# Chapter 12:

# The Second Derivative

Chapter 12 Overview: Review of All Derivative Rules

The emphasis of the previous chapters was graphing the families of functions as they are viewed (mostly) in Analytic Geometry, that is, with traits. The focus of this chapter is sketching graphs of the various functions, but with a Calculus outlook. First, it would be best to review the traits of each kind.

REMEMBER TRAITS:

1. Domain
2. Range
3. $y$-intercept
4. Zeros
5. Vertical Asymptotes
6. Points of Exclusion
7. End Behavior
8. Extreme Points

In this chapter, graphs are determined by the sign patterns of the function, its derivative, and the derivative of the derivative. Rather than the traits above, the sign patterns emphasize what is referred to as KEY TRAITS:

- Zeros and/or VAs
- Extreme Points
- Points of Inflection

The interpretations of the sign patterns of these key traits are connected by way of the First Derivative Test and the Concavity Test. That is, the zeros of the 1st Derivative are the critical values of the "original" function; the increasing and decreasing intervals of the derivative are the intervals of concavity of the "original" function, etc.

**Interpretation of Sign Patterns:**

| Sign | $+$ | 0 / dne | $-$ |
|---|---|---|---|
| $f(x)$ | Curve above $x$-axis | Zero / VA | Curve below $x$-axis |
| $f'(x)$ | Increasing | Critical Value | Decreasing |
| $f''(x)$ | Concave Up | POI | Concave Down |

12-1: Second Derivatives and Concavity

This section explores some of the meanings of the 2nd derivative (that is, the derivative of the derivative). But before that, it is best recap all derivative rules:

The Power Rule: $\frac{d}{dx}u^n = nu^{n-1} \bullet \frac{du}{dx}$

The Product Rule: $\frac{d}{dx}(u \bullet v) = u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v \bullet \frac{du}{dx} - u \bullet \frac{dv}{dx}}{v^2}$

The Chain Rule: $\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \bullet g'(x)$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$
$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$
$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$
$$\frac{d}{dx}\left[a^u\right] = \left(a^u \bullet \ln a\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \bullet \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \bullet D_u \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-1}{|u|\sqrt{u^2-1}} \bullet D_u$$
$$\frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-1}{\sqrt{1-u^2}} \bullet D_u \qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}} \bullet D_u$$
$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{u^2+1} \bullet D_u \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-1}{u^2+1} \bullet D_u$$

What had previously been called "the derivative" is actually the <u>1st derivative</u>. There can be successive uses of the derivative rules, and they have meanings other than the slope of the tangent line. This section will explore the process of finding the 2nd derivative and will explore the meanings in subsequent sections.

*Vocabulary*:
1. ***Second Derivative*** – the derivative of the derivative

Just as with the 1st derivative, there are several symbols for the 2nd derivative:

<u>Second Derivative Symbols:</u>

$\frac{d^2y}{dx^2}$ $\frac{d^2}{dx^2}$

$f''(x) =$ "$f$ double prime of $x$" $y'' =$ "$y$ double prime"

LEARNING OUTCOMES

Find the second derivative.
Apply the Second Derivative Test.

EX 1 $\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]$

$$\begin{aligned}\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right] &= \frac{d}{dx}\left[\frac{d}{dx}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]\right]\\ &= \frac{d}{dx}\left[4x^3 - 21x^2 - 6x + 2\right]\\ &= 12x^2 - 42x - 6\end{aligned}$$

More complicated functions, in particular composite functions, have a complicated process. When the Chain Rule is applied, the answer becomes a product or quotient. Therefore, the 2nd derivative will require the Product or Quotient Rules as well as, possibly, the Chain Rule again.

EX 2 $y = e^{3x^2}$, find $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = e^{3x^2} \bullet 6x = 6xe^{3x^2}$$

$$\frac{d^2y}{dx^2} = 6x\left(e^{3x^2} \bullet 6x\right) + e^{3x^2} \bullet 6$$

$$= 36x^2e^{3x^2} + 6e^{3x^2}$$

$$= 6e^{3x^2}\left(6x^2 + 1\right)$$

EX 3 $y = \sin^4 x$, find $\dfrac{d^2y}{dx^2}$

$$y' = 4\sin^3 x \cdot \cos x$$

$$y'' = 4\sin^3 x\left(-\sin x\right) + \cos x\left(12\sin^2 x \cdot \cos x\right)$$

$$= 4\sin^2 x\left(3\cos^2 x - \sin^2 x\right)$$

EX 4 $f(x)=\ln\left(x^2+3x-1\right)$, find $f''(x)$.

$$f'(x)=\frac{1}{x^2+3x-1}(2x+3)=\frac{2x+3}{x^2+3x-1}$$

$$f''(x)=\frac{\left(x^2+3x-1\right)(2)-(2x+3)(2x+3)}{\left(x^2+3x-1\right)^2}$$

$$=\frac{\left(2x^2+6x-2\right)-\left(4x^2+12x+9\right)}{\left(x^2+3x-1\right)^2}$$

$$=\frac{-2x^2-6x-11}{\left(x^2+3x-1\right)^2}$$

EX 5 $g(x)=\sqrt{4x^2+1}$, find $g''(x)$.

$$g'(x)=\frac{1}{2}\left(4x^2+1\right)^{-1/2}(8x)=\frac{4x}{\left(4x^2+1\right)^{1/2}}$$

$$g''(x)=\frac{\left(4x^2+1\right)^{1/2}(4)-(4x)\left[\frac{1}{2}\left(4x^2+1\right)^{-1/2}(8x)\right]}{\left[\left(4x^2+1\right)^{1/2}\right]^2}$$

$$=\frac{\left(4x^2+1\right)^{1/2}(4)-\frac{16x^2}{\left(4x^2+1\right)^{1/2}}}{\left(4x^2+1\right)}$$

$$=\frac{\left(4x^2+1\right)(4)-16x^2}{\left(4x^2+1\right)^{3/2}}$$

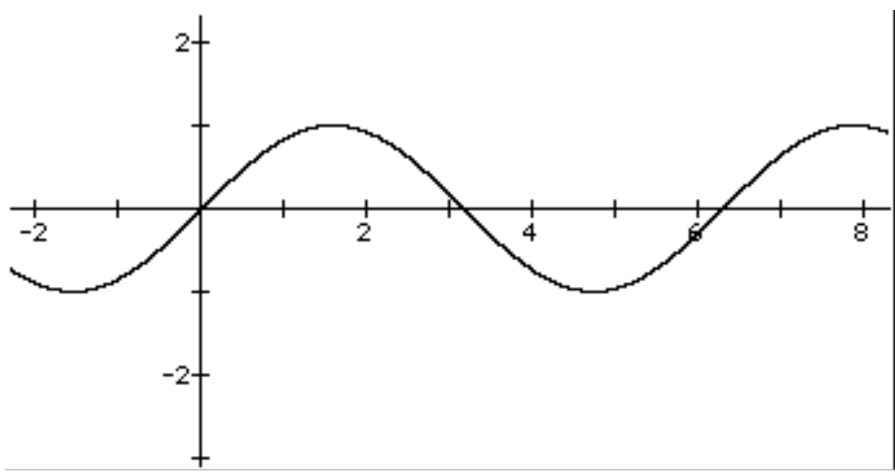
$$=\frac{4}{\left(4x^2+1\right)^{3/2}}$$

**Concavity and Points of Inflection**

One trait of algebraic and transcendental curves, which was not previously explored, is concavity.

*Vocabulary*:
1. ***Concavity*** – the way a curve bends
2. ***Concave Up*** – the curve bends counter-clockwise
3. ***Concave Down*** – the curve bends clockwise
4. ***Inflection Value*** – the $x$-value at which the concavity of the curve switches. This can be a vertical asymptote.
5. ***Point of Inflection*** – the point at which the concavity of the curve switches from up to down or down to up

The easiest way to see this is with a sine wave.




On $x \in (0, \pi)$, the curve is concave down. On $x \in (\pi, 2\pi)$, it is concave up.

The 2nd derivative is what is used to analyze this trait. Points of inflection are to the 2nd derivative as extreme points are to the 1st derivative, for the most part.

If $x = c$ is at a Point of Inflection (POI), then

i) $f''(x) = 0$

or ii) $f''(x)$ does not exist.

**NB. Endpoints on a domain CANNOT be Points of Inflection.**

Also, intervals of concave up or down are comparable to intervals of increasing or decreasing, respectively.

LEARNING OUTCOME

Find points of inflection and intervals of concavity.

EX 1 Given this sign pattern for the second derivative of $F(x)$, what can be determined about $F(x)$?

$$\begin{array}{c|ccccccc} F''(x) & - & 0 & + & 0 & - & 0 & + \\ x & & -1 & & 0 & & 2 & \end{array}$$

$F(x)$ is concave up where the second derivative is positive—namely,

$$x \in (-1,\ 0) \cup (2,\ \infty)$$

$F(x)$ is concave down where the second derivative is negative—namely,

$$x \in (-\infty,\ -1) \cup (0,\ 2)$$

$F(x)$ has inflection values where the signs change—namely,

$$x = -1,\ 0,\ \text{and}\ 2$$

**Note that this use of the sign pattern of the 2nd derivative is known as the Second Derivative Test for Concavity, not the Second Derivative Test.**

EX 2 Given this sign pattern $\begin{array}{c} y'' \\ x \end{array} \xleftrightarrow[\quad -2 \qquad\quad 0 \qquad\quad 2 \quad]{+ \ \text{DNE} \ - \quad 0 \quad + \ \text{DNE} \quad -}$, where is $y$ concave down?

It can be concluded that $y$ is concave down on $x \in (-2, 0)$ and $(2, \infty)$.

EX 3 Find the points of inflection of $y = x^3 + 4x^2 - 3x + 1$.

$$y' = 3x^2 + 8x - 3$$
$$y'' = 6x + 8 = 0$$
$$x = -\frac{4}{3}$$

$y''$ is never DNE.

$$\begin{array}{c} y'' \\ x \end{array} \xleftrightarrow[\qquad -\frac{4}{3} \qquad]{- \qquad 0 \qquad +}$$

$$x = -\frac{4}{3} \to y = 9.741$$

$\left(-\frac{4}{3}, 9.741\right)$ is the POI.

EX 4 Find the points of inflection of $y=\sin x$ on $x\in(0,\ 2\pi)$.

$$\frac{dy}{dx}=\cos x$$
$$\frac{d^2y}{dx^2}=-\sin x=0$$
$$x=0\pm\pi n$$

$\frac{d^2y}{dx^2}$ always exists (result is never DNE).

On the given interval, the only inflection value is $x=\pi$.

$$\begin{array}{c|ccccc} y'' & \text{END} & - & 0 & + & \text{END} \\ \hline x & 0 & & \pi & & 2\pi \end{array}$$

Because of the sign change in the sign pattern $x=\pi$ is the $x$-coordinate of the POI. $x=\pi\rightarrow y=0$, so

$(\pi,\ 0)$ is the POI.

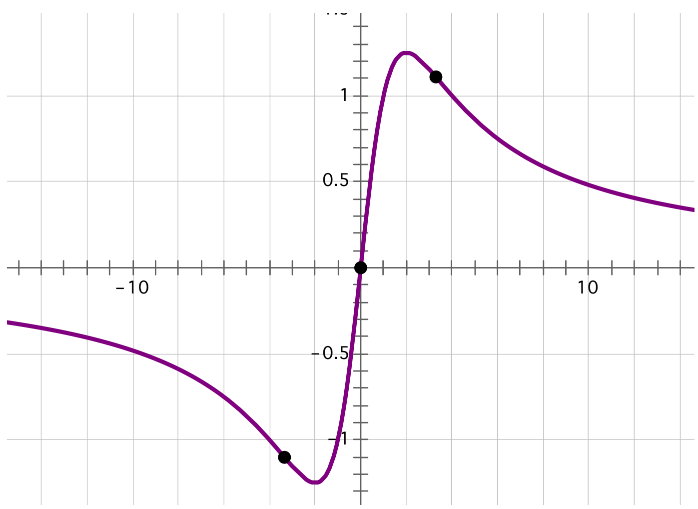
EX 5 Find the points of inflection of $y=\dfrac{5x}{x^2+4}$.

$$y'=\frac{(x^2+4)(5)-(5x)(2x)}{(x^2+4)^2}=\frac{-5x^2+20}{(x^2+4)^2}$$
$$y''=\frac{(x^2+4)^2(-10x)-(-5x^2+20)\left(2(x^2+4)(2x)\right)}{(x^2+4)^4}$$
$$=\frac{(x^2+4)(-10x)\left[(x^2+4)-2(x^2-4)\right]}{(x^2+4)^4}$$
$$=\frac{-10x(-x^2+12)}{(x^2+4)^3}=0\rightarrow x=0,\ \pm2\sqrt{3}$$

$$\begin{array}{c|ccccccc} y'' & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2\sqrt{3} & & 0 & & 2\sqrt{3} & \end{array}$$

Because of the change in the sign pattern, $x = 0,\ \pm 2\sqrt{3}$ are the $x$-coordinates of the POIs.

$$(0,\ 0),\ \left(-2\sqrt{3},\ \frac{-5\sqrt{3}}{8}\right),\text{ and }\left(2\sqrt{3},\ \frac{5\sqrt{3}}{8}\right)\text{ are the POIs.}$$




$$y = \frac{5x}{x^2 + 4}$$

The graph shows that $(0,\ 0)$ is a point of inflection. But it shows something else. Namely, that the concavity switches (as it often does) on either side of the vertical asymptotes. The vertical asymptotes cannot be points of inflection, because they are not points, but one must be aware of this in cases where the question is about concavity rather than about POI. The process is similar to the 1st Derivative Test, including the need to check where the derivative does not exist. It will be referred to as the **<u>2nd Derivative Test for Concavity</u>**.

