Planar Homographies

Most illustrations and slides adopted from:

Robert Collins
CSE486, Penn State

http://www.cse.psu.edu/~rtc12/CSE486/lecture16.pdf

$$^{c}T_{w} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_{x} \\ r_{21} & r_{22} & r_{23} & t_{y} \\ r_{31} & r_{32} & r_{33} & t_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Takes a point from World to Camera
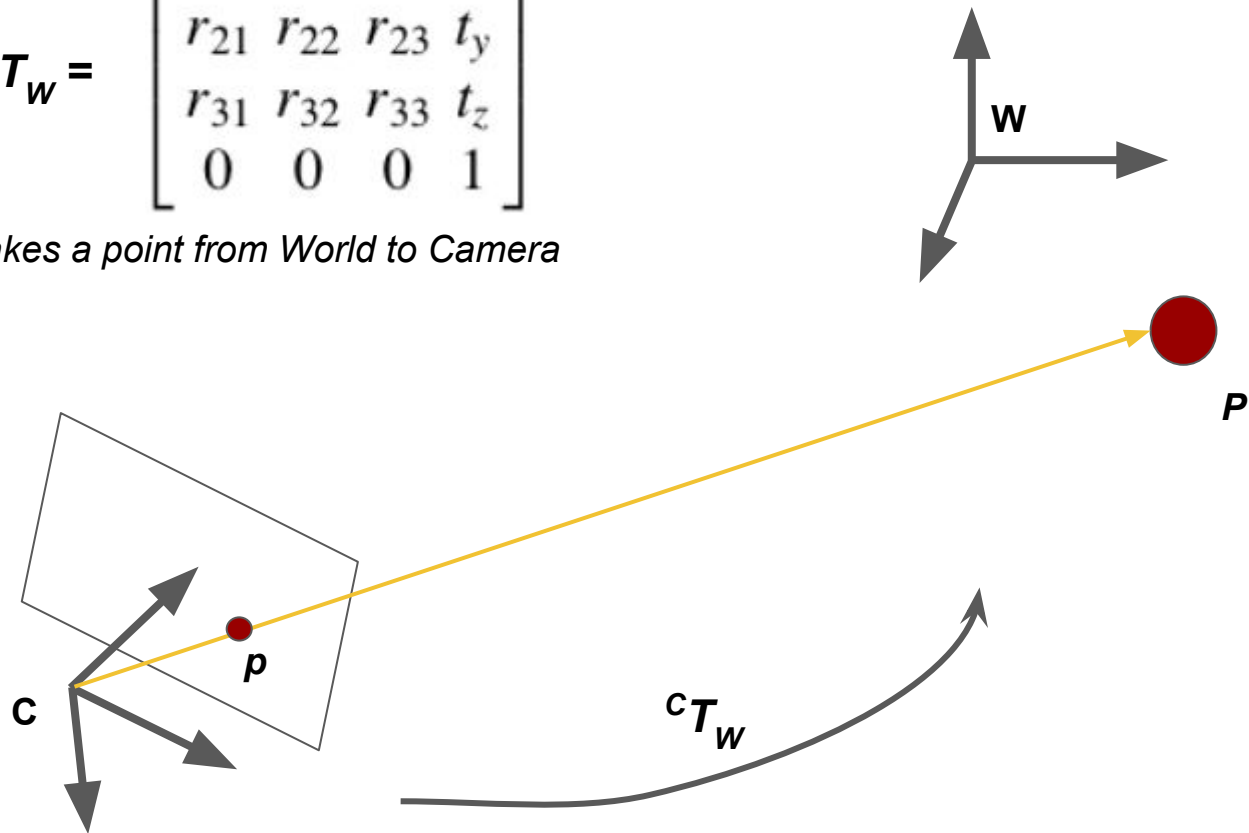
W

