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$$= \frac{1}{40000}\left[\frac{e^{-1/40000\,x}}{-1/40000}\right]_0^{30000}$$

$$= -\left[e^{-1/4000\,x}\right]_0^{30000}$$

$$= -\left[e^{-1/40000\,(30000)} - e^{-1/40000\,(0)}\right]$$

$$= -e^{-3/4} + 1$$

$$= -0.4723 + 1$$

$$= 0.5277 \,//$$

$\log e^x = x$

$\log a^x = x \log a$

$\frac{d}{dx} \log x = \frac{1}{x}$

$\frac{1}{\lambda} = \lambda^{-1}$

$\frac{d}{dx} x^n = n x^{n-1}$

λ, μ, σ^2

Maximum Likelihood Estimators :

Let $x_1, x_2, \dots, x_n$ be independent and identically distributed random variables with pdf $f(x, \theta)$. The likelihood function $L(\theta)$ is defined by

$$L(\theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta).$$

Then the maximum likelihood estimator of θ is given by the solution of the equation.

$$\frac{\partial}{\partial \theta} \log L(\theta) = 0, \text{ provided.}$$

partial different

$$\frac{\partial^2}{\partial \theta^2} \log L(\theta) \leq 0.$$

maximum $\rightarrow \leq 0$

minimum $\rightarrow \geq 0$

Step 1 : L

" 2 : $\log L$

3 : $\frac{\partial}{\partial \theta} \log L = 0$

$\theta = ?$

$x_1 + x_2 + \dots + x_n = \sum_{i=1}^{n} x_i$

$x_1 . x_2 \dots x_n = \prod_{i=1}^{n} x_i$

Q) Let $x_1, x_2, \dots, x_n$ be a random sample from the poisson distribution with parameter λ. Obtain the ML estimator of λ.

soln:

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2 \dots$$

step 1:

$$L = f(x_1) . f(x_2) \dots f(x_n)$$

$$= \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \times \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \times \dots \times \frac{e^{-\lambda} \lambda^{x_n}}{x_n!}$$

$$= \frac{(e^{-\lambda})^n \lambda^{x_1 + x_2 + \dots + x_n}}{\prod_{i=1}^{n} x_i!}$$

$$L = \frac{e^{-\lambda n} . \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

step 2: take log on both side

$\log(AB) = \log A + \log B$ | $\log\left(\frac{A}{B}\right) = \log A - \log B$.

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$$\log L \text{ on } \log\left[\frac{e^{-\lambda n} * \lambda^{\sum_{i=1}^{n} xi}}{\prod_{i=1}^{n} xi!}\right]$$

$$= \log\left(e^{-\lambda n} * \lambda^{\sum_{i=1}^{n} xi}\right) - \log\left(\prod_{i=1}^{n} xi!\right)$$

$$= \log e^{-\lambda n} + \log \lambda^{\sum_{i=1}^{n} xi} - \log\left(\prod_{i=1}^{n} xi!\right)$$

$$= -\lambda n + \sum_{i=1}^{n} xi \log \lambda - \log\left(\prod_{i=1}^{n} xi!\right)$$

step 3 : $\frac{\partial}{\partial \lambda}(\log L) =$

$$-n + \sum_{i=1}^{n} xi \frac{1}{\lambda}$$

$$\Rightarrow \frac{\partial^2}{\partial \lambda^2}(\log L) = 0 + \sum_{i=1}^{n} xi\left(-\frac{1}{\lambda^2}\right)$$

$$= -\frac{1}{\lambda^2}\sum_{i=1}^{n} xi \leq 0$$

λ Now $\frac{\partial}{\partial \lambda} \log L = 0 \Rightarrow$

$\hat{\lambda} =$

$$\frac{-n + \sum_{i=1}^{n} xi}{\lambda} = 0$$

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$$\frac{\sum x_i}{\lambda} = n$$

$$\frac{\sum_{i=1}^{n} x_i}{n} = \lambda$$

$$\bar{x} = \lambda$$

Q) Let $x_1, x_2, \ldots, x_n$ be a random sample from an exponential population with the density function

$f(x) = \lambda e^{-\lambda x}$, $x > 0$. Find the MLE of the parameter λ.

soln :

given,

$$f(x) = \lambda e^{-\lambda x}$$

$e^x \cdot e^y = e^{x+y}$

step 1 :

$L = f(x_1) \cdot f(x_2) \ldots f(x_n)$

$L = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} \cdot \ldots \lambda e^{-\lambda x_n}$

