CS217 - Algorithm Design and Analyis
Homework Assignment 3

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7^{th} Nov, 2016

3 Minimum Spanning Trees

Exercise 3.1.

Lemma 1 (**Cut Lemma**). *Let $X \subseteq E$ be a good set. If (V, X) is not a spanning tree, then (V, X) consists of two or more connected components. Let $V = S \cup \overline{S}$ be a cut of X. That is, no edge of X goes from S to $\overline{S}$. Let $e \in E$ be an edge of minimum cost connecting two connecting components of (V, X). Then $X \cup \{e\}$ is good, too.*

Proof. Suppose the $X \cup \{e\}$ is not a good set. Then all of the minimum spanning trees exclude the minimum edge which connects the two connecting components of (V, X). Denote these two connecting components as C_1 and C_2. According to the property of spanning tree, for each minimum spanning tree T_m, there is exactly one edge connecting C_1 and C_2 which we denote as $e_t \in E(T_m)$. Substitute e_t to e_m and obvirusly the new spanning tree is still a spanning tree T'_m. And using the following argumentation, we get a smaller spanning tree.

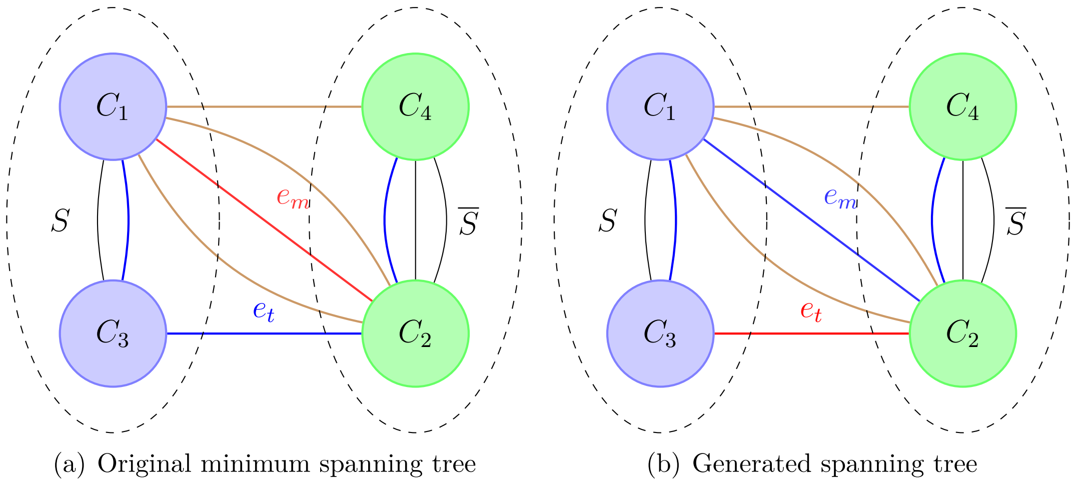


Figure 1: The illustration of the proof

The "costs" of the new spanning tree T'_m is:

$$c(T'_m) = \sum_{e \in E(T')} c(e) = c(T_m) - c(e_t) + c(e_m)$$

Using the condition e_m is the minimum edge connecting C_1 and C_2, we can easily know that:

$$c(e_m) \le c(e_t)$$

Then we obtain a more optimal spanning tree(or we obtain another minimum spanning tree containing the edge e_m) which leads the contradictory according to the inequality below:

$$c(T'_m) = c(T_m) - c(e_t) + c(e_m) \le c(T_m)$$

Hence, $X \cup \{e\}$ is good. □

Lemma 2 (**The Inverse of Cut Lemma**). *If X is good, $e \notin X$, and $X \cup \{e\}$ is good, then there is a cut S, $V \backslash S$ such that the following two holds:*

1. *no edge from X crosses this cut;*
2. *e is minimum weight edge of G crossing the cut.*

