Chapter 11:

General Trigonometric Functions

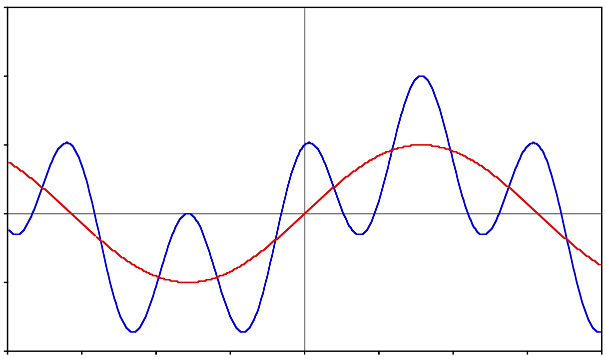
Chapter 11 Overview: Types and Traits of General Trigonometric Functions

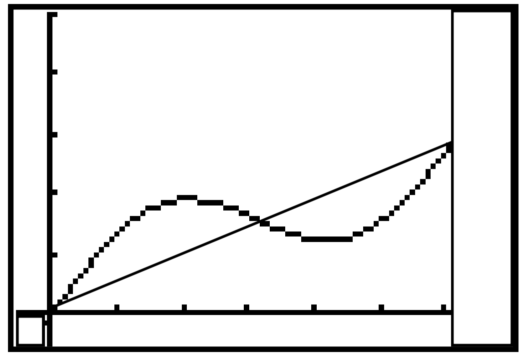
In a previous chapter, the lens for looking at the traits of trigonometric graphs was essentially that of an Algebra course. That is, looking at the transformation of a parent function into the graph of a particular equation via amplitude, period, vertical shift, and horizontal (or phase) shift. The trigonometric functions will also be viewed in terms of the traits that are comparable to those of the functions in previous chapters:

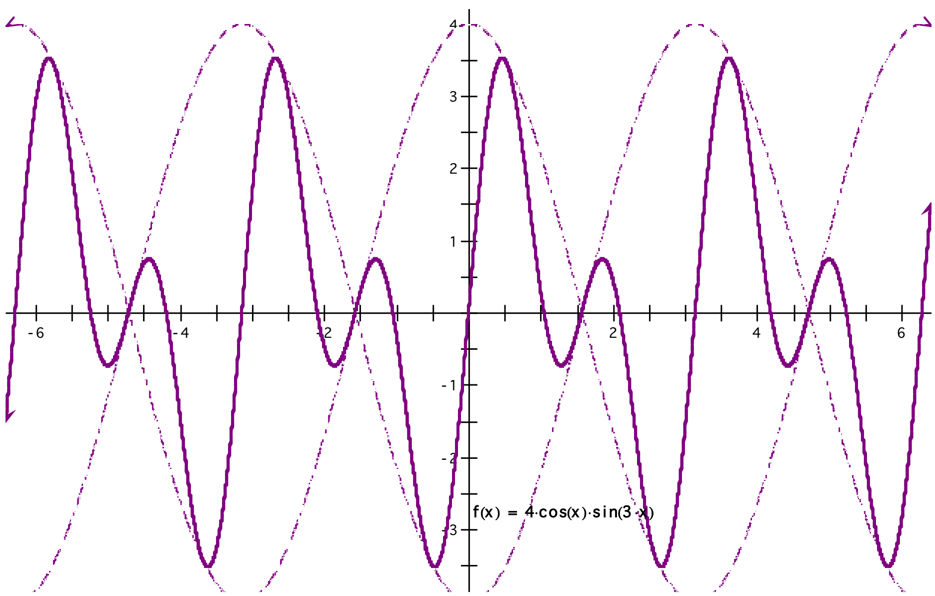
1. Domain
2. Axis Points (instead of zeros)
3. Horizontal/Phase Shift (instead of the y-intercept)
4. Vertical Asymptotes
5. Points of Exclusion
6. Extreme Points
7. Range

Note: Since the trigonometric functions are periodic, it makes no sense to consider end behavior. The curve just continues to repeat. For this reason, most of the problems in this chapter will have a given domain.

This approach is far more effective when there is a variable axis or amplitude. Here are some examples of these curves:







11-1: Trigonometric Derivatives and the Chain Rule

Here are the trigonometric derivative rules:

Trigonometric Derivative Rules*

$$\frac{d}{dx}\left[\sin u\right]=\cos u \bullet D_u \qquad \frac{d}{dx}\left[\csc u\right]=-\csc u\cot u \bullet D_u$$

$$\frac{d}{dx}\left[\cos u\right]=-\sin u \bullet D_u \qquad \frac{d}{dx}\left[\sec u\right]=\sec u\tan u \bullet D_u$$

$$\frac{d}{dx}\left[\tan u\right]=\sec^2 u \bullet D_u \qquad \frac{d}{dx}\left[\cot u\right]=-\csc^2 u \bullet D_u$$

*Note that these formulas are stated using the Chain Rule.

LEARNING OUTCOMES

Find derivatives involving trigonometric functions.
Find the extreme points of trigonometric functions

EX 1 $\frac{d}{dx}\left(\sin x^3\right)$

$$\begin{aligned}\frac{d}{dx}\left(\sin x^3\right)&=\cos x^3\left(3x^2\right)\\&=3x^2\cos x^3\end{aligned}$$

EX 2 $D_x\left(\sin^3 x\right)$

$$D_x\left(\sin^3 x\right)=3\sin^2 x\cos x$$

EX 3 If $y=\sec^5 3x^4$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx}&=5\sec^4 3x^4\left(\sec 3x^4\tan 3x^4\right)\left(12x^3\right)\\&=60x^3\sec^5 3x^4\tan 3x^4\end{aligned}$$

EX 4 $\frac{d}{dx}\left(\tan\sqrt{x}\right)^4$

$$\begin{aligned}\frac{d}{dx}\left(\tan\sqrt{x}\right)^4&=\frac{d}{dx}\left(\tan^4 x^{1/2}\right)\\&=4\tan^3 x^{1/2}\left(\sec^2 x^{1/2}\right)\left(\frac{1}{2}x^{-1/2}\right)\\&=\frac{2\tan^3 x^{1/2}\sec^2 x^{1/2}}{x^{1/2}}\end{aligned}$$

EX 5 Find the extreme values of $y=3+4\cos\left[\frac{\pi}{4}(x-1)\right]$.

$$\begin{aligned}\frac{dy}{dx}=-4\sin\left[\frac{\pi}{4}(x-1)\right]\bullet\frac{\pi}{4}\\-\pi\sin\left[\frac{\pi}{4}(x-1)\right]&=0\\\sin\left[\frac{\pi}{4}(x-1)\right]&=0\\\frac{\pi}{4}(x-1)&=\sin^{-1}0=\begin{cases}0\pm 2\pi n\\\pi\pm 2\pi n\end{cases}\\x-1&=\begin{cases}0\pm 8n\\4\pm 8n\end{cases}\\x&=\begin{cases}1\pm 8n\\5\pm 8n\end{cases}\end{aligned}$$

$\frac{dy}{dx}$ always exists in this problem.

$$x = 1 \pm 8n \rightarrow y = 7$$
$$x = 5 \pm 8n \rightarrow y = -1$$

EX 6 The temperature equation for a certain chemical reaction is $y = 48 + 8\cos\left[\frac{\pi}{12}(t-17)\right]$. How fast is the temperature changing when the temperature was originally 50°?

$$y = 48 + 8\cos\left[\frac{\pi}{12}(t-17)\right] = 50$$
$$t = 11.965$$

$$v(t) = -8\sin\left[\frac{\pi}{12}(t-17)\right] \bullet \frac{\pi}{12} = \frac{-2\pi}{3}\sin\left[\frac{\pi}{12}(t-17)\right]$$
$$v(11.965) = \frac{-2\pi}{3}\sin\left[\frac{\pi}{12}(11.965-17)\right]$$

$$= 2.028\,\frac{\text{deg}}{\text{min}}$$

EX 7 Find the extreme points of $y=\frac{1}{2}x-\cos x$ on $x\in[0,\ 2\pi]$.

i) $$\frac{dy}{dx}=\frac{1}{2}+\sin x$$

$$\frac{1}{2}+\sin x=0$$

$$\sin x=-\frac{1}{2}$$

$$x=\sin^{-1}\left(-\frac{1}{2}\right)=\begin{cases}\frac{7\pi}{6}\pm 2\pi n\\ \frac{11\pi}{6}\pm 2\pi n\end{cases}$$

$$=\frac{7\pi}{6}\ \text{or}\ \frac{11\pi}{6}$$

ii) $\frac{dy}{dx}$ always exists in this problem.

iii) Endpoints: $x=0$ and 2π

Therefore, the extreme points are

$$(0,\ 1),\ \left(\frac{7\pi}{6},\ 2.699\right),\ \left(\frac{11\pi}{6},\ 2.014\right),\ \text{and}\ (2\pi,\ 2.142)$$

11-1 Free Response Homework

Find the derivatives.

1. $\frac{d}{dx}\left(\cos^4 x\right)$

2. $\frac{d}{dx}\left(\cot\sqrt{x}\right)$

3. $\frac{d}{dx}\left(\sin^2 x+\cos^2 x\right)$

4. $\frac{d}{dx}\left(3\csc\sqrt{x}\right)$

5. $\frac{d}{dx}\left(\cos^4 2x\right)$

6. $\frac{d}{dx}\left(-10-3\cos x\right)$

7. $\frac{d}{dx}\left(\sec\frac{x^2}{3}\right)$

8. $\frac{d}{dx}\left(\sqrt{\cot x^3}\right)$

9. $\frac{d}{dx}\left(\csc\sqrt[3]{x^4}\right)$

10. $\frac{d}{dx}\left(1+2\cos 2x\right)^{-4}$

11. $f(x)=\tan x+\sec x$; find the exact value of $f'\left(\frac{\pi}{3}\right)$

12. Prove: a) $\frac{d}{dx}\left[\tan x\right]=\sec^2 x$

    b) $\frac{d}{dx}\left[\csc x\right]=-\csc x\cot x$

Find the exact critical values.

