Section: Senior **Date: 12-06-2020**

Differential Equation: Synopsis

---

1) Differential equation

By a differential equation we mean an equation involving independent variable, dependant variable and the differential coefficients of the dependent variable i.e. it will be an equation in x, y and derivatives of y w.r.t x. eg. $\frac{dy}{dx}+y=xe^x$, $\frac{d^2y}{dx^2}+y=0$

2) Order and Degree of a Differential equation:

1) The order of the highest differential coefficient appearing in the differential equation is called the order of the differential equation, with the highest exponent of the highest differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficients least degree, is known as the degree of the differential equation.

2) The order of the differential equation will be equal to the **minimum** number of **independent parameters** and not equal to the number of all the parameters in the family of curves.

Examples:

a. $y=1+\frac{dy}{dx}+\frac{1}{2!}\left(\frac{dy}{dx}\right)^2+\frac{1}{3!}\left(\frac{dy}{dx}\right)^3+........$ Order = 1, degree = 2 (given differential equation can be re-written as y = $e^{dy/dx}$)

b. $\frac{dy}{dx}=\sqrt{\frac{d^2y}{dx^2}+y}$ order = 2, degree = 1 (given differential equation can be re-written as $\left(\frac{dy}{dx}\right)^2=\frac{d^2y}{dx^2}+y$)

c. Order of the differential equation $\frac{d^2y}{dx^2}=\sin\left(\frac{dy}{dx}\right)+xy$ is 2 and degree is **not defined** (given differential equation cannot be written as a polynomial in all the differential coefficients)

d. Order of the differential equation $\frac{d^2y}{dx^2}=x\ln\left(\frac{dy}{dx}\right)$ is 2 and degree is **not defined** (given differential equation cannot be written as a polynomial in all the differential coefficients)

e. Order of the differential equation corresponding to the family of curve y = $(C_1 + C_2)\,e^x + C_3 e^{x+C_4}$ is 1 where c_1, c_2, c_3, c_4 are arbitrary constants

f. Order of the differential equation corresponding to the family of curve $y=c_1\sin^{-1}x+c_2\cos^{-1}x+c_3\tan^{-1}x+c_4\cot^{-1}x$ is 2 where c_1, c_2, c_3, c_4 are arbitrary constants

g. Order of the differential equation corresponding to the family of curve $y=(c_1+c_2)\cos(x+c_3)+c_4e^{x+c_5}$ is 3 where $y(1)=\frac{®}{2}$ are arbitrary constants

h. The degree of the differential equation satisfying the relation
$\sqrt{1+x^2}+\sqrt{1+y^2}=a\left(x\sqrt{1+x^2}+y\sqrt{1+y^2}\right)$ is 1

i. Order of the differential equation $\left(y^{111}\right)^3+\ln\left(y^{11}-xy^1\right)=0$ is 3 and degree of the differential equation is not defined as it can not be written as a polynomial equation in derivatives

* Solution of the differential equation is the family of curves

3) First Order Differential Equations with Separable Variables:

Let the differential equation be of the form $\frac{dy}{dx}=f(x,y)$ Where f(x, y) denotes a function of x and y, is separable if f(x,y) can be expressed in the form $\frac{M(x)}{N(y)}$ or $M(x).N(y)$ M(x), N(y) are real valued functions of x and y respectively.

Then, we have $\frac{dy}{dx}=\frac{M(x)}{N(y)}$ or $M(x).N(y)\Rightarrow N(y)dy=M(x)dx$ or $\frac{dy}{N(y)}=M(x)dx$

Integrating both sides of (2), we get the solution viz. $\int\frac{dy}{N(y)}=\int M(x)dx+c$

Examples: a. $\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0\Rightarrow\sqrt{1+x^2}+\frac{1}{2}\ln\left(\frac{\sqrt{1-x^2}-1}{\sqrt{1+x^2}+1}\right)=-\sqrt{1+y^2}+c$

b. $y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)\Rightarrow cy=(x+a)(1-ay)$

4) Differentiable Equations Reducible to the Separable Variable Type:

Sometimes differential equation of the first order cannot be solved directly by variable separation. By some substitution we can reduce it to a differential equation with separable variables. A differential equation of the form $\frac{dy}{dx}=f(ax+by+c)$ is solved by ax + by + c = t.

