CS217 - Algorithm Design and Analyis
Homework Assignment 4

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4 Network Flows, Matchings, and Paths

Exercise 4.1. The problem can be easily proved by Max-flow min-cut theorem.

Proof. If there is a flow from s to r of value k, then the minimum cut between s and r must be greater or equal to k(the maximum flow must be greater or equal to k). Similarly, the minimum cut between r to t must be greater or equal to k. Assume we found a cut between s and t which were less than k, then after we removed the edges in the cut, s and r would still be connected(otherwise we get a cut whose value is less than k). Similary, r and t is still connected. Hence s and t is connected, which contradicts to the assumption that there were a cut from s to t whose value is less than k. Thus the maximum flow from s to t is greater or equal to k, which means there is a flow from s to t of value k. □

Exercise 4.5. We will construct a network graph and use the Max-flow min-cut theorem to show the (i) and (ii).

Proof. Considering a network graph:

1. Connect every nodes in L_k to $L_{k+1}(k+1 \leq n-i)$ with capacity ∞.
2. Connect source to every node in L_i with capacity ∞.
3. Connect every nodes in L_{n-i} to sink with capacity ∞.

Then we empow each node except the source and the sink with vertex capacity 1. We will show that the minimum cut of this network graph must be of value $|L_i| = |L_{n-i}|$.

Assume the number of paths from the source to the sink is p. Then if we remove a node in L_k, according to the symmetry of the graph(each node in L_i plays equal roles in the graph), the number of paths from the source to the sink will decrease by $p/\binom{n}{k}$. If we cut the nodes $v_1, v_2, v_3, ..., v_s$ and $v_j \in L_{b_j}$, then we can decrease the paths by

$$\min\left\{p, \sum_{j=1}^{s} \frac{p}{|L_{b_j}|}\right\} \leq \min\left\{p, \sum_{j=1}^{s} \frac{p}{|L_i|}\right\} = \min\left\{p, \frac{sp}{|L_i|}\right\}$$

In other words, if we cut s nodes, we can at most decrease the paths by $\min\{p, sp/|L_i|\}$. And if $v_j \in L_i$ for every $j \in \{1, 2, ..., s\}$, then the equal sign established. Considering the cut of this graph can be exactly vertex, we can easily know that the minimum cut of this graph is minimum s satisfying

$$\frac{sp}{|L_i|} \geq p$$

Hence the minimum cut of this graph is exactly $|L_i|$.

Hence, we know that the maximum flow of this graph is $|L_i|$, which means that there are $|L_i|$ disjoint paths(the vertex capacity being one meets the the disjoint properties). □