \int(1-x)e^{-x}\,dx$$

$$= \int e^{-x}\,dx - \int xe^{-x}\,dx \text{ — } (*)$$

Now $(*) \Rightarrow$.

$$ye^{-x}(x+1) = -e^{-x} - \left[-xe^{-x} - e^{-x}\right] + C$$

$$= \cancel{-e^{-x}} + xe^{-x} + \cancel{e^{-x}} + C$$

$$= xe^{-x} + C$$

Or. $y(x+1) = x + ce^x$

Or $y = \frac{x}{x+1} + \frac{e^x}{x+1} C$ // where C is an arbitary constant. $x \neq -1$

Home work:-

Solve:- $y\,dx - x\,dy + \ln x\,dx = 0$

$$y - x\frac{dy}{dx} + \ln x = 0$$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{\ln x}{x}.$$

Bernoulli Equations:-

A first order DE of the form:-

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n;$$

where $n \in \mathbb{R}$ is called a bernoulli equations.

Note:- (i) when $n = 0$.

The BE becomes,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

which is a linear eq^n

(ii) when $n = 1$

The BE becomes,

$$\frac{dy}{dx} + P(x)y = Q(x)\,y.$$

which is also a linear DE

(iii) when $n \neq 0, 1$

The BE is a non linear DE.

To Solve the BDE we use the transformation $v = y^{1-n}$, where $n \neq 0, 1$

Multiplying the BDE by $v = y^{-n}$ gives,

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = d(x). \quad —①.$$

$$v = y^{1-n} \Rightarrow \frac{dv}{dx} = \frac{d}{dy}\left(y^{1-n}\right)\frac{dy}{dx}$$

$$= (1-n)\,y^{-n}\,\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y^{n}\frac{dv}{dx}.$$

on substituting in ① gives,

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)\,v = Q(x)$$

$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)\,Q(x)$$

is a linear equation in the form,

$$\frac{dy}{dx} + Py = Q(x)$$

To solve we may use the integrating factor methord.

Eg:-①. Solve $\dfrac{dy}{dx} + y = xy^3$

$$\boxed{\begin{array}{l} \dfrac{dy}{dx} + P(x)\,y = Q(x)\,y^{n} \\ \text{Transformation } v = y^{1-n}. \end{array}}$$

No. .... Date. ....

$$\frac{dy}{dx} + y = xy^3$$

Let $v = y^{1-n} = y^{1-3} = y^{-2}$ (since $n = 3$).

$$\frac{dv}{dx} = \frac{dv}{dy}\,\frac{dy}{dx} \quad \text{(Chain rule)}.$$

$$= \frac{d(y^{-2})}{dy}\,\frac{dy}{dx}$$

$$= -2y^{-3}\,\frac{dy}{dx}.$$

Multiplying the given BDE by y^{-3} gives,

$$y^{-3}\frac{dy}{dx} + y^{-2} = x$$

$$-\frac{1}{2}\frac{dv}{dx} + y^{-2} = x$$

$$\Rightarrow \frac{dv}{dx} - 2v = -2x \quad \text{---} \ (2).$$

This is a linear eq^n, with integrating factor

$\therefore$ $I.F = e^{\int p\,dx} = e^{\int -2\,dx} = e^{-2x}$.

Multiplying (2) ~~to~~ by e^{-2x} gives,

$$e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -2xe^{-2x}.$$

$$\frac{d}{dx}\left(e^{-2x}v\right) = -2xe^{-2x}.$$

Integrating by Parts ~~Product rule~~ formula.

$$\int u\,dv = uv - \int v\,du.$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\frac{d}{dx}(e^{-2x}) = -2e^{-2x}$$

$$d(e^{-2x}) = -2e^{-2x}\,dx.$$

$\int^{ing}$ gives

$$\Rightarrow v = e^{2x}\int -2xe^{-2x}\,dx.$$

$$v = e^{2x}\int x\underbrace{(-2e^{-2x}\,dx)}_{de^{-2x}}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$v = e^{2x}\int x\left(-2e^{-2x}\,dx\right).$$

$$\int x e^{-2x}\,dx = \int \underbrace{x}_{u}\, d\underbrace{\left[e^{-2x}\right]}_{v}$$

$$= xe^{-2x} - \int e^{-2x}\,dx$$

$$= xe^{-2x} + \frac{1}{2}e^{-2x} + C$$

Now integrating by parts gives,

$$v = e^{2x}\left(xe^{-2x} + \frac{1}{2}e^{-2x} + c\right)$$

$$= x + \frac{1}{2} + ce^{2x}$$

$$y^{-2} = x + \frac{1}{2} + ce^{-2x} \left(\text{since } v = y^{-2}\right).$$

$$y^2 = \frac{1}{\underbrace{\left(x + \frac{1}{2} + ce^{-2x}\right)}_{\neq 0}}$$

$$\therefore y = \pm\sqrt{\frac{1}{\left(x + \frac{1}{2} + e^{2x}\right)}}$$

$$= \pm\sqrt{\left(x + \frac{1}{2} + e^{2x}\right)^{-1}}$$ where c is an arbitary constant

This valids if $\left(x + \frac{1}{2} + e^{2x}\right) \neq 0$ //.

Home work:-

Solve:- $(1-x^2)\frac{dy}{dx} + xy = xy^2.$

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{x}{1-x^2}y^2 \quad (x \neq \pm 1)$$

No. .... Date. ....

Exact Differential Equations :-

Let $f(x,y)$ be a function of two real variables such that F has continuous first order partial derivatives in a domain D. the total differential, dF, of F is defined by,

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

for all $(x,y) \in D$

Let $F(x,y) = xy^2 + 2x^3y$

$$\frac{\partial F}{\partial x} = y^2 + 6x^2y$$

$$\frac{\partial F}{\partial y} = 2xy + 2x^3$$

Then, $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$

$$= (y^2+6x^2y)\,dx + (2xy+2x^3)\,dy.$$

$=$ total differential of F.

Definition :-

The differential form $M(x,y)\,dx + N(x,y)\,dy$ is said to be exact in a domain D if there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y) \text{ for all}$$

$(x,y) \in D$

$$\boxed{M(x,y)\,dx + N(x,y)\,dy.}$$

$$\frac{\partial F}{\partial x} \qquad\qquad \frac{\partial F}{\partial y}.$$

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF$$

This is, the total differential of F satisfies

$$dF(x,y) = M(x,y)dx + N(x,y)\,dy$$ If

$M(x,y)\,dx + N(x,y)\,dy$ is an ~~form,~~ exact form, then the DE.

