Unit-2 :

Discrete Random variables :

1. Joint PMF (JPMF).

i) $0 \le p(x_i, y_i) \le 1$

ii) $\sum_j \sum_i p(x_i, y_i) = 1$

2. Marginal function :

The marginal distribution function of x is given by.

$P_x(x) = \sum_j p(x_i, y_i)$.

The marginal distribution of y is given by,

$P_y(y) = \sum_p p(x_i, y_i)$.

3) Independent

Let (x, y) be two dim- discrete random variable. use says x & y are independent if

$p(x_i, y_j) = P_x(x_i) \, P_y(y_i)$.

Continuous Random variable :

1. Joint PDF (JPDF).

i) $0 \le f(x, y) \le 1$

ii) $\iint f(x, y) \, dx \, dy = 1$

2. Marginal function :

> The marginal density function of x, y given by.

$f_x(x) = \int f(x, y) \, dy$.

The Marginal density function of Y is given by

$$f_y(y) = \int f(x,y)\,dx.$$

3) Independent:

Let (x,y) be two continuous random variable we say x & y are independent if

$$f(x,y) = f_x(x) \cdot f_y(y)$$

Q) The JPMF of (x,y) is $p(x,y) = K(2x+3y)$; $x = 0,1,2$; $y = 1,2,3$. Find all the Marginal and conditional probability distributions also find the probability distribution of $x+y$.

soln:

$$p(x,y) = K(2x+3y).$$

y \ x	0	1	2	
1	3K	5K	7K	15K
2	6K	8K	10K	24K
3	9K	11K	13K	33K
	18K	24K	30K	72K.

WKT,

$$\sum\sum p(x,y) = 1.$$

$$72K = 1$$

$$\boxed{K = 1/72}$$

y\x	0	1	2	p(y)
1	3/72	5/72	7/72	15/72
2	6/72	8/72	10/72	24/72
3	9/72	11/72	13/72	33/72
p(x)	18/72	24/72	30/72	1

Marginal function:

X	0	1	2
p(x)	18/72	24/72	30/72

Y	p(y)
1	15/72
2	24/72
3	33/72

Conditional Probability:

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

X	0	1	2
$p(x\|y=1)$	$\frac{3/72}{15/72} = 3/15$	$\frac{5/72}{15/72} = 5/15$	7/15
$p(x\|y=2)$	6/24	8/24	10/24
$p(x\|y=3)$	9/33	11/33	13/33

$$p(y|x) = \frac{p(x,y)}{p(x)}.$$

y	$p(y\|x=0)$	$p(y\|x=1)$	$p(y\|x=2$
1	$\frac{3/72}{18/72} = 3/18$	5/24	7/30
2	$\frac{6/72}{18/72} = 6/18$	8/24	10/30
3	9/18	11/24	13/30.

$x+y$ takes the values

1, 2, 3, 4, 5.

$x+y = 1, 2, 3, 4, 5.$

$x+y$	$P(x+y)$
1	$P_{01} = 3/72$.
2	$P_{11} + P_{02} = 11/72$
3	$P_{03} + P_{12} + P_{24} = 24/72$
4	$P_{27} + P_{13} = 21/72$
5	$P_{23} = 13/72$.

Q) The JPDF of R.V, (x,y) is given by $p(x = x, y = y) =$

$$\begin{cases} \frac{x+y}{12}, & x = 1, 2 \ \& \ y = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

1) The Marginal P.M.F x & y.
2) The conditional P.M.F $P(x|y)$.
3) Are x & y independent?
4) $P(x+y \leq 3)$, $P(x>y)$.

Soln:

$y \backslash x$	1	2
1	$2/12$	$3/12$
2	$3/12$	$4/12$.

1) The Marginal pdf is $p(x)$ & $p(y)$.

$p(x)$.

x	1	2
$p(x)$	$5/12$	$7/12$

$p(y)$.

y	$p(y)$
1	$5/12$
2	$7/12$

2) The condition P.M.F $p(x|y) =$

$$\frac{p(x,y)}{p(y)}.$$

y	1	2
$p(x\|y=1)$	$\frac{2/12}{5/12} = 2/5$	$\frac{3/12}{5/12} = 3/5$
$p(x\|y=2)$	$3/7$	$4/7$

4) $p(x+y \le 3)$.

If $y = 1, 2$. $y = 1, 2$

Then the possible value of $x+y$ are

2, 3, 3, 4.

$p(x+y \le 3) = p(x+y=2) + p(x+y=3)$.

$= p_{11} + p_{12} + p_{21}$

$= 2/12 + 3/12 + 3/12$

$= 8/12$.

$p(x>y) = p(x=2, y=1)$.

$= 3/12$.

3) x & y are independent

WKT,

If $p(x,y) = p(x)\, p(y)$.

i) $x = 1,\ y = 1.$

$p(x=1) = 5/12.$

$p(y=1) = 5/12.$

but $p(x=1, y=1) = 2/12.$

$p(x,y) = 2/12 \neq 5/12 * 5/12.$

$\therefore$ X & Y are not independent.

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continuous :

Q) Find K, if $f(x,y) = K(1-x)(1-y).$ $0 < (x,y) < 1$ to be JPDF of $(x,y).$

soln :

W.K.T.

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = 1$$

$$\int_0^1\int_0^1 K(1-x)(1-y)\,dx\,dy = 1$$

$$\cancel{K\int_0^1 \left[(1-x)dx\right]_0^1 (1-y)\,dy}$$

$$K\int_0^1 \left[\int_0^1 (1-x)\,dx\right](1-y)\,dy = 1$$

$$K\int_0^1 \left[x - \frac{x^2}{2}\right]_0^1 (1-y)\,dy = 1$$

$$K\int_0^1 \left[1 - \frac{1}{2} - 0\right](1-y)\,dy = 1$$

$$k \int_0^1 \left[\frac{2-1}{2} \right] (1-y)\, dy = 1$$

$$k \int_0^1 \tfrac{1}{2} (1-y)\, dy = 1$$

$$(k)(\tfrac{1}{2}) \left[y - \frac{y^2}{2} \right]_0^1 = 1$$

$$k(\tfrac{1}{2}) \left[1 - \frac{1}{2} - 0 \right] = 1$$

$$k(\tfrac{1}{2}) \left[\frac{2-1}{2} \right] = 1$$

$$k(\tfrac{1}{2})(\tfrac{1}{2}) = 1$$

$$k(\tfrac{1}{4}) = 1$$

$$k = 1\left(\frac{4}{1}\right) \qquad \boxed{k = 4}$$

Q) The JPDF of (x, y) is given by.

$f(x,y) = e^{-(x+y)}$; $x > 0$, $y > 0$.

Find,

i) Marginal densities of x & y.

ii) Are x & y independent ?

soln :

i) $f_x(x) = \int_0^\infty f(x,y)\, dy$.

$$= \int_0^\infty e^{-(x+y)}\, dy.$$

$$\boxed{\therefore \int e^{ax} dx = \frac{e^{ax}}{a}} \qquad = \int_0^\infty e^{-x} e^{-y}\, dy.$$

$$= e^{-x} \int_0^\infty e^{-y}\, dy.$$

$$= e^{-x} \left[\frac{e^{-y}}{-1} \right]_0^\infty$$

$$= e^{-x}\left[\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1}\right]$$

$$= e^{-x}[0+1].$$

$$= e^{-x}.$$

$f_x(x) = e^{-x}.$

ii> $f_y(y) = \int_0^\infty f(x,y)\, dx.$

$$= \int_0^\infty e^{-(x+y)}\, dx.$$

$$= \int_0^\infty e^{-x} e^{-y}\, dx.$$

$$= e^{-y}\int_0^\infty e^{-x}\, dx.$$

$$= e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= e^{-y}\left[\frac{e^{-\infty}}{-1} - \frac{e^{0}}{-1}\right].$$

$$= e^{-y}[0+1].$$

$$= e^{-y}[1]$$

$f_y(y) = e^{-y}$

ii> W.K.T

we say. x & y independent.

if $f(x,y) = f_x(x) \cdot f_y(y)$.

consider:

$$f_x(x) \cdot f_y(y) = e^{-x} \cdot e^{-y}.$$

$$= e^{-(x+y)}.$$

$$= f(x,y)$$

$\therefore$ x & y are independent.