13. $y=\frac{\sqrt{3}}{2}x+\sin x$ on $x\in\left[-\pi,\ \pi\right]$

14. $y=x-\cos x$ on $x\in\left[0,\ 2\pi\right]$

15. $y=\frac{\sqrt{3}}{2}x+\cos x$ on $x\in\left[0,\ 2\pi\right)$

16. $y=2x-\tan x$ on $x\in\left[0,\ \pi\right]$

17. A particle moves in a straight line according to $x(t)=2+\sin^2 t$. Find the $v(t)$ equation.

18. A particle moves in a straight line according to $x(t)=\cos t+\sin t$. Find the times when the particle stops.

19. A weight is attached to a spring and is bouncing up and down. Its height at time t is described by $H(t)=1+4\cos\left(\frac{\pi}{8}(t-5)\right)$. Describe the motion of the weight between 0 and 20 seconds.

<u>11-1 Multiple Choice Homework</u>

1. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

a) 0 b) $\frac{1}{2}$ c) 1 d) 2 e) DNE

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2. For what value(s) of x will the graph of the function $f(x)=\sin\sqrt{B-x^2}$ have a maximum?

a) $\frac{\pi}{2}$ b) $\sqrt{B-\frac{\pi}{2}}$ c) $\sqrt{B-\left(\frac{\pi}{2}\right)^2}$

d) $\pm\sqrt{B-\frac{\pi}{2}}$ e) $\pm\sqrt{B-\left(\frac{\pi}{2}\right)^2}$

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3. If $y = \cos^2 x - \sin^2 x$, then $y' =$

a) -1 b) 0 c) $-2(\cos x + \sin x)$

d) $2(\cos x + \sin x)$ e) $-4(\cos x)(\sin x)$

---

4. If is a vector-valued function defined by $f(t) = (e^{-t},\ \cos t)$, then $f''(t) =$

a) $-e^{-t} + \sin t$ b) $-e^{-t} + \sin t$ c) $(-e^{-t},\ -\sin t)$

d) $(e^{-t},\ \cos t)$ e) $(e^{-t},\ -\cos t)$

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5. Consider the function defined on $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ by $f(x) = \frac{\tan x}{\sin x}$ f or all $x \ne \pi$. If is continuous at $x = \pi$, then $f(\pi) =$

a) 2 b) 1 c) 0 d) –1 e) –2

11-2: The Product and Quotient Rules Revisited

Derivatives of trigonometric functions are not limited to the Chain Rule as in the last section. All the previous rules must still work.

REMEMBER:

The Product Rule: $\frac{d}{dx}(u \bullet v) = u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \bullet \frac{du}{dx} - u \bullet \frac{dv}{dx}}{v^2}$

LEARNING OUTCOME

Find the derivative of a product or quotient of two functions.

EX 1 $\frac{d}{dx}\left(x^2 \sin x\right)$

$$\frac{d}{dx}\left(x^2 \sin x\right) = x^2 \cos x + \sin x(2x)$$

$$= x^2 \cos x + 2x \sin x$$

EX 2 Find the exact critical values of $y=\sin x\cos x$.

$$y=\sin x\cos x$$
$$=\frac{1}{2}\sin 2x$$

$$\frac{dy}{dx}=\frac{1}{2}\cos 2x(2)=0$$
$$\cos 2x=0$$
$$2x=\cos^{-1}0=\pm\frac{\pi}{2}\pm 2\pi n$$
$$x=\pm\frac{\pi}{4}\pm\pi n$$

$\frac{dy}{dx}$ always exists in this problem.

So the critical values are $x=\pm\frac{\pi}{4}\pm\pi n$.

EX 3 $\frac{d}{dx}\left(\frac{\cot 3x}{x^2+1}\right)$

$$\frac{d}{dx}\left(\frac{\cot 3x}{x^2+1}\right)=\frac{\left(x^2+1\right)\left(-\csc^2 3x\right)(3)-\left(\cot 3x\right)2x}{\left(x^2+1\right)^2}$$
$$=\frac{-3x^2\csc^2 3x-3\csc^2 3x-2x\cot 3x}{\left(x^2+1\right)^2}$$

EX 4 $\frac{d}{dx}\left(e^{4x}\cos 3x\right)$

$$\frac{d}{dx}\left(e^{4x}\cos 3x\right)=e^{4x}\left(-\sin 3x\right)(3)+\cos 3x\left(e^{4x}\right)(4)$$
$$=e^{4x}\left(4\cos 3x-3\sin 3x\right)$$

11-2 Free Response Homework

Find the following derivatives.

1. $\frac{d}{dx}\left(x^3 \sec x\right)$

2. $D_x\left(x^2 \csc x\right)$

3. $D_x\left(x^2 \sin x + 2x \cos x\right)$

4. If $y = x \tan^2 x$, find $\frac{dy}{dx}$

5. If $f(x) = 2\sin^2 x \cos^2 x$, find $f'(x)$

6. $\frac{d}{dx}\left(\frac{\tan x + 5}{\sin x}\right)$

7. If $y = \frac{\tan x}{\cos x - 3}$, find $\frac{dy}{dx}$

8. $\frac{d}{dx}\left(\frac{x^2}{\cos x}\right)$

9. $\frac{d}{dx}\left(\frac{\sin x}{1 - \cos x}\right)$

10. $D_x\left(x^3 \sec x + x^2 \tan x\right)$

11. $f(x) = \sec x \tan x$; find $f'\left(\frac{\pi}{4}\right)$

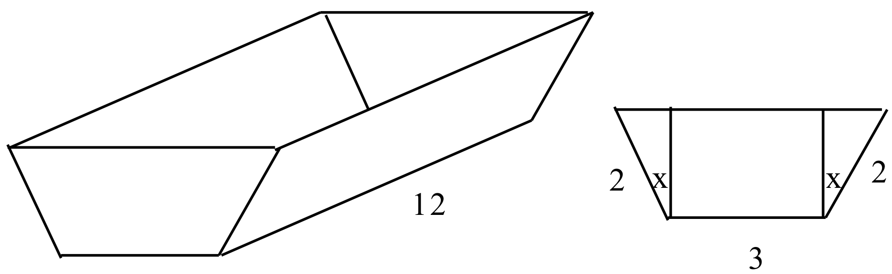
12. $f(x) = \frac{\tan x}{\tan x + 1}$; find $f'\left(\frac{\pi}{4}\right)$

13. $f(x) = x \cos x + x \sin x$; find $f'\left(\frac{\pi}{4}\right)$

14. A particle moves in a straight line according to $x(t) = 4t \bullet \cos\left(\frac{\pi}{2}t\right)$ for $t \geq 0$.

Find the distance from the origin when the particle switches direction the first two times. Use your graphing calculator.

15. A trough with an isosceles trapezoidal cross section (see diagram on next page) has the dimensions below. At what angle x would the trough have maximum volume?



Find a) the velocity vector, b) acceleration vector, and c) the speed when $t=\theta=\pi$ for each of the following parametric equations.

16. $x(t)=\sqrt{t}$, $y(t)=\cos t$

17. $x(t)=5\sin t$, $y(t)=t^2$

18. $x=\sec\theta$, $y=\tan\theta$

19. At time $t\in[0,\ 2\pi]$, the position of a particle moving along a path in the xy-plane is described by $x(t)=e^t\sin t$ and $y(t)=e^t\cos t$. What is its speed at $t=1$?

11-2 Multiple Choice Homework

1. The derivative of the function is given by $f'(x)=x^2\cos(x^2)$. How many maximum and minimum points of have on the open interval $(-2,\ 2)$?

a) 1 b) 2 c) 3 d) 4 e) 5

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2. $\lim_{x\to 0} \frac{\sin x \cos x}{x}$

a) -1 b) 0 c) 1 d) $\frac{\pi}{4}$ e) DNE

---

3. For $x \neq 0$, the slope of the tangent to $y = x\cos x$ equals zero whenever

a) $\tan x = -x$ b) $\tan x = \frac{1}{x}$ c) $\tan x = \frac{1}{x}$

d) $\sin x = x$ e) $\cos x = x$

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4. If $\cos x = e^y$ and $0 < x < \frac{\pi}{2}$, what is $\frac{dy}{dx}$ in terms of x?

a) $-\tan x$ b) $-\cot x$ c) $\cot x$

d) $\tan x$ e) $\csc x$

---

5. Let f and g be differentiable functions with the following properties:

i) $g(x) > 0$ for all x

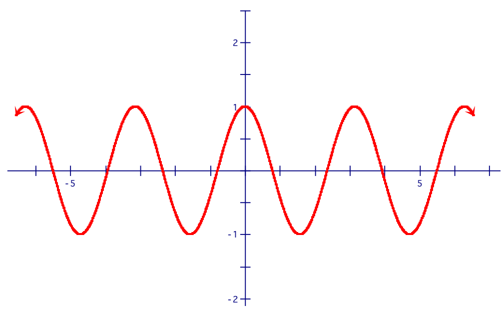
ii) $f(0) = 1$

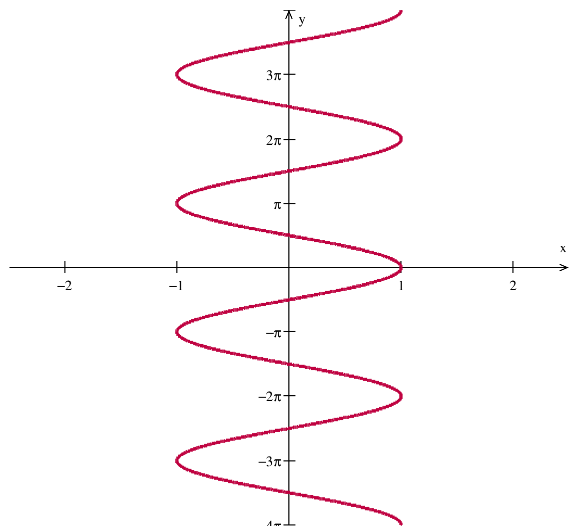
If $h(x) = f(x)g(x)$ and $h'(x) = f(x)g'(x)$, then $f(x) =$

