CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2016

- Handed out: Friday, 2016-10-14
- Due: Wednesday, 2016-10-19
- Feedback: Friday, 2016-10-21
- Revision due: Monday, 2016-10-24

2 Sorting Algorithms

2.1 largest, smallest, and second largest

Exercise 1. Let A be an array of size n, where n is even. Describe how to find both the minimum and the maximum with at most $\frac{3}{2}n - 2$ comparisons. Make sure your solution is *simple*, in describe it in a clear and succinct way!

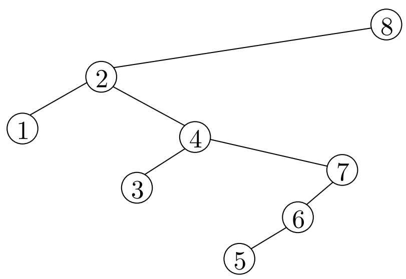
Exercise 2. Given an array A of size $n = 2^k$, find the second largest element element with at most $n + \log_2(n)$ comparisons. Again, your solution should be *simple*, and you should explain it in a clear and succinct way!

Exercise 3. [max needs at least $n - 1$ comparisons] Suppose I propose an algorithm for finding the maximum of A using $n - 2$ comparisons. Show that I am wrong, i.e., my algorithm must fail on certain inputs A.

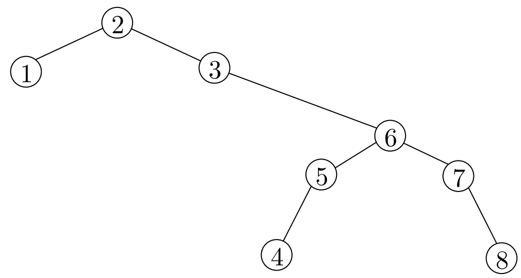
Remark. The only way my algorithm can access the elements is to take two elements and *compare* them pairwise. That is, the algorithm is not allowed to check whether $A[i] > n/2$ or something similar. Think of the elements of A as abstract objects, not necessarily numbers!

Exercise 4. In the lecture we proved that for $n = 2^d$, the number of comparisons performed by `mergesort` is at most $n \log(n) - n + 1$. Find a method that constructs, for every $n = 2^d$, an array A of n integers on which `mergesort` reaches this bound, i.e., performs exactly $n \log(n) - n + 1$ when sorting A. For $n = 8$ we have seen $[0, 4, 2, 6, 1, 5, 3, 7]$ in class.

Recall the quicksort tree defined in the lecture.



quicksort tree of $[8, 2, 4, 1, 7, 6, 5, 3]$



quicksort tree of $[2, 3, 6, 1, 7, 5, 8, 4]$

We denote a specific list (ordering) by π and the tree by $T(\pi)$. $A_{i,j}$ is an indicator variable which is 1 if i is an ancestor of j in the tree $T(\pi)$, and 0 otherwise. In the lecture, we have derived:

$$\begin{aligned} \mathbb{E}[A_{i,j}] &= \frac{1}{|i-j|+1} \\ \text{total number of comparisons } &= \sum_{i \neq j} A_{i,j} \ . \end{aligned}$$

From this we have deduced that the expected number of comparisons of quicksort on a random input is at most $2nH_n$, where $H_n = \sum_{i=1}^{n} \frac{1}{i}$ is called the n^{th} harmonic number.

Exercise 5. Derive a precise formula for the expected number of comparisons made by quicksort. Your formula should be in "closed form", i.e., not contain any $\mathbb{E}$, $\prod$ or $\sum$, but may contain H_n.

2.2 Quickselect

Remember the recursive algorithm QuickSelect from the lecture. I write it here in pseudocode. In analogy to quicksort we define QuickSelect deterministically and assume that the input array is random, or has been randomly

Algorithm 1 Select the k^{th} smallest element from a list A

```
procedure QuickSelect(A, k)
    if |A| = 1 then
        return A[1]
    else:
        p := A[1]
        B := [x ∈ A | x < p]
        C := [x ∈ A | x > p]
        if |B| = k − 1 then
            return p
        else if |B| ≥ k then
            return QuickSelect(B, k)
        else
            Return QuickSelect(C, k − |B| − 1)
        end if
    end if
end procedure
```

shuffled before QuickSelect is called. We assume that A consists of distinct elements and $1 \leq k \leq |A|$.

Let C be the number of comparison made by QuickSelect. In the lecture we proved that $\mathbb{E}[C] \leq O(n)$ when we run it on a random input.

Exercise 6. Explain how QuickSelect can be viewed as a "partial execution" of quicksort with the random pivot selection rule. Draw an example quicksort tree and show which part of this tree is visited by QuickSelect.

Let $B_{i,j,k}$ be an indicator variable which is 1 if i is a common ancestor of j and k in the quicksort tree. That is, if both j and k appear in the subtree of $T(\pi)$ rooted at i.

Exercise 7. What is $\mathbb{E}[B_{i,j,k}]$? Give a succinct formula for this.

Exercise 8. Let $C(\pi)$ be the number of comparisons made by QuickSelect when given π as input. Design a formula for $C(\pi)$ with the help of the indicator variables $A_{i,j}$ and $B_{i,j,k}$ (analogous to the formula $\sum_{i \neq j} A_{i,j}$ for the number of comparisons made by quicksort).

Exercise 9. Derive a precise, closed formula for the expected number of comparisons made by QuickSelect when run on a random input. Again, “closed formula” means no $\mathbb{E}, \prod, \sum$. The harmonic number H_n, however, is allowed.