# Rigid Body Transformations

Robotics Research Center

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# Transformations

*The process of changing something to something else via rules are called “transformations”*

In this course we will be only dealing with Homogeneous, Projective and Euclidean Transformations of frames. The camera matrices both Intrinsic as well as Extrinsic are transformations.

# Some basic transformation matrices

Try to write 2D/3D matrices for some basic transformations commonly used:

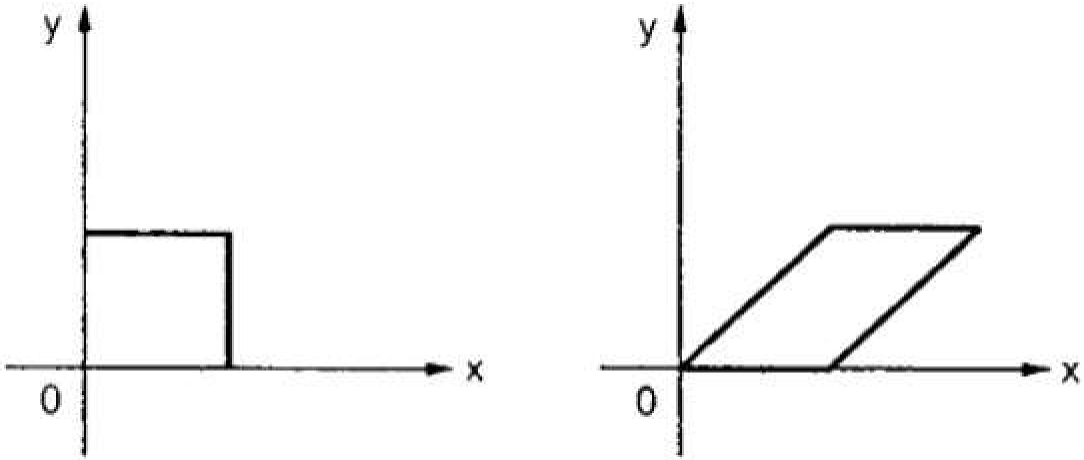
1. Scaling(Isotropic(overall)/ Anisotropic(along selected axes))
2. Shearing
3. Reflection(about different planes)
4. Translation
5. Rotation(about standard axes)
6. Can be many more...

Shear:

$$X_{sh} = \begin{bmatrix} 1 & 0 & 0 \\ shx & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X' = X + Sh_x \,.\, Y$$

$$Y' = Y$$



(a) Original object (b) Object after x shear

# Properties of Rotation Matrices

Rotation matrices belong to a group called “Special Orthonormal Group”

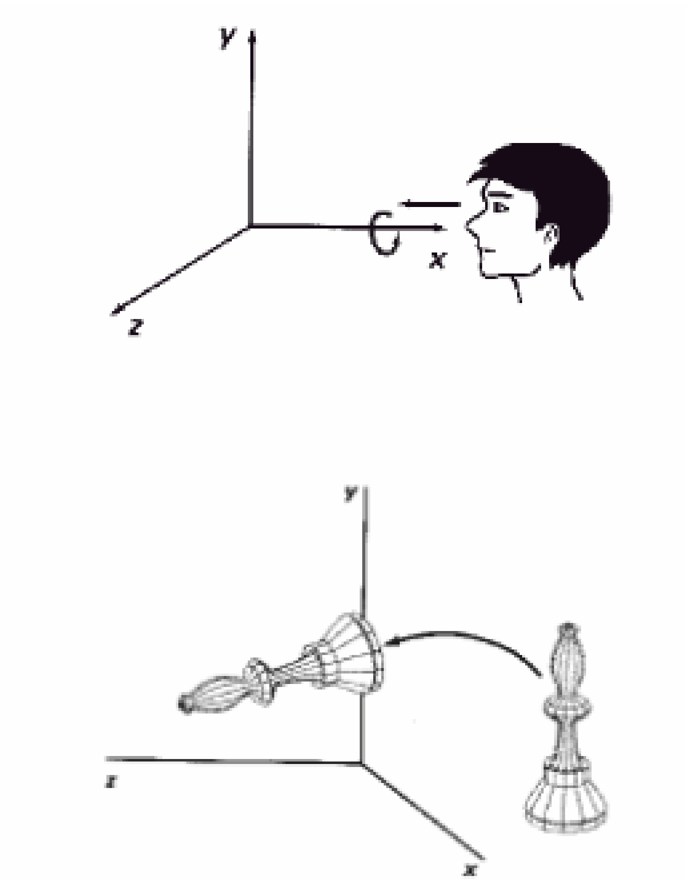
\# Orthonormal Group:
- U is an orthonormal matrix iff $UU^T=I$(Also $UU^T = U^TU = I$)
- This Implies Det(U)=+-1
- These matrices form a Lie group O(s)

\# **Special** Orthonormal Group:
- These are a subset of Orthonormal Group with Det(U)=1
- These also form a Lie group SO(s)

# Rotation about standard axes

Rotatin the scene about any one of the standard axes

# Rotating about X-axis

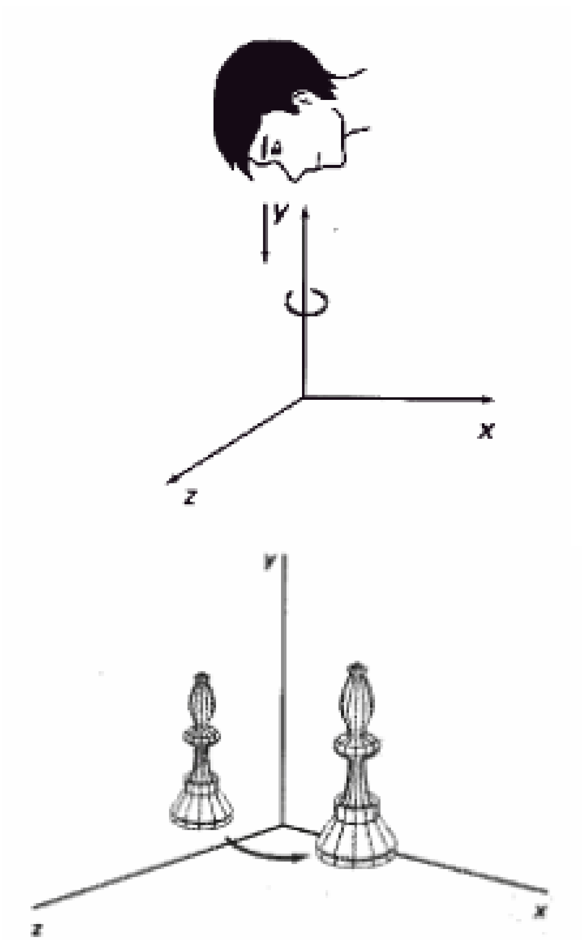



y' = y cos ? - z sin ?,
z' = y sin ? + z cos ?, and
x' = x

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 & 0 \\ & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The X-coordinate remains constant

# Rotating about Y-axis

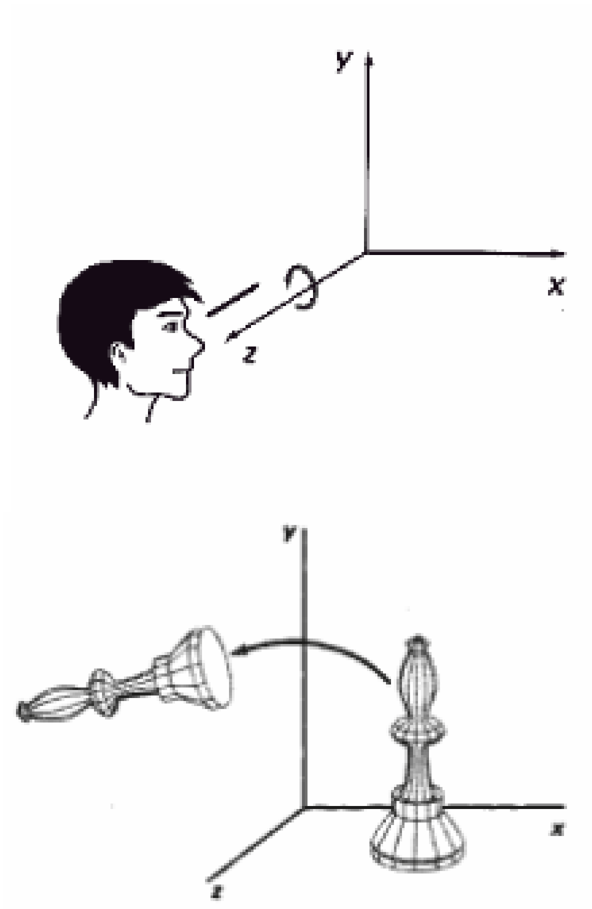



z' = z cos ? - x sin ?,
x' = z sin ? + x cos ?, and
y' = y

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & & \sin\theta & 0 \\ -\sin\theta & & \cos\theta & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The Y-coordinate remains constant

# Rotating about Z-axis



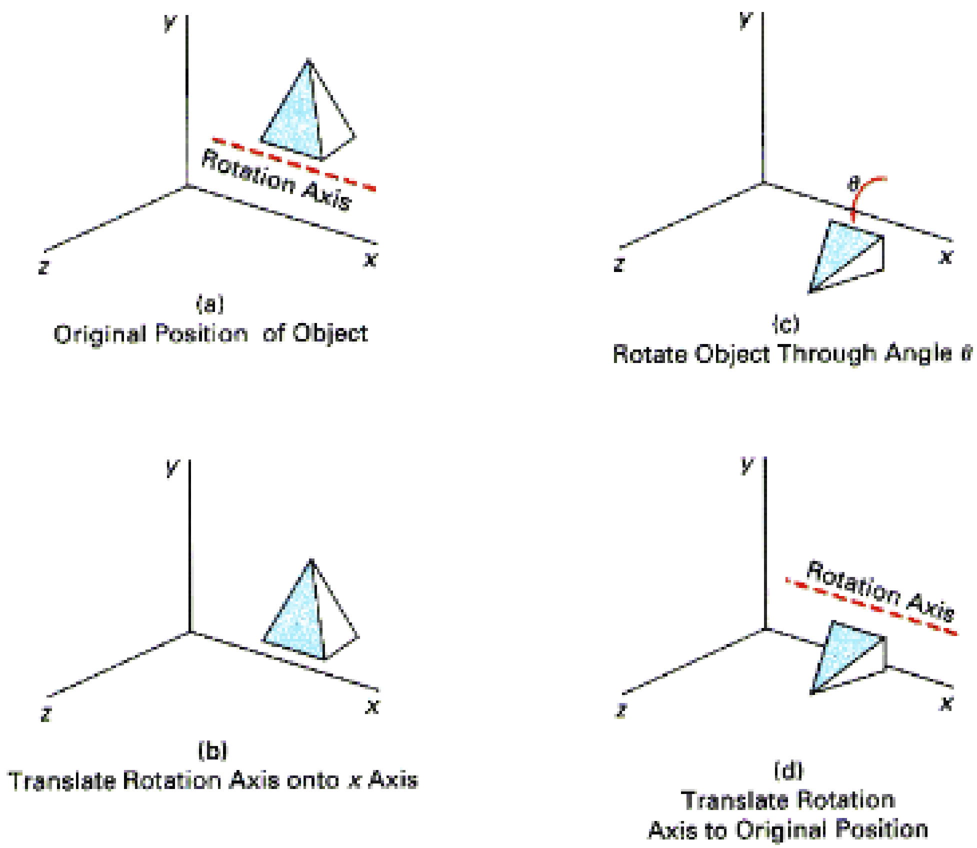

x' = x cos ? - y sin ?,
y' = x sin ? + y cos ?, and
z' = z

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The Z-coordinate remains constant

# Rotation about parallel axes

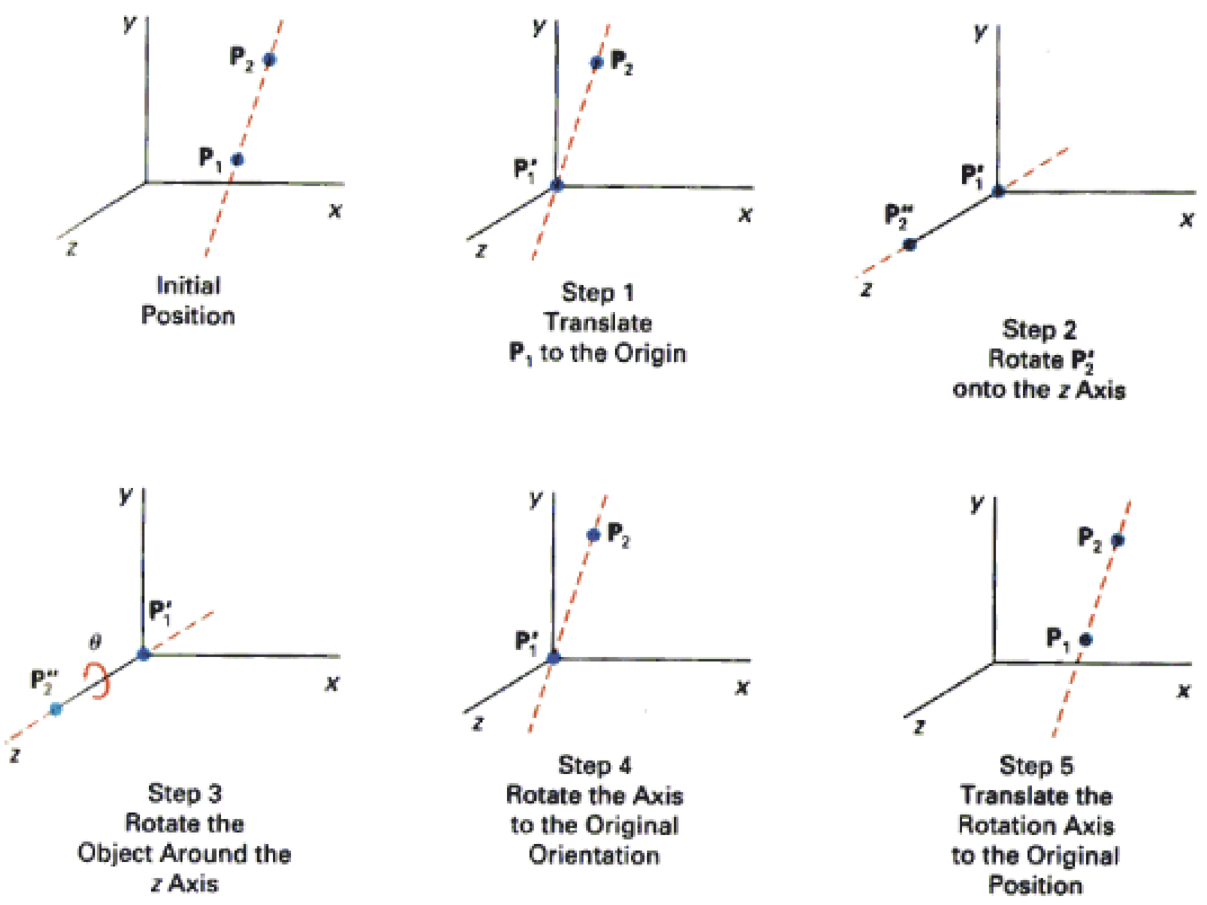
Rotating the scene about any line parallel to one of the standard axes




Step 1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
Step 2. Perform the specified rotation about that axis.
Step 3. Translate the object so that the rotation axis is moved back to its original position.

# Rotating about any axis

Rotating the scene about any general line in the 3D space




Step 1. Translate the object so that the rotation axis passes through the coordinate origin.
Step 2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.
Step 3. Perform the specified rotation about that coordinate axis.
Step 4. Rotate the object so that the rotation axis is brought back to its original orientation.
Step 5. Translate the object so that the rotation axis is brought back to its original position.

# Why Homogeneous Coordinates?

There can be several reasons for this:

It makes calculations easier when we move from Euclediean space to Projective space. The Cartesian coordinate system will make the calculations difficult. How??

* Representation of line and point at infinity becomes tractable.

* Simpler Formulas

* Duality

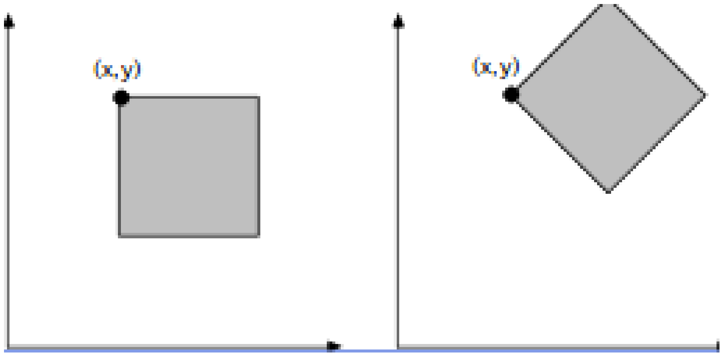
# Why Homogeneous Coordinates?

Facilitates both Rotation and Translations to be represented with a single Transformation matrix. This makes composition of multiple transformation Cleaner and easier to understand(A lot of methods/theories available for matrix manipulation which can be directly used).

Example: Rotation(theta) about any arbitarary point(x,y) in the 2D space:

Steps: Translate (x,y) to origin
Rotate with the angle theta
Translate origin to (x,y)

$$C = \underbrace{\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}}_{T_{x,y}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\theta} \underbrace{\begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}}_{T_{-x,-y}}$$



# Why Homogeneous Coordinates?

Two Parallel lines cannot meet in an Euclidean space but can in Projective space

Euclidean space (or Cartesian space) describe our 2D/3D geometry so well, but they are not sufficient to handle the projective space (Actually, Euclidean geometry is a subset of projective geometry). The Cartesian coordinates of a 2D point can be expressed as (x, y)

What if this point goes far away to infinity? The point at infinity would be (∞,∞), and it becomes meaningless in Euclidean space. The parallel lines should meet at infinity in projective space, but cannot do in Euclidean space.

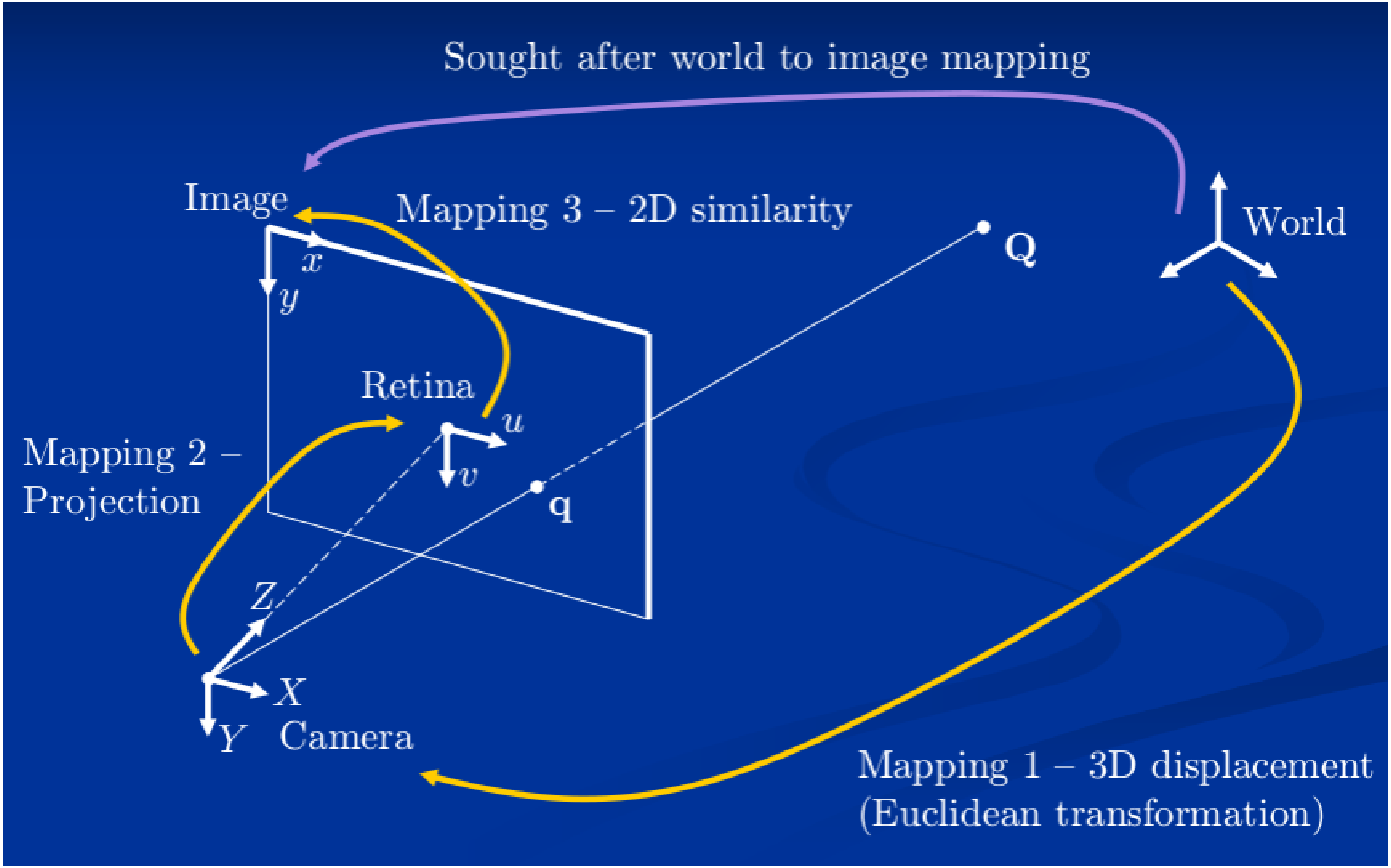
Easy to switch between Homogeneous and Cartesian coordinate

simply divide x and y by w;