$M(x,y)\,dx + N(x,y)\,dy = 0$ is Called an exact eq^n.

Consider the DE.

$$y^2 + 2xy\frac{dy}{dx} = 0.$$

$\Rightarrow$ $y^2\,dx + 2xy\,dy = 0.$

If this is an exact DE, then there must be a function $F(x,y)$ Such that.

$$\frac{\partial F}{\partial x} = y^2 = M(x,y)$$

$$\frac{\partial F}{\partial y} = 2xy = N(x,y).$$

Let ~~Ex~~ $F(x,y) = xy^2.$

$$\frac{\partial F}{\partial x} = y^2 \text{ and } \frac{\partial F}{\partial y} = 2xy.$$

$\therefore$ The DE is exact.

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No .... Date ....

Theorem 01 :-

Suppose that the first partial derivatives of $M(x,y)$ and $N(x,y)$ are Continuous is a domain D the DE,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact in D if and only if (iff),

$$\cancel{\frac{\partial y}{\partial x} = \frac{\partial u}{\partial y}} \qquad \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ far all (x,y) in D.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} \qquad \frac{\partial N}{\partial x}.$$

~~the~~ If $M(x,y)\,dx + N(x,y)\,dy = 0$ if exact

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0.$$

(F needs to be determined)

$\Rightarrow \quad dF = 0$

$F = F(x,y) = c$ (where c is an arbitary Constant).

This is the general solution of the exact DE

$$\boxed{\begin{array}{c} M(x,y)\,dx + N(x,y)\,dy = 0 \\ F(x,y) = c \end{array}}$$

a one parameter ~~formula~~ family of Cureves.

No. Date.

Examples:-

Show that the DE,

$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$ is exact and Solve it.

Solution:-

Here, $M = 3x^2 + 4xy$ and

$N = 2x^2 + 2y$

Then we have,

$\frac{\partial M}{\partial y} = 4x$ and $\frac{\partial N}{\partial x} = 4x$

Therefore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the DE is exact.

To solve the DE, we will find an $F(x,y)$ such that,

$$\frac{\partial F}{\partial x} = M = 3x^2 + 4xy \quad — ① \quad \text{and}$$

$$\frac{\partial F}{\partial y} = N = 2x^2 + 2y \quad — ②.$$

integrating partially ① w.r.t x gives.

$$F = x^3 + 2x^2y + \phi(y) \quad — ③.$$

Differentiating partially ③ w.r.t. y gives,

$$\frac{\partial F}{\partial y} = 2x^2 + \phi'(y)$$

$$= 2x^2 + 2y \quad (\text{by } ②).$$

$\Rightarrow \phi'(y) = 2y.$

No .... Date ....

$$\frac{d\phi}{dy} = 2y.$$

$$\phi = y^2 + C_1$$

Then, from ③ we have

$$F = x^3 + 2x^2y + y^2 + C_1'$$

There fore, the Solution of the DE is given implicitery by $x^3 + 2x^2y + y^2 = C$

∴ C is an arbitary constant.

∴ One parameter family of solution.

Homework :-

Solve 1. $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0.$ subject to the initial ~~Con~~ Condition $y(0) = 2$.

Solve (i) $(2xy + \sec^2 x)dx + (x^2 + 2y)dy = 0$

(ii) $(1 + e^x y + xe^x y)\cdot dx + (xe^x + 2)dy = 0.$

01) $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0.$

Solutions :-

Here :- $M = 2x\cos y + 3x^2y$ and

$N = x^3 - x^2\sin y - y$

Then we have,

$$\frac{\partial M}{\partial y} = 2x(-\sin y)1 + 3x^2$$
$$= -2x\sin y + 3x^2 \quad \text{and.}$$

$$\frac{\partial N}{\partial x} = 3x^2 - \sin y \cdot 2x$$
$$= 3x^2 - 2x\cdot\sin y.$$
$$= -2x\sin y + 3x^2$$

No. Date

There fore, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence, the DE is exact,

to solve the DE, we will find an $F(x,y)$ such that,

$$\frac{\partial F}{\partial x} = M = 2x\cos y + 3x^2 y \quad -① \text{ and.}$$

$$\frac{\partial F}{\partial y} = N = x^3 - x^2 \sin y - y \quad -②.$$

integrating partially ① w.r.t x gives;

$$F = \cos y \cdot x^2 + y \cdot x^3 + \phi(y) \quad -③.$$

Differentiating partially ③ w.r.t y gives.

$$\frac{\partial F}{\partial y} = x^2(-\sin y) + x^3 + \phi'(y)$$

$$= x^3 - x^2 \sin y + \phi'(y) \quad -④.$$

② = ④

$$x^3 - x^2 \sin y - y = x^3 - x^2 \sin y + \phi'(y)$$

$$\therefore \phi'(y) = -y.$$

$$\frac{d\phi}{dy} = -y.$$

$$\phi = \frac{-y^2}{2} + C_1$$

Then from ③ we have,

$$F = x^2\cos y + x^3 y - \frac{y^2}{2} + C_1.$$

$$2F = 2x^2\cos y + 2x^3 y - y^2 + C_1$$

$$\therefore 2x^2\cos y + 2x^3 y - y^2 = C.$$

C is an arbitary Constant.

$y(0) = 2.$

$$\underbrace{2x^2\cos(0)}_{1} + 2x^3(0) - y\,0^2 = 2.$$

$$2x^2 + 0 = 2$$

$$x^2 = 1$$

$$x = \pm 1$$

02) (i) $(2xy + \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0.$

Solutions:-

Here, $M = 2xy + \sec^2 x$ and

$N = x^2 + 2y$

Then we have,

$$\frac{\partial M}{\partial y} = 2x \quad \text{and.}$$

$$\frac{\partial N}{\partial x} = \cancel{2x}\ \cancel{x^2}\ 2x$$

Therefore, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hense, the DE is exact,

to Solve the DE, we will find an $F(x,y)$ Such that,

$$\frac{\partial F}{\partial x} = M = 2xy + \sec^2 x \quad \text{—} ① \quad \text{and.}$$

$$\frac{\partial F}{\partial y} = N = x^2 + 2y \quad \text{—} ②.$$

integrating partially ① w.r.t. x gives.

$$F = x^2 y + \tan x + \phi(y) \quad \text{—} ③.$$

Differentiating partially ③ w.r.t y gives.

$$\frac{\partial F}{\partial y} = \cancel{2x}\ x^2 + \phi'(y) \quad \text{—} ④$$

② = ④ $\Rightarrow$ $x^2 + 2y = x^2 + \phi'(y)$

$$\therefore \phi'(y) = 2y.$$

No. Date

$$\frac{d\phi}{dy} = 2y.$$

$$\phi = y^2 + C_1$$

Then from ③ we have,

$$F = x^2y + \tan x + y^2 + C_1$$

$$x^2y + \tan x + y^2 = C. \quad C \text{ is an arbitary Constant.}$$

(ii). $(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0.$

Solutions :-

Here, $M = 1 + e^x y + xe^x y$ and.

