Formula :-

$E(x) = \sum x p_m$

Finding Mean & Variance from MGF.

If MGF $= M_x(t)$ is known

then,

central Moments

$$\mu_1 = E(x) = \left[\frac{d}{dt}(M_x(t))\right]_{t=0}$$

$$\mu_2 = E(x^2) = \left[\frac{d^2}{dt^2}(M_x(t))\right]_{t=0}$$

$$\mu_3 = E(x^3) = \left[\frac{d^3}{dt^3}(M_x(t))\right]_{t=0}$$

→ discrete

$$M_r(t) = (1 + 2 + \dots),$$

$E(x) = 1! \times$ co-efficient of t.

$E(x^2) = 2! \times$ " " t^2

$E(x^3) = 3! \times$ " " t^3

→ continuous

Binomial Distribution:

Binomial distribution if its probability Mass function is given by.

$$P(X=x) = nC_x \, p^x q^{n-x} \quad , x = 0, 1, 2, 3 \ldots$$

$n < 30$. = Bionamial D.
$n > 30$ = poission.

$P \to$ probability for success.

discreat - p. Mass function
continuous - P. density function.

$q = 1 - p$ [$p + q = 1$]

$n \Rightarrow$ no. of trials.

[Parameters (n, P)] $\Rightarrow$ Aim.

Q) Derive MGF for Binomial distribution and hence find its Mean & variance.

soln:

Let x be a Binomial random variable.

then,

$$P(X=x) = nC_x \, p^x q^{n-x}$$

$$MGF = M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} \, p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \, nC_x \, p^x \, q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x \, (pe^t)^x \, q^{n-x}.$$

W.K.T,

$$(A+B)^n = \sum_{x=0}^{n} nC_x A^x B^{n-x})$$

$$= (B^n + nC_1 B^{n-1} A + nC_2 B^{n-2} A^2 + \cdots + A^n).$$

$$= B^2 + 2AB + A^2$$

$$\boxed{\therefore M_X(t) = (pe^t + q)^n}$$

$$\text{Mean} = E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}(pe^t + q)^n\right]_{t=0}$$

$$= \left[n(pe^t + q)^{n-1}(pe^t + 0)\right]_{t=0}$$

$$= n(pe^0 + q)^{n-1}(pe^0)]$$

$$= [n(p+q)^{n-1} p]$$

$$= n(1)p$$

$$\boxed{E(X) = np}$$

Variance $= E(x^2) - [E(x)]^2$

$$E(x^2) = \left[\frac{d^2}{dt^2}(M_x(t))\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(npe^t(pe^t+q)^{n-1}\right)\right]$$

$$= np\left[\frac{d}{dt}\, e^t(pe^t+q)^{n-1}\right]_{t=0}$$

$$= np\Big[e^t(n-1)(pe^t+q)^{n-2}(pe^t+q) + (pe^t+q)^{n-1}e^t\Big]_{t=0}$$

$$= np\left[(n-1)(p+q)^{n-2}(p) + (p+q)^{n-1}\right]$$

$$= np[(n-1)(p)+1]$$

$n^2p^2 - np^2 + np$

$$= np[np-p+1]$$

$$E(x^2) = n^2p^2 - np^2 + np$$

$\therefore Var(x) = [\cancel{n^2p^2} - np^2 + np] - [\cancel{n^2p^2}]$

$= -np^2 + np.$

$= np[1 - p]$

$p + q = 1$

$q = 1 - p.$

$\boxed{Var(x) = npq.}$

Q) A Machine Manufacturing bolts is known to produce 5% defective. In a random sample of 15 bolts. what is the probability that there are

1) exactly 3 defective bolts
2) not more than 3 defective bolts.

soln:

here $p = 5\% = \frac{5}{100} = 0.05.$

$q = 1 - p = 1 - 0.05 = 0.95$, $n = 15.$

By Bino. dist $p(x) = nc_x \, p^x q^{n-x}.$

(i)

$p(x) = 15C_x (0.05)^x (0.95)^{15-x}$, $x = 0, 1, 2, \ldots 15.$

$P[\text{exactly 3 defective}] = P[x=3]$

$= 15C_3 \ (0.05)^3 \ (0.95)^{12}$

$= 0.0307$

ii) $P[\text{not more than 3 defective}] = p(x \leq 3)$

$= p(x=0) + p(x=1) + p(x=2) + p(x=3)$

$= 15C_0 (0.05)^0 (0.95)^{15} + 15C_1 (0.05)^1 (0.95)^{14} +$

$15C_2 (0.05)^2 (0.95)^{13} + 15C_3 (0.05)^3 (0.95)^{12}$

$= 0.9945.$

objects	x - Coordinate	y - coordinate
1	2	4
2	4	6
3	6	8
4	10	4
5	12	4

object

1 : (2,4)