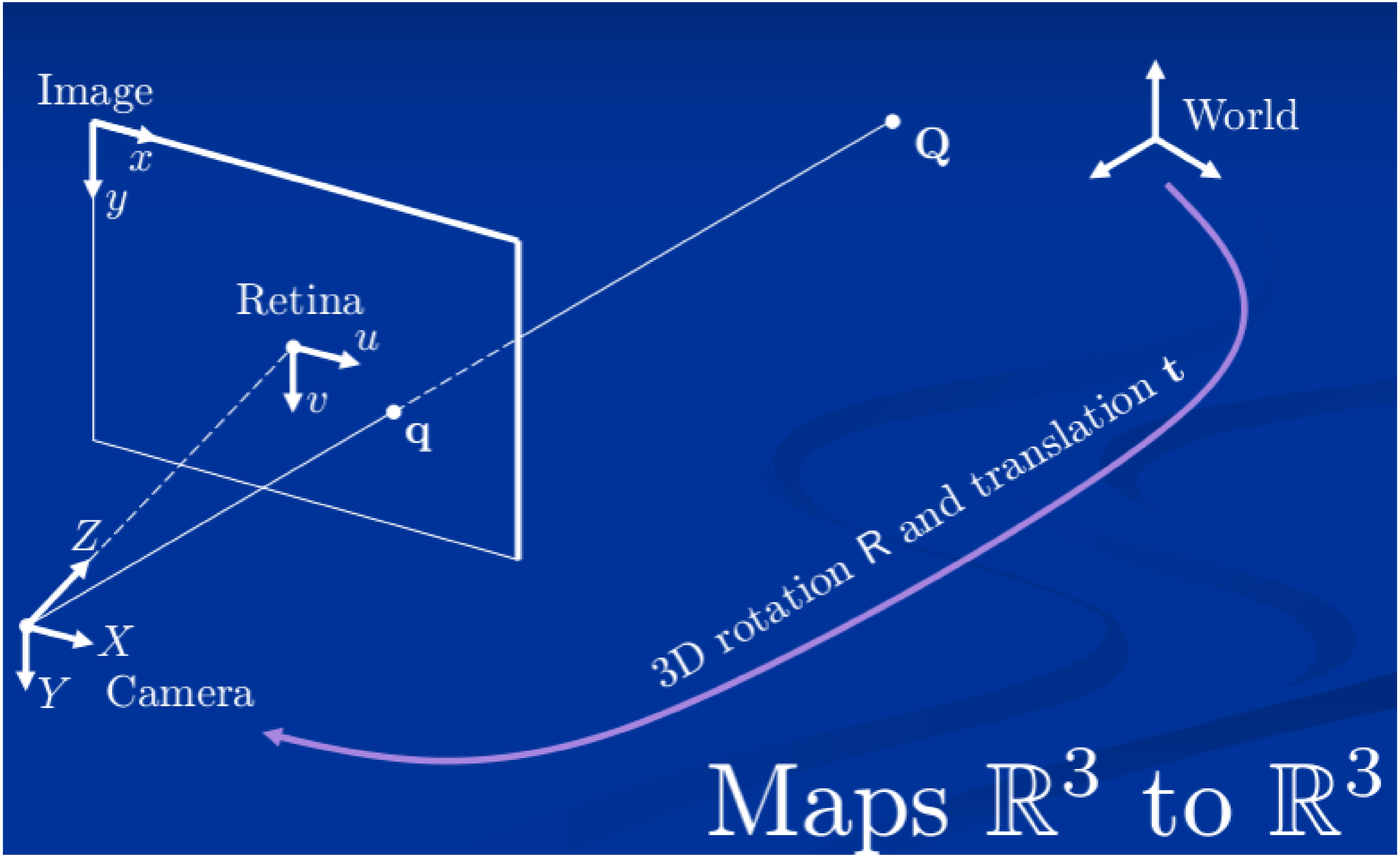
$$\underset{\text{Homogeneous}}{(x, y, w)} \quad \Leftrightarrow \quad \underset{\text{Cartesian}}{\left(\frac{x}{w}, \frac{y}{w}\right)}$$

# Why do we need Transformations?

There can be many reasons but in Machine Vision it is useful for following Mappings:



# Mapping 1: World to Camera



# Mapping 1: World to Camera

- Models camera displacement, *i.e.* its position and orientation
- 3D translation and rotation
- In homogeneous coordinates:

$$\mathbf{Q}_c = \begin{pmatrix} & \mathsf{R} & & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{Q},$$

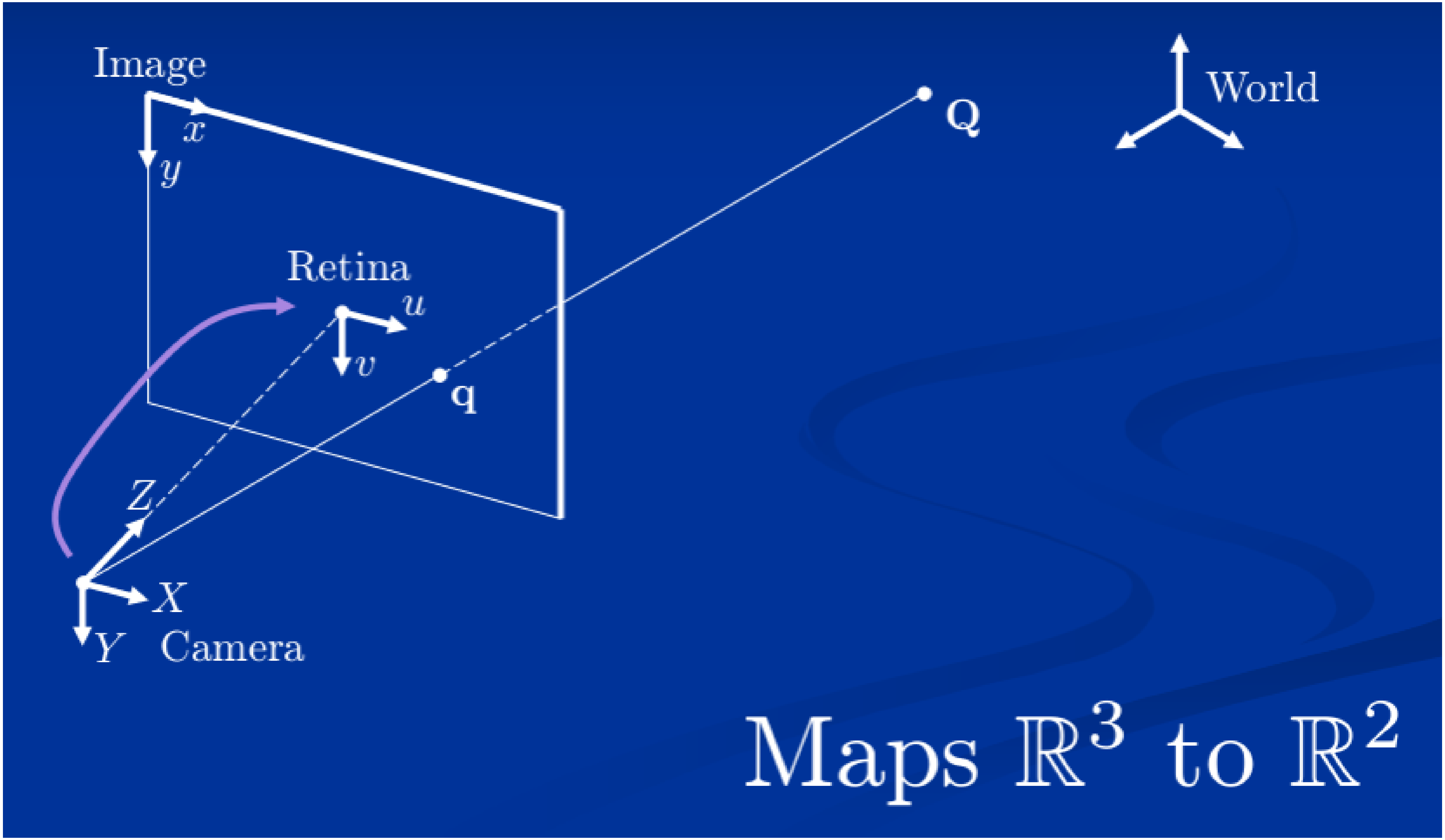
where $\mathsf{R}$ is a $(3 \times 3)$ rotation matrix ($\mathsf{R}^\mathsf{T}\mathsf{R} = \mathrm{I}$ and $\det(\mathsf{R}) = 1$), $\mathbf{t}$ is a 3-vector and $\mathbf{Q}$ is the world homogeneous coordinate vector of the 3D point:

$$\mathbf{Q} \sim \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

- The translation is often expressed in terms of the centre of projection:

$$\mathbf{t} = -\mathsf{R}\bar{\mathbf{C}}$$

# Mapping 2: Camera to Retina



# Mapping 2: Camera to Retina

- Camera-centred 3D point coordinates: $\mathbf{Q}^c \sim (X^c \;\; Y^c \;\; Z^c \;\; 1)^\mathsf{T}$
- Using similar triangles gives:

$$u = f\frac{X^c}{Z^c} \qquad \text{and} \qquad v = f\frac{Y^c}{Z^c},$$

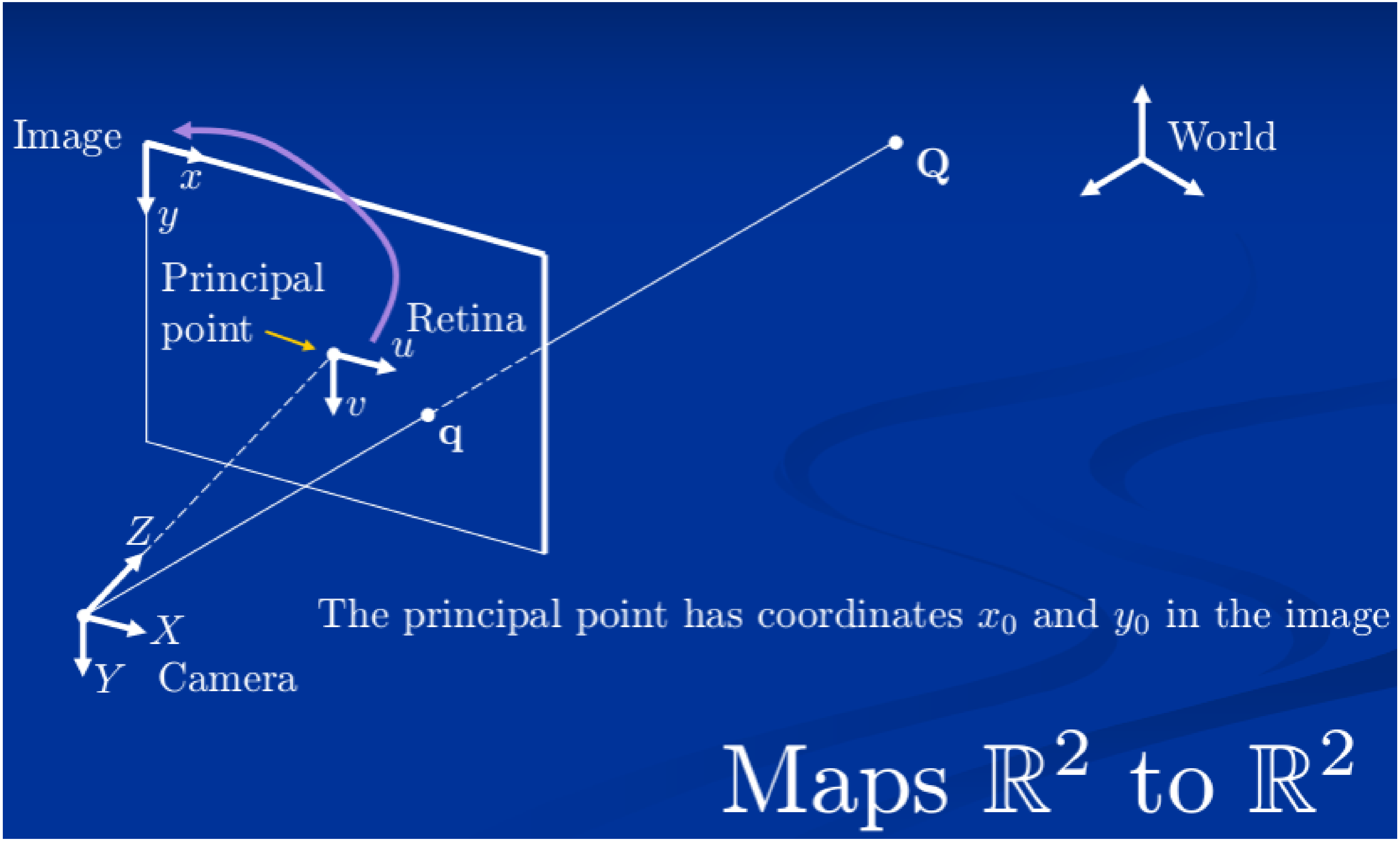
or in homogeneous coordinates:

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim \begin{pmatrix} fX^c \\ fY^c \\ Z^c \end{pmatrix}$$

- The division by $Z^c$ is 'hidden' in the homogeneous coordinates
- In matrix form, we write this projection as:

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^c \\ Y^c \\ Z^c \\ 1 \end{pmatrix}$$

# Mapping 3: Retina to Image



# Mapping 3: Retina to Image

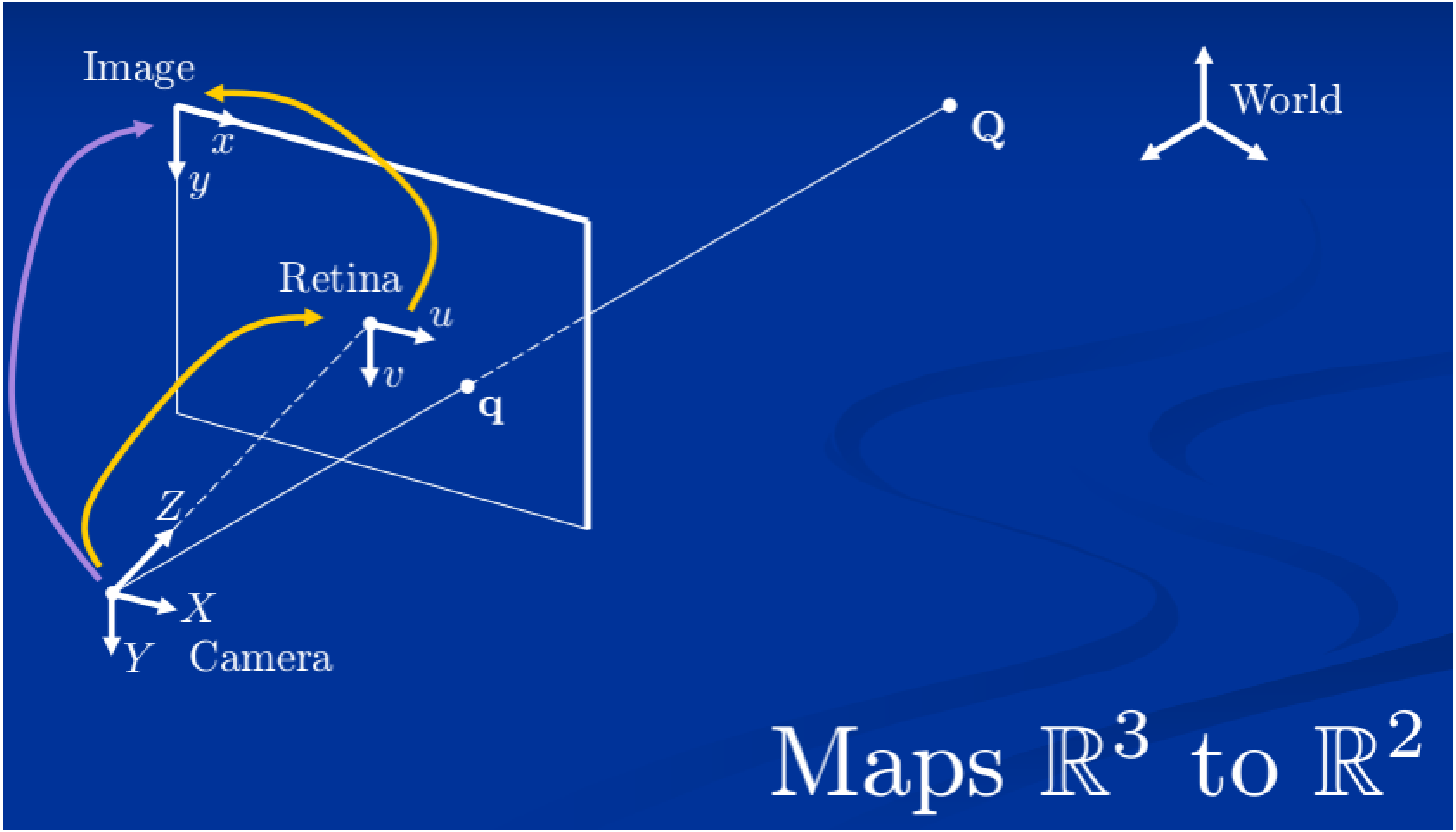
- $k$ is the density of pixels along $u$ and $v$ (*e.g.* in number of pixels per mm)
- We have:

$$x = ku + x_0 \qquad \text{and} \qquad y = kv + y_0,$$

or in matrix form with homogeneous coordinates:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} k & 0 & x_0 \\ 0 & k & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

# Mapping 2+3 : Camera to Image



# Mapping 2+3 : Camera to Image

- We combine:

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^c \\ Y^c \\ Z^c \\ 1 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} k & 0 & x_0 \\ 0 & k & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

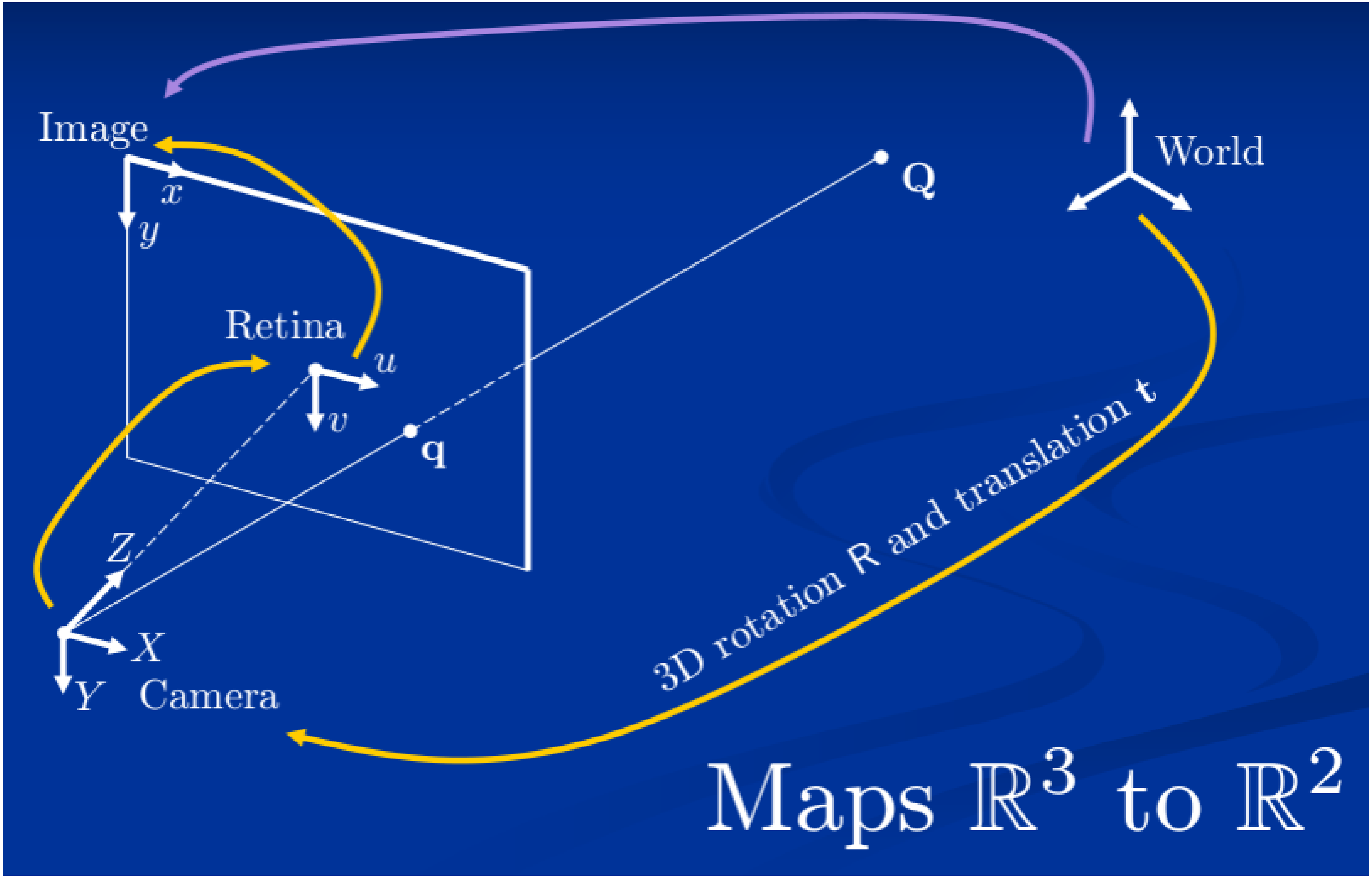
- Giving:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} k & 0 & x_0 \\ 0 & k & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^c \\ Y^c \\ Z^c \\ 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} fk & 0 & x_0 \\ 0 & fk & y_0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathsf{K}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^c \\ Y^c \\ Z^c \\ 1 \end{pmatrix}$$

- K is the *camera calibration matrix*, containing the 'internal' or 'intrinsic' camera parameters

# Mapping 1+2+3:World to Image



# Mapping 1+2+3: World to Image

- Putting all mappings together yields:

$$\mathbf{q} \sim \underbrace{\begin{pmatrix} fk & 0 & x_0 \\ 0 & fk & y_0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathsf{K}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} & \mathsf{R} & & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{Q}$$

