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Brief paper

Range-only measurements based target following for wheeled mobile robots[☆]

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ABSTRACT

We consider the problem of navigation and guidance of a wheeled mobile robot towards a maneuvering target based on the measurements concerning only the distance from the robot to the target. We propose a sliding mode controller that drives the robot to the predefined distance from the target and makes the robot follow the target at this distance. Mathematically rigorous proof of convergence and stability of the proposed guidance law is presented. Simulation results confirm the applicability and performance of the proposed guidance approach.

1. Introduction

The problem of target following by nonholonomic wheeled mobile robots (WMRs) has been widely studied in robotics over the past two decades (see e.g. d'Andrea Novel, Bastin, & Campion, 1995; Gu & Hu, 2005; Soetanto, Lapierre, & Pascoal, 2003; Yang & Kim, 1999 and the references therein). Because of wheels rolling without slipping sideways, these robots have restrictions in mobility and typically are not controllable by linear controllers (Brockett, 1983).

Several sophisticated approaches to target following and trajectory tracking by WMRs have been proposed in the literature. A receding horizon controller was presented in Gu and Hu (2005). The Lyapunov and back-stepping techniques were used in Soetanto et al. (2003) to design a controller steering a WMR along a desired path. A sliding mode controller based on the computed torque method combined with feedback linearization of the system model was proposed in Yang and Kim (1999). Another solution also based on feedback linearization was given in d'Andrea Novel et al. (1995). Interceptor guidance laws ensuring the given impact angle were proposed in Manchester and Savkin (2004, 2006) and Manchester, Savkin, and Faruqi (2008). We refer the reader to Dixon, Dawson, Zergeroglu, and Behal (2001) for an extended survey of the available algorithms for WMR motion control.

Due to the well-known benefits such as stability under large disturbances, robustness against system uncertainties, good dynamic response, simple implementation, etc. (Utkin, 1992), the sliding mode approach attracts an increasing interest in the area of WMRs control. Apart from Yang and Kim (1999), examples include but are not limited to stabilization of a nonholonomic system (Bloch & Drakunov, 1994) and a strategy for trajectory tracking (Solea & Nunes, 2007).

Most of the existing approaches to target following by WMRs assume that both the line-of-sight angle (bearing) and the relative distance (range) are available to the controller. There also is a relatively large literature on navigation based on only bearing measurements. At the same time, only a few publications address guidance towards an unpredictable target on the basis of range-only measurements and no rigorous justifications of the proposed control laws have been provided. The problem of range-only based navigation arises in many areas, e.g., wireless networks, unmanned vehicles, surveillance services etc. (Arora, Dutta, & Bapat, 2004; Gadre & Stilwell, 2004). Many sensors typical for these areas, like sonar or range-only radars, provide only the relative distance between the pursuer and the target via, e.g., measurement of the time-of-flight of an acoustics pulse or the strength of the signal radiated by the target. Range-only based navigation has a potential to reduce the hardware complexity and cost. However this benefit is undermined by the lack of related control design techniques.

In this paper, the problem of range-only based navigation is considered for Dubins-like WMRs. A WMR travels at a constant speed and is controlled by the turning radius, unlike e.g., automotive adaptive cruise control systems (Jurgen, 2006). The objective is to approach a mobile target and then to maintain

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the predefined distance from it. The target may maneuver in an unknown and unpredictable way; the steady target is a particular case. A single sensor provides the relative distance between the robot and the target.[1] A sliding mode control strategy inspired by some ideas from Teimoori and Savkin (2010) is proposed and shown to solve the problem. It is proved via rigorous mathematical analysis that under some minor and partly unavoidable technical assumptions, the control objective is achieved without fail modulo the proper choice of the controller parameters. A method to make such a choice is also offered. The applicability of the proposed controller is confirmed by computer simulations.

2. System description and problem setup

We consider two planar Dubins-like WMRs. They travel with constant speeds and time-varying turning radii, both limited by known constants. One of the robots (say robot 1) plays the role of a target, the objective of the other is to approach the target and maintain a predefined distance d_0 from it. Unlike many works in the area, the pursuer (robot 2) has access to only the current distance $d(t)$ to robot 1 and its derivative $\dot{d}(t)$. It is also aware of the bounds on the speed and turning radius of robot 1. At the same time, it has no access to the actual values of these variables, cannot determine the position of robot 1, and is unaware of both the control rule driving this robot and its future maneuver.

The kinematic models of the robots ($i = 1, 2$) are as follows

$$\begin{matrix} \dot{x}_i = v_i \cos\theta_i \\ \dot{y}_i = v_i \sin\theta_i \end{matrix}, \quad \dot{\theta}_i = u_i \in [-\overline{u}_i, \overline{u}_i], \; v_1 \in [0, \overline{v}_1]. \tag{1}$$

Here $\mathbf{col}(x_i, y_i)$ is the vector of the robot Cartesian coordinates, θ_i gives its orientation, the speed v_i is constant, the angular velocity u_i is time-varying, $\overline{v}_1$, $\overline{u}_1$, and $\overline{u}_2$ are given.

The problem is as follows. Find a controller that asymptotically drives robot 2 to the predefined distance $d(t) \to d_0$ to robot 1, irrespective of what is the motion of the latter, i.e., $x_1(\cdot), y_1(\cdot), u_1(\cdot)$, and v_1 satisfying (1). The situation where the initial distance $d(0) \gg d_0$ is our main concern.

In this paper, we examine the following control algorithm:

$$u_2(t) = \overline{u}_2 \operatorname{sgn}\left\{\dot{d}(t) + \chi[d(t) - d_0]\right\}. \tag{2}$$

Here $\chi(\cdot)$ is a saturated linear function

$$\chi(r) := \begin{cases} \gamma r & \text{if } |r| \le \delta \\ v_* \operatorname{sgn} r & \text{if } |r| > \delta \end{cases}, \quad \text{where } v_* := \gamma\delta. \tag{3}$$

The gain coefficient $\gamma > 0$ and the saturation threshold $\delta > 0$ are to be designed to achieve the control objective.

Remark 2.1. A simpler control law without saturation $u_2(t) = \overline{u}_2 \operatorname{sgn}\left\{\dot{d}(t) + \gamma \cdot [d(t) - d_0]\right\}$ is unsatisfactory since even if the target is steady, it fails to achieve the control goal for large enough initial distances $d(0) \ge d_\dagger(\gamma)$. This holds irrespective of the sign of the gain coefficient $\gamma \in \mathbb{R}$.

Indeed, this law shapes into $u_2(t) = \overline{u}_2 \operatorname{sgn}\gamma$ whenever $d \ge d_\star := d_0 + |\gamma|^{-1}(\overline{v}_1 + v_2)$.[2] Then robot 2 moves along a circle of the radius $R_2 := v_2/\overline{u}_2$. So for $d(0) \ge d_\star + 2R_2$, the robot does not leave the domain $d \ge d_\star$ and maintains circular motion without approaching the target.

[1] This precludes the use of the triangulation method by which the target's position can be accessed via simple trigonometry based on measurements of pair-wise distances between three points, including the robot and target.

