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AMS 203
Homework #2
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DeGroot and Schervish: 3.5.10, 3.6.10, 3.6.12, 3.7.2, 3.7.8, 3.8.14

3.5.10 Suppose that a point (X, Y) is chosen at random from the circle S defined as follows:

$$S = \{(x, y) : x^2 + y^2 \leq 1\}.$$

a. Determine the joint p.d.f. of X and Y, the marginal p.d.f. of X, and the marginal p.d.f. of Y.

b. Are X and Y independent?

a. The points (x, y) are chosen at random, meaning that they are all equally likely, and that $f(x, y)$ is uniform on the unit circle. Thus,

$$f(x, y) = \frac{1}{\pi},$$

since the radius of S is unity. The marginal distribution for a given Y is found by integrating along all possible values for x, namely,

$$f(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2\sqrt{1-y^2}}{\pi}.$$

Similarly for X,

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2\sqrt{1-x^2}}{\pi}.$$

b. If X and Y are independent, $f(x)f(y) = f(x, y)$. However,

$$f(x)f(y) = \frac{4}{\pi^2}\sqrt{1-x^2}\sqrt{1-y^2} \neq \frac{1}{\pi} \text{ in general.}$$

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3.6.10 In a large collection of coins, the probability of X that a head will be obtained when a coin is tossed varies from one coin to another, and the distribution of X in the collection is specified by the following p.d.f.:

$$f_1(x) = \begin{cases} 6x(1-x) & \text{for } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

Suppose that a coin is selected at random from the collectin and tossed once, and that a head is obtained. Determine the conditional p.d.f. of X for this coin.

In this problem we use Bayes' Theorem. Let $T_1 = \{h, t\}$ be the event of tossing a coin for the first time, with the two possible outcomes of head or tail. We seek $f(x|T_1 = h)$. We have

$$\begin{aligned} f(x|T_1 = h) &= \frac{f(T_1 = h|x)f(x)}{\int_0^1 f(T_1 = h|x)f(x)\,dx} = \frac{x\,[6x(1-x)]}{\int_0^1 x\,[6x(1-x)]\,dx} \\ &= \frac{6x^2(1-x)}{\int_0^1 6x^2(1-x)\,dx} = \frac{6x^2(1-x)}{1/2} \\ &= 12x^2(1-x) \end{aligned}$$

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3.6.12 Let Y be the rate (calls per hour) at which calls arrive at a switchboard. Let X be the number of calls during a two-hour period. Suppose that the marginal p.d.f. of Y is

$$f_2(y) = \begin{cases} e^{-y} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

and that the conditional p.f. of X given $Y = y$ is

$$g_1(x|y) = \begin{cases} \frac{(2y)^x}{x!} e^{-2y} & \text{if } x = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the marginal p.f. of X. (You may use the formula $\int_0^\infty y^k e^{-y}\, dy = k!$)

b. Find the conditional p.d.f. $g_2(y|0)$ of Y given $X = 0$.

c. Find the conditional p.d.f. $g_2(y|1)$ of Y given $X = 1$.

d. For what values of y is $g_2(y|1) > g_2(y|0)$? Does this agree with the intuition that the more calls you see, the higher you should think the rate is?

a. The marginal p.f. of X is found by integrating the product $g_1(x|y)f_2(y)$ over all admissible values of y,

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g_1(x|y) f_2(y)\, dy = \int_0^\infty \frac{(2y)^x}{x!} e^{-2y} e^{-y}\, dy \\ &= \frac{2^x}{3^x x!} \int_0^\infty (3y)^x e^{-3y}\, dy, \quad \text{substituting } m = 3y \\ &= \frac{2^x}{3^{x+1} x!} \int_0^\infty (m)^x e^{-m}\, dm = \frac{2^x}{3^{x+1} x!} x! \\ &= \frac{1}{3} \left(\frac{2}{3} \right)^x \end{aligned}$$

b. We have $g_2(y|0) = g_1(0|y) f_2(y) / f(0)$ so that

$$g_2(y|0) = \frac{g_1(0|y) f_2(y)}{f(0)} = \frac{e^{-2y} e^{-y}}{1/3} = 3e^{-3y}$$

c. We have $g_2(y|1) = g_1(1|y) f_2(y) / f(1)$ so that

$$g_2(y|1) = \frac{g_1(1|y) f_2(y)}{f(1)} = \frac{2y e^{-2y} e^{-y}}{2/9} = 9y e^{-3y}$$

d. We seek y from the condition

$$9y e^{-3y} > 3e^{-3y} \quad \Rightarrow \quad y > \frac{1}{3}$$

This agrees with the intuition that that the more calls you see, the higher you should think the rate is.

