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Sec: Sr. **DEFINITE INTEGRATION SYNAPSIS** **Date:12-06-2020**

DEFINITE INTEGRALS: SYNOPSIS

1. Newton – Leibnitz formula:

Let $\frac{d}{dx}(F(x)) = f(x)\,\forall x \in (a,b)$. Then $\int_a^b f(x)dx = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$

Note:

a) If a > b, then $\int_a^b f(x)dx = \lim_{x\to b^+} F(x) - \lim_{x\to a^-} F(x)$.

b. If F(x) is continuous at a and b, then $\int_a^b f(x)dx = F(b) - F(a)$.

c. Leibnitz Theorem: If F(x) = $\int_{g(x)}^{h(x)} f(t)dt$, then $\frac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$

d. If a function f(x) is integrable on a closed interval [a,b] then the function $g(x) = \int_a^x f(t)dt$ is atleast continuous at every point "x" in the interval [a,b]

e. If a function f(x) is continuous on [a,b] then, the function g(x) defined by $g(x) = \int_a^x f(t)dt$ is continuous on [a,b] and differentiable on (a,b) and $g^1(x) = f(x)$

2. Properties of Definite Integral:

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$ i.e. definite integral is independent of variable of integration.

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where c may lie inside or outside the interval [a, b].

4. $\int_{-a}^a f(x)dx = \int_0^a f(x)+f(-x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$.

5. $\int_{\frac{a}{k}}^{\frac{b}{k}} f(kx)dx = \int_a^b f(x)dx$ (Expansion and contraction property)

6. If $f(x) \ge 0$ for all $a \le x \le b$ then $\int_a^b f(x)dx \ge 0$

7. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, (Shift property)

7a. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

7c. $\int_a^b f(x)dx = (b-a)\int_0^1 f(a+(b-a)x)dx$

[example: i) $\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{\frac{1}{3}}^{\frac{2}{3}} e^{9\left(x-\frac{2}{3}\right)^2} dx = 0$ ii) $\int_{-4}^{-5} \sin^2(x^2-3)dx + \int_{-2}^{-1} \sin^2(x^2+12x+33)dx = 0$]

7d. If f (a – x) = f (x) then $\int_0^a x\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$

7e. If f (a + b – x) = f (x) then $\int_a^b x\,f(x)dx = \frac{a+b}{2}\int_a^b f(x)\,dx$

8. $\int\limits_0^{2a} f(x)dx \;= \int\limits_0^{a}\left(f(x)+f(2a-x)\right)dx = \begin{cases} 2\int\limits_0^a f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

$\int_0^{16} f(t)dt = 4\left(\sim^3 f(\sim^4) + \;\; ^3 f(\;\; ^4)\right) \; = \int_0^a \left(f(a+x)+f(a-x)\right)dx = \int_0^a \left(f(x)+f(a+x)\right)dx$

8. If f(x) is a periodic function with period T, then

a. $\int\limits_0^{nT} f(x)dx = n\int\limits_0^{T} f(x)dx, n \in z$

b. $\int\limits_0^{a+nT} f(x)dx = n\int\limits_0^{T} f(x)dx, n \in z, a \in R$

c. $\int\limits_{mT}^{nT} f(x)dx = (n-m)\int\limits_0^{T} f(x)dx, m, n \in z$

d. $\int\limits_{nT}^{a+nT} f(x)dx = \int\limits_0^{a} f(x)dx, n \in z, a \in R$

e. $\int\limits_{a+nT}^{b+nT} f(x)dx = \int\limits_a^{b} f(x)dx, n \in z, a, b \in R$

[example: i) f(x) = sinx + cosx is aperiodic function with period ® , and $\int_0^{2®}[\cos x + \sin x]dx = -®$ and hence $\int_0^{2n®}[\cos x + \sin x]dx = -n®$ where [.] denotes G.I.F

3. Definite Integral as a limit of sum: $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int\limits_0^{p} f(x)dx$

4. Reduction formula:

1. $I_{m,n} = \int\limits_0^1 x^m (1-x)^n\,dx = \frac{m!n!}{(m+n+1)!}$

2. $I_n = \int\limits_0^{\frac{\pi}{2}} \sin^n\, x dx = \int\limits_0^{\frac{\pi}{2}} \cos^n\, x dx$

According as n is even or odd. $I_o = \frac{\pi}{2}, I_1 = 1$

Hence $I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)......\left(\frac{1}{2}\right).\frac{\pi}{2}, & \text{if n is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)......\left(\frac{2}{3}\right).1, & \text{if n is odd} \end{cases}$.

3. If $I_{m,n} = \int\limits_0^{\frac{\pi}{2}} \sin^m x.\cos^n x\,dx$,

$= \frac{(m-1)(m-3)(m-5).....(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4)..........}\frac{®}{2}$ when both m,n are even.

