Sec: Sr. **COMPLEX NUMBERS** **Date:02/05/2020**

I). **Modulus:** If $z_1, z_2, z_2 \in C$ then

1). If $if \ z = x + iy, \ |z| = \sqrt{x^2 + y^2}$

2). $|z| = 0 \Rightarrow z = 0$ & $\text{Re}(z) = \text{Im}(z) = 0.$

3). $|z| = |\bar{z}| = |-z|$.

4). $-|z| \le \text{Re}(z) \le |z|.$ and $-|z| \le \text{Im}(z) \le |z|.$

5). $|z^2| = z\bar{z}$ and $|z_1 z_2 ..... z_n| = |z_1||z_2| ..... |z_n|.$

6). $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}; \ |z_2| \ne 0.$

7). $|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1 + z_2}\right) = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\left(z_1 \bar{z}_2\right).$ **or**

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(where $|z_1| = r_1, \ |z_2| = r_2$ *and* $\arg(z_1) = \theta_1$ *and* $\arg(z_2) = \theta_2$)

7a) If $\theta_1 - \theta_2 = 0 \Rightarrow |z_1 + z_2|^2 = (|z_1| + |z_2|)^2 \Rightarrow |z_1 + z_2| = |z_1| + |z_2|$

i.e. $\arg(z_1) = \arg(z_2) \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = 0 \Rightarrow \frac{z_1}{z_2} =$ purely real and hence $\frac{z_1}{z_2} = \overline{\frac{z_1}{z_2}}$

7b) If $\theta_1 - \theta_2 = \frac{\pi}{2} \Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

i.e. $\arg(z_1) - \arg(z_2) = \frac{\pi}{2} \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \Rightarrow \frac{z_1}{z_2} =$ purely imaginary and hence $\frac{z_1}{z_2} = -\overline{\frac{z_1}{z_2}}$

If $\theta_1 - \theta_2 = \frac{\pi}{2}$, then the complex numbers z_1, z_2, and origin form a right angle triangle, right angled at origin

7c) If $\theta_1 - \theta_2 = \pi \Rightarrow |z_1 + z_2|^2 = (|z_1| - |z_2|)^2 \Rightarrow |z_1 + z_2| = \big||z_1| - |z_2|\big|$

i.e. $\arg(z_1) - \arg(z_2) = \pi \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pi \Rightarrow \frac{z_1}{z_2} =$ purely real and hence $\frac{z_1}{z_2} = \overline{\frac{z_1}{z_2}}$

8). $|z_1 - z_2|^2 = (z_1 - z_2)\left(\overline{z_1 - z_2}\right) = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\left(z_1 \bar{z}_2\right).$ **or**

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(where $|z_1| = r_1,\ |z_2| = r_2$ *and* $\arg(z_1) = \theta_1$ *and* $\arg(z_2) = \theta_2$)

8a) If $\theta_1 - \theta_2 = 0 \Rightarrow |z_1 - z_2|^2 = (|z_1| - |z_2|)^2 \Rightarrow |z_1 - z_2| = ||z_1| - |z_2||$

i.e. $\arg(z_1) = \arg(z_2) \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = 0 \Rightarrow \frac{z_1}{z_2} =$ purely real and hence $\frac{z_1}{z_2} = \frac{\overline{z_1}}{\overline{z_2}}$

8b) If $\theta_1 - \theta_2 = \frac{\pi}{2} \Rightarrow |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

i.e. $\arg(z_1) - \arg(z_2) = \frac{\pi}{2} \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \Rightarrow \frac{z_1}{z_2} =$ purely imaginary and hence $\frac{z_1}{z_2} = -\frac{\overline{z_1}}{\overline{z_2}}$

8c) If $\theta_1 - \theta_2 = \pi \Rightarrow |z_1 - z_2|^2 = (|z_1| + |z_2|)^2 \Rightarrow |z_1 - z_2| = |z_1| + |z_2|$

i.e. $\arg(z_1) - \arg(z_2) = \pi \Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pi \Rightarrow \frac{z_1}{z_2} =$ purely real and hence $\frac{z_1}{z_2} = \frac{\overline{z_1}}{\overline{z_2}}$

9) If $\theta_1 - \theta_2 = \frac{\pi}{2} \Rightarrow |z_1 + z_2| = |z_1 - z_2|$ and $|z_1| = |z_2|$ & $z_1 + z_2 = 0$ *then* $\arg(z_1) + \pi = \arg(z_2)$

10a). $|z_1 + z_2|^2 + |z_1 - z|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$.

10b). $|az_1 - bz_2|^2 + |bz_1 - az_2|^2 = \left(a^2 + b^2\right)\left(|z_1|^2 + |z_2|^2\right)(a, b, \in R)$

11a). $|z_1 + z_2| \le |z_1| + |z_2|$

11b). $|z_1 + z_2| \ge ||z_1| - |z_2||$

11c). $|z_1 - z_2| \le |z_1| + |z_2|$

11d). $|z_1 - z_2| \ge ||z_1| - |z_2||$

II). Conjugate of a complex number:

$a)\ \bar{\bar{z}} = z;$ $\quad b)\ z + \bar{z} = 2\,\mathrm{Re}(z)$ $\quad c)\ z - \bar{z} = 2i.\mathrm{Im}(z)$

$d)\ z = \bar{z} \Rightarrow z$ *is a purely real* $\quad e)\ z + \bar{z} = 0 \Rightarrow z$ *is purely imaginary*

$f)\ z.\bar{z} = (re(z))^2 + (im(z))^2 = |z|^2$ $\quad g)\ \left(\overline{z_1 \pm z_2}\right) = \overline{z_1} \pm \overline{z_2}$

$h)\ \left(\overline{z_1.z_2}\right) = \overline{z_1}.\overline{z_2}$ $\quad i)\ \left(\overline{z_1 / z_2}\right) = \overline{z_1} / \overline{z_2}$

j) $\bar{z}$ is the image of z with respect to the real axis.

k). Sum and product of two complex numbers are real if and only if they are conjugate to each other.