P

p

C

$^{c}T_{w}$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \pi \left(\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \right)$$
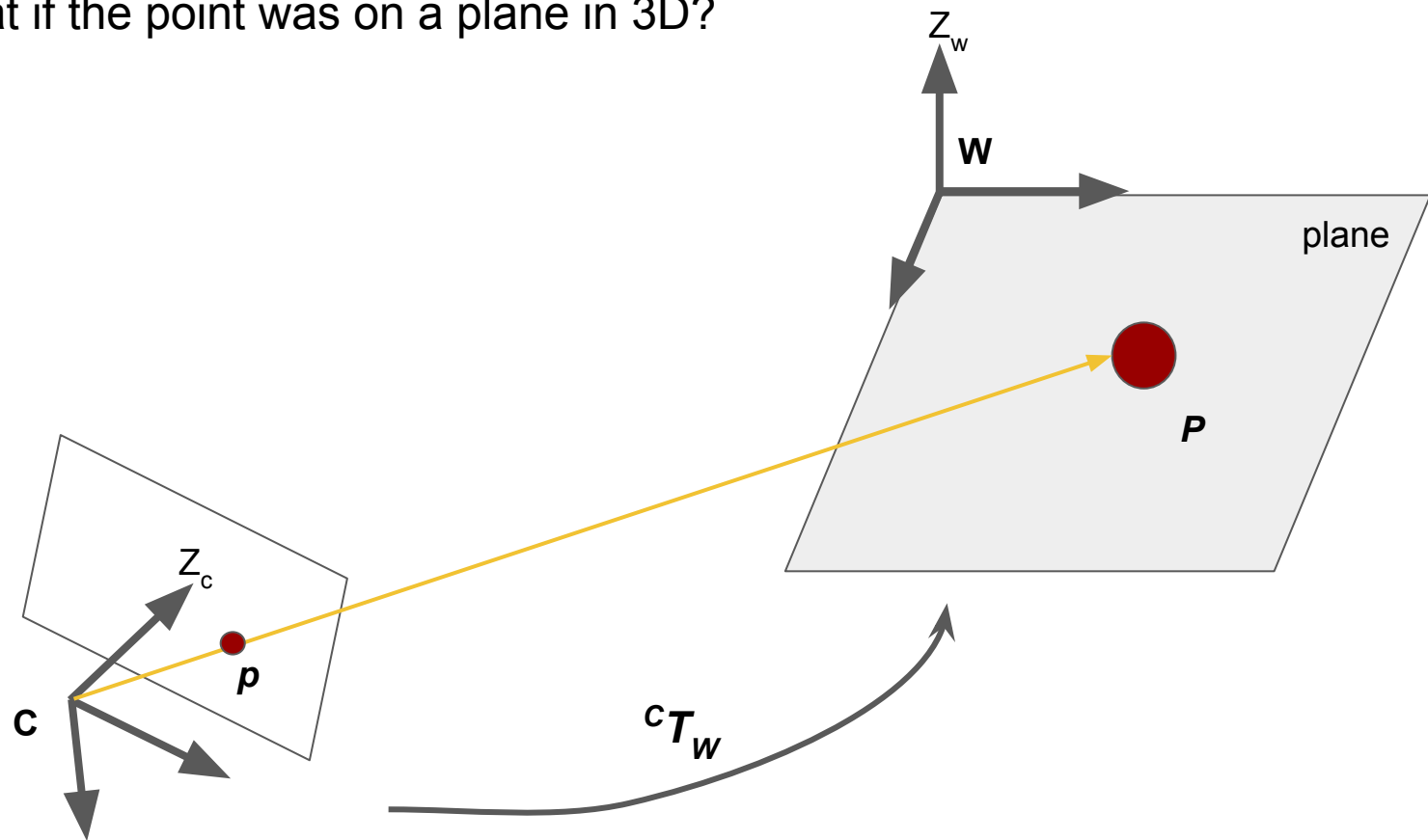
In Image

Projection

Perspective scaling

Taken to camera

In world

What if the point was on a plane in 3D?

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \pi \left(\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} \right)$$