$N = xe^x + 2$

Then we have,

$$\frac{\partial M}{\partial y} = e^x + xe^x \quad \text{and.}$$

$$\frac{\partial N}{\partial x} = x.e^x \times 1 + e^x$$
$$= e^x + xe^x$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hense, the DE is exact,

to ~~Slate~~ Solve the DE, we will find an $F(x,y)$ Such that,

$$\frac{\partial F}{\partial x} = M = y.e^x \times 1 + y[xe^x + e^x]$$
$$= 2e^x y + xye^x. \quad \text{—— ①}.$$

$$\frac{\partial F}{\partial y} = N = x.e^x + e^x \quad \text{—— ②}.$$

Integrating Partially ① w.r.t x gives,

$$F = 2ye^x + y.\int xe^x\, dx.$$

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No. .... Date. ....

Chapter 02 :- Higher Order Linear Differential Equations.

Definition :-

A linear DE of order n is an equation of the form :-

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x) y = Q(x) \quad \text{—} ①$$

where,

$a_0(x), a_1(x), \dots, a_n(x)$ and $Q(x)$ are Continuous real functions on a Common interval I and $a_n(x) \neq 0$ for all $x \in I$

The right side of (01), $Q(x)$ is Called the non homogeneous term of the DE.

If $Q(x)$ is identically zero
($Q(x) \equiv 0$:- $Q(x) = 0$ for all $x \in I$),
then ① is Called a homogeneous DE of order n.

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x) y = 0 \quad \text{—} ②$$

Equation ② is Called a homogeneous DE of order n.

If each Coefficient $a_i(x)$ in ① is Constant, then it is Called a linear D.equations with Constant Coefficients.

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Let 'D' stand for $\frac{d}{dx}$

Then we may have,

D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$, and so on.

$$D \equiv \frac{d}{dx}, \quad D^2 \equiv \frac{d^2}{dx^2}, \quad D^3 \equiv \frac{d^3}{dx^3}, \ldots\ldots, \quad D^n \equiv \frac{d^n}{dx^n}.$$

Let, $y = x^3 + 2x^2 + 1$

$Dy = 3x^2 + 4x$.

$D^2y = DDy = D(3x^2+4x) = 6x+4$.

$D^3y = DD^2y = D(6x+4) = 6$

$D^4y = DD^3y = D(6) = 0$.

Now, ① can be written, using D operator as follows:

$$a_n(x)D^ny + a_{n-1}(x)D^{n-1}y + \ldots + a_1(x)Dy + a_0(x)y = Q(x).$$

$$\underbrace{\left(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x)\right)}\,y = Q(x)$$

$$\boxed{P(D)y = Q(x)};$$

where, $P(D) = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_n(x)D + a_0(x)$

Note:-

i) D is a linear operator.

i.e, $D(\alpha f + \beta g) = \alpha Df + \beta Dg$.

for any scalars (numbers) α and β.

ii) $P(D)$ is a linear operator

i.e. $P(D)(\alpha f + \beta g) = \alpha P(D)f + \beta P(D)g$

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In this Chapter we Consider Constant Coefficient linear Equations

We first Consider Constant Cofficient Homogeneous linear differential equations. That is ;

$$(a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0) y = 0 \quad \text{(1)}$$

$a_n, a_{n-1}, \dots, a_1, a_0$ are constant

$$D^{-1} x = \frac{1}{D} x = \int x \, dx = \frac{x^2}{2}.$$

$$\frac{1}{D^2} = \iint.$$

Suppose that or possible solution of the homogeneous DE :-

$$(a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0) y = 0 \longrightarrow (3).$$

is $y = e^{rx}$, where r is a real/Complex numbers.

Then, $Dy = r e^{rx}$

$D^2 y = r^2 e^{rx}$

$D^3 y = r^3 e^{rx}$ and

$D^k y = r^k e^{rx}, \quad k = 0, 1, 2, \dots$

Then (3) $\Rightarrow$

$$\underbrace{(a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0)}_{\text{A polynomial of Degrees } n.} e^{rx} = 0$$

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Since, $e^{rx} \neq 0$ for any r and x, we have,

$$\underbrace{a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0}_{P(r)} = 0. \quad \text{— (4)}.$$

$P(D) = D^2 + 3D + 1$ A polynomial of Degree n.

$P(r) = r^2 + 3r + 1$

$\Rightarrow$ $P(r) = 0$.

obtained by replacing D in $P(D)$ by r.

$P(r) = a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$

Equations (4) is Called auxiliary / Characteristic equation.

Its roots are Called eigenvalues or characteristic values.

$x^2 + 7x + 12 = 0$

$(x+3)(x+4) = 0$

$x = -3$ or $x = -4$.

Then -3 and -4 are Roots of $x^2 + 7x + 12 = 0$.

we denote the roots of $P(r) = 0$ by $r_1, r_2, r_3, \ldots, r_n$

So then each function $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x_2}$, ...,

$y_n = e^{r_n x}$ is a Solutions of (3), $P(D)y = 0$.

while Solving the auxiliary (A.E) $P(r) = 0$; the following, three Case's may occur.

$x = -3, 4$

$-3 \neq 4$. $x = 1, -1$

$-1 \neq 1$

01. All the roots are distinct and real

02. All the roots are real but some are repeating

03. All the roots are Complex / imaginary.

$x^2 = i^2$

$x = \pm i$.

Case 01 :-

Let $r_1, r_2, r_3, \ldots, r_n$ be n distinct roots of $P(r)=0$

Then, the solutions,

$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$.

are linearly independent.

$$C_1 e^{r_1 x} + C_2 x^{r_2 x} + \ldots + C_n e^{r_n x} = 0$$

$$C_1 = C_2 = C_3 = \ldots = C_n = 0.$$

Then, the general solutions. of $P(D)y = 0$. is given by

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \ldots + C_n e^{r_n x},$$

where, $C_1, C_2, \ldots, C_n$ are arbitrary constants

Sometimes, this solutions is called Complementary function and denoted by y_c (later use).

