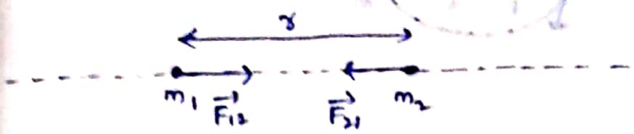
GRAVITATION:-

Newton's law of gravitation:-

[one mass will attract the other mass]



↳ Force acting on mass-2 by mass-1

$$|\vec{F_{12}}| = |\vec{F_{21}}| \propto m_1 m_2$$

$$\propto \frac{1}{r^2}$$

$$|\vec{F_{12}}| \propto \frac{m_1 m_2}{r^2}$$

$$|\vec{F_{12}}| = |\vec{F_{21}}| = \frac{G m_1 m_2}{r^2}$$ ⟶ [inverse square law Force]

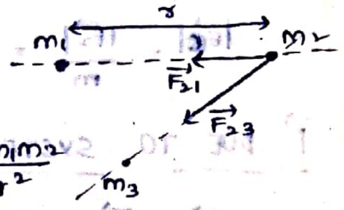
$G \rightarrow$ universal gravitation constant.

$$G = 6.67 \times 10^{-11}\ N \cdot m^2/kg^2$$

$$[G] = [M^{-1} L^3 T^{-2}]$$

→ Force is independent of medium present.

→ Force acting on particle by other particle is independent of other particles Presence.

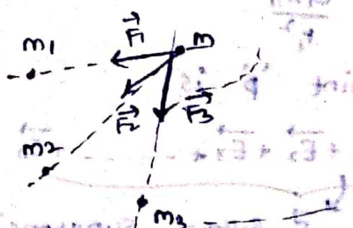


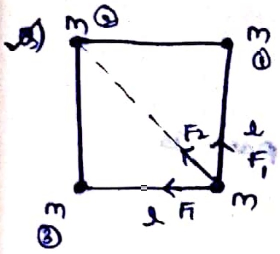
→ Net Force acting on a particle is dependent on presence of particles around it.

$$|\vec{F_{21}}| = \frac{G m_1 m_2}{r^2}$$

→ Net Force acting on particle of mass m is

$$\underbrace{\vec{F_{net}} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \cdots}_{\text{Gravitational Force Follows superposition principle}}$$



Q) 

Find the net Force acting on any one of the particles?

Soln:

$$\vec{F_1} = \frac{Gmm}{l^2} = \frac{Gm^2}{l^2} = \vec{F_3}$$

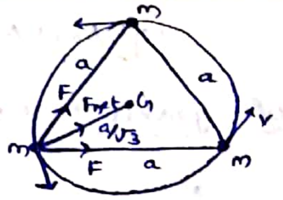
$$\vec{F_2} = \frac{G \cdot m \cdot m}{(\sqrt{2}l)^2} = \frac{Gm^2}{2l^2}$$

Resultant of $\vec{F_1}$ and $\vec{F_3}$ $= \sqrt{2}F_1 = \frac{\sqrt{2}Gm^2}{l^2}$

$\therefore \vec{F_{net}} = F_{res} + F_2 = \frac{Gm^2}{l^2}\left(\sqrt{2} + \frac{1}{2}\right)$ acting towards centre of square.

Q) If 3 identical particles of mass 'm' each are at corners of eq. Δ^{le} of side length 'a' and they revolve in a circle due to their mutual attraction. Find the speed of any one of the particles?

Soln: $F = \frac{Gm^2}{a^2}$



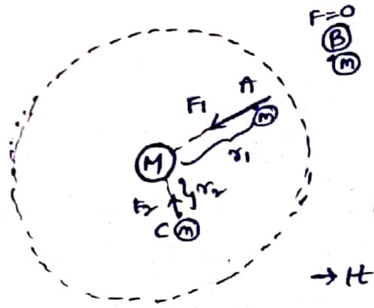
$$F_{net} = \sqrt{F^2+F^2+2F(\cos 60^\circ)} = \sqrt{3}F$$

$$\Rightarrow \frac{\sqrt{3}Gm^2}{a^2} = \frac{mv^2 \times \sqrt{3}}{a}$$

$$\Rightarrow \boxed{v = \sqrt{\frac{Gm}{a}}}$$

GRAVITATIONAL FIELD:-

↳ The region in which gravitational force is present.



$F = \frac{GMm}{r^2}$

To get E_0, $\boxed{r \to \infty}$

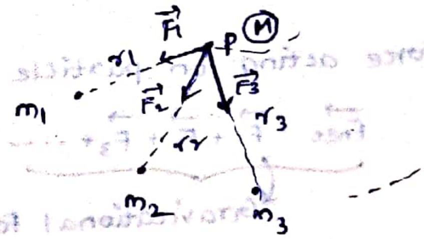
GRAVITATIONAL FIELD INTENSITY AT A POINT: ($\vec{E}$)

→ It is defined as force acting per unit mass at that point. $\boxed{\vec{E} = \frac{\vec{F}}{m}}$

$$|\vec{E}_A| = \frac{|\vec{F}_1|}{m} = \frac{\frac{GMm}{r_1^2}}{m} = \frac{GM}{r_1^2}$$

$$|\vec{E}_C| = \frac{|\vec{F}_2|}{m} = \frac{\frac{GMm}{r_2^2}}{m} = \frac{GM}{r_2^2}$$

$\vec{E}$ DUE TO SYSTEM OF PARTICLES:-



$$|\vec{E}_1| = \frac{\vec{F}_1}{M} = \frac{Gm_1}{r_1^2}$$

∴ Net $\vec{E}$ at point 'P' is

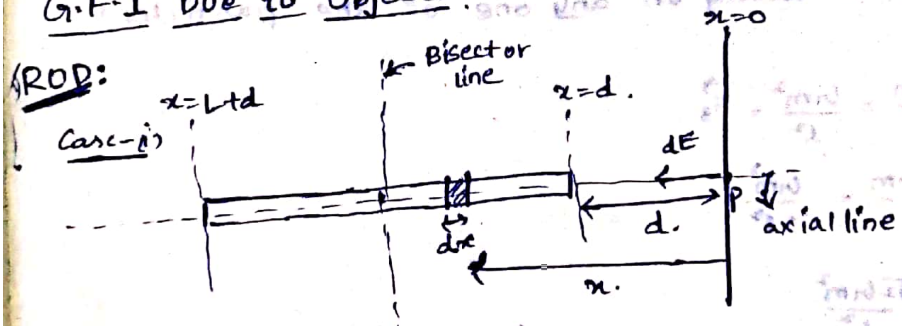
$$\underbrace{\vec{E}_{net} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots + \vec{E}_n}$$

↳ $\vec{E}$ follows superposition principle.

G.F.I Due to Objects:

ROD:

Case-(i)



mass of the small part is

$$dm = \frac{M}{L}\cdot dx$$

GFI due to small part is

$$dE = \frac{G(dm)}{x^2}$$

$$dE = \frac{GM}{L}\cdot\frac{dx}{x^2}$$

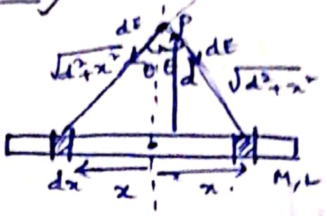
Uniform rod of mass M, length L.