Q) The JPDF of R.V (x, y) is given by $f(x, y) = Kxy \cdot e^{-(x^2+y^2)}$

$x \geq 0$, $y \geq 0$ find.

i) K

ii) The conditional distribution of x given y.

iii) Are x & y independent.

soln:

WKT,

$$\int_0^\infty \int_0^\infty f(x, y)\, dx\, dy = 1$$

$$\int_0^\infty \int_0^\infty Kxy e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$K \int_0^\infty \int_0^\infty xy e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$K \int_0^\infty \left[\int_0^\infty xy e^{-x^2} \cdot e^{-y^2}\, dx \right] dy = 1$$

$$K \int_0^\infty \left[\int_0^\infty x e^{-x^2}\, dx \right] y e^{-y^2} dy = 1$$

put $t = x^2$

$dt = 2x\, dx \Rightarrow x\, dx = \frac{dt}{2}$

x	0	∞
t	0	∞

$$K \int_0^\infty \left(\int_0^\infty e^{-t} \frac{dt}{2} \right] y e^{-y^2} dy = 1$$

$$K \int_0^\infty \left(\frac{1}{2} \left[\frac{e^{-t}}{-1} \right]_0^\infty \right) y e^{-y^2} dy = 1$$

$$K \int_0^\infty \left(\frac{1}{2} \left[\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1} \right] y e^{-y^2} dy = 1$$

$$K\int_0^\infty \left[\tfrac{1}{2}(1)\right] y e^{-y^2} dy = 1.$$

$$K\left(\tfrac{1}{2}\right)\left[\int_0^\infty y e^{-y^2} dy\right] = 1$$

$$K \cdot \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = 1$$

$$\boxed{K = 4}$$

$$\therefore f(x,y) = 4xy e^{-(x^2+y^2)}.$$

Marginal function:

$$f_x(x) = \int_0^\infty f(x,y)\, dy.$$

$$= \int_0^\infty 4xy e^{-(x^2+y^2)} dy.$$

$$= \int_0^\infty 4xy e^{-x^2} \cdot e^{-y^2} dy..$$

$$= 4x e^{-x^2}\left[\underbrace{\int_0^\infty y e^{-y^2}}_{\to 1/2}\right] dy$$

$$= \overset{2}{\cancel{4}} x e^{-x^2}\left[\tfrac{1}{\cancel{2}}\right]$$

$$f_x(x) = 2x e^{-x^2}.$$

Illy $f_y(y) = 2y e^{-y^2}$

ii) $f(x|y) = \dfrac{f(x,y)}{f(y)}$

$$= \frac{\overset{2}{\cancel{4}} xy e^{-x^2} \cdot e^{-y^2}}{\cancel{2} y e^{-y^2}}$$

$$f(x|y) = 2x e^{-x^2}.$$

iii> X & Y are independent

$$f_x(x) \cdot f_y(y) = f(x,y).$$

$$f_x(x) \cdot f_y(y) = (2x e^{-x^2})(2y e^{-y^2})$$

$$= 4xy e^{-(x^2+y^2)}$$

$$= f(x,y).$$

∴ hence x & y are independent.

Q) The JPDF of R.V (x,y) is given by

$$f(x,y) = \begin{cases} c e^{-(2x+3y)} & 0 < y \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find (i) c

ii) Are x & y independent ?

$0 \le y \le x < \infty$

W.K.T.

$$\Rightarrow \int_0^\infty \int_0^\infty f(x,y)\, dx\, dy = 1.$$

$$\Rightarrow \int_0^\infty \int_y^\infty c e^{-(2x+3y)} dx\, dy = 1$$

$$c \int_0^\infty \left[\int_y^\infty e^{-2x} dx \right] \cdot e^{-3y} dy = 1$$

$$c \int_0^\infty \left[\frac{e^{-2x}}{-2} \right]_y^\infty \cdot e^{-3y} dy = 1.$$

$$c \int_0^\infty \left[\frac{e^{-2(\infty)}}{-2} - \frac{e^{-2(y)}}{-2} \right] e^{-3y} dy = 1$$

$$c \int_0^\infty \left(\frac{-e^{2y}}{2} \right) e^{-3y} dy = 1$$

$$\frac{c}{2}\int_0^{\infty} e^{-5y}\,dy = 1$$

$$\frac{c}{2}\left(\left[\frac{e^{-5y}}{-5}\right]_0^{\infty}\right) = 1$$

$$\frac{c}{2}\left[\frac{e^{-5(\infty)}}{-5} - \frac{e^{-5(0)}}{-5}\right] = 1.$$

$$\frac{c}{2}\left(\frac{1}{5}\right) = 1.$$

$$\frac{c}{2} = \frac{5}{1}$$

$$c = 5\times 2$$

$$\boxed{c = 10}$$

$$\therefore f(x,y) = 10\,e^{-2x}\cdot e^{-5y}.$$

ii) We say x and y are independent if $f(x,y) = f(x).f(y)$.

$\boxed{0 \le y \le x} < \infty$

$$\Rightarrow f(x) = \int_0^{x} f(x,y)\,dy.$$

$$= \int_0^{x} 10\,e^{-2x}\cdot e^{-3y}\,dy.$$

$$= 10.e^{-2x}\int_0^{x} e^{-3y}\,dy$$

$$= 10.e^{-2x}\left[\frac{e^{-3y}}{-3}\right]_x^0$$

$$= 10.e^{-2x}\left[\frac{e^{-3(0)}}{-3} + \frac{e^{-3x}}{3}\right]$$

$$f(x) = \frac{10}{3}\,e^{-2x}\left(1 - e^{-3x}.\right).$$

$$f(y) = \int_y^{\infty} f(x,y)\,dx \qquad 0 \le \boxed{y \le x < \infty}$$

$$= 10 \int_y^{\infty} e^{-2x} e^{-3y}\,dx$$

$$= 10 e^{-3y} \left[\frac{e^{-2x}}{-2} \right]_y^{\infty}$$

$$= 10 e^{-3y} \left[\frac{e^{-2(\infty)}}{-2} - \frac{e^{-2y}}{-2} \right]$$

$$= \cancel{10}^{5} e^{-3y} \left[\frac{e^{-2y}}{\cancel{2}} \right]$$

$$f(y) = 5 e^{-5y}$$

consider:

$$f(x) \cdot f(y) = \frac{10}{3} e^{-2x} (1 - e^{-3x}) . 5 e^{-5y}$$

$$= \frac{50}{3} e^{-2x} (1 - e^{-3x}) e^{-5y}$$

$$f(x) f(y) = \frac{50}{3} e^{-2x} (1 - e^{-3x}) e^{-5y} \neq 10 e^{-2x} e^{-3y}$$

$\therefore$ x & y are Not independent.

Q) The JPDF is $f(x,y) =$

$$\begin{cases} \frac{5}{16} x^2 y, & 0 < y < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find.

i) Marginal pdf's

ii) obtain $f(x|y)$

iii) Are x & y ie independent.