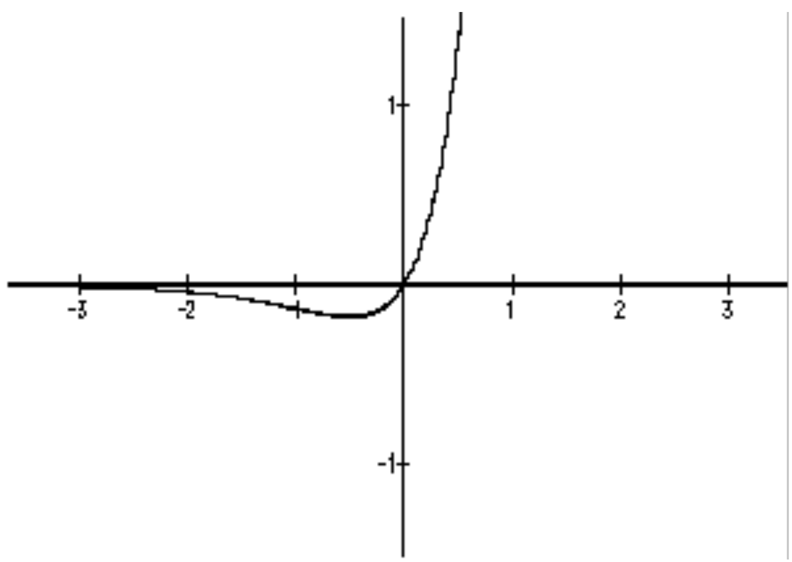
EX 6  Find the intervals of concavity for $y = xe^{2x}$.

$$\frac{dy}{dx} = x\left(e^{2x}(2)\right) + e^{2x}(1) = e^{2x}(2x+1)$$
$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x+1)\left(e^{2x}(2)\right)$$
$$= e^{2x}\left[(2) + (2x+1)(2)\right]$$
$$= e^{2x}(4x+4) = 0$$

The inflection value for the POI is $x = -1$ and the 2nd derivative is never DNE.

So, $\begin{array}{c} y'' \\ x \end{array} \underset{-1}{\overset{-\quad 0 \quad +}{\longleftrightarrow}}$ and

$y$ is concave down on $x \in (-\infty,\ -1)$ and
$y$ is concave up on $x \in (-1,\ \infty)$.




$y = xe^{2x}$

EX 7 Find the points of inflection of $y = \frac{-x}{x^2-4}$.

$$y' = \frac{\left(x^2-4\right)(-1)-(-x)(2x)}{\left(x^2-4\right)^2} = \frac{x^2+4}{\left(x^2-4\right)^2}$$

$$y'' = \frac{\left(x^2-4\right)^2(2x)-\left(x^2+4\right)\left(2\left(x^2-4\right)(2x)\right)}{\left(x^2-4\right)^4}$$

$$= \frac{\left(x^2-4\right)(2x)\left[\left(x^2-4\right)-2\left(x^2+4\right)\right]}{\left(x^2-4\right)^4}$$

$$= \frac{2x\left(-x^2-12\right)}{\left(x^2-4\right)^3} = 0 \rightarrow x = 0$$

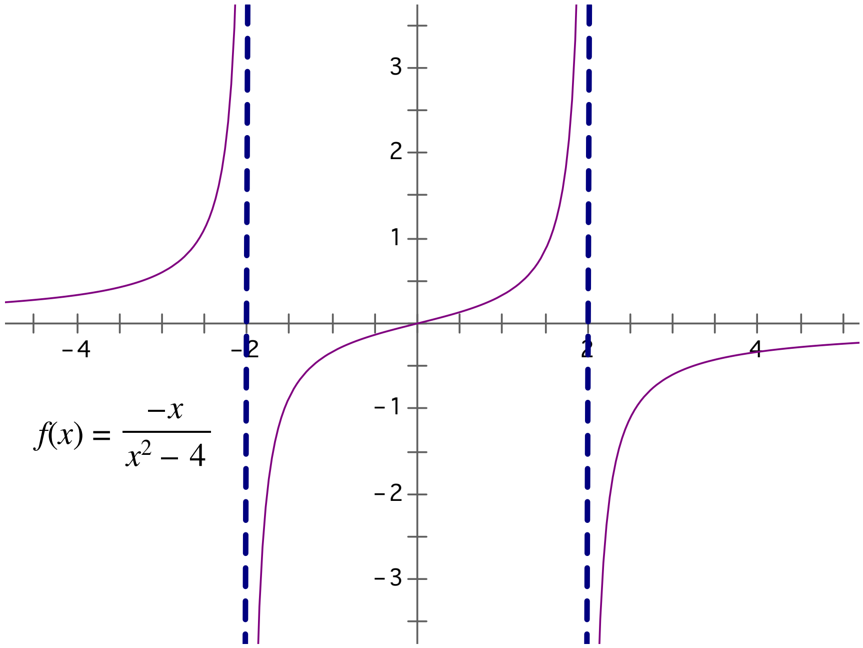
$y''$ DNE $\rightarrow x = \pm 2$

But from a previous chapter, $x = \pm 2$ are vertical asymptotes; hence they cannot be $x$-coordinates for POIs.

| $y''$ | + | DNE | − | 0 | + | DNE | − |
|---|---|---|---|---|---|---|---|
| $x$ | | $-2$ | | $0$ | | $2$ | |

Because of the sign change in the sign pattern we know $x = 0$ is the $x$-coordinate of a POI. $x = 0 \rightarrow y = 0$, so

$(0, 0)$ is the POI.




Summary of Sign Patterns

| Pattern of: | Interpretation |
|---|---|
| $y$ | To determine where the curve is above or below the $x$-axis |
| $\dfrac{dy}{dx}$ | To determine where the curve is increasing or decreasing |
| $\dfrac{d^2y}{dx^2}$ | To determine where the curve is concave up or concave down |
| $y^2$ or $e^y$ | To determine where the function exists |
| $v(t)$ | To determine when the motion is in which direction |
| $a(t)$ | To determine—with $v(t)$—when the motion is speeding up or slowing down |

12-1 Free Response Homework

Evaluate the 2nd derivative.

1. $\dfrac{d^2}{dx^2}\left[5x^4+9x^3-4x^2+x-8\right]$

2. $\dfrac{d^2}{dx^2}\left[4x^7-3x^5+3x^3+6x-1\right]$

3. $y=\cos x^2$, find $y''$

4. $y=\tan^2 x$, find $y''$

5. $y=\sec 3x$, find $\dfrac{d^2y}{dx^2}$

6. $y=xe^{2x}$, find $\dfrac{d^2y}{dx^2}$

7. $f(x)=\ln\left(x^2+3\right)$, find $f''(x)$

8. $g(x)=\ln\left(x^2-4x+4\right)$, find $g''(x)$

9. $h(x)=\sqrt{x^2+5}$, find $h''(x)$

10. $F(x)=\sqrt{3x^2-2x+1}$, find $F''(x)$

11. $y=\dfrac{x^2-3}{x^2-10}$, find $\dfrac{d^2y}{dx^2}$

12. $y=\dfrac{3x+3}{x^3+1}$, find $\dfrac{d^2y}{dx^2}$

13. Given this sign pattern for the second derivative of $F(x)$, on what interval(s) is $F(x)$ concave down?

$$\begin{array}{c|ccccccc} F''(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & 0 & & 3 & \end{array}$$

14. Given this sign pattern for the second derivative of $G(x)$, on what interval(s) is $G(x)$ concave down?

$$\begin{array}{c|ccccccc} G''(x) & + & 0 & + & 0 & - & 0 & - \\ \hline x & & -3 & & \frac{2}{3} & & 2 & \end{array}$$

15. The sign pattern for the second derivative of $H(x)$ is given. a) Is $x=-4$ a point of inflection? b) Is $x=-1$ a point of inflection?

$$\begin{array}{c|ccccccc}
\frac{d^2H}{dx^2} & + & 0 & - & 0 & - & 0 & + \\
\hline
x & & -4 & & -1 & & 2 &
\end{array}$$

16. Given these two sign patterns:

$$\begin{array}{c|ccccccccc}
\frac{dy}{dx} & - & 0 & + & 0 & - & 0 & - & 0 & + \\
\hline
x & & -9 & & -3 & & -1 & & 7 &
\end{array}$$

$$\begin{array}{c|ccccccccc}
\frac{d^2y}{dx^2} & + & 0 & - & 0 & + & 0 & - & 0 & + \\
\hline
x & & -7 & & -2 & & 0 & & 5 &
\end{array}$$

a) On what interval(s) is $y$ decreasing?
b) On what interval(s) is $y$ concave up?
c) On what interval(s) is $y$ both increasing and concave down?

Find the sign pattern of the second derivative and identify the intervals of concavity.

17. $y = x^3 + x^2 - 7x - 15$

18. $y = 3x^4 - 20x^3 + 42x^2 - 36x + 16$

19. $y = \dfrac{-4x}{x^2 + 4}$

20. $y = \dfrac{x^2 - 1}{x^2 - 4}$

21. $y = x\sqrt{8 - x^2}$

22. $y = \dfrac{1}{2}x + \sin x$ on $x \in (0,\ 2\pi)$

23. $y = xe^{-x}$

24. $y = e^{-x^2}$

25. $y = \dfrac{x}{x^2 - 9}$

26. $y = 2x - x^{2/3}$

Find $a(t)$ if the given equation describes the position of a particle at time *t*.

27. $x(t)=t^3-6t^2-63t+4$

28. $x(t)=3t^5-38t^4+141t^3-180t^2+1$

29. $x(t)=4+9\sin\left[\frac{\pi}{4}(t-5)\right]$

30. $x(t)=\frac{t}{t^2+1}$

31. $x(t)=\sqrt{1+e^t}$

Find $a(c)$ for the given position equation.

32. $x(t)=-5+8\cos\left[\frac{\pi}{4}(t-3)\right]; \quad c=6$

33. $x(t)=\sqrt{1+\ln(t^2)}; \quad c=-1$

34. $x(t)=(4-t^2)e^{-\frac{1}{2}t}; \quad c=2$

12-1 Multiple Choice Homework

1. Consider the function $f(x)=(x^2-5)^3$ for all real numbers $x$. The number of inflection points for the graph of $f$ is

a) 1 b) 2 c) 3 d) 4 e) 5

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2. The maximum acceleration attained on the interval $0 \le t \le 3$ by the particle whose velocity is given by $v(t)=t^3-3t^2+4$ is

a) 9 b) 12 c) 14 d) 21 e) 40

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3. If $\frac{d}{dx}[f(x)]=g(x)$ and $\frac{d}{dx}[g(x)]=f(3x)$, then $\frac{d^2}{dx^2}\left[f(x^2)\right]$ is

a) $4x^2f(3x^2)+2g(x^2)$ b) $f(3x^2)$ c) $f(x^4)$

d) $2xf(3x^2)+2g(x^2)$ e) $2xf(3x^2)$

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4. If $y=e^{kx}$, then $\frac{d^5y}{dx^5}=$

a) $k^5e^x$ b) $k^5e^{kx}$ c) $5!e^{kx}$ d) $5!e^x$ e) $5e^{kx}$

---

5. The graph of $y = 3x^5 - 10x^4$ has an inflection point at

a) $(0,\ 0)$ and $(2,\ -64)$ b) $(0,\ 0)$ and $(3,\ -81)$

c) $(0,\ 0)$only d) $(3,\ -81)$only e) $(2,\ -64)$only

---

6. An equation of the line tangent to the graph of $y = x^3 + 3x^2 + 2$ at its point of inflection is

a) $y = -3x + 1$ b) $y = -3x + 7$ c) $y = x + 5$

d) $y = 3x + 1$ e) $y = 3x + 7$

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7. The number of inflection points for the graph of $y = 2x + \cos(x^2)$ in the interval $0 \le x \le 5$ is

a) 6 b) 7 c) 8 d) 9 e) 10

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8. On which of the following intervals is the graph of the curve $y = x^5 - 5x^4 + 10x + 15$ concave up?

I. $x < 0$
II. $0 < x < 3$
III. $x > 3$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

9. Let *f* be a function with a second derivative given $f''(x) = x^2(x-3)(x-6)$. What are the *x*-coordinates of the points of inflection of the graph of *f*?

a) 0 only b) 3 only c) 0 and 6 only

d) 3 and 6 only e) 0, 3, and 6

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10. The derivative of the function is given by $f'(x) = x^2 \cos(x^2)$. How many points of inflection does the graph of have on the open interval $(-2, 2)$?

a) One b) Two c) Three d) Four e) Five

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11. How many points of inflection does the graph of $y = \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x$ have on the interval $0 \le x \le \pi$?

a) 1 b) 2 c) 3 d) 4 e) 5

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## 12-2: The Second Derivative Test:

The 1st Derivative Test (which has been used through previous chapters) is not the only way to test whether critical values are associated with maximum or minimum points. There is a 2nd Derivative Test, which determines the same thing, but it does not use a sign pattern. The sign pattern of the 2nd derivative implies something other than whether the critical value is at a maximum point vs. at a minimum point. What it implies (and why) will be discussed in the next section; for now, here is the Second Derivative Test:

**The Second Derivative Test:**

For a function $f(x)$,

1. If $f'(c)=0$ and $f''(c)>0$, then $f(x)$ has a relative minimum value at $x=c$.

2. If $f'(c)=0$ and $f''(c)<0$, then $f(x)$ has a relative maximum value at $x=c$.

3. If $f'(c)=0$ and $f''(c)=0$, then $f(x)$ has neither a relative maximum nor a relative minimum value at $x=c$.

**Note that the 2$^{nd}$ Derivative Test does NOT work for critical values determined by $f'(c)$ not existing or for endpoints.**

**Steps to Applying the 2nd Derivative Test:**

1. Find the critical values from the $\frac{dy}{dx} = 0$.
2. Find $\frac{d^2y}{dx^2}$.
3. Plug the answers from step 1 into $\frac{d^2y}{dx^2}$ and interpret.