[2] Since $|\dot{d}| \le \overline{v}_1 + v_2$ by (1), the magnitude of the first addend in the sum $\dot{d} + \gamma \cdot [d - d_0]$ does not exceed $|\gamma \cdot [d - d_0]|$ if $d \ge d_\star$. So the sum inherits the sign from the second addend, which equals $\operatorname{sgn}\gamma$ since $d \ge d_\star \Rightarrow d - d_0 > 0$.

3. Assumptions

To be able to achieve the control objective, the pursuer should have a high enough level of maneuverability. This level depends on both the properties of the target and the required distance d_0. To specify this, we introduce the following.

Definition 3.1. The distance d_0 between the robots is said to be *nominally maintainable* if robot 2 is capable of maintaining this distance in any feasible situation, provided that this robot is given access to the exhaustive data: $x_i, y_i, \theta_i, v_i, u_1$.

Not only the relative distance d but also its time derivative $\dot{d}$ are functions of the system state ($d = D(\Xi)$, $\dot{d} = D_1(\Xi)$, where $\Xi = \{x_i, y_i, \theta_i\}_{i=1}^2$). Maintenance of distance d_0 means that the state Ξ remains on the manifold $D(\Xi) = d_0, D_1(\Xi) = 0$. So mathematically speaking, nominal maintainability means existence of a control rule $u_2 = U[\Xi, v_1, v_2, u_1] \in [-\overline{u}_2, \overline{u}_2]$ for robot 2 that makes this manifold invariant irrespective of the constant speed $v_1 \in [0, \overline{v}_1]$ and time-varying control input $u_1(t) \in [-\overline{u}_1, \overline{u}_1]$ of robot 1.

The definition of the nominal maintainability is concerned with the idealized full knowledge scenario. However, we shall show that conditions necessary for the nominal maintainability are nearly sufficient for solvability of the primal problem in which information constraints are imposed on robot 2. To this end, we start with these conditions.

Proposition 3.1. *The distance d_0 is nominally maintainable if and only if the following inequalities hold:*

$$\rho := \frac{v_2}{\overline{v}_1} \ge 1, \qquad \overline{u}_2 v_2 \ge \overline{u}_1 \overline{v}_1 + d_0^{-1}(v_2 + \overline{v}_1)^2. \tag{4}$$

The second of them means that the maximal feasible acceleration of robot 2 is not less than that of robot 1 plus the maximal feasible radial acceleration due to rotation of robot 1 relative to robot 2 at a distance d_0 from it.

The following natural assumption is adopted from now on.

Assumption 3.1. Relations (4) hold with $\ge$ replaced by $>$.

For the steady target, this assumption reduces to $d_0 > R_2$, where $R_2 := v_2/\overline{u}_2$ is the minimal turning radius of robot 2.

4. Main results

It can be shown that the controller (2) cannot ensure global convergence $d \to d_0$ even for the steady target: there always is a domain in the phase space, which may be unbounded and even nearly equal to the entire space, such that after entering this domain, robot 2 keeps moving with the constant control $u_2 \equiv \pm\overline{u}_2$ and $d \not\to d_0$. By the following theorem, a proper choice of the controller parameters makes this domain small so that the convergence is guaranteed whenever the pursuer is launched to the target from a remote location.

Theorem 4.1. *Let Assumption 3.1 hold and the parameters $\gamma > 0$ and $\delta > 0$ of the controller (2) be chosen so that*

$$\rho_* := \delta\gamma\overline{v}_1^{-1} < \rho - 1 \quad (\Leftrightarrow v_2 - \overline{v}_1 > v_* := \gamma\delta), \tag{5}$$

$$d_0 > d_*, \qquad \overline{u}_2\varkappa + \gamma(\rho + 1) < Q := \tag{6}$$

$$:= \begin{cases} \sqrt{\rho^2 - 1}\sqrt{\overline{u}_2^2 - \overline{u}_1^2} & \textit{if } \overline{u}_2 \ge \rho\overline{u}_1 \\ \overline{u}_2\rho - \overline{u}_1 & \textit{otherwise}. \end{cases} \tag{7}$$

Here $\varkappa$ is a one-to-one substitute for ρ_*:

$$\varkappa = \rho\rho_* \left[\rho^2 - (1+\rho_*)^2\right]^{-1/2} \quad \text{and} \tag{8}$$

$$d_* := \overline{v}_1 \max_{0 \le \alpha \le \frac{\pi}{2}} \eta(\alpha|\varkappa, \gamma), \quad \text{where} \tag{9}$$

$$\eta(\alpha|\varkappa, \gamma) := \frac{\left[\sqrt{\rho^2 - \cos^2\alpha} + \sin\alpha + \varkappa\right]^2}{\zeta(\alpha|\varkappa, \gamma)}, \tag{10}$$

$$\zeta(\alpha|\varkappa, \gamma) := \overline{u}_2\left(\sqrt{\rho^2 - \cos^2\alpha} - \varkappa\right) - \overline{u}_1 \sin\alpha - \gamma(\rho+1). \tag{11}$$

Then the controller (2) asymptotically drives robot 2 to the required distance to robot 1 $d(t) \xrightarrow{t\to\infty} d_0$, provided that initially the distance between the robots is large enough:

$$d(0) > d_* + 3R_2\left[1 + \pi\frac{\overline{v}_1}{v_2 - \overline{v}_1}\right], \quad R_2 := v_2/\overline{u}_2. \tag{12}$$

Convergence $d \to d_0$ is exponentially fast $|d - d_0| \le ce^{-\gamma t}$.

Remark 4.1. (i) The max in (9) is achieved since the second inequality in (6) holds if and only if $\zeta(\alpha|\varkappa, \gamma) > 0\ \forall\alpha$.
(ii) The set of the controllers (2) satisfying the convergence requirements (5) and (6) is not empty. This set contains all controllers for which $\gamma > 0$ and $\rho_* > 0$ are small enough.

The proof of this remark is given in Appendix.

In the case of the steady target, the domain of non-convergence $d \not\to d_0$ can be exactly computed for all values of the controller parameters. If the parameter ρ_* exceeds a certain threshold, this domain appears to be infinite: $d \not\to d_0$ whenever the robots are initially far enough from each other. If ρ_* exceeds a larger threshold, $d \not\to d_0$ for almost all initial states. So it is reasonable to avoid large ρ_*, as is recommended by (5) and (6). At the same time, a superfluous decrease of ρ_* may worsen the transient performance.

Inequality (12) provides an upper estimate of the convergence domain; if (12) is violated, non-convergence $d \not\to d_0$ does not necessarily occur.

To maintain both the speed v_2 and the distance d_0 to the steady target, the pursuer has to circulate around it. This behavior is typical in general. To show this, we introduce the algebraic angle ψ from the x-axis of the absolute Cartesian coordinate system to the vector from the target to the pursuer.

Proposition 4.1. *Suppose that the assumptions of Theorem 4.1 hold. Then $\underline{\lim}_{t\to\infty} \sigma\dot{\psi}(t) \ge d_0^{-1}(v_2 - \overline{v}_1)$ for some $\sigma = \pm 1$ and irrespective of the motion of robot 1.*

So the pursuer asymptotically circulates around the target at the required distance d_0 at the speed $\gtrsim v_2 - \overline{v}_1 > 0$.