Example 01.

Consider DE $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$.

This is a second order homogeneous DE.

The DE can be written interms of D ~~of~~ as,

$$(D^2 - D - 6)y = 0.$$

Here $P(D) = D^2 - D - 6$

$\Rightarrow P(r) = r^2 - r - 6 \leftarrow$ Characteristic polynomial.

Then the auxiliary / Characteristic eq^n ~~g~~ is given by

$$r^2 - r - 6 = 0.$$

$$(r-3)(r+2) = 0$$

$$r = -2 \text{ or } r = 3.$$

$(r-3)(r+2) = 0$

$r = 3$ / $r = -2$.

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~~Thus, e^{-2x} and e^{3x}~~

Then we may show that e^{-2x} and e^{3x} are linearly independent solutions of the given DE.

$\therefore$ The general solutions is

$$y = C_1 e^{-2x} + C_2 e^{3x}.$$

where C_1 and C_2 are arbitary constant.

Example:-

Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0.$

Solutions:-

The DE can be written interm of D as;

$$(D^3 - D^2 - 6D)y = 0.$$

Here $P(D) = D^3 - D^2 - 6D$

$\Rightarrow P(r) = r^3 - r^2 - 6r$

Then the auxiliary eq^n is given by.

$$r^3 - r^2 - 6r = 0$$

$$r(r^2 - r - 6) = 0$$

$$r(r-3)(r+2) = 0.$$

$$r = 0, \quad r = 3 \quad \text{or} \quad r = -2.$$

Then we may show that e^0, e^{3x} and e^{-2x} are linearly independent sol^n of the given DE.

$\therefore$ the general solutions is.

$$y = C_1 e^{0x} + C_2 e^{-2x} + C_3 e^{3x}.$$

$$y = C_1 + C_2 e^{-2x} + C_3 e^{3x}.$$

where C_1, C_2, and C_3 are arbitary constant.

No. Date

Case 02 :- Repeated (equal) roots.

Consider the DE :-

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

The auxiliary eq^n is

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

$\Rightarrow$ $r = 3$ or $r = 3$ (double roots).

$f_1, f_2, \dots, f_n$ are function's;

$C_1f_1 + C_2f_2 + \dots + C_nf_n$ is Called a linear Combination of $f_1, f_2, \dots, f_n$.

If $f_1, f_2, f_3, \dots, f_n$ are linear independant.

The,

$C_1f_1 + C_2f_2 + \dots + C_nf_n = 0$

has only the trivial $Solu^n$:-

$C_1 = 0$, $C_2 = 0$, ..., $C_n = 0$.

$P(D)y = 0$ has n linearly independent Solutions. $y_1, y_2, y_3, \dots, y_n$

$y = C_1y_1 + C_2y_2 + \dots + C_ny_n$

$\Rightarrow$ The Solution Corresponding to $r_1 = 3$ and $r_2 = 3$ are e^{3x} and e^{3x} ~~there~~ they are not linearly independent.

Let $y = e^{3x}v$, where $V = V(x)$ is a function of x.

No. Date.

now :- $\dfrac{dy}{dx} = e^{3x}\dfrac{du}{dx} + 3e^{3x}v = Dy$

$$D^2y = \frac{d^2y}{dx^2} = e^{3x}\frac{d^2v}{dx^2} + 3e^{3x}\frac{du}{dx} + 3e^{3x}\frac{du}{dx} + 9e^{3x}v$$

$$= e^{3x}\frac{d^2v}{dx^2} + 6e^{3x}\frac{dv}{dx} + 9e^{3x}v$$

Substituting into the original eq^n gives

$$e^{3x}\frac{d^2v}{dx^2} = 0 \Rightarrow \int\frac{d^2v}{dx^2} = \int 0.$$

$$\int\frac{d}{dx}\left(\frac{dv}{dx}\right) = 0.$$

$$\frac{dv}{dx} = C_1$$

$$\int\frac{dv}{dx} = C_1$$

$$v = xC_1 + C_0$$

where, C_0 and C_1 is an Constant

$\therefore y = e^{3x}(C_1x + C_0)$

It Can be shown that e^{3x} and $e^{3x}(C_1x+C_0)$ are linearly independent. for any constants $C_1, C_0 \neq 0$.

$\therefore$ The general Solu^n is,

$$y = C_2e^{3x} + C_3e^{3x}(C_1x + C_0)$$

$$y = (C_2 + C_3C_1x + C_3C_0)e^{3x}.$$

$$= (A + Bx)e^{3x}.$$

eg:- $r = 2, 2, 2$ $\quad (A + Bx + Cx^2)e^{2x}$

$r = 5, 5, 5, 5$ $\quad (A + Bx + Cx^2 + Dx^3)e^{5x}.$

$r = -2, -2, -2, 5, 5$

$(A + Bx + Cx^2)e^{-2x} + (C + C_1 x)e^{5x}$.

Note :-

Suppose that the auxiliary eq^n $P(r) = 0$ has the root r occurs k times.

Then the general $Solu^n$ Corresponding to this k-fold repeated root is

$$y = (C_1 + C_2 x + C_3 x^2 + \ldots\ldots + C_k x^{k-1})e^{rx}$$

Example:-

Solve the eq^n

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0.$$

The auxiliary eq^n is given by.

$$r^3 - 4r^2 - 3r + 18 = 0$$

Let $P(r) = r^3 - 4r^2 - 3r + 18$.

$P(-2) = -8 - 16 + 6 + 18$

$= 0$

$(r+2)$ is a factor of $P(r)$

$P(3) = 27 - 36 - 9 + 18$.

$= 0$.

$(r-3)$ is a factor of $P(r)$

$(r+2)(r-3)(r+b) = r^3 - 4r^2 - 3r + 18$.

$(r^2 - r - 6)(r+b) =$

$b = -3$

$\Rightarrow$ $r = 3, 3, -2.$

$\therefore$ the general Solu^n is ~~y=~~

$$y = (C_1 + C_2 x)e^{3x} + C_3 e^{-2x}.$$

where C_1, C_2 and C_3 are an arbitary constant//.

04) eg:-

Suppose that a 6^{th} order Homegeneous linear DE with Constant Coefficients has the following ~~rules~~ roots of a auxiliary eq^n

$-1, -1, 2, 3, 3, 3$

$\therefore$ The general Solu^n is.

$$y = (C_1 + C_2 x)e^{-1x} + C_3 e^{2x} + (C_4 + C_5 x + C_6 x^2)e^{3x}.$$

Case 03 :- Complex roots

If the Constant Coefficient of the auxiliary eq^n are real then any Complex / imaginary root it may have must occur in Conjugate pairs.