$= \lambda^n e^{-\lambda x_1 - \lambda x_2 \ldots - \lambda x_n}$

$= \lambda^n e^{-\lambda (x_1 + x_2 + \ldots x_n)}$

$\log(AB) = \log A + \log B.$

$\log e^{x} = x$

$\log a^{x} = x \log a$

$L = \lambda^n \cdot e^{-\lambda \sum_{i=1}^{n} x_i}$ — ①

step 2 :

$\log L = \log\left[\lambda^n \cdot e^{-\lambda \cdot \sum_{i=1}^{n} x_i}\right]$

$= \log \lambda^n + \log e^{-\lambda \sum_{i=1}^{n} x_i}$

$\log L = n \log \lambda + \left(-\lambda \sum_{i=1}^{n} x_i\right)$ — ②

step 3 : $\frac{\partial}{\partial \lambda} \log L = n\left(\frac{1}{\lambda}\right) - \sum_{i=1}^{n} x_i \lambda$

$\frac{\partial}{\partial \lambda} \log L = 0$

$n\left(\frac{1}{\lambda}\right) - \sum_{i=1}^{n} x_i = 0$

$\frac{n}{\lambda} = \sum_{i=1}^{n} x_i$

$\frac{1}{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}$

$\frac{1}{\lambda} = \bar{x} \Rightarrow \lambda = \frac{1}{\bar{x}}$

$n \cdot \lambda^{-1}$

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$$\frac{\partial^2}{\partial \lambda^2} \log L = n x (-1) \lambda^{-1-1}$$

$$= -n \frac{1}{\lambda^2} \leq 0$$

Q) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normal distribution. Obtain the maximum likelihood estimator of μ and σ^2.

soln :

the pdf of a Normal distribution is

$$f(x) = \frac{1}{}$$

Q) Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a Bernoulli random variable with parameter p. Find the maximum likelihood estimator of p.

soln :

$$f(x) = p^x (1-p)^{1-x} \quad , \quad x = 0, 1$$

$\frac{d}{dx}\log(ax) = \frac{1}{ax} \times a$

step 1 :

$L = p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x^2} \times \dots \times$

$p^{x_n}(1-p)^{1-x_n}.$

$= p^{x_1+x_2+\dots+x_n} \times (1-p)^{(1-x_1)+(1-x_2)+\dots+(1-x_n)}$

$= p^{\sum_{i=1}^{n} x_i} (1-p)^{n-(x_1+x_2+\dots x_n)}$

$L = p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum_{i=1}^{n} x_i}$ — (1)

step 2 :

$\log L = \log\left[p^{\sum_{i=1}^{n} x_i} \times (1-p)^{n-\sum_{i=1}^{n} x_i}\right]$

$= \log p^{\sum_{i=1}^{n} x_i} + \log\left((1-p)^{n-\sum_{i=1}^{n} x_i}\right)$

$\log L = \sum_{i=1}^{n} x_i \log p + n - \sum_{i=1}^{n} x_i \log(1-p)$ — (2)

step 3 : $\frac{\partial}{\partial p} \log L =$

$\sum_{i=1}^{n} x_i \frac{1}{p} + n - \sum_{i=1}^{n} x_i \frac{1}{1-p} \times (-1)$

$= \sum_{i=1}^{n} x_i \left(\frac{1}{p}\right) + \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{1}{1-p}\right)$ — (3)

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③ =>

$$\frac{\partial^2}{\partial p^2}\log L =$$

$$\sum_{i=1}^{n} x_i \times (-1)p^{-1-1} -$$

$$(n-\sum_{i=1}^{n} x_i) \times (-1)(1-p)^{-1-1} \times 1$$

$$= -\sum_{i=1}^{n} x_i \frac{1}{p^2} - (n-\sum_{i=1}^{n} x_i)\frac{1}{(1-p)^2}$$

$\rightarrow$ ④

Now, $\frac{\partial}{\partial p}\log L = 0$

$$\sum_{i=1}^{n} x_i \frac{1}{p} - (n-\sum_{i=1}^{n} x_i)\frac{1}{1-p} = 0$$

$$\frac{\sum x_i}{p} = \frac{(n-\sum x_i)}{1-p}$$

$$(\sum_{i=1}^{n} x_i)(1-p) = p(n-\sum_{i=1}^{n} x_i)$$

$$\sum_{i=1}^{n} x_i - \cancel{p\sum_{i=1}^{n} x_i} = pn - \cancel{p\sum_{i=1}^{n} x_i}$$

