

CS217 - Algorithm Design and Analysis

Homework Assignment 5

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4 Linear Programs and Farkas Lemma

Exercise 4.2.

Necessity.

Obviously if $y \in \{0, 1\}$ and the set $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover, then y is a feasible solution (every edge is covered by a node, namely $\forall (u, v) \in E, y_u + y_v \geq 1$). To show the solution is a basic solution, we firstly show that the following lemma is true.

Lemma 1. *If C is a minimal vertex cover, then $\forall u \in C, \exists v \notin C$, such that $(u, v) \in E$.*

Proof. Otherwise if there exists x satisfied that $\forall (u, v) \in E$ and $v \notin C$. Then remove u from C and we can get C' . Obviously C' is a vertex cover, which contradicts to the fact that C is a minimal vertex cover. \square

Considering the nodes u satisfying $u \notin C$, then $m + u \in I(y)$ holds. These nodes donate an identity matrix in $A_{I(y)}$. Then considering the nodes u which is in the vertex cover, then according to the lemma, we can find another adjacent vertex $v \notin C$, and $y_u + y_v = 1$, namely $id(u, v) \in I(y)$ where $id(u, v)$ stands for the serial number of the edge (u, v) . Hence $A_{I(y)}$ is consisted of two kinds of rows, and one of them is $a_{id(u, v)}$ (either u or v is in C and $(u, v) \in E$) and a_{m+i} (i is not in C). By Gauss elimination, it is obviously that $\text{rank}(A_{I(y)}) = n$, hence y is a basic solution.

Sufficiency.

Firstly we will show that given a basic and feasible solution $y, y_u \in \{0, 1\}$ holds. We can construct a graph $G' = (V', E')$ where $V' = \{u \mid y_u \in (0, 1)\}$ and $E' = \{(u, v) \mid u, v \in V', y_u + y_v = 1\}$. Then we know that the degree of each vertex i is at least one, otherwise the i -th column of $A_{I(y)}$ is a zero vector ($y_i \neq 0$ and every edge (u, v) is not "tight"), contradicting to the fact that $A_{I(y)}$ is a non-singular matrix.

Lemma 2. *If a graph G has n vertexes and n edges, then there must be a cycle in G .*

Proof. Otherwise G is a forest and has at most $n - 1$ edges. \square

According to the lemma, G' has a cycle c_1, c_2, \dots, c_t . Let E_c be the set containing the serial number of the edges in the cycle. With the rearrangement of the rows, the submatrix A_{E_c} itself become:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

And obviously the above matrix is singular (and the original matrix A is singular too), which leads the contradictions.

Then we will show that $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover. Showing that $y_u \in \{0, 1\}$, we can easily know that y is a vertex cover. If y is not a minimal vertex cover, then there must be a node $p \in C$, after we remove which we can obtain another vertex cover y' . Consider the p -th column of A , and according to the lemma 1, we can see that there is no adjacent node q such that $y_p + y_q = 1$, which means that the p -th column of A is a zero vector, which leads a contradiction.

Exercise 4.6. FLGV: According to FLGV-2, we have:

$$\exists \mathbf{y} \in \mathbf{R}^m : A^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

if FLGV-1 also holds, there exists $\mathbf{x} \geq 0$, $A\mathbf{x} = \mathbf{b}$.

because $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{b}^T = \mathbf{x}^T A^T$, we have:

$$\mathbf{b}^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T \mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{x}^T A^T \mathbf{y} \geq 0$.

but according to FLGV-2, $\mathbf{b}^T \mathbf{y} < 0$, which is a contradictory, so FLGV-2 implies that FLGV-1 does not hold.

FLLPV: According to FLLPV-2, we have:

$$\exists \mathbf{y} \in \mathbf{R}^m : \mathbf{y} \geq 0, A^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

if FLLPV-1 also holds, there exists $\mathbf{x} \geq 0$, $A\mathbf{x} \leq \mathbf{b}$.

because $A\mathbf{x} \leq \mathbf{b}$ can be written as $\mathbf{b}^T \geq \mathbf{x}^T A^T$, and \mathbf{y} holds $\mathbf{y} \geq 0$, we have:

$$\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T \mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y} \geq 0$.

but according to FLLPV-2, $\mathbf{b}^T \mathbf{y} < 0$, which is a contradictory, so FLLPV-2 implies that FLLPV-1 does not hold.

FLEV: According to FLEV-2, we have:

$$\exists \mathbf{y} \in \mathbf{R}^m : \mathbf{y} \geq 0, A^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

if FLEV-1 also holds, there exists $A\mathbf{x} \leq \mathbf{b}$.

because $A\mathbf{x} \leq \mathbf{b}$ can be written as $\mathbf{b}^T \geq \mathbf{x}^T A^T$, and \mathbf{y} holds $\mathbf{y} \geq 0$, we have:

$$\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T \mathbf{y} = 0$, we can get $\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y} = 0$.

but according to FLEV-2, $\mathbf{b}^T \mathbf{y} < 0$, which is a contradictory, so FLEV-2 implies that FLEV-1 does not hold.

Exercise 4.7.