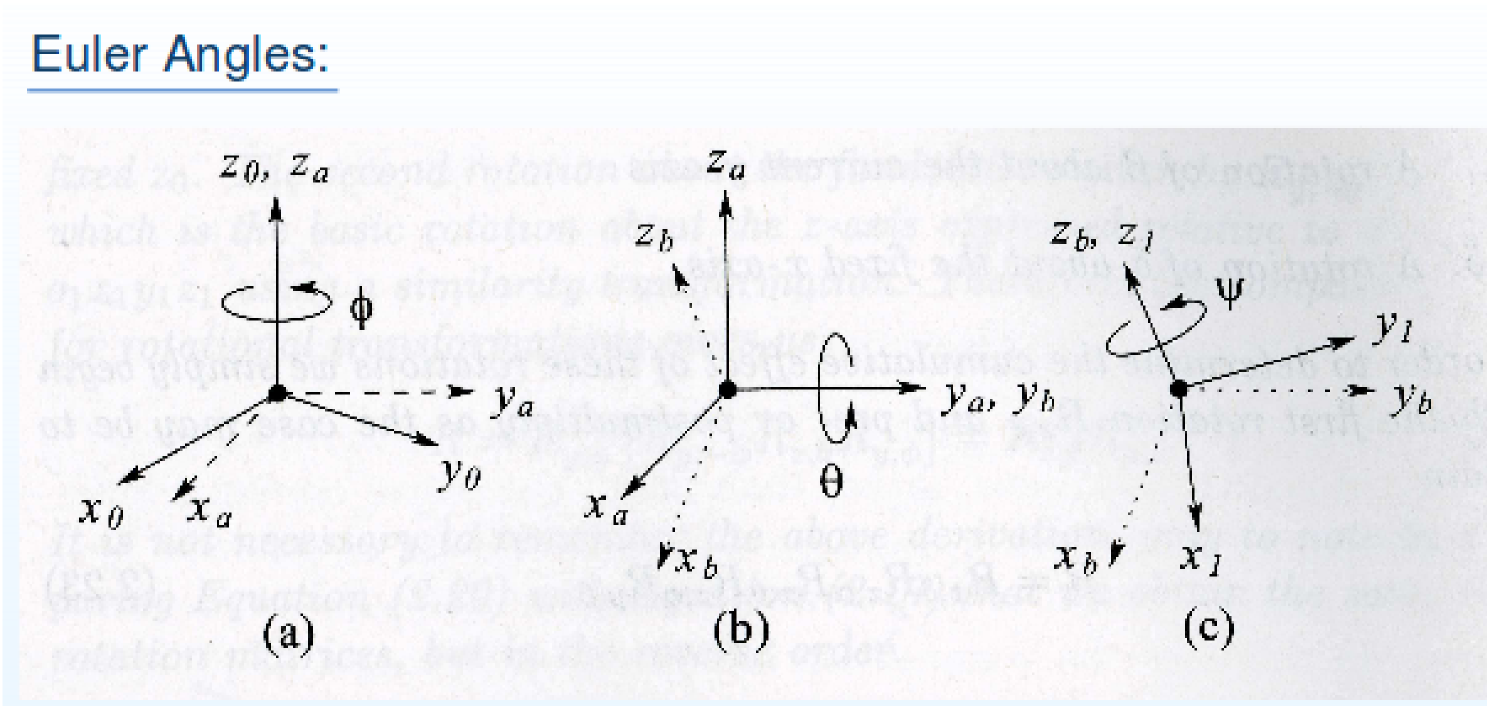
or:

$$\mathbf{q} \sim \underbrace{(\mathsf{KR}\ \ \mathsf{K}\mathbf{t})}_{\mathsf{P}} \mathbf{Q}$$

- The $(3 \times 4)$ matrix $\mathsf{P}$ is the **perpective projection matrix**
- $\mathsf{K}$ contains the 'intrinsic' or 'internal' camera parameters
- $\mathsf{R}$ and $\mathbf{t}$ are the 'extrinsic' or 'external' camera parameters, also called the pose of the camera

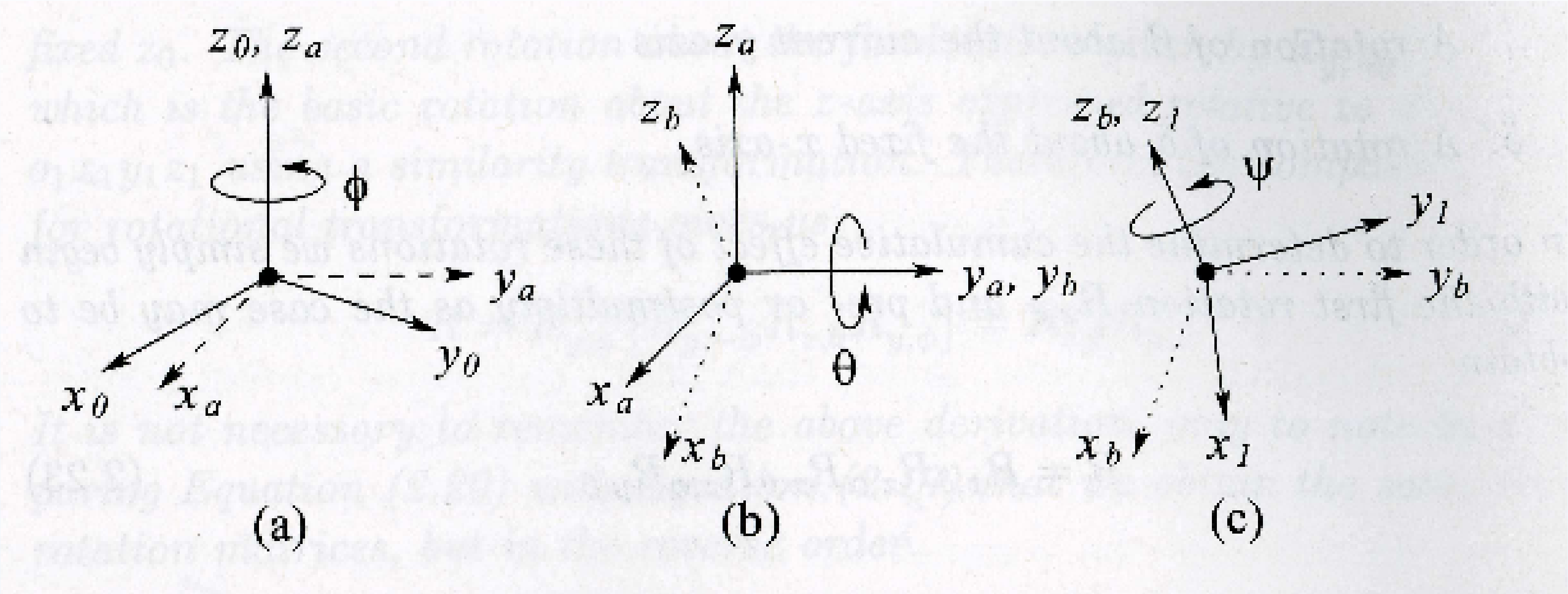
# Current Axis Rotation

Current Axis



# Current Axis Rotation

Euler Angles:

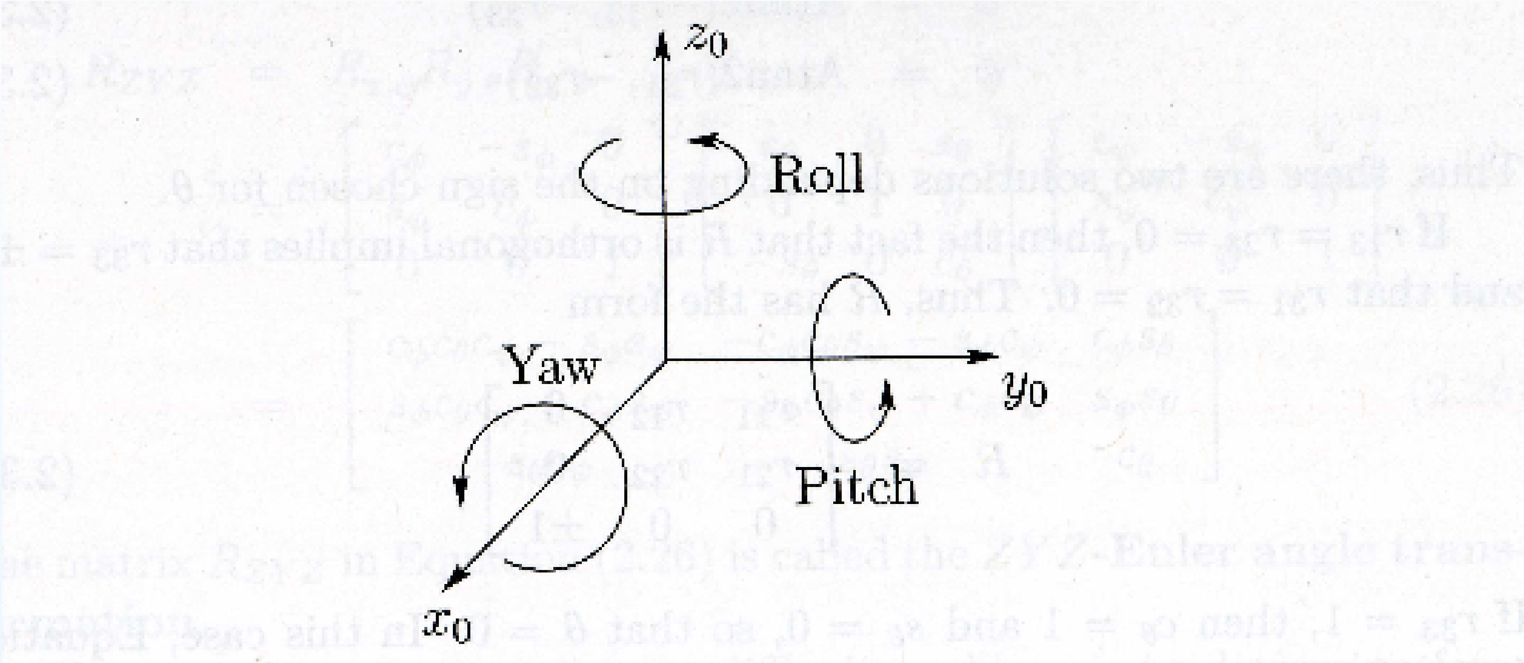



Euler angles are angles of 3 rotations about current axes

$$R_{ZYZ} := \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \cdot \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Fixed Axis Rotation

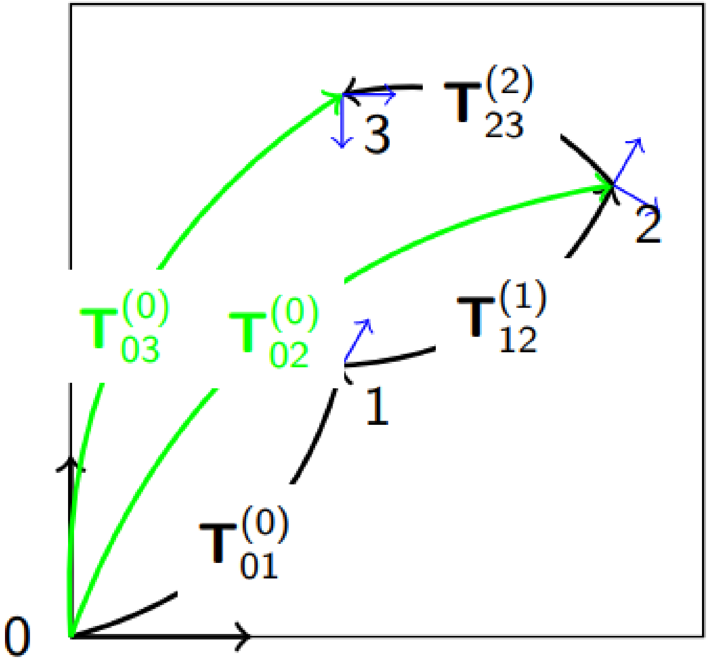
## Roll, Pitch Yaw Angles:




Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes $x$, $y$ and $z$

$$R_{xyz} := R_{z,\phi} \cdot R_{y,\theta} \cdot R_{x,\psi}$$

# Current Axis & Fixed Axis




- $\mathbf{T}_{01}^{(0)}$: pose of 1 w.r.t. 0
- $\mathbf{T}_{12}^{(1)}$: pose of 2 w.r.t. 1
- $\mathbf{T}_{02}^{(0)} = \mathbf{T}_{01}^{(0)}\mathbf{T}_{12}^{(1)}$: pose of 2 w.r.t. 0
- $\mathbf{T}_{23}^{(2)}$: pose of 3 w.r.t. 2
- $\mathbf{T}_{03}^{(0)} = \mathbf{T}_{01}^{(0)}\mathbf{T}_{12}^{(1)}\mathbf{T}_{23}^{(2)} = \mathbf{T}_{02}^{(0)}\mathbf{T}_{23}^{(2)}$: pose of 3 w.r.t. 0

# Generalized Transformation Matrix

- Generalized 4 x 4 transformation matrix in homogeneous coordinates

$$[T] = \left[\begin{array}{ccc|c} a & d & g & l \\ b & e & i & m \\ c & f & j & n \\ \hline p & q & r & s \end{array}\right]$$

Translations l, m, n along x, y, and z axis

Linear transformations – local scaling, shear, rotation reflection

Perspective transformations

Overall scaling

# Properties of Rotation Matrices

Rotation matrices belong to a group called “Special Orthonormal Group”

# Orthonormal Group:
- U is an orthonormal matrix iff $UU^T$=I(Also $UU^T$ = $U^TU$ = I)
- This Implies Det(U)=+-1
- These matrices form a Lie group O(s)

# **Special** Orthonormal Group:
- These are a subset of Orthonormal Group with Det(U)=1
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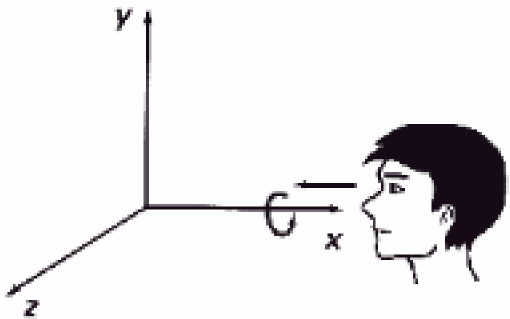
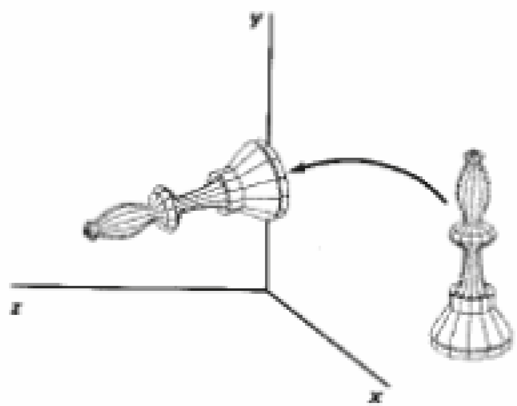
# Rotation about standard axes

Rotatin the scene about any one of the standard axes

# Rotating about X-axis



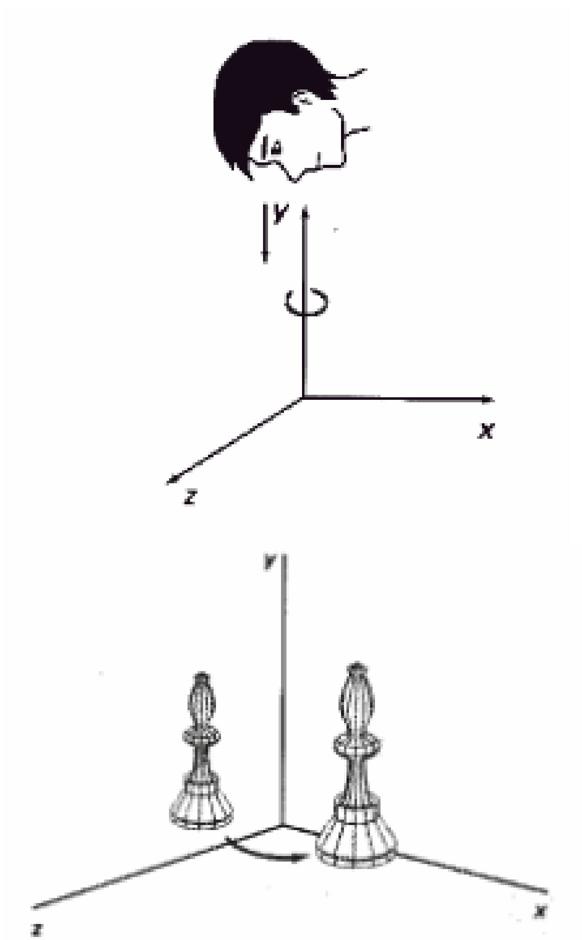




y' = y cos ? - z sin ?,
z' = y sin ? + z cos ?, and
x' = x

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 & 0 \\ & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The X-coordinate remains constant

# Rotating about Y-axis




z' = z cos ? - x sin ?,
x' = z sin ? + x cos ?, and
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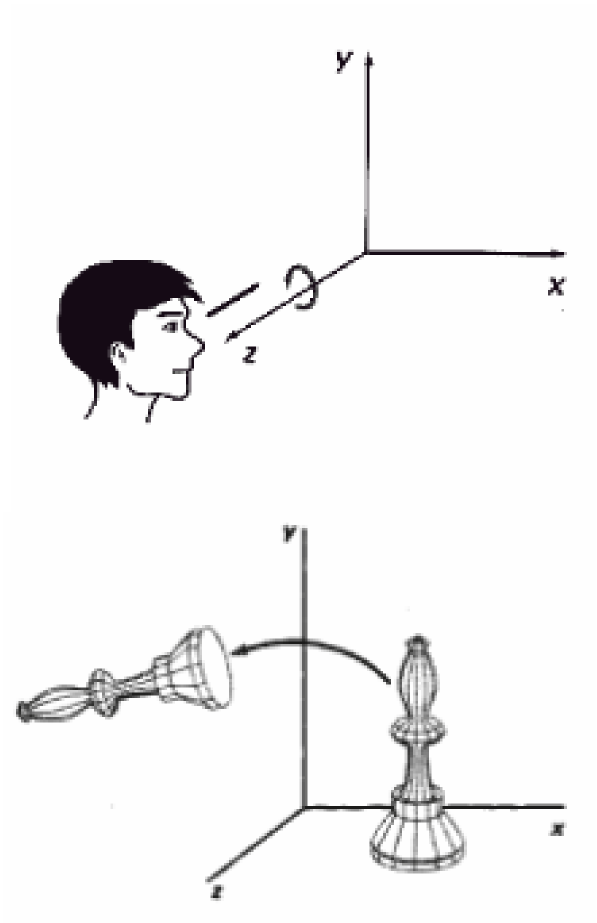
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & & \sin\theta & 0 \\ -\sin\theta & & \cos\theta & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The Y-coordinate remains constant

# Rotating about Z-axis



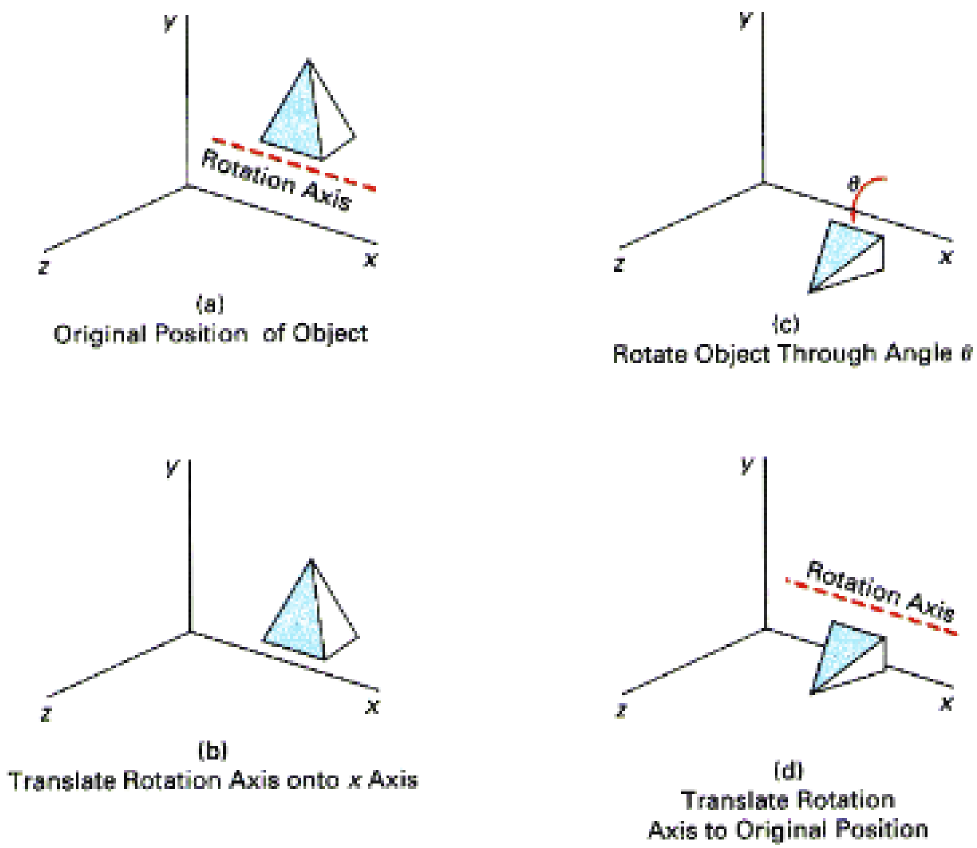

x' = x cos ? - y sin ?,
y' = x sin ? + y cos ?, and
z' = z

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The Z-coordinate remains constant

# Rotation about parallel axes

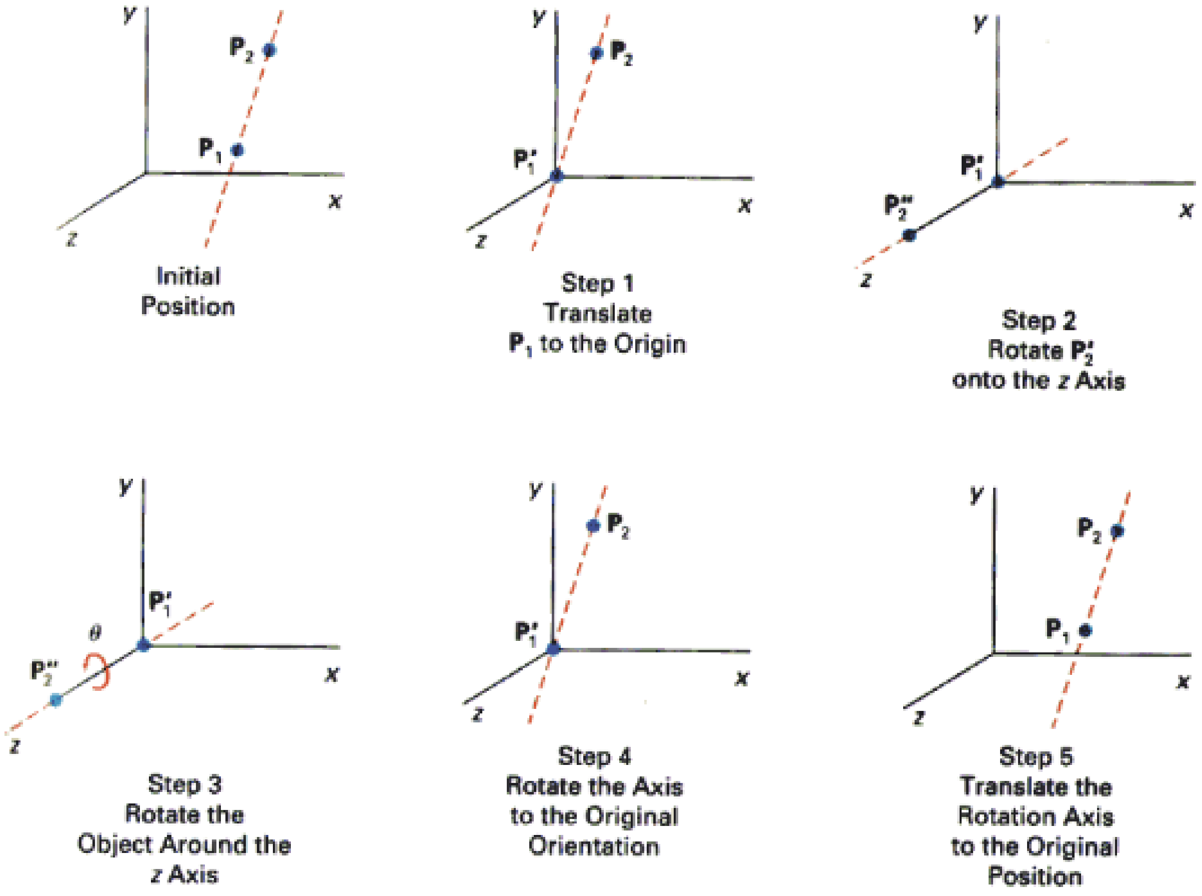
Rotating the scene about any line parallel to one of the standard axes




Step 1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
Step 2. Perform the specified rotation about that axis.
Step 3. Translate the object so that the rotation axis is moved back to its original position.