$$\sum_{i=1}^{n} x_i = pn$$

$$\sum_{i=1}^{n} \frac{x_i}{n} = p$$

$$\boxed{p = \bar{x}}$$

$$\frac{\partial^2}{\partial p^2} \log L = -\sum_{i=1}^{n} x_i \left(\frac{1}{p^2}\right) - \left(n - \sum_{i=1}^{n} x_i\right) \frac{1}{(1-p)^2}$$

$$= -\sum_{i=1}^{n} x_i \cdot \frac{1}{\bar{x}^2} - \left(n - \sum x_i\right) \cdot \frac{1}{(1-\bar{x})^2}$$

$$= -n\cancel{\bar{x}} \cdot \frac{1}{\bar{x}^{\cancel{2}}} - \frac{(n - n\bar{x})}{(1-\bar{x})^2}$$

$$= \frac{-n}{\bar{x}} - \frac{n\cancel{(1-\bar{x})}}{(1-\bar{x})^{\cancel{2}}}$$

$$= -n\left[\frac{1}{x^2} + \frac{1}{1-\bar{x}}\right]$$

$$= -n\left[\frac{(1-\cancel{\bar{x}}) + \cancel{\bar{x}}}{\bar{x}(1-\bar{x})}\right]$$

$$= -n\left[\frac{1}{\bar{x}(1-\bar{x})}\right] \leq 0$$

$n \to m, \quad \theta \to p$

Q) Maximum likelihood estimator for Binomial distribution.

soln:
The probability distribution function of Binomial distribution is

$$p(x) = mC_x \, \theta^x \, (1-\theta)^{m-x}$$

step 1:

$$L = p(x_1) \cdot p(x_2) \cdots p(x_n)$$

$$= mC_{x_1} \theta^{x_1} \cdot (1-\theta)^{m-x_1} \times mC_{x_2} \theta^{x_2} \cdot (1-\theta)^{m-x_2} \times \cdots mC_{x_n} \theta^{x_n} \cdot (1-\theta)^{m-x_n}$$

$$= \prod_{i=1}^{n} mC_{x_i} \, \theta^{x_1+x_2+\cdots+x_n} (1-\theta)^{(m-x_1)+(m-x_2)+\cdots+(m-x_n)}$$

$$= \prod_{i=1}^{n} mC_{x_i} \, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm-(x_1+x_2+\cdots+x_n)}$$

$$= \prod_{i=1}^{n} mC_{x_i} \, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm-\sum_{i=1}^{n} x_i} \quad \text{--- (1)}$$

$$\log(ABC) = \log A + \log B + \log C.$$

step 2 :

$$\log L = \log\left[\prod_{i=1}^{n} mCx_i * \theta^{\sum_{i=1}^{n} x_i} * (1-\theta)^{nm-\sum_{i=1}^{n} x_i}\right]$$

$$= \log\left(\prod_{i=1}^{n} mCx_i\right) + \log\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \log\left((1-\theta)^{nm-\sum_{i=1}^{n} x_i}\right)$$

$$\log L = \log\left(\prod_{i=1}^{n} mCx_i\right) + \sum_{i=1}^{n} x_i \log\theta + nm - \sum_{i=1}^{n} x_i \log(1-\theta) \quad \text{②}$$

step3 : $\frac{\partial}{\partial\theta}\log(L) =$

Now 'diff' p.w.r. to 'θ'

$$\sum_{i=1}^{n} x_i \cdot \frac{1}{\theta} + \cancel{0} + (nm - \sum x_i) \cdot \frac{1}{(1-\theta)} \times 1$$

$$\frac{\partial}{\partial\theta}\log L = \sum_{i=1}^{n} x_i \left(\frac{1}{\theta}\right) - \left(nm - \sum_{i=1}^{n} x_i\right) \cdot \left(\frac{1}{1-\theta}\right)$$

$\frac{\partial}{\partial\theta}$

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$$\frac{\partial^2}{\partial\theta^2}\log L =$$

$$\sum_{i=1}^{n} x_i \times [(-1)\theta^{-1-1}] + (nm - \sum x_i) \times (-1)(1-\theta)^{-1-1} \times (-1)$$

$$= -\frac{\sum x_i}{\theta^2} - \frac{(nm - \sum x_i)}{(1-\theta)^2} \longrightarrow (4)$$

$$\frac{\partial}{\partial\theta}\log L = 0$$

$$\sum_{i=1}^{n} x_i \left(\frac{1}{\theta}\right) - (nm - \sum x_i)\frac{1}{1-\theta} = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{\theta} = \frac{(nm - \sum_{i} x_i)}{1-\theta}$$