FLGV \Rightarrow FLLPV

Construct $\mathbf{A}' = (\mathbf{A} \ \mathbf{I})$ where \mathbf{I} is identity matrix. Use algorithm for FLGV on \mathbf{A}' and \mathbf{b} . Then we have exact one of the following:

1. $\exists \mathbf{x}' \in \mathbb{R}^{n+m} : \mathbf{x}' \geq 0, \mathbf{A}' \mathbf{x}' = \mathbf{b}$.
 Let $\mathbf{x}' = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}'' \end{pmatrix}, \mathbf{x} \in \mathbb{R}^n, \mathbf{x}'' \in \mathbb{R}^m$.
 So we have $\mathbf{b} = \mathbf{A}' \mathbf{x}' = \mathbf{A} \mathbf{x} + \mathbf{I} \mathbf{x}'' = \mathbf{A} \mathbf{x} + \mathbf{x}'' \geq \mathbf{A} \mathbf{x}$, thus $\mathbf{x} \geq 0, \mathbf{A} \mathbf{x} \leq \mathbf{b}$
2. $\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{A}'^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$.
 So we have $\begin{cases} \mathbf{A}^T \mathbf{y} \geq 0 \\ \mathbf{I}^T \mathbf{y} \geq 0 \end{cases}$, thus $\mathbf{y} \geq 0, \mathbf{A}^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$

FLLPV \Rightarrow FLGV

Construct $\mathbf{A}' = \begin{pmatrix} \mathbf{A} \\ -\mathbf{A} \end{pmatrix}, \mathbf{b}' = \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \end{pmatrix}$. Use algorithm for FLLPV on \mathbf{A}' and \mathbf{b}' . Then we have exact one of the following:

1. $\exists \mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \mathbf{A}' \mathbf{x} \leq \mathbf{b}'$.
 So we have $\begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ -\mathbf{A} \mathbf{x} \leq -\mathbf{b} \end{cases}$, thus $\mathbf{x} \geq 0, \mathbf{A} \mathbf{x} = \mathbf{b}$
2. $\exists \mathbf{y}' \in \mathbb{R}^{2m} : \mathbf{y}' \geq 0, \mathbf{A}'^T \mathbf{y}' \geq 0, \mathbf{b}'^T \mathbf{y}' < 0$.
 Let $\mathbf{y}' = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}$, where $\mathbf{y}_1 \in \mathbb{R}^m, \mathbf{y}_2 \in \mathbb{R}^m$.
 So we have $\mathbf{A}'^T \mathbf{y}' = \mathbf{A}^T \mathbf{y}_1 - \mathbf{A}^T \mathbf{y}_2 = \mathbf{A}^T (\mathbf{y}_1 - \mathbf{y}_2) \geq 0, \mathbf{b}'^T \mathbf{y}' = \mathbf{b}^T \mathbf{y}_1 - \mathbf{b}^T \mathbf{y}_2 = \mathbf{b}^T (\mathbf{y}_1 - \mathbf{y}_2) < 0$.
 Let $\mathbf{y} \in \mathbb{R}^m, \mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$, thus $\mathbf{A}^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$

FLEV \Rightarrow FLLPV

Construct $\mathbf{A}' = \begin{pmatrix} \mathbf{A} \\ -\mathbf{I} \end{pmatrix}$ where \mathbf{I} is identity matrix, $\mathbf{b}' = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$. Use algorithm for FLEV on \mathbf{A}' and \mathbf{b}' . Then we have exact one of the following:

1. $\exists \mathbf{x} \in \mathbb{R}^n : \mathbf{A}' \mathbf{x} \leq \mathbf{b}'$.
 So we have $\begin{cases} \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ -\mathbf{I} \mathbf{x} \leq 0 \end{cases}$, thus $\mathbf{x} \geq 0, \mathbf{A} \mathbf{x} \leq \mathbf{b}$

$$2. \exists \mathbf{y}' \in \mathbb{R}^{n+m} : \mathbf{y}' \geq 0, \mathbf{A}'^T \mathbf{y}' = 0, \mathbf{b}'^T \mathbf{y}' < 0.$$

$$\text{Let } \mathbf{y}' = \begin{pmatrix} \mathbf{y} \\ \mathbf{y}'' \end{pmatrix}, \mathbf{y} \in \mathbb{R}^m, \mathbf{y}'' \in \mathbb{R}^n.$$

$$\text{So we have } \mathbf{A}'^T \mathbf{y}' = \mathbf{A}^T \mathbf{y} - \mathbf{I} \mathbf{y}'' = 0, \mathbf{b}'^T \mathbf{y}' = \mathbf{b}^T \mathbf{y} < 0, \text{ thus } \mathbf{y} \geq 0, \mathbf{A}^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

FLLPV \Rightarrow FLEV

Construct $\mathbf{A}' = (\mathbf{A} \quad -\mathbf{A})$. Use algorithm for FLLPV on \mathbf{A}' and \mathbf{b} . Then we have exact one of the following:

$$1. \exists \mathbf{x}' \in \mathbb{R}^{2n} : \mathbf{x}' \geq 0, \mathbf{A}' \mathbf{x}' \leq \mathbf{b}.$$

$$\text{Let } \mathbf{x}' = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}, \text{ where } \mathbf{x}_1 \in \mathbb{R}^n, \mathbf{x}_2 \in \mathbb{R}^n.$$

$$\text{So we have } \mathbf{A}' \mathbf{x}' = \mathbf{A} \mathbf{x}_1 - \mathbf{A} \mathbf{x}_2 = \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{b}.$$

$$\text{Let } \mathbf{x} \in \mathbb{R}^n, \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \text{ thus } \mathbf{A} \mathbf{x} \leq \mathbf{b}$$

$$2. \exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y} \geq 0, \mathbf{A}^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0.$$

$$\text{So we have } \begin{cases} \mathbf{A}^T \mathbf{y} \geq 0 \\ -\mathbf{A}^T \mathbf{y} \geq 0 \end{cases}, \text{ thus } \mathbf{A}^T \mathbf{y} = 0, \mathbf{b}^T \mathbf{y} < 0$$