2 : (4,6)

3 : (6,8)

4 : (10,4)

5 : (12,4)

cluster 1 : $\underset{x_1 \; y_1}{(4,6)}$

cluster 2 : $\underset{x_2 \; y_2}{(12,4)}$

$= |x_2 - x_1| + |y_2 - y_1|$

object 1

$c_1 = (4-2) + (6-4) = 2+2 = 4$

$c_2 = (12-2) + (4-4) = 10 = 10$

$\boxed{c_1 = 4}$

object 2:

$c_1 = (4-4) + (6-6) = 0+0 = 0$

$c_2 = (12-6) + (4-6) = 6-2 = 4$

$\boxed{c_1 = 0}$

Binomial distributions:

$n, p \rightarrow$ Binomial, poission

$p \rightarrow$ geomentry

$\lambda \rightarrow$ poission.

$\boxed{p \rightarrow \text{probability of success.}}$

soln:

~~[illegible]~~

Q) A coin is biased so that a head is twice as likely to appear as a tail. If the coin is tossed 6 times, find the probabilities of getting (1) exactly 2 heads. (2) atleast 3 heads (3) atmost 4 heads.

soln:

Let p be the probability of getting head & q be the pro. of r/ Tail.

given,

$$p = 2q \quad \text{---} (1)$$

WKT, $p + q = 1 \Rightarrow q = 1 - p.$

$\therefore (1) \Rightarrow \quad p = 2(1-p) = 2 - 2p.$

ie $\quad p + 2p = 2.$

$$3p = 2$$

$$\therefore p = \frac{2}{3}$$

If $= \frac{2}{3}$, then $q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}.$

ie here $p = \frac{2}{3}, q = \frac{1}{3}$ & $n = 6.$

By Bino. dis, $P(x) = nC_x \, p^x q^{n-x}.$

ie, $P(x) = 6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}, \quad x = 0, 1, 2, -$

① $p[\text{exactly 2 heads}] = p(x=2)$

$= 6c_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{4}\right)^4$

$= 0.0823.$

② $p[\underbrace{\text{atleast}}_{\text{minimum.}} \text{ 3 heads}] = p(x \geq 3).$

$= p(x=3) + p(x=4) + p(x=5) + p(x=6).$

$= 6c_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3 + 6c_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 +$

$6c_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^1 + 6c_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0$

$= 0.8999.$

③ $\underbrace{\text{atmost}}_{\text{maximum.}}$ 4 heads $:= p[x \leq 4]$

$$= 1 - p(x > 4)$$

$$= 1 - [p(x=5) + p(x=6)].$$

$$= 1 - \left[{}^6C_5 \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^1 + {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0 \right]$$

$$= 0.6488.$$

unbiased estimator :

Q) If $x_1, x_2, x_3, \ldots x_n$ be random variables with mean μ and variance σ^2. Prove that $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ is an unbiased estimator of σ^2.

soln : $E(x_i) = \mu$ $\quad E(x) = y$

$var(x_i) = \sigma^2$

given :

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

T.P $E(s^2) = \sigma^2$.

$$E(s^2) = E\left[\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\right]$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$$

$$= \frac{1}{n-1} E\left\{\sum_{i=1}^{n} (x_i)^2 - \sum_{i=1}^{n} (2x_i\bar{x}) + \sum_{i=1}^{n} (\bar{x}^2)\right\}$$

$$= \frac{1}{n-1} E\left\{\sum_{i=1}^{n} (x_i)^2 - 2\bar{x} \sum_{i=1}^{n} x_i + \bar{x}^2 \sum_{i=1}^{n} 1\right\}.$$

$\because \bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n}$.