# Rotating about any axis

Rotating the scene about any general line in the 3D space




Step 1. Translate the object so that the rotation axis passes through the coordinate origin.
Step 2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.
Step 3. Perform the specified rotation about that coordinate axis.
Step 4. Rotate the object so that the rotation axis is brought back to its original orientation.
Step 5. Translate the object so that the rotation axis is brought back to its original position.

# Similarity Transformations

The matrix representation of a general linear transformation is transformed from one frame to another using a so-called **similarity transformation**.

**For example**, if $\boldsymbol{A}$ is the matrix representation of a given linear transformation in $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ and $\boldsymbol{B}$ is the representation of the same linear transformation in $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ then $\boldsymbol{A}$ and $\boldsymbol{B}$ are related as:

$$B = (R_1^0)^{-1} A R_1^0$$

where $\boldsymbol{R_1^0}$ is the coordinate transformation between frames $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ and $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ . In particular, if $\boldsymbol{A}$ itself is a rotation, then so is $\boldsymbol{B}$, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.

# Example

$$B = (R_1^0)^{-1} A R_1^0$$

Suppose frames $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ and $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ are related by the rotation

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If $\boldsymbol{A = R_z}$ relative to the frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$, then, relative to frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ we have

$$B \quad = \quad (R_1^0)^{-1} A^0 R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

$\boldsymbol{B}$ is a rotation about the $\boldsymbol{z0 - axis}$ but expressed relative to the frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ .

# Rotation With Respect To The Current Frame

The matrix $\boldsymbol{R_1^0}$ represents a rotational transformation between the frames $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ and $\boldsymbol{o_1\, x_1\, y_1\, z_1}$.

Suppose we now add a third coordinate frame $\boldsymbol{o_2\, x_2\, y_2\, z_2}$ related to the frames $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ and $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ by rotational transformations.
A given point $\boldsymbol{p}$ can then be represented by coordinates specified with respect to any of these three frames: $\boldsymbol{p^0}$, $\boldsymbol{p^1}$ and $\boldsymbol{p^2}$.

The relationship among these representations of $\boldsymbol{p}$ is:

$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
p^1 &= R_2^1 p^2 \\
p^0 &= R_2^0 p^2
\end{aligned}
$$

where each $\boldsymbol{R_j^i}$ is a rotation matrix

$$p^0 = R_1^0 R_2^1 p^2$$

$$R_2^0 = R_1^0 R_2^1$$

# Composition Law for Rotational Transformations

In order to transform the coordinates of a point $\boldsymbol{p}$ from its representation $\boldsymbol{p}^2$ in the frame $\boldsymbol{o}_2\, \boldsymbol{x}_2\, \boldsymbol{y}_2\, \boldsymbol{z}_2$ to its representation $\boldsymbol{p}^0$ in the frame $\boldsymbol{o}_0\, \boldsymbol{x}_0\, \boldsymbol{y}_0\, \boldsymbol{z}_0$, we may first transform to its coordinates $\boldsymbol{p}^1$ in the frame $\boldsymbol{o}_1\, \boldsymbol{x}_1\, \boldsymbol{y}_1\, \boldsymbol{z}_1$ using $\boldsymbol{R}_2^1$ and then transform $\boldsymbol{p}^1$ to $\boldsymbol{p}^0$ using $\boldsymbol{R}_1^0$.

$$\begin{aligned} p^0 &= R_1^0 p^1 \\ p^1 &= R_2^1 p^2 \\ p^0 &= R_2^0 p^2 \end{aligned}$$

$$p^0 = R_1^0 R_2^1 p^2$$

$$R_2^0 = R_1^0 R_2^1$$

# Composition Law for Rotational Transformations

$$R_2^0 = R_1^0 R_2^1$$

Suppose initially that all three of the coordinate frames are coincide.

We first rotate the frame $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ relative to $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ according to the transformation $\boldsymbol{R_1^0}$ .

Then, with the frames $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ and $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ coincident, we rotate $\boldsymbol{o_2\ x_2\ y_2\ z_2}$ relative to $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ according to the transformation $\boldsymbol{R_2^1}$ .

In each case we call the frame relative to which the rotation occurs the **current frame**.

# Example

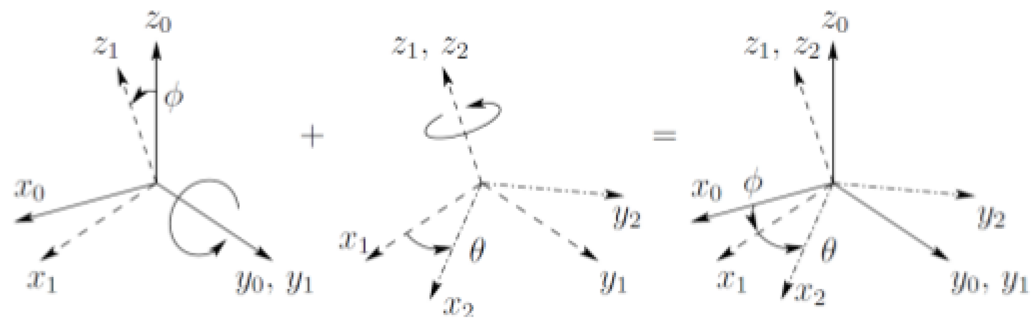
Suppose a rotation matrix R represents
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$.

$$R = R_{y,\phi} R_{z,\theta}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Example

Suppose a rotation matrix R represents

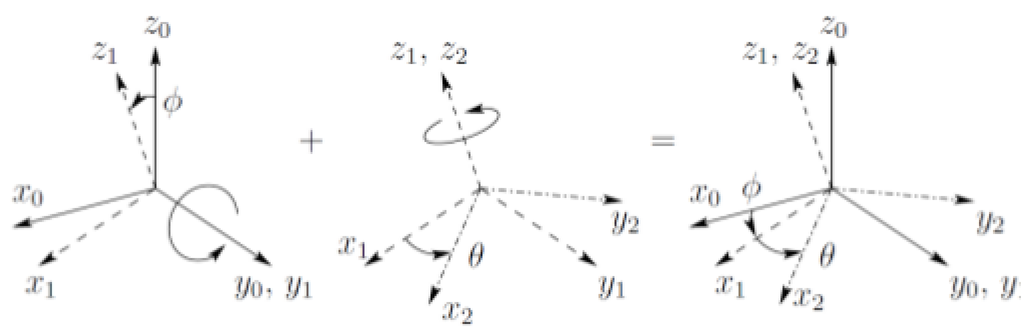
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$.

$$
\begin{aligned}
R &= R_{y,\phi}R_{z,\theta} \\
&= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

$$
R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$



# Example

Suppose a rotation matrix R represents

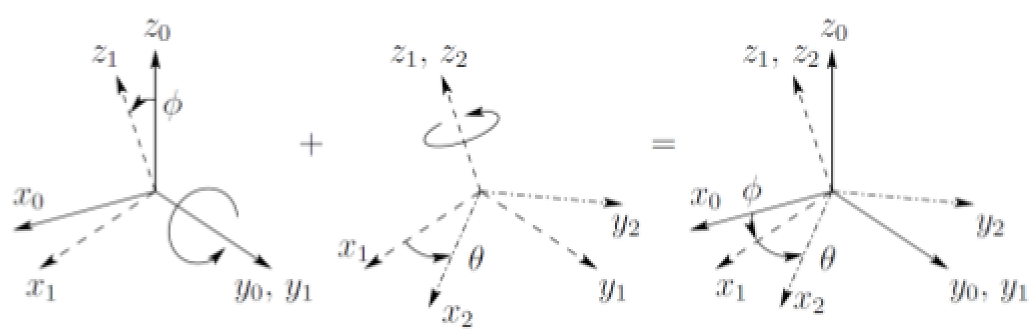
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$.

$$
\begin{aligned}
R &= R_{y,\phi}R_{z,\theta} \\
&= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}
\end{aligned}
$$

$$
R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

$$
R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$



# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$ followed by
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$

$$R' =$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$ followed by
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R' &= R_{z,\theta} R_{y,\phi} \\ &= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\ &= \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \end{aligned}$$

# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$ followed by
- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y-axis}$

$$
\begin{aligned}
R' &= R_{z,\theta}R_{y,\phi} \\
&= \begin{bmatrix} c_\theta & -s_\phi & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}
\end{aligned}
$$

$$
R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

$$
R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

**Rotational transformations do not commute**

$$
R \neq R'
$$

# Rotation With Respect To The Fixed Frame

Performing a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames.

For example we may wish to perform a rotation about $\boldsymbol{x_0}$ followed by a rotation about $\boldsymbol{y_0}$ (and not $\boldsymbol{y_1}$!). We will refer to $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ as the **fixed frame**. In this case the composition law given before is not valid.

$$R_2^0 \cancel{=} R_1^0 R_2^1$$

The composition law that was obtained by multiplying the successive rotation matrices in the reverse order from that given by is not valid.

# Rotation with Respect to the Fixed Frame

Suppose we have two frames $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ and $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ related by the rotational transformation $\boldsymbol{R_1^0}$.

If $\boldsymbol{R}$ represents a rotation relative to $\boldsymbol{o_0\, x_0\, y_0\, z_0}$, the representation for $\boldsymbol{R}$ in the current frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ is given by:

$$(R_1^0)^{-1} R R_1^0$$

With applying the composition law for rotations about the current axis:

$$R_2^0 \quad = \quad R_1^0 \left[ (R_1^0)^{-1} R R_1^0 \right] = R R_1^0$$

**Reminder:**

Similarity Transformations

$$B = (R_1^0)^{-1} A R_1^0$$

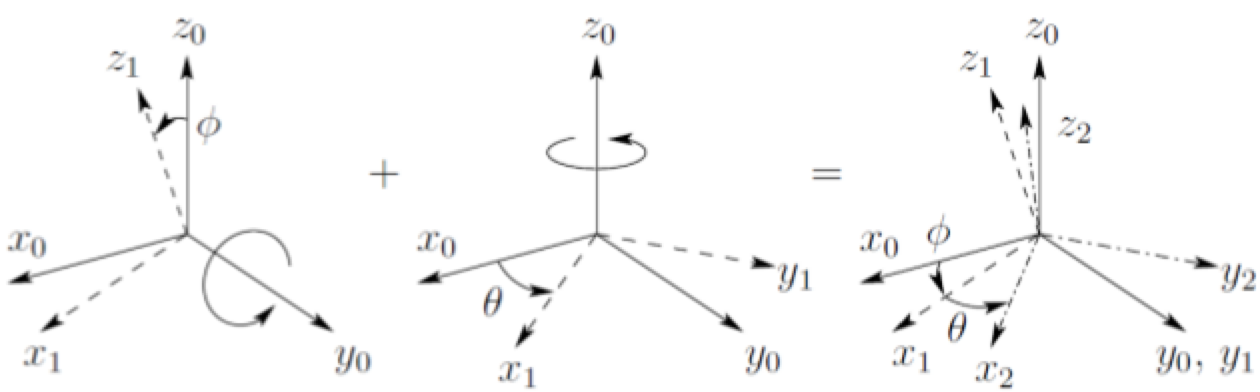
composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$

# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0 - axis}$ followed by
- a rotation of angle$\boldsymbol{\theta}$about the fixed $\boldsymbol{z_0 - axis}$




The **second** rotation about the fixed axis is given by

$$R_{y,-\phi}R_{z,\theta}R_{y,\phi}$$

which is the basic rotation about **the z-axis** expressed relative to the frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ using a similarity transformation.

**Reminder:**

Similarity Transformations

$$B = (R_1^0)^{-1} A R_1^0$$

composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$

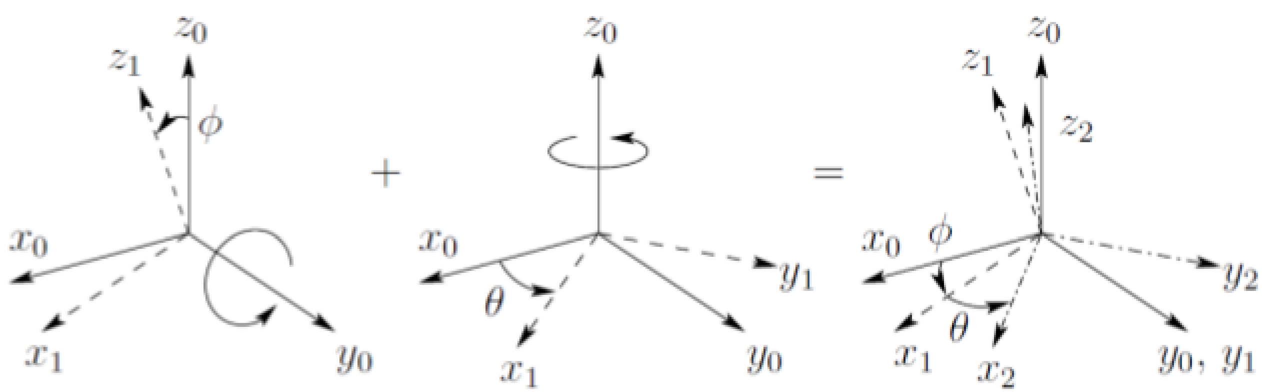
composition law for rotations about the fixed axis

$$\begin{aligned} R_2^0 &= R_1^0 \left[(R_1^0)^{-1} R R_1^0\right] \\ &= R R_1^0 \end{aligned}$$

# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0 - axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the fixed $\boldsymbol{z_0 - axis}$




Therefore, the composition rule for rotational transformations

$$
\begin{aligned}
p^0 &= \boxed{\quad} \, p^1 \\
&= \boxed{\qquad\qquad\qquad} \, p^2 \\
&= \boxed{\quad} \, p^2
\end{aligned}
$$

**Reminder:**

Similarity Transformations

$$B = (R_1^0)^{-1} A R_1^0$$

composition law for rotations about the current axis

$$R_2^0 = R_1^0 R_2^1$$

composition law for rotations about the fixed axis

$$
\begin{aligned}
R_2^0 &= R_1^0 \left[ (R_1^0)^{-1} R R_1^0 \right] \\
&= R R_1^0
\end{aligned}
$$

# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0 - axis}$ followed by
- a rotation of angle$\boldsymbol{\theta}$about the fixed $\boldsymbol{z_0 - axis}$

$$
\begin{aligned}
p^0 &= R_{y,\phi} p^1 \\
&= R_{y,\phi} \left[ R_{y,-\phi} R_{z,\theta} R_{y,\phi} \right] p^2 \\
&= R_{z,\theta} R_{y,\phi} p^2
\end{aligned}
$$

# Example

Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about $\boldsymbol{y_0} - \boldsymbol{axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the fixed $\boldsymbol{z_0} - \boldsymbol{axis}$

$$\begin{aligned} p^0 &= R_{y,\phi} p^1 \\ &= R_{y,\phi} \left[ R_{y,-\phi} R_{z,\theta} R_{y,\phi} \right] p^2 \\ &= R_{z,\theta} R_{y,\phi} p^2 \end{aligned}$$

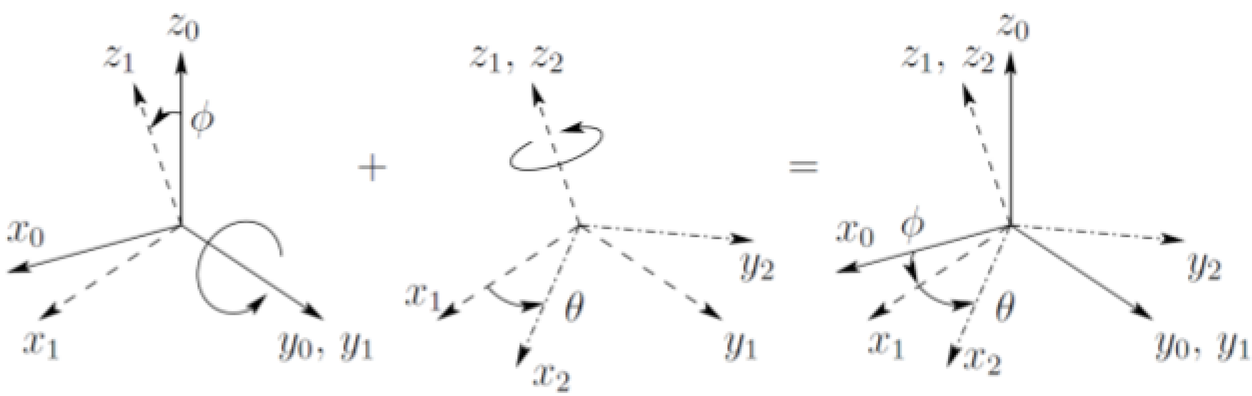
Suppose a rotation matrix R represents

- a rotation of angle $\boldsymbol{\phi}$ about the current $\boldsymbol{y} - \boldsymbol{axis}$ followed by
- a rotation of angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z} - \boldsymbol{axis}$.

$$R = R_{y,\phi} R_{z,\theta}$$

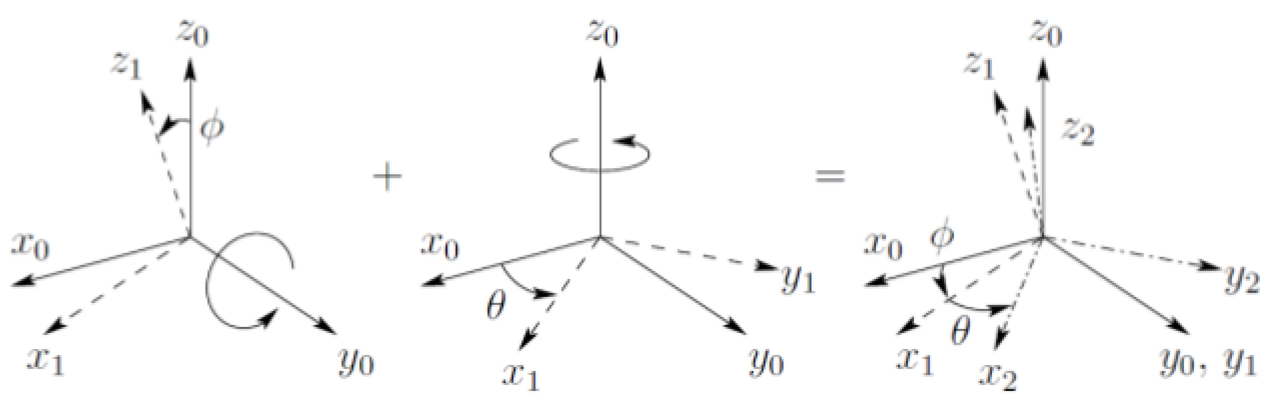
# Summary

To note that we obtain the same basic rotation matrices, but in the reverse order.




Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$




Rotation with Respect to the Fixed Frame

$$R_2^0 = \cancel{R_2^1} R_1^0$$

$$R_2^0 = R\, R_1^0$$

# Rules for Composition of Rotational Transformations

We can summarize the rule of composition of rotational transformations by:

Given a fixed frame $\boldsymbol{o_0\,x_0\,y_0\,z_0}$ a current frame $\boldsymbol{o_1\,x_1\,y_1\,z_1}$, together with rotation matrix $\boldsymbol{R_1^0}$ relating them, if a third frame $\boldsymbol{o_2\,x_2\,y_2\,z_2}$ is obtained by a rotation $\boldsymbol{R}$ performed relative to the current frame then post-multiply $\boldsymbol{R_1^0}$ by $\boldsymbol{R = R_2^1}$ to obtain

$$R_2^0 = R_1^0 R_2^1$$

If the second rotation is to be performed relative to the fixed frame then it is both confusing and inappropriate to use the notation $\boldsymbol{R_2^1}$ to represent this rotation. Therefore, if we represent the rotation by $\boldsymbol{R}$, we pre-multiply $\boldsymbol{R_1^0}$ by $\boldsymbol{R}$ to obtain

$$R_2^0 = R\,R_1^0$$

In each case $\boldsymbol{R_2^0}$ represents the transformation between the frames $\boldsymbol{o_0\,x_0\,y_0\,z_0}$ and $\boldsymbol{o_2\,x_2\,y_2\,z_2}$.