a) $f'(x)$ b) $g(x)$ c) e^x d) 0 e) 1

11-3: Inverse Trigonometric Functions and Their Derivatives

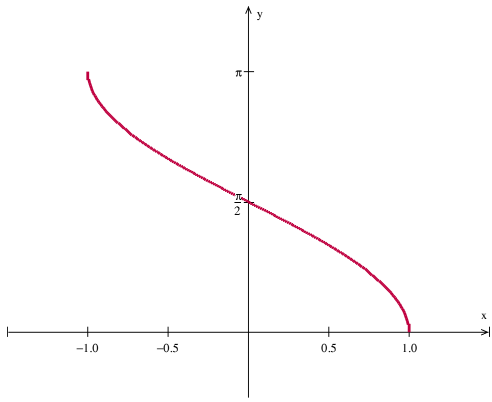
Before the inverse curves are analyzed, their meaning must be considered. The inverse OPERATIONS are those that cancel a given operation. So $y = \sin x$ leads to $x = \sin^{-1} y$. To consider the curve of $y = \sin^{-1} x$, switch the variables. On a graph this means flipping the curve along the diagonal line $y = x$. So,





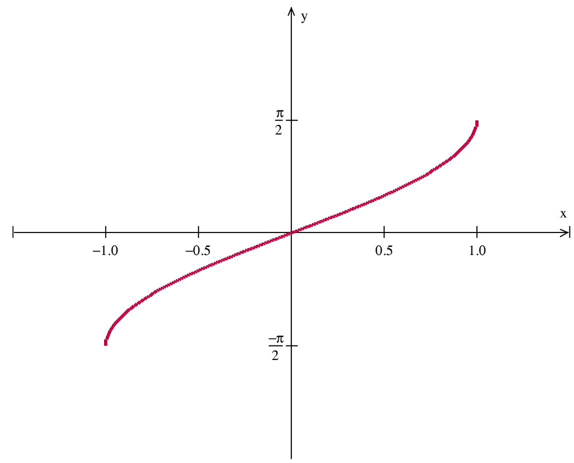
$y = \cos x$ becomes $y = \cos^{-1} x$

$y = \cos^{-1} x$ is not a function; it does not pass the Vertical Line Test. To make this a function, the range must be restricted and capitalize $y = \cos^{-1} x$ to denote the function (vs. the relation). Therefore, when the graphs of the inverse trigonometric functions are considered, there is an implied limit on the ranges, as stated below.



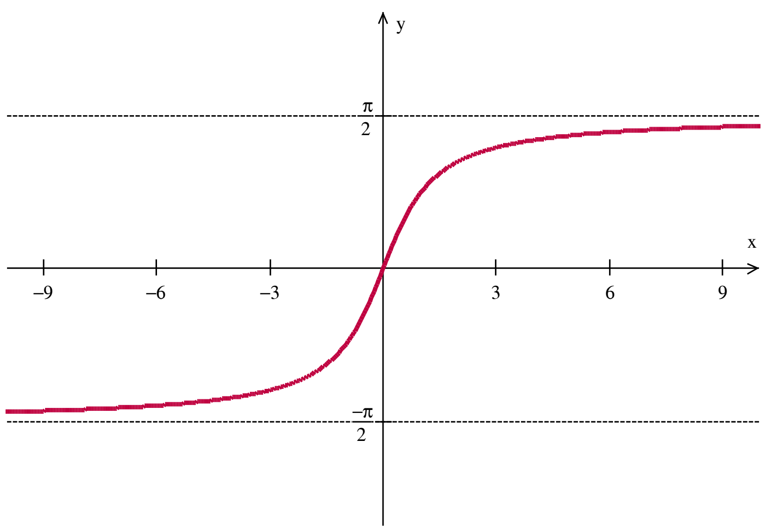
$y = \cos^{-1} x$

$0 \le y \le \pi$



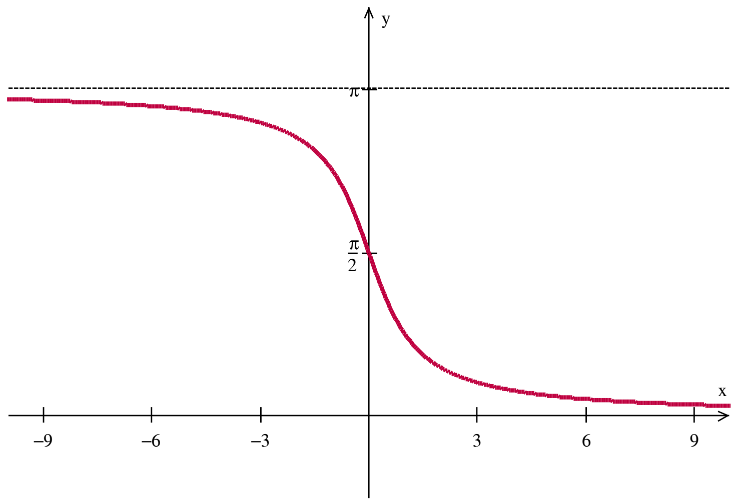
$y = \sin^{-1} x$

$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$



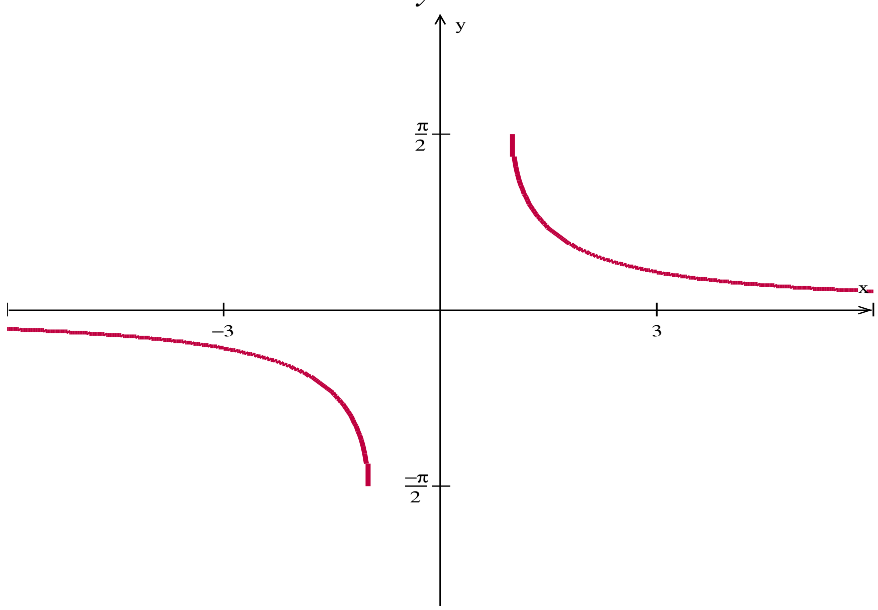
$y = \tan^{-1} x$

$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$



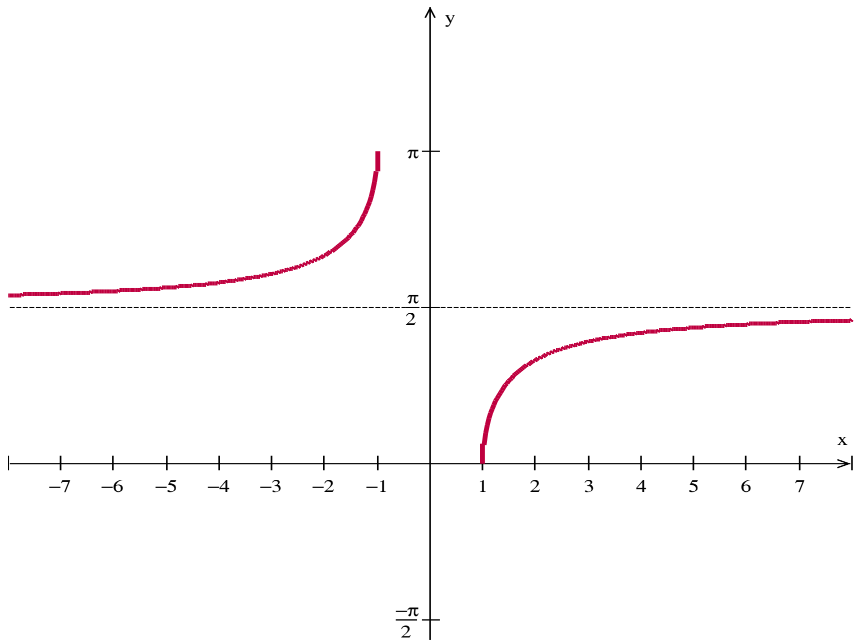
$y = \cot^{-1} x$

$0 \le y \le \pi$



$y = \csc^{-1} x$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



$$y = \sec^{-1} x$$
$$0 \le y \le \pi$$

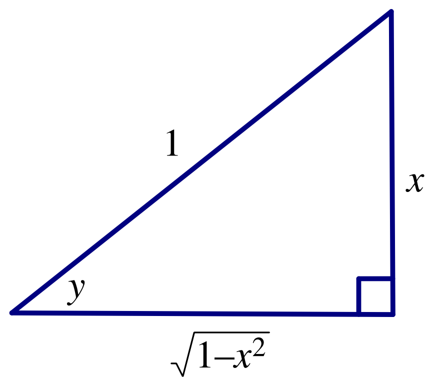
All these graphs are subject to the same shifts and stretches as the non-inverse trigonometric curves and extreme points occur at the endpoints imposed on their ranges. The derivatives of these curves are not particularly interesting as they apply to the graphs, but they are interesting in their process.