→ we can integrate only if all the GFI are in same direction.

$$E_{net} = \int dE = \frac{GM}{L}\int\frac{dx}{x^2} = \frac{GM}{L}\left[\frac{-1}{x}\right]_d^{L+d} = \frac{GM}{\cancel{L}}\left(\frac{\cancel{L}}{d(L+d)}\right)$$

$$\therefore \boxed{|\vec{E}_{net}| = \frac{GM}{d(L+d)}}$$

Case - (ii)



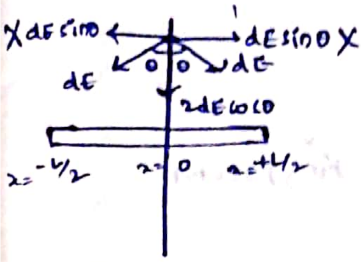
$dm = \frac{M}{L}(dx)$

GFI at pt. point 'P' due to small part

$$dE = \frac{G(dm)}{(\sqrt{x^2+d^2})^2}$$

$$dE = \frac{GM \cdot dx}{L(x^2+d^2)}$$

$\cancel{\boxed{E_{net} = \int dE}}$ (since diff. directions)



$\therefore E_{net} = 2\int_{-L/2}^{L/2} dE\cos\theta$, where $\cos\theta = \frac{d}{\sqrt{d^2+x^2}}$

$= 2\int^{L/2} dE\cos\theta$

$$E = 2\int_{0}^{L/2} \frac{GM \cdot dx \cdot d}{L\cdot(d^2+x^2)^{3/2}} = \frac{2GMd}{L}\int_0^{L/2}\frac{dx}{(x^2+d^2)^{3/2}}$$

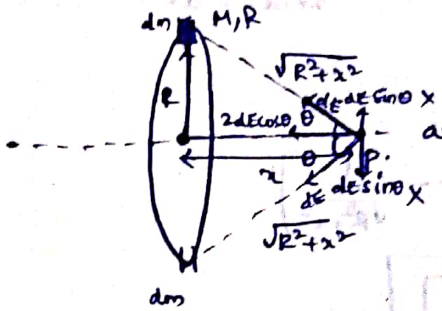
$[x = d\cdot\tan\theta]$.

$$\Rightarrow E_{net} = \frac{2GMd}{L}\int_0^{L/2}\frac{\sec^2\theta\cdot d\cdot(d\theta)}{\sec^3\theta\cdot d^3}$$

$$= \frac{2MdG}{L}\int_0^{L/2} d^{-2}\cos\theta\cdot d\theta = \frac{2MGd^2}{L\cdot d^3}\int_0^{L/2}\cos\theta\cdot d\theta = \frac{2MG}{L\cdot d}[\sin\theta]_0^{L/2}.$$

$$= \frac{2MG}{L\cdot d}\left[\frac{x}{\sqrt{x^2+d^2}}\right]_0^{L/2} = \frac{2MG\,(L/2)}{L\cdot d\sqrt{\left(\frac{L^2}{4}+d^2\right)}} \Rightarrow \boxed{\frac{2MG}{d\sqrt{(L^2+4d^2)}} = E_{net}}$$

2) GFI due to Ring:



$E_{net} = \int dE\cos\theta$

$$= \int \frac{G(dm)}{R^2+x^2}\cdot\frac{x}{\sqrt{R^2+x^2}}$$

$$= \frac{G\cdot x}{(R^2+x^2)^{3/2}}\int dm$$

mass of the small part = dm

GFI at point 'P' due to small part $\Rightarrow dE = \frac{G\cdot(dm)}{R^2+x^2}$

$$\boxed{E_{net} = \frac{G\cdot M\cdot x}{(R^2+x^2)^{3/2}}}$$

* GFI at Centre of Ring $(x=0)$ is $E_{net} = 0$

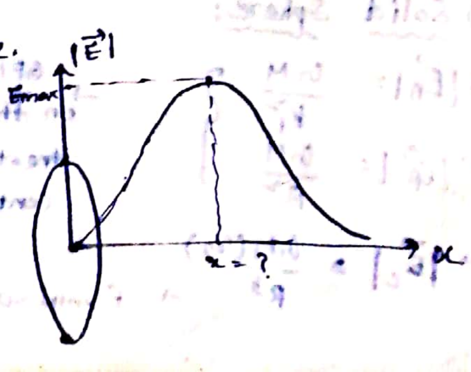
* GFI at point P which is very closer to centre $(x \ll R)$

$$E = \frac{GM\cdot x}{R^3} \Rightarrow E \propto x \quad x\uparrow \Rightarrow E\uparrow.$$

* If $R \ll x$, Point 'P' is far away from centre.

$$E = \frac{GMx}{x^3} = \frac{GM}{x^2}$$

$\Rightarrow E \propto \frac{1}{x^2}$ as $x\uparrow$, $E\downarrow$



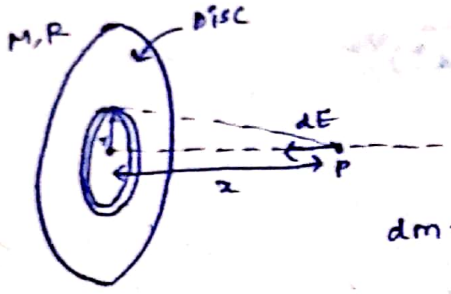
$E = \frac{GMx}{(R^2+x^2)^{3/2}} \Rightarrow \frac{dE}{dx} = \frac{GM\left[(R^2+x^2)^{3/2} - 3x/_2 (R^2+x^2)^{1/2} \cdot 2x\right]}{R^2+x^2} = 0$

$\Rightarrow dE \quad R^2+x^2 - \frac{3x^2}{2} = 0$

$x^2 - \frac{3x^2}{2} + R^2 = 0 \Rightarrow R^2 = 2x^2 \Rightarrow \boxed{x = \pm R/\sqrt{2}}$

$E_{max} = \frac{GMR}{\sqrt{2}\,(R)^3 (3/2)^{3/2}} = \boxed{\frac{2GM}{3\sqrt{3}\,R^2}}$

3) GFI due to Disc:



small part → ring / sector

$dE = \frac{G(dm)x}{(r^2+x^2)^{3/2}}$

$E_{net} = \int dE = GMx \int \frac{dm}{(r^2+x^2)^{3/2}}$

$dm = \frac{2M}{\cancel{2\pi} R^2} \cdot (\cancel{2\pi} r \cdot dr)$

$\Rightarrow \int dE = G \cdot x \cdot \int \frac{2M r \cdot dr}{R^2 \cdot (r^2+x^2)^{3/2}}$

$= \frac{2GMx}{R^2} \int_0^R \frac{r \cdot dr}{(r^2+x^2)^{3/2}}$ Let $r^2+x^2 = t$, $2r \cdot dr = dt$

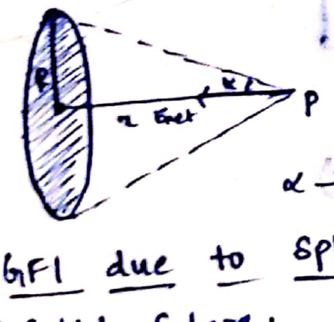
$= \cancel{\frac{2GMx}{R^2} \left[\frac{-1}{2\sqrt{r^2+x^2}} \right]_0^R}$