1) The Marginal pdf of $f_X(x)$

$$f_X(x) = \int_0^x f(x,y)\,dy \quad ; \boxed{0<y<x<2}$$

$$= \int_0^x \frac{5}{16} x^2 y\,dy$$

$$= \frac{5}{16} x^2 \int_0^x y\,dy$$

$$= \frac{5}{16} x^2 \left[\frac{y^2}{2}\right]_0^x$$

$$= \frac{5}{16} x^2 \left[\frac{x^2}{2} - \frac{0^2}{2}\right]$$

$$= \frac{5}{16} x^2 \left(\frac{x^2}{2}\right)$$

$$= \frac{5}{32} x^4$$

The Marginal p.d.f of y $f_Y(y)$ -

$$f_Y(y) = \int_y^2 f(x,y)\,dx \quad ; 0<\boxed{y<x<2}$$

$$= \int_y^2 \frac{5}{16} x^2 y\,dx$$

$$= \frac{5}{16} y \int_y^2 x^2\,dx$$

$$= \frac{5}{16} y \left(\frac{x^3}{3}\right]_y^2$$

$$= \frac{5}{16} y \left[\frac{8}{3} - \frac{y^3}{3}\right]$$

$$= \frac{5}{16} y \cdot \frac{8-y^3}{3}$$

$$= \frac{5}{48} y(8-y^3) \qquad = \frac{5}{16} y\left[\frac{8}{3} - \frac{1}{3}y^3\right]$$

ii) $f(x|y) = \dfrac{f(x,y)}{f(y)}$

$= \dfrac{5/16\, x^2 y}{5/16\, y\left(\frac{8}{3} - \frac{1}{3} y^3\right)}$

$= \dfrac{x^2}{8/3 - y^3/3}$

iv) If x & y are independent if

$f(x,y) = f(x) \cdot f(y)$

consider

$f(x) \cdot f(y) = \frac{5}{32} x^4 \cdot \frac{5}{16} y\left[\frac{8}{3} - \frac{y^3}{3}\right]$

$\neq f(x,y)$

$\therefore$ x & y are not independent.

Q) The JPDF of a r.v x, y is given by $f(x,y) = K(xy + y^2)$.

$0 \le x \le 1$, $0 \le y \le 2$.

Find,

i) $P(Y > 1)$

ii) $P(X > 1/2, Y < 1)$

iii) $P(X + Y \le 1)$

i) $P(Y>1) = \int_1^2 \int_0^1 f(x,y)\, dx\, dy$

$= K \int_1^2 \int_0^1 (xy + y^2)\, dx\, dy$

$= K \int_1^2 \left[y\frac{x^2}{2} + y^2 x \right]_0^1 dy$

$= K \int_1^2 \left[\frac{y}{2} + y^2 \right] dy$

$= K \left[\frac{y^2}{4} + \frac{y^3}{3} \right]_1^2$

$K\left[\cancel{\frac{3}{2}}\ \frac{7}{4}\right.$

$= K\left[\frac{3}{4} + \frac{7}{3}\right]$

$= K\left[\frac{9+28}{12}\right.$

$= \frac{37}{12}K$

$P(y>1) = \frac{37}{12}K$

ii) $P(x>1/2,\ y<1) = \int_0^1\int_{1/2}^1 f(x,y)\,dx\,dy$

$= \int_0^1\int_{1/2}^1 K(xy+y^2)\,dx\,dy$

$= K\int_0^1\int_{1/2}^1 (xy+y^2)\,dx\,dy$

$= K\int_0^1\left[\int_{1/2}^1 xy\,dx + \int_{1/2}^1 y^2\,dx\right]dy$

$= K\int_0^1\left[y\left[\frac{x^2}{2}\right]_{1/2}^1 + y^2\left[x\right]_{1/2}^1\right.\ dy$

$= K\int_0^1\left[y\left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{8}\right] + y^2\left[1-\frac{1}{2}\right]\right.\ dy$

$= K\int_0^1 y\left[\frac{1}{2} - \frac{1}{8}\right) + y^2\left(\frac{1}{2}\right)dy$

$= K\int_0^1 y\left[\frac{4}{8} - \frac{1}{8}\right] + y^2\left(\frac{1}{2}\right)dy$

$= K\int_0^1 \frac{3}{8}y + \frac{1}{2}y^2\ dy$

$= K\left[\frac{3}{8}\cdot\frac{y^2}{2} + \frac{y^3}{6}\right]_0^1$

$= k\left[\frac{3}{16} + \frac{1}{6} - 0\right]$

$= k\left[\cancel{\frac{9 \quad 48}{48}}\right] \quad k\left(\frac{18+16}{96} = \frac{34}{96}\right)$

$= k \cdot \frac{17}{48}$

$p(x > 1/2, y < 1) = \frac{17}{48}k$

iii) $p(x + y \leq 1)$.

(or)

$p(p\, x > y)$.

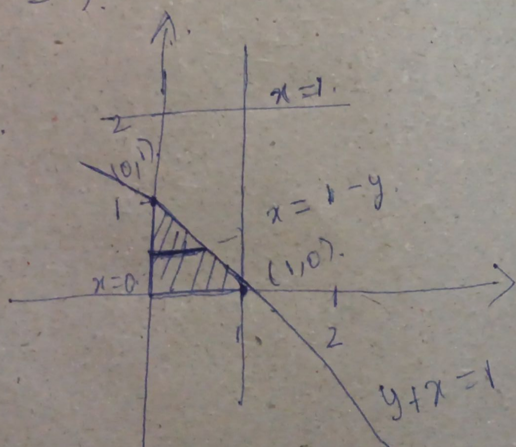
⇓

Drawing Must.

$x + y = 1$.

$(0, 1)$

$(1, 0)$.



$= \int_0^1 \int_0^{1-y} f(x, y)\, dx\, dy.$

Q) If the joint p.d.f of x & y is given by $f(x,y) = \begin{cases} \left(x^2 + \frac{xy}{3}\right); & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$

Find (i) $P(X > \frac{1}{2})$

ii) $P[Y < X]$.

iii) $P[Y < \frac{1}{2} / X < \frac{1}{2}]$. Also check whether the conditional density functions are valid. $f(x|y)$ & $f(y|x)$.

Soln:

First find $f(x)$ & $f(y)$.

$$f(x) = \int_0^2 f(x,y)\, dy$$

$$= \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \left[x^2 y + \frac{x}{3}\left(\frac{y^2}{2}\right)\right]_0^2$$

$$f(x) = 2x^2 + \frac{2x}{3} = \frac{6x^2 + 2x}{3}$$

$$f(y) = \int_0^1 f(x,y)\,dx.$$

$$= \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx.$$

$$= \left[\frac{x^3}{3} + \frac{y}{3}\left(\frac{x^2}{2}\right)\right]_0^1$$

$$= \frac{1 \times 2}{3 \times 2} + \frac{y}{6}.$$

$$f(y) = \frac{2+y}{6}$$

i) Find $p(x > 1/2)$

$$= \int_{1/2}^1 f(x)\,dx$$

$$= \int_{1/2}^1 \frac{6x^2 + 2x}{3}\,dx$$

$$= \frac{1}{3}\left[{}_2 \cancel{6}\left(\frac{x^3}{\cancel{3}}\right) + \cancel{2}\left(\frac{x^2}{\cancel{2}}\right)\right]_{1/2}^1$$

$$= \frac{1}{3}\left[(2+1) - \left(\cancel{2}\left(\frac{1}{\cancel{8}_4}\right) + \frac{1}{4}\right)\right].$$

$$= \frac{1}{3}\left[3 - \frac{1}{2}\right].$$

$$= \frac{1}{3}\left[\frac{6-1}{2}\right].$$

$$= \frac{5}{6}.$$

ii) Now Find $p(y < x)$.