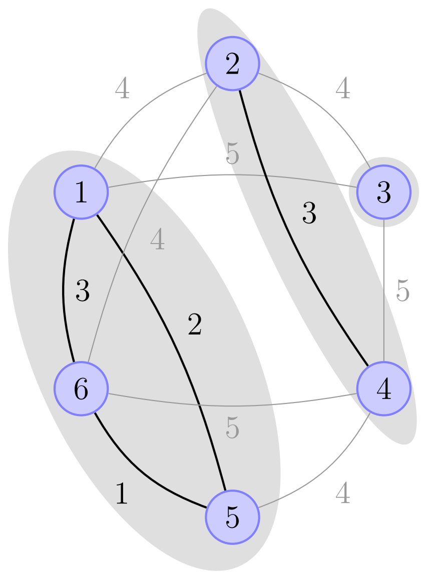
Proof. Because $X \cup \{e\}$ is good, there exists a minimum spanning tree T of G such that $X \cup \{e\} \subseteq E(T)$. In this spanning tree T, $\{e\}$ connects two components, denoted by $S, V \setminus S$.
Consider the cut of G: $S, V \setminus S$.

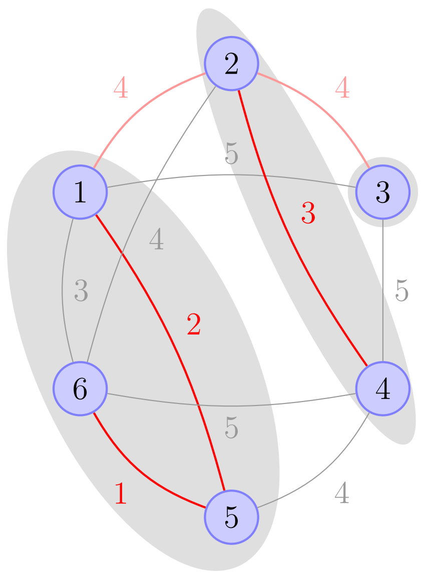
(i) Obviously, for each edge $e(u,v)$ in X, either $u, v \in S$ holds or $u, v \in V \setminus S$ holds; that is to say it can't crosss the cut.

(ii) If e is't a minimum weight edge of G crossing this cut; that is to say, there exist a smaller edge crossing G, denoted by e'.
Then $T \setminus \{e\} \cup \{e'\}$ forms a smaller spanning tree, which contradicts the condition that T is the minimum spanning tree.
So e is a minimum weight edge of G crossing this cut.

□

Exercise 3.4. The example is illustrated below, in which we can see that no matter which minimum spanning trees we choose, the connected components is indentical.


(a) The original graph and its connect components only considering the edges whose weight is less than 4.


(b) A possible minimum spanning tree and its connect components only considering the edges whose weight is less than 4.

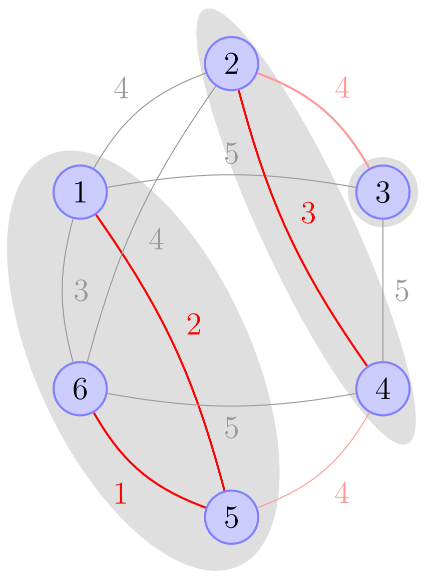

(c) Another possible minimum spanning tree and its connect components only considering the edges whose weight is less than 4.