In Image   Projection   Perspective scaling   Taken to camera   In world

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \boxed{r_{13}} & t_x \\ r_{21} & r_{22} & \boxed{r_{23}} & t_y \\ r_{31} & r_{32} & \boxed{r_{33}} & t_z \\ 0 & 0 & \boxed{0} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ \mathbf{0} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & \boxed{0} \\ 0 & f & 0 & \boxed{0} \\ 0 & 0 & 1 & \boxed{0} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \\ \boxed{0 \quad 0 \quad 1} \end{bmatrix} \begin{bmatrix} X \\ Y \\ \mathbf{0} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{1} \end{bmatrix}$$
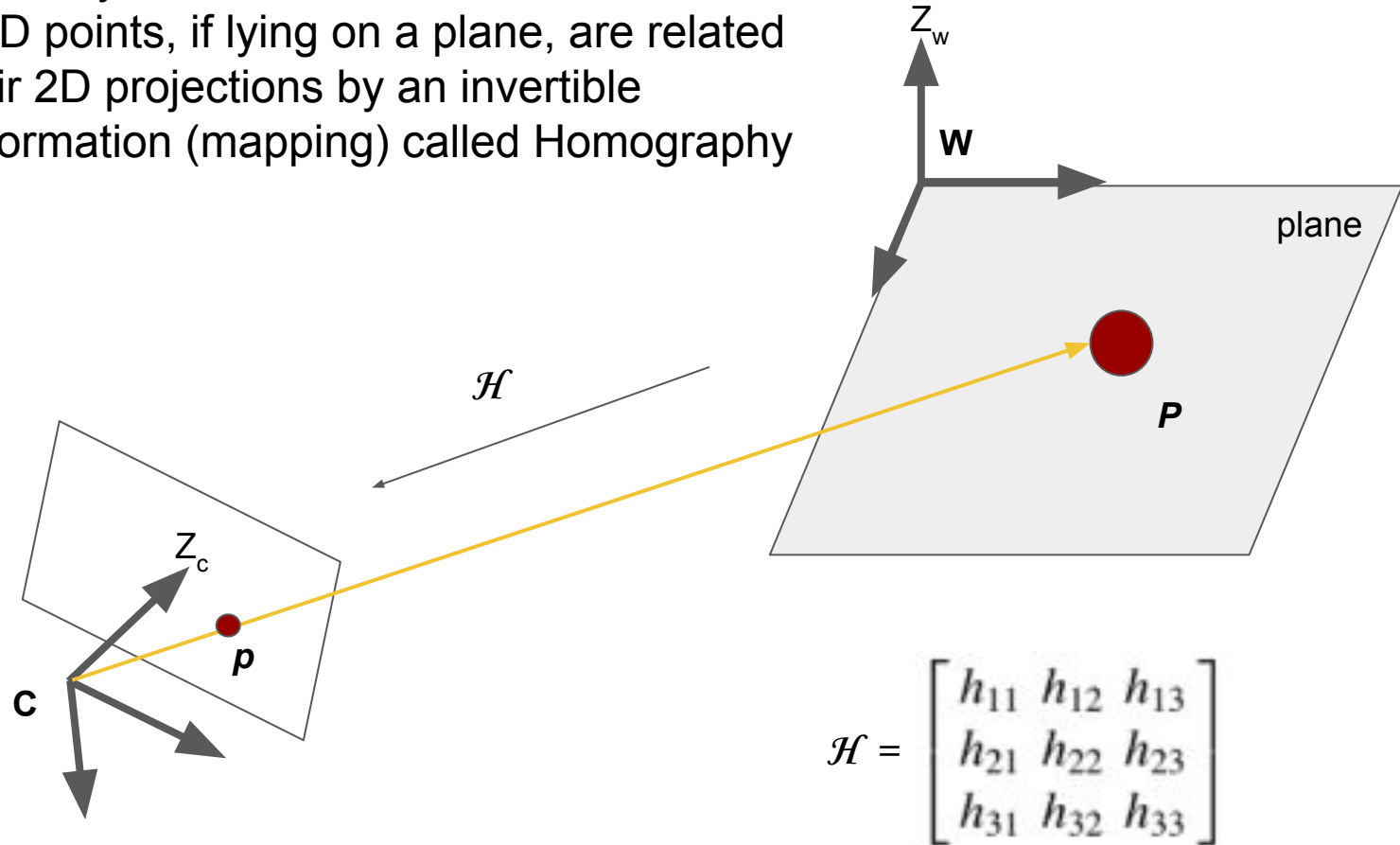
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$
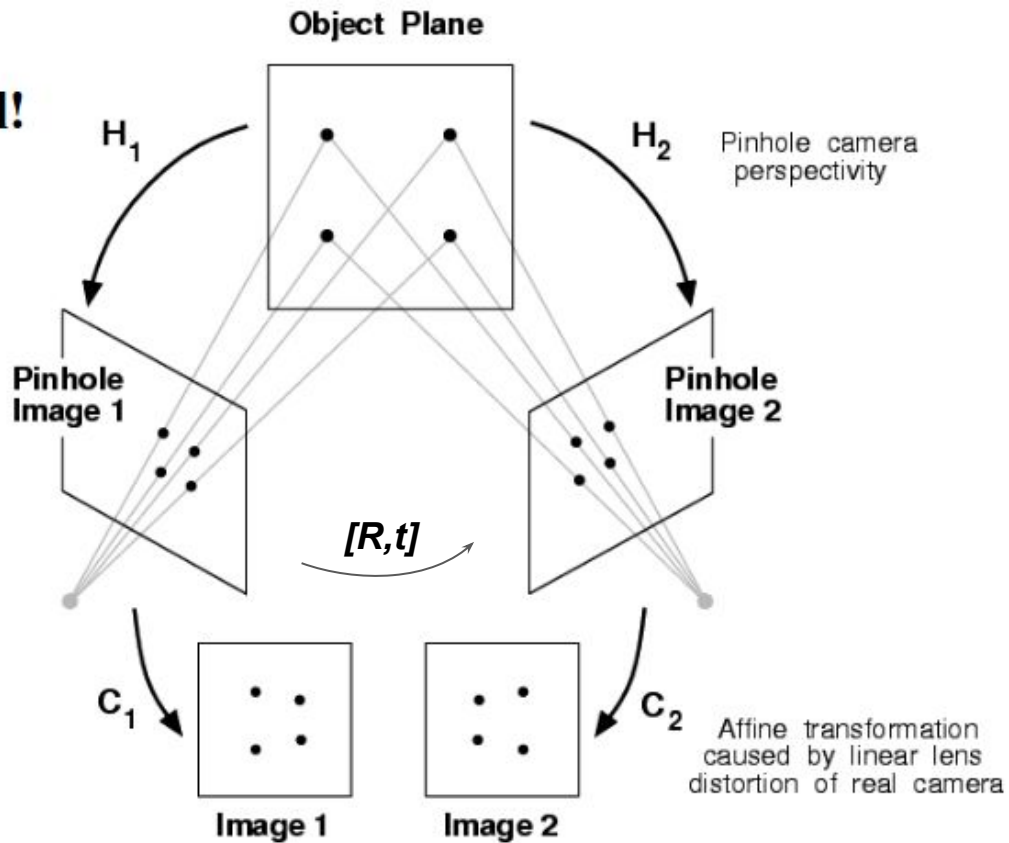
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$ **Homography H (planar projective transformation)**
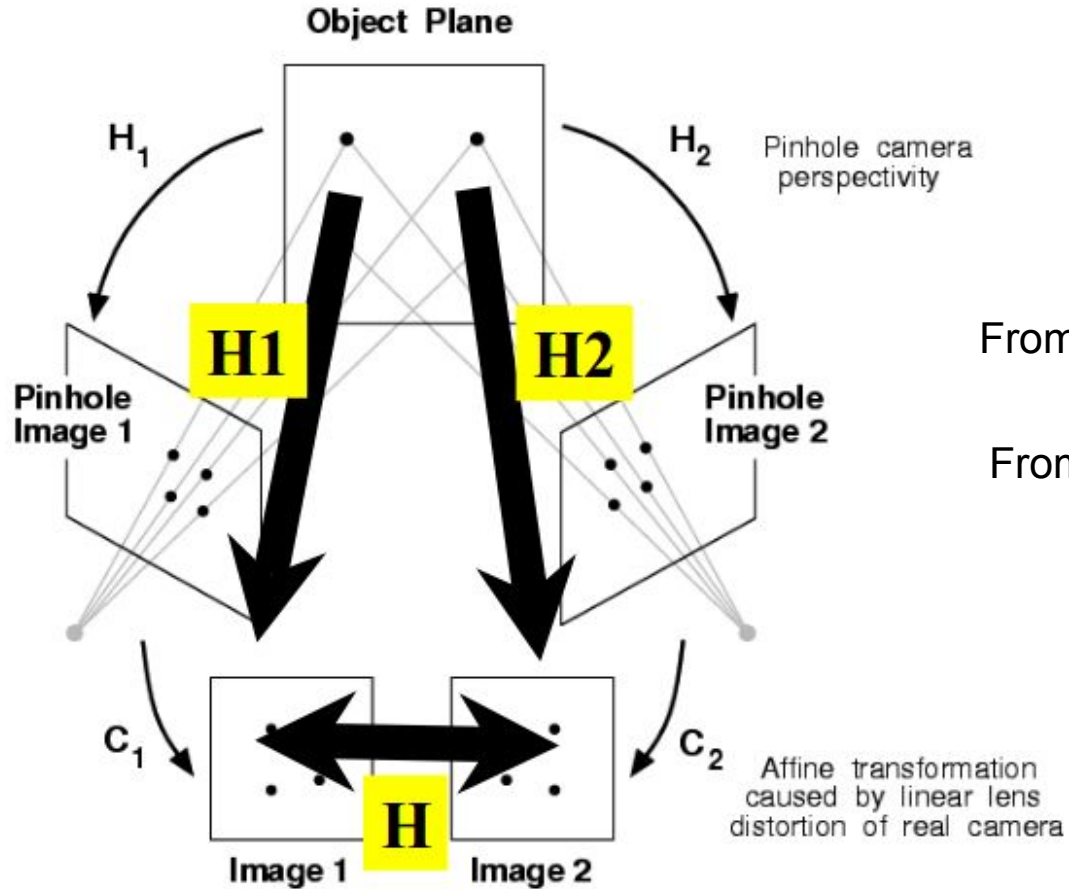
We can say:
The 3D points, if lying on a plane, are related to their 2D projections by an invertible transformation (mapping) called Homography

