CS217 - Algorithm Design and Analyis
Homework Assignment 5

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4 Linear Programs and Farkas Lemma

Exercise 4.2.
Necessity.

Obvirusly if $y \in \{0,1\}$ and the set $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover, then y is a feasible solution(every edge is covered by a node, namely $\forall (u,v) \in E, y_u + y_v \geq 1$). To show the solution is a basic solution, we firstly show that the following lemma is true.

Lemma 1. *If C is a minimal vertex cover, then $\forall u \in C$, $\exists v \notin C$, such that $(u,v) \in E$.*

Proof. Otherwise if there exists x satisfied that $\forall (u,v) \in E$ and $v \notin C$. Then remove u from C and we can get C'. Obvirusly C' is a vertex cover, which contradicts to the fact that C is a minimal vertex cover. $\square$

Considering the nodes u satisfying $u \notin C$, then $m + u \in I(y)$ holds. These nodes donate an identity matrix in $A_{I(y)}$. Then considering the nodes u which is in the vertex cover, then according to the lemma, we can find another adjacent vertex $v \notin C$, and $y_u + y_v = 1$, namely $id(u,v) \in I(y)$ where $id(u,v)$ stands for the serial number of the edge (u,v). Hence $A_{I(y)}$ is consisted of two kinds of rows, and one of them is $a_{id(u,v)}$(either u or v is in C and $(u,v) \in E$) and a_{m+i} (i is not in C). By Gauss elimination, it is obvirusly that $\text{rank}\left(A_{I(y)}\right) = n$, hence y is a basic solution.

Sufficiency.

Firstly we will show that given a basic and feasible solution y, $y_u \in \{0,1\}$ holds. We can construct a graph $G' = (V', E')$ where $V' = \{u \mid y_u \in (0,1)\}$ and $E' = \{(u,v) \mid u, v \in V', y_u + y_v = 1\}$. Then we know that the degree of each vertex i is at least one, otherwise the i-th column of $A_{I(y)}$ is a zero vector($y_i \neq 0$ and every edge (u,v) is not "tight"), contradicting to the fact that $A_{I(y)}$ is a non-singular matrix.

Lemma 2. *If a graph G has n vertexes and n edges, then there must be a cycle in G.*

Proof. Otherwise G is a forest and has at most $n - 1$ edges. □

According to the lemma, G' has a cycle $c_1, c_2, \ldots, c_t$. Let E_c be the set containing the serial number of the edges in the cycle. With the rearrangement of the rows, the submatrix A_{E_c} itself become:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

And obvirusly the above matrix is singular(and the original matrix A is singular too), which leads the contradictions.

Then we will show that $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover. Showing that $y_u \in \{0, 1\}$, we can easily know that y is a vertex cover. If y is not a minimal vertex cover, then there must be a node $p \in C$, after we remove which we can obtain another vertex cover y'. Consider the p-th column of A, and according to the lemma 1, we can see that there is no adjacent node q such that $y_p + y_q = 1$, which means that the p-th column of A is a zero vector, which leads a contradiction.

Exercise 4.6.

FLGV

According to **FLGV-2**, we have:

$$\exists \mathbf{y} \in \mathbf{R}^m : A^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

if FLGV-1 also holds, there exists $\mathbf{x} \geq 0$, $A\mathbf{x} = \mathbf{b}$. because $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{b}^T = \mathbf{x}^T A^T$, we have:

$$\mathbf{b}^T \mathbf{y} = \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T \mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{x}^T A^T \mathbf{y} \geq 0$. But according to **FLGV-2**, $\mathbf{b}^T \mathbf{y} < 0$, which is a contradictory, so FLGV-2 implies that **FLGV-1** does not hold.

FLLPV

According to **FLLPV-2**, we have:

$$\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y} \geq 0, A^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

if **FLLPV-1** also holds, there exists $\mathbf{x} \geq 0$, $A\mathbf{x} \leq \mathbf{b}$. Because $A\mathbf{x} \leq \mathbf{b}$ can be written as $\mathbf{b}^T \geq \mathbf{x}^T A^T$, and $\mathbf{y}$ holds $\mathbf{y} \geq 0$, we have:

$$\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T \mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y} \geq 0$. But according to **FLLPV-2**, $\mathbf{b}^T \mathbf{y} < 0$, which is a contradictory, so **FLLPV-2** implies that FLLPV-1 does not hold.

FLEV

According to **FLEV-2**, we have:

$$\exists \mathbf{y} \in \mathbb{R}^m : \mathbf{y} \geq 0, A^T \mathbf{y} \geq 0, \mathbf{b}^T \mathbf{y} < 0$$

If **FLEV-1** also holds, there exists $A\mathbf{x} \leq \mathbf{b}$. Because $A\mathbf{x} \leq \mathbf{b}$ can be written as $\mathbf{b}^T \geq \mathbf{x}^T A^T$, and $\mathbf{y}$ holds $\mathbf{y} \geq 0$, we have:

$$\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y}$$

Because $A^T \mathbf{y} \geq 0$, $\mathbf{x} \geq 0$, we can get $\mathbf{b}^T \mathbf{y} \geq \mathbf{x}^T A^T \mathbf{y} \geq 0$. But according to **FLEV-2**, $\mathbf{b}^T \mathbf{y} < 0$, which is a contradictory, so **FLEV-2** implies that FLEV-1 does not hold.

Exercise 4.7.