$\Rightarrow E = \frac{2GMx}{R^2} \int \frac{dt}{2t^{3/2}}$

$= \frac{2GMx}{R^2} \left[\frac{-1}{\sqrt{r^2+x^2}}\right]_0^R = \frac{2GMx}{R^2}\left[\frac{1}{x} - \frac{1}{\sqrt{R^2+x^2}}\right]$

$= \frac{2GM \cancel{x} \left[\sqrt{R^2+x^2} - x\right]}{R^2 \cdot \sqrt{R^2+x^2}} = \boxed{\frac{2GM}{R^2}\left[1 - \frac{x}{\sqrt{R^2+x^2}}\right]}$

$\Rightarrow$ Enet at centre of Disc $= \frac{2GM}{R^2}$



$\cos\alpha = \frac{x}{\sqrt{R^2+x^2}} \Rightarrow \boxed{E_{net} = \frac{2GM}{R^2}(1-\cos\alpha)}$

α → semi vertical angle

4) GFI due to Spheres:

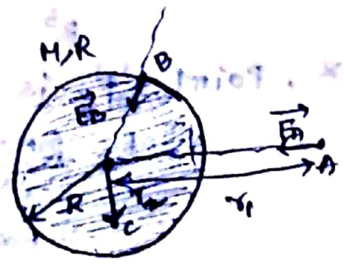
i) Solid Sphere:

$\rightarrow |\vec{E}_A| = \frac{GM}{r_1^2}$

$\rightarrow |\vec{E}_B| = \frac{GM}{R^2}$

→ GFI at any point outside & on the surface, sphere to be treated as pt. mass at the centre.

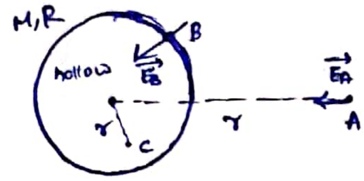
$\rightarrow |\vec{E}_c| = \frac{GM(r_2)}{R^3}$ → Ecentre = 0



ii) Hollow Sphere:

(Spherical shell of zero thickness)

M, R

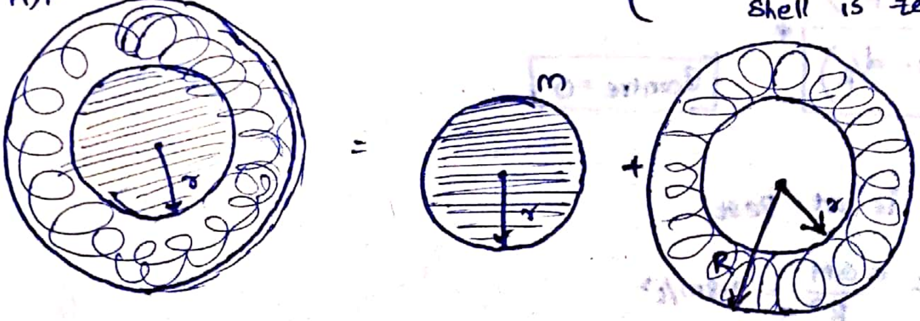


$|\vec{E_A}| = \frac{GM}{r^2}$ $(r>R)$.

$|\vec{E_B}| = \frac{GM}{R^2}$ $(r=R)$

$|\vec{E_C}| = 0$ (G.F.I in any spherical Shell is zero.)

M, R



Net G.F.I at point C

$= \begin{pmatrix}\text{GFI at 'c' due to}\\ \text{solid sphere of}\\ \text{radius} - r\end{pmatrix} + \begin{pmatrix}\text{GFI at C due to}\\ \text{spherical shell of}\\ \text{Thickness } (R-r)\end{pmatrix}$

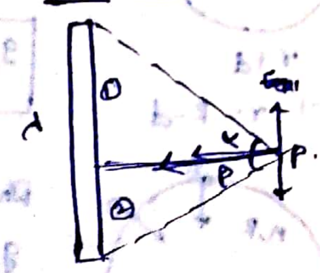
$\vec{E}_{net} = \vec{E_1} + \cancel{\vec{E_2}}^{\,0}$

$|\vec{E}_{net}| = |\vec{E_1}| = \frac{G(m)}{r^2} = \frac{G\left(\rho \cdot \frac{4}{3}\pi r^3\right)}{r^2} = \frac{GM.(r)}{R^3}$

GFI due to arc of an angle 'α':

$E_{net} = \frac{2GM}{R^2 \alpha} \cdot \sin(\alpha/2)$

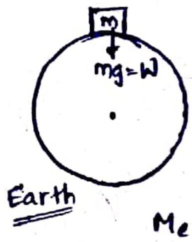
GFI due to Rod:-



$E_{\perp \text{ to rod}} = \frac{G\lambda}{d}(S\alpha + S\beta)$

$E_{\parallel \text{ to rod}} = \frac{G\lambda}{d}(C\alpha - C\beta)$

GFI due to Earth:



Gravitational Field Intensity at any point on the Earth surface

$|\vec{E}| = \frac{|\vec{F}|}{m} = \frac{mg}{m} = g$
↳ acc. due to gravity

$M_e = 6 \times 10^{24}$ kg

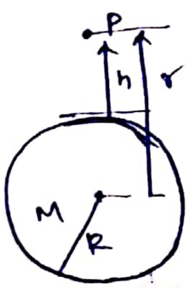
$R_e = 6400 Km = 64 \times 10^5 m$

→ GFI at any point on the surface of Earth

$|\vec{E}| = \frac{GM_e}{R_e^2} = g_s$
↳ 'g' value on the surface.

$g_s = \frac{GM}{R^2} = \frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{(64 \times 10^5)^2} = \boxed{9.8\, m/sec^2}$

g-value at height h:-



$g = \frac{\vec{F}}{m} = |\vec{E}|$

$g_h = \frac{GM}{r^2} = \boxed{\frac{GM}{(R+h)^2}}$

$g_h = \frac{GM}{R^2(1+h/R)^2} = g_s (1+h/R)^{-2}$

$$\boxed{g_h = g_s(1 - 2h/R)} \quad \text{when } h << R$$

$h < 100km$

'g' value at a depth d:

$R = r + d$

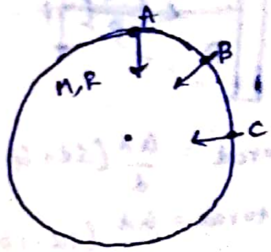
$\Rightarrow r = R - d$

$$g_d = |\vec{E_P}| = \frac{GM}{R^3} \cdot r.$$

$$= \frac{GM}{R^2}\left(\frac{R-d}{R}\right)$$

$$\boxed{g_d = g_s(1 - d/R)} \, *$$

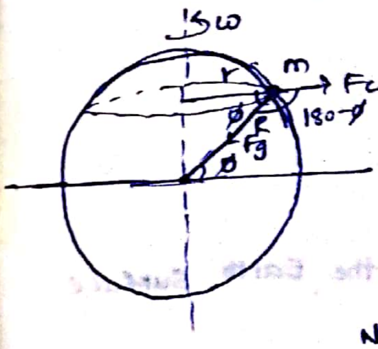
$$\boxed{g_{centre} = 0.}$$



When Earth is at Rest

$$g_A = g_B = g_C = \frac{GM}{R^2} = 9.8 m/s^2$$

'g' variation due to earth's rot. around itself:



$\phi \rightarrow$ latitude

$\cos\phi = \frac{r}{R}$

$r = R\cos\phi$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{86400}$ rad/sec (small)

$$F_g = m \cdot g_s = \frac{GMm}{R^2}$$

$$F_c = mr\omega^2 = m(R\cos\phi)\omega^2$$

Net Force acting on the particle is:

$$F_{net} = \sqrt{F_g^2 + F_c^2 + 2F_g \cdot F_c \cos(180 - \phi)}$$

$$= \sqrt{(mg_s)^2 + \cancel{(mR\cos\phi\omega^2)^2} - 2(mg_s)(mR\omega^2\cos\phi)\cos\phi}$$

Neglected.

$$= \sqrt{m^2 g_s^2 - 2m^2 g_s R\omega^2\cos^2\phi}$$

$$F_{net} = m\sqrt{g_s^2 - 2g_s R\omega^2\cos^2\phi}$$

$$\frac{F_{net}}{m} = g_\phi = g_s\sqrt{1 - \frac{2R\omega^2\cos^2\phi}{g_s}}$$

$$= g_s\left(1 - \frac{1}{2} \times \frac{2R\omega^2\cos^2\phi}{g_s}\right) = \boxed{g_s - R\omega^2\cos^2\phi} \, *$$

At equator: ($\phi = 0$)

$$g_{eq} = g_{min} = g_s - R\omega^2$$

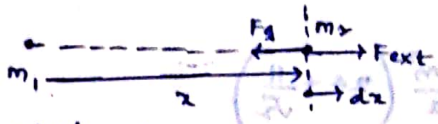
At poles: ($\phi = 90°$)

$$g_{poles} = g_{max} = g_s = 9.8 m/s^2$$

GRAVITATIONAL POTENTIAL ENERGY:-

$U_{at\ r=r_1}$ m_1 —— r_1 —— m_2

$U_{at\ r=r_2}$ m_1 —— r_2 —— m_2



$$F_g = \frac{Gm_1m_2}{x^2} = F_{ext}$$

Small work done $(dW_{ext}) = F_{ext} \cdot dx \cos 0°$

$$dW_{ext} = \frac{Gm_1m_2}{x^2} \cdot dx$$

Total work done in changing separation from r_1 to r_2 is:

$$W = \int dW = \int_{r_1}^{r_2} \frac{Gm_1m_2}{x^2} \cdot dx$$

$$W_{ext} = Gm_1m_2\left[-\frac{1}{r}\right]_{r_1}^{r_2} = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \Delta U$$

$$\therefore \boxed{U_{at\ r=r_2} - U_{at\ r=r_1} = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

m_1 —— ∞ —— m_2

$U_\infty = 0$ (assumption)

m_1 ←— r —→ m_2 $\quad r_1 = r$, $r_2 = \infty$

$$Gm_1m_2\left(\frac{1}{r} - \cancel{\frac{1}{\infty}}^{\,0}\right) = \cancel{U_{at\ r=\infty}}^{\,0} - U_{at\ r=r}$$

$$\Rightarrow \boxed{U_{at\ r=r} = -\frac{Gm_1m_2}{r}}$$

m_1 —— ∞ —— m_2

m_1 ←— r —→ m_2

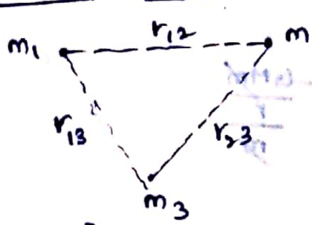
$W_{ext} = \Delta U$

$W_{ext} = U_{at\ r=r} - \cancel{U_{at\ r=\infty}}^{\,0}$

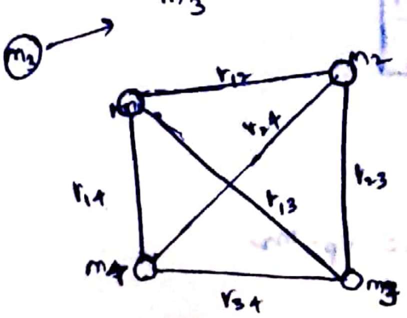
$$\boxed{W_{ext} = -\frac{Gm_1m_2}{r}}$$

→ GPE of system of 2 particles is given by work done by external agent in changing the dist b/w them from ∞ to 'r'.

GPE of 3, 4 ~~system~~ particles system:-

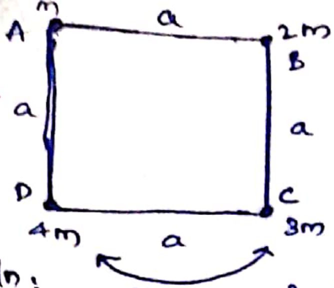


$$U = W_{ext} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$$



$$U = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_3m_4}{r_{34}} - \frac{Gm_2m_4}{r_{24}} - \frac{Gm_1m_4}{r_{14}}$$

Q) Find the minimum work required to exchange the Particles at C & D positions?



Soln:

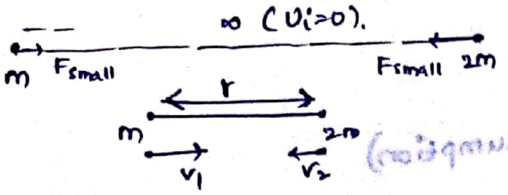
$$U_i = -\frac{Gm^2}{a}\left(2+6+4+12+\frac{3}{\sqrt{2}}+\frac{8}{\sqrt{2}}\right) = -\frac{Gm^2}{a}\left(24+\frac{11}{\sqrt{2}}\right)$$

$$U_f = -\frac{Gm^2}{a}\left(2+8+3+12+\frac{4}{\sqrt{2}}+\frac{6}{\sqrt{2}}\right) = -\frac{Gm^2}{a}\left(25+\frac{10}{\sqrt{2}}\right)$$

$$(W_{min})_{ext} = \Delta U = \frac{Gm^2}{a}\left(\frac{1}{\sqrt{2}}-1\right) = -\frac{Gm^2(\sqrt{2}-1)}{a(\sqrt{2})}$$

Q) If 2 particles of masses m & 2m are released from rest at infinite sep (large sep.), then find the speed of particle of mass 'm' when their separation is decreased to 'r'?

Soln:



$F_{ext} = 0$

$\boxed{T.E_i = T.E_f}$ By COLM $\Rightarrow$ $mv_1 = 2mv_2 \Rightarrow v_1 = 2v_2$

$$0 = -\frac{Gm(2m)}{r} + \frac{1}{2}m(v_1)^2 + \frac{1}{2}\cdot 2m(v_2)^2$$

$$\Rightarrow \frac{2Gm^2}{r} = \frac{1}{2}m(v_1)^2 + \frac{mv_1^2}{4}$$

$$\frac{8Gm^2}{r} = 3mv_1^2 \Rightarrow \boxed{v_1 = \sqrt{\frac{8Gm}{3R}}}$$

Relative velocity of approach =

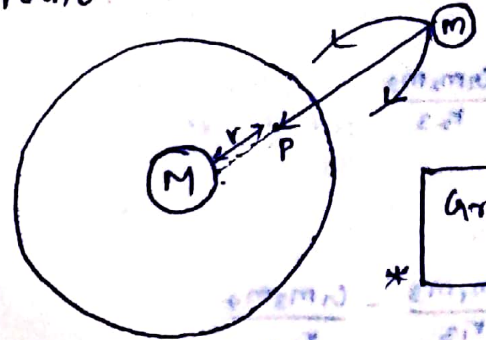
$$v_1 + v_2 = \frac{3v_1}{2}$$

$$= \frac{3}{2}\sqrt{\frac{8GM}{3r}}$$

$$\boxed{v_{rel} = \sqrt{\frac{6GM}{r}}}$$

GRAVITATIONAL POTENTIAL (V) (Scalar quantity)