Z_w

W

plane

$\mathcal{H}$

P

Z_c

p

C

$$\mathcal{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Here's where transformation groups get useful!

Object Plane

H_1

H_2

Pinhole camera perspectivity

Pinhole Image 1

Pinhole Image 2

[R,t]

C_1

C_2

Affine transformation caused by linear lens distortion of real camera

Image 1

Image 2

Object Plane

H_1 Pinhole camera perspectivity

H_2

H1   **H2**

Pinhole Image 1   **Pinhole Image 2**

C_1 C_2 Affine transformation caused by linear lens distortion of real camera
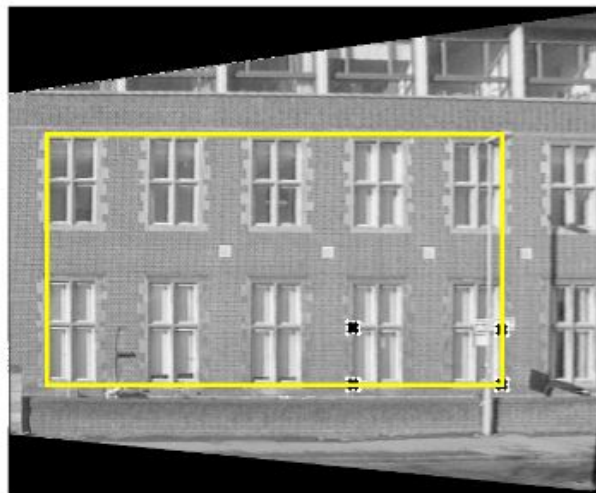
H

Image 1   Image 2

From Image1 to Image2: $\mathcal{H} = \mathcal{H}1^{-1} * \mathcal{H}2$

From Image2 to Image1: $\mathcal{H} = \mathcal{H}2^{-1} * \mathcal{H}1$
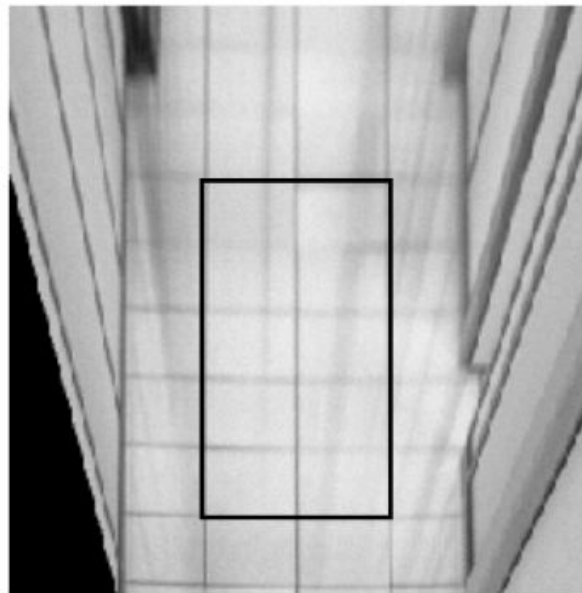
Applying Homographies to Remove Perspective Distortion

from Hartley & Zisserman

4 point correspondences suffice for
the planar building facade

Robert Collins
CSE486, Penn State

Homographies for
Bird's-eye Views

from Hartley & Zisserman

Estimating a Homography

Matrix Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Equations:

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Degrees of Freedom?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

There are 9 numbers $h_{11},\ldots,h_{33}$, so are there 9 DOF?

No. Note that we can multiply all h_{ij} by nonzero k without changing the equations:

$$x' = \frac{k h_{11} x + k h_{12} y + k h_{13}}{k h_{31} x + k h_{32} y + k h_{33}}$$

$$y' = \frac{k h_{21} x + k h_{22} y + k h_{23}}{k h_{31} x + k h_{32} y + k h_{33}}$$

$$x' = \frac{h_{11} x + h_{12} y + h_{13}}{h_{31} x + h_{32} y + h_{33}}$$

$$y' = \frac{h_{21} x + h_{22} y + h_{23}}{h_{31} x + h_{32} y + h_{33}}$$

Enforcing 8 DOF

One approach: Set h$_{33}$ = 1.