$z = x + iy$

the Conjugate of z is $x - iy = \bar{z}$

$\left.\begin{array}{l} 3+2i = z \\ 3-2i = \bar{z} \end{array}\right\}$ Conjugate Pairs.

* Thus if $\alpha + i\beta$ is one root, then $\alpha - i\beta$ must be another root.

If $\alpha + i\beta$ and $\alpha - i\beta$ are the two Complex roots of an AE of a second order Homogeneous DE with Constant Coefficient.

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$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \leftarrow \text{Euler formula.}$$

* The general sol^{n} is,

$$y = Ae^{(\alpha+i\beta)x} + \cancel{Be} Be^{(\alpha-i\beta)x}.$$

$$= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[\left[A(\cos\beta x + i\sin\beta x)\right] + B\left[\cos\beta x - i\sin\beta x\right]\right]$$

$$y = e^{\alpha x}\left\{\underbrace{(A+B)}\cos\beta x + \underbrace{(Ai - Bi)}\sin\beta x.\right\}$$

$$y = e^{\alpha x}\left(C_1\cos\beta x + C_2\sin\beta x\right) \Leftarrow r = \alpha + i\beta.$$

eg:- If $\alpha+i\beta$ and $\alpha-i\beta$ each occurs twise as a root, then,

$$y = e^{\alpha x}\left[(C_1 + C_2x)\cos\beta x + (C_3 + C_4x)\sin\beta x\right]$$

$$\Rightarrow r = \begin{matrix}\alpha+i\beta, & \alpha+i\beta \\ \alpha-i\beta, & \alpha-i\beta.\end{matrix}$$

$\alpha + i\beta \to 4$

$\alpha - i\beta \to 4.$

$$y = e^{\alpha x}\left[(C_1 + C_2x + C_3x^2 + C_4x^3)\cos\beta x + (C_5 + C_6x + C_7x^2 + C_8x^3)\sin\beta x.\right.$$

eg:- Solve :-

$$(D^4 + 8D^2 + 16)y = 0.$$

The AE given by.

$$r^4 + 8r^2 + 16 = 0.$$

$$(r^2+4)(r^2+4) = 0$$

$$r^2 + 4 = r^2 - 4i^2$$
$$= r^2 - (2i)^2$$
$$= (r-2i)(r+2i)$$

No. .... Date. ....

$(r+2i)(r-2i)(r+2i)(r-2i) = 0.$

$r = -2i, -2i, 2i, 2i$

$-2i = 0+-2i \qquad 2i = 0+2i.$

The general sol^n is,

$$y = e^{0x}\left[(C_1+C_2x)\cos 2x + (C_3+C_4x)\sin 2x + (C_5+C_6x)\cos(-2x) + (C_7+C_8x)\sin(-2x)\right].$$

$$= (A+Bx)\cos 2x + (C+Dx)\sin 2x.$$

Home work :-

Solve :- $(D^3+1)y = 0$

01). Solve $\left(\frac{dy}{dx}-y\right)^2\left(\frac{d^2y}{dx^2}+y\right)^2=0$.

$$\left(\frac{dy}{dx}-y\right) = (D-1)y \Rightarrow \left(\frac{dy}{dx}-y\right)^2$$

$$= (D-1)^2 y$$
$$= (D^2-2D+1)y.$$

$$\frac{d^2y}{dx^2}+y = (D^2+1)y.$$

$$\left(\frac{d^2y}{dx^2}+y\right)^2 = (D^2+1)^2 y.$$
$$= (D^4+2D^2+1)y.$$

$$\therefore \left(\frac{dy}{dx}-y\right)^2\left(\frac{d^2y}{dx^2}+y\right)^2 =$$

$$= (D-1)^2 y \ (D^2+1)^2 y.$$
$$= (D-1)^2 (D^2+1)^2 y.$$

The AE is given by.

$$(r-1)^2(r^2+1)^2 = 0$$

$$r = 1, 1, \quad r = -i, -i, i, i$$

$$\therefore y = (C+C_1x)e^{x} + e^{0i}\left[(C_2+C_3x)\cos x + (C_4+C_5x)\sin x\right] + e^{0i}\left[(C_6+C_7x)\cos(-x) + (C_8+C_9x)\sin(-x)\right]$$

$$= (C+C_1x)e^{x} + (A+Bx)\cos x\,(E+Fx)\sin x.$$

Non-homogeneous Linear Differential Equations with Constant Coefficients.

Consider a non-homogeneous DE of the form:-

$P(D)y = Q(x)$, where, —— ①

$P(D) = \cancel{a_n} \; a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$

with constant a_i, for all $a_i = 0, 1, 2, \dots, n$.

Let $y = u$ be a solution of ①.

$\therefore \; P(D)u = Q(x)$ —— ②.

Let $y = u + v$ and put it into ①

$P(D)(u+v) = P(D)u + P(D)v = Q(x)$ —— ③.

③ − ② $\Rightarrow$ $P(D)v = 0$.

That is v is a solution of the homogeneous DE, $P(D)y = 0$.

Suppose that $v = F(x)$ is the general solution of the homogeneous DE, $P(D)y = 0$.

That is $v = F(x)$ contains n arbitrary constants.

Then the general solution of ① is

$$\boxed{y = u + F(x)}$$

u is called particular solution or particular integral of ①

P.I (c) particular integral

$(2D^2+5D+2)x = 5+2x$
$2D^2x + 5Dx + 2x$
$0 + 5 + 2x = R.H.$

Consider the DE,
$(2D^2 + 5D + 2)y = 5+2x$.
It can be seen that $y = x$ is a Sol^n of the above DE.
That is, $y = x$ is a particular Solution/integral of the DE.
Next, we must Solve the Corrosponding HDE
$(2D^2+5D+2)y = 0$.

The Ax. is,
$2r^2+5r+2 = 0$.
$(r+2)(2r+1) = 0$. $\Rightarrow r = -2, -\frac{1}{2}$.
$\therefore$ The Sol^n of HDE is,

$y = y_c = C_1 e^{-2x} + C_2 e^{-\frac{1}{2}x}$.

The solution of the HDE is the Complementary function (y_c)
$\therefore$ The general Sol^n of the NHDE is,
$y = y_c + P.I$
$= C_1 e^{-2x} + C_2 e^{-x/2} + x$ where C_1 and C_2 are arbitrary Constants.