EX 1 Use the 2nd Derivative Test to determine if the critical values of $g(t) = 27t - t^3$ are at a maximum or minimum value.

$$g'(t) = 27 - 3t^2 = 0$$
$$t = \pm 3$$

So $t = -3$ and $t = 3$ are critical values.

Next, find the $g''(t)$ and substitute the critical values:

$$g''(t) = -6t$$
$$g''(-3) = -6(-3) = 18$$
$$g''(3) = -6(3) = -18$$

Therefore, $g$ has a minimum value at $t = -3$ and has a maximum value at $t = 3$. (Note that the numerical value “18” is irrelevant. Only the sign matters.)

EX 2 Determine if the critical values of $y=4x^3-x^4$ are at a maximum or a minimum value.

First, find the critical values:

$$y=4x^3-x^4$$
$$\frac{dy}{dx}=12x^2-4x^3=4x^2(3-x)=0$$

The derivative is never DNE.

Critical values: $x=0, 3$

Find the 2nd derivative and substitute the critical values:

$$\frac{dy}{dx}=12x^2-4x^3$$
$$\frac{d^2y}{dx^2}=24x-12x^2=12x(2-x)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0}=0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=3}=-$$

Therefore, $x=0$ is not at an extreme and $x=3$ is at a maximum value.

EX 3 Determine if the critical values of $y=\left(3-x^2\right)e^x$ are at a maximum or a minimum value.

First, find the critical values:

$$\begin{aligned}\frac{dy}{dx}&=\left(3-x^2\right)e^x+e^x(-2x)\\&=-e^x\left(x^2+2x-3\right)=0\end{aligned}$$

The derivative is never DNE.

Critical values: $x=-3,1$

Find the 2nd derivative and substitute the critical values:

$$\begin{aligned}\frac{dy}{dx}&=-e^x\left(x^2+2x-3\right)\\\frac{d^2y}{dx^2}&=-e^x(2x+2)+\left(x^2+2x-3\right)\left(-e^x\right)\\&=-e^x\left(x^2+4x-1\right)\end{aligned}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1}=-4e \qquad \left.\frac{d^2y}{dx^2}\right|_{x=-3}=4e^{-3}$$

Therefore, $x=1$ is at a maximum value and $x=-3$ is at a minimum value.

EX 4  Determine if the critical values of $y=\dfrac{-4x}{4+x^2}$ are at a maximum or a minimum value.

First, find the critical values:

$$\frac{dy}{dx}=\frac{\left(4+x^2\right)(-4)-(-4x)(2x)}{\left(4+x^2\right)^2}$$
$$=\frac{4x^2-4}{\left(4+x^2\right)^2}=0$$

The derivative is never DNE.

Critical values: $x=\pm1$

Find the 2nd derivative and substitute the critical values:

$$\frac{dy}{dx}=\frac{4x^2-4}{\left(4+x^2\right)^2}$$
$$\frac{d^2y}{dx^2}=\frac{\left(4+x^2\right)^2(8x)-\left(4x^2-4\right)2\left(4+x^2\right)(2x)}{\left(4+x^2\right)^4}$$
$$=\frac{\left(4+x^2\right)(8x)-4x\left(4x^2-4\right)}{\left(4+x^2\right)^3}$$
$$=\frac{-8x^3+48x}{\left(4+x^2\right)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1}=+ \qquad\qquad \left.\frac{d^2y}{dx^2}\right|_{x=-1}=-$$

Therefore, $x=1$ is at a minimum value and $x=-1$ is at a maximum value.

The time and place for the 2nd Derivative Test instead of the first is when a sign pattern cannot be created. This happens in two ways: 1) if the derivative information is not found from an equation (we will see this with Series in BC Calculus), and 2) with Implicit Differentiation where *y* variable cannot easily be isolated to determine the sign pattern..

Ex 5 Consider $x^2+4y^2=7+3xy$. Find the point on the curve with $x=3$, where the tangent line is horizontal. Then determine if that point is at a maximum, a minimum, or neither.

First, we need to find :

$$\frac{d}{dx}\left[x^2+4y^2=7+3xy\right]$$

$$\begin{aligned}2x+8y\frac{dy}{dx}&=3x\frac{dy}{dx}+y(3)\\(8y-3x)\frac{dy}{dx}&=3y-2x\\\frac{dy}{dx}&=\frac{3y-2x}{8y-3x}\end{aligned}$$

Second, we need to find the *y*-value for $x=3$ and $\frac{dy}{dx}=0$:

$$\begin{aligned}\frac{dy}{dx}&=\frac{3y-2(3)}{8y-3(3)}=0\\y&=2\end{aligned}$$

and make sure the point $(3,2)$ is on the curve:

$$\begin{aligned}(3)^2+4(2)^2&=7+3(3)(2)\\9+16&=7+18\end{aligned}$$

Finally, find $\frac{d^2y}{dx^2}$ at $(3,2)$:

$$\frac{d}{dx}\left[\frac{dy}{dx}=\frac{3y-2x}{8y-3x}\right]$$

$$\frac{d^2y}{dx^2}=\frac{(8y-3x)\left(3\frac{dy}{dx}-2\right)-(3y-2x)\left(8\frac{dy}{dx}-3\right)}{(8y-3x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,2)}=\frac{\big(8(2)-3(3)\big)\big(3(0)-2\big)-\big(3(2)-2(3)\big)\big(8(0)-3\big)}{\big(8(2)-3(3)\big)^2}=\frac{-14}{49}$$


$$\frac{d}{dx}\left[\frac{dy}{dx}=\frac{3y-2x}{8y-3x}\right]$$

$$\frac{d^2y}{dx^2}=\frac{(8y-3x)\left(3\frac{dy}{dx}-2\right)-(3y-2x)\left(8\frac{dy}{dx}-3\right)}{(8y-3x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,2)}=\frac{\big(8(2)-3(3)\big)\big(3(0)-2\big)-\big(3(2)-2(3)\big)\big(8(0)-3\big)}{\big(8(2)-3(3)\big)^2}=\frac{-14}{49}$$


Since $\left.\frac{d^2y}{dx^2}\right|_{(3,2)}<0$, $(3,2)$ is at a maximum.

Summary:

**The First Derivative Test***

As the sign pattern of the 1st Derivative is viewed left to right, the critical value represents the *x*-value of a

i) relative maximum point if the sign changes from + to –
ii) relative minimum point if the sign changes from – to +
or iii) non-extreme if the sign does not change

*Note that $\frac{dy}{dx}=0$ is necessary but not sufficient for a critical value to be at an extreme point.

**The Second Derivative Test:**

For a function $f(x)$,

1. If $f'(c)=0$ and $f''(c)>0$, then $f(x)$ has a relative minimum value at $x=c$.

2. If $f'(c)=0$ and $f''(c)<0$, then $f(x)$ has a relative maximum value at $x=c$.

3. If $f'(c)=0$ and $f''(c)=0$, then $f(x)$ has neither a relative maximum nor a relative minimum value at $x=c$.

**Note that the 2nd Derivative Test does NOT work for critical values determined by $f'(c)$ not existing or for endpoints.**

12-2 Free Response Homework

Use the 2nd Derivative Test to determine which critical values are at minimums, maximums, or neither.

1. $y=2x^3+9x^2-168x$

2. $y=\dfrac{x}{x^2+9}$

3. $y=\sqrt{9x^3-4x^2-27x+12}$

4. $y=\sqrt{\dfrac{3x}{9-x^2}}$

5. $y=x^2\bullet\sqrt[3]{9-x^2}$

6. $y=\left(x-x^2\right)e^x$

7. $y=x^3-12x$

8. $y=x^3-36x$

9. $y=2x^3-9x^2-24x+11$

10. $y=x^3+5x^2+3x-4$

11. $y=3x^4-15x^2+7$

12. $y=x^4-17x^2+16$

13. $y=\dfrac{3x}{x^2+9}$

14. $y=\dfrac{3x}{x^2-9}$

15. $y=\dfrac{x^2-2x}{x-3}$

16. $y=\dfrac{x^2-1}{x^2+x-6}$

17. $y=\left(x-x^2\right)e^x$

18. $y=x^2e^{-x}$

19. $y=\left(x^2-4\right)\sqrt{16-x^2}$

Show that, at the given points, there is a horizontal tangent line. Determine if the point is at a maximum, at a minimum, or neither.

20. $x^2-xy+y^3=7;\ (1,2)$

21. $x^2-xy+y^2=3;\ (1,2)\ and\ (-1,-2)$

22. $x^2+4xy+4y^2=0; (2,-1)\ and\ (-2,1)$

23. $x^2+3xy+y^2+5=0; (-3,2)\ and\ (3,-2)$

12-2 Multiple Choice Homework

1. Assuming $g(x)$ has critical values at $x=\pm 2$ and that $g''(x)=\left(4x-x^2\right)e^{-x}$. Which of the following statements is true?

a) $g(2)$ is a maximum and $g(-2)$ is a minimum.
b) $g(2)$ is a minimum and $g(-2)$ is a maximum.
c) $x=\pm 2$ are neither at maximums nor minimums.
d) $g(2)$ is a maximum and $g(-2)$ is neither a max or min.
e) $g(2)$ is a minimum and $g(-2)$ is neither a max or min.

---

2. Assuming $g(x)$ has critical values at $x=\pm 2$ and that $g''(x)=\ln\left(\frac{3}{4x-x^2}\right)$. Which of the following statements is true?

a) $g(2)$ is a maximum and $g(-2)$ is a minimum.
b) $g(2)$ is a minimum and $g(-2)$ is a maximum.
c) $x=\pm 2$ are neither at maximums nor minimums.
d) $g(2)$ is a maximum and $g(-2)$ is neither a max or min.
e) $g(2)$ is a minimum and $g(-2)$ is neither a max or min.

---

3. Given this sign pattern, $\begin{array}{c|ccccccccc} h''(x) & + & 0 & - & & dne & & - & 0 & + \\ \hline x & & -\sqrt{3} & & & 0 & & & \sqrt{3} & \end{array}$, and $h'\left(-\sqrt{3}\right)=h'(0)=h'(3)=0$, then $h(x)$ has a minimum at

a) $x=-\sqrt{3}$ b) $x=0$ c) $x=3$ d) $x=\pm\sqrt{3}$
e) $x=0\ and\ \pm\sqrt{3}$

---

4. Given this sign pattern, $\begin{array}{c|ccccccc} h''(x) & + & 0 & - & dne & - & 0 & + \\ \hline x & & -\sqrt{3} & & 0 & & \sqrt{3} & \end{array}$ and

$h'(\pm1)=h'(0)=h'(3)=0$, then $h(x)$ has a maximum at

a) $x=-\sqrt{3}$
b) $x=0$
c) $x=\sqrt{3}$
d) $x=\pm1$
e) none of these

---

5. Given this sign pattern, $\begin{array}{c|ccccccc} h''(x) & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -\sqrt{3} & & 0 & & \sqrt{3} & \end{array}$, $h(x)$ has a POI at

a) $x=-\sqrt{3}$
b) $x=0$
c) $x=\sqrt{3}$
d) $x=\pm\sqrt{3}$
e) $x=0 \ and \ \pm\sqrt{3}$

---

6. Given this sign pattern, $\begin{array}{c|ccccccc} F''(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & -1 & & 1 & \end{array}$ and that

$F'(-3)=F'(-1)=F'(1)=0$, which of the following statements must be true?

I. $\mathrm{F}(x)$ has a local minimum at $x=0$
II. $\mathrm{F}(x)$ has a local maximum at $x=-3$
III. $\mathrm{F}(x)$ has a point of inflection at $x=1$

a) I only  b) II only  c) III only

d) I and III only  e) I, II, and III

---

7. Given this sign pattern

$$\begin{array}{c|ccccccc} F''(x) & - & 0 & + & 0 & - & 0 & - \\ \hline x & & -1 & & 0 & & 2 & \end{array}$$

and that $F'(-1)=F'(1)=F'(2)=0$, which of the following statements must be true?

I. $\mathrm{F}(x)$ has a local maximum at $x=1$

II. $\mathrm{F}(x)$ has a local minimum at $x=-1$

III. $\mathrm{F}(x)$ has a point of inflection at $x=2$

a) I only
b) II only
c) III only
d) I and III only
e) I, II, and III

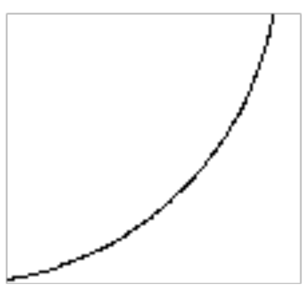
---

12-3:  Advanced Curve Sketching

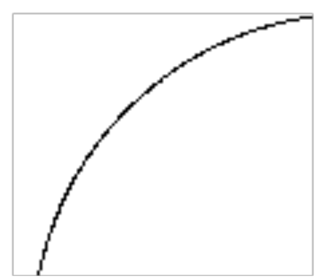
One now has just about everything to do an accurate, complete curve sketch without a calculator.