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R =$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \qquad\qquad R_{x,\theta}$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\,R_1^0$$

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \qquad R_{x,\theta} R_{z,\phi}$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \qquad R_{z,\alpha} R_{x,\theta} R_{z,\phi}$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = \quad R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of Θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3. A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta}$$

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\, R_1^0$$

# Example

Find **R** for the following sequence of basic rotations:

1. A rotation of δ about the fixed x-axis
2. A rotation of β about the current y-axis
3. A rotation of α about the fixed z-axis
4. A rotation of ϕ about the current z-axis
5. A rotation of Θ about the current x-axis

**Reminder:**

Rotation with Respect to the Current Frame

$$R_2^0 = R_1^0 R_2^1$$

Rotation with Respect to the Fixed Frame

$$R_2^0 = R\,R_1^0$$

**Reminder:**

# Rotations in Three Dimensions

Each axis of the frame $\boldsymbol{o_1}\ \boldsymbol{x_1}\ \boldsymbol{y_1}\ \boldsymbol{z_1}$ is projected onto the coordinate frame $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$.

The resulting rotation matrix is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

The nine elements $\boldsymbol{r_{ij}}$ in a general rotational transformation **R** are not independent quantities.

# Rotations In Three Dimensions

For any $R \epsilon \ SO(n)$ The following properties hold

- $R^T = R^{-1} \ \epsilon \ SO(n)$
- The columns and the rows of $R$ are mutually orthogonal
- Each column and each row of $R$ is a unit vector
- $\det R \ = \ 1$ (the determinant)

Where $SO$(n) denotes the Special Orthogonal group of order n.

**Example for $\boldsymbol{R^T = R^{-1} \ \epsilon \ SO(2)}$:**

$$\begin{aligned} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \end{aligned}$$

# Parameterizations Of Rotations

The nine elements $\boldsymbol{r_{ij}}$ in a general rotational transformation R are not independent quantities.

$$R \in SO(3)$$

Where $SO$(n) denotes the **Special Orthogonal** group of order n.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

As each column of $R$ is a unit vector, then we can write:

$$\sum_i r_{ij}^2 = 1, \quad j \in \{1, 2, 3\}$$

3 Equations

As the columns of $R$ are mutually orthogonal, then we can write:

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \quad i \neq j$$

3 Equations

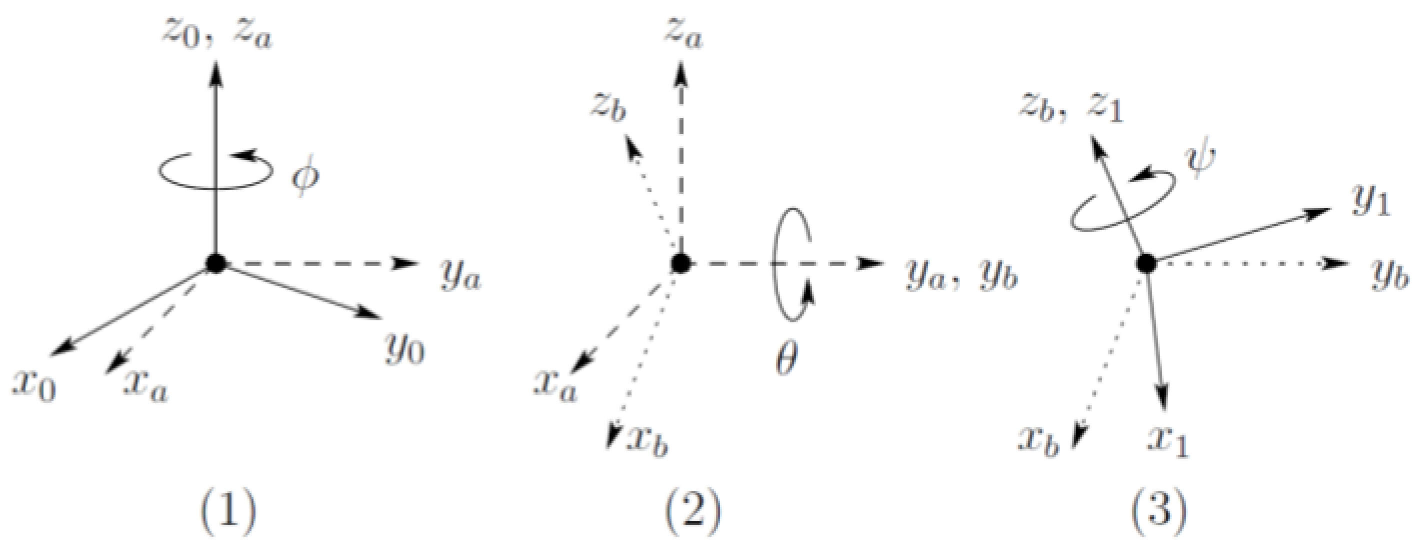
Together, these constraints define **six independent equations with nine unknowns**, which implies that there are **three free variables**.

# Parameterizations of Rotations

We present three ways in which an arbitrary rotation can be represented using only three independent quantities:

- **Euler Angles** representation
- **Roll-Pitch-Yaw** representation
- **Axis/Angle** representation

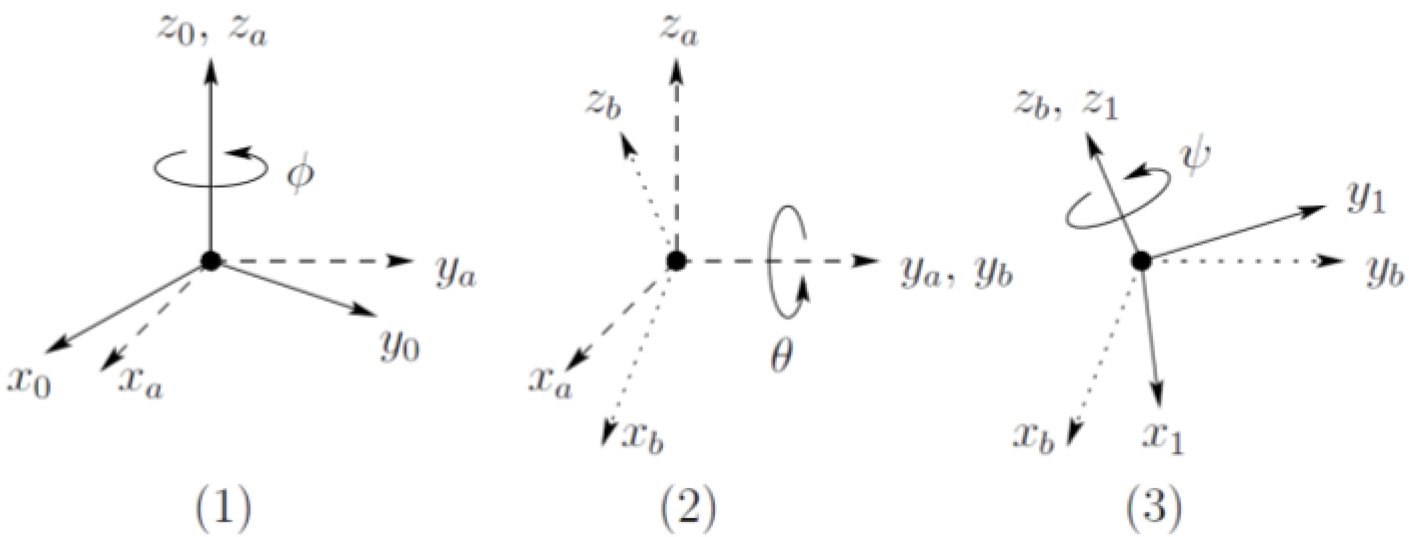
# Euler Angles Representation




We can specify the orientation of the frame $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ relative to the frame$\boldsymbol{o_0\ x_0\ y_0\ z_0}$ by three angles $\boldsymbol{(\phi, \theta, \psi)}$, known as Euler Angles, and obtained by three successive rotations as follows:

1. rotation about the ***z-axis*** by the angle $\boldsymbol{\phi}$
2. rotation about the ***current y−axis*** by the angle $\boldsymbol{\theta}$
3. rotation about the ***current z−axis*** by the angle $\boldsymbol{\psi}$

# Euler Angles Representation

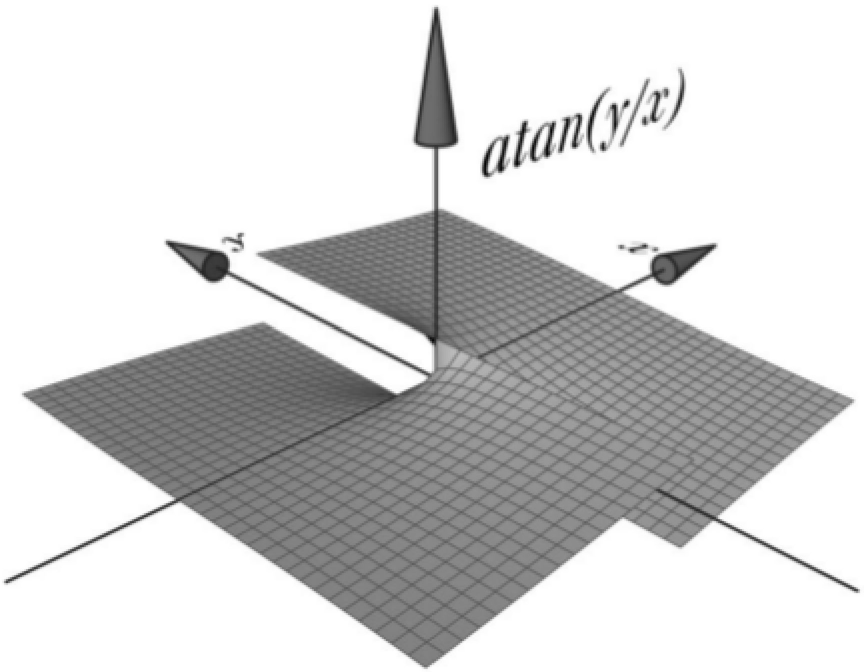


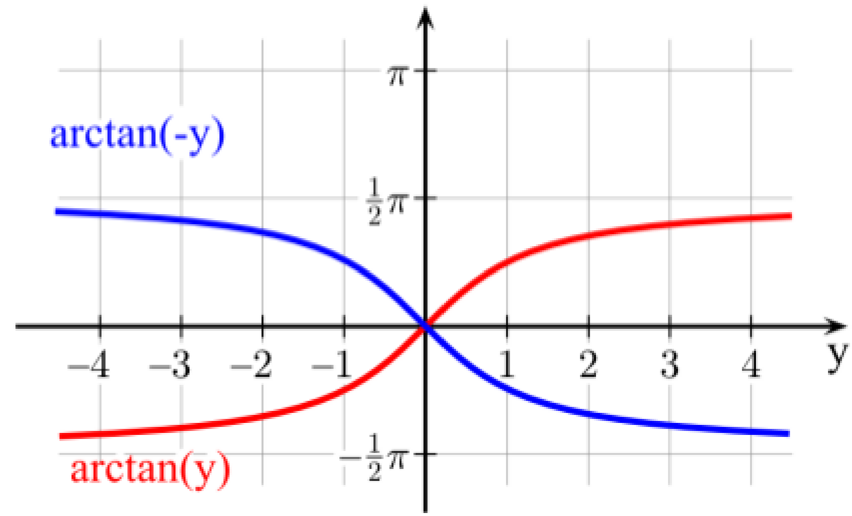

ZYZ-Euler Angle Transformation

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

**Reminder:**

# Trigonometry (Atan vs. Atan2)

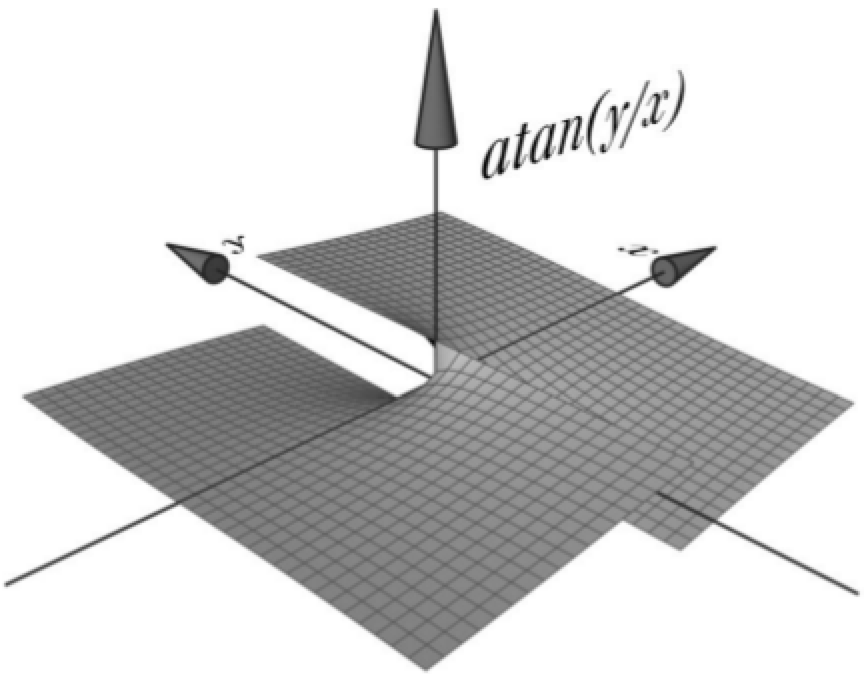


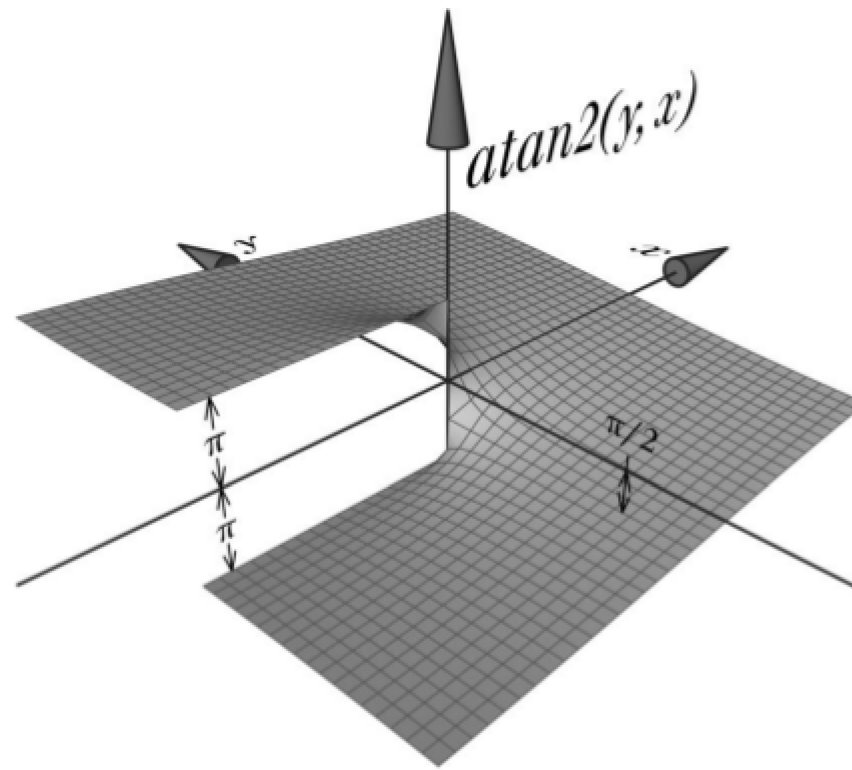



Reminder:

# Trigonometry (Atan vs. Atan2)

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\frac{y}{x} & x > 0 \\ \arctan\frac{y}{x} + \pi & y \geq 0, x < 0 \\ \arctan\frac{y}{x} - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$



# Euler Angles Representation

Given a matrix $R \epsilon\, SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

**Determine** a set of Euler angles $\boldsymbol{\phi}, \boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{ZYZ}}$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

If $\boldsymbol{r_{13} \neq 0}$ and $\boldsymbol{r_{23} \neq 0}$,
it follow that: $\boldsymbol{\phi = Atan2(r_{23}, r_{13})}$
where the function $Atan2(y, x)$ computes the arctangent of the ration $y/x$.
Then squaring the summing of the elements (1,3) and (2,3) and using the element (3,3) yields:

$$\boldsymbol{\theta = Atan2\left(+\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)} \quad \text{or} \quad \boldsymbol{\theta = Atan2\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)}$$

If we consider the first choice then $\boldsymbol{sin(\theta) > 0}$ then: $\boldsymbol{\psi = Atan2(r_{32}, -r_{31})}$

If we consider the second choice then $\boldsymbol{si\,n(\theta) < 0}$ then: $\boldsymbol{\psi = Atan2(-r_{32}, r_{31})}$
and $\boldsymbol{\phi = Atan2(-r_{23}, -r_{13})}$

# Euler Angles Representation

Given a matrix $R \epsilon\, SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

**Determine** a set of Euler angles $\boldsymbol{\phi}$, $\boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{ZYZ}}$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

If $\boldsymbol{r_{13}} = \boldsymbol{r_{23}} = \boldsymbol{0}$, then the fact that R is orthogonal implies that $\boldsymbol{r_{33}} = \pm\boldsymbol{1}$ and that $\boldsymbol{r_{31}} = \boldsymbol{r_{32}} = \boldsymbol{0}$ thus R has the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

If $\boldsymbol{r_{33}} = +\boldsymbol{1}$ then $c_\theta = 1$ and $s_\theta = 0$, so that $\theta = 0$.

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the sum $\boldsymbol{\phi} + \boldsymbol{\psi}$ can be determined as $\boldsymbol{\phi} + \boldsymbol{\psi}$ ***= Atan2(*** $\boldsymbol{r_{21}}, \boldsymbol{r_{11}}$ ***) = Atan2(*** $-\boldsymbol{r_{12}}, \boldsymbol{r_{22}}$ ***)***
There is infinity of solutions.

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$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

**Determine** a set of Euler angles $\boldsymbol{\phi}$, $\boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{ZYZ}}$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

If $\boldsymbol{r_{13} = r_{23} = 0}$, then the fact that R is orthogonal implies that $\boldsymbol{r_{33} = \pm 1}$ and that $\boldsymbol{r_{31} = r_{32} = 0}$ thus R has the form:

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

If $\boldsymbol{r_{33} = -1}$ then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$.

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

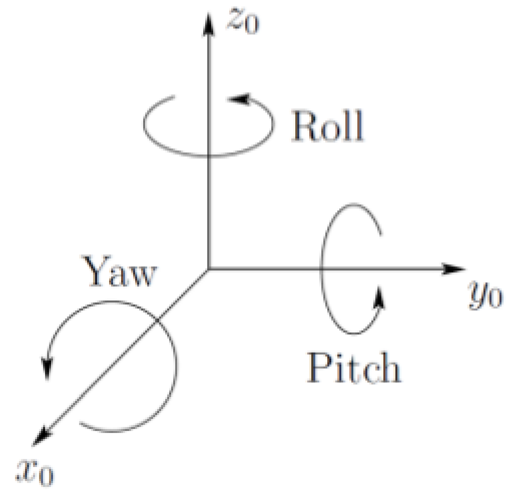
Thus, the $\boldsymbol{\phi - \psi}$ can be determined as $\boldsymbol{\phi - \psi}$ ***= Atan2(*** $\boldsymbol{-r_{12}, -r_{11}}$ ***) = Atan2(*** $\boldsymbol{r_{21}, r_{22}}$ ***)***
As before there is infinity of solutions.

# Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$




Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

$$R_{XYZ} \quad =$$

# Yaw-Pitch-Roll Representation

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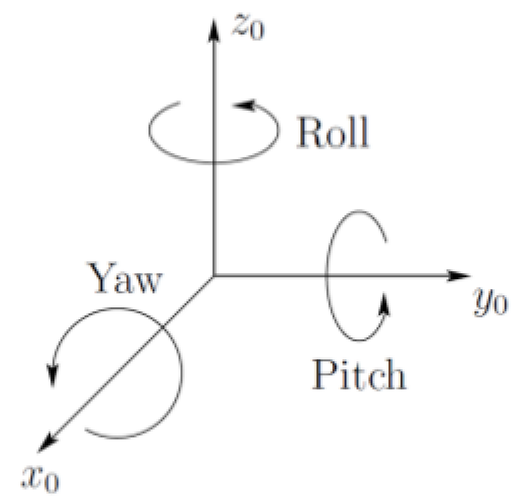
We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ −***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ − ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ − ***axis*** by the angle $\boldsymbol{\phi}$




Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

$$R_{XYZ} \quad = \quad R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

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A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

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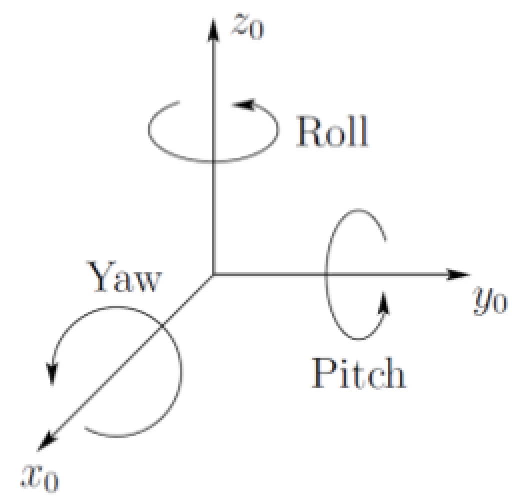
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$




Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

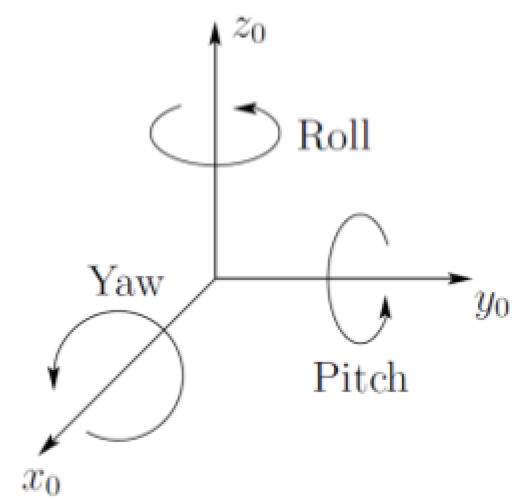
$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}
\end{aligned}
$$

# Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$




Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

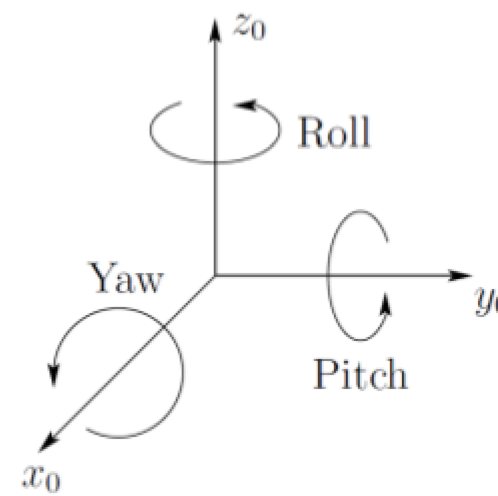
$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$

# Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$




Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

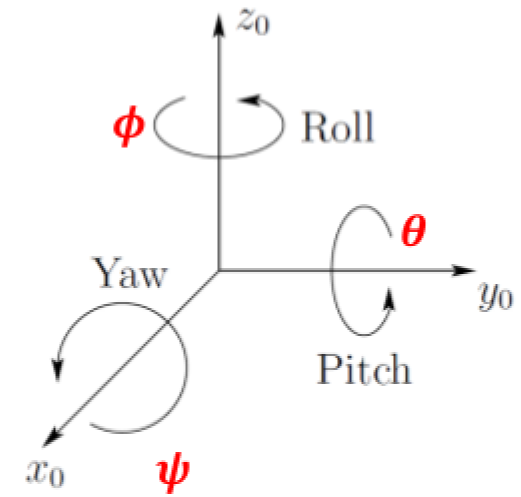
$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

# Yaw-Pitch-Roll Representation

A rotation matrix $\boldsymbol{R}$ can also be described as a product of successive rotations about the principal coordinate axes $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote $(\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\psi})$

We specify the order in three successive rotations as follows:

1. Yaw rotation about $\boldsymbol{x_0}$ $-$***axis*** by the angle $\boldsymbol{\psi}$
2. Pitch rotation about $\boldsymbol{y_0}$ $-$ ***axis*** by the angle $\boldsymbol{\theta}$
3. Roll rotation about $\boldsymbol{z_0}$ $-$ ***axis*** by the angle $\boldsymbol{\phi}$




Since the successive rotations are relative to the **fixed frame**, the resulting transformation matrix is given by:

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

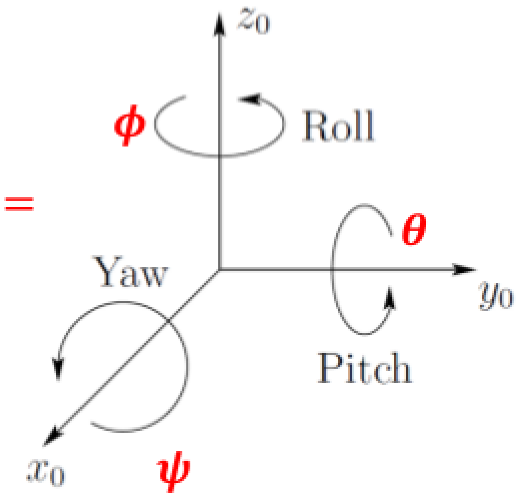
Instead of **yaw-pitch-roll** relative to the **fixed** frames we could also interpret the above transformation as **roll-pitch-yaw**, in that order, each taken with respect to the **current** frame.
The end result is the same matrix.

