CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2016

- Handed out: Wednesday, 2016-11-09
- Due: Tuesday, 2016-11-15
- Feedback: Thursday, 2016-11-17
- Revision due: Sunday, 2016-11-20

4 Network Flows, Matchings, and Paths

4.1 Flows

Exercise 1. Let $G = (V, E, c)$ be a flow network. Prove that flow is "transitive" in the following sense: If there is a flow from s to r of value k, and a flow from r to t of value k, then there is a flow from s to t of value k.

Hint. The solution is extremely short. If you are trying something that needs more than 3 lines to write, you are on the wrong path.

4.2 Matchings

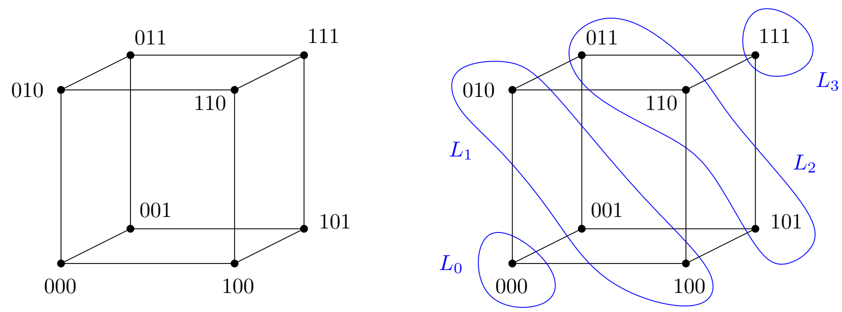
Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subseteq E$ of edges such that every $v \in V$ is incident to at most one edge of M.

Exercise 2. [Matchings in Regular Bipartite Graphs] Let $G = (V, E)$ be a bipartite graph and $V = V_1 \cup V_2$ be a bipartition. Suppose there are numbers d_1, d_2 such that $\deg_G(v) = d_1$ for all $v \in V_1$ and $\deg_G(v) = d_2$ for all $v \in V_2$. Show that G contains a matching of size $\min(|V_1|, |V_2|)$.

Let $V = \{0,1\}^n$. The n-dimensional Hamming cube H_n is the graph (V, E) where $\{u, v\} \in E$ if u, v differ in exactly one coordinate. Define the i^{th} level of H_n as

$$L_i := \{u \in V \mid \|u\|_1 = i\} \ ,$$

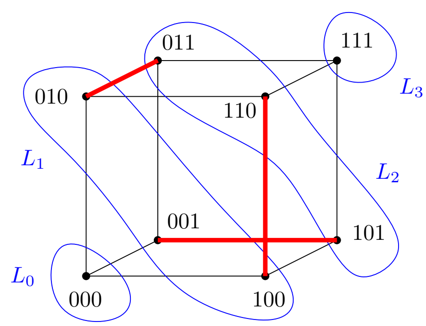
i.e., those vertices u having exactly i coordinates which are 1.



The 3-dimensional Hamming cube and the four sets L_0, L_1, L_2, L_3.

Exercise 3. [Matchings in H_n] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in H_n.

Show that for $i \leq n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.



A matching of size 3 between L_1 and L_2.

4.3 Vertex Disjoint Paths

Suppose we have a directed graph $G = (V, E)$ but instead of *edge capacities* we have *vertex capacities* $c : V \to \mathbb{R}$. Now a flow f should observe the *vertex* *capacity constraints*, i.e., the outflow from a vertex u should not exceed $c(u)$:

$$\forall u \in V : \sum_{v \in V, f(u,v) > 0} f(u,v) \leq c(u) \ .$$

Exercise 4. Consider networks with vertex capacities.

1. Show how to model networks with vertex capacities by networks with edge capacities. More precisely, show how to transform $G = (V, E, c)$ with $c : V \to \mathbb{R}^+$ into a network $G' = (V', E', c')$ with $c' : E' \to \mathbb{R}^+$ such that every s-t-flow f in G that respects the vertex capacities corresponds to an s-t-flow f' (of same value) in G' that respects edge capacities, and vice versa.

2. Draw a picture illustrating your solution.

3. Show that there is a polynomial time algorithm solving the following problem: Given a directed graph $G = (V, E)$ and two vertices $s, t \in V$. Are there k paths $p_1, \ldots, p_k$, each from s to t, such that the paths are *internally vertex disjoint*? Here, internally vertex disjoint means that for $i \neq j$ the paths p_i, p_j share no vertices besides s and t.