EX 1 Find the derivative of $y = \sin^{-1} x$.

$$y = \sin^{-1} x \rightarrow \sin y = x$$

$$D_x(\sin y = x)$$
$$(\cos y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

But this derivative is not in terms of x, so the problem is not done. Consider the right triangle that would yield this SOHCAHTOA relationship:



Note that for $\sin y$ to equal x, x must be the opposite leg and the hypotenuse is 1 (SOH). The Pythagorean theorem gives us the adjacent leg. By CAH,

$$\cos y = \sqrt{1 - x^2}$$

Therefore,

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\cos y}$$
$$= \frac{1}{\sqrt{1 - x^2}}$$

As with the logarithmic functions, the derivatives of these transcendental functions become algebraic functions.

Inverse Trigonometric Derivative Rules:

$$\frac{d}{dx}\left[\sin^{-1}u\right]=\frac{1}{\sqrt{1-u^2}}\bullet D_u \qquad \frac{d}{dx}\left[\csc^{-1}u\right]=\frac{-1}{|u|\sqrt{u^2-1}}\bullet D_u$$

$$\frac{d}{dx}\left[\cos^{-1}u\right]=\frac{-1}{\sqrt{1-u^2}}\bullet D_u \qquad \frac{d}{dx}\left[\sec^{-1}u\right]=\frac{1}{|u|\sqrt{u^2-1}}\bullet D_u$$

$$\frac{d}{dx}\left[\tan^{-1}u\right]=\frac{1}{u^2+1}\bullet D_u \qquad \frac{d}{dx}\left[\cot^{-1}u\right]=\frac{-1}{u^2+1}\bullet D_u$$

LEARNING OUTCOME

Find the derivatives of inverse trigonometric functions.

EX 2 $\frac{d}{dx}\left[\tan^{-1}3x^4\right]$

$$\frac{d}{dx}\left[\tan^{-1}3x^4\right]=\frac{1}{\left(3x^4\right)^2+1}\bullet\left(12x^3\right)$$

$$=\frac{12x^3}{9x^8+1}$$

EX 3 $\frac{d}{dx}\left[\sec^{-1}x^2\right]$

$$\frac{d}{dx}\left[\sec^{-1}x^2\right]=\frac{1}{|\,x^2\,|\sqrt{\left(x^2\right)^2-1}}\bullet 2x$$

$$=\frac{2x}{\left(x^2\right)\sqrt{\left(x^2\right)^2-1}}$$

$$=\frac{2}{x\sqrt{x^4-1}}$$

11-3 Free Response Homework

Find the derivatives of these functions.

1. $y = \sin^{-1}\left(x\sqrt{2}\right)$

2. $y = \csc^{-1}\left(x^2 + 1\right)$

3. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x$

4. $y = \cos^{-1} x + x\sqrt{1-x^2}$

5. $y = \frac{\sec^{-1} x}{x}$ for $x > 1$

6. $y = \ln\left(x^2 + 4\right) - x\tan^{-1}\left(\frac{x}{2}\right)$

7. $y = \csc^{-1}\left(2e^{3x}\right)$

8. $y = \sin^{-1}\left(\frac{t-1}{t+1}\right)$

9. $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

10. $y = x^2 \sec^{-1}\left(\frac{1}{x}\right)$ for $x > 1$

11. At time $t \geq 0$, the **velocity** of a particle moving along a path in the xy-plane is described by $x'(t) = \tan^{-1}\left(\frac{t}{t+1}\right)$ and $y'(t) = \ln\left(t^2 + 1\right)$. What are its speed and the acceleration vector at $t = 4$?

11-3 Multiple Choice Homework

1. If $\arcsin x = 2\arccos x$, then $x =$

a) 0.5 b) 0.9 c) 0 d) -0.9 e) 0.5

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2. $\frac{d}{dx}[\arctan 3x] =$

a) $\frac{1}{1+9x^2}$ b) $\frac{3}{1+9x^2}$ c) $\frac{3}{\sqrt{4x^2-1}}$

d) $\frac{3}{1+3x}$ e) None of these

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3. If $g(x) = \arcsin 2x$, then $g'(x) =$

a) $2\arccos 2x$ b) $2\csc 2x \cot 2x$ c) $\frac{2}{1+4x^2}$

d) $\frac{2}{\sqrt{4x^2-1}}$ e) $\frac{2}{\sqrt{1-4x^2}}$

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4. What is the area of the largest rectangle that can be inscribed under the graph of $y = 2\cos x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$?

a) 2.20 units^2 b) 2.24 units^2 c) 2.28 units^2

d) 2.32 units^2 e) 2.36 units^2

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5. The slope of the line tangent to the curve $y^2 + (xy+1)^3 = 0$ at $(2, -1)$ is

a) $-\frac{3}{2}$ b) $-\frac{3}{4}$ c) 0 d) $\frac{3}{4}$ e) $\frac{3}{2}$

11-4: General Trigonometric Curve Sketching

For the parent functions, finding extreme points was simple because their x-values were evenly spaced along their cycle and their y-values were always the amplitude above and below the sinusoidal axis. The algebraic traits of simple trigonometric functions do not need to be examined because it is much easier to just use the trigonometric traits. But ***when the trigonometric functions are combined with other functions*** (whether as products, quotients, or composites), finding the algebraic traits is a better approach.

LEARNING OUTCOME
Find the traits and sketch composite functions involving trigonometric operations.

Algebraic Traits of Trigonometric Functions: The trigonometric traits (amplitude, period, horizontal and vertical shift) were examined with A, B, h, and k of the general equation. That information can be translated into an algebraic trait format.

1. Domain: sine and cosine = All Reals
   others = All Reals, except the vertical asymptotes

2. Axis Points: The points where the curve crosses the **sinusoidal axis**. Found by setting the trigonometric term equal to zero. (These are more valuable for graphing trigonometric functions but are comparable to the zeros of the other functions.)

3. Horizontal (Phase) Shift: h
   (This is more valuable for graphing trigonometric functions but is comparable to the y-intercept of the other functions.)

4. VAs: where the denominator of the rational trigonometric function equals zero. (Cosine is the denominator of tangent and secant, and sine is the denominator of cosecant and cotangent.)

5. Extreme Points: $\frac{dy}{dx} = 0$ or DNE, or the endpoints on a given domain

6. Range: y-values

EX 1 Find the traits and sketch $y=\frac{1}{2}x+\sin x$ on $x\in\left[0,\ 2\pi\right]$.

1. Domain: $x\in\left[0,\ 2\pi\right]$

2. Axis Points: $\sin x=0$

$$x=0\pm\pi n$$

$$\left(0,\ 0\right),\ \left(\pi,\ \frac{\pi}{2}\right),\text{ and }\left(2\pi,\ \pi\right)$$

3. Phase Shift: $h=0$

4. VA: None

5. Extreme Points: $\frac{dy}{dx}=\frac{1}{2}+\cos x=0$

$$\cos x=-\frac{1}{2}$$

$$x=\pm\frac{2\pi}{3}\pm 2\pi n$$

$$x=\frac{2\pi}{3}\text{ or }\frac{4\pi}{3}$$

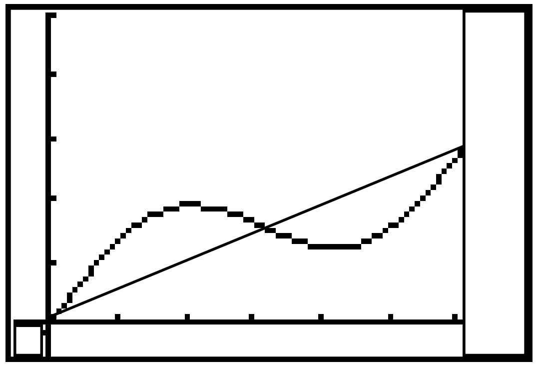
$\frac{dy}{dx}$ is never DNE on its domain.

Remember, the endpoints are also critical values. Therefore, the extreme points are

$$\left(0,\ 0\right),\ \left(\frac{2\pi}{3},\ 1.913\right),\ \left(\frac{4\pi}{3},\ 1.228\right),\text{ and }\left(2\pi,\ \pi\right)$$

6. Range: $y\in\left[0,\ \pi\right]$

Sketch:



$y=\frac{1}{2}x+\sin x$

Note that $y=\frac{1}{2}x$ is in the position of "k" in the general equation and, as such, serves as the sinusoidal axis. Unlike the simpler equations before, the sinusoidal axis is now a slanted line. The same occurs with $-2x$ in EX 2.

EX 2 Find the traits and sketch $y=-2x+\tan x$ on $x\in\left[-\pi,\ \pi\right]$.

1. Domain: $x\in\left[-\pi,\ -\frac{\pi}{2}\right)\cup\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\ \pi\right]$

2. Axis Points: $\tan x=0$
$$x=0\pm\pi n$$
$$(0,\ 0),\ (-\pi,\ 2\pi),\text{ and }(\pi,\ -2\pi)$$

3. Phase Shift: $h=0$

4. VA: $\tan x=\frac{\sin x}{\cos x}$ therefore, the VAs are where $\cos x=0$
$$\cos x=0$$
$$x=\frac{\pi}{2}\pm 2\pi n$$
$$x=-\frac{\pi}{2},\ \frac{\pi}{2}$$

5. Extreme Points: $\frac{dy}{dx} = -2 + \sec^2 x = 0$

$$\sec^2 x = 2$$
$$\sec x = \pm\sqrt{2}$$
$$\cos x = \frac{\pm 1}{\sqrt{2}}$$
$$x = \pm\frac{\pi}{4} \pm 2\pi n \text{ or } \pm\frac{3\pi}{4} \pm 2\pi n$$

$x = \pm\frac{\pi}{4}$ and $\pm\frac{3\pi}{4}$ are the domain critical values.