Finding a particular Sol^n/ integral of NHDE.

Note:-
01). $P(D)e^{ax} = e^{ax} \cdot P(a)$
Since $D.e^{ax} = a \cdot e^{ax}$, $D^2e^{ax} = a^2e^{ax}$ and so on we have,
$P(D)e^{ax} = (a_0 + a_1D + \dots + a_{n-1}D^{n-1} + a_nD^n)e^{ax}$
$= (a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n)e^{ax} = P(a)e^{ax}$

No.

Date

02) For any function $u = u(x)$ of x,

$$P(D)\{e^{ax} u\} = e^{ax} P(D+a) u.$$

03) $P(D^2)\cos ax = P(-a)^2 \cos ax$.

Rules to find a particular solution / integral.

note :- A particular integral / solⁿ of $P(D)y = Q(x)$ is denoted by :-

$$P.I / P.S = \frac{1}{P(D)}\{Q(x)\}.$$

$D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$.

$\frac{1}{D} = \int$

$\frac{1}{D^2} = \int\int$.

$\frac{1}{D}\{x^2\} = \int x^2 d$

$= \frac{x^3}{3}$

$\frac{1}{D^2}\{x^2\} = \int \frac{x^3}{3} d$

$= \frac{x^4}{12}$.

01) $\frac{1}{P(D)}\{c x^n\} = c \frac{1}{P(D)}\{x^n\}$.

$= c\left[P(D)^{-1}\right]\{x^n\}$

where c is a constant.

02). $\frac{1}{P(D)}\{e^{ax}\} = e^{ax}\frac{1}{P(a)}$ if $P(a) \neq 0$.

If $P(a) = 0$, then $(D-a)$ is a factor of $P(D)$.

Suppose $(D-a)$ occurs m times.

Then $P(D) = (D-a)^m \phi(D)$ where $\phi(a) \neq 0$.

$$\frac{1}{P(D)}\{e^{ax}\} = \frac{1}{(D-a)^n \phi(D)}\{e^{ax}\}.$$

$$= \frac{1}{(D-a)^m}\frac{e^{ax}}{\phi(a)}\{1\}$$

$= e^{ax}$.

No.

Date.

02) For any function $v = v(x)$ of x,

$$P(D)\{e^{ax} v\} = e^{ax} P(D+a) v.$$

03) $P(D^2)\cos ax = P(-a)^2 \cos ax$.

Rules to find a particular solution / integral.

note :- A particular integral / sol^n of $P(D)y = Q(x)$ is denoted by :-

$$P.I / P.S = \frac{1}{P(D)}\{Q(x)\}.$$

$D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$.

$\frac{1}{D} = \int$

$\frac{1}{D^2} = \iint$.

$\frac{1}{D}\{x^2\} = \int x^2 dx$

$= \frac{x^3}{3}$.

$\frac{1}{D^2}\{x^2\} = \int \frac{x^3}{3} dx$.

$= \frac{x^4}{12}$.

01) $$\frac{1}{P(D)}\{Cx^n\} = C\frac{1}{P(D)}\{x^n\}.$$

$$= C\left[P(D)^{-1}\right]\{x^n\}$$

= where C is a Constant.

02). $$\frac{1}{P(D)}\{e^{ax}\} = e^{ax}\frac{1}{P(a)} \quad \text{if } P(a) \neq 0.$$

If $P(a) = 0$, then $(D-a)$ is a factor of $P(D)$.

Suppose $(D-a)$ occurs m times.

Then $P(D) = (D-a)^m \phi(D)$ where $\phi(a) \neq 0$.

$$\frac{1}{P(D)}\{e^{ax}\} = \frac{1}{(D-a)^n \phi(D)}\{e^{ax}\}.$$

$$= \frac{1}{(D-a)^m}\frac{e^{ax}}{\phi(a)}\{1\}$$

$= e^{ax}$.

Note :- $\frac{1}{P(D)}\left\{e^{ax}\, v(x)\right\} = e^{ax}\frac{1}{P(D+a)}\left\{v(x)\right\}$

$\frac{1}{(D-a)^m}\,\frac{e^{ax}}{\phi(a)}\left\{1\right\}$.

$\frac{e^{ax}}{\phi(a)}\,\frac{1}{(D+a-a)^m}\left\{1\right\}$.

$\frac{e^{ax}}{\phi(a)}\,\frac{1}{D^m}\left\{1\right\}$.

$= \frac{e^{ax}}{\phi(a)}\,\frac{x^m}{m}$

03). $\frac{1}{P(D^2)}\left\{\sin ax\right\} = \frac{1}{P(-a^2)}\sin ax$

$\frac{1}{P(D^2)}\left\{\cos ax\right\} = \frac{1}{P(-a^2)}\cos ax$

$\Bigg\}$ if $P(-a^2) \neq 0$

$D^2 = -a^2$

$D^3 = D \cdot D^2$

$D^4 = D^2 \cdot D^2$

$D^5 = D \cdot D^4 = D \cdot D^2 \cdot D^2$

$D^6 = D^2 \cdot D^2 \cdot D^2$ and so on

No .... (C.F) $\Rightarrow$ Complementary function. Date ....

eg:- Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 5.$

solution:-

The DE can be written as:-

$$(D^2 - 2D - 3)y = 5.$$

The AE is,

$r^2 - 2r - 3 = 0$ and solving gives $r = -1, 3$

$\therefore$ C.F $= y_c =$ The general Sol^n of HDE.

$$= C_1 e^{-x} + C_2 e^{3x}.$$

A particular Sol^n of NHDE is.

$$P.I = P.S = y_p = \frac{1}{D^2 - 2D - 3}\{5\}.$$

$$= \frac{1}{(D+1)(D-3)}\{5\}$$

$$\boxed{\begin{aligned} \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \dots \end{aligned}}$$

- using partial fractions.

$$= \left[\frac{-1/4}{D+1} + \frac{1/4}{D-3}\right]\{5\}$$

$$= -\frac{1}{4}\,\frac{1}{D+1}\{5\} + \frac{1}{4}\,\frac{1}{D-3}\{5\}.$$

$$\frac{1}{D-3} = \frac{1}{-3\left(\frac{-D}{3}+1\right)} = \frac{1}{-3\left(1-\frac{D}{3}\right)}$$

$$\frac{1}{D+1} = 1 - D + D^2 - D^3 + \ldots\ldots$$

$$\frac{1}{1-\frac{D}{3}} = 1 + \frac{D}{3} + \left(\frac{D}{3}\right)^2 + \left(\frac{D}{3}\right)^3 + \ldots\ldots$$

$$-\frac{1}{4}\left[1 - D + \ldots\right]\{5\} + \frac{1}{4}\left[x \frac{1}{-3}\left(1 + \frac{D}{3} + \ldots\right)\right]\{5\}$$

$$-\frac{1}{4}\left[5 - 0\right] - \frac{1}{12}\left[5 + 0\right]$$

$$-\frac{5}{4} - \frac{5}{12} = \frac{-15-5}{12}$$

$$-\frac{\cancel{20}^{5}}{\cancel{12}_{3}} = -\frac{5}{3}$$

Therefore,
The general Sol^n of the NHDE is

$$y = y_c + P.I$$

$$= c_1 e^{-x} + c_2 e^{3x} - \frac{5}{3},$$ where c_1 and c_2 are arbitrary Constant.