It should be noted that the inverse of the transformation (8)

$$\rho_* = \frac{\varkappa}{\rho^2 + \varkappa^2}\left[\sqrt{\varkappa^2 + (\rho^2 - 1)(\rho^2 + \varkappa^2)} - \varkappa\right], \quad \varkappa > 0$$

produces ρ_* that satisfies (5). So in terms of $\varkappa > 0$ and $\gamma > 0$, (6) exhausts all requirements to the controller parameters.

To ensure (ii) of Remark 4.1, (9) employs $\max_\alpha$. Practical controller design may involve upper estimation or computation of this max in an analytical form. The latter is possible e.g., for the steady target, which gives rise to the following.

Corollary 1. *Let the target remain at a fixed position, $d_0 > R_2$, and in (2), $\gamma > 0$ and δ be chosen so that $\delta = \frac{v_2}{\gamma}\frac{\omega}{\sqrt{1+\omega^2}}$, where*

$$d_0^{-1}R_2(1+\omega)^2 + \omega + \overline{u}_2^{-1}\gamma < 1, \quad \omega > 0.$$

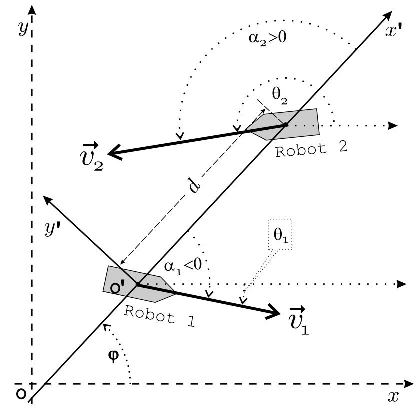

Fig. 1. The coordinate frames.

Then $d(t) \to d_0$ and $\dot{\psi}(t) \to w_\infty$ as $t \to \infty$, with $|w_\infty| = d_0^{-1}v_2$, if initially the robots are far enough from each other:

$$d(0) > R_2\left[3 + (1+\omega)^2(1 - \omega - \overline{u}_2^{-1}\gamma)^{-1}\right].$$

Here ω is an auxiliary variable (a one-to-one substitute for $v_* = \gamma\delta$). In general, an example of analytical though non-conservative sufficient condition for stability is based on the fact that the numerator in (10) does not exceed $(\rho + 1 + \varkappa)^2$, whereas the denominator is not less than $Q - \varkappa\overline{u}_2 - \gamma(\rho + 1)$, as is shown in (i) of Appendix. So the stability requirements (6) are satisfied whenever the point $(\gamma, \varkappa)$ belongs to the following set bounded by a parabola:

$$d_0^{-1}\overline{v}_1(\rho + 1 + \varkappa)^2 + \gamma(\rho + 1) + \overline{u}_2\varkappa < Q.$$

Since the feedback (2) may exhibit a sliding motion, the equivalent dynamics (Filippov, 1988) are addressed by the above results. The major obstacle to implementation of sliding mode controllers is a harmful phenomenon called "chattering" (Utkin, 1992), i.e., undesirable finite frequency oscillations around the ideal trajectory due to un-modeled system dynamics and constraints. The problem of chattering elimination and reduction has an extensive literature (see e.g., Lee, Utkin, & Malinin, 2009, for a survey). It offers a variety of effective approaches, including smooth approximation of the discontinuity, inserting low-pass filters/observers into the control loop, combining sliding mode and adaptive control techniques, higher order sliding modes, etc. When chattering is encountered in applications of the proposed controller, it can be subjected to treatment under the framework of this general discipline.

5. Proof of Proposition 3.1

We introduce the normally oriented Cartesian coordinate frame $0'x'y'$ centered at robot 1, with the abscissa axis directed towards robot 2. (See Fig. 1, where the world frame $0xy$ is depicted by the dashed lines and the dotted axes are parallel with the x-axis.) Let $\alpha_i \in (-\pi, \pi]$ be the directed angle from the axis $0'x'$ to the velocity vector $\vec{v}_i$ of the ith robot, and φ be the angle between the $0x$ and $0'x'$ axes. Then $\alpha_i = \theta_i - \varphi$ and with regard to (1), we see that

$$\dot{d} = v_2\cos\alpha_2 - v_1\cos\alpha_1, \qquad \dot{\alpha}_i = u_i - \dot{\varphi}, \quad \dot{\varphi} = d^{-1}[v_2\sin\alpha_2 - v_1\sin\alpha_1]. \tag{13}$$

Differentiation of the first equation yields that

$$\begin{aligned}
\ddot{d} &= -v_2\dot{\alpha}_2\sin\alpha_2 + v_1\dot{\alpha}_1\sin\alpha_1 \\
&\overset{(13)}{=\!=} -v_2(u_2 - \dot{\varphi})\sin\alpha_2 + v_1(u_1 - \dot{\varphi})\sin\alpha_1 \\
&= \dot{\varphi}\underbrace{\left[v_2\sin\alpha_2 - v_1\sin\alpha_1\right]}_{=d\dot{\varphi}\text{ by }(13)} + v_1u_1\sin\alpha_1 - v_2u_2\sin\alpha_2 \\
&= d\dot{\varphi}^2 + v_1u_1\sin\alpha_1 - v_2u_2\sin\alpha_2.
\end{aligned} \tag{14}$$

Nominal maintainability of d_0 means that by applying a proper control, robot 2 can keep the system state on the surface $d = d_0, \dot{d} = 0$ in the configuration space. This should be true at any point on the surface and irrespective of the motion of robot 1. To remain on the surface, it is necessary and sufficient to ensure that $\ddot{d} = 0$. Given a point on the surface and v_1, the sum of the first two addends in the last expression from (14) may take any value from the interval $\Delta_{\text{uncont}} := [d_0\dot{\varphi}^2 - v_1\overline{u}_1|\sin\alpha_1|, d_0\dot{\varphi}^2 + v_1\overline{u}_1|\sin\alpha_1|]$ depending on the feasible motion of robot 1, since the only constraint on u_1 is $|u_1| \le \overline{u}_1$. Robot 2 can manipulate only the third addend in the above expression. This addend ranges over $\Delta_{\text{cont}} := [-v_2\overline{u}_2|\sin\alpha_2|, v_2\overline{u}_2|\sin\alpha_2|]$ as the control u_2 runs over the admissible set $u_2 \in [-\overline{u}_2, \overline{u}_2]$. So $\ddot{d} = 0$ can be ensured by robot 2 irrespective of the control inputs u_1 of robot 1 iff $\Delta_{\text{uncont}} \subset \Delta_{\text{cont}}$. With regard to the last formula from (13), this is equivalent to the following inequality

$$d_0^{-1}\big[v_2\sin\alpha_2 - v_1\sin\alpha_1\big]^2 + v_1\overline{u}_1|\sin\alpha_1| \le v_2\overline{u}_2|\sin\alpha_2|, \qquad (15)$$

which thus should hold for all $v_1 \in [0, \overline{v}_1]$ and all points on the surface $d = d_0, \dot{d} = 0$. Here by (13), $\dot{d} = 0$ means that