# Yaw-Pitch-Roll Representation

Find the inverse solution to a given rotation matrix R.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$R_{XYZ} = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

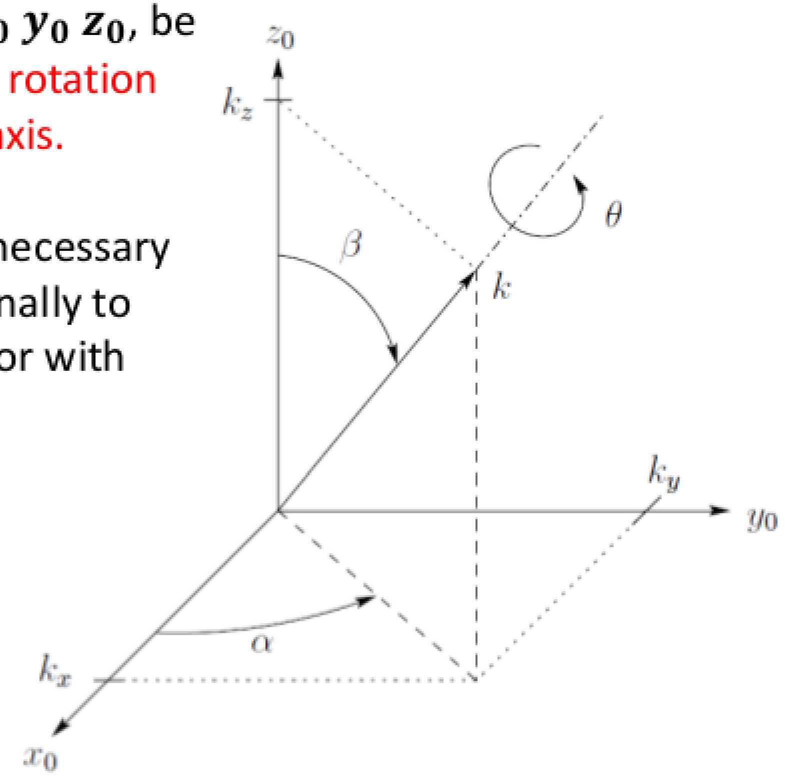



**Determine** a set of **Roll-Pitch-Yaw** angles $\boldsymbol{\phi}$, $\boldsymbol{\theta}$, and $\boldsymbol{\psi}$ so that $\boldsymbol{R} = \boldsymbol{R_{XYZ}}$

# Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

Let $\boldsymbol{k} = [\boldsymbol{k_x}, \boldsymbol{k_y}, \boldsymbol{k_z}]^{\boldsymbol{T}}$, expressed in the frame $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$, be a unit vector defining an axis. We wish to derive the rotation matrix $\boldsymbol{R_{k,\theta}}$ representing a rotation of $\boldsymbol{\theta}$ about this axis.




A possible solution is to rotate first $\boldsymbol{k}$ by the angles necessary to align it with $\boldsymbol{z}$, then to rotate by $\boldsymbol{\theta}$ about $\boldsymbol{z}$, and finally to rotate by the angels necessary to align the unit vector with the initial direction.

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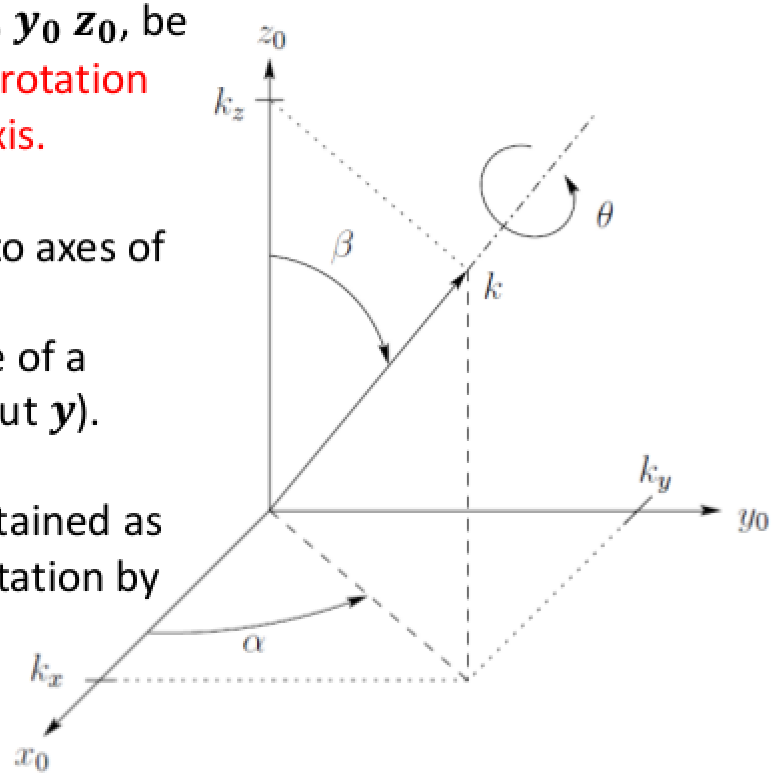



The sequence of rotations to be made with respect to axes of **fixed frame** is the following:

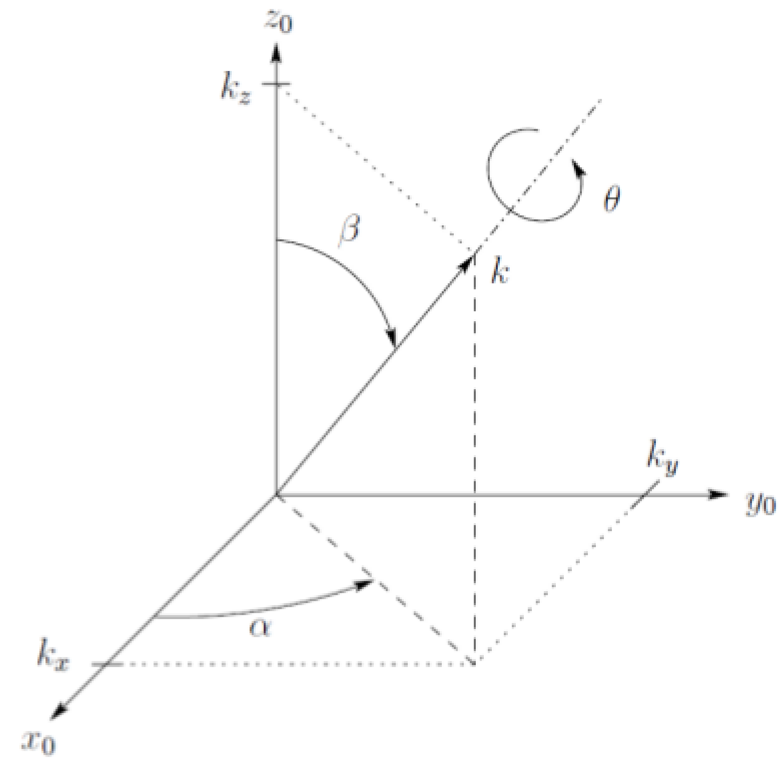
- Align $\boldsymbol{k}$ with $\boldsymbol{z}$ (which is obtained as the sequence of a rotation by $-\boldsymbol{\alpha}$ about $\boldsymbol{z}$ and a rotation of $-\boldsymbol{\beta}$ about $\boldsymbol{y}$).
- Rotate by $\boldsymbol{\theta}$ about $\boldsymbol{z}$.
- Realign with the initial direction of $\boldsymbol{k}$, which is obtained as the sequence of a rotation by $\boldsymbol{\beta}$ about $\boldsymbol{y}$ and a rotation by $\boldsymbol{\alpha}$ about $\boldsymbol{z}$.

$$\boldsymbol{R_{k,\theta}} = \boldsymbol{R_{z,\alpha}}\, \boldsymbol{R_{y,\beta}} \boldsymbol{R_{z,\theta}} \boldsymbol{R_{y,-\beta}} \boldsymbol{R_{z,-\alpha}}$$

# Axis/Angle Representation

$$R_{k,\theta} = R_{z,\alpha}\, R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$

$$\begin{aligned}
\sin\alpha &= \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \\
\cos\alpha &= \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \\
\sin\beta &= \sqrt{k_x^2 + k_y^2} \\
\cos\beta &= k_z
\end{aligned}$$



# Axis/Angle Representation

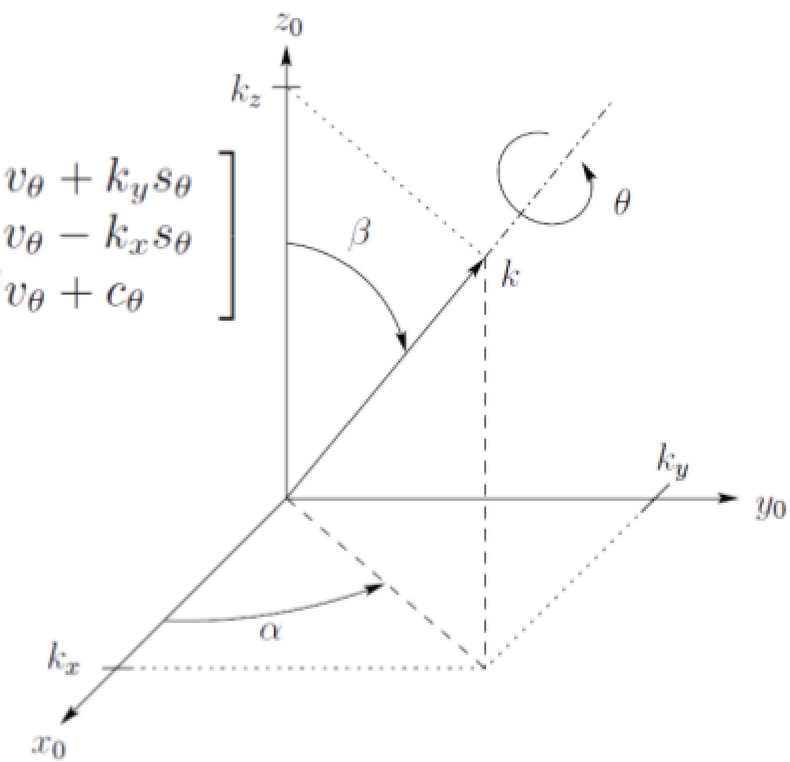
Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices.

$$\boldsymbol{R}_{k,\theta} = \boldsymbol{R}_{z,\alpha}\,\boldsymbol{R}_{y,\beta}\boldsymbol{R}_{z,\theta}\boldsymbol{R}_{y,-\beta}\boldsymbol{R}_{z,-\alpha}$$

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

$$v_\theta = \text{ vers } \theta = 1 - c_\theta.$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$



# Axis/Angle Representation

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right] \qquad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

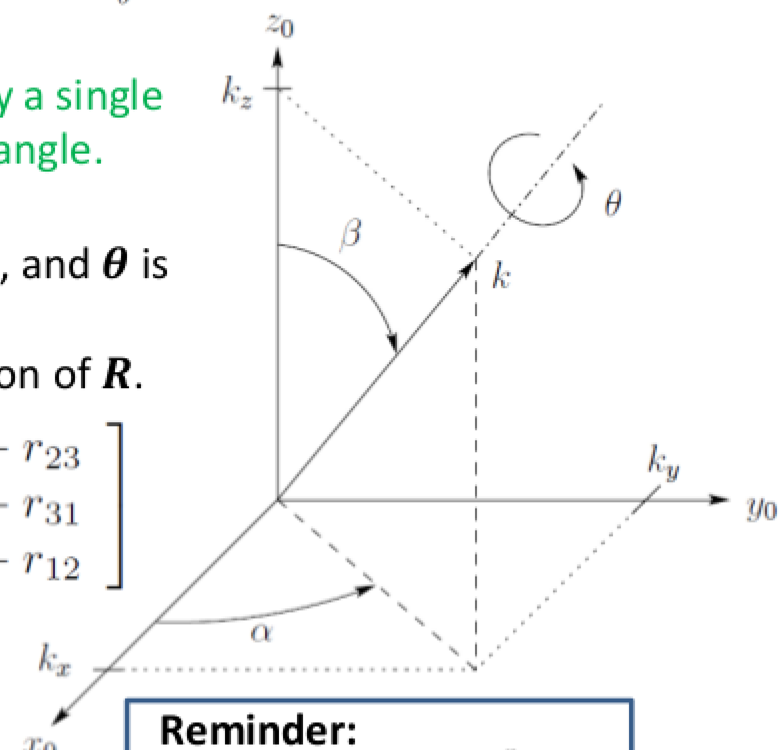
$$v_\theta = \text{vers}\,\theta = 1 - c_\theta.$$

Any rotation matrix $R \epsilon\, SO(3)$ can be represented by a single rotation about a suitable axis in space by a suitable angle.

$$\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{k},\boldsymbol{\theta}}$$

where $\boldsymbol{k}$ is a unit vector defining the axis of rotation, and $\boldsymbol{\theta}$ is the angle of rotation about $\boldsymbol{k}$.

The matrix $\boldsymbol{R}_{\boldsymbol{k},\boldsymbol{\theta}}$ is called the axis/angle representation of $\boldsymbol{R}$.




Given $\boldsymbol{R}$ find $\boldsymbol{\theta}$ and $\boldsymbol{k}$:

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \end{aligned}$$

**Reminder:**

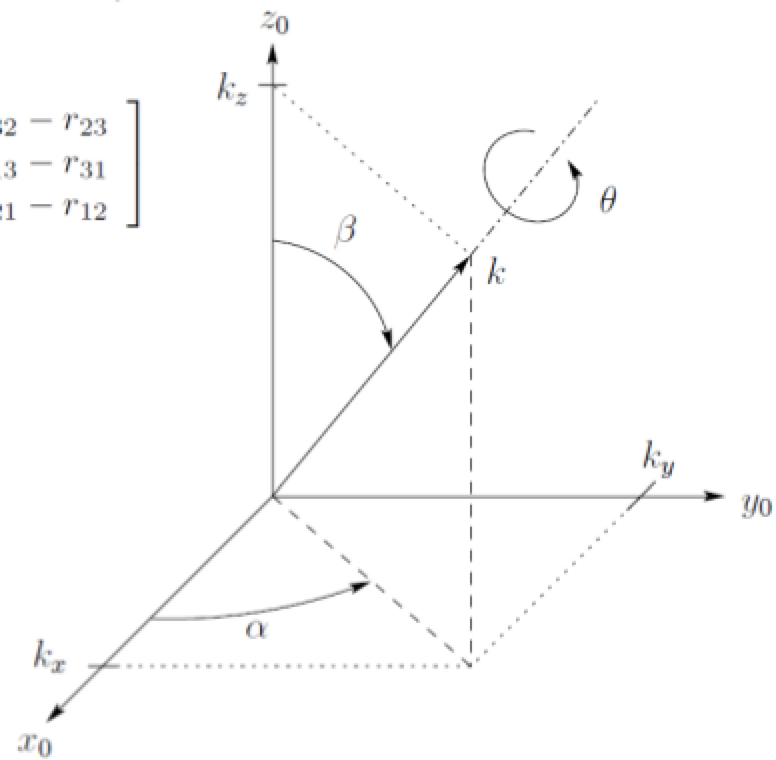
$$trace(\mathbf{A}) = tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

# Axis/Angle Representation

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

$$v_\theta = \text{vers}\,\theta = 1 - c_\theta.$$

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \end{aligned} \qquad k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$




The axis/angle representation is not unique since a rotation of $-\boldsymbol{\theta}$ about $-\boldsymbol{k}$ is the same as a rotation of $\boldsymbol{\theta}$ about $\boldsymbol{k}$.

$$\boldsymbol{R}_{\boldsymbol{k},\boldsymbol{\theta}} = \boldsymbol{R}_{-\boldsymbol{k},-\boldsymbol{\theta}}$$

If $\boldsymbol{\theta} = \boldsymbol{0}$ then $\boldsymbol{R}$ is the identity matrix and the axis of rotation is undefined.

# Example

Suppose $\boldsymbol{R}$ is generated by a rotation of $90^{\circ}$ about $z_0$ followed by a rotation of $30^{\circ}$ about $y_0$ followed by a rotation of $60^{\circ}$ about $x_0$. Find the axis/angle representation of $\boldsymbol{R}$

$$R = R_{x,60}R_{y,30}R_{z,90}$$

**Reminder:**

The axis/angle representation of $\boldsymbol{R}$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right)\\ &= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)\\ k &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23}\\ r_{13}-r_{31}\\ r_{21}-r_{12}\end{bmatrix}\end{aligned}$$

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Suppose $\boldsymbol{R}$ is generated by a rotation of $90^{\circ}$ about $z_0$ followed by a rotation of $30^{\circ}$ about $y_0$ followed by a rotation of $60^{\circ}$ about $y_0$. Find the axis/angle representation of $\boldsymbol{R}$

$$
\begin{aligned}
R &= R_{x,60}R_{y,30}R_{z,90} \\
&= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}
\end{aligned}
$$

$$Tr(R) =$$

$$\theta \ =$$

$$k \ =$$

**Reminder:**
The axis/angle representation of $\boldsymbol{R}$

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\
&= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \\
k &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}
\end{aligned}
$$

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$$
\begin{aligned}
R &= R_{x,60}R_{y,30}R_{z,90} \\
&= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}
\end{aligned}
$$

$$Tr(R) = 0$$

$$\theta =$$

$$k =$$

**Reminder:**
The axis/angle representation of $\boldsymbol{R}$

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\
&= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \\
k &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}
\end{aligned}
$$

# Example

Suppose $\boldsymbol{R}$ is generated by a rotation of 90° about $z_0$ followed by a rotation of 30° about $y_0$ followed by a rotation of 60° about $y_0$. Find the axis/angle representation of $\boldsymbol{R}$

$$
\begin{aligned}
R &= R_{x,60}R_{y,30}R_{z,90} \\
&= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}
\end{aligned}
$$

$$Tr(R) = 0$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

$$k =$$

**Reminder:**

The axis/angle representation of $\boldsymbol{R}$

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\
&= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \\
k &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}
\end{aligned}
$$

# Example

Suppose ***R*** is generated by a rotation of $90^{\circ}$ about $z_0$ followed by a rotation of $30^{\circ}$ about $y_0$ followed by a rotation of $60^{\circ}$ about $x_0$. Find the axis/angle representation of ***R***

$$
\begin{aligned}
R &= R_{x,60}R_{y,30}R_{z,90} \\
&= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}
\end{aligned}
$$

$$Tr(R) = 0$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^{\circ}$$

$$k = \left(\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right)^T$$

**Reminder:**
The axis/angle representation of ***R***

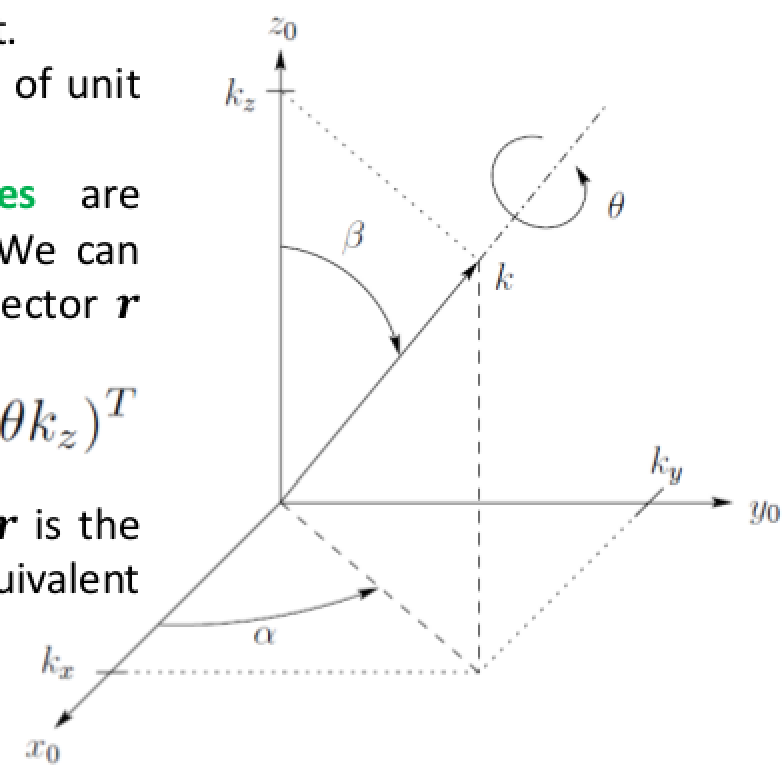
$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\
&= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \\
k &= \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}
\end{aligned}
$$

# Axis/Angle Representation

The above axis/angle representation characterizes a given rotation by four quantities, namely the three components of the equivalent axis $\boldsymbol{k}$ and the equivalent angle $\boldsymbol{\theta}$.
However, since the equivalent axis $\boldsymbol{k}$ is given as a unit vector only two of its components are independent.
The third is constrained by the condition that $\boldsymbol{k}$ is of unit length.
Therefore, only **three independent quantities** are required in this representation of a rotation $\boldsymbol{R}$. We can represent the equivalent axis/angle by a single vector $\boldsymbol{r}$ as:

$$r \;=\; (r_x, r_y, r_z)^T = (\theta k_x, \theta k_y, \theta k_z)^T$$

since $\boldsymbol{k}$ is a unit vector, the length of the vector $\boldsymbol{r}$ is the equivalent angle $\boldsymbol{\theta}$ and the direction of $\boldsymbol{r}$ is the equivalent axis $\boldsymbol{k}$.