$n\bar{x} = \sum x_i \Leftarrow \bar{x} = \frac{\sum x_i}{n}$

$$= \frac{1}{n-1} E\left\{\sum_{i=1}^{n} (x_i^2) - 2\bar{x}(n\bar{x}) + \bar{x}^2(1+1+1+\cdots+1)\right\}$$

$$= \frac{1}{n-1} E\left\{\sum_{i=1}^{n} (x_i^2) - 2n\bar{x}^2 + N\bar{x}^2\right\}.$$

$$= \frac{1}{n-1} E\left\{\sum_{i=1}^{n} (x_i^2) - n\bar{x}^2\right\}.$$

$$= \frac{1}{n-1} \left\{ E\left(\sum_{i=1}^{n} (x_i^2)\right) - E(n\bar{x}^2)\right\}$$

$$E(s^2) = \frac{1}{n-1} \left\{\sum_{i=1}^{n} E(x_i^2) - nE(\bar{x}^2)\right\}. \quad \text{—— (1).}$$

To find $E(x_i^2)$ $\qquad V(x) = E(x^2) - [E(x)]^2.$

WKT, $V(x_i) = E(x_i^2) - [E(x_i)]^2.$

$$\sigma^2 = E(x_i^2) - \mu^2$$

$$\sigma^2 + \mu^2 = E(x_i^2). \quad \text{—— (2).}$$

to find, $E(\bar{x})^2.$

$$Var(\bar{x}) = E(\bar{x}^2) - [E(\bar{x})]^2.$$

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$$Var\left(\frac{x_1+x_2+\dots+x_n}{n}\right) = E(\bar{x}^2) - \left[E\left(\frac{x_1+x_2+\dots+x_n}{n}\right)\right]^2.$$

$$V(ax+by) = a^2 V(x) + b^2 V(y)$$

$$\because E(ax+by) = E(ax) + E(by)$$
$$aE(x) + bE(y)$$

$$\frac{1}{n^2}\{\overset{\sigma^2}{Var(x_1)} + Var(x_2) + \dots + Var(x_n)\} =$$

$$E(\bar{x}^2) - \left[\frac{1}{n}(E(x_1) + E(x_2) + \dots + E(x_n))\right]^2.$$

$$\frac{1}{n^2}\left[\sigma^2 + \sigma^2 + \dots \sigma^2\right] = E(\bar{x}^2) - \frac{1}{n}\left[\mu + \mu + \dots + \mu\right]^2.$$

$$\frac{1}{n^{\cancel{2}}}\left[\cancel{n}\sigma^2\right] = E(\bar{x}^2) - \left[\frac{1}{\cancel{n}}(\cancel{n}\mu)\right]^2$$

$$\therefore \frac{\sigma^2}{n} = E(\bar{x}^2) - \mu^2.$$

$$\frac{\sigma^2}{n} + \mu^2 = E(\bar{x}^2) \quad \text{--- (3)}.$$

Now (1) $\Rightarrow$.

$$E(S^2) = \frac{1}{n-1}\left[\sum_{i=1}^{n}(\sigma^2+\mu^2) - n\left[\frac{\sigma^2}{n}+\mu^2\right)\right]$$

$$= \frac{1}{n-1}\left[(\sigma^2+\mu^2)\boxed{\sum_{i=1}^{n} 1}^{\Rightarrow N} - \sigma^2 - n\mu^2\right]$$

$$= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - \sigma^2 - n\mu^2\right]$$

$$= \frac{1}{n-1}\left(n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}\right]$$

$$\Rightarrow \frac{1}{(\cancel{n-1})}\left[\sigma^2(\cancel{n-1})\right]$$

$$= \sigma^2.$$

$$E(S^2) = \sigma^2.$$

It is an unbiased estimates of σ^2

Q) Let $y_1, y_2, y_3 \ldots y_n$ be independent and identically distributed random variables with mean m and variance σ^2. Verify the given Quantity $\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$ unbiased or not for the variance σ^2.

$E(x) = y_i$

soln:

given;

$$\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

Verify that $E(\sigma_y^2) = \sigma^2$ or not

$$E(y_i) = m \quad , \quad Var(y_i) = \sigma^2 .$$

$(a-b)^2 = (a^2 - 2ab + b^2)$

$$E(\sigma_{y^2}) = E\left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]$$

$$= \frac{1}{n} E\left[\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2)\right]$$

$$= \frac{1}{n} E\left[\sum_{i=1}^{n}(y_i^2) - \sum_{i=1}^{n} 2y_i\bar{y} + \sum_{i=1}^{n}\bar{y}^2\right]$$

$$= \frac{1}{n}\left[E\left[\sum_{i=1}^{n}(y_i)^2 - 2\bar{y}(n\bar{y}) + \bar{y}^2 n\right]\right]$$

$$\bar{y} = \frac{y_1 + y_2 + \quad y_n}{n}$$

$$\bar{y} = \frac{\Sigma y_i}{N} = N\bar{y} = \Sigma y_i$$

$$= \frac{1}{n}\left(E\left(\sum_{i=1}^{n}(y_i^2) - 2n\bar{y}^2 + \bar{y}^2 n\right)\right)$$

$$= \frac{1}{n}\left[E\left(\sum_{i=1}^{n}(y_i^2) - n\bar{y}^2\right)\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}E(y_i^2) - nE(\bar{y}^2)\right] \quad \text{—— (1)}$$