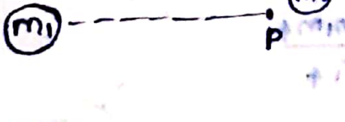
→ Gravitational Potential at a point inside the field is defined as Work (external) required to move unit mass from ∞ to required point.



$$W_{ext} = -\frac{GMm}{r}$$

$$W_{ext} \text{ per unit mass} = \frac{W_{ext}}{m} = \frac{-\frac{GMm}{r}}{m}$$

$$\boxed{\text{Grav. potential at 'P'} = V_p = -\frac{GM}{r}}$$ *



$$V_p = -\frac{Gm_1}{r}$$

$$GPE = -\frac{Gm_1m_2}{r}$$

$$U = -\left(\frac{Gm_1}{r}\right)m_2 = V_p \cdot m_2$$

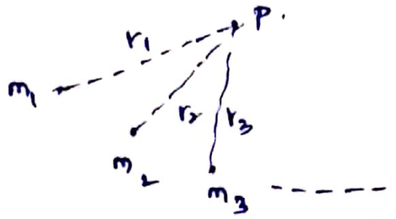
$$\boxed{\begin{aligned} U &= m(V) \\ \Delta U &= m\cdot(\Delta V) \end{aligned}}$$ *

$$E = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$$

$$E = \left(-\frac{dV}{dr}\right)$$

$$\int dV = -\int E\cdot dx$$

* Gravitational potential due to sys. of particles:



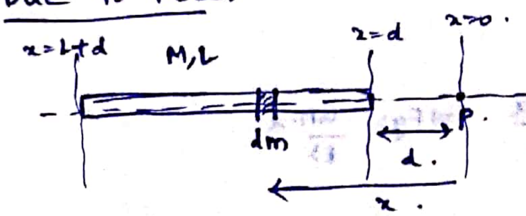
Net potential at 'P' is

$$V_{net} = V_1 + V_2 + V_3 + \cdots$$

$$V_{net} = \frac{-Gm_1}{r_1} - \frac{Gm_2}{r_2} - \frac{Gm_3}{r_3} - \cdots$$

G. potential due to objects:

i) Due to Rod:



$$dV = -\frac{G\,dm}{x}$$

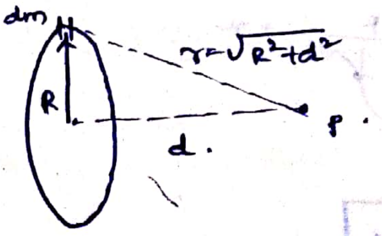
$$dV = -\frac{G\cdot M\cdot dx}{L\cdot x}$$

$$V_{tot} = \int dV = \frac{-GM}{L}\int\frac{dx}{x}$$

$$= \frac{-GM}{L}\left[\ln x\right]_d^{L+d}$$

$$\Rightarrow \boxed{V = \frac{-GM}{L}\ln\left(\frac{L+d}{d}\right)}$$

ii) Due to Ring:

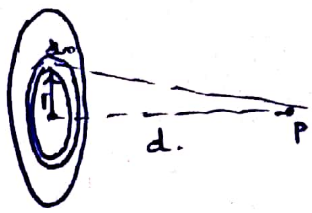


$$dV = -\frac{G\,dm}{\sqrt{R^2+d^2}}$$

$$V = \frac{-G}{\sqrt{R^2+d^2}}\int dm = \boxed{\frac{-GM}{\sqrt{R^2+d^2}}}$$

$$\boxed{V_{centre} = \frac{-GM}{\sqrt{R^2}}} = \frac{-GM}{R}$$

iii) Due to Disc:-



$$dm = \frac{M(2\pi r\cdot dr)}{\pi R^2} = \frac{2M\cdot dr\cdot r}{R^2}$$

$$dV = \frac{-G\,dm}{\sqrt{r^2+d^2}} = \frac{-2GM\cdot dr\cdot r}{R^2\sqrt{r^2+d^2}}$$

$$V = \int dV = \frac{-2GM}{R^2}\int\frac{r\cdot dr}{\sqrt{r^2+d^2}}$$

$r^2 + d^2 = t$

$2r\cdot dr = dt$

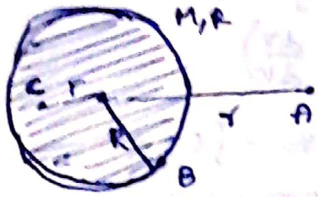
$$= \frac{-2GM}{R^2}\int\frac{dt}{2\sqrt{r^2+d^2}} = \cancel{\frac{-2GM}{R^2}\int\frac{dt}{(t-d^2)(t)}} \quad \frac{-GM}{R^2}\int\frac{dt}{\sqrt{t}}$$

$$= \frac{-GM}{R^2}\left[2\sqrt{t}\right]_0^R \Rightarrow \frac{-GM}{R^2}\left[2\sqrt{r^2+d^2}\right]_0^R$$

$$\Rightarrow \boxed{V = \frac{-2GM}{R^2}\left[\sqrt{R^2+d^2} - d\right]}$$

$$\boxed{V_{centre} = \frac{-2GM}{R}} \quad (d=0)$$

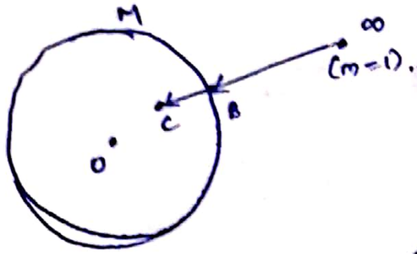
G.P due to solid sphere:



$V_A = -\frac{GM}{r}$ $(r>R)$

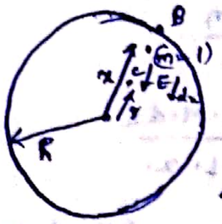
$V_B = -\frac{GM}{R}$ $(r=R)$

$V_C = ?$ $(r<R)$



$W_{ext\,\infty\to C} = W_{ext\,\infty\to B} + W_{ext\,B\to C}$

$V_C = -\frac{GM}{R} + W_{ext\,B\to C}$



$F_g = m\cdot E$

$F_g = \cancel{m}\cdot\frac{GM\cdot x}{R^3}$ ① $\Rightarrow F_g = \frac{GM\cdot x}{R^3}$

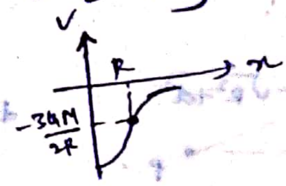
$dW_g = \frac{GM}{R^3}\cdot x\cdot dx$. Reason: opp direction to x

$W_{ext} = \int dW_a = \ominus\frac{GM}{R^3}\int x\cdot dx$ $\boxed{W_{ext} = -W_{cons}}$

$= -\frac{GM}{R^3}\left[\frac{x^2}{2}\right]_R^r = +\frac{GM}{R^3}\left[\frac{R^2-r^2}{2}\right]$

$\Rightarrow V_C = -\frac{GM}{R} - \left(\frac{GM}{R^3}\left[\frac{R^2}{2}\right] - \frac{GM}{R^3}\left[\frac{r^2}{2}\right]\right)$.