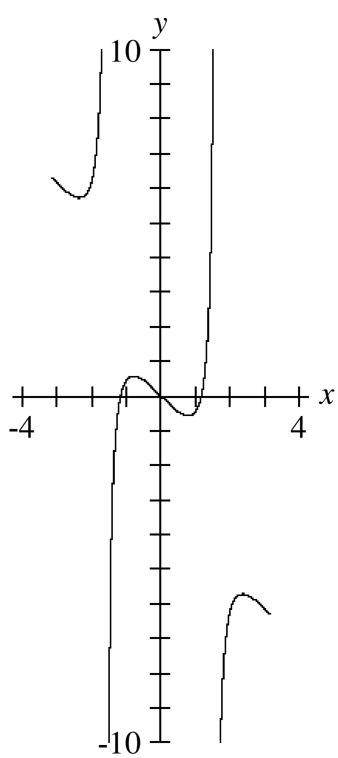
$\frac{dy}{dx}$ DNE $\rightarrow x = \pm\frac{\pi}{2}$, but $x = \pm\frac{\pi}{2}$ are VAs and cannot be critical values.

The extreme points are

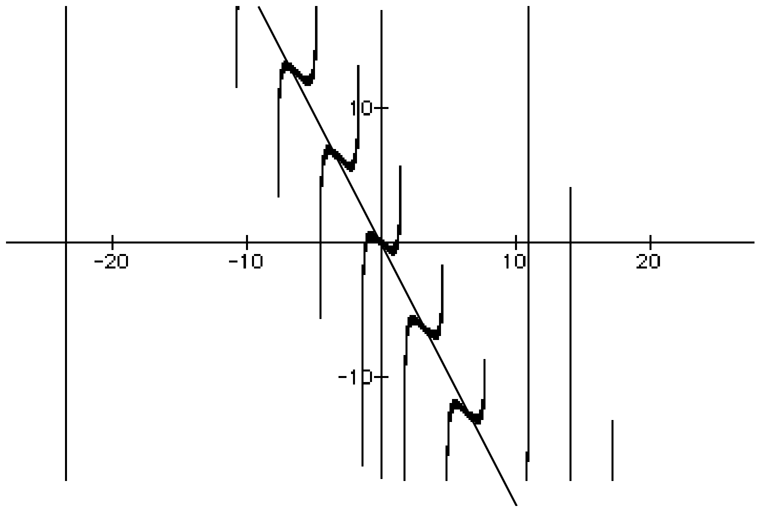
$\left(\frac{\pi}{4},\ -0.571\right)$, $\left(-\frac{\pi}{4},\ 0.571\right)$, $\left(\frac{3\pi}{4},\ -5.713\right)$, $\left(-\frac{3\pi}{4},\ 5.713\right)$, $\left(-\pi,\ 2\pi\right)$ and $\left(\pi,\ -2\pi\right)$

6. Range: All Reals

Sketch:



A wider view (larger domain) more clearly shows the effect $-2x$ has on the graph:



$y=-2x+\tan x$

This is what happens with the sum of two functions. $y=\sin x+\cos 4x$ is an even more dramatic example. In EX 3, the sinusoidal axis is a sinusoid itself:

EX 3 Find the traits and sketch $y=\sin x+\cos 4x$ on $x\in\left[0,\ 2\pi\right]$.

1. Domain: $x\in\left[0,\ 2\pi\right]$

2. Axis Pts: $\cos 4x=0$

$$4x=\frac{\pi}{2}\pm\pi n$$

$$x=\frac{\pi}{8}\pm\frac{\pi}{4}n$$

$$\left(\frac{\pi}{8},\ 0.383\right),\ \left(\frac{3\pi}{8},\ 0.924\right),\left(\frac{5\pi}{8},\ 0.924\right),\ \left(\frac{7\pi}{8},\ 0.383\right),$$
$$\left(\frac{9\pi}{8},\ -0.383\right),\ \left(\frac{11\pi}{8},\ -0.924\right),\left(\frac{13\pi}{8},\ -0.924\right),\ \left(\frac{15\pi}{8},\ -0.383\right)$$

3. Phase Shift: $h=0$

4. VA: None

5. Extreme Points: $\frac{dy}{dx} = \cos x - 4\sin\ 4x = 0$. This equation would be somewhat difficult to solve algebraically, so it will be done graphically.

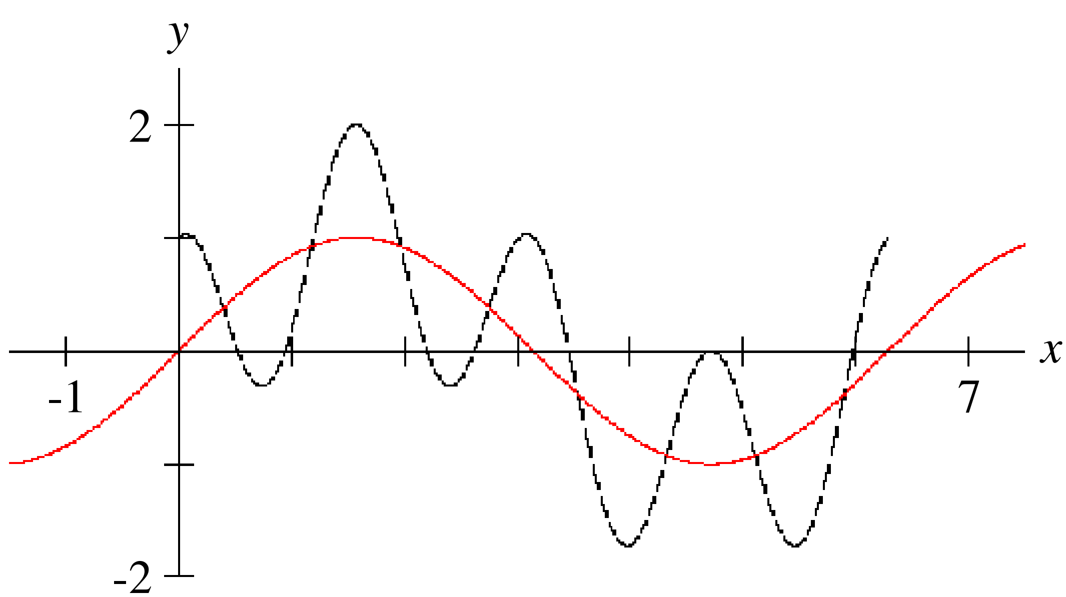
$$(0.063,\ 1.031),\ (0.739,\ -0.309),\ \left(\frac{\pi}{2},\ 2\right),\ (2.403,\ -0.309),$$
$$(3.079,\ 1.031),\ (3.969,\ -1.722),\ (4.712,\ 0),\ (5.455,\ -1.722)$$

Don't forget that the endpoints are extreme points as well:

$$(0, 1) \text{ and } (2\pi,\ 1)$$

6. Range: $y \in [-1.722,\ 2]$

Sketch:



$y = \sin x + \cos 4x$

EX 4 Find the traits and sketch $y=\sqrt{\cos 4x}$ on $x\in\left[0,\ \pi\right]$.

1. Domain:

$$\cos 4x \ge 0$$
$$\cos 4x = 0$$
$$4x = \pm\frac{\pi}{2} \pm 2\pi n$$
$$x = \pm\frac{\pi}{8} \pm \frac{\pi}{2}n$$
$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \text{ or } \frac{7\pi}{8}$$

$$\cos 4x \ge 0 \to x \in \left[0,\ \frac{\pi}{8}\right] \cup \left[\frac{3\pi}{8},\ \frac{5\pi}{8}\right] \cup \left[\frac{7\pi}{8},\ \pi\right]$$

2. Axis Pts: The zeros were found while finding the domain:

$$\left(\frac{\pi}{8},\ 0\right), \left(\frac{3\pi}{8},\ 0\right), \left(\frac{5\pi}{8},\ 0\right) \text{ and } \left(\frac{7\pi}{8},\ 0\right)$$

3. Phase Shift: $h=0$

4. VA: None

5. Extreme Points:

$$\frac{dy}{dx} = \left(\frac{1}{2}\cos^{-1/2} 4x\right)\left(-\sin 4x\right)\left(4\right)$$
$$= \frac{-2\sin 4x}{\sqrt{\cos 4x}}$$

$$\frac{dy}{dx} = 0$$
$$\sin 4x = 0$$
$$x = 0 \pm \frac{\pi}{4}n \to x = 0,\ \frac{\pi}{2},\ \pi$$

Note: $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are not in the domain.

$$\frac{dy}{dx} \text{ DNE} \to \cos 4x = 0 \Rightarrow x = \frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{5\pi}{8},\ \text{or } \frac{7\pi}{8}$$

The endpoints of the domain are $x = 0,\ \pi$.