**The zeros, POEs, extreme points, POIs, and vertical asymptotes are specific values that are used to plot the graph.  These are referred to as the key traits.**

What was not explored previously is how to connect the dots more accurately.  The intervals and how they interrelate dictate how the specific values are connected. Below are the ways that the intervals of increasing and decreasing can mix with the intervals of concave up and concave down as well as how they affect the graph.

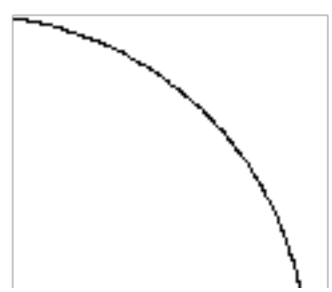


Increasing and Concave Up



Increasing and Concave Down

Decreasing and Concave Up



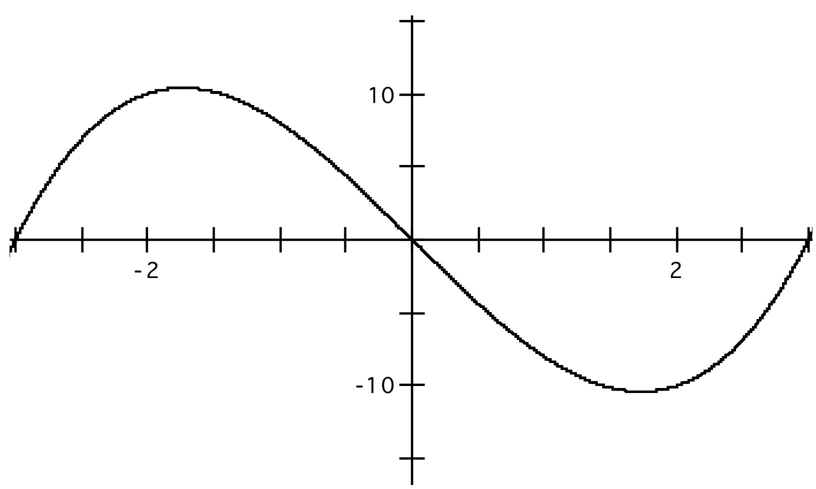
Decreasing and Concave Down

As seen in the homework of the last section, the sign patterns are not necessarily the best way to organize all the information. A table, with the 1st and 2nd derivative signs and conclusions as the column (or row) headings and the intervals and key traits in the first row (or column), often helps. The columns with the 1st and 2nd derivative signs can be read the same as the sign patterns from earlier. Whether the table is oriented horizontally (see EX 2), like the sign patterns, or vertically (see EX 3) is a matter of personal choice, but, a table with many traits probably should be oriented vertically for all the information to fit.

| LEARNING OUTCOME |
|---|
| Find the key traits and a more complete sketch of any function. |

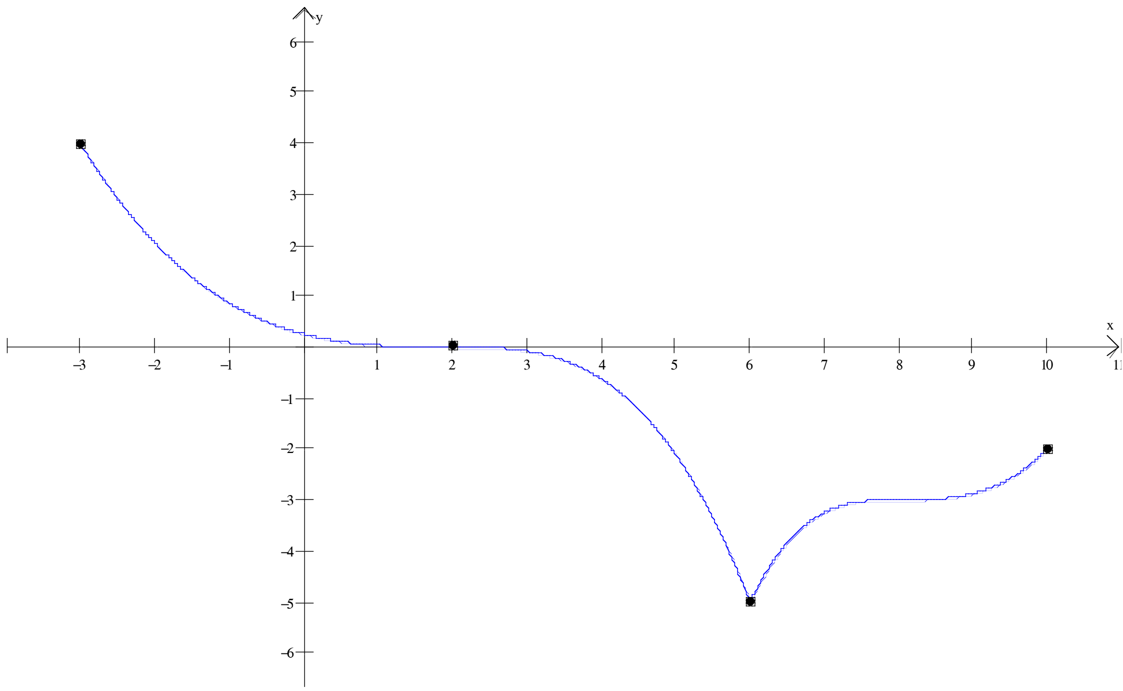
EX 1 Sketch a graph that matches the traits given in this table:

| $x$ | $x=-3$ | $\left(-3,-\sqrt{3}\right)$ | $x=-\sqrt{3}$ | $\left(-\sqrt{3},0\right)$ | $x=0$ | $\left(0,\sqrt{3}\right)$ | $x=\sqrt{3}$ | $\left(\sqrt{3},3\right)$ | $x=3$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | + | + | + | 0 | – | – | – | 0 |
| $f'(x)$ | + | + | 0 | – | – | – | 0 | + | + |
| $f''(x)$ | – | – | – | – | 0 | + | + | + | + |



EX 2  Sketch the graph of the function whose traits are given below.

| $x$ | –3 | | 2 | | 6 | | 8 | | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 4 | + | 0 | – | –5 | – | –3 | – | –2 |
| $f'(x)$ | DNE | – | 0 | – | DNE | + | + | + | DNE |
| $f''(x)$ | DNE | + | 0 | – | DNE | – | 0 | + | DNE |




Note that the information given in the table was not from a given equation; therefore, it is not a problem that this graph is not smooth.

EX 3 Make a table of key traits and a complete sketch of $y = xe^{2x}$.

Zeros: $xe^{2x} = 0 \to x = 0 \to (0,\ 0)$

$y$-int: $x = 0 \to y = 0 \to (0,\ 0)$

Extreme Points:

$$\frac{dy}{dx} = xe^{2x}(2) + e^{2x}(1)$$

$$xe^{2x}(2) + e^{2x}(1) = 0$$

$$e^{2x}(2x+1) = 0$$

$$x = -\frac{1}{2} \to y = -0.184$$

$$\left(-\frac{1}{2},\ -0.184\right)$$

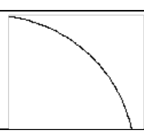
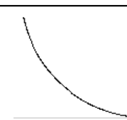
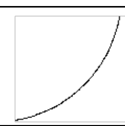
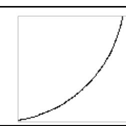
POI:

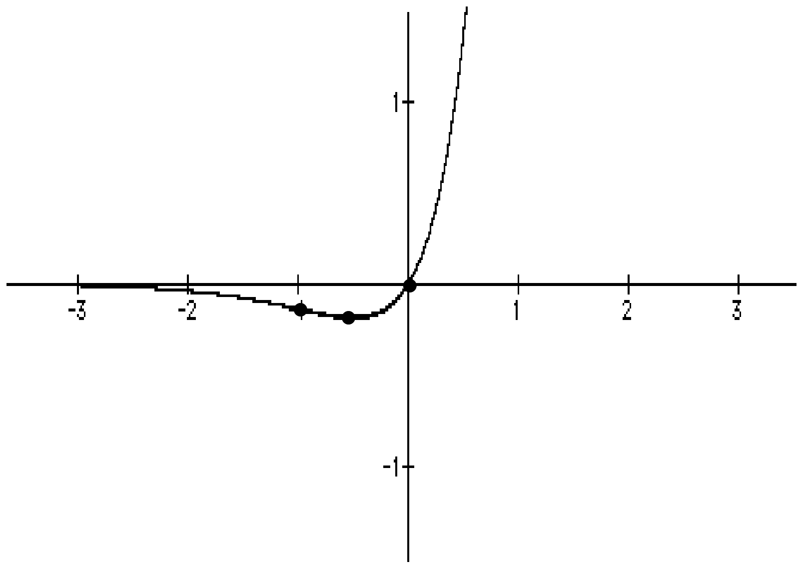
$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x+1)\left(e^{2x}(2)\right)$$

$$= e^{2x}(4x+4) = 0$$

$$x = -1 \to y = -0.135$$

$$(-1,\ -0.135)$$

| $x$ | $(-\infty,\ -1)$ | $x=-1$ | $\left(-1,\ -\frac{1}{2}\right)$ | $x=-\frac{1}{2}$ | $\left(-\frac{1}{2},\ 0\right)$ | $x=0$ | $(0,\ \infty)$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | – | –0.135 | – | –0.184 | – | 0 | + |
| $f'(x)$ | – | – | – | 0 | + | + | + |
| $f''(x)$ | – | 0 | + | + | + | + | + |
| Conclusion |  | POI |  | min pt. |  | zero |  |



$y = xe^{2x}$

Many prefer to make the table horizontal, since this way it matches the sign patterns more directly. The only disadvantage to the horizontal table is whether it fits on the page or not. (The homework answers have vertical tables for that reason.)

EX 4 Make a table of key traits and a complete sketch of $y = x^4 - 2x^3$.

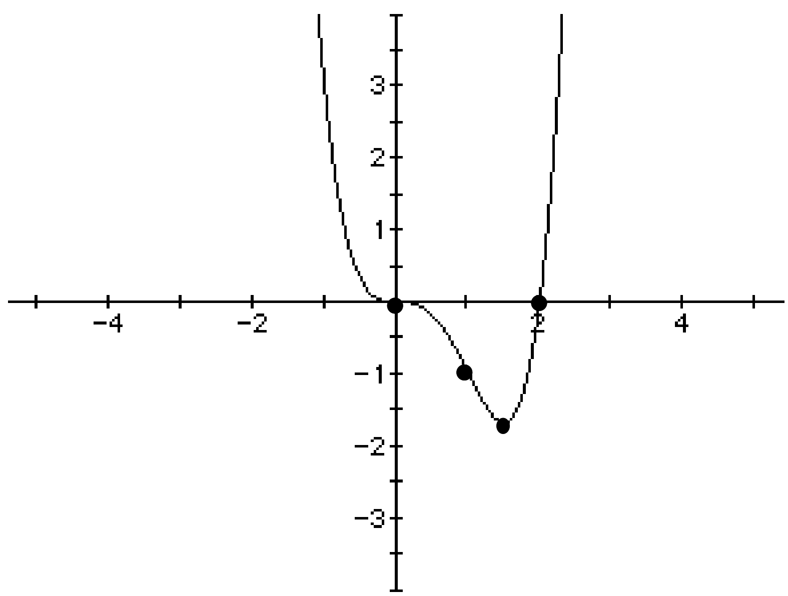
Zeros: $x^4 - 2x^3 = 0 \rightarrow x = 0 \text{ or } 2 \rightarrow (0,\ 0),\ (2,\ 0)$

Extreme Points: $\dfrac{dy}{dx} = 4x^3 - 6x^2 = 0 \rightarrow x = 0 \text{ or } \dfrac{3}{2} \rightarrow (0,\ 0),\ \left(\dfrac{3}{2},\ -\dfrac{27}{16}\right)$

POI: $\dfrac{d^2y}{dx^2} = 12x^2 - 12x = 0 \rightarrow x = 0 \text{ or } 1 \rightarrow (0,\ 0),\ (1,\ -1)$

There are several key trait values, so a vertical table might be better.

| Trait | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $x<0$ | + | – | + | Above the $x$-axis, decreasing and concave up |
| $x=0$ | 0 | 0 | 0 | Zero and POI |
| $0<x<1$ | – | – | – | Below the $x$-axis, decreasing and concave down |
| $x=1$ | –1 | – | 0 | POI |
| $1<x<\frac{3}{2}$ | – | – | + | Below the $x$-axis, decreasing and concave up |
| $x=\frac{3}{2}$ | $-\frac{27}{16}$ | 0 | + | Minimum Point |
| $\frac{3}{2}<x<2$ | – | – | – | Below the $x$-axis, decreasing and concave down |
| $x=2$ | 0 | – | – | Zero |
| $x>2$ | + | + | + | Above the $x$-axis, increasing and concave up |




$y=x^4-2x^3$

EX 5 Make a table of key traits and a complete sketch of $y=\dfrac{x+1}{x^2-x-6}$.