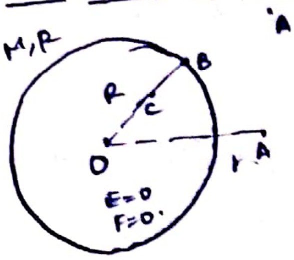
$\Rightarrow -\frac{GM}{R} - \frac{GM}{2R} + \frac{GMr^2}{2R^3} = -\frac{3GM}{2R} + \frac{GMr^2}{2R^3}$.



$\boxed{V_C = -\frac{GM}{2R^3}[3R^2 - r^2]}$ ✱

$\boxed{V_{centre}\ (r=0) = -\frac{3GM}{2R}}$

G.P due to Hollow sphere:



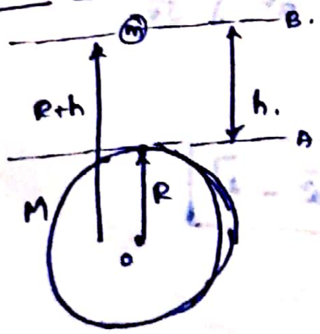
$V_A = -\frac{GM}{r}$ $(r>R)$

$V_B = -\frac{GM}{R}$ $(r=R)$

$V_C = -\frac{GM}{R}$ $(r<R)$

inside the hollow sphere.

GPE DUE TO EARTH:



$U_B > U_A$

GPE of particle at point 'B' is

$U = m\cdot V$

$U_B = m\left(\frac{-GM}{R+h}\right) = -\frac{GMm}{R+h}$

GPE of particle at point A

$U_A = m(V_A) = m\left(\frac{-GM}{R}\right) = -\frac{GMm}{R}$

$$U_B - U_A = \frac{GMm}{R} - \frac{GMm}{R+h} = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$\boxed{U_B - U_A = \frac{GMm(h)}{R(R+h)}}$$

If $h \ll R$.

$$\Rightarrow \Delta U = \frac{GMm(h)}{R(R)} = \frac{GMm\,h}{R^2} = mgh$$

Q) If a particle is projected vertically upwards with velocity V_0, then Find the max. height reached by the particle from the Earth surface.

from the Earth surface

Soln:

$\Delta U = \Delta KE$

$$\frac{GM\cancel{m}(h)}{R(R+h)} = \frac{1}{2}\cancel{m}V_0^2 \Rightarrow \frac{2g\cdot Rh}{R+h} = V_0^2 \qquad \left[\frac{GM}{R^2} = g\right]$$

$$2gRh = V_0^2 R + V_0^2 h.$$

$$\Rightarrow \boxed{h = \frac{V_0^2 R}{2gR - V_0^2}}$$

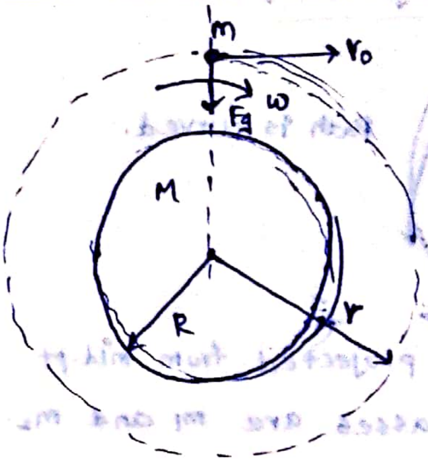
If V_0 is small

$$2gR - V_0^2 \approx 2gR \Rightarrow \boxed{h = \frac{V_0^2 \cancel{R}}{2g\cancel{R}} = \frac{V_0^2}{2g}}$$

ORBITAL SPEED:

~~Motoin Motoin~~

Motion of Satellites in Circular orbits:



$$F_g = m(E) = m\left(\frac{GM}{r^2}\right) = \frac{GMm}{r^2}$$

↳ acts a centripetal Force.

$$\frac{GM\cancel{m}}{r^2} = \frac{\cancel{m}V_0^2}{(R+h)}$$

$$\Rightarrow \boxed{V_0 = \sqrt{\frac{GM}{r}}}$$

$$V_0 = \sqrt{\frac{GM}{R+h}}$$

If $h \ll R$.

$$\boxed{V_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR}}$$

For earth
$V_0 = 7.9$ km/s

Period of Revolution: (365)

$$T = \frac{2\pi r}{V_0} = \frac{2\pi r}{\sqrt{\frac{GM}{r}}} = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\boxed{T = 2\pi\sqrt{\frac{r^3}{gR^2}}}$$

Angular speed of Satellite:

$$\omega = \frac{V}{(R+h)} = \frac{V}{r} = \frac{1}{r}\sqrt{\frac{GM}{r}} \Rightarrow \boxed{\sqrt{\frac{GM}{r^3}} = \omega}$$

Kinetic energy of Satellite:

$$K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$$

$$\boxed{K = \frac{GMm}{2r}}$$

Potential Energy of Satellite: $U = m(V) = m\left(\frac{-GM}{r}\right) = \frac{-GMm}{r}$

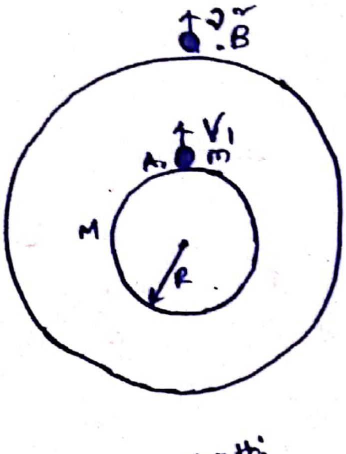
Total Energy of satellite: $K+U = \boxed{\frac{-GMm}{2r}}$

$$\left\{\begin{array}{l} TE = -KE \\ 2KE = |PE| \end{array}\right\}$$

Escape Speed:

→ Min. speed req. for a particle to escape from gravitational Field.

→ To escape from the gravitational pull, the total mech. energy = 0



By COME, [Cons. of mech energy]

$$TE_A = TE_B$$

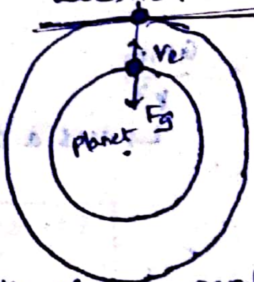
$$\left(\frac{-GM}{R}\right)m + \frac{1}{2}mv_1^2 = 0 + \frac{1}{2}mv_2^2$$

$$\frac{v_{min}^2}{2} - \frac{GM}{R} = \cancel{\frac{v_2^2 min}{2}}\ 0$$

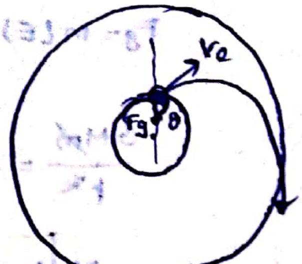
Escape speed ← $\boxed{v_{min} = \sqrt{\frac{2GM}{R}}}$

Earth: $v_e = \sqrt{2g \cdot R} = \sqrt{2 \times 9.8 \times 6.4 \times 10^5} = 11.2\,km/sec$

→ Escape speed doesnot depend on angle of projection / direction of projection.



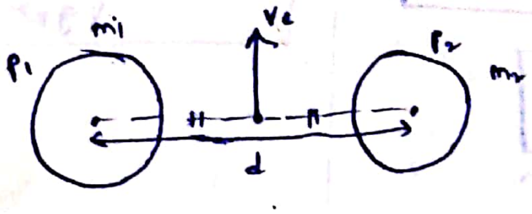
Path of the particle is st line



Path is curved.