III). Argument of a complex number:

1) $\arg(z_1.z_2) = \arg(z_1) + \arg(z_2)$ and $\arg(z_1.z_2.z_3.....z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + ... + \arg(z_n)$

and $\arg(z^n) = n\arg(z),\ n \in I$

2) $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2)$

3) $\arg(\bar{z}) = -\arg(z) = \arg\left(\frac{1}{z}\right)$

4) $\arg(z-\bar{z}) = \pm\frac{\pi}{2}$ and $\arg(z+\bar{z}) = 0$ *or* π

5) $\arg(-z) = \arg(z) \pm \pi$ and $\arg(z) + \arg(-\bar{z}) = \pi$

6) If $\arg(z) < 0 \Rightarrow \arg(-z) - \arg(z) = \pi$ and If $\arg(z) > 0 \Rightarrow \arg(-z) - \arg(z) = -\pi$

7) $\arg(z_1.z_2) = \arg(z_1) + \arg(z_2)$ is true for general value of arguments of $z_{1,}\ z_{2,}$, but is not true for principal argument of $z_{1,}\ z_{2,}$ **example**: $\left[z_1 = -\sqrt{3}+i,\ z_2 = -1+i\sqrt{3}\right]$

8). Angle BAC = $\arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = \arg\left(\frac{\overline{OC}-\overline{OA}}{\overline{OB}-\overline{OA}}\right)$

9). Locus of z, such that $\arg\left(\frac{z-1}{z+1}\right) = \theta.$ and A (1, 0) and B (-1, 0)

If $\theta = 0$, P lies either above or below A figure.........

If $\theta = \pi$, P lies on the line segment AB figure..........

If $\theta = \alpha$, then the locus of z is part of a circle.figure......

IV). Equilateral triangle/Area of a triangle:

1). Area of a triangle whose vertices are $z_1, z_{2,} z_3$ is $\frac{i}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$.

2). If $|z_1| = |z_2| = |z_3| = 1$ then its area is $\left|\frac{(z_1-z_2)(z_2-z_3)(z_3-z_1)}{4z_1z_2z_3}\right|$

$\left[z_1\bar{z}_1 = 1 \Rightarrow \bar{z}_1 = \frac{1}{z_1}\ \&\ z_2\bar{z}_2 = 1 \Rightarrow \bar{z}_2 = \frac{1}{z_2}\ \&\ z_3\bar{z}_3 = 1 \Rightarrow \bar{z}_3 = \frac{1}{z_3}\right]$

3). Vertices of a triangle are $\frac{z-z_2}{\bar{z}_1-\bar{z}_2} - \frac{a}{\bar{a}} = 0$ then its area is $\frac{1}{2}|z|^2$. also, the triangle is right angled triangle.

$\left(let\ z_1 = z\ \&\ z_2 = iz\ \&\ z_2 = z_1 e^{\frac{i\pi}{2}} \Rightarrow |z_1| = |z_2|\ \ \&\ \ |z_3| = \sqrt{2}|z_1|\right)$.

4). Area of the triangle, formed $z,\ iz,\ z-iz$ *is* $\frac{3}{2}|z|^2$.

5). Area of the triangle, formed by the points $z,\ wz,\ wz+z$ is....

6) $z_{1,}z_{2,}z_3$ are the vertices of an equilateral triangle, then

a). $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_{1.}$

b). if z_0 is the centroid of triangle then $z_1 + z_2 + z_3 = 3z_0$, & $z_1^2 + z_2^2 + z_3^2 = 3z_o^2$

c). $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

d) $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$

e). $z_1, z_2,$ and origin, form an equilateral triangle then $z_1^2 + z_2^2 - z_1 z_2 = 0$

7). z_1, z_2, z_3, z_4 are the complex numbers representing the vertices of a square, having centre at" z_o " then.

a) $\sum_{i=1}^{4} z_i = 4z_0$

b) $\sum_{i=1}^{4} (z_i)^2 = 4(z_0)^2$

c) $(z_1 - z_o)^2 + (z_2 - z_0)^2 + (z_3 - z_o)^2 + (z_4 - z_o)^2 = 0.$

8). $z_{1,} z_2 \ldots\ldots z_n$ be the complex numbers representing the vertices of n – sided regular polygon, and " z_o " is its centre. Then.

a) $\sum_{i=1}^{n} z_i = n z_o$

b) $\sum_{i=1}^{n} z_1^2 = n z_0^2$

9). If z_1, z_2, z_3 are the vertices of a triangle and satisfies $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ then the triangle is equilateral triangle. (taking modulus and argument on both sides)

V). Line

1) Equation of a line, passing through two points z_1 and $z_{2.}$ is,

$$z(\overline{z_2} - \overline{z_1}) - \overline{z}(z_2 - z_1) + z_1 \overline{z_2} - \overline{z_1} z_2 = 0 \ (or) \ \arg\left(\frac{z - z_1}{z_1 - z_2}\right) = 0 \ or \ \pi. (or) \begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$$

2). $\overline{a}z + a\overline{z} + b = 0$ where $b \in R$ and a = $-i(z_1 - z_2)$.

3). If z_1 and $z_{2.}$ are two points on a line, its complex stope is defined as $\frac{z_1 - z_2}{\overline{z_1} - \overline{z_2}}$.

4). Complex slope of the line $a\overline{z} + \overline{a}z + b = 0$ is $\frac{-a}{\overline{a}}$

5). If M, is the complex slope of a line joining the points z_1 and $z_{2.}$

m, is real slope of the line $m = \tan\theta.(\theta =$ inclination of the line, with +ve direction of x – axis (real axis).