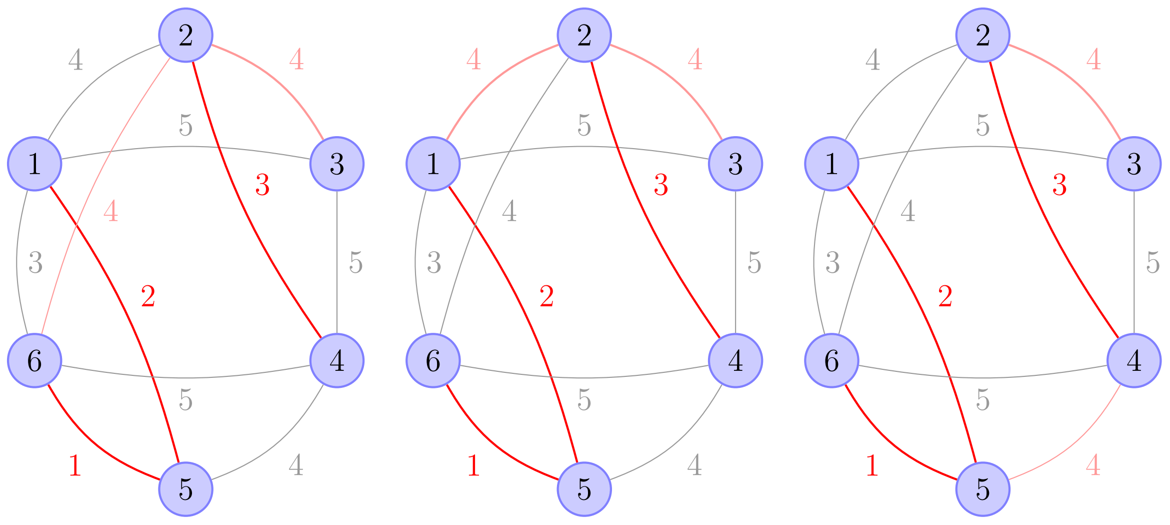
Figure 2: The illustration of the proof

Consider T itself is a subgraph of G, so the only case is: there exists $u, v \in V$ such that u, v are not connected in T_c, but are connected in G_c. Because the u, v are not connected in T_c, so there must be a edge e_1 which $w(e_1) > c$ on the path from u to v in T. This edge e_1 will link two components of T together, let's call them T_1 and T_2.

Proof. There must exists a edge $e_2 = (x, y)$ in G_c which x is belong to T_1 and y is belong to T_2. Consider u, v are connected in G_c, so there must exist a edge which can combine these two components together, otherwise u, v will in two different components, and be isolated.

So, after adding this edge e_2 into T, there must form a circle in T. According the Cut Lemma, this edges can replace e_1 (for $w(e_2) \le c < w(e_1)$), so T is not the Minimum Spanning Tree of G, which is a contradictory. so the Lemma 3 is correct. □

Exercise 3.7. The example is illustrated below, in which we can see that no matter which minimum spanning trees we choose, the m_3 of each minimum spanning trees is indentical. The edges colored red are the minimum spanning tree.



(a) A possible minimum spanning tree (b) Another possible version of the minimum spanning tree (c) The remained possible minimum spanning tree

Figure 3: Three possible minimum spanning trees and its m_3

The proof is showed below:

Proof. Because T and T' are two minimum spanning trees of G, according to Lemma 3, T_c and T'_c and G_c have exactly the same components. Consider the number of components equals to $V - m_c(T)$, which V is the total number of vertexs in G. if $m_c(T') \neq m_c(T)$, the number of components is not equal too, which is a contradictory. so $m_c(T')$ must equal to $m_c(T)$. □

Exercise 3.8. Suppose that there are two different minimum spanning tree, called T, T'.

We can find the edge which belongs to only one of the minimum spanning tree and its weight is minimal. Because no two edges of G have the same weight, the above edge is unique, denoted as e. (Without loss of generality, let e belong to T)

By Lemma 3, we known that

$$m_{c_e}\{T\} = m_{c_e}\{T'\} \text{ and } m_{c_e-1}\{T\} = m_{c_e-1}\{T'\}$$

So

$$m_{c_e}\{T\} - m_{c_e-1}\{T\} = m_{c_e}\{T'\} - m_{c_e-1}\{T'\}$$

However, e belongs to only T, in other word

$$m_{c_e}\{T\} - m_{c_e-1}\{T\} = 1 \text{ and } m_{c_e}\{T'\} - m_{c_e-1}\{T'\} = 0$$

which contradicts the equation above.