Q) Find the escape speed of a particle if it is projected from mid.pt of line joining centres of 2 planets whose masses are m_1 and m_2. (Dist b/w 2 centres of planets is 'd')

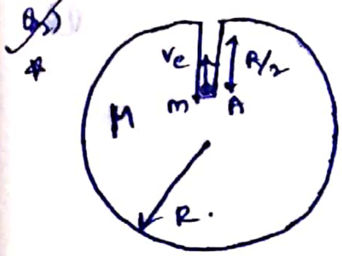
Soln:



$$\frac{-Gm_1\cancel{m}}{(d/2)} - \frac{Gm_2\cancel{m}}{(d/2)} + \frac{1}{2}\cancel{m}v_{min}^2 = 0 + 0$$

$$\sqrt{\frac{4G(m_1+m_2)}{d}} = v_{min}$$

35)



Find the escape velocity of the particle if it is projected from point A?

Soln: $U = m(V)$.

$$= m\left(-\frac{GM}{2R^3}\left(3R^2 - \left(\frac{R}{2}\right)^2\right)\right)$$

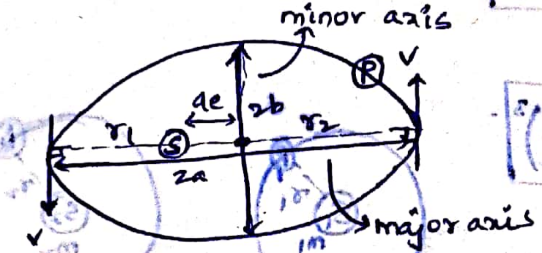
$$= m\left(-\frac{GM}{2R^3}\left(\frac{11R^2}{4}\right)\right) = \frac{-11GMm}{8R}$$

$$K.E = \frac{1}{2}mv_e^2 = \frac{11GMm}{8R}$$

$$\Rightarrow \boxed{v_e = \sqrt{\frac{11GM}{4R}}} \boxed{= \sqrt{\frac{11gR}{4}}}$$

Kepler's Law:-

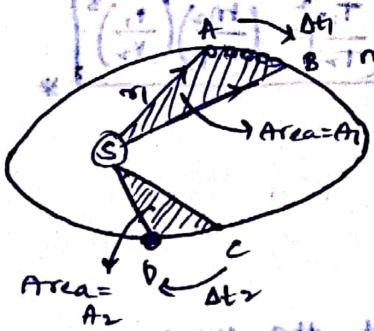
1) All planents revolve around the sun in elliptical orbits with sun at one of it's Focii.



$r_1 + ae = a$ | $r_2 = a + ae$

$r_1 = a(1-e)$ | $r_2 = a(1+e)$

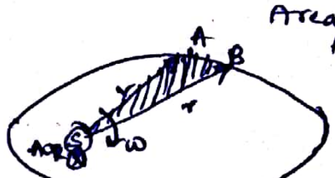
2) Area swept by radian vector in equal time intervals would bo equal.



m - mass of Planet

If $\Delta t_1 = \Delta t_2 \Rightarrow A_1 = A_2$

$\frac{dA}{dt}$ = constant



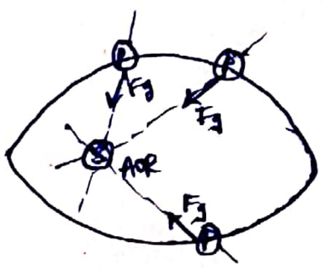
$$\text{area}(A) = \frac{1}{2}r^2\theta$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{mr^2\omega}{2m} = \frac{I.\omega}{2m}$$

$$\boxed{\frac{dA}{dt} = \frac{L}{2m}} \Rightarrow \text{angular momentum is constant through out the motion.}$$

↳ areal velocity



→ ~~As the potential~~ gravitational Force on the planet directed towards the Sun, Torque produced by gravitational force is zero.

⇓

Angular momentum of planet is conserved about the sun.

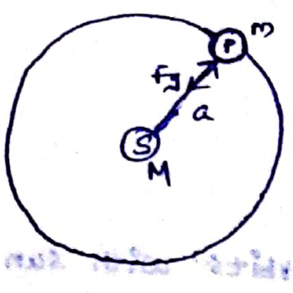
$\vec{L}$ = constant.

$$\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{constant}}$$

3) Square of time period of revolution of planet around the sun is proportional to cube of semi-major axis.

$$T^2 \propto a^3$$

For circular orbits:

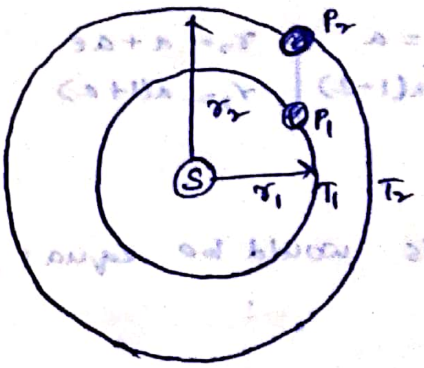


$$F_g = \frac{GM\cancel{m}}{a^2} = \cancel{m}a\omega^2$$

$$\Rightarrow \omega^2 = \frac{GM}{a^3}.$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{GM}{a^3} \Rightarrow T^2 = \frac{4\pi^2}{GM_s}\cdot a^3$$

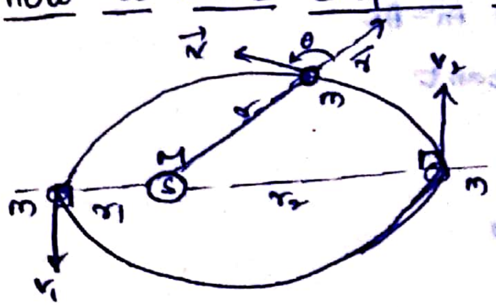
$$* \boxed{T^2 \propto a^3}$$



$$\boxed{\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3}.$$

$$* \boxed{\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{m_2}{m_1}\right)\left(\frac{r_1}{r_2}\right)^3} *$$

How to solve elliptical orbits:



$$\vec{L} = m(\vec{r}\times\vec{v})$$
$$= mvr\sin\theta$$

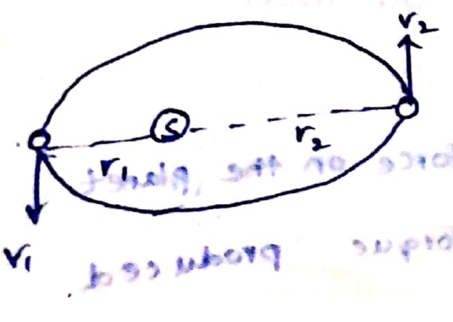
By C.O.A.M about the sun:

$$mv_1r_1 = mv_2r_2 = mvr\sin\theta$$

$$v_1r_1 = v_2r_2 = vr\sin\theta \longrightarrow ①$$

By C.O.M.E:

$$-\frac{GMm}{r_1} + \frac{1}{2}mv_1^2 = -\frac{GMm}{r_2} + \frac{1}{2}mv_2^2 = -\frac{GMm}{r} + \frac{1}{2}mv^2 \longrightarrow ②$$