Examples: a. $\frac{dy}{dx}=\sin^2(x+3y)+5$ b. $\frac{dy}{dx}=\frac{2x-y+2}{2y-4x+1}\Rightarrow x+2y+\ln|2x-y|+c=0$

5) Homogeneous Differential Equations:

Suppose we have a differential equation of the form $\frac{dy}{dx}=$ f(x, y),

Where f(x, y) is a function of x and y and is of the form $F\left(\frac{y}{x}\right)$ or $F\left(\frac{x}{y}\right)$.

These equations are solved by putting y = vx, & $\frac{dy}{dx}=v+x\frac{dv}{dx}$ where v $\equiv$v(x) is a function of x. **or** by puttingx = vy& $\frac{dx}{dy}=v+y\frac{dv}{dy}$

Example: a. $\left(x^3-2y^3\right)dx+3xy^2dy=0$. b. $\left(x\frac{dy}{dx}-y\right)\tan^{-1}\left(\frac{y}{x}\right)=x$ given that y(1) = 0

$\Rightarrow\sqrt{x^2+y^2}=e^{\frac{y}{x}\tan^{-1}\left(\frac{y}{x}\right)}$

c. $\frac{dy}{dx}=\frac{(x+y)^2}{(x+2)(y-2)}$ d. $\frac{dy}{dx}=\frac{-\cos x(3\cos y-7\sin x-3)}{\sin y(3\sin x-7\cos y+7)}$

6) Equations Reducible to the Homogeneous form:

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+c}(aB \neq Ab)$ can be reduced to a homogeneous form by changing the variables x,y to X, Y by writing x = X + h and y = Y + k; where h, k are constants to be chosen so as to make the given equation homogeneous. We have

$\frac{dy}{dx} = \frac{d(Y+k)}{d(X+h)} = \frac{dY}{dX}$ Hence the given equation becomes $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$

Let h and k be so chosen as to satisfy the relation ah + bk + c = 0 and Ah + Bk + C = 0.These give $h = \frac{bC-Bc}{aB-Ab}, k = \frac{Ac-aC}{aB-Ab}$

Which are meaningful except when aB = Ab.

$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$ can now be solved by means of the substitution Y = VX.

7) In case aB = Ab, we write ax + by = t. This reduces the differential equation to the separable variable type.

Examples:

a. $\frac{dY}{dX} = \frac{x+2y+3}{2x+3y+4}$

b. $\frac{dy}{dx} = \frac{(x-1)^2+(y-2)^2\tan^{-1}\left(\frac{y-2}{x-1}\right)}{(xy-2x-y+2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$

8) First Order Linear Differential Equations:

The most general form of this category of differential equations is $\frac{dy}{dx} + P(x).y = Q(x)$, where P(x) and Q(x) are functions of x alone or constants.

We use here an **integrating factor**, namely $e^{\int P(x)dx}$ and the solution is obtained in the form $y.e^{\int P(x)dx} = \int Q(x)e^{\int P(x)\,dx}dx + c$, where c is an arbitrary constant.

Note: In some cases a linear differential equation may be of the form $\frac{dx}{dy} + P(y)x = Q(y)$ where $P(y)$ and $Q(y)$ are function of y alone or constants.

the solution is obtained in the form $x.e^{\int P(y)dx} = \int Q(y)e^{\int P(y)\,dx}dy + c$, where c is an arbitrary constant.

In such a case the integrating factor is $e^{\int P(y)dy}$

Examples:

a. $\frac{dx}{dy} + x\cos y = \sin 2y$. $\Rightarrow x.e^{\sin y} = \int \sin 2y.e^{\sin y}dy = 2e^{\sin y}(\sin y - 1) + c$

b. $(1+y^2)dx + \left(x - e^{-\tan^{-1}y}\right)dy = 0$ and $y(1) = \frac{®}{2}$ then $xe^{\tan^{-1}y} = \tan^{-1}y$

9) Extended form of linear Differential Equations:

Sometimes a differential equation is not linear but it can be converted into a linear differential equation by a suitable substitution.

Bernoulli's Equation:

a. $\frac{dy}{dx} + P(x).y = Q(x)y^n$ ......(1)

$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$ and let $t = \frac{1}{y^{n-1}}$ & $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{n-1}\frac{dt}{dx}$

Then equation 1 becomes $\frac{dt}{dx} + (1-n)tP(x) = Q(x)(1-n)$, which is linear

b. $\frac{dx}{dy}+P(y).x=Q(y)x^n$ ……(1)

$\frac{1}{x^n}\frac{dx}{dy}+\frac{P(y)}{x^{n-1}}=Q(y)$ and let $t=\frac{1}{x^{n-1}}$ & $\frac{1}{x^n}\frac{dx}{dy}=\frac{1}{n-1}\frac{dt}{dy}$

