CS217 - Algorithm Design and Analyis
Homework Assignment 5

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4 Linear Programs and Farkas Lemma

Exercise 4.2.
Necessity.

Obviously if $y \in \{0, 1\}$ and the set $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover, then y is a feasible solution(every edge is covered by a node, namely $\forall (u, v) \in E, y_u + y_v \geq 1$). To show the solution is a basic solution, we firstly show that the following lemma is true.

Lemma 1. *If C is a minimal vertex cover, then $\forall u \in C$, $\exists v \notin C$, such that $(u, v) \in E$.*

Proof. Otherwise if there exists x satisfied that $\forall (u, v) \in E$ and $v \notin C$. Then remove u from C and we can get C'. Obviously C' is a vertex cover, which contradicts to the fact that C is a minimal vertex cover. □

Considering the nodes u satisfying $u \notin C$, then $m + u \in I(y)$ holds. These nodes donate an identity matrix in $A_{I(y)}$. Then considering the nodes u which is in the vertex cover, then according to the lemma, we can find another adjacent vertex $v \notin C$, and $y_u + y_v = 1$, namely $id(u, v) \in I(y)$ where $id(u, v)$ stands for the serial number of the edge (u, v). Hence $A_{I(y)}$ is consisted of two kinds of rows, and one of them is $a_{id(u,v)}$(either u or v is in C and $(u, v) \in E$) and a_{m+i} (i is not in C). By Gauss elimination, it is obvirusly that $\operatorname{rank}\left(A_{I(y)}\right) = n$, hence y is a basic solution.

Sufficiency.

Firstly we will show that given a basic and feasible solution y, $y_u \in \{0, 1\}$ holds. We can construct a graph $G' = (V', E')$ where $V' = \{u \mid y_u \in (0, 1)\}$ and $E' = \{(u, v) \mid u, v \in V', y_u + y_v = 1\}$. Then we know that the degree of each vertex i is at least one, otherwise the i-th column of $A_{I(y)}$ is a zero vector($y_i \neq 0$ and every edge (u, v) is not "tight"), contradicting to the fact that $A_{I(y)}$ is a non-singular matrix.

Lemma 2. *If a graph G has n vertexes and n edges, then there must be a cycle in G.*

Proof. Otherwise G is a forest and has at most $n-1$ edges. $\square$

According to the lemma, G' has a cycle $c_1, c_2, \ldots, c_t$. Let E_c be the set containing the serial number of the edges in the cycle. With the rearrangement of the rows, the submatrix A_{E_c} itself become:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

And obviously the above matrix is singular(and the original matrix A is singular too), which leads the contradictions.

Then we will show that $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover. Showing that $y_u \in \{0, 1\}$, we can easily know that y is a vertex cover. If y is not a minimal vertex cover, then there must be a node $p \in C$, after we remove which we can obtain another vertex cover y'. Consider the p-th column of A, and according to the lemma 1, we can see that there is no adjacent node q such that $y_p + y_q = 1$, which means that the p-th column of A is a zero vector, which leads a contradiction.

Exercise 4.6.

FLGV

According to **FLGV-2**, we have:

$$\exists \mathbf{y} \in \mathbb{R}^m : A^T\mathbf{y} \geq 0, \mathbf{b}^T\mathbf{y} < 0$$

if FLGV-1 also holds, there exists $\mathbf{x} \geq 0$, $A\mathbf{x} = \mathbf{b}$. because $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{b}^T = \mathbf{x}^T A^T$, we have:

$$\mathbf{b}^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y}$$

Because $A^T\mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{x}^T A^T\mathbf{y} \geq 0$. But according to **FLGV-2**, $\mathbf{b}^T\mathbf{y} < 0$, which is a contradictory, so FLGV-2 implies that **FLGV-1** does not hold.

FLLPV

According to **FLLPV-2**, we have:

$$\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y} \geq 0, A^T\mathbf{y} \geq 0, \mathbf{b}^T\mathbf{y} < 0$$

if **FLLPV-1** also holds, there exists $\mathbf{x} \geq 0$, $A\mathbf{x} \leq \mathbf{b}$. Because $A\mathbf{x} \leq \mathbf{b}$ can be written as $\mathbf{b}^T \geq \mathbf{x}^T A^T$, and $\mathbf{y}$ holds $\mathbf{y} \geq 0$, we have:

$$\mathbf{b}^T\mathbf{y} \geq \mathbf{x}^T A^T\mathbf{y}$$

Because $A^T\mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{b}^T\mathbf{y} \geq \mathbf{x}^T A^T\mathbf{y} \geq 0$. But according to **FLLPV-2**, $\mathbf{b}^T\mathbf{y} < 0$, which is a contradictory, so **FLLPV-2** implies that FLLPV-1 does not hold.