Then $M = \frac{z_1 - z_2}{\overline{z_1} - \overline{z_2}} = \frac{1+im}{1-im}$ & $|M| = 1$ and M = m is never possible

$M = \cos 2\theta + i \sin 2\theta \quad \left(i.e.\ M = e^{2i\theta}\right)$

6). If w_1 *and* w_2 are the complex slopes of the two parallel lines in the argand plane , then $w_1 = w_2$. and if the lines are perpendicular, then $w_1 + w_2 = 0$.

7). a) Equatioin of a line parallel to the line $a\bar{z} + \bar{a}z + b = 0$ *is* $a\bar{z} + \bar{a}z + \lambda = 0 \qquad (\lambda \in R)$

b). If the line $a\bar{z} + \bar{a}z + b = 0$ is parallel to the real axis then Re(a) = 0

c). If the lines $a_1\bar{z} + \overline{a_1}z + b_1 = 0$ *and* $a_2\bar{z} + \overline{a_2}z + b_2 = 0$ are parallel, then $\frac{a_1}{a_2}$ is real.

$$\left[because\ \frac{-a_1}{\overline{a}_1} = \frac{-a_2}{\overline{a}_2} \Rightarrow \frac{a_1}{a_2} = \frac{\overline{a}_1}{\overline{a}_2} \right]$$

8). Equation of a line, perpendicular to the line $a\bar{z} + \bar{a}z + b = o$ *is* $a\bar{z} - \bar{a}z + i\lambda = 0.$

9) The length of the perpendicular from a point $P(z_o)$ to the line

$a\bar{z} + \bar{a}z + b = 0\, is \left| \frac{a\overline{z_o} + \bar{a}z_o + b}{2|a|} \right|$

10). The algebraic sum of the perpendicular distances calculated from the points,represented by the complex numbers, $1, \alpha, \alpha^2, \alpha^3, ..., \alpha^{n-1}$ (n^{th} roots of unity) to the line $a\bar{z} + \overline{az} + b = o$ is $\frac{nb}{2|a|}$

11). Equation a line joining the points represented by the complex numbers a and ib $(a, b \in R,\ \&\ a, b \neq 0).is,\ z\left(\frac{1}{2a} - \frac{i}{2b}\right) + \bar{z}\left(\frac{1}{2a} + \frac{i}{2b}\right) = 1.$ (i.e. intercepts made by the line on the axes)

Consider the line $a\bar{z} + \bar{a}z + b = 0$

a). Intercept of the line on imaginary axis is, $\frac{-ib}{a - \bar{a}}$

b). Intercept of the line on imaginary axis is, $\frac{-b}{a + \bar{a}}$

$$\left[\begin{array}{l} a = \alpha + i\beta, b = \alpha - i\beta \text{ and put } z = x + iy \text{ in } a\bar{z} + \bar{a}z + b = 0 \\ \text{and it becomes } x\alpha + y\beta = b,\ x = \frac{-b}{\alpha},\ y = \frac{-b}{\beta} \end{array} \right]$$

12). Intersection of two lines $\overline{a_1}z + a_1\bar{z} + b_1 = 0$ & $\overline{a_2}z + a_2\bar{z} + b_2 = 0.$ is obtained, solving for z. (i.e. eliminate z or $\bar{z}$)

13). a) z_2 is the reflection of z_1 with respect to the line $a\bar{z}+\bar{a}z+b=0$,then

$\overline{z_2}a+\bar{a}z_1+b=0.$

$\left[\dfrac{z_1-z_2}{\bar{z}_1-\bar{z}_2}+\dfrac{-a}{\bar{a}}=0 \text{ \& } \dfrac{z_1+z_2}{2} \textit{ lies on the line}\right]$

b). $\arg(z_1-z_0)+\arg(z_2-z_0)=0$, where z_0 is foot of the perpendicular, drawn from z_1 to the line.

c). z_2 is the reflection of z_1 with respect to the line $a\bar{z}+\bar{a}z+b=0$,then $\operatorname{Re}(a(z_1+z_2))=-b$

$\left[\begin{array}{l}\overline{z_2}a+\bar{a}z_1+b=0.\rightarrow 1 \quad z_2\bar{a}+a\bar{z}_1+b=0.\rightarrow 2 \quad \textit{and adding both we get} \\ a(\bar{z}_1+\bar{z}_2)+\bar{a}(z_1+z_2)=-2b \quad \Rightarrow \quad 2\operatorname{Re}(a(z_1+z_2))=-2b\end{array}\right]$

d) $\dfrac{z-z_2}{\bar{z}_1-\bar{z}_2}-\dfrac{a}{\bar{a}}=0$

Some examples:

1). reflection of $\dfrac{4+zi}{1+zi}$ in the line $iz=\bar{z}$ $is=(1-2i)$

2). reflection of $\left(\dfrac{2-i}{3+i}\right)$ in the line of $z(1+i)=\bar{z}(i-1)$ $is=\left(\dfrac{i-1}{2}\right)$.

3). reflection of point $z_{1,}$ in the line $\theta=\alpha$ through origin, is z_2 and reflection of z_2 in the line

$\theta=\beta$, through origin is z_3. Then

a) $z_2=\overline{z_1}.\ e^{2i\alpha}$

b) $z_3=z_1.e^{2i(\beta-\alpha)}$

c) if $\alpha=\beta$ then $z_3=z_{1.}$ { $\dfrac{z-0}{\bar{z}-0}=M\Rightarrow z=M\bar{z}$ and $\dfrac{z_1+z_2}{2}$ lies in the line &

$\dfrac{z_1-z_2}{\bar{z}_1-\bar{z}_2}+M=0\}\Rightarrow z_2=\overline{z_1}e^{2i\alpha}\}$

VI. CIRCLE:

1). Equation of a circle whose centre is at origin and radius is r, is $|z|=r$

2). Equation of a circle whose centre at a point (z_o) and radius "r" is $|z-z_o|=r$

$z\bar{z}+a\bar{z}+\bar{a}z+b=o(b\in R)$.