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Second approach: Impose unit vector constraint

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Subject to the constraint: $h_{11}^2 + h_{12}^2 + h_{13}^2 + h_{21}^2 + h_{22}^2 + h_{23}^2 + h_{31}^2 + h_{32}^2 + h_{33}^2 = 1$

Algebraic Distance, ||h||=1

$$\|\mathbf{h}\| = 1 \qquad x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + h_{33})x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + h_{33})y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - h_{33}x' = 0$$

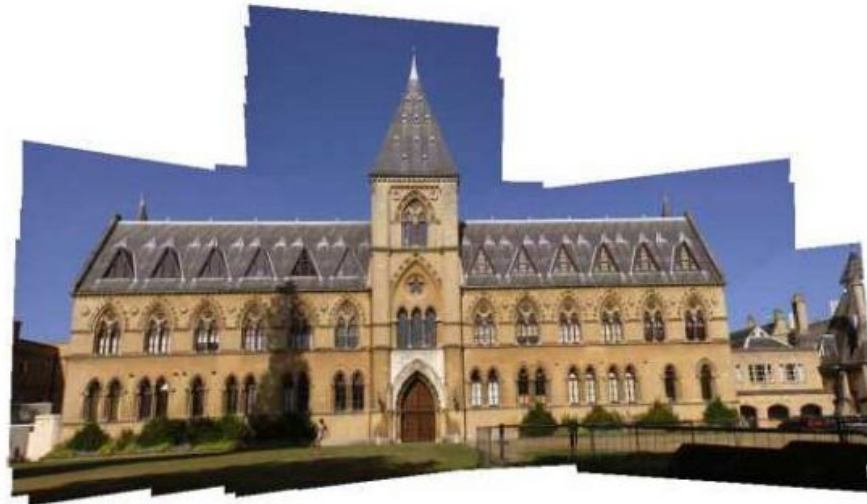$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - h_{33}y' = 0$$

Algebraic Distance, ||h||=1 (cont)

2N x 9   **9 x 1**   **2N x 1**

4
P
O
I
N
T
S

$$
\begin{bmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x_1' & -y_1x_1' & -x_1' \\
0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y_1' & -y_1y_1' & -y_1'
\end{bmatrix}
\begin{bmatrix}
h_{11} \\
h_{12} \\
h_{13} \\
h_{21} \\
h_{22} \\
h_{23} \\
h_{31} \\
h_{32} \\
h_{33}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\\
\\
\\
\\
\\
\\
\end{bmatrix}
$$

additional
points

Use SVD to estimate H
(On board)

Mosaics (example)

As seen in the example on the right, the intensity value at the pixel computed to be at row 20.2, column 14.5 can be calculated by first linearly interpolating between the values at column 14 and 15 on each rows 20 and 21, giving
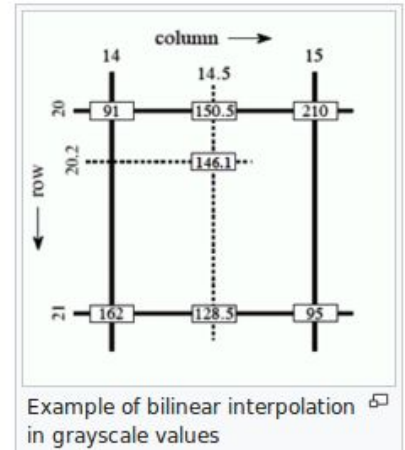
$$I_{20,14.5} = \frac{15-14.5}{15-14} \cdot 91 + \frac{14.5-14}{15-14} \cdot 210 = 150.5,$$

$$I_{21,14.5} = \frac{15-14.5}{15-14} \cdot 162 + \frac{14.5-14}{15-14} \cdot 95 = 128.5,$$

and then interpolating linearly between these values, giving

$$I_{20.2,14.5} = \frac{21-20.2}{21-20} \cdot 150.5 + \frac{20.2-20}{21-20} \cdot 128.5 = 146.1.$$

This algorithm reduces some of the visual distortion caused by resizing an image to a non-integral zoom factor, as opposed to nearest-neighbor interpolation, which will make some pixels appear larger than others in the resized image.

Example of bilinear interpolation in grayscale values

Source: wikipedia