To find $E(y_i^2)$ $\quad var(y) = E(y_i^2) - [E(y_i)]^2$

$$\sigma^2 = E(y_i^2) - m^2$$

$$E(y_i^2) = \sigma^2 + m^2 \quad \text{—— (2)}$$

To find $E(\bar{y}^2)$:

$$var(y) = E(\bar{y}^2) - [E(\bar{y})]^2$$

$$\mathrm{var}\left(\frac{y_1+y_2+\dots y_n}{n}\right) = E(\bar{y}^2) - \left[E\left(\frac{y_1+y_2+\dots y_n}{n}\right)\right]^2$$

$$\mathrm{var}(ax+by) = a^2 V(x) + b^2 V(y)$$

$$E(ax+by) = E(ax) + E(by) = aE(x) + bE(y)$$

$$\frac{1}{n^2}\left\{\mathrm{var}(y_1) + \mathrm{var}(y_2) \dots \mathrm{var}(y_n)\right\} = E(\bar{y}^2) - \left[\frac{1}{n}\left(E(y_1) + E(y_2) \dots + E(y_n)\right)\right]^2$$

$$\frac{1}{n^2}\left[\sigma^2 + \sigma^2 + \dots \sigma^2\right] = E(\bar{y}^2) - \left[\frac{1}{n}\left[m + m + \dots + m\right]\right]^2$$

$$\frac{1}{n^{\cancel{2}}}\left[\cancel{n}\sigma^2\right] = E(\bar{y}^2) - \left(\frac{1}{\cancel{n}}\left[\cancel{n}m\right]\right)^2$$

$$\frac{\sigma^2}{n} + m^2 = E(\bar{y}^2) \quad \text{--- (3)}$$

(1) $\Rightarrow$

$$E(\sigma y^2) = \frac{1}{n}\left[\sum_{i=1}^{n} \sigma^2 + m^2 - n\left[\frac{\sigma^2}{n} + m^2\right]\right]$$

$$= \frac{1}{n}\left[\sigma^2 + m^2 \sum_{i=1}^{n} 1 - \cancel{n}\frac{\sigma^2}{\cancel{n}} - nm^2\right]$$

$$= \frac{1}{n}\left[n(\sigma^2 + m^2) - \sigma^2 - nm^2\right]$$

$$= \frac{1}{n}\left[n\sigma^2 + \cancel{nm^2} - \sigma^2 - \cancel{nm^2}\right]$$

$$= \frac{1}{n}\sigma^2(n-1)$$

$$E(\sigma y^2) = \frac{\sigma^2(n-1)}{n}$$

$\neq \sigma^2$. so,

σy^2 is not an unbiased estimator of σ^2.

Q) If $x_1, x_2, x_3, \ldots, x_n$ constitutes a random sample from the population given by $f(x) = \begin{cases} e^{-(x-\delta)} & \text{for } x > \delta \\ 0 & \text{otherwise} \end{cases}$

show that $\bar{x}$ is a biased estimator of δ.

$E(\bar{x}) \neq \delta$

soln:

T.P $E(\bar{x}) \neq \delta$

i.e T.P $E(x) \neq \delta$

Theorem:

Let $x_1, x_2, \ldots, x_n$ be a simple random sample from a population then the sample mean $\bar{x}$ is an unbiased estimator of the population mean"

$E(\bar{x}) = \mu$

$\mu = E(x)$.