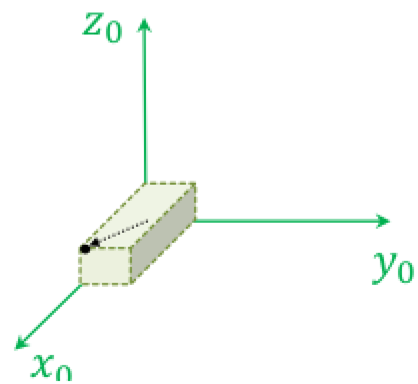
# Rigid Motions

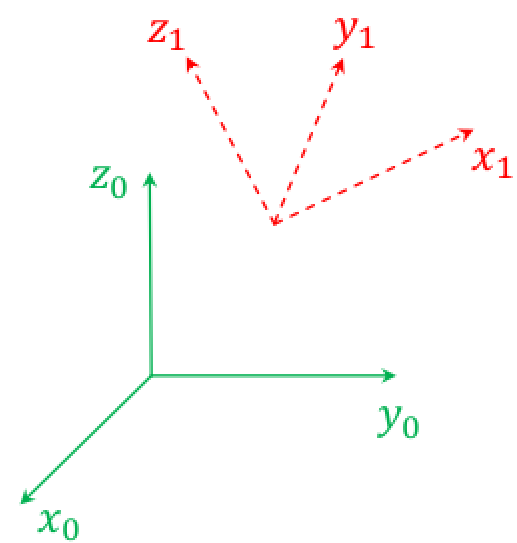
A rigid motion is a pure translation together with a pure rotation.

A rigid motion is an ordered pair $(\boldsymbol{d}, \boldsymbol{R})$ where $\boldsymbol{d} \in \mathbb{R}^{\mathbf{3}}$ and $\boldsymbol{R} \in \boldsymbol{SO}(\mathbf{3})$. The group of all rigid motions is known as the **Special Euclidean Group** and is denoted by $\boldsymbol{SE}(\mathbf{3})$. We see then that $\boldsymbol{SE}(\mathbf{3}) = \mathbb{R}^{\mathbf{3}} \times \boldsymbol{SO}(\mathbf{3})$.

# One Rigid Motion

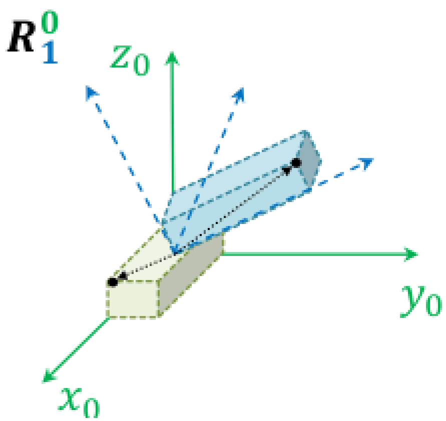
If frame $\boldsymbol{o_1\, x_1\, y_1\, z_1}$ is obtained from frame $\boldsymbol{o_0\, x_0\, y_0\, z_0}$ by first applying a rotation specified by $\boldsymbol{R_1^0}$

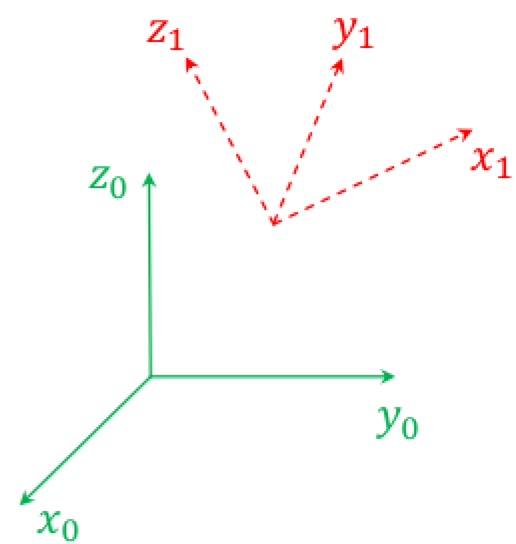


# One Rigid Motion

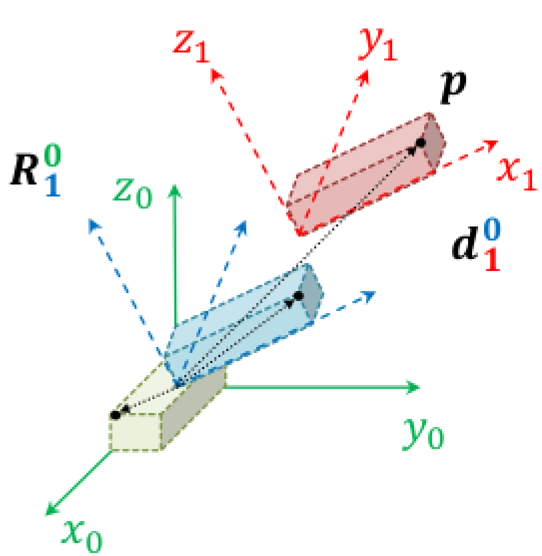
If frame $\boldsymbol{o_1\ x_1\ y_1\ z_1}$ is obtained from frame $\boldsymbol{o_0\ x_0\ y_0\ z_0}$ by first applying a rotation specified by $\boldsymbol{R_1^0}$ followed by a translation given (with respect to $\boldsymbol{o_0\ x_0\ y_0\ z_0}$) by $\boldsymbol{d_1^0}$

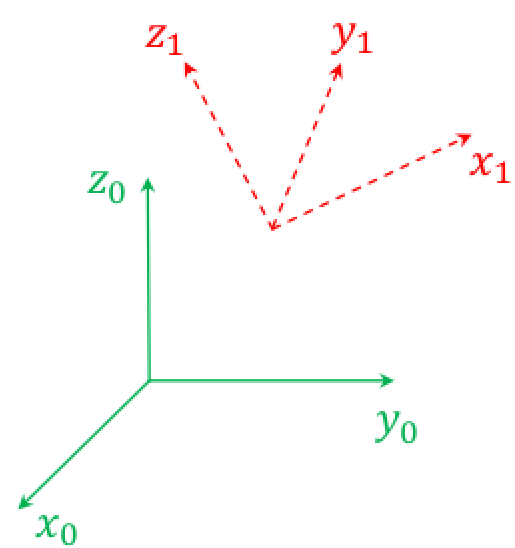


# One Rigid Motion

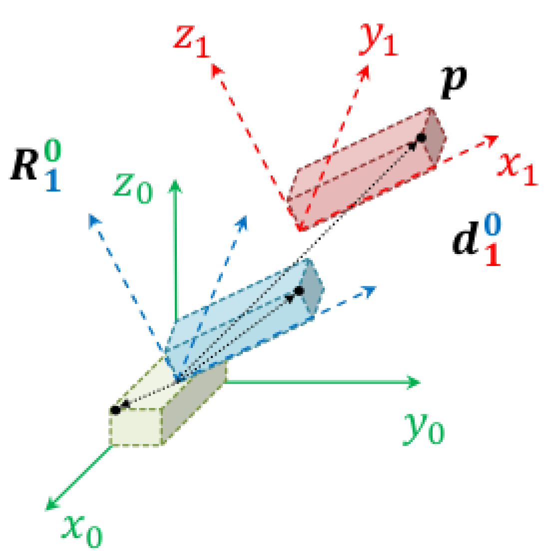
If frame $o_1\ x_1\ y_1\ z_1$ is obtained from frame $o_0\ x_0\ y_0\ z_0$ by first applying a rotation specified by $R_1^0$ followed by a translation given (with respect to $o_0\ x_0\ y_0\ z_0$) by $d_1^0$

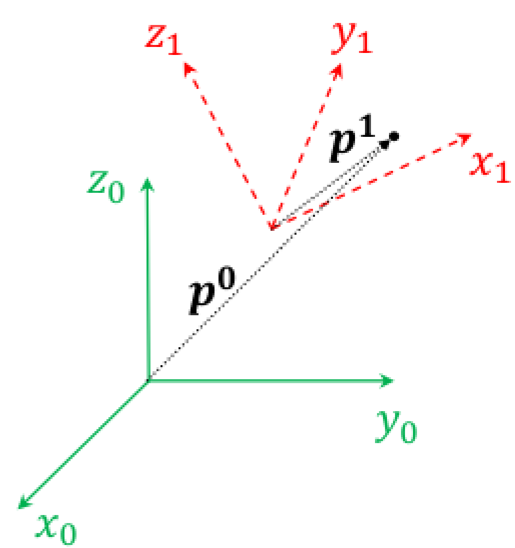


# One Rigid Motion

If frame $o_1\, x_1\, y_1\, z_1$ is obtained from frame $o_0\, x_0\, y_0\, z_0$ by first applying a rotation specified by $R_1^0$ followed by a translation given (with respect to $o_0\, x_0\, y_0\, z_0$) by $d_1^0$ , then the coordinates $p^0$ are given by:

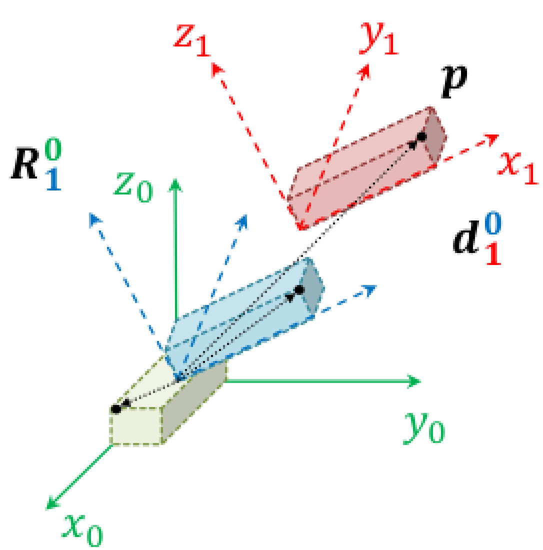


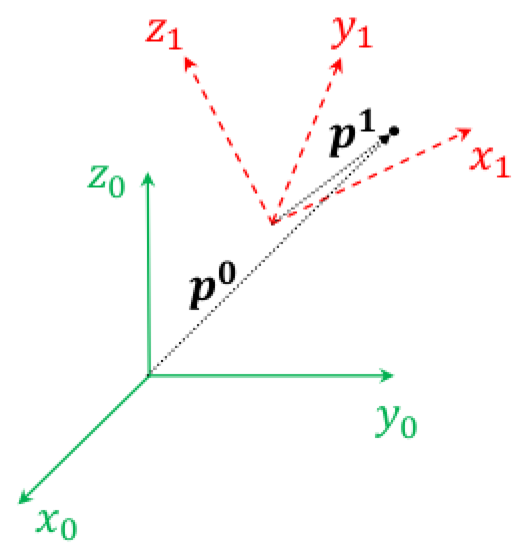



# One Rigid Motion

If frame $o_1\, x_1\, y_1\, z_1$ is obtained from frame $o_0\, x_0\, y_0\, z_0$ by first applying a rotation specified by $R_1^0$ followed by a translation given (with respect to $o_0\, x_0\, y_0\, z_0$) by $d_1^0$ , then the coordinates $p^0$ are given by:

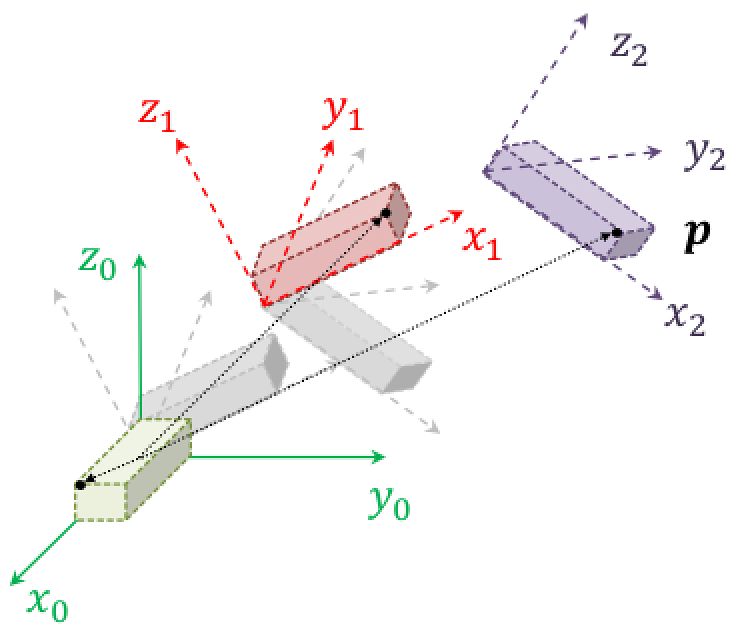
$$p^0 = R_1^0\, p^1 + d_1^0$$



# Two Rigid Motions

If frame $o_2\, x_2\, y_2\, z_2$ is obtained from frame $o_1\, x_1\, y_1\, z_1$ by first applying a rotation specified by $R_2^1$ followed by a translation given (with respect to $o_1\, x_1\, y_1\, z_1$) by $d_2^1$ .
If frame $o_1\, x_1\, y_1\, z_1$ is obtained from frame $o_0\, x_0\, y_0\, z_0$ by first applying a rotation specified by $R_1^0$ followed by a translation given (with respect to $o_0\, x_0\, y_0\, z_0$) by $d_1^0$ , find the coordinates $p^0$.




For the first rigid motion:

$$p^0 = R_1^0\; p^1 + d_1^0$$

For the second rigid motion:

$$p^1 = R_2^1\; p^2 + d_2^1$$

Both rigid motions can be described as one rigid motion:

$$p^0 = R_2^0\; p^2 + d_2^0$$

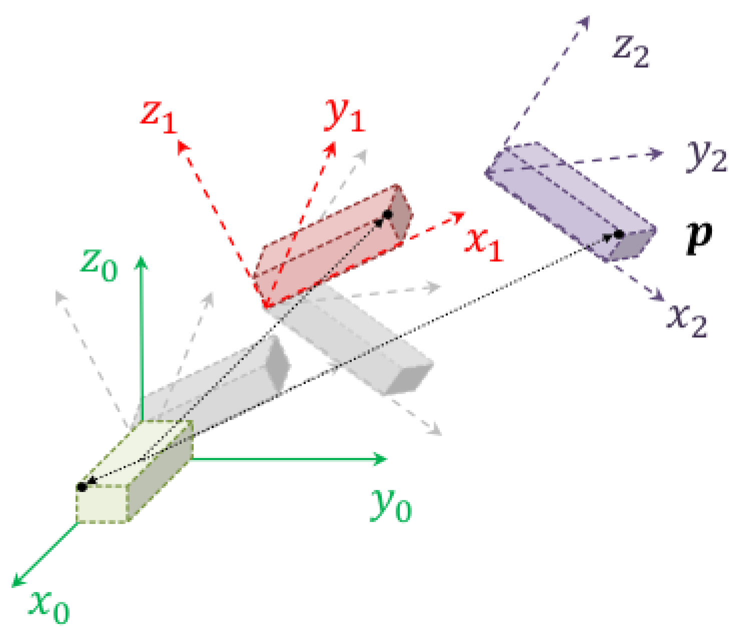
$$p^0 = R_1^0\; R_2^1\; p^2 + R_1^0\; d_2^1 + d_1^0$$

# Two Rigid Motions

$R_2^0$ The orientation transformations can simply be multiplied together.

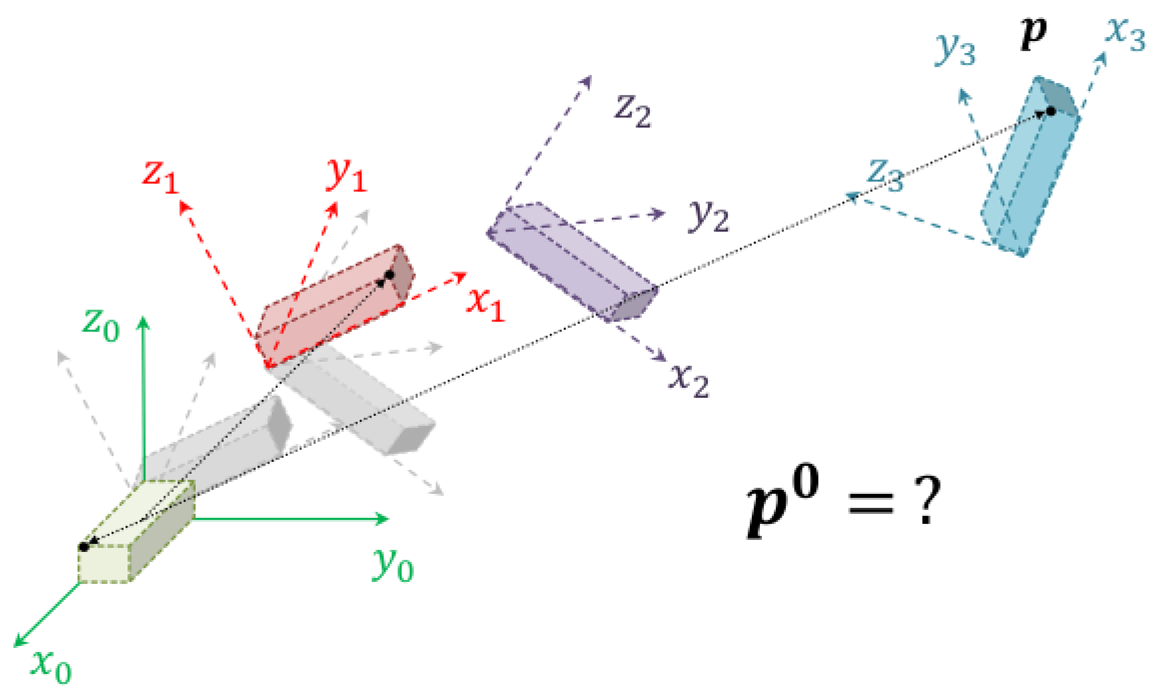
$d_2^0$ The translation transformation is the sum of:

- $d_1^0$ the vector from the origin $o_0$ to the origin $o_1$ expressed with respect to $o_0\, x_0\, y_0\, z_0$.
- $R_1^0\ d_2^1$ the vector from $o_1$ to $o_2$ expressed in the orientation of the coordinate system $o_0\, x_0\, y_0\, z_0$.