$$p(y < x) = \int_0^1 \int_0^x f(x,y)\,dy\,dx.$$

$$= \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\,dx.$$

$$= \int_0^1 \left[x^2 (y) + \frac{x}{3}\left(\frac{y^2}{2}\right) \right]_0^x dx.$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{6}\right) dx.$$

$$= \left[\frac{x^4}{4} + \frac{x^4}{24} \right]_0^1$$

$$= \left(\frac{1 \times 6}{4 \times 6} + \frac{1}{24} - 0 \right)$$

$$= \frac{6+1}{24} = \frac{7}{24}.$$

iii). Now find $p(Y < 1/2 \mid X < 1/2)$

We have $p(Y < 1/2 \mid X < 1/2) =$

$$\frac{p(X < 1/2,\ Y < 1/2)}{p(X < 1/2)} \quad \text{—①.}$$

$$p(X < 1/2,\ Y < 1/2) = \int_0^{1/2}\int_0^{1/2} f(x,y)\, dx\, dy$$

$$= \int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^{1/2} \left[\frac{x^3}{3} + \frac{x^2}{2}\cdot\frac{(y)}{3} \right]_0^{1/2} dy$$

$$= \int_0^{1/2} \left[\frac{1}{24} + \frac{y}{24} \right] dy.$$

$$\cancel{= \frac{1}{24}\left[\frac{1+y^2}{24}\right]_0^{1/2}} = \frac{1}{24}\int_0^{1/2} (1+y)\, dy.$$

$$\cancel{= \frac{1}{24}\left(\frac{1/4}{24} - 0\right)} = \frac{1}{24}\left[y + \frac{y^2}{2} \right]_0^{1/2}$$

$$= \frac{1}{24}\left[\frac{1}{2} + \frac{(1/2)^2}{2} \right]$$

$$\cancel{= \frac{1}{24}\left(\frac{1}{96}\right)} = \frac{1}{24}\left(\frac{1}{2} + \frac{1}{8} \right)$$

$$= \frac{1}{24}\left[\frac{4+1}{8}\right]$$

$$= \frac{1}{24}\left(\frac{5}{8}\right)$$

$$= \frac{5}{192}.$$

Now $p(x < 1/2) = \int_0^{1/2} f(x)\,dx.$

$$= \int_0^{1/2} \left(2x^2 + \frac{2x}{3}\right) dx.$$

$$= \left[2\left(\frac{x^3}{3}\right) + \frac{2}{3}\left(\frac{x^2}{2}\right)\right]_0^{1/2}$$

$$= \left[\frac{\cancel{2}\,1}{\cancel{24}\,12} + \frac{1}{12}\right] = \frac{\cancel{2}\,1}{\cancel{12}\,6}$$

$$= \frac{1}{6}$$

$$① \Rightarrow \frac{5/\cancel{192}\,32}{1/\cancel{6}.}$$

$$= 5/32.$$

Now find $f(x/y)$ & $(y|x)$.

WKT, $f(x|y) = \dfrac{f(x,y)}{f(y)} = \dfrac{x^2 + \frac{xy}{3}}{\frac{2+y}{6}}$

$$= \frac{\left(\frac{3x^2+xy}{\cancel{3}}\right)}{\frac{2+y}{\cancel{6}\,2}} = \frac{2(3x^2+xy)}{2+y.}$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x^2 + \frac{xy}{3}}{\frac{6x^2+2x}{3}}$$

$$\frac{\frac{3x^2+xy}{\cancel{3}}}{\frac{6x^2+2x}{\cancel{3}}} = \frac{\cancel{x}(3x+y)}{\cancel{x}(6x+2)}$$

$$f(x+y) = \frac{f(x,y)}{f(y)}$$

$$f(y/x) = \frac{3x+y}{6x+2} \qquad f(y) = \int f(x,y)\,dx$$

checking the conditional density functions are valid.

Now, $$\int_0^1 f(x/y)\,dx = \int \frac{2(3x^2+xy)}{(2+y)}\,dx$$

$$= \frac{2}{2+y}\left[\frac{3x^3}{3} + y\left(\frac{x^2}{2}\right)\right]_0^1$$

$$= \frac{2}{2+y}\left[1 + \frac{y}{2}\right]$$

$$f(x/y)\,dx = \frac{\cancel{2}}{2+y}\left(\frac{2+y}{\cancel{2}}\right) = 1$$

Now $$\int_0^2 f(y/x)\,dy = \int_0^2 \frac{3x+y}{6x+2}\,dy$$

$$= \frac{1}{6x+2}\left[3xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{6x+\cancel{2}}\left[\cancel{6}x + 2\right]$$

$$f(y/x)\,dy = 1$$

$\therefore$ the conditions are satisfied.

Coefficient of correlation :- (Karl pearson's coeff. of correlation

x & y → r.v.

⇓

correlation

σ^2 = Variance

S.D $= \sqrt{\sigma^2} = \sigma$

$$r_{x,y} = \frac{\text{cov}(x,y)}{\sigma x \, \sigma y},$$

S.D.

$$\text{cov}(x,y) = E(xy) - E(x) \cdot E(y)$$

$$\sigma x = \sqrt{\text{var}(x)}$$

$$\sigma y = \sqrt{\text{var}(y)}$$

Dis	Contin
$E(x) = \frac{\sum x}{n}$	$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$
$E(y) = \frac{\sum y}{n}$	$E(y) = \int_{-\infty}^{\infty} y \cdot f(y) \cdot dy$

② x 10 15 20

y 12

↓

dis

$f(x,y) =$

⇓

const

↓

continuous

$E(xy) = \frac{\sum(xy)}{n}$	$E(xy) = \int_{-\infty}^{\infty}\int xy \cdot f(x,y)\, dx\, dy$
$E(x^2) = \frac{\sum x^2}{n}$	$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$
$E(y^2) = \frac{\sum y^2}{n}$	$E(y^2) = \int_{-\infty}^{\infty} y^2 \cdot f(y)\, dy$

$$\left.\begin{aligned} \text{var}(x) &= E(x)^2 - [E(x)]^2 \\ \sigma(x) &= \sqrt{\text{var}(x)} \\ \text{var}(y) &= E(y)^2 - [E(y)]^2 \\ \sigma(y) &= \sqrt{\text{var}(y)} \end{aligned}\right\} \rightarrow \text{common.}$$

Note:

1. If X & Y are independent, then

$Cov(x,y) = 0$. $(\because E(xy) = E(x).E(y))$

2. $-1 \leq r \leq 1$

3. when $r=1$, the correlation is perfect & positive.

4. If $r(x,y) = 0$ then x & y are uncorrelated.

Q) Calculate the correlation coefficient for the following heights (in inches) of father x their sons y.

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

soln:

We know that correlation coefficient b/w x & y is

$E(x) = \bar{x}$

$E(y) = \bar{y}$

$$r(x,y) = \frac{cov(x,y)}{\sigma x \; \sigma y}, \text{ where}$$

$$cov(x,y) = E(xy) - E(x).E(y) \;\&$$

$$\sigma x = \sqrt{var(x)} \quad \& \; \sigma y \; \sqrt{var(y)}$$

$$var(x) = E(x^2) - [E(x)]^2$$

x	y	xy	x^2	y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma xy = 37560$	37028	38132
			$\Sigma = x^2$	Σy^2

$(\therefore n = 8)$

$E(x) = \bar{x} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$

$E(y) = \bar{y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$

$E(xy) = \frac{\Sigma xy}{n} = \frac{37560}{8} = 4695$

$E(x^2) = \frac{\Sigma x^2}{n} = \frac{37028}{8} = 4628.5$

$E(y^2) = \frac{\Sigma y^2}{n} = \frac{38132}{8} = 4766.5$

w.k.T. $var(x) = E(x^2) - (E(x))^2$

$= 4628.5 - (68)^2$

$= 4.5$

$\sigma x = \sqrt{var(x)} = \sqrt{4.5} = 2.121$

$var(y) = E(y^2) - [E(y)]^2$

$= 4766.5 - (69)^2$

$= 5.5$

$\sigma y = \sqrt{var(y)} = \sqrt{5.5} = 2.345$

x	y	xy	x^2	y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma xy = 37560$	37028	38132
			$\Sigma = x^2$	Σy^2

$(\because n = 8)$

$$E(x) = \bar{x} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$$

$$E(y) = \bar{y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$$

$$E(xy) = \frac{\Sigma xy}{n} = \frac{37560}{8} = 4695$$

$$E(x^2) = \frac{\Sigma x^2}{n} = \frac{37028}{8} = 4628.5$$

$$E(y^2) = \frac{\Sigma y^2}{n} = \frac{38132}{8} = 4766.5$$

w.k.T. $Var(x) = E(x^2) - (E(x))^2$

$= 4628.5 - (68)^2$

$= 4.5$

$$\sigma x = \sqrt{var(x)} = \sqrt{4.5} = 2.121$$

$Var(y) = E(y^2) - [E(y)]^2$

$= 4766.5 - (69)^2$

$= 5.5$

$$\sigma y = \sqrt{var(y)} = \sqrt{5.5} = 2.345$$

$$r(x,y) = \frac{cov(x,y)}{\sigma x \, \sigma y}$$

$$= \frac{E(xy) - E(x) \cdot E(y)}{\sigma x \quad \sigma y}$$

$$= \frac{4195 - 68 \times 69}{2.121 \times 2.345}$$

$$= 0.6032$$

Calculator Method:

1) clear.