$$\sum_{i=1}^{n} x_i (1-\theta) = \theta(nm - \sum_{i=1} x_i)$$

$$\sum_{i=1}^{n} x_i - \cancel{\theta\sum_{i=1}^{n} x_i} = \theta nm - \cancel{\theta\sum_{i=1}^{n} x_i}$$

$$\sum_{i=1}^{n} x_i = nm\theta.$$

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Page $(a-b)^2 = a^2 - 2ab + b^2$

$$m\theta = \frac{\sum xi}{n}.$$

(ii) $m\theta = \bar{x}$

$$\theta = \frac{\bar{x}}{m}$$

$$\frac{d^2}{d\theta^2}\log L = -\frac{\sum_{i=1}^{n} xi}{\theta^2} - \frac{nm - \sum_{i=1}^{n} xi}{(1-\theta)^2}$$

$$= \frac{-(1-\theta)^2 \sum xi - \theta^2(nm - \sum xi)}{\theta^2(1-\theta)^2}$$

$$= \frac{-(1 - 2\theta + \theta^2)\sum xi - \theta^2 nm + \theta^2 \sum xi}{\theta^2(1-\theta)^2}$$

$$= \frac{-\sum_{i=1}^{n} xi + 2\theta \sum_{i=1}^{n} xi - mn\theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{-n\bar{x} + 2\theta(n\bar{x}) - mn\theta^2}{\theta^2(1-\theta)^2}$$

Date $\theta = \frac{x}{m} = x = \theta m$
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$$= \frac{-n\bar{x} + 2\theta(n\bar{x}) - mn\theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{-nm\theta + 2nm\theta^2 - mn\theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{-nm\theta + nm\theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{-nm\theta\,\cancel{(1-\theta)}}{\theta^2(1-\theta)^{\cancel{2}}}$$

$$\frac{\partial^2}{\partial\theta^2}\log L = -\left[\frac{nm}{\theta(1-\theta)}\right] \leq 0$$

Q.) Suppose $x_1, x_2, \ldots x_n$ be a random sample from an uniform population defined in the interval $[\alpha, \beta]$, where α & β are unknown parameters. Find the MLE of α & β.

soln :

$$f(x) = \frac{1}{\beta-\alpha}, \quad \alpha \leq x \leq \beta$$

$\log a^n = n \log a$

$\log\left(\frac{A}{B}\right) = \log A - \log B$

step1:

$$L = f(x_1) \cdot f(x_2) \cdots\cdots f(x_n)$$

$$= \left(\frac{1}{\beta-\alpha}\right) \times \left(\frac{1}{\beta-\alpha}\right) \times \cdots\cdots \times \left(\frac{1}{\beta-\alpha}\right)$$

$$L = \left(\frac{1}{\beta-\alpha}\right)^n \quad \text{——} \; (1)$$

step 2:

$$\log L = \log\left(\frac{1}{\beta-\alpha}\right)^n$$

$$= n \log\left(\frac{1}{\beta-\alpha}\right)$$

$$= n\left[\log 1 - \log(\beta-\alpha)\right]$$

$$\log L = -n\left[\log(\beta-\alpha)\right] \longrightarrow (2)$$

α MLE	β MLE
$\frac{\partial}{\partial\alpha} \log L = 0$	$\frac{\partial}{\partial\beta} \log L = 0$
$= -n\left[\frac{1}{\beta-\alpha} \times (-1)\right]$	$= -n\left[\frac{1}{\beta-\alpha} \times (+1)\right]$
$\frac{\partial}{\partial\alpha} \log L = \frac{n}{\beta-\alpha}$	$= -\frac{n}{\beta-\alpha}$

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Now, $\frac{\partial}{\partial \alpha} \log L = 0$

$$\frac{n}{\beta-\alpha} = 0$$

$$\infty \left\{ \frac{n}{0} = \beta - \alpha \right.$$

$$\beta - \alpha = \infty$$

Now, $\frac{\partial}{\partial \beta} \log L = 0$

$$\frac{-n}{\beta-\alpha} = 0$$

$$\infty \left\{ \frac{-n}{0} = \beta - \alpha \right.$$

$$\beta - \alpha = \infty$$

$$\beta = \infty$$

$$\underset{\text{first order}}{\alpha} \le x_i \le \underset{\text{nth order}}{\beta}$$

$$\alpha \le x(1) \le x(2) \le x(3) \cdots \le x(n) \le \beta$$

$$\underset{\text{min}}{\beta} - \underset{\text{maxim}}{\alpha}$$

$$\beta = x(n)$$

$$\alpha = x(1)$$