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3.7.2 Suppose that three random variables X_1, X_2, and X_3 have mixed joint distribution with p.f./p.d.f.:

$$f(x_1, x_2, x_3) = \begin{cases} c x_1^{1+x_2+x_3}(1-x_1)^{3-x_2-x_3} & \text{if } 0 < x_1 < 1 \text{ and } x_2, x_3 \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

(Notice that X_1 has a continuous distribution and X_2 and X_3 have discrete distributions.) Determine

a. the value of the constant c;

b. the marginal joint p.f. of X_2 and X_3;

c. the conditional p.d.f. of X_1 given $X_2 = 1$ and $X_3 = 1$.

a. We need c such taht $\int f = 1$. This integral takes on the form

$$\begin{aligned} \int f(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 &= c\left[\int_0^1 x_1(1-x_1)^3 dx_1 + \int_0^1 x_1^2(1-x_1)^2 dx_1 \right.\\ &\quad \left. + \int_0^1 x_1^2(1-x_1)^2 dx_1 + \int_0^1 x_1^3(1-x_1) dx_1\right] \\ &= c\left[\frac{1}{20} + \frac{2}{30} + \frac{1}{20}\right] = c\frac{1}{6} \\ &\Rightarrow c = 6 \end{aligned}$$

b. The marginal joint p.f. of X_2 and X_3 is given by

$$f_2(x_2, x_3) = \int_0^1 f(x_1, x_2, x_3) dx_1 = 6\int_0^1 x_1^{1+x_2+x_3}(1-x_1)^{3-x_2-x_3} dx_1$$

This integral is difficult to compute, so we will rely on the fact that (X_1, X_2) have discrete distributions. The integrals in Part **a** will be used to compute the probabilities.

$$\begin{aligned} f_2(0,0) &= 6\left[\int_0^1 x_1(1-x_1)^3 dx_1\right] = 6\left[\frac{1}{20}\right] = 0.3 \\ f_2(0,1) &= 6\left[\int_0^1 x_1^2(1-x_1)^2 dx_1\right] = 6\left[\frac{1}{30}\right] = 0.2 \\ f_2(1,0) &= 6\left[\int_0^1 x_1^2(1-x_1)^2 dx_1\right] = 6\left[\frac{1}{30}\right] = 0.2 \\ f_2(1,1) &= 6\left[\int_0^1 x_1^3(1-x_1) dx_1\right] = 6\left[\frac{1}{20}\right] = 0.3 \end{aligned}$$

Thus,

$$f_2(x_2, x_3) = \begin{cases} 0.2 & \text{if } (x_2, x_3) \in \{(1,0), (0,1)\} \\ 0.3 & \text{if } (x_2, x_3) \in \{(0,0), (1,1)\} \end{cases}$$

c. If $x \in (0,1)$,

$$\begin{aligned} f_3(x_1|x_2=1,x_3=1) &= \frac{f(x_1,x_2=1,x_3=1)}{f_2(1,1)} = \frac{3/2(x_1^3(1-x_1)^1)}{3/40} \\ &= 20\left(x_1^3(1-x_1)^1\right) \end{aligned}$$

Else, $f_3(x_1|x_2=1,x_3=1)=0$.

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3.7.8 Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{1}{n!}x^n e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose also that for any given value $X = x (x > 0)$, the n random variables $Y_1, \ldots, Y_n$ are i.i.d. and the conditional p.d.f. g of each of them is as follows:

$$g(y|x) = \begin{cases} \frac{1}{x} & \text{for } 0 < y < x \\ 0 & \text{otherwise.} \end{cases}$$

Determine

a. the marginal joint p.d.f. of $Y_1, \ldots, Y_n$ and

b. the conditional pdf of X for any given values of $Y_1, \ldots, Y_n$.

For any given value of X, the random variables $Y_1, \ldots, Y_n$ are iid, each with a pdf $g(y|x)$. Therefore, the conditional joind pdf of $Y_1, \ldots, Y_n$ given that $X = x$ is

$$h(y_1, \ldots, y_n|x) = g(y_1|x) \cdots g(y_n|x) = \begin{cases} \frac{1}{x^n} & \text{for } 0 < y_i < x, i = 1, \ldots, n, \\ 0 & \text{otherwise} \end{cases}$$

The joing pdf is positive if and only if each $y_i > 0$ and x is greater than every y_i, that is $x > m = \max\{y_1, \ldots, y_n\}$

a. For $y_1 > 0 \ \ (i = 1, \ldots, n)$, the marginal pdf of $Y_1, \ldots, Y_n$ is

$$g_0(y_1, \ldots, y_n) = \int_{-\infty}^{\infty} f(x)h(y_1, \ldots, y_n|x)\, dx = \int_m^{\infty} \frac{1}{n!}e^{-x}\, dx = \frac{1}{n!}e^{-m}$$

b. For $y_i > 0 \ \ (i = 1, \ldots, n)$, the conditional pdf of X given that $Y_i = y_i \ \ (i = 1, \ldots, n)$ is

$$g_1(x|y_1, \ldots, y_n) = \frac{f(x)h(y_1, \ldots, y_n|x)}{g_0(y_1, \ldots, y_n)} = \begin{cases} e^{-(x-m)} & \text{for } x > m, \\ 0 & \text{otherwise.} \end{cases}$$

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3.8.14 Let X have the uniform distribution on the interval $[a, b]$, and let $c > 0$. Prove that $cX + d$ has the uniform distribution on the interval $[ca + d, cb + d]$.

Let $Y = cX + d$. The inverse transformation is $x = (y - d)/c$. Assume that $c > 0$. The derivative of the inverse is $1/c$. The pdf of Y is

$$g(y) = f([y - d]/c)/c = [c(b - a)]^{-1}, \quad \text{for } a \leq (y - d)/c \leq b.$$

It is easy to see that $a \leq (y - d)/c \leq b$ if and only if $ca + d \leq y \leq cb + d$, so g is the pdf of the uniform distribution on the interval $[ca + d, cb + d]$. If $c < 0$, the distribution of Y would be uniform on the interval $[cb + d, ca + d]$. If $c = 0$, the distribution of Y is degenerate at the value d, ie $P(Y = d) = 1$. ∎