Zeros: $y=0 \rightarrow x=-1 \rightarrow (-1,\ 0)$

VAs: $y=\infty \rightarrow x=3 \text{ or } -2$

Extreme Points: $\dfrac{dy}{dx}=\dfrac{(x^2-x-6)(1)-(x+1)(2x-1)}{(x^2-x-6)^2}=0$

$-x^2-2x-5=0$

$x=\dfrac{2\pm\sqrt{4-20}}{2}=\text{no real solutions}$

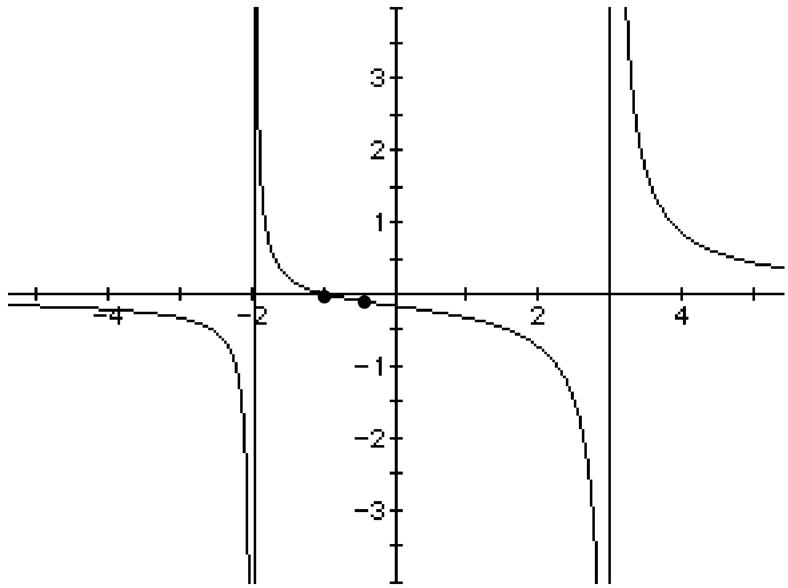
POI:

$$\frac{d^2y}{dx^2}=\frac{(x^2-x-6)^2(-2x-2)-(-x^2-2x-5)\left(2(x^2-x-6)(2x-1)\right)}{(x^2-x-6)^4}$$

$$\frac{d^2y}{dx^2}=\frac{2x^3+6x^2+30x+2}{(x^2-x-6)^3}=0$$

$x=-0.068$ (Must be solved with graphing calculator.)

| Trait | $f(x)$ | $f'(x)$ | $f''(x)$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -2)$ | – | – | – | Below axis, decreasing, concave down |
| $x=-2$ | $\infty$ | $\infty$ | $\infty$ | VA |
| $(-2,\ -1)$ | + | – | + | Above axis, decreasing, concave up |
| $x=-1$ | 0 | – | + | Zero |
| $(-1,\ -0.068)$ | – | – | + | Below axis, decreasing, concave up |
| $x=-0.068$ | –0.157 | – | 0 | POI |
| $(-0.068,\ 3)$ | – | – | – | Below axis, decreasing, concave down |
| $x=3$ | $\infty$ | $\infty$ | $\infty$ | VA |
| $(3,\ \infty)$ | + | – | + | Above axis, decreasing, concave up |




$$y = \frac{x+1}{x^2 - x - 6}$$

Of course, the function's equation need not be given in order to find the graph. The table or sign patterns would suffice, especially if the function is piece-wise defined.

## 12-3 Free Response Homework

Sketch the possible graph of a function that has the traits shown below

1.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|---|
| $x<2$ | + | – | + |
| $x=2$ | 0 | – | + |
| $2<x<3$ | – | – | + |
| $x=3$ | –5 | 0 | + |
| $3<x<5$ | – | + | + |
| $x=5$ | 0 | + | 0 |
| $5<x<7$ | + | + | – |
| $x=7$ | 9 | 0 | – |
| $7<x<9$ | + | – | – |
| $x=9$ | 1 | – | 0 |
| $9<x<10$ | + | – | + |
| $x=10$ | 0 | – | + |
| $x>10$ | – | – | + |

2.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|---|
| $x<-1$ | + | + | + |
| $x=-1$ | 3 | DNE | DNE |
| $-1<x<4$ | + | – | + |
| $x=4$ | 1 | 0 | + |
| $4<x<9$ | + | + | + |
| $x=9$ | 3 | DNE | DNE |
| $x>9$ | + | + | + |

Find the key traits, make a table, and sketch.

3. $y=x^3-12x$

4. $y=x^3-36x$

5. $y=2x^3-9x^2-24x+11$

6. $y=x^3+5x^2+3x-4$

7. $y=3x^4-15x^2+7$

8. $y=x^4-17x^2+16$

9. $y=\dfrac{3x}{x^2+9}$

10. $y=\dfrac{3x}{x^2-9}$

11. $y=\dfrac{x+1}{x^2-2x-3}$

12. $y=\dfrac{x^2-1}{x^2+x-6}$

13. $y=5x^{2/3}-x^{5/3}$

14. $y=x^2e^{-x}$

15. $y=\left(x-x^2\right)e^x$

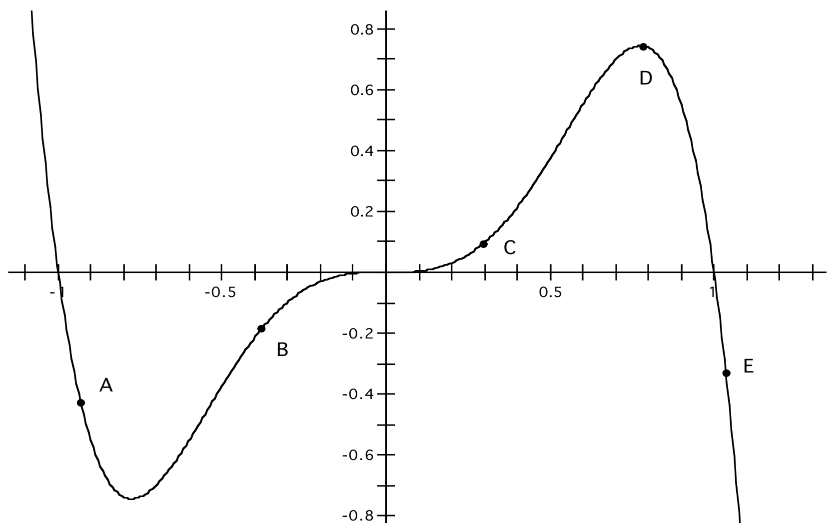
16. $y=\left(x^2\right)\sqrt{4-x}$

17. $y=\left(x^2-4\right)\sqrt{16-x^2}$

18. $y=\dfrac{x^2-9}{x^2-1}$

12-3 Multiple Choice Homework

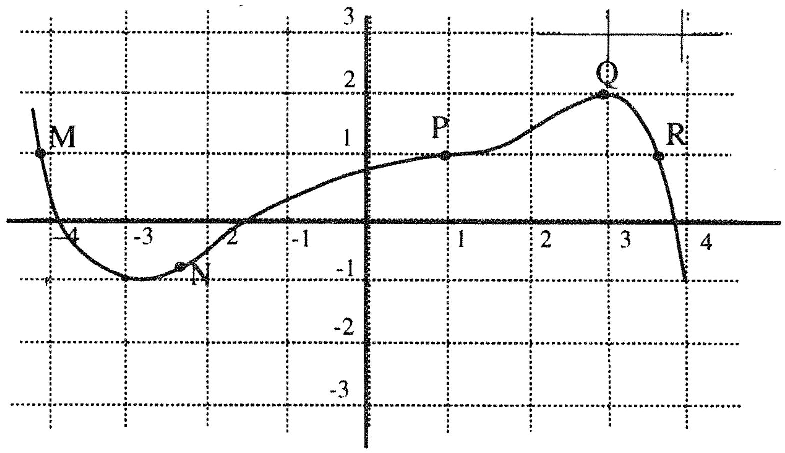
1. The graph of the function $f(x)$ is shown below. At which point on the graph of $f(x)$ is $f'(x)>0$ and $f''(x)>0$?




a) A b) B c) C d) D e) E

2. At what point on the graph of $f$ below is , $f' < 0$, and $f'' < 0$

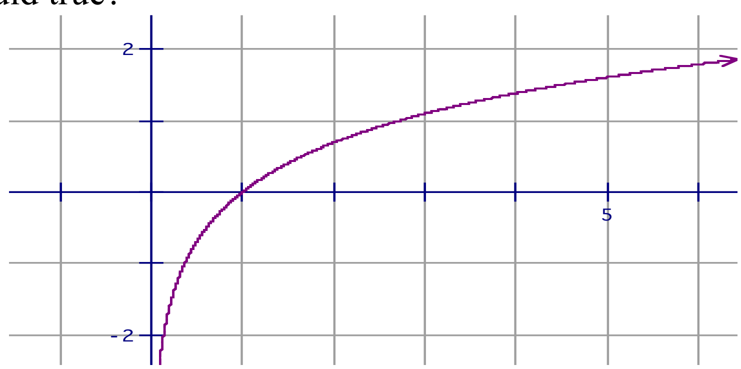
the graph of $f$




a) M b) N c) P d) Q e) R

---

3. The graph of a twice differentiable function $f$ is shown below. Which of the following could true?




(a) $f''(1) < f(1) < f'(1)$

(b) $f(1) < f''(1) < f'(1)$

(c) $f(1) < f'(1) < f''(1)$

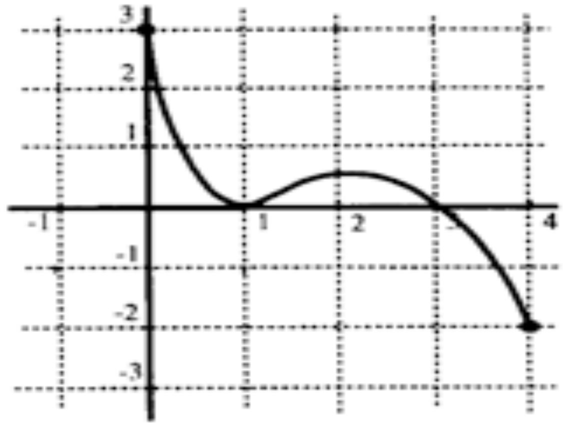
(d) $f'(1) < f''(1) < f(1)$

(e) $f'(1) < f(1) < f''(1)$

---

4. The graph of the second derivative of a function $g(x)$ is shown here. Use the graph to determine which of the statements below is/are true.

Graph of $g''$




I. $g(x)$ has points of inflection at $x=1$ and $x=3$

II. The graph of $g(x)$ is concave down on the interval $(3, 4)$.

III. If $g'(0)=0$, then $g(x)$ is increasing on $x \in (2, 3)$

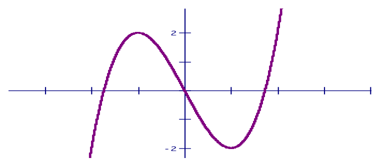
a) I only b) II only c) I and II only

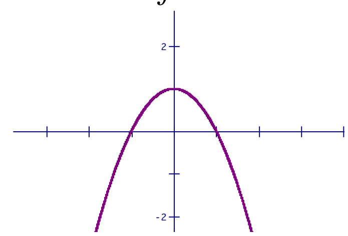
d) II and III only e) I, II, and III

---

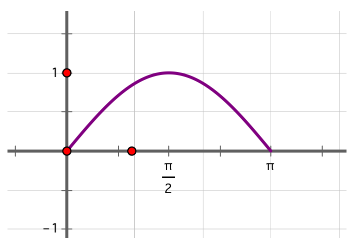
5. Which of the following pairs of graphs represent the graph of *f* and its derivative $f'$?

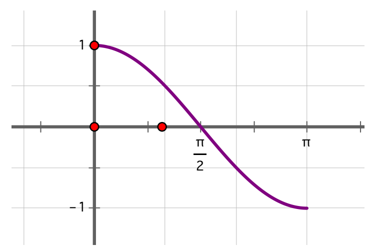
I. $f$ $f'$



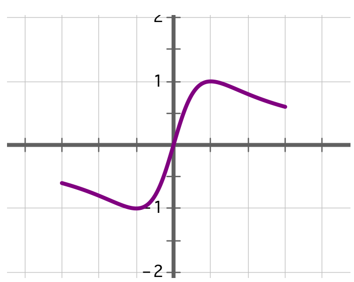



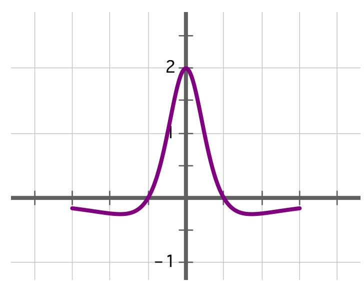

II.