$E(x) = \int x \cdot f(x)\, dx$

$= \sum x \cdot p(x)$

$$E(x) = \int x \cdot f(x)\, dx \qquad \because e^{x+y} = e^{x} \cdot e^{y}$$

$$= \int_{\delta}^{\infty} x \cdot e^{-(x-\delta)}\, dx$$

$$= \int_{\delta}^{\infty} x \cdot e^{-x+\delta}\, dx \qquad \int e^{ax} dx = \frac{e^{ax}}{a}$$

$$= \int_{\delta}^{\infty} x \cdot e^{-x} \cdot e^{\delta}\, dx$$

$$= e^{\delta} \int_{\delta}^{\infty} \underset{u}{x}\, \underset{v}{e^{-x}}\, dx \quad \Big(\because \int u \cdot v\, dx = u \cdot v_1 - u' \cdot v_2 + u'' \cdot v_3 - u''' \cdot v_4 + \cdots \Big) \text{ different value} = 0$$

$$= e^{\delta} \left[(x) \cdot \left(\frac{e^{-x}}{-1} \right) - (1) \left(\frac{e^{-x}}{-1 \times -1} \right) + 0 \right]_{\delta}^{\infty}$$

$$\because e^{-\infty} = 0$$

$$= e^{\delta} \left[0 - \left[-\delta \cdot e^{-\delta} - e^{-\delta} \right] \right]$$

$$= \delta\, \underline{e^{\delta} \cdot e^{-\delta}} + \underline{e^{\delta}\, e^{-\delta}} \qquad e^{\delta} \cdot e^{-\delta} = e^{\delta - \delta} = e^{0} = 1$$

$$E(x) = \delta + 1$$

$E(\bar{x}) = \delta + 1 \neq \delta$

i.e. $E(\bar{x}) \neq \delta$

so it is biased estimator.

Q) A random sample of size 5 is draw from a normal population with unknown mean μ. consider $t_1 = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$, $t_2 = \frac{x_1 + x_2}{2} + x_3$, $t_3 = \frac{2x_1 + x_2 + \lambda x_3}{3}$

i) Find λ, if t_3 is unbiased estimator of $\bar{\mu}$.

ii) Are t_1, t_2 unbiased estimator.

iii) which is the best among t_1, t_2, t_3.

soln:

$n=5 \qquad x_1, x_2 \ldots x_5$

$E(x_1) = \mu \qquad Var(x_1) = \sigma^2$

$\vdots \qquad \vdots \qquad \vdots$

$E(x_5) \quad \vdots \qquad var(x_5) = \sigma^2$

i) Find λ.

$E(t_3) = \mu$.

$$E\left[\frac{2x_1 + x_2 + \lambda x_3}{3}\right] = \mu.$$

$$\frac{1}{3}\left[E(2x_1 + x_2 + \lambda x_3)\right] = \mu.$$

$$\frac{1}{3}\left(2E(x_1) + E(x_2) + \lambda E(x_3)\right] = \mu$$

$$\frac{1}{3}\left[2(\mu) + \mu + \lambda \mu\right] = \mu$$

$$\frac{1\cancel{\mu}}{3}\left[3 + \lambda\right] = \cancel{\mu}$$

$$3 + \lambda = 3$$

$$\lambda = 3 - 3$$

$$\boxed{\lambda = 0}$$

ii) t_1, t_2 are unbiased estimator.

$$E(t_1) = \mu$$
$$E(t_2) = \mu.$$

$$E(t_1) = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$$

$$= E\left[\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}\right]$$

$$= \frac{1}{5}\left[E(x_1 + x_2 + x_3 + x_4 + x_5)\right]$$

$$= \frac{1}{5}\left[E(x_1) + E(x_2) + E(x_3) + E(x_4) + E(x_5)\right].$$

$$= \frac{1}{5}\left[\mu + \mu + \mu + \mu + \mu\right].$$

$$= \frac{1}{\cancel{5}}(\cancel{5}\mu)$$

$$= \mu.$$

so, it is unbiased estimator.

$$t_2 = \frac{x_1 + x_2}{2} + x_3.$$

$$E(t_2) = E\left[\frac{x_1 + x_2}{2} + x_3\right]$$

$$= \frac{1}{2}\left[E(x_1 + x_2) + E(x_3)\right]$$

$$= \frac{1}{2}\left(E(x_1) + E(x_2) + E(x_3)\right]$$

$$= \frac{1}{2}\left[(\mu + \mu) + \mu\right]$$

$$= \frac{1}{\cancel{2}}(\cancel{2}\mu + \mu)$$

$$= \mu + \mu$$

$$= 2\mu.$$

$$2\mu \neq \mu$$

so it is not an unbiased estimator.