[$a=-z_0$ and $b=|z_0|^2-r^2\Rightarrow r^2=|z_0|^2-b$ and if $|z_0|^2-b\geq 0$, represents a real circle]

3) equation of circle described on a line segment joining

$A(z_1)\&B(z_2)$ as diameter is $(z-z_1)(\overline{z}-\overline{z_2})+(z-z_2)(\overline{z}-\overline{z_1})=0.$

$\left[because \ \dfrac{z-z_1}{\overline{z}-\overline{z_1}}+\dfrac{z-z_2}{\overline{z}-\overline{z_2}}=0.\right]$

4). z_1 and z_2, the two given points and the locus of z, if

$|z-z_1|=k|z-z_2| \quad (k\in R \ \ and \ \neq 1).$

is a circle $\quad (i.e. \ \ PA=k.PB) \quad .$

5) If $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$ $(i.e. \ \ PA^2+PB^2=AB^2)$.

locus of z is a circle.with centre $\dfrac{z_1+z_2}{2}$, radius $\dfrac{|z_1-z_2|}{2}$

6). $|z-z_1|=|z-z_2|=|z-z_3|$. $(ie \ PA=PB=PC)$.Then locus of z is a circle with centre at "z" and radius= $|z-z_1|$.

7). If $|z_1|=|z_2|=|z_3|=r$, then the complex numbers z_1, z_2, z_3, lie on a circle with centre at origin and radius 'r'

8). If $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\theta$ (where θ fixed), then locus of z is a part of the circle $(\underline{|APB}.=\theta)$

If θ is acute, then locus of z, is a major part of the circle, and

if θ is obtuse then locus of z is a minor part of a circle, and if $\theta=\pm\dfrac{\pi}{2}$ is a full circle.

9) If t, c $\in$C such that $|t|\neq|c|, |t|=1 \ and \ z=\dfrac{at+b}{t-c}$, $Z=x+iy$ Then locus of z, is a circle (a,b are given real numbers)

10) Equation of the tangent, drawn to the circle $|z|=r$ at , z_1 is $z\overline{z_1}+\overline{z}z_1=2r^2$.

11) Locus of z, through which perpendicular tangents are drawn to the circle $|z|=r$ is.

$|z|=\sqrt{2}\,r$

Some examples:

1) Centre and radius of $z\overline{z}+(1-i)\overline{z}+(1+i)z-7=0$ is $(-1+i)$ and 3

2) Two different non- parallel lines AB and CD meet the circle $|z|=r$ at A(a) & B(b) ,

C(c) & D(d).

respectively. Then these lines meet at the point P(z), then P(z) = $\dfrac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$

2a) A tangent PC and a secant PAB are drawn to a circle $|z|=r$ where A(a) & B(b) , C(c) then

P(z) = $\dfrac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-(c^{-1})^2}$

2b) Two tangents PA and PB are drawn to a circle $|z| = r$ where A(a) and B(b) ten P(z) =

$$\frac{2a^{-1}-2b^{-1}}{\left(a^{-1}\right)^2-\left(b^{-1}\right)} = \frac{2ab}{a+b}$$

3) If the vertices of a triangle ABC, are o, z_1 *and* z_2, then orthocentre of triangle ABC, satisfies $z\overline{z_2} + \bar{z}z_2 - z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

4) If points $A(z_1), B(z_2)$ are two non – zero complex numbers such that

$|z_1 + z_2| = |z_1 - z_2|$ and O - origin then.

a) $z_1\overline{z_2} + z_2\overline{z_1} = 0$

b) triangle OAB is right angled at origin and hence its orthocentre is at origin

c) circumcentre of triangle OAB is $\frac{z_1+z_2}{2}$

VII). Standard Locus:

1).If $|z - z_1| = |z - z_2|$ Then locus of "z" is the perpendicular bisector of the line joining z_1 and z_2. $(i.e. PA = PB)$.the equation of the perpendicular bisector is

$$z\left(\bar{z}_1 - \bar{z}_2\right) + \bar{z}\left(z_1 - z_2\right) - z_1\bar{z}_1 + z_2\bar{z}_2 = 0$$

2). If $|z - z_1| + |z - z_2| = |z_1 - z_2|$ The locus of z is a line segment, joining the two given points z_1 and z_2 $(i.e. PA + PB = AB)$

3). If $|z - z_1| - |z - z_2| = |z_1 - z_2|$. Then locus of z is a ray, where z_1 and z_2 .the two given points $(i.e.\ PA - PB = AB)$.

4). If $|PA - PB| = PB,$ i.e. $\big||z - z_1| - |z - z_2|\big| = |z_1 - z_2|$ then locus of z is a pair of ray.

