CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2016

- Handed out: Wednesday, 2016-12-07
- Due: Sunday, 2016-12-11
- Feedback: Tuesday, 2016-12-13
- Revision due: Friday, 2016-12-16

5 Linear Programs and Farkas Lemma

5.1 Basic Solutions of a Linear Program

Let P be the following linear program.

$$P: \quad \begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \ \leq \mathbf{b} \end{array}$$

where we translated the constraint $\mathbf{x} \geq 0$ into n constraints $-x_i \leq 0$ and integrated them into A, so the n last rows of A form the negative identity matrix $-I_n$. We introduce some notation: $\mathbf{a}_i$ is the i^{th} row of i; for $I \subseteq [m+n]$ let A_I be the matrix consisting of the rows $\mathbf{a}_i$ for $i \in I$.

Definition 1. *For $\mathbf{x} \in \mathbb{R}^n$ let $I(\mathbf{x}) := \{i \in [m+n] \mid \mathbf{a}_i\mathbf{x} = b_i\}$ be the set of indices of the constraints that are "tight", i.e., satisfied with equality. We call $\mathbf{x} \in \mathbb{R}^n$ a* basic point *if $\operatorname{rank}(A_{I(x)}) = n$. If $\mathbf{x}$ is a basic point and feasible, we call it a* basic feasible solution *or simply a* basic solution.

We can define the same concept for minimization programs. Let $G = (V, E)$ be a bipartite graph and consider the Vertex Cover Linear Program:

$$\text{VCLP}: \quad \begin{array}{lrl} \text{minimize} & \sum_{u \in V} y_u & \\ \text{subject to} & y_u + y_v & \geq 1 \quad \forall \{u, v\} \in E \\ & \mathbf{y} & \geq \mathbf{0} \end{array}$$

We say a set $C \subseteq V$ is a minimal vertex cover of $G = (V, E)$ if (1) it is a vertex cover and (2) it is minimal, i.e., for every $u \in C$ the set $C \setminus \{u\}$ is not a vertex cover anymore.

Exercise 2. Show that $\mathbf{y} \in \mathbb{R}^V$ is a basic solution of VCLP if and only if (1) $y_u \in \{0, 1\}$ for all $u \in V$ and (2) the set $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover.

Hint. Suppose $e_1, \ldots, e_k$ form a cycle in G. Note that every edge corresponds to a constraint of the VCLP, and thus this cycle corresponds to a submatrix A_I with $|I| = k$. Show that the k rows of A_I are linearly dependent.

Hint. Suppose C is a minimal vertex cover. Let F be the set of "tight" edges, i.e., the edges $e \in E$ incident to exactly one $u \in C$. What does minimality of C say about the relation between C and F? Does this help you to show that the set of tight constraints of VCLP has rank n?

Hint. Conversely, suppose $\mathbf{y}$ is a basic solution. Look at the vertices u with $y_0 = 0$ and the "tight edges", those $e = \{u, v\}$ for which $y_u + y_v = 1$.

5.2 Different Versions of Farkas Lemma

In the following three lemmas, let A be a matrix of width n and height m. Let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Let $\mathbb{R}_{\geq 0}$ denote the set of all non-negative real numbers.

Lemma 3 (Farkas Lemma, Geometric Version)**.** *Exactly one of the following holds:*

$$\exists \mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} \geq \mathbf{0}, \ A\mathbf{x} = \mathbf{b}, \tag{FLGV-1}$$

$$\exists \mathbf{y} \in \mathbf{R}^m : \ A^T \mathbf{y} \geq \mathbf{0}, \ \mathbf{b}^T \mathbf{y} < 0 \ . \tag{FLGV-2}$$

Lemma 4 (Farkas Lemma, Linear Programming Version)**.**

$$\exists \mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} \geq \mathbf{0}, \ A\mathbf{x} \leq \mathbf{b} \tag{FLLPV-1}$$

$$\exists \mathbf{y} \in \mathbb{R}^m : \ \mathbf{y} \geq \mathbf{0}, \ A^T\mathbf{y} \geq \mathbf{0}, \mathbf{b}^T\mathbf{y} < 0 \ . \tag{FLLPV-2}$$

Lemma 5 (Farkas Lemma, Elimination Version (Version we Proved in Class))**.**

$$\exists \mathbf{x} \in \mathbb{R}^n : \ A\mathbf{x} \leq \mathbf{b} \tag{FLEV-1}$$

$$\exists \mathbf{y} \in \mathbb{R}^m : \ \mathbf{y} \geq \mathbf{0}, \ A^T\mathbf{y} = \mathbf{0}, \ \mathbf{b}^T\mathbf{y} < 0 \ . \tag{FLEV-2}$$

Exercise 6. For each version (*) of Farkas Lemma, prove the "easy" direction, i.e., prove that (*-2) implies that (*-1) does not hold.

For the next exercise, we consider the algorithmic problem posed by Farkas Lemma. We say an algorithm A *solves Farkas Lemma Version X (FLX)* if for every $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ it either returns some $\mathbf{x} \in \mathbb{R}$ satisfying (FLX-1) or some $\mathbf{y} \in \mathbb{R}$ satisfying (FLX-2). Here FLX can stand for FLGV, FLLPV, or FLEV.

Exercise 7. Prove that all three lemmas are *algorithmically equivalent*. That is, if you have an algorithm A for some version, you can use this as a subroutine to obtain algorithms A' and A'' solving the other two versions. The running time of A' and A'' should not be much larger than that of A.

Hint. Review your notes about dualizing linear programs with $=$ and unbounded variables $y \in \mathbb{R}$. From then on, it will not be too difficult.