$= \dfrac{(m-1)(m-3)(m-5).....(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4)..........}$, otherwise

[Examples: $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^8 x dx \;=\; \dfrac{5.3.1.7.5.3.1}{14.12.10.8.6.4.2}\dfrac{\pi}{2}$]

4. Reduction formula - some examples:

1. $I_n = \int_0^{\frac{\pi}{2}} \dfrac{\sin nx}{\sin x} dx = \dfrac{\pi}{2}$, for all positive odd positive integers n.

2. $I_n = \int_0^{\pi} \dfrac{\sin(2n+1)x}{\sin x} dx = \pi$, for all natural number n.

3. $I_n = \int_0^{\pi} \dfrac{\sin^2 x}{\sin x} dx = n\pi$, for all natural numbers n

4. $I_n = \int_0^{\frac{\pi}{2}} \dfrac{\sin^2 x}{\sin x} dx = 1 + \dfrac{1}{3} + \dfrac{1}{5} + \ldots + \dfrac{1}{2n-1}$ for all natural numbers n

5. If $I_n = \int \tan^n x dx$, then $I_n + I_{n-2} = \dfrac{\tan^{n-1} x}{n-1}, n \geq 2$

$I_0 + I_1 + 2\left(I_2 + ... + I_8\right) + I_9 + I_{10}$, is equal to $\sum_{n=1}^{9} \dfrac{\tan^n x}{n}$

6. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ then

i) $I_n + I_{n-2} = \dfrac{1}{n+1}$, for all n = 2,3,4,.......

ii) $\dfrac{1}{I_2 + I_4}, \dfrac{1}{I_3 + I_5}, \dfrac{1}{I_4 + I_6}, ...$ are in A.P

iii) $I_{n+1} + I_{n-1} = \dfrac{1}{n}$

iv) $\dfrac{1}{n+1} < 2I_n < \dfrac{1}{n-1}$ for all natural numbers greater than one.

5. Maximum and Minimum Inequality:

1. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then m(b – a) $\leq \int_a^b f(x) dx \leq M(b-a)$, where m and M are absolute minimum and maximum values of the function f(x) in [a,b]

Further if f(x) is monotonically decreasing in (a, b), thenf(b) (b – a) $< \int_a^b f(x) dx < f(a)(b-a)$

and if f(x) is monotonically increasing in (a,b), then f(a) (b – a) $< \int_a^b f(x) dx < f(b)(b-a)$

[examples:

i) $1 < \int_0^{\frac{\pi}{2}} \dfrac{\sin x}{x} dx < \dfrac{\pi}{2}$ {because f(x) decreases and $f\left(\dfrac{\pi}{2}\right) = \dfrac{2}{\pi}$ & $f\left(0^+\right) = 1$}

ii) $\dfrac{\sqrt{3}}{8} < \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{\sin x}{x} dx < \dfrac{\sqrt{2}}{6}$ {because f(x) decreases and $f\left(\dfrac{\pi}{3}\right) = \dfrac{3\sqrt{3}}{2\pi}$ & $f\left(\dfrac{\pi}{4}\right) = \dfrac{2\sqrt{2}}{\pi}$}

iii) $1 < \int_0^2 \left(\dfrac{5-x}{9-x^2}\right) dx < \dfrac{6}{5}$ { absolute maximum and minimum values of f(x) in [0,2] are f(2) and f(1)}

6. Schwartz inequality:

For any two integrable functions f(x) and g(x) on the interval (a,b), then

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx.\int_a^b g^2(x)dx}$$

[example: The maximum value of $\int_0^1 \sqrt{(1+x)(1+x^3)}dx$ is $\sqrt{\frac{15}{8}}$

7. Other inequalities:

1. If the function f(x) increases and has a concave up graph in the interval [a,b] then

$$(b-a)f(a) < \int_a^b f(x)dx < (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

1a. If the function f(x) increases and has a concave down graph in the interval [a,b] then

$$(b-a)\left(\frac{f(a)+f(b)}{2}\right) < \int_a^b f(x)dx < (b-a)f(b)$$

[example: $1 < \int_0^1 e^{x^2}dx < \frac{e+1}{2}$

2. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then $\int_a^b \psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)dx$

[example:

i) $\frac{\text{®}}{3\sqrt{3}} \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le \frac{\text{®}}{4}$

[$1+x^2+2x^5 < 1+x^2+2x^2 = 1+3x^2$ which implies $\int_0^1 \frac{1}{1+3x^2}dx < \int_0^1 \frac{1}{1+x^2+2x^5}dx < \int_0^1 \frac{1}{1+x^2}dx$]

3. If $f(x) \le g(x)$ for all $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b g(x)dx$

8. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

9. Integral of an inverse function:

If f(x) is invertible function and f(x) is continuous then a definite integral of can be expressed in terms of f(x) i.e. $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy$ or

$$\int_a^b f(x)dx + \int_c^d f^{-1}(y)dy = bd - ac$$

Examples: i) $\int_0^1 e^{\sqrt{e^x}}dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln(x))dx = e^{\sqrt{e}}$

ii) If the value of $\int_1^2 e^{x^2}dx$ is k, then the value of $\int_e^{e^4} \sqrt{\ln x}\,dx$ is $2e^4 - e - k$