5). If $|z - z_1| + |z - z_2| = k.$ *where* $k > |z_1 - z_2|$ Then the locus of z is an ellipse, with foci at z_1 and z_2.

$$\left[\begin{array}{l} PA + PB = k,\ \ k > AB.\ \ and\ \ SP + S'P = K,\ \ K > SS' \\ because\ in\ an\ ellipse\ SS' = 2ae\ \&\ SP + S'P = 2a\ \&\ 2a > 2ae \end{array}\right]$$

6). If $\big||z - z_1| - |z - z_2|\big| = k$ $k < |z_1 - z_2|$ Then the locus of z is a hyperbola, with foci at z_1 and z_2.

$$\left[\begin{array}{l} |PA - PB| = k,\ \ k < AB.\ \ and\ \ |SP - S'P| = k,\ \ k < SS' \\ because\ in\ a\ hyperbola\ SS' = 2ae\ \&|SP - S'P| = 2a\ \&\ 2a < 2ae \end{array}\right]$$

7). If $|z - z_1| - |z - z_2| = k$ where $k < |z_1 - z_2|$ Then the locus of z is a branch of a hyperbola,

8). If arg $\arg(z) = \frac{\pi}{4}$, then locus of z is, a ray

9. Some examples: (Locus)

1). Locus of z, if $\left|z-|z+1|\right|=\left|z+|z-1|\right|$ is a line segment passing through the points (-1,0) and (1,0)

2). Locus of z if $|iz-1|+|z-i|=2$ is a line segment AB where $A(0,1), B(0-1).(\because PA+PB=AB)$.

3). Locus of z if $\operatorname{Re}(z^2)=0$ *is* $(y=\pm x)$ a pair of lines.

4). Locus of z if $w=\dfrac{1-iz}{3-i}$ & $|w|=1$ is perpendicular bisector of a line joining the points $(0,1)\&(0,-1)$.

5). Locus of z if $|z^2-1|=|z|^2+1$ is imaginary axis $(x=0)$.

6). Locus of $w=\dfrac{z}{1-z^2}$ & *if* $|z|=1$ & $z\neq\pm1$ is, imaginary axis $(x=0(or)\operatorname{Re}(z)=0)$.

7). Locus of z if $\left|\dfrac{z-5i}{z+5i}\right|=1$ is a real axis.

8). i)Locus of z if $|z-2+2i|=1$ is a circle with centre $(2,-2)$ radius 1.

9). Locus of z if $3=2+t+i\sqrt{3-t^2}$ $(t\in R)$ is a circle with radius $\sqrt{3}$.

10). Locus of $w=-1+4z$ *if* $|z|=3$ is a circle of radius 12.

11). Locus of $w=\dfrac{1}{4+i-z}$ if $|z+1|=3$ is a circle with centre $\left(\dfrac{5}{17},\dfrac{-1}{17}\right)$ and radius $\dfrac{3}{17}$

12). Arg $\left(\dfrac{z-z_1}{z-z_2}\right)=\dfrac{\pi}{4}$ $\& z_1=10+6i,\ z_2=4+2i$ Then locus of z is a part of circle with centre $9+i$ and radius $\sqrt{26}$.

13). If z satisfies $|z|^2+2(z+\bar{z})+3i(z-\bar{z})+4=0$ then the complex numberz+3+2i lies on a circle with centre 1+5i and radius 3.

14). Locus of z if $|i-1-2z|>9\Rightarrow\left|z-\dfrac{1}{2}(-1+i)\right|>\dfrac{9}{2}$ exterior of a circle with centre $\left(\dfrac{1}{2},\dfrac{-1}{2}\right)$ & radius $\dfrac{9}{2}$.

15). Locus of z if $|z-1|^2+|z+1|^2=4$ is a circle. ($PA^2+PB^2=AB^2$)

16). Locus of z if $\sin(|z|)>0$ is circular region (or) annular regions, i.e. circular region between the circles with center at origin and radii $2n\pi\,\&\,(2n+1)\pi$

17). If $|z-2-i| = |z|\left|\sin\left(\frac{\pi}{4}-\arg(z)\right)\right|$, then locus of z is, a parabola with focus (2, 1) and directrix x – y = 0

$$\left(because\ \sqrt{(x-2)^2+(y-1)^2} = \frac{1}{2}|x-y|\right)$$

18). If $|z-(3+2i)| = \left|z.(\cos\left(\frac{\pi}{4}-\arg(z)\right)\right|$, then locus of z is, a parabola with focus (3,2) and directrix x + y = 0

xviii). Locus of z if $z+\bar{z} = 2|z-1|$ is a parabola ($y^2 = 2x - 1$)

$3 = t+it^2 \quad (t \in R)$ is a parabola.

19). Locus of z if $|z-1|+|z+1| \le 3$ ellipse.(i.e. the points z lie on and inside the ellipse)

20). Locus of z if $|z-1|+|z+1| = 4$ is ellipse with foci $(\pm 1, 0)$ with area $2\sqrt{2}\pi$

VIII). Cube roots of unity:

1. Cube roots of one are 1,w,w^2

$$\left[z^3-1 = (z-1)(z-w)(z-w^2) = 0 \Rightarrow z = 1, w, w^2 \ \ and\ \ w = \frac{-1+i\sqrt{3}}{2},\ w^2 = \frac{-1-i\sqrt{3}}{2}\right]$$

2) Cube roots of one, z_1, z_2, z_3 lie on the unit circle $|z| = 1$, and divide the circumference of the circle in to 3 equal parts and $|z_1| = |z_2| = |z_3| = 1$

$$z_1 = e^{i0}, z_2 = e^{\frac{2\pi i}{3}}, z_2 = e^{\frac{2\pi i}{3}}\ and\ \arg(z_1) = 0,\ \arg(z_2) = \frac{2\pi}{3},\ \arg(z_3) = \frac{-2\pi}{3}$$

$$\left[\arg(1) = 0,\ \arg(w) = \frac{2\pi}{3},\ \arg(w^2) = \frac{-2\pi}{3}\right]$$

3) Product of the cube roots of one is 1 and sum of the roots of one is zero. ($1+w+w^2$) = 0

4) If z_1, z_2, z_3 complex numbers represents the vertices of an equilateral triangle with side length $\sqrt{3}$, then its area is $\frac{3\sqrt{3}}{4}$