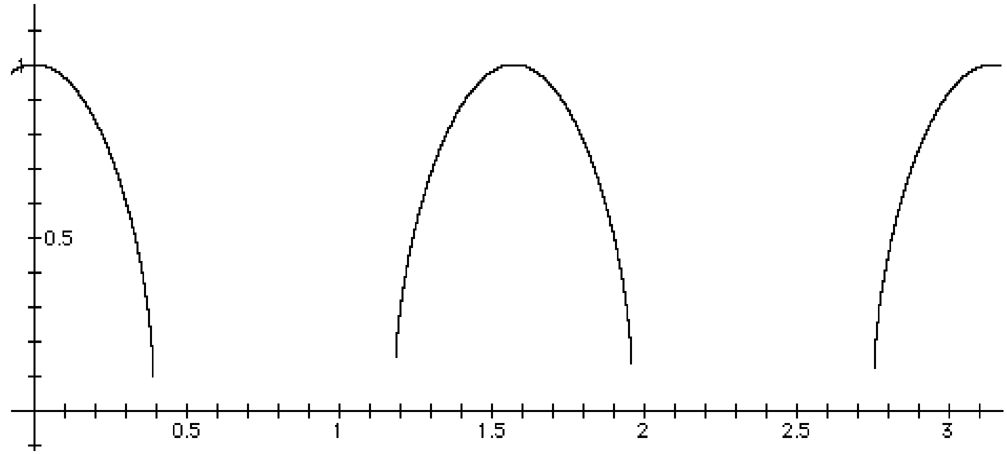
Critical values: $x = 0,\ \dfrac{\pi}{8},\ \dfrac{3\pi}{8},\ \dfrac{\pi}{2},\ \dfrac{5\pi}{8}, \dfrac{7\pi}{8},\ \pi$

Extreme Points:

$(0,\ 1),\ \left(\dfrac{\pi}{8},\ 0\right),\ \left(\dfrac{3\pi}{8},\ 0\right),\ \left(\dfrac{\pi}{2},\ 1\right),\ \left(\dfrac{5\pi}{8},\ 0\right), \left(\dfrac{7\pi}{8},\ 0\right),\ (\pi,\ 1)$

6. Range: $y \in [0,\ 1]$

Sketch:



$y = \sqrt{\cos 4x}$

Notice that the calculator does not show the curve going all the way down to the x-axis, *even though it must and does because zeros were found*. **The calculator is not infallible!!** The sketch should show the curve finished off properly (i.e., stopping at the x-axis).

11-4 Free Response Homework

Find the critical values.

1. $y=(\sin 3x)(\cos 3x)$

2. $y=(\sin 4x)+(\cos 4x)$

3. $y=\cot^2 5x$

4. $y=\cos\sqrt{x}$

5. $y=\frac{\sqrt{3}}{2}x+\cos x$

Find the algebraic traits and sketch.

6. $y=\frac{\sqrt{3}}{2}x+\cos x$ on $x\in[-2\pi,\ 2\pi]$

7. $y=\frac{1}{2}x-\sin x$ on $x\in[0,\ 2\pi]$

8. $y=4x+\cot x$ on $x\in(0,\ \pi)$

9. $y=x\sin x$ on $x\in[0,\ 2\pi]$

10. $y=\sqrt[4]{\sin \pi x}$ on $x\in[0,1]$

11. $y=e^x\sin x$ on $x\in[-\pi,\ \pi]$

12. $y=\ln(\tan x)$ on $x\in[0,\ 2\pi]$

11-4 Multiple Choice Homework

1. For $f(x)=\sin^2 x$ and $g(x)=0.5x^2$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the instantaneous rate of is *f* greater than the instantaneous rate of change of *g* for which values of *x*?

a) –0.8 b) 0 c) 0.9 d) 1.2 e) 1.5

---

2. A point moves so that *x*, its distance from the origin at time *t*, $t \geq 0$ is given by $x(t)=\cos^3 t$. The first time interval in which the point is moving to the right is

a) $0<t<\frac{\pi}{2}$ b) $\frac{\pi}{2}<t<\pi$ c) $\pi<t<\frac{3\pi}{2}$

d) $\frac{3\pi}{2}<t<2\pi$ e) None of these

---

3. The graph of $y=\frac{\sin x}{x}$ has

I. a vertical asymptote at $x=0$
II. a horizontal asymptote at $y=0$
III. an infinite number of zeros

a) I only b) II only c) III only

d) I and III only e) II and III only

---

4. At how many points on the interval $-2\pi \le x \le 2\pi$ does the tangent to the graph of the curve $y = x\cos x$ have slope $\frac{\pi}{2}$?

a) 5 b) 4 c) 3 d) 2 e) 1

---

5. If $y = \cos^2(2x)$, then $\frac{dy}{dx} =$

a) $2\cos 2x \sin 2x$ b) $-4\cos 2x \sin 2x$ c) $2\cos 2x$

d) $-2\cos 2x$ e) $4\cos 2x$

General Trigonometric Functions Practice Test
Part 1: CALCULATOR REQUIRED

Round to 3 decimal places. Show all work.

Multiple Choice (3 pts. each)

1. If $f(x)=\sin(e^{-x})$, then $f'(x)=$

(a) $-\cos(e^{-x})$
(b) $\cos(e^{-x})+e^{-x}$
(c) $\cos(e^{-x})-e^{-x}$
(d) $e^{-x}\cos(e^{-x})$
(e) $-e^{-x}\cos(e^{-x})$

2. An equation of the line tangent to the graph of $y=x+\cos x$ at the point $(0, 1)$ is

(a) $y=2x+1$
(b) $y=x+1$
(c) $y=x$
(d) $y=x-1$
(e) $y=0$

3. The first derivative of the function $f(x)$ is given by $f'(x)=\dfrac{\cos^2 x}{x}-\dfrac{1}{5}$. How many critical values does $f(x)$ have on the open interval $(0, 10)$?

(a) One
(b) Three
(c) Four
(d) Five
(e) Seven

4. Administrators at Massachusetts General Hospital believe that the hospital's expenditures $E(B)$, measured in dollars, are a function of how many beds B are in use with $E(B) = 14000 + (B+1)^2$. On the other hand, the number of beds B is a function of time t, measured in days, and is estimated that $B(t) = 20\sin\left(\frac{t}{10}\right) + 50$. At what rate are the expenditures decreasing when $t = 100$?

(a) 120 dollars/day
(b) 125 dollars/day
(c) 130 dollars/day
(d) 135 dollars/day
(e) 140 dollars/day

Free Response (10 pts. each)

1. Find the extreme points of $y = \frac{\sqrt{3}}{2}x + \cos x$ on $x \in [-2\pi, 2\pi]$.

2. Find the extreme points of $y = \ln(\tan x)$ on $x \in [0, 2\pi]$.

General Trigonometric Functions Practice Test
Part 2: CALCULATOR NOT ALLOWED

Round to 3 decimal places. Show all work.

Multiple Choice (3 pts. each)

5. If $f(x) = \tan(2x)$, then $f'\left(\frac{\pi}{6}\right) =$

(a) $\sqrt{3}$
(b) $2\sqrt{3}$
(c) 4
(d) $4\sqrt{3}$
(e) 8

6. If $\sin(xy) = x$, then $\frac{dy}{dx} =$

(a) $\dfrac{1}{\cos(xy)}$
(b) $\dfrac{1}{x\cos(xy)}$
(c) $\dfrac{1-\cos(xy)}{\cos(xy)}$
(d) $\dfrac{1-y\cos(xy)}{x\cos(xy)}$
(e) $\dfrac{y(1-\cos(xy))}{\cos(xy)}$

7. An equation for a tangent to the graph of $y = \arctan\frac{x}{3}$ at the origin is:

(a) $x - 3y = 0$
(b) $x - y = 0$
(c) $x = 0$
(d) $y = 0$
(e) $3x - y = 0$

8. $\lim_{h\to 0}\dfrac{\cos(x+h)-\cos x}{h}=$

(a) $\sin x$
(b) $-\sin x$
(c) $\cos x$
(d) $-\cos x$
(e) DNE

Free Response (10 pts. each)

3. List the traits and **sketch** $y=\dfrac{\sqrt{3}}{2}x+\cos x$ on $x\in[-2\pi,\ 2\pi]$.

Domain:

Range:

y-int:

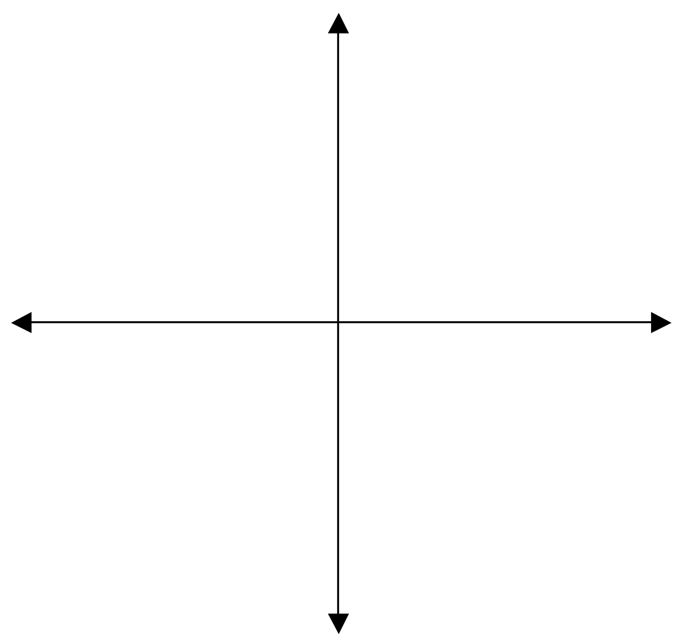
Axis Points:

Extreme Points:

VAs:

POEs:

End Behavior:



4. List the traits **and** $y = \ln(\tan x)$ on $x \in [0,\ 2\pi]$.

Domain:

Range:

y-int:

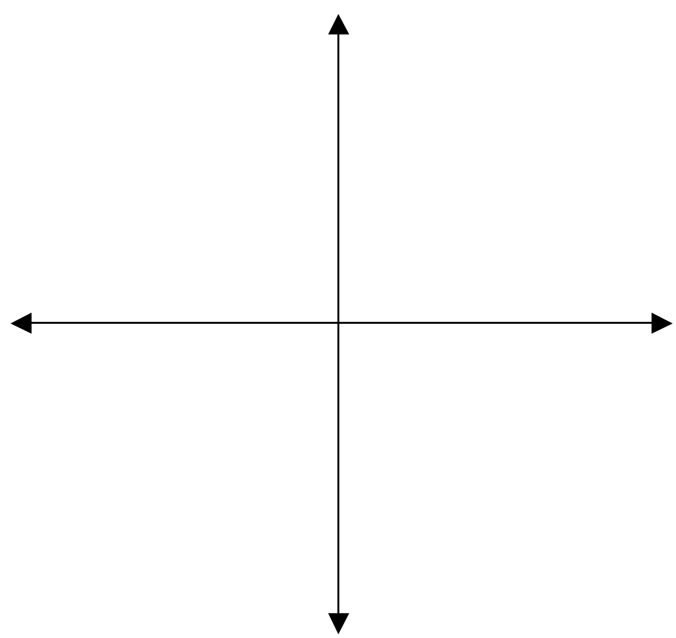
Zeros:

