Algorithm Design and Analysis
Homework Assignment 1

Group Name: Static
Group Members: Zhengjia Xu, JiaCheng Yang
Zhuolin Yang, Wenda Qiu

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1 Recursion and Dynamic Programming

Exercise 1.1.

1. Proof by introduction

   Proof. We can easily prove the lemma using induction:

   (a) Basic step
   When $n = 2^1$, the lemma obvirusly holds because

   $$\binom{2}{1} = 0$$

   (b) Induction step
   Assume that the lemma holds where n equals to 2^d, namely

   $$\binom{n}{k} = 0$$

   when $1 \le k \le n - 1$.
   Then we will show that the lemma holds when n equals to 2^{d+1}.
   According to the hints, on the one hand, the number of paths to the n-th line and the k-th grid equals to

   $$\binom{n}{k}$$

   On the other hand, we can also calculate the number by Multiplication Principle and Addition Principle, namely

   $$\binom{n}{k} = \sum_{i=0}^{n/2} \binom{n/2}{i} * \binom{n/2}{k-i} \tag{1}$$

   Then our proof divides into three cases:

- When $1 \le k \le n/2 - 1$, it is obvirusly that

  $$\binom{n/2}{k} = 0$$

  and

  $$\binom{n/2}{0} = \binom{n/2}{n/2} = 1$$

  According to the equation (1), we have

  $$\binom{n}{k} = \binom{n/2}{k-0} + \binom{n/2}{k-n/2}$$

  By assumption, when $1 \le k < n/2$, we have

  $$\binom{n/2}{k} = 0$$

  Notice that $k - n/2 < n/2$, so we have

  $$\binom{n/2}{k-n/2} = 0$$

  Therefore

  $$\binom{n}{k} = 0$$

- When $k = n/2$, it is obvirusly that

  $$\binom{n/2}{k} = \binom{n/2}{k-n/2} = 1$$

  Therefore

  $$\binom{n}{k} = (1+1) = 2 \equiv 0 \pmod 2$$

- When $n/2 \le k < n$, it is easy to see that

  $$\binom{n/2}{k} = 0$$

  And by assumption

  $$\binom{n/2}{k-n/2} = 0$$

  So that we have

  $$\binom{n}{k} = 0$$

(c) Conclusion By induction, we have

$$\binom{n}{k} \equiv 0 \pmod 2$$

where $1 \le k \le n-1$ and $n = 2^d$ $(d \ge 1)$.

$\square$

2. Direct Proof

Proof. We can denote the binomial coefficient as $2^p * q$, namely

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = 2^p * q \tag{2}$$

where $2 \nmid q$, $n = 2^d (d \geq 1)$ and $1 \leq k \leq (n-1)$.
We can easily calculate the exponent p with the formular below

$$p = \sum_{i=1}^{d} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i=1}^{d} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i=1}^{d} \left\lfloor \frac{n-k}{2^i} \right\rfloor$$

Specially when $i = d$, we have

$$\left\lfloor \frac{n}{2^i} \right\rfloor = 1 \text{ and } \left\lfloor \frac{k}{2^i} \right\rfloor = \left\lfloor \frac{n-k}{2^i} \right\rfloor = 0$$

According to the equation (2), we have

$$\begin{aligned} p &= 1 - 0 - 0 + \sum_{i=1}^{d-1} \left\lfloor \frac{n}{2^i} \right\rfloor - \left\lfloor \frac{k}{2^i} \right\rfloor - \left\lfloor \frac{n-k}{2^i} \right\rfloor \\ &\geq 1 + \sum_{i=1}^{d-1} \frac{n}{2^i} - \frac{k}{2^i} - \frac{n-k}{2^i} = 1 \end{aligned}$$

Hence we have

$$2 \mid \binom{n}{k}$$

That is

$$\binom{n}{k} \equiv \pmod 2$$

$\square$

Exercise 1.2. **Exercise 1.3.** **Exercise 1.4.**