FLGV$\Rightarrow$ FLLPV

Construct $\boldsymbol{A}' = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{I} \end{pmatrix}$ where $\boldsymbol{I}$ is identity matrix. Use algorithm for FLGV on $\boldsymbol{A}'$ and $\boldsymbol{b}$. Then we have exact one of the following:

1. $\exists \boldsymbol{x}' \in \mathbb{R}^{n+m} : \boldsymbol{x}' \geq 0, \boldsymbol{A}'\boldsymbol{x}' = \boldsymbol{b}$.
   Let $\boldsymbol{x}' = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}'' \end{pmatrix}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x}'' \in \mathbb{R}^m$.
   So we have $\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{x}'' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}'' \geq \boldsymbol{A}\boldsymbol{x}$, thus $\boldsymbol{x} \geq 0, \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$

2. $\exists \boldsymbol{y} \in \mathbb{R}^m : \boldsymbol{A}'^T \boldsymbol{y} \geq 0, \boldsymbol{b}^T \boldsymbol{y} < 0$.
   So we have $\begin{cases} \boldsymbol{A}^T \boldsymbol{y} \geq 0 \\ \boldsymbol{I}^T \boldsymbol{y} \geq 0 \end{cases}$, thus $\boldsymbol{y} \geq 0, \boldsymbol{A}^T \boldsymbol{y} \geq 0, \boldsymbol{b}^T \boldsymbol{y} < 0$

FLLCP$\Rightarrow$ FLGV

Construct $\boldsymbol{A}' = \begin{pmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \end{pmatrix}, \boldsymbol{b}' = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{pmatrix}$. Use algorithm for FLLCP on $\boldsymbol{A}'$ and $\boldsymbol{b}'$. Then we have exact one of the following:

1. $\exists \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0, \boldsymbol{A}'\boldsymbol{x} \leq \boldsymbol{b}'$.
   So we have $\begin{cases} \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ -\boldsymbol{A}\boldsymbol{x} \leq -\boldsymbol{b} \end{cases}$, thus $\boldsymbol{x} \geq 0, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

2. $\exists \boldsymbol{y}' \in \mathbb{R}^{2m} : \boldsymbol{y}' \geq 0, \boldsymbol{A}'^T \boldsymbol{y}' \geq 0, \boldsymbol{b}^T \boldsymbol{y}' < 0$.
   Let $\boldsymbol{y}' = \begin{pmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \end{pmatrix}$, where $\boldsymbol{y_1} \in \mathbb{R}^m, \boldsymbol{y_2} \in \mathbb{R}^m$.
   So we have $\boldsymbol{A}'^T \boldsymbol{y}' = \boldsymbol{A}^T \boldsymbol{y_1} - \boldsymbol{A}^T \boldsymbol{y_2} = \boldsymbol{A}^T(\boldsymbol{y_1} - \boldsymbol{y_2}) \geq 0$, $\boldsymbol{b}'^T \boldsymbol{y}' = \boldsymbol{b}^T \boldsymbol{y_1} - \boldsymbol{b}^T \boldsymbol{y_2} = \boldsymbol{b}^T(\boldsymbol{y_1} - \boldsymbol{y_2}) < 0$.
   Let $\boldsymbol{y} \in \mathbb{R}^m, \boldsymbol{y} = \boldsymbol{y_1} - \boldsymbol{y_2}$, thus $\boldsymbol{A}^T \boldsymbol{y} \geq 0, \boldsymbol{b}^T \boldsymbol{y} < 0$

FLEV$\Rightarrow$ FLLPV

Construct $\boldsymbol{A}' = \begin{pmatrix} \boldsymbol{A} \\ -\boldsymbol{I} \end{pmatrix}$ where $\boldsymbol{I}$ is identity matrix, $\boldsymbol{b}' = \begin{pmatrix} \boldsymbol{b} \\ 0 \end{pmatrix}$. Use algorithm for FLEV on $\boldsymbol{A}'$ and $\boldsymbol{b}'$. Then we have exact one of the following:

1. $\exists x \in \mathbb{R}^n : A'x \le b'$.

   So we have $\begin{cases} Ax \le b \\ -Ix \le 0 \end{cases}$, thus $x \ge 0, Ax \le b$

2. $\exists y' \in \mathbb{R}^{n+m} : y' \ge 0, A'^T y' = 0, b'^T y' < 0$.

   Let $y' = \begin{pmatrix} y \\ y'' \end{pmatrix}, y \in \mathbb{R}^m, y'' \in \mathbb{R}^n$.

   So we have $A'^T y' = A^T y - I y'' = 0$, $b'^T y' = b^T y < 0$, thus $y \ge 0, A^T y \ge 0, b^T y < 0$

FLLPV$\Rightarrow$ FLEV

Construct $A' = \begin{pmatrix} A & -A \end{pmatrix}$. Use algorithm for FLLCP on A' and b. Then we have exact one of the following:

1. $\exists x' \in \mathbb{R}^{2n} : x' \ge 0, A'x' \le b$.

   Let $x' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^n$.

   So we have $A'x' = Ax_1 - Ax_2 = A(x_1 - x_2) \le b$.

   Let $x \in \mathbb{R}^n, x = x_1 - x_2$, thus $Ax \le b$

2. $\exists y \in \mathbb{R}^m : y \ge 0, A'^T y \ge 0, b^T y < 0$.

   So we have $\begin{cases} A^T y \ge 0 \\ -A^T y \ge 0 \end{cases}$, thus $A^T y = 0, b^T y < 0$