2) Mode → 2 times press.

SD	$\boxed{REG}$	BASE
1	$\boxed{2}$	3

3)

$\boxed{Lin}$	Log	EXP.
$\boxed{1}$	2	3

4) Enter the value.

x	10	20	30	
y	10	15	15	(Eg)

10, 10 (press M+) n=1

20, 15 // n=2

30, 15 // n=3.

5) on button press.

6) shift press 2.

We can find $\bar{x}$, σx, $\bar{y}$, σy, r

$\bar{x}$

1 press and equal to press.

Value will come.

Q) Find the coefficient of correlation b/w industrial production and export using the following data.

Production x	55	56	58	59	60	60	62
Export y	35	38	37	39	44	43	44

Soln.

x	y	$u = x-58$	$v = y-40$	u^2	v^2	uv
55	35	-3	-5	9	25	15
56	38	-2	-2	4	4	4
58	37	0	-3	0	9	0
59	39	1	-1	1	1	-1
60	44	2	4	4	16	8
60	43	2	3	4	9	6
62	44	4	4	16	16	16
		$\Sigma u = 4$	$\Sigma v = 0$	$\Sigma u^2 = 38$	$\Sigma v^2 = 80$	$\Sigma uv = 48$

$r_{x,y} = r_{u,v}$

here $n = 7$.

$$r_{(x,y)} = r_{(u,v)} = \frac{cov(u,v)}{\sigma u \; \sigma v} \quad \text{— (1)}$$

where, $cov(u,v) = E(uv) - E(u).E(v)$ &

$\sigma u = \sqrt{var(u)}$, $\sigma v = \sqrt{var(v)}$

Now $E(u) = \frac{4}{7} = 0.5714$

$E(v) = \frac{\Sigma v}{n} = 0$.

$E(uv) = \frac{\Sigma uv}{n} = \frac{48}{7} = 6.8571$

$E(u^2) = \frac{\Sigma u^2}{n} = \frac{38}{7} = 5.4286$

$E(v^2) = \frac{\Sigma v^2}{n} = \frac{80}{7} = 11.4286$

$\therefore \; Var(u) = E(u^2) - [E(u)]^2$

$= 5.4286 - (0.5714)^2$

$= 5.1021$

$\sigma_u = \sqrt{var(u)} = 2.2588$

Now, $Var(v) = E(v^2) - [E(v)]^2$

$= 11.4286 - 0^2$

$= 11.4286$

$\therefore \; \sigma_v = \sqrt{var(v)} = 3.3806$

$$r(x,y) = \gamma(u,v) = \frac{E(uv) - E(u).E(v)}{\sigma_u . \sigma_v}$$

$$= \frac{6.8571 - 0}{2.2588 \times 3.3806}$$

$= 0.898$

If the joint p.d.f of x & y is given by
$$f(x,y) = \begin{cases} (x+y); & 0 \le x \le 1, \; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the correlation coefficient b/w x & y.

soln:

$\sigma = S.D$

$\sigma = \sqrt{var}$

$$\gamma(x,y) = \frac{cov(x,y)}{\sigma_x \; \sigma_y},$$

$$cov(x,y) = \frac{E(xy) - E(x).E(y)}{\sqrt{var(x)} . \sqrt{var(y)}}$$

W.K.T

$$f(x) = \int_0^1 f(x,y)\, dy \;\; \&$$

$$f(y) = \int_0^1 f(x,y)\, dx.$$

$$= \int_0^1 (x+y)\,dy \quad \& \int_0^1 (x+y)\,dx$$

$$= \left[xy + \frac{y^2}{2} \right]_0^1 \quad \& \; xy \left[\frac{x^2}{2} \right]_0^1$$

$$\cancel{= xy\left[\frac{1}{2}\right] \quad \& \; xy\left[\frac{1}{2}\right]}$$

$$\cancel{= x \quad \& \; (1 \; y/2)}$$

$$f(x) = \left(x + \frac{1}{2}\right) \quad \& \quad \left(\frac{1}{2} + y\right) = f(y)$$

$$E(x) = \int_0^1 x\, f(x)\,dx \quad \& \quad E(y) = \int_0^1 y\, f(y)\,dy$$

$$= \int_0^1 x \cdot \left(x + \frac{1}{2}\right) dx \quad \& \quad \int_0^1 y \cdot \left(\frac{1}{2} + y\right) dy$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right) dx \qquad \int_0^1 \left(\frac{y}{2} + y^2\right) dy$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 \qquad \left[\frac{y^2}{4} + \frac{y^3}{3} \right]_0^1$$

$$= \left[\frac{1}{3} + \frac{1}{4} \right] \qquad \left[\frac{1}{4} + \frac{1}{3} \right]$$

$$E(x) = \frac{4+3}{12} \qquad E(y) = \frac{3+4}{12}$$

$$E(x) = \frac{7}{12} \qquad E(y) = \frac{17}{12}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \cdot \left(x + \frac{1}{2}\right) dx$$

$$= \int_0^1 x^3 + \frac{x^2}{2}\,dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$$

$$= \left[\frac{1}{4} + \frac{1}{6}\right]$$

$$= \frac{6+4}{24}$$

$$= \frac{\cancel{10}5}{\cancel{24}12} = \frac{5}{12}$$

$$E(y^2) = \int_0^1 y^2 f(y)\,dy$$

$$= \int_0^1 y^2 \cdot \left(\frac{1}{2} + y\right) dy$$

$$= \int_0^1 \frac{y^2}{2} + y^3\,dy$$

$$= \left[\frac{y^3}{6} + \frac{y^4}{4}\right]_0^1$$

$$= \left[\frac{1}{6} + \frac{1}{4}\right]$$

$$= 5/12$$

$$E(xy) = \int_0^1\int_0^1 xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy\,(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x^2y + y^2x\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3}(y) + y^2\left(\frac{x^2}{2}\right)\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{1}{3}\cdot y + y^2\cdot\frac{1}{2}\right] dy$$

$$= \left[\frac{1}{3}\left(\frac{y^2}{2}\right) + \frac{y^3}{3}\cdot\frac{1}{2}\right]_0^1$$

$$= \left[\frac{1}{3}\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\cdot\frac{1}{2}\right] - 0$$

$$E(x,y) = \frac{1}{6} + \frac{1}{6} = 2/6 = 1/3$$

$$Var.(x) = E(x^2) - [E(x)]^2$$

$$= 5/12 - \left(\frac{7}{12}\right)^2$$

$$= \frac{5 \times 12}{12 \times 12} - \frac{49}{144}$$

$$= \frac{60}{144} - \frac{49}{144}$$

$$= \frac{11}{144}.$$

$$Var(y) = E(y^2) - [E(y)]^2$$

$$= 5/12 - (7/12)^2$$

$$= 11/144.$$

$$\sigma x = \sqrt{\frac{11}{144}} \quad ; \quad \sigma y = \sqrt{\frac{11}{144}}$$