iii)

$$var(t_1) = var\left[\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}\right]$$

$$= \frac{1}{25}\left[var(x_1 + x_2 + x_3 + x_4 + x_5)\right]$$

$$= \frac{1}{25}\left[V(x_1) + V(x_2) + V(x_3) + V(x_4) + V(x_5)\right].$$

$$= \frac{1}{25}\left[\sigma^2 + \sigma^2 + \sigma^2 + \sigma^2 + \sigma^2\right].$$

$$= \frac{1}{\cancel{25}_5}\left[\cancel{5}\sigma^2\right]$$

$$= 1/5\ \sigma^2.$$

$$\boxed{t_1 = 0.25\ \sigma^2}$$

$$var(t_2) = var\left(\frac{x_1 + x_2}{2} + x_3\right)$$

$$= \frac{1}{4}\left[var(x_1 + x_2) + var(x_3)\right]$$

$$= \frac{1}{4} \left[(\sigma^2 + \sigma^2) + \sigma^2 \right]$$

$$= \frac{1}{\cancel{4}_2} \left(\cancel{2} \sigma^2 \right) + \sigma^2 \right]$$

$$= \cancel{\frac{1}{2} 2\sigma^2}$$

$$= \frac{1}{2} \left[\sigma^2 \right) + \sigma^2 \right]$$

$$= \sigma^2 \left[\frac{1}{2} + 1 \right]$$

$$= \sigma^2 \left(\frac{3}{2} \right).$$

$$\boxed{t_2 = 1.5 \sigma^2.}$$

$$t_3 = Var \left(\frac{2x_1 + x_2 + \lambda x_3}{3} \right)$$

$$= \frac{1}{9} \left[var(2x_1 + x_2 + \lambda x_3) \right]$$

$$= \frac{1}{9} \left[var(2x_1) + var(x_2) + var(\lambda x_3) \right]$$

$$= \frac{1}{9}\left[4(\sigma^2) + \sigma^2 + \lambda\,\sigma^2\right]$$

$$= \frac{1}{9}\left[4\sigma^2 + \sigma^2 + 0\right]$$

$$= \frac{1}{9}\left[5\sigma^2\right].$$

$$= \frac{5}{9}\sigma^2.$$

$$\boxed{t_3 = 0.55\,\sigma^2}$$

t_1 is the best ~~estim~~ estimator.

Method of Moments:

population | sample. $\bar{x}$

$\mu_1' = E(x)$ | $m_1' = \boxed{\frac{\sum x_i}{n}}$

$\mu_2' = E(x^2)$ | $m_2' = \boxed{\frac{\sum x_i^2}{n}}$ S^2.

$\vdots$ | $\vdots$

$\mu_r' = E(x^r)$ | $m_r' = \frac{\sum x_i^r}{n}.$

---

1) parameter $\mu_1' = m_1'$

2 parameter :

$$\mu_1' = m_1' \left[\because \int x \cdot f(x)\,dx, \; \sum x \cdot p(x) \right.$$

$$\mu_2' = m_2' \left[\because \int x^2 \cdot f(x)\,dx, \; \sum x^2 \cdot p(x) \right]$$

$$\mu_3' = m_3'.$$

Q) Find the estimator of θ in the population with density function

$$f(x,\theta) = \theta\, x^{(\theta-1)} \; ; \; 0 < x < 1 ; \; \theta > 0,$$

$\bar{x}$

by the method of moments.

soln:

given $f(x) = \theta x^{\theta-1}$

By Method of moments $\mu_1' = m_1'$

Now, LHS,

$$\mu_1' = E(x) = \int x \cdot f(x)\,dx.$$

$$= \int_0^1 x \cdot \theta x^{\theta-1} \cdot dx$$

$x^m \cdot x^n = x^{m+n}$.

$$= \theta \int_0^1 x^{\theta}\,dx$$

$$1^{\theta+1} = 1.$$

$$= \theta \left[\frac{x^{\theta+1}}{\theta+1} \right]_0^1$$

$$= \theta \left[\frac{1}{\theta+1} - 0 \right].$$

$$\boxed{\mu_1' = \frac{\theta}{\theta+1}}$$

$$\therefore \mu_1' = m_1' \Rightarrow \frac{\theta}{\theta+1} = \bar{x}$$

$$\theta = \bar{x}(\theta+1).$$

$$\theta = \bar{x}\theta + \bar{x}$$

$$\text{i.e. } \theta - \bar{x}\theta = \bar{x}$$

$$\theta(1-\bar{x}) = \bar{x}$$

$$\therefore \theta = \frac{\bar{x}}{1-\bar{x}}$$

$\therefore$ The estimator of θ is $\frac{\bar{x}}{1-\bar{x}}$ //

Q) Let $(x_1, x_2, \ldots, x_n)$ be a random sample from the uniform population with the density function $f(x, a, b) = \frac{1}{b-a}, a < x < b$. Find the estimators of a and b by the method of moments.

soln:

WKT, MOM $= \mu_r' = m_r'$

$\mu_1' = m_1'$ & $\mu_2' = m_2'$
$\hookrightarrow (1)$ $\hookrightarrow (2)$.