10. Some standard results:

1. If f(x) $\ge$ 0 on [a, b], then $\int_a^b f(x)dx \ge 0$.

2. If $\int_a^b f(x)dx = 0$ and f(x) is continuous on [a, b], then the equation f(x) = 0 has atleast one root in the interval [a, b]

3. If f(x) $\ge$ 0 and continuous on [a, b], and $\int_a^b (ax^2+bx+c)f(x)dx = 0$, then $ax^2+bx+c=0$ has one root in the interval [a, b]

Example: Let $\int_0^1 f(x)dx = 1$, $\int_0^1 xf(x)dx = 2$ & $\int_0^1 x^2 f(x)dx = 4$ then no such function f(x), exists for all real x in [0, 1] such that f(x) > 0 and continuous for all x in [0, 1] …….

4. $\int_0^{\frac{\pi}{2}} \log(\sin x)dx = \int_0^{\frac{\pi}{2}} \log(\cos x)dx = = \int_0^{\frac{\pi}{2}} \log(\sin 2x)dx = \int_0^{\frac{\pi}{2}} \log(\cos 2x)dx = -\frac{\pi}{2}\log 2$

4a. $\int_0^{\frac{\pi}{2}} \log(\cos x + \sin x)dx = -\frac{\pi}{4}\log 2$

5. $\int_0^1 \cot^{-1}(1-x+x^2)dx = 2\int_0^1 \tan^{-1} xdx$

6. $\int_0^{\sin^2 x} \sin^{-1}\left(\sqrt{t}\right)dt + \int_0^{\cos^2 x} \cos^{-1}\left(\sqrt{t}\right)dt = \frac{®}{4}$ (a constant)

7. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^{\frac{®}{4}} \ln(1+\tan x)dx = \frac{®}{8}\ln 2$

8. For n > 1, $0 < \int_0^{\frac{®}{2}} \sin^{n+1} xdx < \int_0^{\frac{®}{2}} \sin^n xdx$ and For n > 1, $0 < \int_0^{\frac{®}{4}} \tan^{n+1} xdx < \int_0^{\frac{®}{4}} \tan^n xdx$

9. Suppose f(x) is a real differentiable function such that $f(x)+f^1(x) \le 1$ for all x and f(0) = 0 then the maximum value of f(1) is equal to $\frac{e-1}{e}$ [$f(x)+f^1(x) \le 1 \Rightarrow \int_0^1 \frac{d}{dx}\left(e^x f(x)\right) \le \int_0^1 e^x dx \Rightarrow$

$f(1) \le \frac{e-1}{e}$]

10. Let f(x) be a continuous function with continuous first derivative on (a,b) and let $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to b^-} f(x) = -\infty$ and $f^2(x)+f^1(x) \ge -1$ for all x in (a, b) then the minimum value of b – a is ®

$[1 \ge \frac{-f^1(x)}{1+f^2(x)} \Rightarrow \int_a^b dx \ge -\int_a^b \frac{f^1(x)}{1+f^2(x)}dx \Rightarrow b-a \ge ®$]

11. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that

$f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x) \ge (f(x))^3 + (f(x))^{-1}$, then, maximum value of a is $\frac{\pi}{24}$

12. If a function f(x) : $[0,16] \to R$, is differentiable. If $0 < \sim < 1$ and $1 < \quad < 2$ then

$\int_0^{16} f(t)dt = 4\left(\sim^3 f(\sim^4) + \quad ^3 f(\quad ^4)\right)$ [Apply LMVT for the function g(x) = $\int_0^{x^4} f(t)dt$, in the interval [0,1] and [1,2] and add to get the result]

11. Mean value theorem: If f(x) is continuous on the interval [a,b] then there exists atleast one number c between a and b such that $\int_a^b f(x)dx = f(c)(a-b)$

[If $m \le f(x) \le M$ for $a \le x \le b$, then m(b – a) $\le \int_a^b f(x)dx \le M(b-a)$, where m and M are absolute minimum and maximum values of the function f(x) in [a,b] and f(x) is contionus on the interval [a,b] , the number $\frac{1}{b-a}\int_a^b f(x)dx$ lies between m and M .By Intermediate value theorem there exists c between a and b such that f(c) = $\frac{1}{b-a}\int_a^b f(x)dx$]

[example:

i) Suppose f(x) and g(x) are two continuous functions defined on [a,b] such that $\int_a^b f(x)-g(x)=0$, then the equation f(x) = g(x) is true for atleast one value of x in [a,b].

ii) If f(x) is continuous function such that $\int_0^x f(t)dt \to \infty$ as $x \to \infty$, then every line y = mx intersects the curve $y^2 + \int_0^x f(t)dt = a$ where a is a positive real number.

(g(x) = is a continuous function as f(x) is continuous and g(0) = 0 and g(x) $\to \infty$ as $x \to \infty$. Thus by intermediate value theorem there must be some x $\in (0,\infty)$ such that g(x) = a)]