Algorithm Design and Analysis
Homework Assignment 1

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1 Recursion and Dynamic Programming

Exercise 1.1.

1. Proof by introduction

   Proof. We can easily prove the lemma using induction:

   (a) Basic step
   When $n = 2^1$, the lemma obvirusly holds because

   $$\binom{2}{1} = 0$$

   (b) Induction step
   Assume that the lemma holds where n equals to 2^d, namely

   $$\binom{n}{k} = 0$$

   when $1 \le k \le n - 1$.
   Then we will show that the lemma holds when n equals to 2^{d+1}.
   According to the hints, on the one hand, the number of paths to the n-th line and the k-th grid equals to

   $$\binom{n}{k}$$

   On the other hand, we can also calculate the number by Multiplication Principle and Addition Principle, namely

   $$\binom{n}{k} = \sum_{i=0}^{n/2} \binom{n/2}{i} * \binom{n/2}{k-i} \tag{1}$$

   Then our proof divides into three cases:

- When $1 \leq k \leq n/2 - 1$, it is obvirusly that

$$\binom{n/2}{k} = 0$$

and

$$\binom{n/2}{0} = \binom{n/2}{n/2} = 1$$

According to the equation (1), we have

$$\binom{n}{k} = \binom{n/2}{k-0} + \binom{n/2}{k-n/2}$$

By assumption, when $1 \leq k < n/2$, we have

$$\binom{n/2}{k} = 0$$

Notice that $k - n/2 < n/2$, so we have

$$\binom{n/2}{k-n/2} = 0$$

Therefore

$$\binom{n}{k} = 0$$

- When $k = n/2$, it is obvirusly that

$$\binom{n/2}{k} = \binom{n/2}{k-n/2} = 1$$

Therefore

$$\binom{n}{k} = (1+1) = 2 \equiv 0 \pmod 2$$

- When $n/2 \leq k < n$, it is easy to see that

$$\binom{n/2}{k} = 0$$

And by assumption

$$\binom{n/2}{k-n/2} = 0$$

So that we have

$$\binom{n}{k} = 0$$

(c) Conclusion By induction, we have

$$\binom{n}{k} \equiv 0 \pmod 2$$

where $1 \leq k \leq n-1$ and $n = 2^d$ $(d \geq 1)$.

$\square$

2. Direct Proof

Proof. $\forall n = 2^d,\ 1 \le k \le (n-1)$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = 2^p * q,\ p = \sum_{i=1}^{d} \lfloor \frac{n}{2^i} \rfloor - \sum_{i=1}^{d} \lfloor \frac{k}{2^i} \rfloor - \sum_{i=1}^{d} \lfloor \frac{n-k}{2^i} \rfloor$$

if $i = d$ then $\lfloor \frac{n}{2^i} \rfloor = 1,\ \lfloor \frac{k}{2^i} \rfloor = \lfloor \frac{n-k}{2^i} \rfloor = 0$

$$p = 1 - 0 - 0 + \sum_{i=1}^{d-1} (\lfloor \frac{n}{2^i} \rfloor - \lfloor \frac{k}{2^i} \rfloor - \lfloor \frac{n-k}{2^i} \rfloor) \ge 1 + \sum_{i=1}^{d-1} (\frac{n}{2^i} - \frac{k}{2^i} - \frac{n-k}{2^i}) = 1$$

so $2 \mid \binom{n}{k}$ that is $\binom{n}{k}$ $mod\ 2 = 0$ $\square$

Exercise 1.2. Exercise 1.3. Exercise 1.4.