$$= \frac{\sqrt{11}}{12} \quad , \quad = \frac{\sqrt{11}}{12}$$

$$\therefore \gamma(x,y) = \frac{cov(x,y)}{\sigma x \ \sigma y}$$

$$= \frac{E(xy) - E(x) \cdot E(y)}{\frac{\sqrt{11}}{12} \times \frac{\sqrt{11}}{12}}$$

$$= \frac{1/3 - 7/12 \times 7/12}{\cancel{144} \ 11} \times 144$$

$$= \frac{\frac{1}{3} - \frac{49}{144}}{11} \times 144$$

$$= \frac{\frac{144 - 3 \times 49}{3 \times 144}}{11} \times 144$$

$$= \frac{144 - 147}{3 \times \cancel{144} \times 11} \times \cancel{144}$$

$$= \frac{-3}{3 \times 11} = -1/11 = -0.0909$$

Q) Two random variables X & Y have the joint p.d.f $f(x,y) = \begin{cases} (2-x-y) ; & 0<x<1, \; 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$

show that $cor(x,y) = -1/11$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$\int a\,dx = a\int dx = ax.$

soln:

$$r(x,y) = \frac{cov(x,y)}{\sigma x \, \sigma y.}$$

$cov(x,y) = E(xy) - E(x).E(y).$

$\sigma x = \sqrt{var(x)}$, $\sigma y = \sqrt{var(y)}.$

$$f(x) = \int_0^1 f(x,y)\, dy$$

$$= \int_0^1 (2-x-y)\, dy.$$

$$= \left[2y - xy - \frac{y^2}{2} \right]_0^1$$

$$= \left[2 - x - \frac{1}{2} \right]$$

$$f(y) = \int_0^1 f(x,y)\, dx.$$

$$= \int_0^1 (2-x-y) dx.$$

$$= \left[2x - \frac{x^2}{2} - xy \right]_0^1$$

$$= \left[2 - \frac{1}{2} - y \right].$$

$$f(x) = \left(\frac{3}{2} - x \right)$$

$$E(x) = \int_0^1 x \cdot f(x)\, dx$$

$$= \int_0^1 x \cdot \left(\frac{3}{2} - x \right) dx.$$

$$= \int_0^1 \frac{3x}{2} - x^2\, dx.$$

$$= \left[\frac{3}{2} \cdot \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \left[\frac{3}{4} - \frac{1}{3} \right]$$

$$= \frac{9-4}{12}$$

$$E(x) = \frac{5}{12}.$$

$$f(y) = \left(\frac{3}{2} - y \right)$$

$$E(y) = \int_0^1 y \cdot f(y) \cdot dy.$$

$$E(y) = \int_0^1 y \left(\frac{3}{2} - y \right) dy.$$

$$= \int_0^1 \frac{3y}{2} - y^2\, dy.$$

$$= \left[\frac{3y^2}{4} - \frac{y^3}{3} \right]_0^1$$

$$= \left[\frac{3}{4} - \frac{1}{3} \right]$$

$$= \frac{9-4}{12}$$

$$E(y) = \frac{5}{12}$$

$$E(x^2) = \int_0^1 x^2 (f(x))\, dx$$

$$= \int_0^1 x^2\left(\frac{3}{2} - x\right) dx$$

$$= \int_0^1 \frac{3x^2}{2} - x^3\, dx$$

$$= \left[\frac{3x^3}{6} - \frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{3}{6} - \frac{1}{4}\right)$$

$$= \frac{12-6}{24}$$

$$= \frac{6}{24}$$

$$E(x^2) = 1/4$$

$$E(y^2) = \int_0^1 y^2 (f(y))\, dy$$

$$= \int_0^1 y^2\left(\frac{3}{2} - y\right) dy$$

$$= \int_0^1 \frac{3y^2}{2} - \frac{y^3}{3}\, dy$$

$$= \left[\frac{3}{2}\cdot\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= \left[\frac{3}{6} - \frac{1}{4}\right]$$

$$E(y^2) = 1/4$$

$$\mathrm{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{4} - \left[\frac{5}{12}\right]^2 = \frac{1}{4} - \frac{25}{144}$$

$$= \frac{144-100}{4\times 144} = \frac{\cancel{44}\,11}{\cancel{4}\times 144} = \frac{11}{144}$$

$$\mathrm{Var}(y) = E(y^2) - [E(y)]^2$$

IIly.

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144}$$

$$\sigma x = \sqrt{\mathrm{var}\, x} = \frac{\sqrt{11}}{\sqrt{144}} = \frac{\sqrt{11}}{12}$$

$$\sigma y = \sqrt{\mathrm{var}\, y} = \frac{\sqrt{11}}{12}$$

$$E(x, y) = \int_0^1 \int_0^1 xy\, f(x, y)\, dx\, dy$$

Q) Let x, y, z be uncorrelated random variables with zero means and standard deviations 5, 12, & 9 respectively. If $u = x+y$ & $v = y+z$, find the correlation b/w u & v.

soln :

given, $cov(x,y) = 0$ & $cov(y,z) = 0$ &

$cov(x,z) = 0$.

$E(x) = 0$ — ④, $E(y) = 0$ — ⑤ & $E(z) = 0$ — ⑥

$\sigma_x = 5$, $\sigma_y = 12$, $\sigma_z = 9$

$E(xy) - E(x) \cdot E(y) = 0$

$E(xy) = 0$ — ①

$\|^{ly}$ $E(yz) - E(y) \cdot E(z) = 0$

$E(yz) = 0$ — ②

$\|^{ly}$ $E(x, z) = 0$ — ③

$var\, x = (\sigma x)^2 = 25$.

$var\, y = (\sigma y)^2 = 144$.

$var\, z = (\sigma z)^2 = 81$

$\Rightarrow E(x^2) - [E(x)]^2 = 25$. $E(y^2) - [E(y)]^2 = 144$

$E(x^2) = 25$ — ⑦ $E(y^2) = 144$ → ⑧

$\|^{ly}$ $E(z^2) = 81$ — ⑨

Cor. coeff b/w u & v $= \gamma(u,v) =$

$$\frac{cov(u,v)}{\sigma u \cdot \sigma v}$$

$$= \frac{E(uv - E(u) \cdot E(v)}{\sqrt{var(u)} \sqrt{var(v)}} \quad — ⑩$$

$E(uv) = E[(x+y)(y+z)]$

$= E[xy + xz + y^2 + yz]$

$= \underset{0}{E(xy)} + \underset{0}{E(xz)} + E(y^2) + \underset{0}{E(yz)}$

$E(uv) = 144$ — (11) (by 1, 2, 3, 8)

$E(u) = E(x+y)$

$= \overset{0}{E(x)} + \overset{0}{E(y)}$

$E(u) = 0.$

$E(v) = E(y+z)$

$= E(y) + E(z)$

$E(v) = 0.$

$\text{var } u = E(u^2) - [E(u)]^2$

$= E[(x+y)^2] - 0$

$= E(x^2 + 2xy + y^2)$

$= E(x^2) + E(2xy) + E(y^2)$

$= E(x^2) + 2\underset{0}{E(xy)} + E(y^2)$

$= 25 + 0 + 144$

$\text{var}(u) = 169.$

$\sqrt{\text{var } u} = \sqrt{169} = 13.$

$\text{var}(v) = E(v^2) - (E(v))^2$

$= E((y+z)^2) - 0$

$= E(y^2 + 2yz + z^2)$

$= E(y^2) + 2\underset{0}{E(yz)} + E(z^2)$

$\text{var}(v) = 144 + 81 = 225$

$\sqrt{\text{var}(v)} = \sqrt{225} = 15.$

$$\therefore (10) \Rightarrow r(u,v) = \frac{144 - 0\times 0}{13 \times 15}$$

$$= \frac{\cancel{144}\,48}{13 \times \cancel{15}\,5}$$

$$= \frac{48}{65}.$$

Two independent random variables X & Y are defined by $f(x) = \begin{cases} 4ax\,; & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$f(y) = \begin{cases} 4by\,; & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$ show that