Then equation 1 becomes $\frac{dt}{dy}+(1-n)tP(y)=Q(y)(1-n)$, which is linear

Examples:

a. $2y\sin x\frac{dy}{dx}=2\sin x\cos x-y^2\cos x \Rightarrow y^2=\sin x$ b.If $y-\cos x\frac{dy}{dx}=y^2(1-\sin x)\cos x$ and $y(0)=1$ then the value of $y\left(\frac{®}{3}\right)=2$

Solving differential equation of the form: $f^1(y)\frac{dy}{dx}+P(x)f(y)=Q(x)$

$f^1(y)\frac{dy}{dx}+P(x)f(y)=Q(x)$ can be converted to $\frac{du}{dx}+P(x)u=Q(x)$ by taking u = f(y)

[examples: a. $\sec^2 y\frac{dy}{dx}+x\tan y=x^3$ **or** $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y \Rightarrow \tan y=\frac{1}{2}\left(x^2-1\right)+ce^{-x^2}$

b. $\frac{dy}{dx}=1-x(y-x)-x^3(y-x)^3 \Rightarrow (y-x)^{-2}=ce^{x^2}-\left(1+x^2\right)$ (take y-x = v)

10) Inspection Method:

1) $\int xdy+ydx = xy+c$

2) $\int \frac{ydx+xdy}{xy}=\log_e\left(xy\right)+c$

3) $\int xdx+ydy=\frac{1}{2}(x^2+y^2)+c$

4) $\int \frac{xdx+ydy}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}+c$

5) $\int d(x+y)=x+y+c$

6) $\int \frac{f^1(x,y)}{\left(f(x,y)\right)^n}=\frac{\left(f(x,y)\right)^{1-n}}{1-n}+c$

7) $\int \frac{xdy-ydx}{x^2}=\frac{y}{x}+c$

8) $\int \frac{xdy-ydx}{y^2}=-\frac{x}{y}+c$

9) $\int \frac{xdy-ydx}{x^2+y^2}=\tan^{-1}\left(\frac{y}{x}\right)+c$

10) $\int \frac{xdy-ydx}{xy}=\log_e\left(\frac{y}{x}\right)+c$

11) $\int \frac{xdy-ydx}{x\sqrt{x^2-y^2}}=\sin^{-1}\left(\frac{y}{x}\right)+c$

12) $\int \frac{xdy-ydx}{x^2-y^2}=\frac{1}{2}\log_e\left(\frac{x+y}{x-y}\right)+c$

13) $\int \frac{ye^x dx-e^x dy}{y^2}=\frac{e^x}{y}+c$

14) $\int \frac{2xydx - x^2dy}{y^2} = \frac{x^2}{y} + c$

Examples:

a. $\left\{\frac{1}{x} - \frac{y^2}{(x-y)^2}\right\}dx + \left\{\frac{x^2}{(x-y)^2} - \frac{1}{y}\right\}dy = 0 \Rightarrow \ln\left|\frac{x}{y}\right| + \frac{xy}{(x-y)} = c$

b. $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2} \Rightarrow$

$\left(\frac{y}{x}\right) - \frac{1}{2\left(x^2+y^2\right)} = c$

c. $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y} \Rightarrow \sin^2 y = x\sin y + \frac{x^2}{2} + c$

d. $\left(y + x\sqrt{xy}(x+y)\right)dx + \left(y\sqrt{xy}(x+y) - x\right)dy = 0 \Rightarrow$

$\frac{x^2+y^2}{2} + 2\tan^{-1}\left(\sqrt{\frac{x}{y}}\right) = c$

e. The solution of the differential equation $\frac{\sqrt{x}dx + \sqrt{y}dy}{\sqrt{x}dx - \sqrt{y}dy} = \sqrt{\frac{y^3}{x^3}}$ is given by

$\frac{1}{2}\log\left(x^3+y^3\right) + \tan^{-1}\left(\frac{y}{x}\right)^{\frac{3}{2}} = \frac{c}{2}$ [use $u = x^{3/2}$ and $v = y^{3/2}$]

11) ORTHOGONAL TRAJECTORY

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family. For example, each straight line passing through the origin, i.e. y = kx is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for finding the orthogonal trajectory:

(i) Let f(x, y, c) = 0 be the equation of the given family of curves, where c is an arbitrary parameter.

(ii) Differentiate f = 0, w.r.t. x and eliminate c hence form a differential equation.