$$v_2\cos\alpha_2 = v_1\cos\alpha_1. \qquad (16)$$

The criterion (15) for maintainability can be simplified by noting that any substitution $\alpha_1 \mapsto -\alpha_1$, $(\alpha_1, \alpha_2) \mapsto (\pi - \alpha_1, \pi - \alpha_2)$, $\alpha_2 \mapsto -\alpha_2$ keeps (16) true and that α_1 and α_2 can exceed $\pi/2$ only simultaneously by (16). Reversing the sign of α_1 followed by reversing the signs of both α_1 and α_2 if necessary, shapes (15) into

$$d_0^{-1}\big[v_2\sin\alpha_2 + v_1\sin\alpha_1\big]^2 + v_1\overline{u}_1\sin\alpha_1 \le v_2\overline{u}_2\sin\alpha_2, \qquad (17)$$

where $\alpha_i \in [0, \pi]$. Due to invariance under $(\alpha_1, \alpha_2) \mapsto (\pi - \alpha_1, \pi - \alpha_2)$, the attention can be limited to $\alpha_i \in [0, \pi/2]$.

Thus d_0 is nominally maintainable iff (17) is valid whenever $\alpha_i \in [0, \pi/2]$, $v_1 \in [0, \overline{v}_1]$, and (16) holds. If $\rho = \frac{v_2}{\overline{v}_1} < 1$, (17) is violated for $\alpha_2 := 0, v_1 := \overline{v}_1, \alpha_1 := \arccos\rho$. So the first inequality in (4) is necessary for maintainability of d_0, and further analysis can be focused on the case where $\rho \ge 1$ (and so $v_2 \ge \overline{v}_1 \ge v_1$). Then (16) is compatible with any $v_1 \in [0, \overline{v}_1]$, $\alpha_1 \in [0, \pi/2]$ and implies that $v_2\sin\alpha_2 = \sqrt{v_2^2 - v_2^2\cos^2\alpha_2} = \sqrt{v_2^2 - p^2}$, where $p := v_1\cos\alpha_1$, $q := v_1\sin\alpha_1$. So the criterion (17) takes the form:

$$F := d_0^{-1}\left[\sqrt{v_2^2 - p^2} + q\right]^2 + \overline{u}_1 q \le \overline{u}_2\sqrt{v_2^2 - p^2} \qquad (18)$$

whenever $p, q \ge 0$ and $p^2 + q^2 \le \overline{v}_1^2$. Since F increases as $q \uparrow$, the worst scenario for (18) to be true occurs when q attains its maximum $\sqrt{\overline{v}_1^2 - p^2}$ for a given p. It follows that (18) can be examined only on the arc $p, q \ge 0$, $p^2 + q^2 = \overline{v}_1^2$. In its turn, this arc can be parameterized as $p = \overline{v}_1\cos\alpha$, $q = \overline{v}_1\sin\alpha$, $\alpha \in [0, \pi/2]$, which shapes (18) into

$$\frac{\overline{v}_1}{d_0}\Big[\underbrace{\sqrt{\rho^2 - \cos^2\alpha}}_{m} + \sin\alpha\Big]^2 \le \overline{u}_2 m - \overline{u}_1\sin\alpha, \qquad (19)$$

where α runs over $[0, \pi/2]$. A prerequisite for (19) to be true is that the right-hand side is positive, i.e.,

$$\overline{u}_1\sin\alpha < \overline{u}_2\sqrt{\rho^2 - \cos^2\alpha} \quad \forall\alpha \in (0, \pi/2]$$
$$\overset{\rho\ge1}{\Longleftrightarrow} \overline{u}_1^2\sin^2\alpha < \overline{u}_2^2\left(\rho^2 - \cos^2\alpha\right) \quad \forall\alpha \in (0, \pi/2]$$
$$\Leftrightarrow \overline{u}_2^2\rho^2 > \overline{u}_1^2\sin^2\alpha + \overline{u}_2^2\cos^2\alpha \quad \forall\alpha \in (0, \pi/2]$$
$$\overset{\rho\ge1}{\Longleftrightarrow} \overline{u}_2^2\rho^2 > \overline{u}_1^2 \Leftrightarrow \overline{u}_2\rho > \overline{u}_1 \overset{\rho = v_2/\overline{v}_1}{\Longleftrightarrow} \overline{u}_2 v_2 > \overline{u}_1\overline{v}_1.$$

The last inequality is the second condition on the right in (A.1). Under this condition, (19) takes the form of the last inequality from (A.1). Formula (A.1) completes the proof.

6. Proofs of Theorem 4.1 and Proposition 4.1

We assume that robot 2 is driven by the controller (2). We first show that under some circumstances, sliding motion is commenced (Lemma 6.1), then is never terminated, and results in achievement of the control objective (Lemma 6.2). Finally we show that sooner or later, these circumstances are encountered if (12) holds (Lemmas 6.3 and 6.4).

Lemma 6.1. *In the domain $d > d_*$, the surface $S := \dot{d} + \chi(d - d_0) = 0$ is sliding if $\sin\alpha_2 > 0$, and repelling if $\sin\alpha_2 < 0$, where d_* is given by (9). On this surface,*

$$|\sin\alpha_2| \ge \frac{\sqrt{\rho^2 - 1} - \varkappa}{\rho}, \qquad |\dot{\varphi}| \ge \frac{\overline{v}_1}{d}(\rho - 1 - \varkappa). \qquad (20)$$

Remark 6.1. In (20), $\sqrt{\rho^2 - 1} - \varkappa > 0$.

Indeed, putting $w := \overline{u}_1/\overline{u}_2$, we see that

$$\varkappa \overset{(6)}{<} \frac{Q}{\overline{u}_2} = \begin{cases} \sqrt{\rho^2 - 1}\sqrt{1 - w^2} \le \sqrt{\rho^2 - 1} & \text{if } \rho^{-1} > w \\ \rho - w \le \rho - \rho^{-1} & \text{if } w \ge \rho^{-1}, \end{cases}$$

where $\rho - \rho^{-1} = \sqrt{\rho^2 - 1}\sqrt{1 - \rho^{-2}} < \sqrt{\rho^2 - 1}$.

Proof of Lemma 6.1. For $v(t) := -\chi(d - d_0)$,

$$|v(t)| \overset{(3)}{\le} v_* \overset{(5)}{=} \overline{v}_1\rho_*, \qquad (21)$$

$$|\dot{v}| \overset{(3)}{\le} \gamma|\dot{d}| \overset{(13)}{\le} \gamma(v_1 + v_2) \overset{(1)}{\le} \gamma(v_2 + \overline{v}_1). \qquad (22)$$

On the surface $\dot{d} + \chi(d - d_0) = 0$, the first relation in (13) implies that $v_2\cos\alpha_2 = p + v$, where $p := v_1\cos\alpha_1$ and $q := v_1|\sin\alpha_1|$. Hence $|v_2\sin\alpha_2| = \sqrt{v_2^2 - v_2^2\cos^2\alpha_2} = \sqrt{v_2^2 - (p + v)^2}$. Due to the final expression for $\ddot{d}$ in (14),

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \ddot{d} - \dot{v} = v_2|\sin\alpha_2|\left[b - u_2\,\mathrm{sgn}(\sin\alpha_2)\right], \qquad (23)$$

where $b = \dfrac{\frac{\{v_2\sin\alpha_2 - v_1\sin\alpha_1\}^2}{d} + v_1u_1\sin\alpha_1 - \dot{v}}{v_2|\sin\alpha_2|}$.