Extreme Points:

VAs:

POEs:

End Behavior:



General Trigonometric Functions Answer Key

11-1 Free Response Homework

1. $-4\cos^3 x \sin x$

2. $\dfrac{-\csc^2\sqrt{x}}{2\sqrt{x}}$

3. 0

4. $\dfrac{-3\csc\sqrt{x}\cot\sqrt{x}}{2\sqrt{x}}$

5. $-8\cos^3 2x\sin 2x$

6. $3\sin x$

7. $\dfrac{2x}{3}\sec\dfrac{x^2}{3}\tan\dfrac{x^2}{3}$

8. $\dfrac{-3x^2\csc^2 x^3}{2\sqrt{\cot x^3}}$

9. $\dfrac{-4}{3}\sqrt[3]{x}\csc\sqrt[3]{x^4}\cot\sqrt[3]{x^4}$

10. $\dfrac{16\sin 2x}{(1+2\cos 2x)^5}$

11. $f'\left(\dfrac{\pi}{3}\right)=4+2\sqrt{3}$

13. $x=\pm\dfrac{5\pi}{6},\ \pm\pi$

14. $x=\dfrac{3\pi}{2},\ 0,\ 2\pi$

15. $x=0,\ \dfrac{\pi}{3},\ \dfrac{2\pi}{3}$

16. $x=0,\ \dfrac{\pi}{4},\ \dfrac{3\pi}{4},\ \pi$

17. $v(t)=\sin 2t$

18. $x=\dfrac{\pi}{4}\pm\pi n$

19.

v	END	+	0	–	0	+	END
t	0		5		13		20

11-1 Multiple Choice Homework

1. B 2. E 3. E 4. E 5. D

11-2 Free Response Homework

1. $x^2\sec x(x\tan x+3)$

2. $x\csc x(2-x\cot x)$

3. $\cos x\left(2+x^2\right)$

4. $\tan x\left(2x\sec^2 x+\tan x\right)$

5. $\sin 4x$

6. $\dfrac{\sin x-\sin x\cos^2 x-5\cos^3 x}{\sin^2 x\cos^2 x}$

7. $\dfrac{1-3\sec x+\sin^2 x}{\cos x\left(\cos x+3\right)^2}$

8. $\dfrac{2x\cos x+x^2\sin x}{\cos^2 x}$ or $2x\sec x+x^2\sec x\tan x$

9. $-\dfrac{1}{2}\csc^2\dfrac{1}{2}x$

10. $x^3\sec x\tan x+3x^2\sec x+x^2\sec^2 x+2x\tan x$

11. $f'\left(\dfrac{\pi}{4}\right)=3\sqrt{2}$

12. $f'(t)=\dfrac{\sec^2 t}{\left(1+\tan t\right)^2}$; $f'\left(\dfrac{\pi}{4}\right)=\dfrac{1}{2}$

13. $f'\left(\dfrac{\pi}{4}\right)=\sqrt{2}$

14. $x'(0.548)=1.429$; $x'(2.181)=-8.374$

15. $x=25.175°$

16a. $\left\langle \dfrac{1}{2\sqrt{t}},\ -\sin t\right\rangle$ b. $\left\langle \dfrac{-1}{4t\sqrt{t}},\ -\cos t\right\rangle$ c. $\dfrac{1}{2\sqrt{\pi}}$

17a. $\langle 5\cos t,\ 2t\rangle$ b. $\langle -5\sin t,\ 2\rangle$ c. 8.030

18a. $\langle \sec\theta\tan\theta,\ \sec^2\theta\rangle$ b. $\langle \sec^3\theta+\sec\theta\tan^2\theta,\ 2\sec^2\theta\tan\theta\rangle$

c. 1

19. $e\sqrt{2}$

11-2 Multiple Choice Homework

1. B 2. C 3. B 4. A 5. E

11-3 Free Response Homework

1. $\frac{\sqrt{2}}{\sqrt{1-2x^2}}$ 2. $\frac{-2}{\left(x^2+1\right)\sqrt{x^2+2}}$

3. 0 4. $\frac{-2x^2}{\sqrt{1-x^2}}$ 5. $\frac{1-\sqrt{x^2-1}\sec^{-1}x}{\left(x^2\right)\sqrt{x^2-1}}$

6. $-\tan^{-1}\frac{x}{2}$ 7. $\frac{-3}{\sqrt{4e^{6x}-1}}$ 8. $\frac{1}{(t+1)\sqrt{2}}$

9. $\frac{2}{x^2+1}$ 10. $\frac{-x^2}{\sqrt{x^2-1}}+2x\sec^{-1}\frac{1}{x}$

11. Speed = 2.912, $a(t)=\left\langle\frac{1}{41}, \frac{8}{17}\right\rangle$

11-3 Multiple Choice Homework

1. B 2. B 3. E 4. B 5. D

11-4 Free Response Homework

1. $x=\pm\frac{\pi}{12}\pm\frac{\pi}{3}n$ 2. $x=\frac{\pi}{16}\pm\frac{\pi}{4}n$

3. $x = \pm\frac{\pi}{10} \pm \frac{2\pi}{5}n,\ 0 \pm \frac{2\pi}{5}n$

4. $x = (0 \pm \pi n)^2$

5. $x = \begin{cases} \frac{\pi}{3} \pm 2\pi n \\ \frac{2\pi}{3} \pm 2\pi n \end{cases}$

6. Domain: $x \in [-2\pi,\ 2\pi]$

$h = 0$

Range: $y \in [-4.441,\ 6.441]$

EB: None

VAs: None

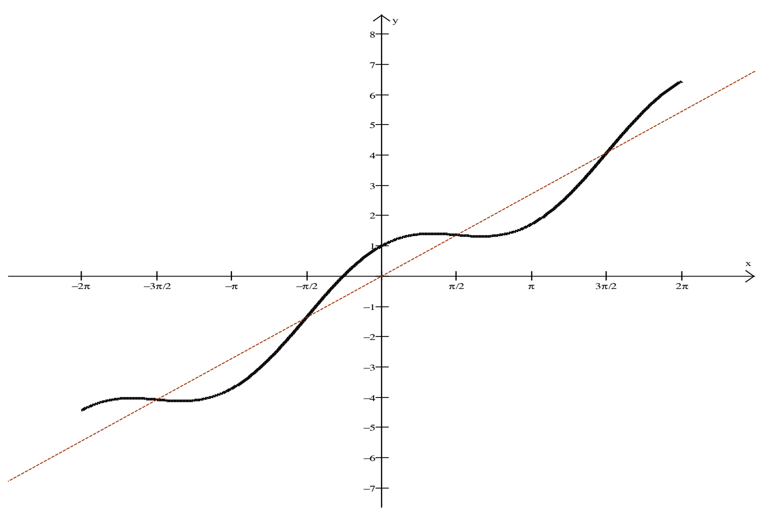
POEs: None

Axis Points: $\left(\frac{\pi}{2},\ 1.360\right),\ \left(-\frac{\pi}{2},\ -1.360\right),\ \left(\frac{3\pi}{2},\ 4.081\right),\ \left(-\frac{3\pi}{2},\ -4.081\right)$

Extreme Points:

$\left(\frac{\pi}{3},\ 1.407\right),\ \left(\frac{2\pi}{3},\ 1.314\right),\ \left(-\frac{5\pi}{3},\ -4.441\right),\ \left(-\frac{4\pi}{3},\ -4.128\right)$

$(-2\pi,\ -4.441),\ (2\pi,\ 6.441)$



7. Domain: $x \in [0,\ 2\pi]$

$h = 0$

Range: $y \in [-0.342,\ 3.484]$

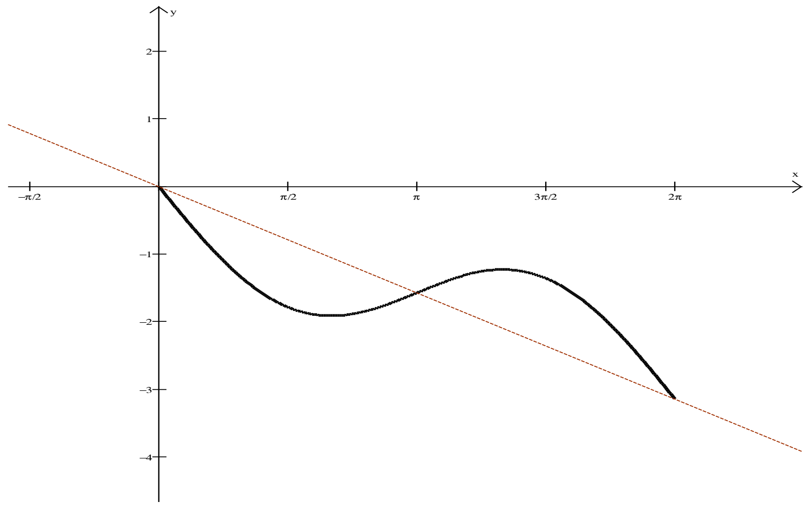
EB: None

POEs: None

VAs: None

Axis Points: $(0,\ 0)$, $\left(\pi,\ \frac{\pi}{2}\right)$, $(2\pi,\ \pi)$

Extreme Points: $\left(\frac{\pi}{3},\ -0.342\right)$, $\left(\frac{5\pi}{3},\ 3.484\right)$, $(0,\ 0)$, $(2\pi,\ \pi)$