III.




a) I only b) II only c) III only

d) I and II e) II and III only

6. Is $f'(x) > 0$ and $f''(x) < 0$, which of the following could be a table of values for $f(x)$?

(A)

| $x$ | $f(x)$ |
|---|---|
| –1 | 4 |
| 0 | 3 |
| 1 | 1 |

(B)

| $x$ | $f(x)$ |
|---|---|
| –1 | 4 |
| 0 | 4 |
| 1 | 4 |

(C)

| $x$ | $f(x)$ |
|---|---|
| –1 | 4 |
| 0 | 5 |
| 1 | 6 |

(D)

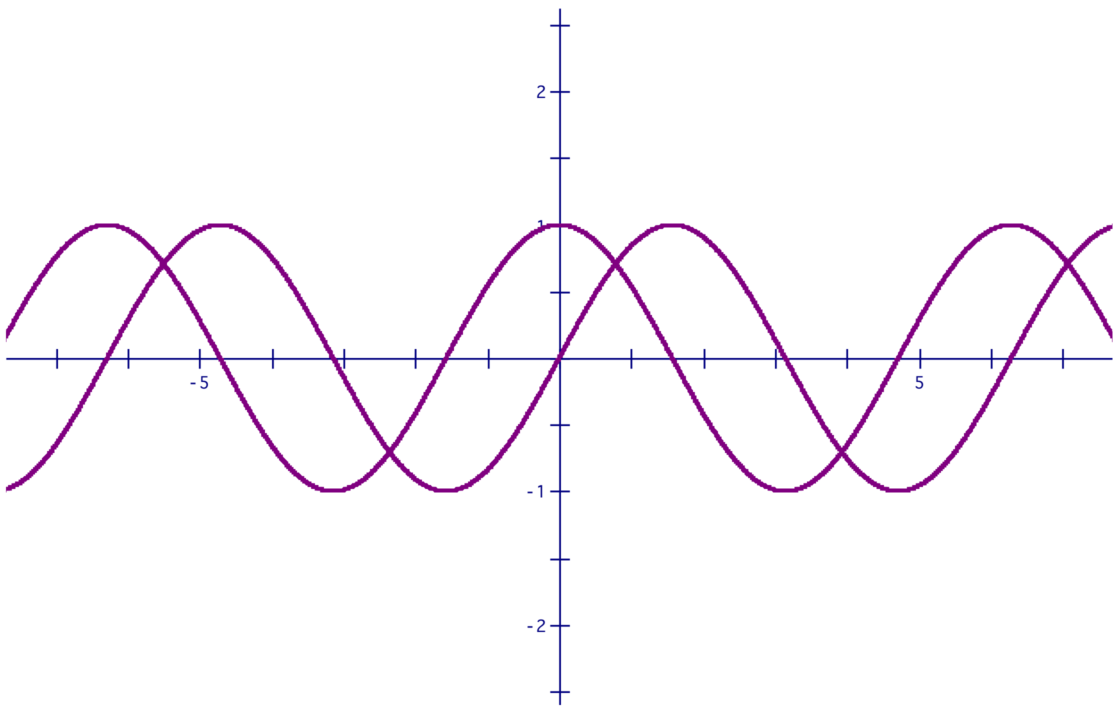
| $x$ | $f(x)$ |
|---|---|
| –1 | 4 |
| 0 | 5 |
| 1 | 7 |

(E)

| $x$ | $f(x)$ |
|---|---|
| –1 | 4 |
| 0 | 6 |
| 1 | 7 |

12-4: Graphical Analysis

Until now, the derivative has been treated as a by-product of some "original" function. But the derivative can be considered a function itself. As such, it has all the traits that any function of its type would have. For example, there is no graphical distinction between $y = \cos x$ as a function and $y = \cos x$ as the derivative of $y = \sin x$. In fact, viewing the graphs shows that some of the traits of $y = \cos x$ correspond directly to traits of the $y = \sin x$.




Most obvious is the fact that the zeros of one curve are the critical values of the other. This should make sense since the way to find a critical value is to set the derivative equal to zero. There are several other comparisons between the curves that can be made:

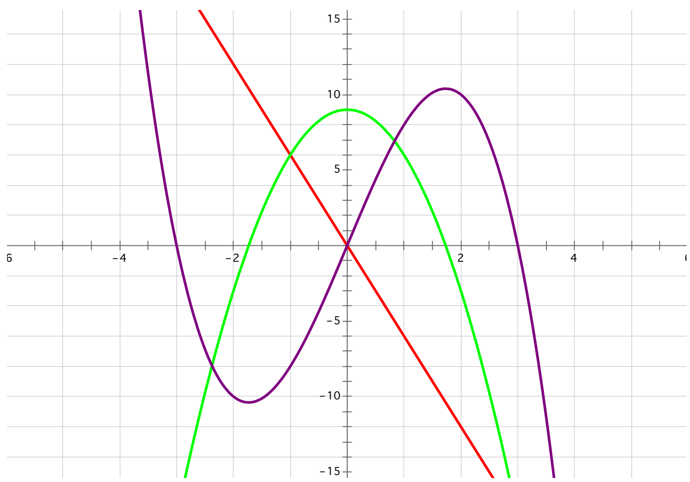
EX 1 Find the sign patterns of $f(x)=9x-x^3$, $f'(x)=9-3x^2$, and $f''(x)=-6x$ and compare them.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $y$ | $+$ | $0$ | $-$ | $0$ | $+$ | $0$ | $-$ |
| $x$ | | $-3$ | | $0$ | | $3$ | |

| | | | | | |
|---|---|---|---|---|---|
| $y'$ | $-$ | $0$ | $+$ | $0$ | $-$ |
| $x$ | | $-\sqrt{3}$ | | $\sqrt{3}$ | |

| | | | |
|---|---|---|---|
| $y''$ | $+$ | $0$ | $-$ |
| $x$ | | $0$ | |

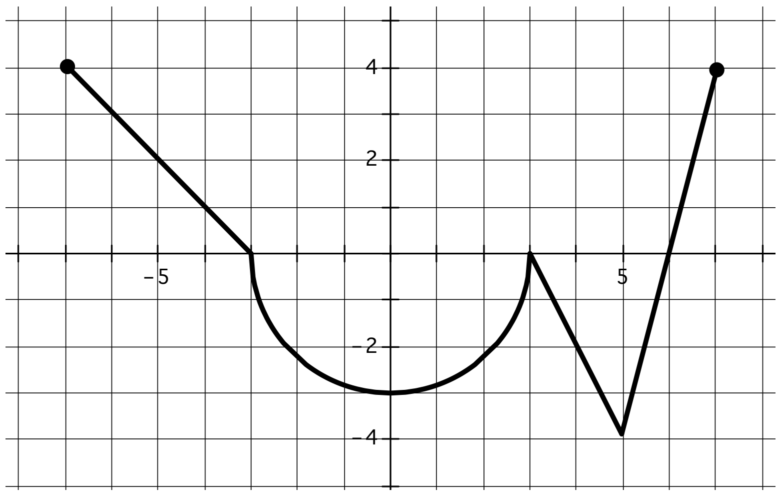
Compare the graphs below of $f(x)=9x-x^3$, $f'(x)=9-3x^2$, and $f''(x)=-6x$.




It can be seen that the extreme points of $f(x)$ line up with the zeros of $f'(x)$, and the extreme point of $f'(x)$ lines up with the zero of $f''(x)$ and the POI of $f(x)$. Similarly, the positive parts of $f''(x)$ match the increasing part of $f'(x)$ and the concave up part of $f(x)$.

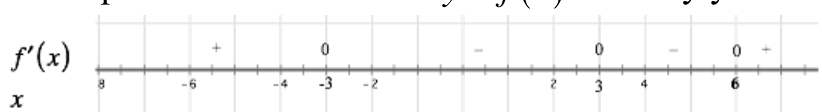
Almost any trait can be found from a known first derivative curve, but, unfortunately, the derivative is not enough to tell the exact original curve, because, though the critical values and $x$-values of the POI are known, their $y$-values cannot be found. Similarly, the zeros of the original curve cannot be found from its derivative.

EX 2 The graph below is of $f'(x)$ on $x \in [-7, 7]$.




a) Where are the relative maximums and relative minimums of $y = f(x)$?

b) Where are the points of inflection of $y = f(x)$? Justify your answer.

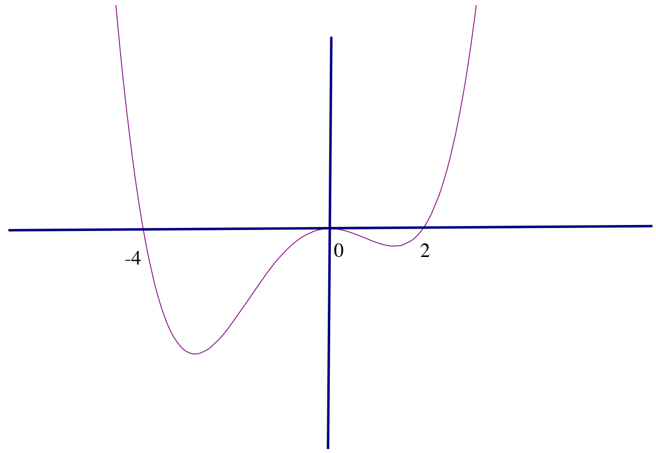



a) $y = f(x)$ has a relative maximum at $x = -3$ because $f'(x)$ switches from positive to negative, and $y = f(x)$ has a relative maximum at $x = 7$ because $f(x)$ is increasing— $f'(x)$ is positive— and then ends.

$y = f(x)$ has a relative minimum at $x = 6$ because $f'(x)$ switches from negative to positive, and $y = f(x)$ has a relative minimum at $x = -7$ because $f(x)$ starts and is decreasing— $f'(x)$ is negative.


b) $y = f(x)$ has POIs at $x = 0, 3,$ and $5$ because $f'(x)$ switches from increasing to decreasing or vice versa.

Ex 3 The graph below is of $f'(x)$. Find the sign patterns of $f'(x)$ and $f''(x)$.




$$\begin{array}{c|ccccccc}
\dfrac{dy}{dx} & + & 0 & - & 0 & - & 0 & + \\
x & & -4 & & 0 & & 2 &
\end{array}$$

$$\begin{array}{c|ccccccc}
\dfrac{d^2y}{dx^2} & - & 0 & + & 0 & - & 0 & + \\
x & & -3 & & 0 & & 1.7 &
\end{array}$$

These would then tell us the intervals of increasing and decreasing and of concave up and concave down for $f(x)$. Those relationships can be summarized in the table below.

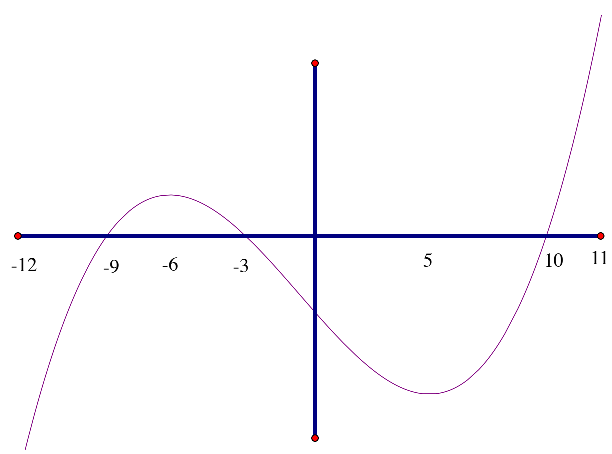
**The Chart: This can help organize what each piece of information can reveal.**

| | $f(x)$ | | | $f'(x)$ | | | $f''(x)$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | **+** | **0** | **-** | **+** | **0** | **-** | **+** | **0** | **-** |
| $f(x)$ | $y$ value positive | $y$ value zero | $y$ value negative | Interval of increasing | max. or min. | Interval of decreasing | Concave up | POI | Concave down |
| $f'(x)$ | | | | $y'$ value positive | $y'$ value zero | $y'$ value negative | Interval of increasing | max. or min. | Interval of decreasing |
| $f''(x)$ | | | | | | | $y''$ value positive | $y''$ value zero | $y''$ value negative |

LEARNING OUTCOME

Determine information about a function from the graph of its derivative.

EX 4 If the curve below is $y = f'(x)$, a) where are the relative maximums and relative minimums of $y = f(x)$ and b) where are the points of inflection of $y = f(x)$? Justify your answer.



What is pictured here is a two dimensional representation of the sign patterns of the first AND second derivatives.

$$\begin{array}{c|ccccccc} y' & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -9 & & -3 & & 10 & \end{array}$$

$$\begin{array}{c|ccccc} y'' & + & 0 & - & 0 & + \\ \hline x & & -6 & & 5 & \end{array}$$

a) Where are the relative maximums and relative minimums of $f(x)$? Justify your answer.

Relative maximums are at $x = -12, -3,$ and $11$.

- At $x = -12$, $f(x)$ starts and is decreasing because $f'(x)$ is negative.
- At $x = -3$, the derivative switches from positive to negative.
- At $x = 11$, $f(x)$ has been increasing before hitting the boundary because the derivative has been positive.

Relative minimums are at $x = -9$, and $10$.

- At $x = -9$ and $10$, the derivative switches from negative to positive.

b) Where are the points of inflection of $f(x)$? Justify your answer.

POIs at $x = -6$ and 5

- $f(x)$ has at points of inflection where $f'(x)$ switches from increasing to decreasing or vice versa.

NB. Endpoints cannot be points of inflection.

EX 5 Given the same graph of $y = f'(x)$ in EX 4 and $f(0) = 0$, sketch a likely curve for $f(x)$ on $x \in [-2, 2]$.

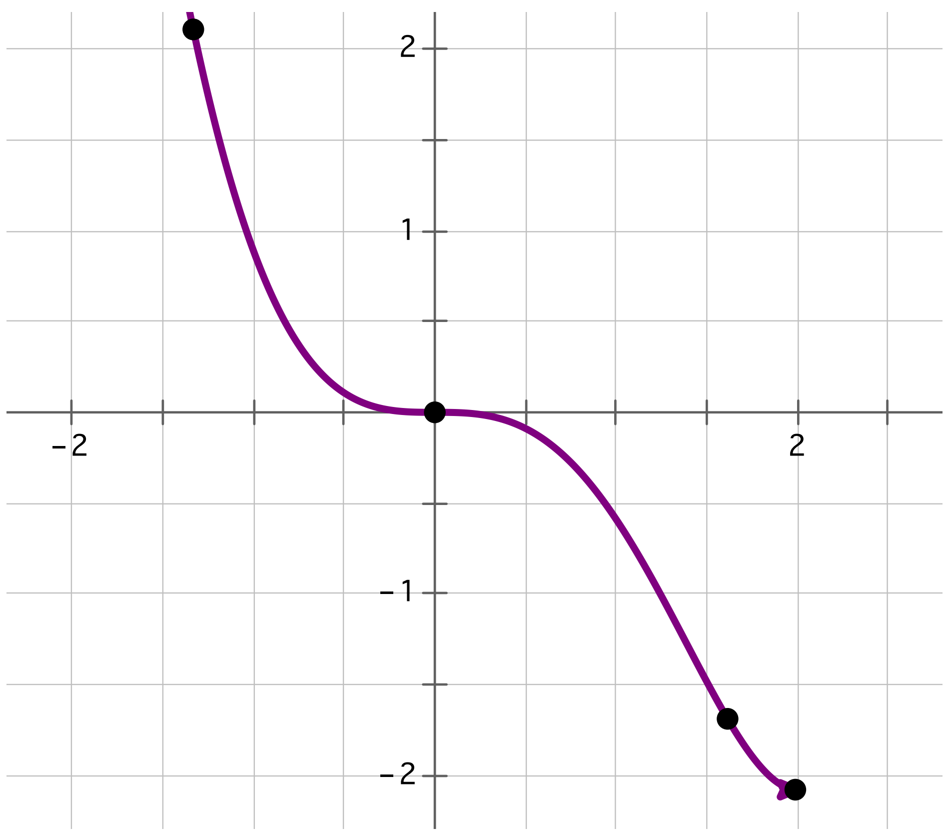
On $x \in [-2, 2]$, $f'(x)$ is negative so $f(x)$ in decreasing on that interval.