Home work :-
Solve $(D^2 - 4)y = x^2$

Solution :-
$$(D^2-4)y = x^2 \Rightarrow \text{HDE :- } (D^2-4)y = 0.$$

The AE is,
$$(r^2 - 4) = 0$$
$$(r-2)(r+2) = 0$$
$$r = 2 \text{ or } r = -2.$$

No. ....

Date. ....

$\therefore$ C.F $= y_c =$ The general solution of HDE,

$$y_c = C_1 e^{-2x} + C_2 e^{2x}.$$

A particular solution of NHDE is,

$$P.I = P.S = y_p = \frac{1}{(D^2-4)}\{x^2\}$$

$$= \frac{1}{(D-2)(D+2)}\{x^2\}.$$

using partial fractions :-

$$\frac{1}{(D-2)(D+2)} = \frac{A}{D-2} + \frac{B}{D+2}$$

$D=2 \Rightarrow 1 = A(4)$

$A = 1/4.$

$$= \frac{A(D+2) + B(D-2)}{(D-2)(D+2)}.$$

$D=-2 \Rightarrow 1 = B(-4)$

$B = -1/4.$

$$1 = A(D+2) + B(D-2)$$

$$\therefore \quad y_p = \left[\frac{1/4}{D-2} - \frac{1/4}{D+2}\right]\{x^2\}$$

$$= \frac{1}{4}\,\frac{1}{D-2}\{x^2\} - \frac{1}{4}\,\frac{1}{D+2}\{x^2\}.$$

$$= \frac{1}{4} \times \frac{1}{-2}\left(\frac{1}{1-\frac{D}{2}}\right)\{x^2\} - \frac{1}{4} \times \frac{1}{2}\left[\frac{1}{\frac{D}{2}+1}\right]\{x^2\}.$$

$$= -\frac{1}{8}\left[1+\frac{D}{2}+\left(\frac{D}{2}\right)^2+\ldots\right]\{x^2\} - \frac{1}{8}\left[1-\frac{D}{2}+\left(\frac{D}{2}\right)^2+\ldots\right]\{x^2\}$$

$$= -\frac{1}{8}\left[x^2 + \right.$$

$$y_p = \frac{1}{(D^2-4)}\{x^2\}.$$

$$= \frac{1}{-4\left[1-\left(\frac{D}{2}\right)^2\right]}$$

$$= \frac{-1}{4}\left[1-\left(\frac{D}{2}\right)^2+\left(\frac{D}{2}\right)^4+\ldots\right]\{x^2\}$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4}\cdot 2 + 0\right]$$

$$= \frac{-1}{4}\left(x^2 + \frac{1}{2}\right)$$

$\left(\frac{D}{2}\right)^2 = x.$

$\frac{1}{2} D\{x^2\}$

$D^2\{x^2\}$

$\frac{1}{\cancel{2}}\cancel{2}x$

$D = 2x$

2

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No .... Date ....

General Soluⁿ :-

$y = y_c + y_p$

$= C_1 e^{-2x} + C_2 e^{2x} - \frac{1}{4}(x^2 + \frac{1}{2})$, where C_1 and C_2 are arbidary constant.

01) Example :-

Solve :-

$(D+3)^2 y = 50e^{2x}$.

AE is $(r+3)^2 = 0$.

Solving the AE gives

$r = -3, -3$ (Double root).

There fore,

The Complementary function :-

$y_c = (C_1 + C_2 x)e^{-3x}$.

The particular solⁿ / Integral :

$P.I = y_p = \frac{1}{(D+3)^2}\{50e^{2x}\}$

$\boxed{\frac{1}{P(D)}\{e^{ax}\} = e^{ax}\frac{1}{P(a)} \quad = \text{if } P(a) \neq 0.}$

$= 50e^{2x}\frac{1}{(2+3)^2}$

$= 2e^{2x}$.

Therefore, general solⁿ

$y = (C_1 + C_2 x)e^{-3x} + 2e^{2x}$, where C_1 and C_2 is are arbitary Constant

No. .... Date. ....

02) Solve :- $(D-2)^2 y = 50e^{2x}$.

AE is $(r-2)^2 = 0$.

Solving the AE gives,

$r = 2, 2$

Therefore,

The Complementary function :-

$y_c = (C_1 + C_2 x)e^{2x}$.

The particular Sol^n / Integral.

$$P.I = y_p = \frac{1}{(D-2)^2}\{50e^{2x}\}.$$

$$= 50e^{2x}\frac{1}{[(D+2)-2]^2}\{1\cancel{50}\}.$$

$$= 50e^{2x}\frac{1}{D^2}\{1\}$$

$$= \frac{50x^2 e^{2x}}{2} = 25x^2 e^{2x}.$$

$$\frac{1}{P(D)}\{e^{ax} v\}$$

$$= e^{ax}\frac{1}{P(D+a)}\{v\}.$$

$\boxed{D \to D+a}$

$\frac{1}{D}\{1\} = \int 1\,dx = x$

$\int x\,dx = \frac{x^2}{2}$.

Therefore, General Sol^n :-

$y = (C_1 + C_2 x)e^{2x} + 25x^2 e^{2x}$.

where C_1 and C_2 are arbitary Constant //.

03). Solve, $(D^2 - 2D + 5)y = e^{-x}$.

Solution:-

AE is $r^2 - 2r + 5 = 0$.

Solving the AE gives,

$r = 1+2i, 1-2i$

Therefore,

The Complementary function :-

$y_c = e^x[C_1 \cos 2x + C_2 \sin 2x]$

~~$+e^x[C_3 \cos 2x + C_4 \sin 2x]$~~.