$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

# Three Rigid Motions

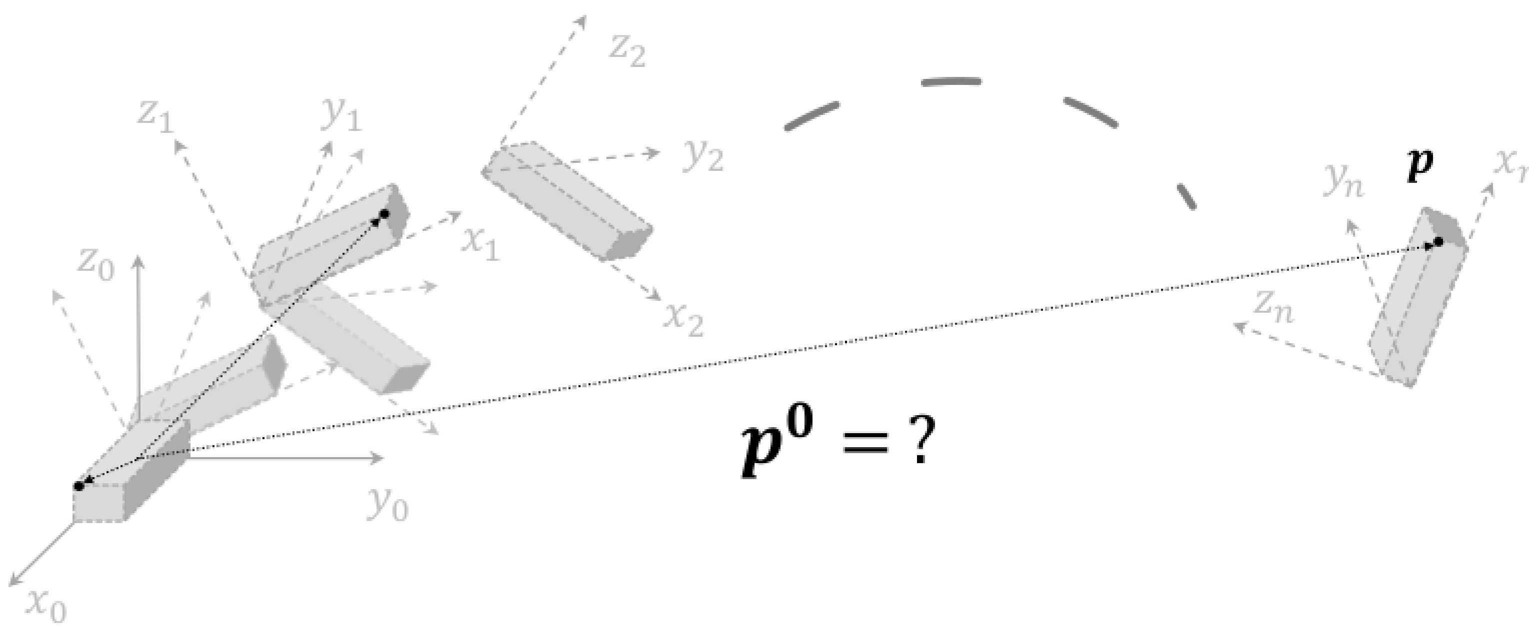



$$\boldsymbol{p}^{\mathbf{0}} = ?$$

# Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p^0}$.

$$\boldsymbol{p^0 = R_n^0\ p^n + d_n^0}$$

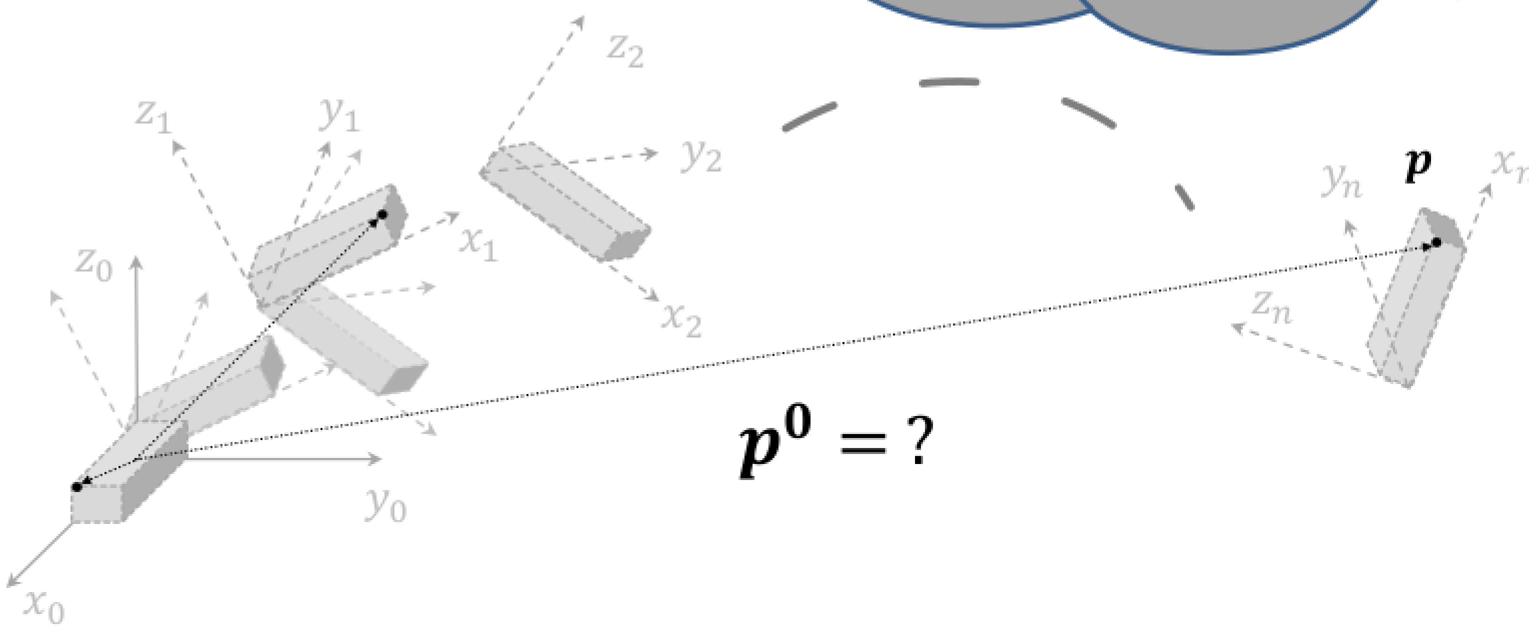


# Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p}^0$.

$$\boldsymbol{p}^0 = \boldsymbol{R}_n^0\ \boldsymbol{p}^n + \boldsymbol{d}_n^0$$

**Represent rigid motions in matrix** so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations



# Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p^0}$.

**Represent rigid motions in matrix** so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations

$$\boldsymbol{p^0} = \boldsymbol{R_n^0}\ \boldsymbol{p^n} + \boldsymbol{d_n^0}$$

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \mathbf{0} & \mathbf{1} \end{bmatrix};\ \boldsymbol{d} \in \mathbb{R}^{3}, \boldsymbol{R} \in \boldsymbol{SO}(3)$$

Transformation matrices of the form **H** are called **homogeneous transformations**.

A **homogeneous transformation** is therefore a matrix representation of a rigid motion.

# Homogeneous Transformations

A long sequence of rigid motions, find $\boldsymbol{p}^{\mathbf{0}}$.

**Represent rigid motions in matrix** so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations

$$\boldsymbol{p}^{\mathbf{0}} = \boldsymbol{R}_{\boldsymbol{n}}^{\mathbf{0}}\ \boldsymbol{p}^{\boldsymbol{n}} + \boldsymbol{d}_{\boldsymbol{n}}^{\mathbf{0}}$$

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \mathbf{0} & \mathbf{1} \end{bmatrix};\ \boldsymbol{d} \in \mathbb{R}^{3}, \boldsymbol{R} \in \boldsymbol{SO}(3)$$

The inverse transformation $\boldsymbol{H}^{-1}$ is given by

$$\mathbf{H}^{-1} = \begin{bmatrix} \boldsymbol{R}^{T} & -\boldsymbol{R}^{T}\boldsymbol{d} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

# Ex. :Two Rigid Motions

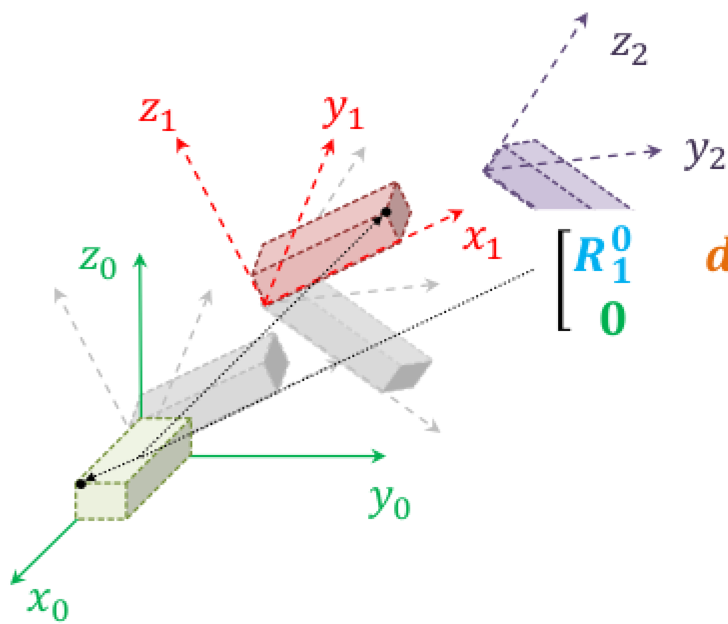
$R_2^0$ The orientation transformations can simply be multiplied together.

$d_2^0$ The translation transformation is the sum of:

- $d_1^0$ the vector from the origin $o_0$ to the origin $o_1$ expressed with respect to $o_0\ x_0\ y_0\ z_0$.
- $R_1^0\ d_2^1$ the vector from $o_1$ to $o_2$ expressed in the orientation of the coordinate system $o_0\ x_0\ y_0\ z_0$.

$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$




$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

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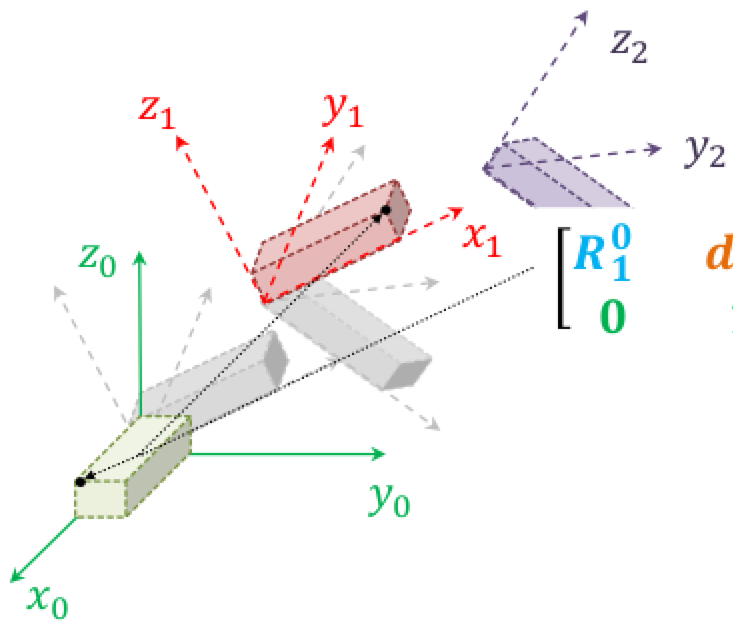
$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$




$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0\ R_2^1 & \\ 0 & \end{bmatrix}$$

# Ex. :Two Rigid Motions

$R_2^0$ The orientation transformations can simply be multiplied together.

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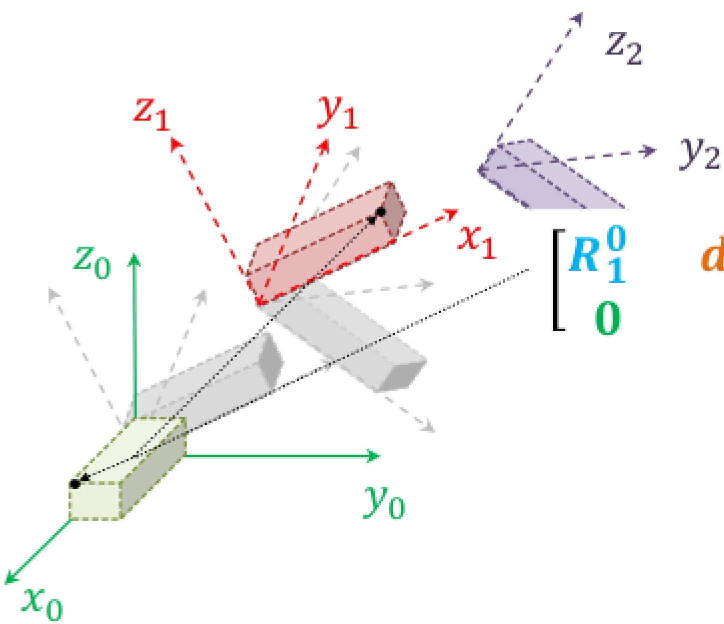
- $d_1^0$ the vector from the origin $o_0$ to the origin $o_1$ expressed with respect to $o_0\, x_0\, y_0\, z_0$.
- $R_1^0\ d_2^1$ the vector from $o_1$ to $o_2$ expressed in the orientation of the coordinate system $o_0\, x_0\, y_0\, z_0$.

$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$




$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0\ R_2^1 & R_1^0\ d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

# Ex. :Two Rigid Motions

$R_2^0$ The orientation transformations can simply be multiplied together.

$d_2^0$ The translation transformation is the sum of:

- $d_1^0$ the vector from the origin $o_0$ to the origin $o_1$ expressed with respect to $o_0\, x_0\, y_0\, z_0$.
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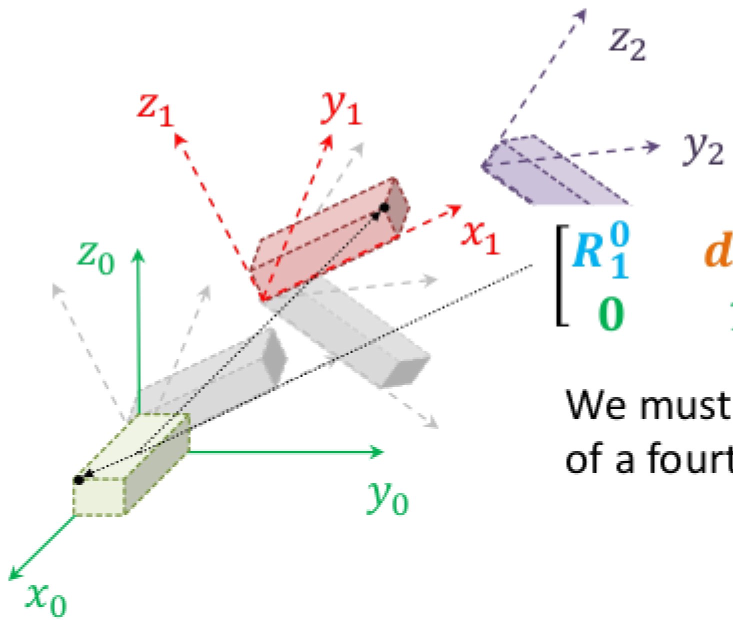
$$p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0$$

$$\mathbf{H} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix};\ d \in \mathbb{R}^3, R \in SO(3)$$




$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0\ R_2^1 & R_1^0\ d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

We must augment the vectors $p^0$, $p^1$ and $p^2$ by the addition of a fourth component of 1:

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}, P^2 = \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$$

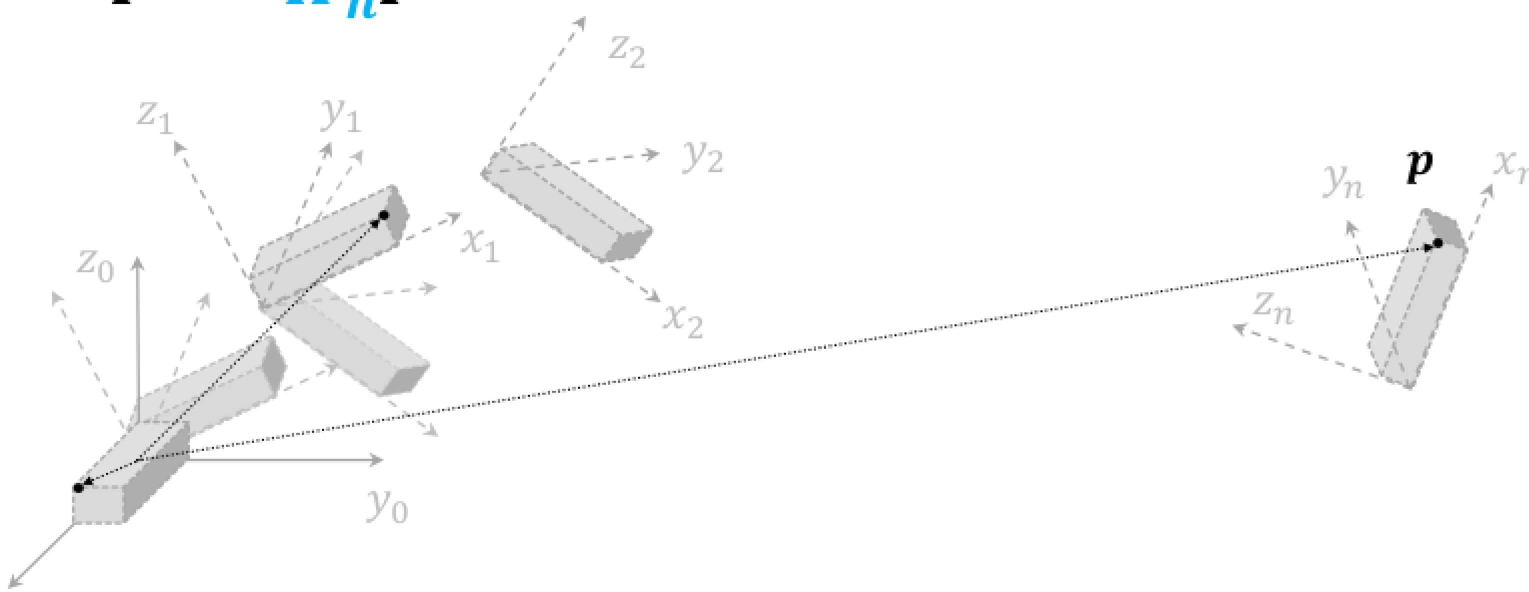
# Homogeneous Transformations

$$P^0 = H_1^0 P^1$$
$$P^0 = H_2^0 P^2$$

.....

$$P^0 = H_n^0 P^n$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}$$



# Basic Homogeneous Transformations

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{\mathrm{z},\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Homogeneous Transformations

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{n}$ is a vector representing the direction of $\boldsymbol{x_1}$ in the $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ system

$\boldsymbol{s}$ is a vector representing the direction of $\boldsymbol{y_1}$ in the $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ system

$\boldsymbol{a}$ is a vector representing the direction of $\boldsymbol{z_1}$ in the $\boldsymbol{o_0}\ \boldsymbol{x_0}\ \boldsymbol{y_0}\ \boldsymbol{z_0}$ system

# Composition Rule For Homogeneous Transformations

Given a homogeneous transformation $\boldsymbol{H_1^0}$ relating two frames, if a second rigid motion, represented by $\boldsymbol{H}$ is performed relative to the **current frame**, then:

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the **fixed frame**, then:

$$H_2^0 = H H_1^0$$

# Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x - axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x - axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z - axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z - axis}$

$$H \quad =$$

**Reminder:**

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

# Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H \quad = \quad Rot_{x,\alpha}$$

**Reminder:**

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

# Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha}\, Trans_{x,b}$$

**Reminder:**

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

# Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha}Trans_{x,b}Trans_{z,d}$$

**Reminder:**

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0 H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H H_1^0}$$

# Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H \quad = \quad Rot_{x,\alpha}\, Trans_{x,b}\, Trans_{z,d}\, Rot_{z,\theta}$$

**Reminder:**

Transformation with respect to the **current** frame

$$\boldsymbol{H_2^0 = H_1^0\, H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H\, H_1^0}$$

# Example

Find $\boldsymbol{H}$ for the following sequence of

1. a rotation by $\boldsymbol{\alpha}$ about the current $\boldsymbol{x-axis}$, followed by
2. a translation of $\boldsymbol{b}$ units along the current $\boldsymbol{x-axis}$, followed by
3. a translation of $\boldsymbol{d}$ units along the current $\boldsymbol{z-axis}$, followed by
4. a rotation by angle $\boldsymbol{\Theta}$ about the current $\boldsymbol{z-axis}$

$$H = Rot_{x,\alpha}Trans_{x,b}Trans_{z,d}Rot_{z,\theta}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Reminder:**

Transformation with respect to the **current** frame

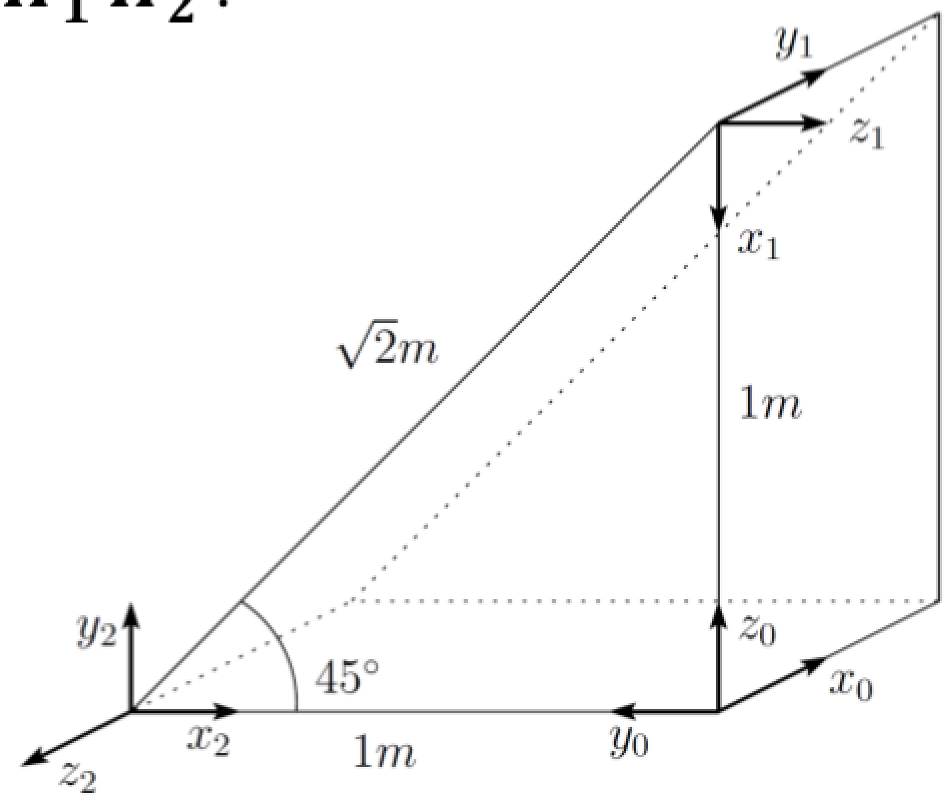
$$\boldsymbol{H_2^0 = H_1^0 H}$$

Transformation with respect to the **fixed** frame

$$\boldsymbol{H_2^0 = H H_1^0}$$

# Example

Find the homogeneous transformations $\boldsymbol{H_1^0}$ , $\boldsymbol{H_2^0}$ , $\boldsymbol{H_2^1}$ representing the transformations among the three frames Shown. Show that $\boldsymbol{H_2^0} = \boldsymbol{H_1^0}\,\boldsymbol{H_2^1}$ .



# Credits

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1. Google Images
2. http://www.cs.cityu.edu.hk/~helena/cs31622000A/Notes03.pdf
3. https://courses.cs.vt.edu/~cs4204/lectures/transformations.pdf
4. http://web.iitd.ac.in/~hegde/cad/lecture/L7_3dtransproj.pdf
5. http://www.songho.ca/math/homogeneous/homogeneous.html
6. http://igt.ip.uca.fr/~ab/Classes/DIKU-3DCV2/index.html
7. http://home.deib.polimi.it/restelli/MyWebSite/pdf/ese-01-handout.pdf
8.