Since $f(x)$ is decreasing and $f(0)$ is the zero, the curve must be above the $x$-axis on $x \in [-2, 0)$ and below the $x$-axis on $x \in (0, 2]$.

On $x \in (-2, 0)$ and $x \in (1.7, 2)$, the slope of $f'(x)$ (which is $f''(x)$) is positive, so $f(x)$ is concave up. Similarly, on $x \in (0, 1.7)$, $f''(x)$ is negative since $f'(x)$ is decreasing, so $f(x)$ is concave down.

$(0, 0)$ and $x = 1.7$are points of inflection.

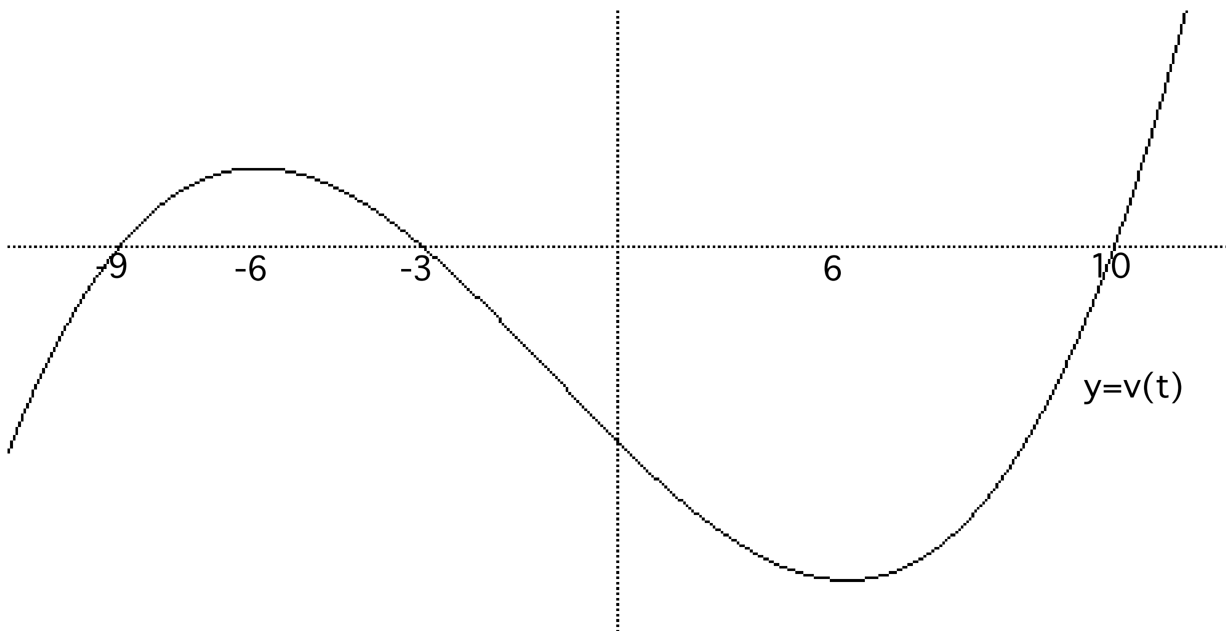
Putting it all together, this is a likely sketch:



Note that the markings for scale on the $y$-axis are arbitrary. This is because the $y$-values of the endpoints of the given domain cannot be known, from the given information.

The graph of $y = f'(x)$ need not be a member of any family of functions either. The sign patterns of the first and second derivatives can be deduced from any graph.

EX 6 If the following graph is the velocity of a particle in rectilinear motion, what can be deduced about the acceleration and the distance?




The particle is accelerating until $t = -6$, decelerating from $t = -6$ to $t = 6$, and then accelerating again. The distance is at a relative maximum value at $t = -3$ and at a relative minimum value at $t = -9$ and $t = 10$, but what the distances from the origin are cannot be known.

Key Phrases:

$f(x)$ is increasing when $f'(x)$ is positive.

$f(x)$ is decreasing when $f'(x)$ is negative.

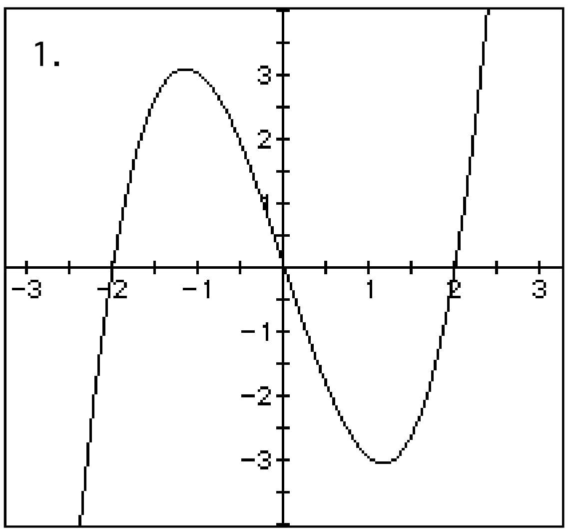
$f(x)$ is concave up when $f'(x)$ is increasing.

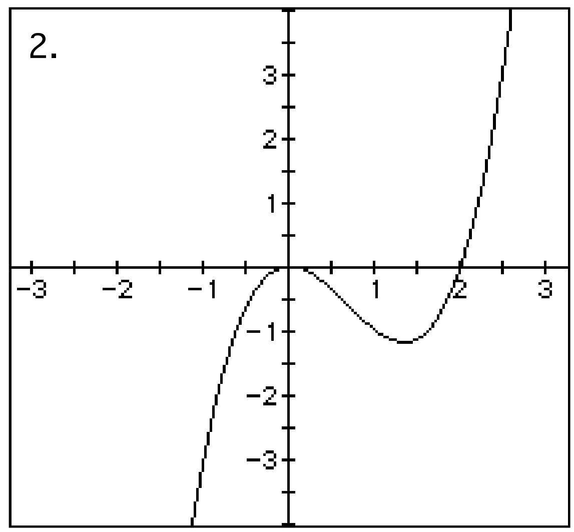
$f(x)$ is concave down when $f'(x)$ is decreasing.

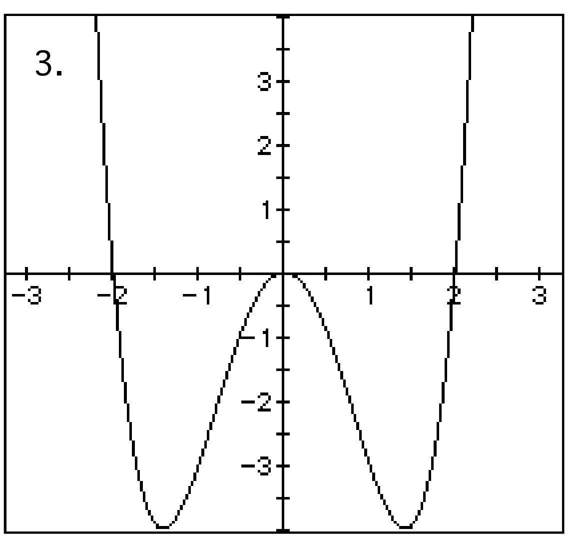
12-4 Free Response Homework

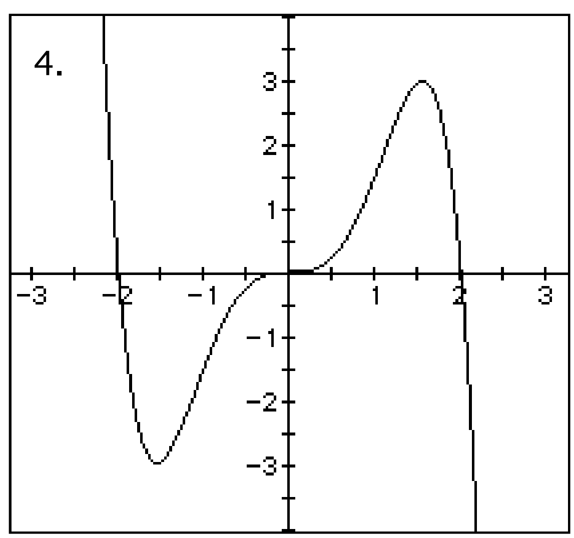
If the curve below is $y = f'(x)$ on $x \in [-3, 3]$, show the sign patterns of the first and second derivatives. Then find

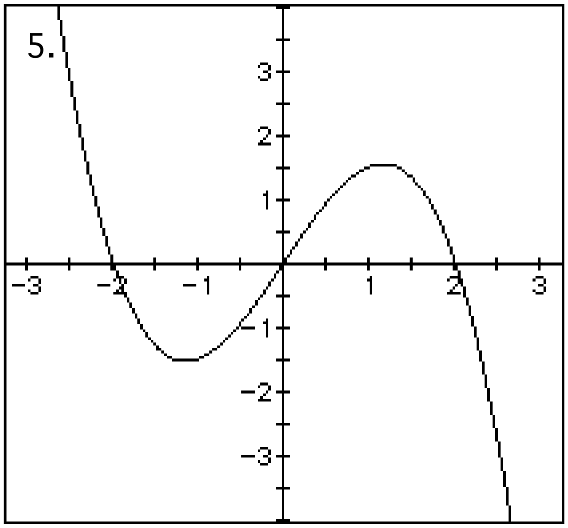
a) the critical values for the maximum and minimum points,
b) $x$-coordinates of the POIs,
c) intervals of increasing and decreasing,
d) the intervals of concavity of $y = f(x)$, and
e) sketch a possible curve for $f(x)$ with $y$-intercept $(0, 0)$.

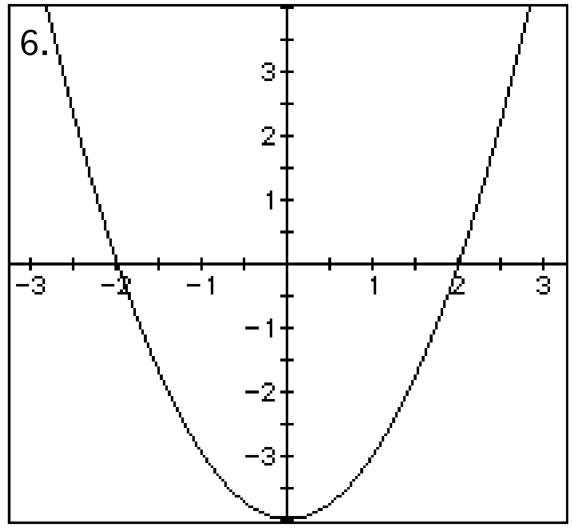


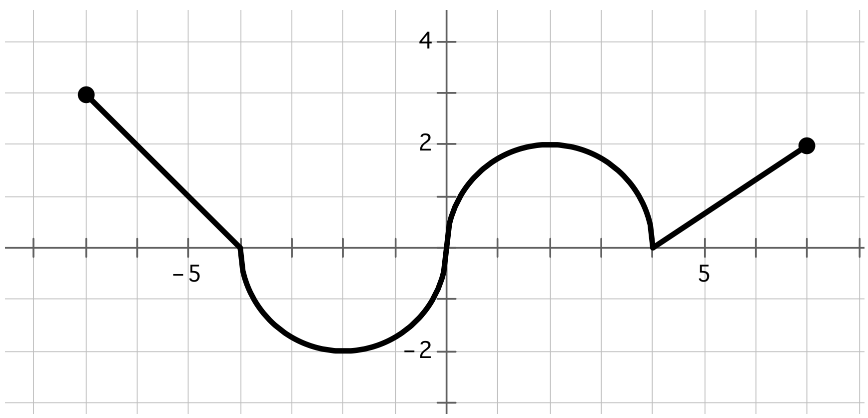





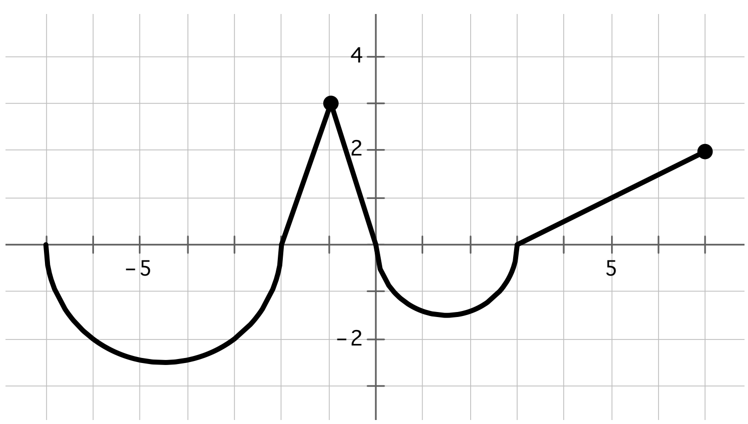




The graphs below is of $f'(x)$ on $x \in [-7, 7]$.