Now

$$\mu_1' = E(x) = \int x \cdot f(x) \, dx.$$

$$= \int_a^b x \cdot \frac{1}{b-a} \, dx.$$

$$= \frac{1}{b-a} \int_a^b x \, dx. = \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$a^2 - b^2 = (a-b)(a+b)$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{\cancel{(b-a)}(b+a)}{2}\right]$$

$$\mu_1' = \frac{b+a}{2}$$

Now $\quad m_1' = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \dots + x_n}{n}$

$$= \bar{x}$$

① $\Rightarrow \frac{b+a}{2} = \bar{x} = b+a = 2\bar{x}$ — ③

Now $\quad \mu_2' = E(x^2) = \int x^2 \cdot f(x)\, dx.$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_a^b x^2\, dx.$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b.$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right].$$

$$= \frac{1}{(b-a)}\left[\frac{b^3 - a^3}{3}\right]$$

$$= \frac{1}{\cancel{(b-a)}}\left[\frac{\cancel{(b-a)}(b^2+ba+a^2)}{3}\right]$$

$$\mu_2' = \frac{b^2+ba+a^2}{3}.$$

$\therefore$ ③ $\Rightarrow$ $\dfrac{b^2+ba+a^2}{3} = s^2$

$$b^2+ba+a^2 = 3s^2 \quad \text{—— ④}.$$

by ③ $\Rightarrow$ $b+a = 2\bar{x}$

Now squaring eqn ③, we get

$$(b+a)^2 = 4\bar{x}^2$$

i.e. $b^2+2ab+a^2 = 4\bar{x}^2$ —— ⑤

Now ⑤ − ④ ⇒

$$(b^2+2ab+a^2)-(b^2+ab+a^2) = (4\bar{x}^2)-(3s^2)$$

$$ab = 4\bar{x}^2 - 3s^2$$

$$\therefore \quad a = \frac{4\bar{x}^2-3s^2}{b}$$

put a value in ③.

$$b + a = 2\bar{x}$$

$$b + \frac{(4\bar{x}^2-3s^2)}{b} = 2\bar{x}$$

$$\frac{b^2+4\bar{x}^2-3s^2}{b} = 2\bar{x}$$

$$b^2+4\bar{x}^2-3s^2 = 2\bar{x}\,b$$

$$\underbrace{b^2}_{x} - \underbrace{2\bar{x}}_{b}\, b + \underbrace{(4\bar{x}^2 - 3s^2)}_{c} = 0$$

$$ax^2 + bx + c = 0.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\cancel{b = -(-2\bar{x}) \pm \sqrt{4\bar{x}^2 - 4(1)(4\bar{x}^2 - 3s^2)}}$$

$$b = \frac{2\bar{x} \pm \sqrt{4\bar{x}^2 - 4(1)(4\bar{x}^2 - 3s^2)}}{2}.$$

$$= \frac{2\bar{x} \pm \sqrt{4(\bar{x}^2 - 4\bar{x}^2 + 3s^2)}}{2}$$

$$= \frac{2\bar{x} \pm 2\cdot\sqrt{3s^2 - 3\bar{x}^2}}{2}.$$

$$= \frac{\cancel{2}\left[\bar{x} \pm \sqrt{3s^2 - 3\bar{x}^2}\right]}{\cancel{2}}$$

$$b = \bar{x} \pm \sqrt{3s^2 - 3\bar{x}^2}.$$

put b value in (3).

$$b + a = 2\bar{x}$$

$$\bar{x} \pm \sqrt{3s^2 - 3\bar{x}^2} + a = 2\bar{x}.$$

$$a = 2\bar{x} - \left(\bar{x} \pm \sqrt{3s^2 - 3x^{-2}}\right).$$

$$= 2\bar{x} - \bar{x} \mp \sqrt{3s^2 - 3\bar{x}^2}$$

$$\boxed{a = \bar{x} \mp \sqrt{3s^2 - 3\bar{x}^2}}$$

$$a = \bar{x} - \sqrt{3s^2 - 3\bar{x}^2} \qquad b = \bar{x} + \sqrt{3s^2 - 3\bar{x}^2}$$