Here $|u_1| \le \overline{u}_1$ by (1), $|\dot{v}|$ is estimated in (22), and $|\{\ldots\}| \le v_2|\sin\alpha_2| + v_1|\sin\alpha_1| = \sqrt{v_2^2 - (p + v)^2} + q$. Hence

$$|b| \le \frac{\frac{\left[\sqrt{v_2^2 - (p+v)^2} + q\right]^2}{d} + \overline{u}_1 q + \gamma(v_2 + \overline{v}_1)}{\sqrt{v_2^2 - (p + v)^2}}.$$

The right-hand side of this inequality increases as $q \uparrow$, whereas $q = v_1|\sin\alpha_1| = \sqrt{v_1^2 - v_1^2\cos^2\alpha_1} = \sqrt{v_1^2 - p^2} \le \overline{q} := \sqrt{\overline{v}_1^2 - p^2}$ since $v_1 \le \overline{v}_1$ by (1). So replacement of q by $\overline{q}$ keeps the inequality true. By noting that $p^2 + \overline{q}^2 = \overline{v}_1^2$ and so $p = \overline{v}_1\cos\alpha$, $\overline{q} := \overline{v}_1\sin\alpha$ for some $\alpha \in [0, \pi/2]$, the resultant inequality shapes into

$$|b| \le \frac{\frac{\overline{v}_1}{d}\left[f(r) + \sin\alpha\right]^2 + \overline{u}_1\sin\alpha + \gamma(\rho + 1)}{f(r)},$$

$$\text{where } f(r) := \sqrt{\rho^2 - (\cos\alpha + r)^2} = \rho|\sin\alpha_2|, \qquad (24)$$

$r := v/\overline{v}_1$, $\rho = \frac{v_2}{\overline{v}_1}$, and $|r| \le \rho_*$ by (21). So

$$|\cos\alpha + r| \le 1 + \rho_* \overset{(5)}{<} \rho \Rightarrow \left|f'(r)\right| = \frac{|\cos\alpha + r|}{\sqrt{\rho^2 - (\cos\alpha + r)^2}} \le \frac{\rho}{\sqrt{\rho^2 - (1 + \rho_*)^2}} \overset{(8)}{=} \frac{\varkappa}{\rho_*}.$$

Due to Remark 6.1, $f(0)-\varkappa \geq \sqrt{\rho^2-1}-\varkappa > 0$. Hence

$$f(0)-\varkappa \leq f(r) \leq f(0)+\varkappa;\ |b|-\overline{u}_2$$
$$\leq \frac{\frac{\overline{v}_1}{d}\left[f(0)+\varkappa+\sin\alpha\right]^2+\overline{u}_1\sin\alpha+\gamma(\rho+1)}{f(0)-\varkappa}-\overline{u}_2$$
$$\overset{(11)}{=\!=} \frac{\frac{\overline{v}_1}{d}\left[f(0)+\varkappa+\sin\alpha\right]^2-\zeta(\alpha|\rho_*,\gamma)}{f(0)-\varkappa}$$
$$\overset{(10)}{=\!=} \frac{\zeta(\alpha|\rho_*,\gamma)}{d\left(\sqrt{\rho^2-\cos^2\alpha}-\varkappa\right)}\times\left[\overline{v}_1\eta(\alpha|\rho_*,\gamma)-d\right]. \quad (25)$$

The last expression is strictly negative for $d > d_*$ due to (9) and (i) of Remark 4.1. So for $u_2 = \pm\overline{u}_2$, the sign of the left-hand side of (23) equals $-\operatorname{sgn}(\sin\alpha_2)\operatorname{sgn} u_2$. This and (2) imply the first claim of the lemma. Thanks to (24),

$$|\sin\alpha_2| = \frac{f(r)}{\rho} \overset{(25)}{\geq} \frac{f(0)-\varkappa}{\rho} \overset{(24)}{=\!=} \frac{\sqrt{\rho^2-\cos^2\alpha}-\varkappa}{\rho},$$

which implies the first inequality in (20). Furthermore,

$$|\dot{\varphi}| \overset{(13)}{=\!=} d^{-1}|v_2\sin\alpha_2 - v_1\sin\alpha_1|$$
$$\geq \frac{1}{d}\Big[v_2|\sin\alpha_2| - \underbrace{v_1|\sin\alpha_1|}_{=q\leq\overline{q}=\overline{v}_1\sin\alpha}\Big] \overset{(24)}{\geq} \frac{\overline{v}_1}{d}\left[f(r)-\sin\alpha\right]$$
$$\overset{(25)}{\geq} d^{-1}\overline{v}_1\left[\sqrt{\rho^2-\cos^2\alpha}-\sin\alpha-\varkappa\right] =: h(\alpha).$$

It is easy to check that $h'(\alpha) \leq 0\ \forall\alpha\in[0,\pi/2]$. So $|\dot{\varphi}| \geq h(\alpha) \geq h(\pi/2) = d^{-1}\overline{v}_1(\rho-1-\varkappa)$, i.e., (20) does hold. □

Lemma 6.2. *If $\dot{d}+\chi(d-d_0) = 0$ at some time instant t_0 when $d > d_*$ and $\sin\alpha_2 > 0$, then $d \to d_0$ as $t\to\infty$ and*

$$\varliminf_{t\to\infty} |\dot{\varphi}| \geq d_0^{-1}(v_2-\overline{v}_1). \quad (26)$$

Convergence $d \to d_0$ is exponentially fast $|d-d_0| \leq ce^{-\gamma t}$.

Proof. For $t \geq t_0$, Lemma 6.1 and Remark 6.1 imply that a sliding motion occurs and $\sin\alpha_2 > 0$ while $d > d_*$. During this motion, $\dot{y} = -\chi(y)$, where $y := d-d_0$. Since $y\chi(y) > 0\ \forall y\neq 0$, $\chi(0) = 0$, any solution d of the sliding mode differential equation monotonically converges to d_0, where $d_0 > d_*$ by (6). So $d > d_*\ \forall t \geq t_0$, the sliding mode will never be terminated, and $d\to d_0$ as $t\to\infty$. So for large enough t, the equation $\dot{y} = -\chi(y)$ takes the form $\dot{y} = -\gamma y$ due to (3), which implies that $|d-d_0| \leq ce^{-\gamma t}$.