So, the hypothesis is wrong; that is to say G has only one minimum spanning tree.

Exercise 3.10. The two parts of the graph are independent, so we only need to caltulate the minimal spanning tree of the two parts respectively.

- Left part:

  If we don't choose any of the parallel edges(containing three edges), we need to choose the other two edges.(1 choice).

  If we choose one of the parallel edges(3 choices), we only need to choose one edge from the other two edges(2 choices).

  In summary, we have $1 + 3 \times 2 = 7$ minimal spanning trees in the left part.

- Right part:

  Using the similar method, the right part have $1 + 2 \times 3 = 7$ minimal spanning trees in the right part.

- Combining two parts:

  Using Multiplication rule, we can get the total amount of the minimal spanning forests, namely $7 \times 7 = 49$

We also justify the answer by programming(enumerating every set of the edges, and check if it can form a minimal spanning forest):

```python
u = [1, 1, 2, 2, 2, 4, 4, 5, 6, 4] # Edge set
v = [2, 3, 3, 3, 3, 5, 5, 6, 7, 7]

def getfa(x) :
    if (f[x] == x) :
        return x
    else:
        f[x] = getfa(f[x])
        return f[x]

if __name__ == '__main__' :
    answer = 0
    for mask in range(1, 1 << 10) : # Enumerate every possible edge sets.
        flag = 1
        number = 0
        f = [i for i in range(8)]
        for i in range(0, 10) :
            if (((mask >> i) & 1) == 1) :
                number = number + 1
                if (getfa(u[i]) == getfa(v[i])) :
                    flag = 0
                else:
                    f[getfa(u[i])] = getfa(v[i])
        if (number == 5 and flag) :
            answer = answer + 1
    print answer
```

Exercise 3.11. Similar like Kruskal, we sort all the edges by their weight $w(e)$, then we process all the edges with the same weight in increasing order. We repeat the following works until all the edges have been processed.

- Define the smallest unprocessed edge weight as c. Label all the edges with $w(e) = c$ as processed.
- Use the given algorithm to caculate the number of spanning forests N_c for a subgraph $g = (V, \{e \in E, w(e) = c\})$.
- According to Lemma.3, the connected components of a subgraph $G_c = (V, e \in E, w(e) \leq c)$ are the same in spanning tree T_c. We can arbitrarily pick a spanning forest of g and combine all the nodes in the same connected component into one node.
  More specificly, we define a function:
  $$f(u) = u'$$
  where u' is the node which has the smallest index in connected component containing u.
  Then we modify G into G':
  $$\begin{aligned} G' = (&\{v \in V, f(v) = v\}, \\ &\{e' = (u', v', w') | e = (u, v, w) \in E, w' = w, u' = f(u), v' = f(v), u' \neq v'\}) \end{aligned}$$

We call a combination of the spanning forests we picked in every step a solution. Finally, we get the number of solutions, as well as the number of minimum spanning trees(MST) of a weighted tree:

$$N = \prod N_c$$

Now we are going to show the correctness of the equality between the number of solutions and the number of MST.

Since the algorithm without counting works the same as Kruskal, every solution will be an MST.

Also, it's easy to see that the edges never disappear or created, so there is a bijection between the edges in the graph we compressed step by step and the edges in the orignal graph all the time. Then, for the edges with the same weight c in a specific MST, we can find the step when it would be processed. Since the spanning forest we picked is arbitrary, we can choose the spanning forest which has excatly these edges we want. According to Lemma.3, those edges will always form a spanning forest with the same connected components, despite the choose of the MST. So every spanning tree is actually the same as one of the solutions.

Finally, we get a bijection between N solutions and all of the MST. Then the equality between the number of them holds.