8. Domain: $x \in (0,\ \pi)$

$h = 0$

Range: $y \in$ All Reals

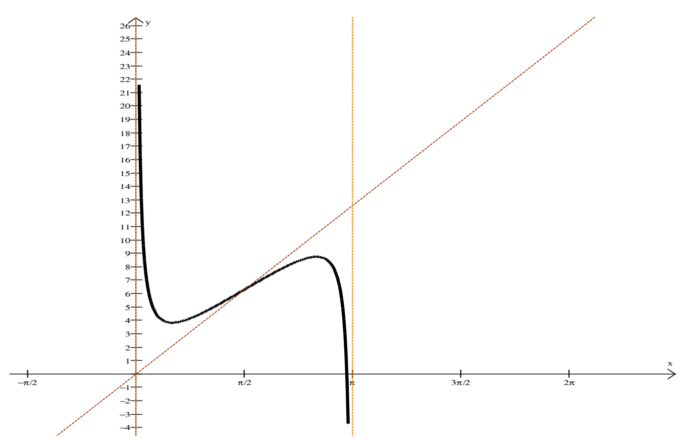
EB: None

VAs: $x = 0$, $x = \pi$

POEs: None

Axis Points: $(\pi,\ 4\pi)$

Extreme Points: $(0.524,\ 3.826)$, $(2.618,\ 8.740)$



9. Domain: $x \in [0,\ 2\pi]$ Range: $y \in [-4.814,\ 1.820]$

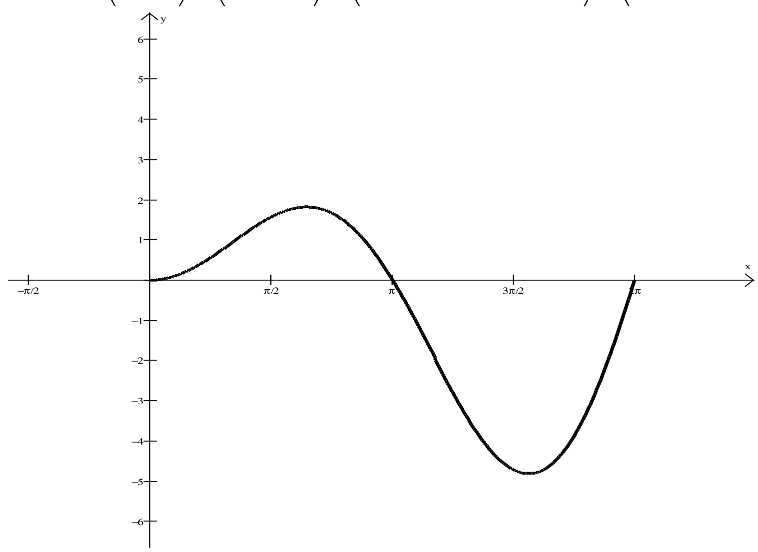
$h = 0$

Axis Points: $(0,\ 0)$, $(\pi,\ 0)$, $(2\pi,\ 0)$

POEs: None

VAs: None

Extreme Points: $(0,\ 0)$, $(2\pi,\ 0)$, $(4.913,\ -4.814)$, $(2.029,\ 1.820)$



10. Domain: $x \in [0,\ 1]$ Range: $y \in [0,\ 1]$

$h = 0$

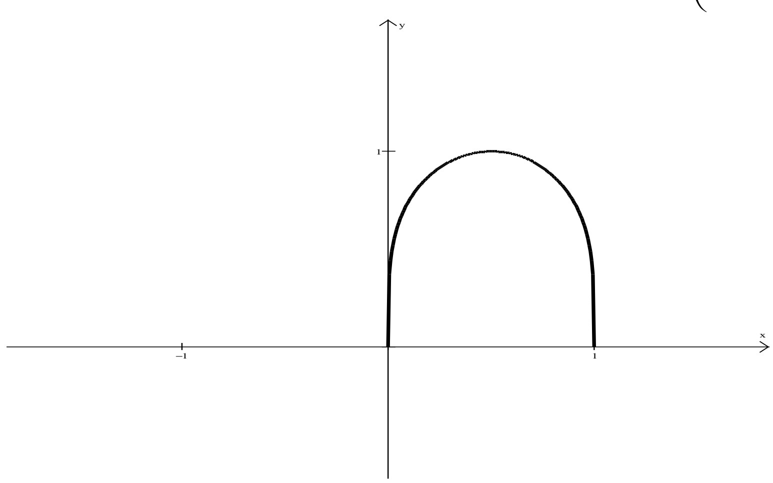
EB: None

POEs: None

VAs: None

Axis Points: $(0,\ 0)$, $(1,\ 0)$

Extreme Points: $(0,\ 0)$, $\left(\frac{1}{2},\ 1\right)$, $(1,\ 0)$



11. Domain: $x \in [-\pi, \pi]$

$h = 0$

POEs: None

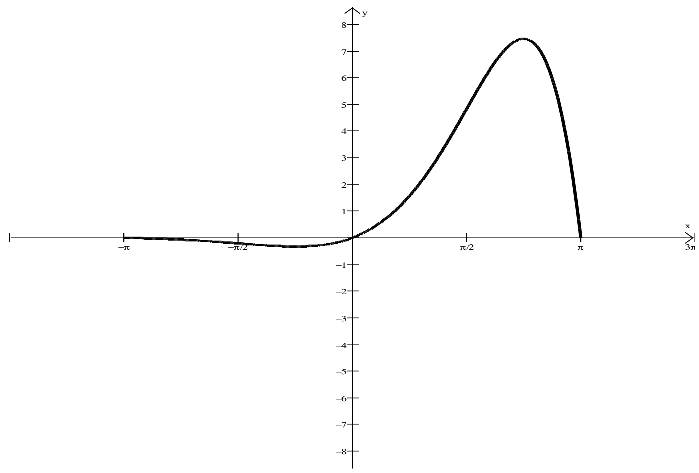
Range: $y \in [0, 1]$

EB: None

VAs: None

Axis Points: $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$

Extreme Points: $\left(-\dfrac{\pi}{4}, -0.322\right)$, $\left(\dfrac{3\pi}{4}, 7.460\right)$, $(-\pi, 0)$, $(\pi, 0)$



12. Domain: $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$

$h = 0$

POEs: None

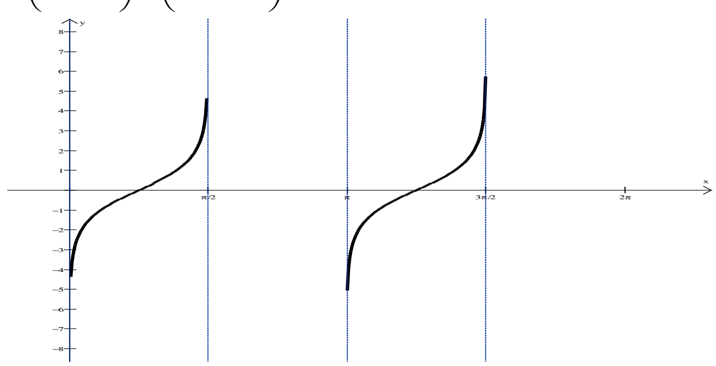
Range: $y \in (-\infty, \infty)$

EB: None

VAs: $x = 0,\ x = \dfrac{\pi}{2},\ x = \pi,\ x = \dfrac{3\pi}{2}$

Axis Points: $\left(\dfrac{\pi}{4}, 0\right)$, $\left(\dfrac{5\pi}{4}, 0\right)$

Extreme Points: None



11-4 Multiple Choice Homework

1. C 2. C 3. E 4. B 5. B

General Trigonometric Functions Practice Test Answer Key

Multiple Choice

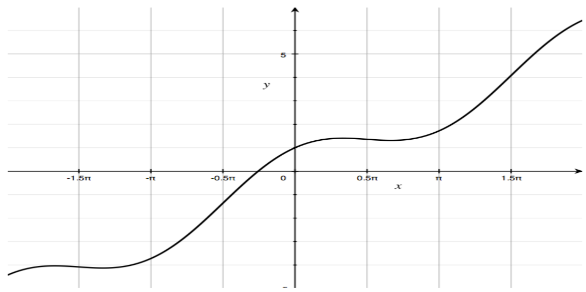
1.	E	2.	B	3.	B	4.	D
5.	E	6.	D	7.	A	8.	B

Free Response

1. Extreme Points: $(-2\pi,\ -4.441)$, $(2\pi, 6.441)$, $\left(\frac{\pi}{3},\ 1.407\right)$, $\left(\frac{2\pi}{3},\ 1.314\right)$, $\left(-\frac{5\pi}{3},\ -4.034\right)$, $\left(-\frac{4\pi}{3},\ -4.128\right)$

2. No extreme points

3. Domain: $x \in [-2\pi,\ 2\pi]$ Range: $y \in [-4.441,\ 6.441]$ y-int: $(0,\ 1)$
Axis Points: $\left(\frac{\pi}{2},\ \frac{\sqrt{3}\pi}{4}\right)$, $\left(\frac{\pi}{2},\ -\frac{\sqrt{3}\pi}{4}\right)$, $\left(\frac{3\pi}{2},\ \frac{3\sqrt{3}\pi}{4}\right)$, $\left(-\frac{3\pi}{2},\ -\frac{3\sqrt{3}\pi}{4}\right)$
Extreme Points: $(-2\pi,\ -4.441)$, $(2\pi,\ 6.441)$, $\left(\frac{\pi}{3},\ 1.407\right)$, $\left(\frac{2\pi}{3},\ 1.314\right)$, $\left(-\frac{5\pi}{3},\ -4.034\right)$, $\left(-\frac{4\pi}{3},\ -4.128\right)$
VAs: None POEs: None End Behavior: None



4. Domain: $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right)$ Range: $y \in$ All Reals

Zeros: $\left(\dfrac{\pi}{4}, 0\right), \left(\dfrac{5\pi}{4}, 0\right)$ y-int: None

Extreme Points: None VAs: $x = 0,\ x = \dfrac{\pi}{2},\ x = \pi,\ x = \dfrac{3\pi}{2}$

POEs: None End Behavior: None