Hence for any $\varepsilon\in(0,\delta)$ (where δ is the controller parameter from (3)), there exists $t_* \geq t_0$ such that $|d(t)-d_0| < \varepsilon\ \forall t \geq t_*$. For such t, the output u_2 equals that of the controller (2) based on the function (3) with the same γ but altered $\delta := \varepsilon$. It follows that for large t, the angle φ obeys the second inequality in (20) written for the altered controller: $|\dot{\varphi}| \geq \frac{\overline{v}_1}{d}[\rho-1-\varkappa(\varepsilon)]$, where $\rho = v_2/\overline{v}_1$ by (4) and the alteration influences only the parameter $\varkappa = \varkappa(\varepsilon)$. Letting $t\to\infty$ yields that $\varliminf_{t\to\infty}|\dot{\varphi}| \geq d_0^{-1}(v_2-\overline{v}_1)-d_0^{-1}\overline{v}_1\varkappa(\varepsilon)$. To obtain (26), it remains to let $\varepsilon\to 0$ and note that $\varkappa(\varepsilon)\to 0$ due to (8) and (5) (where $\rho_* = \varepsilon\gamma\overline{v}_1^{-1}$ for $\delta := \varepsilon$). □

To complete the proof of Theorem 4.1, it basically remains to show that both relations $S := \dot{d}+\chi(d-d_0) = 0$ and $\sin\alpha_2 > 0$ become true sooner or later. To this end, we note that in the domain $\pm S > 0$, robot 2 is driven by $u_2 \equiv \pm\overline{u}_2$ due to (2). The next lemma shows that during the related motion, the system inevitably arrives at the surface $S = 0$, irrespective of the motion of robot 1.

Lemma 6.3. *Let (12) hold and robot 2 be driven by the constant control $u_2 \equiv \pm\overline{u}_2$. Then $S = 0$ at a time instant*

$$t_0 \leq \frac{3\pi}{\beta_*}, \quad \text{where } \beta_* := R_2^{-1}\left[v_2-\overline{v}_1\right]. \quad (27)$$

Proof. Let $u_2 \equiv \overline{u}_2$ for the definiteness. It suffices to show that $\pm\dot{d}(t_\pm) > v_* = \gamma\delta$ for some $t_+, t_- \leq 3\pi/\beta_*$. Indeed, then $\pm S > 0$ at time $t = t_\pm$ thanks to (3), where S continuously evolves over time. Hence $S = 0$ at some time instant between t_- and t_+, which completes the proof.

Under $u_2 \equiv \overline{u}_2$, robot 2 moves along a circle with the radius R_2. The inequalities $\dot{d} > v_*$ and $\dot{d} < -v_*$ hold if the straight line connecting the robots is tangential to this circle and robot 2 moves outwards and towards robot 1, respectively:

$$\cos\alpha_2 = 1 \overset{(13)}{\Rightarrow} \dot{d} = v_2 - v_1\cos\alpha_1 \geq v_2-\overline{v}_1 \overset{(5)}{>} v_* \quad (28)$$

and $\cos\alpha_2 = -1 \Rightarrow \dot{d} < -v_*$. We introduce the variables

$$p := d\cos\alpha_2, \qquad q := d\sin\alpha_2, \qquad q_- := q - R_2 \quad (29)$$

and note that in terms of them,

$$\cos\alpha_2 = \pm 1 \Leftrightarrow p = \pm d \quad\text{and}\quad q_- = -R_2, \quad (30)$$

whereas (13) yields that

$$\dot{p} = -u_2q_- - v_1\cos\Delta, \quad \text{where } \Delta := \alpha_2-\alpha_1$$
$$\dot{q}_- = u_2p - v_1\sin\Delta, \quad \text{where } \dot{\Delta} = u_2-u_1.$$

In their turn, these relations imply the following equations for the polar coordinates l, β of $\mathbf{col}(p, q_-)$:

$$\begin{array}{l}\dot{l} = -v_1\cos(\beta-\Delta)\\ \dot{\beta} = u_2 + \dfrac{v_1}{l}\sin(\beta-\Delta)\end{array}, \quad \dot{\Delta} = u_2-u_1. \quad (31)$$

We are going to show first that whenever $d > R_2$,

$$\begin{array}{l}\beta = \mu_+ := -\arcsin\dfrac{R_2}{\sqrt{R_2^2+d^2}} \Rightarrow \dot{d} > v_*\\ \beta = \pi+\mu_+ \Rightarrow \dot{d} < -v_*.\end{array} \quad (32)$$

The focus in the proof is on the first entailment; the second one is established likewise. If $\beta = \mu_+$, we have

$$p = l\cos\beta = l\sqrt{1-\sin^2\beta} = \frac{ld}{\sqrt{R_2^2+d^2}};$$

$$q = R_2 + q_- = R_2 + l\sin\beta = R_2 - R_2l\left(\sqrt{R_2^2+d^2}\right)^{-1/2};$$

$$d^2 = p^2+q^2 = l^2 - 2R_2^2\left(\sqrt{R_2^2+d^2}\right)^{-1} l + R_2^2.$$

The product of the roots l_1, l_2 of the last quadratic equation amounts to $l_1l_2 = R_2^2-d^2$ by the Viéta's formula. For $d > R_2$, this product is negative and so l equals the positive root $\sqrt{R_2^2+d^2}$. Then $p = d$, $q = 0$, $q_- = q-R_2 = -R_2$ and thus $\cos\alpha_2 = 1$ by (30), which implies $\dot{d} > v_*$ by (28).

Now we show that the antecedents of the entailments (32) do hold at some time instants. Due to (29), $d = \|\mathbf{col}(p,q)\|$ and $l = \|\mathbf{col}(p,q_-)\|$, where $\mathbf{col}(p,q)-\mathbf{col}(p,q_-) = \mathbf{col}(0,R_2)$. Hence

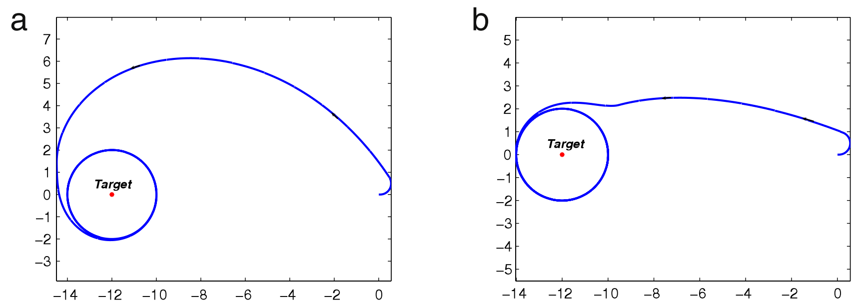

Fig. 2. Steady target: (a) $v_* = 0.25$; (b) $v_* = 0.45$.

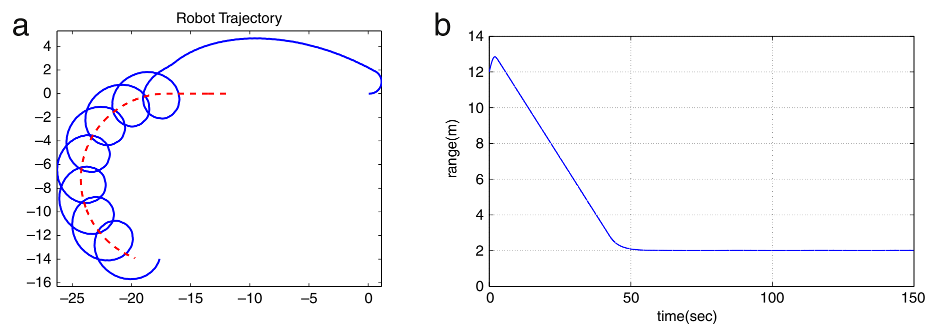

Fig. 3. Moving target; (a) trajectories; (b) relative range.