FLEV

According to **FLEV-2**, we have:

$$\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y} \geq 0, A^T\mathbf{y} \geq 0, \mathbf{b}^T\mathbf{y} < 0$$

If **FLEV-1** also holds, there exists $A\mathbf{x} \leq \mathbf{b}$. Because $A\mathbf{x} \leq \mathbf{b}$ can be written as $\mathbf{b}^T \geq \mathbf{x}^T A^T$, and $\mathbf{y}$ holds $\mathbf{y} \geq 0$, we have:

$$\mathbf{b}^T\mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T\mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{b}^T\mathbf{y} \geq \mathbf{x}^T A^T\mathbf{y} \geq 0$. But according to **FLEV-2**, $\mathbf{b}^T\mathbf{y} < 0$, which is a contradictory, so **FLEV-2** implies that FLEV-1 does not hold.

Exercise 4.7.

FLGV⇒ FLLPV

Construct $\boldsymbol{A'} = \begin{pmatrix}\boldsymbol{A} & \boldsymbol{I}\end{pmatrix}$ where $\boldsymbol{I}$ is identity matrix. Use algorithm for FLGV on $\boldsymbol{A'}$ and $\boldsymbol{b}$. Then we have exact one of the following:

1. $\exists \boldsymbol{x'} \in \mathbb{R}^{\boldsymbol{n+m}} \boldsymbol{:} \boldsymbol{x'} \geq \boldsymbol{0}, \boldsymbol{A'x'} = \boldsymbol{b}$.
   Let $\boldsymbol{x'} = \begin{pmatrix}\boldsymbol{x} \\ \boldsymbol{x''}\end{pmatrix}, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{x''} \in \mathbb{R}^{\boldsymbol{m}}$.
   So we have $\boldsymbol{b} = \boldsymbol{A'x'} = \boldsymbol{Ax} + \boldsymbol{Ix''} = \boldsymbol{Ax} + \boldsymbol{x''} \geq \boldsymbol{Ax}$, thus $\boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{Ax} \leq \boldsymbol{b}$
2. $\exists \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}} \boldsymbol{:} \boldsymbol{A'^T y} \geq \boldsymbol{0}, \boldsymbol{b^T y} < \boldsymbol{0}$.
   So we have $\begin{cases} \boldsymbol{A^T y} \geq \boldsymbol{0} \\ \boldsymbol{I^T y} \geq \boldsymbol{0} \end{cases}$, thus $\boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{A^T y} \geq \boldsymbol{0}, \boldsymbol{b^T y} < \boldsymbol{0}$

FLLCP⇒ FLGV

Construct $\boldsymbol{A'} = \begin{pmatrix}\boldsymbol{A} \\ -\boldsymbol{A}\end{pmatrix}, \boldsymbol{b'} = \begin{pmatrix}\boldsymbol{b} \\ -\boldsymbol{b}\end{pmatrix}$. Use algorithm for FLLCP on $\boldsymbol{A'}$ and $\boldsymbol{b'}$. Then we have exact one of the following:

1. $\exists \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \boldsymbol{:} \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{A'x} \leq \boldsymbol{b'}$.
   So we have $\begin{cases} \boldsymbol{Ax} \leq \boldsymbol{b} \\ -\boldsymbol{Ax} \leq -\boldsymbol{b} \end{cases}$, thus $\boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{Ax} = \boldsymbol{b}$
2. $\exists \boldsymbol{y'} \in \mathbb{R}^{\boldsymbol{2m}} \boldsymbol{:} \boldsymbol{y'} \geq \boldsymbol{0}, \boldsymbol{A'^T y'} \geq \boldsymbol{0}, \boldsymbol{b^T y'} < \boldsymbol{0}$.
   Let $\boldsymbol{y'} = \begin{pmatrix}\boldsymbol{y_1} \\ \boldsymbol{y_2}\end{pmatrix}$, where $\boldsymbol{y_1} \in \mathbb{R}^{\boldsymbol{m}}, \boldsymbol{y_2} \in \mathbb{R}^{\boldsymbol{m}}$.
   So we have $\boldsymbol{A'^T y'} = \boldsymbol{A^T y_1} - \boldsymbol{A^T y_2} = \boldsymbol{A^T(y_1 - y_2)} \geq \boldsymbol{0}$, $\boldsymbol{b'^T y'} = \boldsymbol{b^T y_1} - \boldsymbol{b^T y_2} = \boldsymbol{b^T(y_1 - y_2)} < \boldsymbol{0}$.
   Let $\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}, \boldsymbol{y} = \boldsymbol{y_1} - \boldsymbol{y_2}$, thus $\boldsymbol{A^T y} \geq \boldsymbol{0}, \boldsymbol{b^T y} < \boldsymbol{0}$

FLEV⇒ FLLPV

Construct $\boldsymbol{A'} = \begin{pmatrix}\boldsymbol{A} \\ -\boldsymbol{I}\end{pmatrix}$ where $\boldsymbol{I}$ is identity matrix, $\boldsymbol{b'} = \begin{pmatrix}\boldsymbol{b} \\ \boldsymbol{0}\end{pmatrix}$. Use algorithm for FLEV on $\boldsymbol{A'}$ and $\boldsymbol{b'}$. Then we have exact one of the following:

1. $\exists \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} : \boldsymbol{A'x} \leq \boldsymbol{b'}$.
   So we have $\begin{cases} \boldsymbol{Ax} \leq \boldsymbol{b} \\ -\boldsymbol{Ix} \leq \boldsymbol{0} \end{cases}$, thus $\boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{Ax} \leq \boldsymbol{b}$

2. $\exists \boldsymbol{y'} \in \mathbb{R}^{\boldsymbol{n+m}} : \boldsymbol{y'} \geq \boldsymbol{0}, \boldsymbol{A'^T y'} = \boldsymbol{0}, \boldsymbol{b'^T y'} < \boldsymbol{0}$.
   Let $\boldsymbol{y'} = \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{y''} \end{pmatrix}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}, \boldsymbol{y''} \in \mathbb{R}^{\boldsymbol{n}}$.
   So we have $\boldsymbol{A'^T y'} = \boldsymbol{A^T y} - \boldsymbol{Iy''} = \boldsymbol{0}$, $\boldsymbol{b'^T y'} = \boldsymbol{b^T y} < \boldsymbol{0}$, thus $\boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{A^T y} \geq \boldsymbol{0}, \boldsymbol{b^T y} < \boldsymbol{0}$

FLLPV⇒ FLEV

Construct $\boldsymbol{A'} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{A} \end{pmatrix}$. Use algorithm for FLLCP on $\boldsymbol{A'}$ and $\boldsymbol{b}$. Then we have exact one of the following:

1. $\exists \boldsymbol{x'} \in \mathbb{R}^{\boldsymbol{2n}} : \boldsymbol{x'} \geq \boldsymbol{0}, \boldsymbol{A'x'} \leq \boldsymbol{b}$.
   Let $\boldsymbol{x'} = \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{pmatrix}$, where$\boldsymbol{x_1} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{x_2} \in \mathbb{R}^{\boldsymbol{n}}$.
   So we have $\boldsymbol{A'x'} = \boldsymbol{Ax_1} - \boldsymbol{Ax_2} = \boldsymbol{A(x_1 - x_2)} \leq \boldsymbol{b}$.
   Let $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{x} = \boldsymbol{x_1} - \boldsymbol{x_2}$, thus $\boldsymbol{Ax} \leq \boldsymbol{b}$

2. $\exists \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}} : \boldsymbol{y} \geq \boldsymbol{0}, \boldsymbol{A'^T y} \geq \boldsymbol{0}, \boldsymbol{b^T y} < \boldsymbol{0}$.
   So we have $\begin{cases} \boldsymbol{A^T y} \geq \boldsymbol{0} \\ -\boldsymbol{A^T y} \geq \boldsymbol{0} \end{cases}$, thus $\boldsymbol{A^T y} = \boldsymbol{0}, \boldsymbol{b^T y} < \boldsymbol{0}$