5) $$\begin{aligned} 1+(w)^n+(w^2)^{2n} &= 0,\ if\ \ n \ne 3k,\ k \in I \\ &= 3\ \ if\ \ n = 3k,\ k \in I \end{aligned}$$

6) cube roots of 8 are 2, 2w, $2w^2$ and cube roots of k^3 are k, kw, kw^2

7) cube roots of **-1** are **-1, -w, -w^2**

$$\left[z^3+1 = (z+1)(z+w)(z+w^2) = 0 \Rightarrow z = -1, -w, -w^2\right]$$

$$\left[\arg(-1) = \pi,\ \arg(-w) = \frac{-\pi}{3},\ \arg(-w^2) = \frac{\pi}{3}\right]$$

8) $\overline{w} = w^2;\ \ \overline{w}^2 = w;\ \ \frac{1}{w} = w^2;\ \frac{1}{w^2} = w$

9) a) $a^3 + b^3 = (a+b)(a+bw)(a+bw^2)$

b) $a^3 - b^3 = (a-b)(a-bw)(a-bw^2)$

c) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+bw+cw^2)(a+bw^2+cw)$

d) $a^2 + b^2 + c^2 - ab - bc - ca = (a+bw+cw^2)(a+bw^2+cw)$

10. Some examples:

a) roots of (x - 1)^3 = 8 are 1+2, 1+2w, 1+2w^2

b) Length of the sides of a triagle whose vertices are given by the roots of $(z+\alpha\beta)^3 = \alpha^3$ are $\sqrt{3}\alpha$ and the triangle is an equilateral triangle

c) Fourth roots of unity are **1, -1, i, -i**, and sum of these roots is 0 and their product is **-1**

IX). nth roots of unity:

1). nth roots of unity are

$$1, \alpha, \alpha^2, \alpha^3 ..... \alpha^{n-1} \ \ or \ \ \alpha^r \ \ where \ r = 0,1,2,3,....n-1 \ \ and \ \ \alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = e^{\frac{2\pi i}{n}}$$

2). nth roots of unity are in G.P. with common ratio α

3). Sum of n^{th} *roots of* 1 is zero. $1 + \alpha + \alpha^2 + \alpha^3 + ... + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} = 0 \ \ (\ \alpha^n = 1\ )$

4). Sum of the pth powers of nth roots of unity is 0, or n

$$\left[\begin{array}{l} 1 + (\alpha)^p + (\alpha^2)^p + (\alpha^3)^p + ... + (\alpha^{n-1})^p = 0, \ \ if \ \ p \neq kn, \ k \in I \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = n \ \ if \ \ p = kn, \ k \in I \end{array}\right]$$

5). Product of nth roots of unity is --1 or 1 depends on n is even or odd.

$$.\left[1.\alpha.\alpha^2.\alpha^3......\alpha^{n-1} = \left(\alpha^{\frac{(n-1)n}{2}}\right) = \left(\alpha^{\frac{n}{2}}\right)^{n-1} = \left(e^{\frac{2\pi i.n}{n.2}}\right)^{n-1} = (\sin\pi + i\sin\pi)^{n-1} = (-1)^{n-1}\right]$$

6). nth roots of unity, lie on the unit circle $|z| = 1$, and divide the circumference of the circle in to n equal parts $\left[\arg(1) = 0, \ \arg(\alpha) = \frac{2\pi}{n}, \ \ \arg(\alpha^2) = \frac{4\pi}{n}, ..., \arg(\alpha^{n-1}) = \frac{2\pi(n-1)}{n}\right]$.

and difference between the argumnets of any two consecutive nth roots of 1 is $\frac{2\pi}{n}$

and nth roots of 1 are the vertices of a n-sided regular polygon inscribed in a circle of radius 1 with centre at origin.

7). A) $z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2)........(z-\alpha^{n-1})$

b) $1 + z + z^2 + z^3 + ... + z^{n-1} = (z-\alpha)(z-\alpha^2)........(z-\alpha^{n-1})$

c) $(1-\alpha)(1-\alpha^2)........(1-\alpha^{n-1})=n \quad (z=1)$

$$(1+\alpha)(1+\alpha^2)........(1+\alpha^{n-1})=1, \text{ if } n \text{ is odd natural}$$
$$=0, \text{ if } n \text{ is even natural.}$$

7d) $(z-\alpha)(z-\alpha^2)........(z-\alpha^{n-1})=0, \text{ if } n \text{ is a multiple of } 3$

$$=1, \text{ if } n \text{ is } 3m+1,\ m\in N$$
$$=1+w, \text{ if } n \text{ is } 3m+2,\ m\in N$$

$$\left[z=w \Rightarrow \frac{w^{3m}-1}{w-1}=0, \frac{w^{3m+1}-1}{w-1}=1, \frac{w^{3m+2}-1}{w-1}=1+w,\right]$$

8). n**th** roots of unity are

$$1,\alpha,\alpha^2,\alpha^3.....\alpha^{n-1} \text{ and } \sum_{r=0}^{n-1}\alpha^r=0 \Rightarrow \sum_{r=0}^{n-1}\cos\left(\frac{2\pi}{r}\right)=0 \ \& \ \sum_{r=0}^{n-1}\sin\left(\frac{2\pi}{r}\right)=0$$

9). $z_1, z_2, z_3, ..., z_n$ are the vertices of a "n" sided regular polygon and its centre is at origin and

one of its vertices is known to be z_1, then $z_k=z_1\alpha^{K-1};\quad \alpha=e^{\frac{2\pi i}{n}},\ k=1,2,.....n.$

10). $A_{1,}A_2.....$ An. Be the vertices of an n – sided regular polygon, inseribed in a circle of radius one, then.