$|l-d| \le R_2$. Since $\dot{l} \ge -v_1 \ge -\overline{v}_1$ by (31) and (1), we have for $0 \le t \le t_l = \overline{v}_1^{-1}\big[d(0)-3R_2\big]$:

$$l(t) = l(0) + \int_0^t \dot{l}(\theta)\,\mathrm{d}\theta \ge l(0) - \overline{v}_1 t \ge \big[d(0)-R_2\big] - \overline{v}_1 t_l \ge 2R_2 > R_2; \quad d(t) \ge l(t) - R_2 \ge R_2; \tag{33}$$

$$\dot{\beta} \overset{(31),u_2\equiv\overline{u}_2,l\ge R_2}{\ge} \overline{u}_2 - \frac{\overline{v}_1}{R_2} \overset{R_2=v_2/\overline{u}_2}{=\!=\!=} R_2^{-1}\big[v_2-\overline{v}_1\big] = \beta_* > 0.$$

So the angle β increases as t runs over $[0, t_\rho]$. Since

$$t_l = \overline{v}_1^{-1}\big[d(0)-3R_2\big] \overset{(12)}{>} \frac{3R_2}{v_2-\overline{v}_1}\pi = \frac{3\pi}{\beta_*},$$

this angle ranges over an interval whose length exceeds 3π. Moreover, this holds as t runs over the smaller interval $[0, 3\pi/\beta_*] \subset [0, t_l]$. The same is thus true for the angle $\beta_+ := \beta - 2\pi j_+$, where the integer j_+ is such that $\beta_+(0) \in [-2\pi-\pi/2, -\pi/2)$. It follows that β_+ continuously runs across the entire interval $[-\pi/2, 0]$, which accommodates all values of the continuous function $\mu_+(t)$ from (32). Hence at some time $t_+ \in [0, 3\pi/\beta_*]$, the functions $\beta_+(t)$ and $\mu_+(t)$ take a common value: $\beta(t_+) = \mu_+(t_+) + 2\pi j_+$. Similarly, $\beta(t_-) = \mu_-(t_-) + 2\pi j_-$ for some $t_- \in [0, 3\pi/\beta_*]$ and an integer j_-, where $\mu_-(t) := \pi + \mu_+(t) \in [\pi/2, \pi]$. By (32), $\dot{d}(t_+) > v_*$ and $\dot{d}(t_-) < -v_*$, which completes the proof. □

Lemma 6.4. *Both relations $S := \dot{d} + \chi(d-d_0) = 0$ and $\sin\alpha_2 > 0$ become true at some time instant t_0 provided that (12) holds. For the first such time,*

$$d(t_0) \ge d(0) - 3R_2\left[1 + \pi\overline{v}_1(v_2-\overline{v}_1)^{-1}\right]. \tag{34}$$

Proof. Suppose that these relations are not true initially. Since $d(0) > d_*$ by (12), we conclude (with regard to the second claim from Lemma 6.1) that $S \neq 0$ for $t > 0$, $t \approx 0$. So until the first time t_0 when $S = 0$ (if there is no such a time, we put $t_0 := \infty$), robot 2 moves with $u_2 \equiv \pm\overline{u}_2$ due to (2). By Lemma 6.3, $t_0 < \infty$ and (27) holds, whereas $\sin\alpha_2(t_0) > 0$ by Lemma 6.1. Furthermore,

$$\begin{aligned} d(t_0) &\overset{(33)}{\ge} l(t_0) - R_2 \overset{(33)}{\ge} \big[d(0)-R_2\big] - \overline{v}_1 t_0 - R_2 \\ &\overset{(27)}{\ge} d(0) - 2R_2 - \overline{v}_1\frac{3\pi R_2}{v_2-\overline{v}_1} \\ &\ge d(0) - 3R_2\left[1 + \pi\frac{\overline{v}_1}{v_2-\overline{v}_1}\right]. \quad \square \end{aligned}$$

Proof of Theorem 4.1 comes to the reference to Lemmas 6.2 and 6.4 since (12) and (34) ensure the assumption $d(t_0) > d_*$ of Lemma 6.2.

Proof of Proposition 4.1. Since $\psi = -\varphi$, this claim follows from (26), Lemma 6.4, and the inequality $d(t_0) > d_*$.

7. Simulation results

The simulation data are given in the Table 1, where $\mathbf{x}_i := \mathbf{col}(x_i, y_i, \theta_i)$. The required distance $d_0 = 2m$. The controller parameters $\delta = 0.83$, $\gamma = 0.3$ meet (5) and (6). Fig. 2 concerns the case of the steady target: the pursuer approaches the target along a spiral-like trajectory and ultimately circulates around it with $d(t) \to d_0$ as $t \to \infty$. As $v_* \to (v_2 - v_1)$, the spiral approaches a straight line, i.e.,the optimal transient trajectory, as is illustrated by Fig. 2(b). Fig. 3 deals with a maneuvering target whose path is represented by the dashed curve. After a transient, the pursuer moves at the predefined speed exceeding that of the target along a curve wheeling round the target trajectory so that the required range margin is kept true, as is shown in Fig. 3(b).

Measurement noise implications were also examined, with a focus on worst case scenarios. The derivative $\dot{d}$ was computed from the range measurements corrupted by an additive Gaussian white noise. Although many modern sensors reach distances of several tens of meters (concerned in simulations) with accuracies of several mm, the focus was on much worse sensing. Fig. 4 is concerned with a steady target and the simplest two-point finite difference derivative estimate. The robot very closely follows the entire 'noise-free' trajectory for the noise level $\sigma = 6$ mm and its

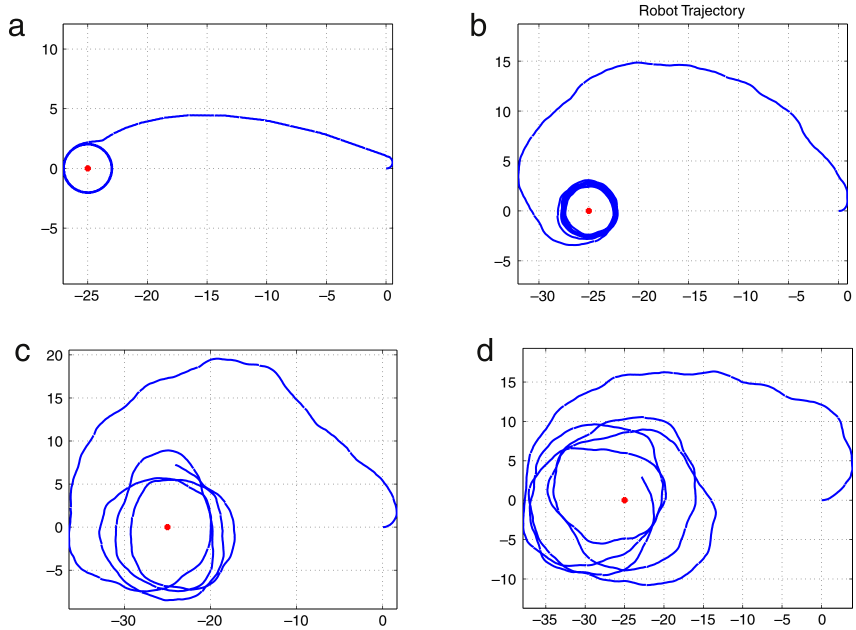

Fig. 4. Noise level: (a) 6 mm; (b) 0.1 m; (c) 0.3 m; (d) 0.5 m.