(and &

$$\bar{x} + \sqrt{3s^2 - 3\bar{x}^2} \qquad b = \bar{x} - \sqrt{3s^2 - 3\bar{x}^2}$$

$\boxed{a = b.}$ $\quad a < x < b.$

here, $a < b.$

$$\therefore a = \bar{x} - \sqrt{3s^2 - 3\bar{x}^2} \quad \&$$

$$b = \bar{x} + \sqrt{3s^2 - 3\bar{x}^2}.$$

Q.) Let $(x_1, x_2, \ldots x_n)$ be a random sample from a population with density function $f(x; \theta, \mu) = \theta e^{-\theta(x-\mu)}$; $x > \mu$. Find the estimators of θ and μ by the method of moments.

soln :

Answer : θ is $\dfrac{1}{\pm\sqrt{s^2 - \bar{x}^2}}$ &

μ is $\bar{x} \mp \sqrt{s^2 - \bar{x}^2}$

1) Let $x_1, x_2, \ldots x_n$ be a random sample of size n from a Poisson population with parameter λ. obtain the estimator of λ by the method of moments.

soln :

Let $x_1, x_2 \ldots\ldots x_n$ be a random sample from the poisson population then,

$$m_1' = \frac{\sum_{i=1}^{n}}{n} = \bar{x}.$$

~~We have~~ $\cancel{\mu_1' = E(x) = \int x f(x) dx.}$

$$\cancel{\sum x \; p(x).}$$

$$\mu_1' = E(x) = \lambda$$

Now,

$\mu_1' = m_1' \Rightarrow \lambda = \bar{x}$.

$\therefore$ ~~Thus estimation~~ the estimator of λ

is $\bar{x}$

2) obtain the estimator of the parameter p of the Binomial population using the method of Moments.

soln;

By the MOM $\mu_r' = m_r'$, where.

$\mu_r' \rightarrow$ rth moment in the population

$m_r' \rightarrow$ " " " " sample.

the probability distribution function of Binomial distribution is $p(x) =$

$nC_x \, p^x q^{n-x}$, $x = 0, 1, 2, \ldots n$.

Let $x_1, x_2, \ldots x_n$ be a random sample from the binomial population then,

$$m_1' = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

We have $\mu_1' = E(x) = np$.

Now $\mu_1' = m_1' \Rightarrow np = \bar{x}$

$$p = \frac{\bar{x}}{n}.$$

$\therefore$ The estimator of p is $\frac{\bar{x}}{n}$.

3) obtain the estimator of the parameter λ of the exponential population using the method of moment

(or)

The no. of hours an electron tube will work is assumed to be an exponential variable with parameter λ. given a sample of n life tubes of this sort, compute the method of moments estimator of λ.

soln:-

the pdf of exponential distribution is $f(x) = \lambda e^{-\lambda x}, \quad x > 0.$

By (

Let $x_1, x_2, \ldots x_n$ be a random sample from the exponential population then,

$$m_1' = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

we have $\mu_1' = E(x)$

Mean value derivation :

$$E(x) = 1/\lambda$$

$$\mu_1' = m_1'$$

$$\frac{1}{\lambda} = \bar{x}$$

reciprocal

$$\Rightarrow \lambda = 1/\bar{x}$$

$\therefore$ the estimator of λ is $1/\bar{x}$.

4) obtain the estimators of μ and σ^2 in normal distribution by the method of moments.

soln :

$$\mu_r' = m_r'$$

By ( )

The pdf of Normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2},$$

$$-\infty < x < \infty$$

$$\mu_1' = m_1' \quad \& \quad \mu_2' = m_2'^2$$

$\hookrightarrow$ (1) $\qquad\qquad \hookrightarrow$ (2)

Let $x_1, x_2, \ldots x_n$ be a random sample from the exponential population then,

$$m_1' = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \quad \& \quad m_2' \frac{\sum_{i=1}^{n} x_i^2}{n} = s^2.$$

we have $\mu_1' = E(x). = \mu$

$\&$ $\mu_2^2 = E(x^2) = \mu^2 + \sigma^2$

Now,

$$\mu_1' = m_1' \Rightarrow \mu = \bar{x}$$

$$\& \quad \mu_2' = m_2' \Rightarrow \mu^2 + \sigma^2 = s^2.$$

$$\sigma^2 = s^2 - \mu^2.$$

$$\sigma^2 = s^2 - \bar{x}^2$$

$\therefore$ the estimator of μ is $\bar{x}$

and the estimator of σ^2 is $s^2 - \bar{x}^2$