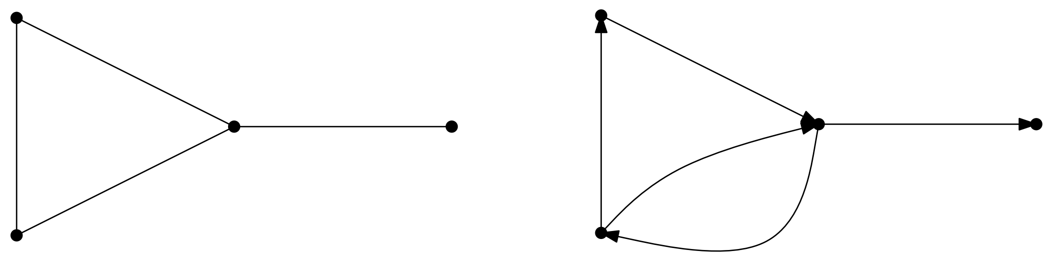
Exercise 5. Let H_n be the n-dimensional Hamming cube. For $i < n/2$ consider L_i and L_{n-i}. Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = L_{n-i}$, so the L_i and L_{n-i} have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \ldots, p_{\binom{n}{i}}$ in H_n such that (i) each p_i starts in L_i and ends in L_{n-i}; (ii) two different paths p_i, p_j do not share any vertices.

Hint. Model this problem as a network flow with vertex capacities. Then define a (non-integral) flow of value $\binom{n}{i}$.

4.4 Graph Orientations With Degree Constraints

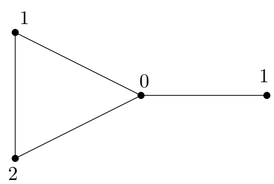
Let $G = (V, E)$ be an undirected graph (i.e., $E \subseteq \binom{V}{2}$). A directed graph $\vec{G} = (V, F)$ (i.e. with $F \subseteq V \times V$) is an *orientation* of G if

$$\forall u, v \in V : \quad \{u, v\} \in E \text{ if and only if } (u, v) \text{ or } (v, u) \in F \ .$$

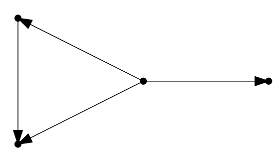


A graph G and an orientation $\vec{G}$.

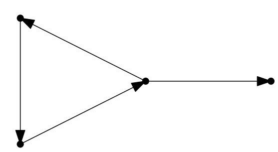
For a directed graph $H = (V, F)$ let $\deg_H^{\text{in}}(v)$ denote the "in-degree", i.e., $|\{u \in V | (u, v) \in F\}|$. Let G be an undirected graph and $c : V \to \mathbb{N}_0$. We say H is a "feasible orientation of G with respect to the degree constraints c" if (1) H is an orientation of G and (2) $\deg_H^{\text{in}}(v) \leq c(v)$ for all $v \in V$.



A graph G with degree constraints.



An feasible orientation.



An infeasible orientation: The degree constraint of the vertex in the middle is violated.

Exercise 6. Design a polynomial-time algorithm for solving the following problem: Given an undirected graph $G = (V, E)$ with degree constraints $c : V \to \mathbb{N}_0$, find a feasible orientation H or determine that there is none. In particular:

1. Reduce it to a max flow problem. Prove that a max flow of a certain value can be converted into an orientation (and vice versa).

2. Reduce it to maximum matching. Show how certain matchings can be converted into orientations (and vice versa).

For maximum matching Hall's Theorem states: If there is no matching of size $|U|$, then some set $X \subseteq U$ has $|X| > |\Gamma(X)|$. The set X is a "witness" for the fact that no matching has size U.

The max flow min cut theorem states: If there is no s-t-flow of value k, then there is an s-t-cut $(S, V \setminus S)$ with $c(S, V \setminus S) < k$. Here, the cut $(S, V \setminus S)$ is a "witness" for the non-existence of a flow of value k.

Exercise 7. Prove "something like Hall's Theorem" for the feasible orientation problem. What could be a witness for the non-existence of a feasible orientation? In particular:

1. Give an example graph with degree constraints which has no feasible orientation. Your example should be non-obvious, i.e., it should not be completely clear that it has no feasible orientation. Try to find a concise argument why it does not have one!

2. Try to make a statement like this: "G has no feasible orientation with respect to c if and only if an object X with a certain property P exists". It should hold that we can easily verify whether P holds for a given object X.

3. Prove the statement you made in Point 2.

Exercise 8. n teams have been playing a foosball (table football) tournament the whole night. In a foosball match there is no draw, so exactly one team wins and the other loses. The winning team gets 1 score, the losing team nothing. At 4a.m. the statistics show that team i has s_i scores, and the tournament schedule shows that for each i, j, team i and j still have to play $m_{i,j} \geq 0$ matches. Team 1 consists of Xiaoming and Thomas. Both are super ambitious and care only about coming up as the unique winner team. Xiaoming says: "Damn, it did not go too well for us. No matter what happens from now on, we won't come up as the unique winners, so let's go home already". Thomas replies "No, wait, if we win all $\sum_{i=2}^{n} m_{1,i}$ matches we still have to play, then we are better than all teams." Xiaoming says "Yes but then, who knows how..."

Design a polynomial-time algorithm for the following problem: Given the tournament statistics (i.e., the numbers $s_1, \ldots, s_n$ and the number of remaining matches $m_{i,j}$ between team i and j, $1 \leq i < j \leq n$, determines whether it is possible (if all goes well) that team 1 comes up as the unique winner (i.e., has higher score than anyone else when the tournament ends).

Hint. First, solve the problem for the case $m_{i,j} \in \{0, 1\}$.