$$v_1r_1 = v_2r_2$$

$$-\frac{GM\cancel{m}}{r_1} + \frac{1}{2}\cancel{m}v_1^2 = -\frac{GM\cancel{m}}{r_2} + \frac{1}{2}\cancel{m}v_2^2$$

$$\frac{v_1^2 - v_2^2}{2} = GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\frac{v_1^2 - v_2^2}{2} = 2GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad v_2 = \frac{v_1 r_1}{r_2}$$

$$v_1^2 - \frac{v_1^2 r_1^2}{r_2^2} = \frac{2GM(r_2 - r_1)}{r_1 \cdot r_2}$$

$$\frac{v_1^2(r_1+r_2)\cancel{(r_2-r_1)}}{r_2^{\cancel{2}}} = \frac{2GM\cancel{(r_2-r_1)}}{r_1 \cdot \cancel{r_2}} \Rightarrow v_1^2 = \frac{2GMr_2}{r_1(r_1+r_2)}$$

$$\Rightarrow \boxed{v_1 = \sqrt{\frac{2GMr_2}{r_1(r_1+r_2)}}} \qquad \boxed{v_2 = \sqrt{\frac{2GMr_1}{r_2(r_1+r_2)}}} \qquad \boxed{\frac{r_1}{r_2} = \frac{1+e}{1-e}}$$

$$v_1 = \sqrt{\frac{\cancel{2}GM\cancel{a}(1+e)}{\cancel{a}(1-e)[\cancel{2}a]}} \Rightarrow \boxed{\sqrt{\frac{GM(1+e)}{a(1-e)}} = v_1}$$

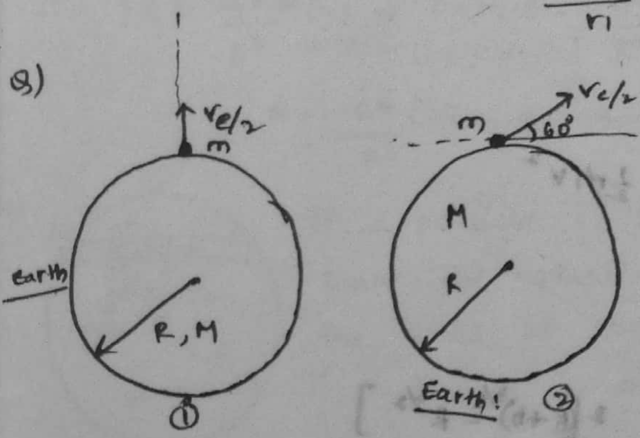
$$v_2 = \sqrt{\frac{2GM\cancel{a}(1-e)}{\cancel{a}(1+e)(2a)}} \Rightarrow \boxed{\sqrt{\frac{GM(1-e)}{a(1+e)}} = v_2}$$

TE of a planet in elliptical orbit :-

$$TE = \frac{-GMm}{r_1} + \frac{1}{2}mv_1^2 = \frac{-GMm}{r_1} + \frac{1}{\cancel{2}}m.\frac{\cancel{2}GMr_2}{(r_1+r_2)r_1}$$

$$= \frac{GMm}{r_1}\left[\frac{r_2}{r_1+r_2} - 1\right] \Rightarrow \frac{-GMm}{r_1+r_2} \Rightarrow \boxed{\frac{-GMm}{2a} = T.E}$$

Q)



Find the max. height reached by Particles from the earth's surface in 2 cases.

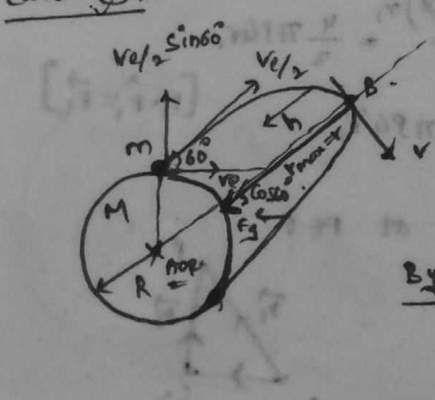
$$\left[v_e = \sqrt{\frac{GM \times 2}{R}}\right].$$

soln: case (i) : $-\frac{GMm}{R} + \frac{1}{2}m(v_e/2)^2 = 0 + \frac{-GMm}{(R+h)}$.

$$\Rightarrow \frac{1}{2}\cancel{m}\left(\frac{\cancel{GM}}{2\cancel{R}}\right) = \cancel{GMm}\left(\frac{h}{\cancel{R}(R+h)}\right)$$

$$\Rightarrow 4h = R + h \Rightarrow \boxed{h = R/3}.$$

Case - ②:



$r_{max} = R + h$.

By C.O.AM:

$L_A = L_B$

$$\cancel{m}\left(\frac{v_e}{2}\cos 60°\right).R = \cancel{m}.v.r \Rightarrow \frac{v_e.R}{4} = v.r \Rightarrow$$

$$v = \frac{v_e R}{4r}$$

By C.O.ME,

T.EA = T.EB.

$$\frac{-GMm}{R} + \frac{1}{2}m\left(\frac{v_e}{2}\right)^2 = \frac{-GMm}{r} + \frac{1}{2}mv^2$$

$$\frac{-GM}{R} + \frac{1}{2}\left(\frac{GM}{2R}\right) = \frac{-GM}{r} + \frac{1}{2}\left(\frac{2GM}{R}\cdot\frac{R^2}{16r^2}\right).$$

$$\frac{1}{4R} - \frac{1}{R} = \frac{R}{16r^2} - \frac{1}{r}.$$

$$\Rightarrow \frac{-3}{4R} = \frac{R-16r}{16r^2} \Rightarrow -48r^2 = 4R^2 - 64Rr.$$

$$\Rightarrow 48r^2 - 64Rr + 4R^2 = 0.$$

$$\Rightarrow 12r^2 - 16Rr + R^2 = 0.$$

$$r = \left(\frac{16 \pm \sqrt{256-48}}{24}\right)R = \left(\frac{16 \pm \sqrt{208}}{24}\right)R.$$

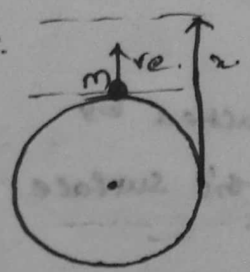
$$r = \left(\frac{16 \pm 4\sqrt{13}}{24}\right)R = \left(\frac{4 \pm \sqrt{13}}{6}\right)R$$

$$\therefore r_{max} = \frac{(4+\sqrt{13})R}{6};\ r.$$

$$h_{max} = r_{max} - R = \boxed{\frac{R}{6}(\sqrt{13}-2)}.$$

Q) If a particle is projected vertically upward from the surface of earth with escape velocity, then find the time taken by the particle to reach height 'h' from the surface?

<u>Soln</u>:



By C.O.M.E

$$\frac{-GMm}{R} + \frac{1}{2}mv_e^2 = -\frac{GMm}{x} + \frac{1}{2}mv^2$$

$$\Rightarrow v = \sqrt{\frac{2GM}{x}}.$$

$$\frac{dx}{dt} = \sqrt{\frac{2GM}{x}} \Rightarrow \int_R^{R+h} dx\cdot\sqrt{x} = \sqrt{2GM}\int_0^t dt.$$

$$\frac{2}{3}\left[(x)^{3/2}\right]_R^{R+h} = \sqrt{2GM}\,(t) \Rightarrow t = \frac{2\left[(R+h)^{3/2} - R^{3/2}\right]}{3\sqrt{2GM}}$$