a) $|A_1A_2|^2+|A_1A_3|^2+|A_1A_4|^2+......+|A_1A_n|^2=2n$

b) $|A_1A_2|.|A_1A_3|.|A_1A_4|......|A_1A_n|=n.$

c) If $\dfrac{1}{A_1A_2}=\dfrac{1}{A_1A_3}+\dfrac{1}{A_1A_4}$ then $n=7.$

11).

a) $\sin\left(\dfrac{\pi}{n}\right).\sin\left(\dfrac{2\pi}{n}\right).\sin\left(\dfrac{3\pi}{n}\right).......\sin\left(\dfrac{(n-1)\pi}{n}\right).=\dfrac{n}{2^{n-1}}$

b) $\left|\cos\left(\dfrac{\pi}{n}\right).\cos\left(\dfrac{2\pi}{n}\right).\cos\left(\dfrac{3\pi}{n}\right).......\cos\left(\dfrac{(n-1)\pi}{n}\right)\right|.=\dfrac{1-(-1)^n}{2^n}$

$$\left[1+z+z^2+z^3+.......+z^{n-1}=(z-\alpha)(z-\alpha^2).........(z-\alpha^{n-1})\ \&\ \alpha=e^{\frac{2\pi i}{n}} \text{ and put } z=1 \text{ and } -1\right]$$

12).

a). If α be the angle, which each side of a regular polygon of n,subtends at its centre then $1+\cos\alpha+\cos 2\alpha+\cos 3\alpha+......\cos(n-1)\alpha=0$

b). If $a_1,a_2,a_3,......,a_n$ are the real numbers and $\alpha=\cos\alpha+i\sin\alpha$ is a root of

$z^n+a_1z^{n-1}+a_2+z^{n-2}a_3+......+a_n=0$ then $a_1\cos\alpha+a_2\cos 2\alpha+a_3\cos 3\alpha+......+a_n\cos n\alpha=0$

13). If $(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+......+a_nx^n,\ n\in N$, then a). $a_0+a_4+a_8+a_{12}+......=2^{n-2}+2^{\frac{n-2}{2}}\cos\left(\frac{n\pi}{4}\right)$

b) $a_0+a_3+a_6+a_9+......=\frac{1}{3}\left(2^n+2\cos\left(\frac{n\pi}{3}\right)\right)$

c) $a_1+a_4+a_7+a_{10}+......=\frac{1}{3}\left(2^n+2\cos\left(\frac{(n-2)\pi}{3}\right)\right)$

d) $a_2+a_5+a_8+a_{11}+......=\frac{1}{3}\left(2^n+2\cos\left(\frac{(n+2)\pi}{3}\right)\right)$

e) $a_0+a_4+a_8+a_{12}+......=2^{n-2}+2^{\frac{n-2}{2}}\cos\left(\frac{n\pi}{4}\right)$

f) $a_0+a_4+a_8+a_{12}+......=2^{n-2}+2^{\frac{n-2}{2}}\cos\left(\frac{n\pi}{4}\right)$

g) $a_0+a_4+a_8+a_{12}+......=2^{n-2}+2^{\frac{n-2}{2}}\cos\left(\frac{n\pi}{4}\right)$

h) $a_0+a_4+a_8+a_{12}+......=2^{n-2}+2^{\frac{n-2}{2}}\cos\left(\frac{n\pi}{4}\right)$

14). If $\left(1+x+x^2\right)^n=a_0+a_1x+a_2x^2+a_3x^3+......+a_{2n}x^{2n},\ n\in N,\ then$

a) $a_0+a_1+a_2+a_3+......=3^n$

b) $a_0+a_3+a_6+a_9+......=3^{n-1}$

c) $a_1+a_4+a_7+a_{10}+......=3^{n-1}$

d) $a_2+a_5+a_8+a_{11}+......=3^{n-1}$

X). De'Moivre's theorem:

1. If $n\in Z,\ (\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ and If $n\in Q,\ \cos n\theta+i\sin n\theta$ is one of the value of $(\cos\theta+i\sin\theta)^n$

a). $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$

b). $\frac{1}{(\cos\theta+i\sin\theta)^n}=\cos n\theta-i\sin n\theta$

2. If $z=\cos\theta+i\sin\theta$ then $z+\frac{1}{z}=2\cos\theta$ and $z-\frac{1}{z}=2i\sin\theta$

3. If $x+\frac{1}{x}=2\cos\theta$ *and* $y+\frac{1}{y}=2\cos\phi$ then

a) $\frac{x}{y}+\frac{y}{x}=2\cos(\theta-\phi)$

b) $\frac{x^m}{y^n}+\frac{y^n}{x^m}=2\cos(m\theta-n\phi)$

c) $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos\left(m\theta + n\phi\right)$

4. If $\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$ then

a) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(3\alpha+3\beta+3\gamma)$

b) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(3\alpha+3\beta+3\gamma)$

c) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=\sin 2\alpha+\sin 2\beta+\sin 2\gamma$

5. If n is a positive integer, then $\left(\sqrt{3}+i\right)^n+\left(\sqrt{3}-i\right)^n=2^{n+1}\cos\left(\dfrac{n\pi}{6}\right)$

XI). Quadratic equations with non real co efficients:

consider $az^2+bz+c=0$ *where* $a,b,c\in C$

1). If equation has one purely imaginary root, then $\left(b\bar{c}+c\bar{b}\right)\left(a\bar{b}+\bar{a}b\right)+\left(\bar{a}c-a\bar{c}\right)^2=0$

$\left[because,\ \overline{az^2+bz+c}=0\ \&\ \bar{a}z^2-\bar{b}z+\bar{c}=0\ have\ a\ common\ root\ \&\ z_1=-\bar{z}_1\right]$

2). a). If equation has one purely real root, then $\left(b\bar{c}-c\bar{b}\right)\left(a\bar{b}-\bar{a}b\right)+\left(\bar{a}c-a\bar{c}\right)^2=0$