$u = X+Y$ & $V = X-Y$ are uncorrelated.

soln:

T.P. $\operatorname{cov}(u,v) = 0$.

$$E(uv) - E(u)\cdot E(v) = 0$$

$$E(u) = E(x+y) = E(x) + E(y).$$

$$E(v) = E(x-y) = E(x) - E(y).$$

$$E(uv) = E((x+y)(x-y)) = E(x^2-y^2)$$
$$= E(x^2) - E(y^2).$$

$\int_0^1 f(x)\,dx = 1$ & $\int_0^1 f(y)\,dy = 1$

$\int_0^1 4ax\,dx = 1$ & $\int_0^1 4by\,dy = 1$

$4a\int_0^1 x^1\,dx = 1$ $\quad$ $4b\int_0^1 y^1\,dy = 1$

$\cancel{4}^2 a\left(\frac{x^2}{\cancel{2}}\right)_0^1 = 1$ $\quad$ $\cancel{4}^2 b\left[\frac{y^2}{\cancel{2}}\right]_0^1 = 1$

$2a(1) = 1$ $\quad$ $2b(1) = 1.$

$\boxed{a = \frac{1}{2}}$ $\quad$ $2b = 1$

$\boxed{b = \frac{1}{2}.}$

$$E(x) = \int_0^1 x \cdot f(x)\, dx \qquad E(y) = \int_0^1 y \cdot g(y)\, dy$$

$$= \int_0^1 x \cdot (2x)\, dx \qquad \int_0^1 y \cdot (2y)\, dy$$

$$= 2\int_0^1 x^2\, dx \qquad = 2\int_0^1 y^2\, dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 \qquad = 2\left[\frac{y^3}{3}\right]_0^1$$

$$E(x) = \frac{2}{3} \qquad E(y) = \frac{2}{3}$$

$$E(x^2) = \int_0^1 x^2 f(x)\, dx. \qquad E(y^2) = \int_0^1 y^2 \cdot f(y)\, dy$$

$$= \int_0^1 x^2 (2x)\, dx \qquad = \int_0^1 y^2 (2y)\, dy$$

$$= 2\int_0^1 x^3\, dx \qquad = 2\int_0^1 y^3\, dy$$

$$= 2\left[\frac{x^4}{4}\right]_0^1 \qquad = 2\left[\frac{y^4}{4}\right]_0^1$$

$$= \cancel{2}\left(\frac{1}{\cancel{4}_2}\right) \qquad = \cancel{2}\left(\frac{1}{\cancel{4}_2}\right)$$

$$E(x)^2 = 1/2 \qquad E(y)^2 = 1/2$$

T.P

$$\operatorname{cov}(u,v) = 0.$$

LHS:

$$\operatorname{cov}(u,v) = E(uv) - E(u) \cdot E(v).$$

$$= E(x^2) - E(y^2) - [E(x) + E(y)]\,(E(x) - E(y))$$

$$= \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{2}{3} + \frac{2}{3}\right] \times \overbrace{\left(\frac{2}{3} - \frac{2}{3}\right)}^{0}$$

$$= 0.$$

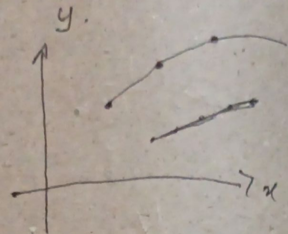
(ii) $\operatorname{cov}(u,v) = 0$

u & v are uncorrelated.

Lines of Regression :

x : 2 5 7

y : 5 8 10



y on x

x on y

$$y-\bar{y} = r\frac{\sigma y}{\sigma x}(x-\bar{x}) \qquad x-\bar{x} = r\frac{\sigma x}{\sigma y}(y-\bar{y})$$

$$byx = \frac{r\sigma y}{\sigma x} \qquad bxy = \frac{r\sigma x}{\sigma y}$$

y on x & x on y $\Rightarrow r = ?$

$$byx \times bxy = r\frac{\cancel{\sigma y}}{\cancel{\sigma x}} \times r\frac{\cancel{\sigma x}}{\cancel{\sigma y}}$$

$$= r^2$$

2 Regression lines given (correlation coefficient)

$$r^2 = byx \times bxy.$$

$$r = \pm\sqrt{byx \times bxy}$$

byx & bxy both +'

$\downarrow$ $\downarrow$

$-$ $-$

$-'$

y on x $\quad y = 5x - 3$ & x on y.

$x = 3y + 5.$

$byx = 5.$

$bxy = 3.$

$$-1 \le r \le 1$$

x on y. $\qquad$ y on x

$3y + 4x = 1$ & $8y + 2x = 4.$

less high $\qquad$ high less

y on x ? $\qquad$ x on y ?

Angle b/w the regression lines is defined by

$$\tan\theta = \frac{\sigma x\,\sigma y}{\sigma x^2 + \sigma y^2}\left[\frac{1-r^2}{r}\right]$$

Properties of regression lines:

1. The regression lines pass through $(\bar{x}, \bar{y})$.

2. If $\theta = \frac{\pi}{2}$ $(r=0)$ then the regression lines are perpendicular.

3. If $\theta = 0$ (or) π $(r = \pm 1)$ then the regression lines are coincide.

Q) In a partially destroyed laboratory record only the lines of regressions & variance of x are available. The regression equations are $8x - 10y + 66 = 0$; $40x - 18y = 214$ and variance of x is 9. Find i) mean value of x and y ii) correlation coefficient b/w x & y iii) variance of y.

soln:

given the two lines of regression are

$$8x - 10y = -66 \quad \text{—} \quad ①$$

$$40x - 18y = 214 \quad \text{—} \quad ②$$

i) Find $\bar{x}$ & $\bar{y}$.

since the two regression lines passing through the points $(\bar{x}, \bar{y})$.

$\therefore$ $\bar{x}$ & $\bar{y}$ satisfies the eqn ① & ②.

i.e $8\bar{x} - 10\bar{y} = -66$ — ③ & $40\bar{x} - 18\bar{y} = 214$ — ④

Now solving eqn ③ & ④ we get $\bar{x} = 13$ & $\bar{y} = 17$.

ii) Find r $\qquad r = \pm\sqrt{b_{yx} * b_{xy}}$

Let the regression line of y on x is $8x - 10y = -66$.

$\Rightarrow -10y = -8x - 66$

(i.e) $y = \frac{8}{10}x + \frac{66}{10}$

$\therefore b_{yx} = \frac{8}{10}$ — (4)

& the regression line of x on y is

$40x - 18y = 214.$

$\Rightarrow 40x = 18y + 214$

(i.e) $x = \frac{18}{40}y + \frac{214}{40}$

$b_{xy} = \frac{18}{40}$ — (5)

~~the b~~

WKT, $r = \pm\sqrt{b_{xy} \times b_{yx}}.$

$= \pm\sqrt{\frac{\cancel{18}^{9}}{\cancel{40}_{5}} \times \frac{\cancel{8}}{\cancel{10}_{5}}}$

$r = \pm\sqrt{\frac{9}{25}}$

$r = \pm\frac{3}{5}$

$\Rightarrow r = \frac{3}{5}$ ($\because b_{xy}$ & b_{yx} both are positive).

iii) find variance of y (i.e) σ_y^2.

given variance of $x = 9$.

i.e $\sigma_x^2 = 9$. $\sigma_x = 3$ — (6)

by (4) $b_{yx} = \frac{8}{10}$ ($\because b_{yx} = \frac{\sigma_y}{\sigma_x} \cdot r$)

(i.e) $\frac{\sigma_y}{\sigma_x} r = \frac{8}{10}$

(ii) $\frac{\sigma_y}{3} \times \frac{3}{5} = \frac{3}{10}$ $\left(\because r = \frac{3}{5}\right)$

$\sigma_y = \frac{3 \times 8}{10 \cdot 2}$ $= 4$.

variance of $y = \sigma_y^2 = 4^2 = 16$.