(iii) Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in the above differential equation. This will give the differential equation of the orthogonal trajectory.

(iv) By solving this differential equation we get the required orthogonal trajectory.

12) Formation of differential equation:

The differential equation to a family of curves is obtained by using the following steps:
a) Identify the number of essential arbitrary constants (say n) in the equation of the curve. b) Differentiate the equation n times c) Eliminate the arbitary constants from the equation of the curve and n additional equations obtained in step (b)

1) Differential equation corresponding to the family of curves, whose equation is $y = Ae^{m_1x} + Be^{m_2x}$ where A and B are arbitrary constants, is $\frac{d^2y}{dx^2} - (m_1+m_2)\frac{dy}{dx} + m_1m_2y = 0$

2) Differential equation corresponding to the family of curves, whose equation is $y = Ae^{m_1x} + Be^{m_2x} + Ce^{m_3x}$ where A,B and C are arbitrary constants, is

$$\frac{d^3y}{dx^3} - (m_1+m_2+m_3)\frac{d^2y}{dx^2} + (m_1m_2+m_2m_3+m_3m_1)\frac{dy}{dx} - m_1m_2m_3y = 0$$

3) Differential equation corresponding to the family of curves, whose equation is $y = \left(Ax+B\right)e^{m_1x}$ where A and B are arbitrary constants, is $\frac{d^2y}{dx^2} - (m_1+m_1)\frac{dy}{dx} + m_1m_1y = 0$

4) Differential equation corresponding to the family of curves, whose equation is $y=\left(Ax^2+Bx+C\right)e^{m_1x}$ where A,B and C are arbitrary constants, is

$$\frac{d^3y}{dx^3}-(m_1+m_1+m_1)\frac{d^2y}{dx^2}+(m_1m_1+m_1m_1+m_1m_1)\frac{dy}{dx}-m_1m_1m_1y=0$$

5) Differential equation corresponding to the family of curves, whose equation is $y=\left(A\sin nx+B\cos nx\right)e^{mx}$ where A and B are arbitrary constants, is

$$\frac{d^2y}{dx^2}-\left((m+in)+(m-in)\right)\frac{dy}{dx}+(m+in).(m-in)y=0$$

examples:a. Differential equation corresponding to the equation $y=ae^{2x}+be^{-2x}$ (wherea,b are arbitrary constants) is $\frac{d^2y}{dx^2}-4y=0$

b. Differential equation corresponding to the equation $y=ae^{x}+be^{2x}+ce^{-3x}$ (where a, b, c are arbitrary constants) is $\frac{d^3y}{dx^3}-7\frac{dy}{dx}+6y=0$.

Sometimes transformation to the polar coordinates facilitates separation of variables.

a) Let $x=r\cos\theta$, and $y=r\sin\theta$ then i) $xdx+ydy=rdr$ ii) $xdy-ydx=r^2d\theta$

b) Let $x=r\sec\theta$ and $y=r\tan\theta$ then i) $\frac{dy}{dx}=\frac{2x-y+2}{2y-4x+1}$ ii) $xdy-ydx=r^2\sec\theta d\theta$

[examples: a. $\frac{xdx+ydy}{xdy-ydx}=\sqrt{\frac{1+x^2+y^2}{x^2-y^2}}\frac{dy}{dx}\Rightarrow\sqrt{x^2-y^2}+\sqrt{1+x^2+y^2}=c\sqrt{\frac{x+y}{x-y}}$ b.

$x\frac{dy}{dx}-y=x\left(\sqrt{x^2+y^2}\right)\Rightarrow\sqrt{1+\frac{y^2}{x^2}}+\frac{y}{x}=ce^{x}$

Miscellaneous examples:

a) 1 b) 2 c) 3 d) none of these

Key. A

S1: The differential equation of parabolas having their vertices at the origin and foci on the x-axis is an equation whose variables are separable

S2: The differential equation of the straight lines which are at a fixed distance p from the origin is an equation of degree 2

S3: The differential equation of all conics whose both axes coincide with the axes of coordinates is an equation of order 2

a) TTT b) TFT c) FFT d) TTF

Key. A

		Order of the differential equation
Circle	$(x-a)^2+(y-a)^2=r^2$, a,b,r are arbitrary Constants	3
Circle, of given radius	$(x-a)^2+(y-b)^2=r^2$	
Circle whose centre is on x axis	$(x-a)^2+(y)^2=r^2$, a, r are arbitrary constants.	2
Circle whose centre is on y axis	$(x)^2+(y-b)^2=r^2$	2
Circle whose centre, is at origin		