$\left[because,\ \overline{az^2+bz+c}=0\ \&\ \bar{a}z^2+\bar{b}z+\bar{c}=0\ have\ a\ common\ root\ \&\ z_1=\bar{z}_1\right]$

3). If equation hastwo purely imaginary roots, then $\dfrac{a}{\bar{a}}=\dfrac{b}{-\bar{b}}=\dfrac{c}{\bar{c}}$

$\left[because,\ \overline{az^2+bz+c}=0\ \&\ \bar{a}z^2-\bar{b}z+\bar{c}=0\ have\ both\ common\ roots\ \&\ z_1=-\bar{z}_1,z_2=-\bar{z}_2\right]$

4). If equation hastwo purely imaginary roots, then $\dfrac{a}{\bar{a}}=\dfrac{b}{\bar{b}}=\dfrac{c}{\bar{c}}$

$\left[because,\ \overline{az^2+bz+c}=0\ \&\ \bar{a}z^2+\bar{b}z+\bar{c}=0\ have\ both\ common\ roots\ \&\ z_1=\bar{z}_1,z_2=\bar{z}_2\right]$

Examples:

a). If α *is a non-real root of* $x^2+\alpha x+\bar{\alpha}=0$, then the real root of the equation is, 1

b). One of the roots of $z^2+\left(a_1+ia_2\right)z+\left(b_1+ib_2\right)=0$ is real then, $a_1a_2b_2=b_1a_2^{\,2}+b^2{}_2$

c). roots of $z^2+\left(a_1+ia_2\right)z+\left(b_1+ib_2\right)=0$ are equal then, $2b_2=a_1a_2$ *or* $4b_1=a_1^{\,2}-b_2^{\,2}$

XII). Some standard locus/results:

1). $A(z_1),B(z_2).$ *and* $C(z)$ is a point which divides the line segment AB in the ratio m:n internally then $C(z)=\dfrac{mz_2+nz_1}{m+n}$. If C is a mid point of AB, then $C(z)=\dfrac{z_2+z_1}{2}$.

2). If $z_1^2+2z_2^2+z_3^2=2z_2\left(z_1+z_3\right)$ *where* z_1,z_2,z_3 are the vertices of a triangle then the triangle is isosceles right triangle $\left[\begin{array}{l} because\quad (z_1^2-2z_1z_2+z_2^2)+(z_2^2-2z_2z_3+z_3^2)=0 \\ \Rightarrow (z_1-z_2)=\pm i(z_3-z_2)\Rightarrow (z_1-z_2)=e^{\frac{\pm i}{2}}(z_3-z_2) \end{array}\right]$

3). If z_1, z_2, z_3 are complex numbers then $z_1 \operatorname{Im}(\overline{z_2} z_3) + z_2 \operatorname{Im}(\overline{z_3} z_1) + z_3 \operatorname{Im}(\overline{z_1} z_2) = 0$

4). If $z_1,= e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ then $\arg(z_1 + z_2) = \dfrac{\theta_1 + \theta_2}{2}$ & $|z_1 + z_2| = \left|\cos\left(\dfrac{\theta_1 - \theta_2}{2}\right)\right|$

5). If $z_1^2 + z_2^2 - z_1 z_2 = 0$, then z_1, z_2 and origin form an equilateral triangle.

6). If $z_1^2 + z_2^2 + z_1 z_2 = 0$, then z_1, z_2 and origin form an isosceles triangle.

7). If z_1, z_2, z_3 are the three complex numbers such that $z_1, + z_2 + z_3 = 0$ & $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = 0$,

then z_1, z_2, z_3 represent vertices of an equilateral triangle.

8). If $z_1 = a + ib,\ z_2 = a' + ib',\ z_1 - z_2$ are collinear points, then $ab' - a'b = 0$

9). $A(z_1), B(z_2), C(z_3)$ are connected by $az_1 + bz_2 + cz_3 = 0$ $(a,b,c \in R)$ such that

$a + b + c = 0$ then A,B,C are colliear points (where a,b,c are all not zero).

10). If $z_1, z_2, z_3 \in C$, such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then z_1, z_2, z_3 represent

vertices of an equilateral triangle.

11). If $z_1, z_2, z_3, z_4 \in C$, are distinct complex numbers on $|z| = r$, then $\dfrac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ is

real (i.e. concyclic points condition)

12). If $P(e^{i\theta_1}), Q(e^{i\theta_2}), R(e^{i\theta_3})$ are the vertices of a triangle PQR, then its orthocenter is

$e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}$ (because circumcentre is at origin and centroid is $\dfrac{e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}}{3}$)

13). If $|z_1| = |z_2| = ... = |z_n| = 1$ & $|z_1 + z_2 + z_3 + ... + z_n| = 1$ *then* $\left|\dfrac{1}{z_1} + \dfrac{1}{z_1} + \dfrac{1}{z_1} + ... + \dfrac{1}{z_n}\right| = 1$

14). If $|z_1| + |z_2| + |z_3| = |z_1 + z_2 + z_3|$, *then* $\dfrac{z_1 z_2}{z_3^{\,2}} + \dfrac{z_2 z_3}{z_1^{\,2}} + \dfrac{z_3 z_1}{z_2^{\,2}}$ is purely real number.

15). Number of distinct elements in the set S=$\{i^n + i^{-n}, n \in Z\}$ is 3 (i.e. 0,2,-2)

16a) If α_i be the roots of the equation $(1+z)^6 + z^6 = 0$, then real parts of all α_i is equal to

$-\dfrac{1}{2}$

16b) All the roots of the equation $(1-z)^{10} = z^{10}$ lie on the line $x = \dfrac{1}{2}$