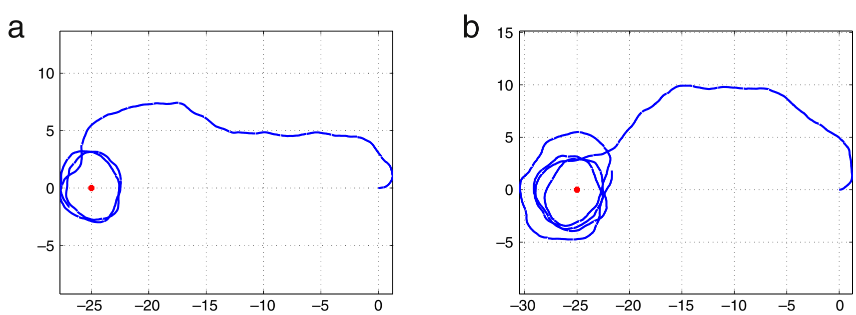

Fig. 5. Noise level (a) 0.5 m; (b) 1 m.

Table 1
Simulation data.

Parameter	Value	Comments
$\mathbf{x}_2(0)$	$[0; 0; 0]'$	Pursuer initial position
v_2	0.5 m/s	Pursuer linear velocity
$\overline{u}_2$	1 rad/s	Pursuer maximum angular velocity
$\mathbf{x}_1(0)$	$[-12$ m; 0; $\pi]'$	Target initial position
v_1	0.15 m/s	Target linear velocity
$\overline{u}_1$	0.5 rad/s	Target maximum angular velocity

key final 'circle' part for $\sigma = 10$ cm. Problems with keeping the distance $d_0 = 2$ m are visible for $\sigma = 30$ and 50 cm (bad sensors with 15% and 25% range errors). Merely averaging the observations over the sliding window of 10 samples puts the ultimate range accuracy approximately within the sensor resolution, which holds up to $\sigma = 1$ m (50% range error) as is illustrated in Fig. 5. This confirms the reputation of sliding-mode controllers as highly tolerable to noise.

8. Conclusions

Range-only navigation of a WMR towards a maneuvering target was addressed. A new control law was proposed and rigorously justified. The proposed controller drives the robot to the target and ultimately makes it circulate around the target at the required distance with the given speed. The performance of the proposed controller was confirmed via computer simulations. Future work includes extensions on the case where the speed is also a control parameter, as well as theoretical analysis of the robustness of the proposed control law against disturbances and model errors and uncertainties.

Appendix. Proof of Remark 4.1

(i) Since $\zeta(\alpha|\varkappa, \gamma) = f[\sin(\alpha)] - \overline{u}_2\varkappa - \gamma(\rho + 1)$ by (11), where $f(z) := \overline{u}_2\sqrt{\nu^2 + z^2} - \overline{u}_1 z$ and $\nu := \sqrt{\rho^2 - 1}$, it suffices to show that $Q := \min_{z \in [0,1]} f(z)$ is given by (7). Since $f'(z) = \overline{u}_2 z(\nu^2 + z^2)^{-1/2} - \overline{u}_1 = 0 \Leftrightarrow (\overline{u}_2^2 - \overline{u}_1^2)z^2 = \overline{u}_1^2\nu^2$ and $f'(0) < 0$, there are no roots of $f'(\cdot)$ in $[0, 1]$ and so $\mu = f(1)$ if $\rho\overline{u}_1 > \overline{u}_2$. Otherwise, $[0, 1]$ contains the unique root $z_* = \overline{u}_1\nu(\overline{u}_2^2 - \overline{u}_1^2)^{-1/2}$ and so $Q = f(z_*)$. The proof is completed by computation of $f(0)$ and $f(z_*)$.

(ii) We first show that

$$(4) \Leftrightarrow \underline{\rho \geq 1 \text{ and } \overline{u}_2\rho > \overline{u}_1} \quad \text{and} \quad d_0 \geq \overline{v}_1 \max_{\alpha \in [0,\pi/2]} \eta(\alpha), \tag{A.1}$$

$$\text{where } \eta(\alpha) := \frac{\left[\sqrt{\rho^2 - \cos^2\alpha} + \sin\alpha\right]^2}{\overline{u}_2\sqrt{\rho^2 - \cos^2\alpha} - \overline{u}_1 \sin\alpha}. \tag{A.2}$$

In (A.1), the underlined inequalities are clearly implied by (4). Let they be true. The denominator $\mathfrak{D}$ in (A.2) equals $f[\sin(\alpha)]$. So $\mathfrak{D} \geq Q$ by the foregoing. Due to (7), $\mathfrak{D} \geq 0$ and $\mathfrak{D} = 0 \Leftrightarrow \rho = 1$ and $\alpha = 0$. (In the last case, formula (A.2) should be modified into $\eta(0) := \lim_{\alpha\to+0}\eta(\alpha) = 0$.) It follows that max in (A.1) does exist and for $\rho = 1$, is attained at $\alpha > 0$.

For $z = \sin\alpha$, elementary differential calculus yields that

$$\frac{\mathrm{d}\eta}{\mathrm{d}z}(z)\frac{f(z)^2}{\left(z+\sqrt{\nu^2+z^2}\right)^2} = \underbrace{2\overline{u}_2 + \overline{u}_1 - (2\overline{u}_1+\overline{u}_2)q(z)}_{c},$$

where $q(z) := z(\nu^2+z^2)^{-1/2} \leq q(1) = \rho^{-1}$. Hence

$$\begin{aligned}\rho c \geq\ & (2\rho-1)\overline{u}_2 + (\rho-2)\overline{u}_1 \\ \overset{\overline{u}_2\rho>\overline{u}_1}{>}\ & \left[(2\rho-1)\rho^{-1}+(\rho-2)\right]\overline{u}_1 = \left[\rho-\rho^{-1}\right]\overline{u}_1 \geq 0.\end{aligned}$$

Hence max in (A.1) is attained at $z = 1 \Leftrightarrow \alpha = \pi/2$ and so equals $\eta(\pi/2) = (\rho+1)^2/(\overline{u}_2\rho - \overline{u}_1) = (v_2+\overline{v}_1)^2(\overline{u}_2 v_2 - \overline{u}_1\overline{v}_1)^{-1}\overline{v}_1^{-1}$, which completes the proof of (A.1).

To prove (ii), we note that the inequalities in (A.1) hold with $\geq$ replaced by $>$ due to Assumption 3.1, and $\eta(\alpha|0,0) = \eta(\alpha)$ by (10). This means that (5), (6) are true for $\gamma = \rho_* = 0$. The proof is completed by the continuity argument.

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