$\alpha = 1$

$\beta = \pm 2$.

~~$= e^x\{(C_1 + C_3 x) \cos 2x +$~~

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4\times1\times5}}{2}$$

$$= \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}.$$

$$= \frac{1 \pm 4i}{2}$$

$= 1 \pm 2i$

$= 1+2i, 1-2i$.

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The particular Integral is,

$$P.I = \frac{1}{(D^2-2D+5)}\{e^{-x}\}$$

$$= e^{-x} \frac{1}{(-1)^2-2(-1)+5}$$

$$= \frac{e^{-x}}{8}.$$

$\therefore$ The general Sol^{n}.

$$y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{e^{-x}}{8}$$

where c_1 and c_2 are arbitary constant //

Home work :-

(i) Solve :- $(D^3 - D^2 - 4D + 4)y = e^{3x}$.

(ii) $(D^2+6D+9)y = 2e^{-3x}$.

04). Solve :- $(D^2-3D+2)y = 3 \sin 2x$.

$P(D^2)y = \sin ax$.

$$P.I = \frac{1}{P(D^2)}\{\sin ax\}$$

$$= \frac{1}{P(-a^2)} \sin ax. \quad \text{if } P(-a^2) \neq 0.$$

$P(D^2)y = \cos ax$.

$$P.I = \frac{1}{P(-a^2)} \cos ax. \quad \text{if } P(-a^2) \neq 0$$

$D^2 \leftarrow -a^2$.

$D^3 \leftarrow D.D^2 = D(-a^2)$

$D^4 \leftarrow D^2 D^2 = (-a^2)(-a^2)$

$= a^4$

No. .... Date. ....

Solutions :-

The AE is $r^2 - 3r + 2 = 0$ and hence,

$r = -1, 2$.

$\therefore$ The Complementary function is

$C.F = c_1 e^{-x} + C_2 e^{2x}$.

The Particular integral is,

$$P.I = \frac{1}{D^2 - 3D + 2} \left\{ 3 \sin 2x \right\}.$$

$$= 3 \frac{1}{(-2^2) - 3D + 2} \left\{ \sin 2x \right\}. \Rightarrow \left\{ \begin{array}{l} 3 \frac{1}{-4 - 3D + 2} \left\{ \sin 2x \right\} \\ 3 \frac{1}{-2 - 3D} \left\{ \sin 2x \right\}. \\ (-3) \frac{1}{(3D + 2)} \left\{ \sin 2x \right\}. \end{array} \right.$$

$$= (-3) \frac{1}{3D + 2} \left\{ \sin 2x \right\}.$$

$$= (-3) \frac{(3D - 2)}{(3 \cdot D + 2)(3D - 2)} \left\{ \sin 2x \right\}.$$

$$= (-3) \frac{(3D - 2)}{9D^2 - 4} \left\{ \sin 2x \right\}$$

$$= (-3) \frac{(3D - 2)}{-4 + 9(-2^2)} \left\{ \sin 2x \right\}$$

$$= \frac{-3}{-40} \left[6 \cos 2x - 2 \sin 2x \right).$$

Therefore, General Sol^n :-

$$y = C_1 e^{-x} + C_2 e^{2x} + \frac{3}{40} \left[6 \cos 2x - 2 \sin 2x \right].$$

where c_1 and c_2 are arbitary Constant.

05). Ex:-

Solve:- $(D^3+D^2-D-1)y = \cos 2x$.

The AE is $r^3+r^2-r+1 = 0$.

$f(r) = r^3+r^2-r+1$

$f(1) = 1+1-1-1 = 0$,

~~$\therefore (r-1)$ is factor~~

$\Rightarrow r = -1, -1, 1$

$\therefore$ The Complementary function is

$$y_c = (C_1+C_2x)e^{-x}+C_3e^{x}$$

The particular Integral is,

$$P.I. = \frac{1}{D^3+D^2-D-1}\{\cos 2x\}.$$

$$= \frac{1}{(-2^2)D-2^2-D-1}\{\cos 2x\}$$

$$= \frac{1}{-5D-5}\{\cos 2x\}.$$

$$= \frac{-1}{(5D+5)}\{\cos 2x\}.$$

$$= \frac{-1}{5}\frac{D-1}{(D+1)(D-1)}\{\cos 2x\}.$$

$$= \frac{-1}{5}\frac{D-1}{D^2-1}\{\cos 2x\} = \frac{-1}{5}\frac{D-1}{-2^2-1}\{\cos 2x\}.$$

$$= \frac{1}{25}[-2\sin 2x - \cos 2x]$$

$$= \frac{-1}{25}[2\sin 2x + \cos 2x].$$

No. .... Date. ....

Therefore, general solu^n.

$$y = (C_1 + C_2 x)e^{-x} + C_3 e^{x} - \frac{1}{25}(2\sin 2x + \cos 2x)$$

where C is an arbitary Constant.

Homework :-

Solve:- (i) $(D^3+1)y = \cos 2x$

(ii) $(D^2+4)y = \cos 2x$.

Ex:- solve :- $(D^2+a^2)y = \sin ax$.

AE :- $r^2 + a^2 = 0$.

$r = ia, -ia$

~~The~~ Here, $P(D) = D^2 + a^2$ and $P(-a^2) = -a^2 + a^2 = 0$.

Consider $\frac{1}{D^2+a^2} e^{iax} = \frac{1}{D^2+a^2}\cos ax + i\frac{1}{D^2+a^2}\sin ax$

$$\Rightarrow \frac{1}{D^2+a^2}\cos ax = Re\left[\frac{1}{D^2+a^2} e^{iax}\right]$$

and,

$$\frac{1}{D^2+a^2}\sin ax = Im\left[\frac{1}{D^2+a^2} e^{iax}\right].$$

Now :- $\frac{1}{D^2+a^2} e^{iax}$

$$= \frac{1}{(D-ai)(D+ai)}\left\{e^{iax}\right\}$$

$$= \frac{1}{D-ai}\left[\frac{e^{iax}}{iax+iax}\right] = \frac{1}{D-ai}\left\{\frac{e^{iax}}{2aix}\right\}$$

$$= \frac{e^{iax}}{2aix}\frac{1}{D+ai-ai}\{1\} = \frac{e^{iax}}{2aix}\frac{1}{D}\{1\}$$

$$= \frac{e^{iax}}{2aix}x = \frac{e^{iax}}{2ai} = \frac{xe^{iax}}{2ai} = \frac{x}{2ai}[\cos ax + i\sin ax]$$