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(2) Two random variables x & y have the joint p.d.f. $f(x,y) = \frac{1}{3}(x+y)$, $0 \le x \le 1$, $0 \le y \le 2$. Find i) the correlation coefficient ii) the two regression lines.

soln:

i) $r(x,y)$.

wkt

$f(x) = \int f(x,y)\,dy$ & $f(y) = \int f(x,y)\,dx$

$= \int_0^2 \frac{1}{3}(x+y)\,dy$ $= \int_0^1 \frac{1}{3}(x+y)\,dx$.

$= \frac{1}{3}\left[x.y + \frac{y^2}{2}\right]_0^2$ $\frac{1}{3}\left[y.x + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{3}\left[2x + \frac{4}{2}\right]$ $f(y) = \frac{1}{3}\left[\frac{1}{2} + y\right]$

$= \frac{1}{3}(2x+2)$

$f(x) = \frac{2}{3}(x+1)$.

$E(x) = \int_0^1 x\, f(x)\,dx$ $E(y) = \int_0^2 y\, f(y)\,dy$.

$= \int_0^1 x\left[\frac{2}{3}(x+1)\right]dx$ $= \int_0^2 y\left[\frac{1}{3}\left(\frac{1}{2}+y\right)\right]dy$.

$= \frac{2}{3}\int_0^1 (x^2+x)\,dx$ $= \frac{1}{3}\int_0^2 \left(\frac{1}{2}y + y^2\right)dy$.

$= \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$ $= \frac{1}{3}\left[\frac{1}{2}\left(\frac{y^2}{2}\right) + \frac{y^3}{3}\right]_0^2$

$$= \frac{2}{3}\left[\frac{1}{3} + \frac{1}{2}\right]$$

$$= \frac{2}{3}\left(\frac{2+3}{6}\right)$$

$$E(x) = \frac{\cancel{2}}{3}\left[\frac{5}{\cancel{6}_3}\right] = 5/9$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \left[\frac{2}{3}(x+1)\right] dx$$

$$= \frac{2}{3}\int_0^1 (x^3 + x^2)\,dx$$

$$= \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{3}\left[\frac{4}{4} + \frac{8}{3}\right]$$

$$= \frac{1}{3}\left[\frac{3+8}{3}\right]$$

$$E(y) = \frac{11}{9}$$

$$E(y^2) = \int_0^2 y^2 f(y)\,dy$$

$$= \int_0^2 y^2 \left[\frac{1}{3}\left(\frac{1}{2} + y\right)\right] dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{1}{2}\cdot y^2 + y^3\right) dy$$

$$= \frac{1}{3}\left[\frac{1}{2}\cdot\frac{y^3}{3} + \frac{y^4}{4}\right]_0^2$$

$$= \frac{1}{3}\left[\frac{1}{2}\cdot\frac{8}{3} + \frac{16}{4}\right]$$

$$= \frac{\cancel{2}}{3}\left(\frac{3+4}{\cancel{12}_6}\right)$$

$$E(x^2) = \frac{7}{18}$$

Now

$$\mathrm{var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{7}{18} - \left(\frac{5}{9}\right)^2$$

$$= \frac{7}{18} - \frac{25}{81}$$

$$\mathrm{var}(x) = \frac{13}{162}$$

$$= \frac{1}{3}\left[\frac{\cancel{8}^{4}}{\cancel{6}_3} + \frac{\cancel{16}^{4}}{\cancel{4}}\right]$$

$$= \frac{1}{3}\left[\frac{4}{3} + 4\right]$$

$$= \frac{1}{3}\left[\frac{4+12}{3}\right]$$

$$E(y^2) = \frac{16}{9}$$

$$\mathrm{var}(y) = E(y^2) - [E(y)]^2$$

$$= \frac{16}{9} - \left(\frac{11}{9}\right)^2$$

$$= \frac{16}{9} - \frac{121}{81}$$

$$\mathrm{var}(y) = \frac{23}{81}$$

$$E(xy) = \int_0^2 \int_0^1 xy\, f(x,y)\, dx\, dy.$$

$$= \int_0^2 \int_0^1 xy \left[\frac{1}{3}(x+y)\right] dx\, dy.$$

$$= \frac{1}{3} \int_0^2 \int_0^1 (x^2 y + xy^2)\, dx\, dy$$

$$= \frac{1}{3} \int_0^2 \left[\frac{x^3}{3} \cdot y + y^2 \cdot \frac{x^2}{2}\right]_0^1 dy.$$

$$= \frac{1}{3} \int_0^2 \left[\frac{1}{3} \cdot y + y^2 \cdot \frac{1}{2}\right] dy.$$

$$= \frac{1}{3} \left[\frac{1}{3} \cdot \frac{y^2}{2} + \frac{y^3}{3} \cdot \frac{1}{2}\right]_0^2$$

$$= \frac{1}{3} \left[\frac{1}{3} \cdot \frac{\cancel{4}^{2}}{\cancel{2}} + \frac{8}{3} \cdot \frac{1}{2}\right].$$

$$= \frac{1}{3} \left[\frac{4}{6} + \frac{8}{6}\right] = \frac{1}{3} \cdot \left[\frac{\cancel{12}^{2}}{\cancel{6}}\right] = \frac{2}{3}$$

$$\therefore \gamma(x,y) = \frac{\text{cov}(x,y)}{\sigma x \cdot \sigma y} = \frac{E(xy) - E(x).E(y)}{\sqrt{\text{var}(x)}\ \sqrt{\text{var}(y)}}$$

$$= \frac{\frac{2}{3} - \frac{5}{9} \times \frac{11}{9}}{\sqrt{\frac{13}{162}}\ \sqrt{\frac{23}{81}}}$$

$$= \frac{\frac{2 \times 81 - 3 \times 55}{3 \times 81}}{\sqrt{\frac{13}{162} \times \frac{23}{81}}}$$

$81 = \sqrt{81^2}$

$$= \frac{\frac{-\cancel{3}}{\cancel{3} \times 81}}{\sqrt{\frac{13 \times 23}{162 \times 81}}} = \frac{-1}{\sqrt{\frac{\cancel{81} \times \cancel{81} \times 13 \times 23}{\cancel{162}_{2} \times \cancel{81}}}}$$

$$\gamma(x,y) = \frac{-1}{\sqrt{\frac{299}{2}}} = -\sqrt{\frac{2}{299}}$$

ii) The line of regression of y on x is

$$y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

$\because \bar{x} = E(x) = \frac{5}{9}$

$\bar{y} = E(y) = \frac{11}{9}$

$\sigma_x = \sqrt{\frac{13}{162}}$

$\sigma_y = \sqrt{\frac{23}{81}}$

(i.e) $y-\frac{11}{9} = -\sqrt{\frac{2}{299}}\left(\frac{\sqrt{\frac{23}{81}}}{\sqrt{\frac{13}{162}}}\right)(x-\frac{5}{9})$.

(i.e) $y-\frac{11}{9} = -\sqrt{\frac{2}{\underset{13}{\cancel{299}}}\times\frac{\cancel{23}}{\cancel{81}}\times\frac{\overset{2}{\cancel{162}}}{13}}\left(x-\frac{5}{9}\right)$

$$y-\frac{11}{9} = -\frac{2}{13}\left(x-\frac{5}{9}\right)$$

The line of regression of x on y.

is $x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$

(i.e) $x-\frac{5}{9} = -\sqrt{\frac{2}{299}}\;\frac{\sqrt{13/162}}{\sqrt{\frac{23}{81}}}\left(y-\frac{11}{9}\right)$

$= -\sqrt{\frac{\cancel{2}}{\underset{23}{\cancel{299}}}\times\frac{\cancel{13}}{\underset{2}{\cancel{162}}}\times\frac{\cancel{81}}{23}}\left(y-\frac{11}{9}\right)$.

(i.e) $x-\frac{5}{9} = -\frac{1}{23}\left(y-\frac{11}{9}\right)$