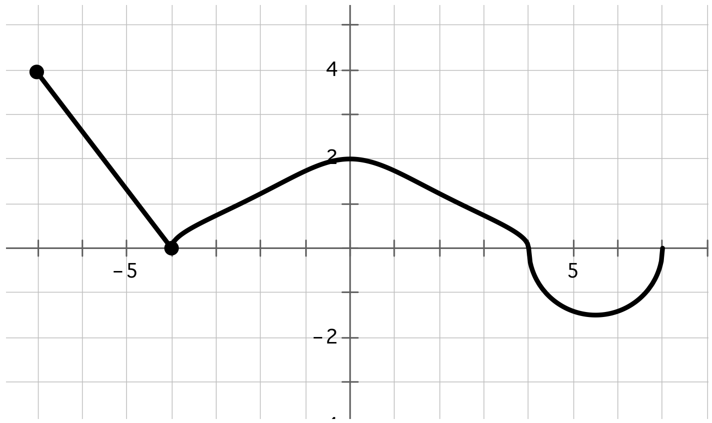
7a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

7b) Where are the points of inflection of $y = f(x)$? Justify your answer.



8a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

8b) Where are the points of inflection of $y = f(x)$? Justify your answer.

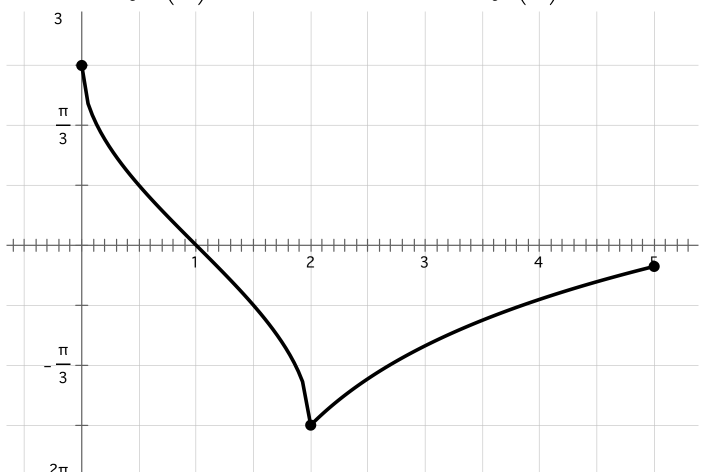



9a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

9b) Where are the points of inflection of $y = f(x)$? Justify your answer.

12-4 Multiple Choice Homework

1. This is the graph of $f'(x)$, the derivative of $f(x)$.




Which of the following sign patterns are hidden with the graph.

I. $$\begin{array}{c|ccc} f'(x) & + & 0 & - \\ x & & 1 & \end{array}$$

II. $$\begin{array}{c|ccccc} f''(x) & + & 0 & - & dne & - \\ x & & 1 & & 2 & \end{array}$$

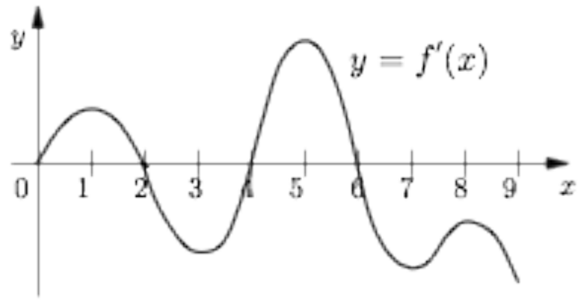
III. $$\begin{array}{c|ccc} f''(x) & - & dne & + \\ x & & 2 & \end{array}$$

a) I only b) II only c) I and II only

d) I and III only e) I, II, and III

---

2. The graph of the derivative $f'(x)$ on the interval $[0, 9]$ is shown below.



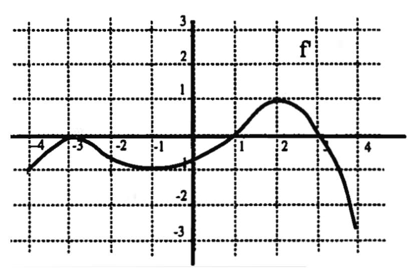

If $g'(x) = -3f'(x)$, how many maxima will $g(x)$ have on the interval $[0, 9]$?

a) None b) One c) Two d) Three e) Four

---

3. The figure shows the graph of $f'$, the *derivative* of a function *f*. The domain of is the interval $-4 \le x \le 4$. Which of the following are true about the graph of *f*?

**graph of the derivative of *f***

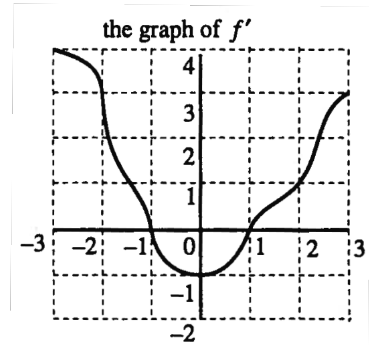



I. At the points where $x = -3$ and $x = 2$ there are horizontal tangents.
II. At the point where $x = 1$ there is a relative minimum point.
III. At the point where $x = -3$ there is an inflection point.

a) None b) II only c) III only d) II and III only e) I, II, and III

---

4. The graph of $f'$, the derivative of a function $f$, is shown below. Which of the following statements are true about the function $f$?




I. $f$ is increasing on the interval $(-2, -1)$.

II. $f$ has an inflection point at $x = 0$.

III. $f$ is concave up on the interval $(-1, 0)$.

a) I only b) II only c) III only d) I and II only e) I, II, and III

---

5. The derivative of the function $g$ is $g'(x) = \cos(\sin x)$. At the point where $x = 0$ the graph of $g$

I. is increasing

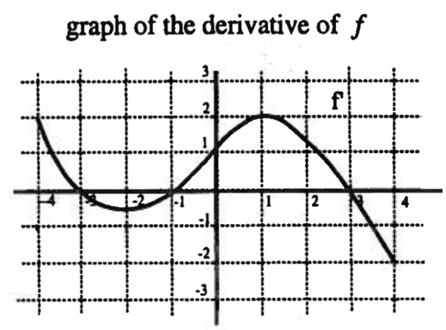
II. is concave down

III. attains a relative maximum point

a) I only b) II only c) III only d) I and III only e) I, II, and III

---

6. The graph of the **derivative** of a function is $f$ shown below. Which of the following is true about the function $f$?




I. $f$ is increasing on the interval $(-2,1)$.

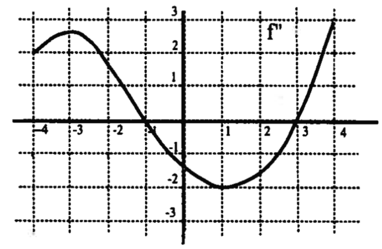
II. $f$ is continuous at $x = 0$.

III. $f$ has an inflection point at $x = -2$.

a) I only b) II only c) III only d) II and III only e) I, II, and III

---

7. The graph of the **second derivative** of a function *f* is shown at right. Which of the following is true?




I. The graph of *f* has an inflection point at $x = -1$.

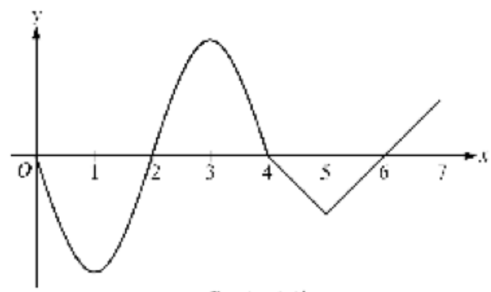
II. The graph of *f* is concave down on the interval $(-1,3)$.

III. The graph of the derivative function $f'$ is increasing at $x = 1$.

a) I only  b) II only  c) III only  d) I and II only  e) I, II, and III

---

8. The graph of the derivative of *f* is shown below. On which of the following intervals is $f$ decreasing?




Graph of $f'$

a) $(2, 4)$ only

b) $(3, 5)$ only

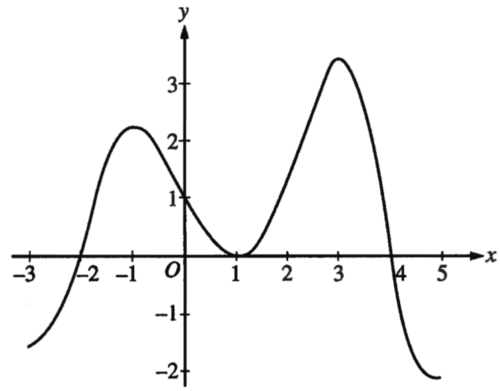
c) $(0, 1)$ and $(3, 5)$

d) $(2, 4)$ and $(6, 7)$

e) $(0, 2)$ and $(4, 6)$

---

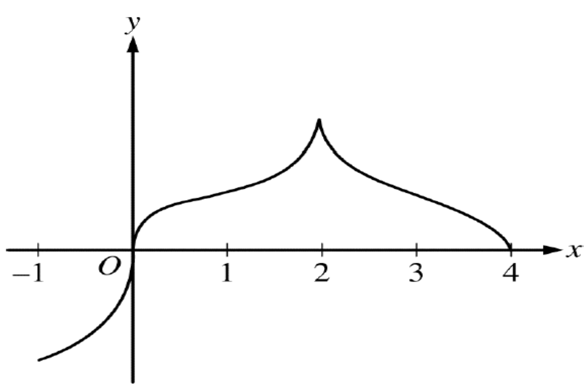
9. The graph of the derivative of a function $f$ is shown in the figure at right. The graph has horizontal tangent lines at $x=-1$, $x=1$ and $x=3$. At which of the following values of $x$ does $f$ have a relative maximum?



Graph of $f'$

a) –2 only
b) 1 only
c) 4 only
d) –1 and 3 only
e) –2, 1, and 4

---



Graph of $f'$

10. The graph of $f'$ is shown above. The line tangent to $f'$ at $x=0$ is vertical, and $f'$ is not differentiable at $x=2$. Which of the following statements is TRUE?

a.) $f'$ does not exist at $x=2$.

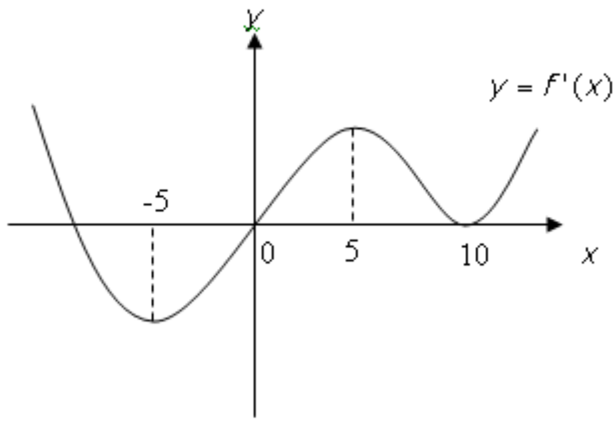
b) $f$ is decreasing on the interval (2, 4)

c) The graph of $f$ has a point of inflection at $x=0$

d) The graph of $f$ has a relative maximum at $x=0$

e) The graph of $f$ has a point of inflection at $x=2$

11. Below is the graph of $f'(x)$. For what value(s) of $x$ does $f(x)$ have a minimum?




a) 0 only

b) 0 and 10

c) -5 and 5

d) -5 and 10

e) None of these

The Second Derivative Practice Test
Part 1: CALCULATOR REQUIRED

Round to 3 decimals places. Show all work.

**Multiple Choice** (3 pts. each)

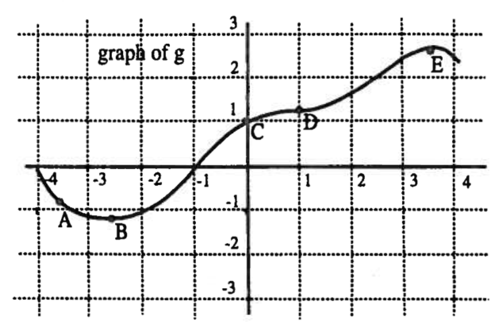
1. On which interval is the graph of $f(x) = 4x^{3/2} - 3x^2$ both concave down and increasing?

(a) $(0, 1)$
(b) $\left(0, \frac{1}{2}\right)$
(c) $\left(0, \frac{1}{4}\right)$
(d) $\left(\frac{1}{4}, \frac{1}{2}\right)$
(e) $\left(\frac{1}{4}, 1\right)$

2. Consider the function $f(x) = \frac{x^4}{2} - \frac{x^5}{10}$. The *derivative* of *f* attains its maximum values at

(a) 3
(b) 4
(c) 5
(d) 0
(e) There is no maximum value

3. At which point on the graph of $y = g(x)$, shown to the right, is $g'(x) = 0$ and $g''(x) = 0$?




(a) A (b) B (c) C (d) D (e) E

**Free Response** (10 pts. each)

1. Find zeros and intervals above/below the $x$-axis for $y = \ln\left(\frac{-3x}{x^2+9}\right)$.

2. Find the extreme points and intervals of increasing/decreasing of $y = \ln\left(\frac{-3x}{x^2+9}\right)$.

3. Find the points of inflection and intervals of concavity for $y = \ln\left(\frac{-3x}{x^2+9}\right)$.

The Second Derivative Practice Test
Part 2: NO CALCULATOR ALLOWED

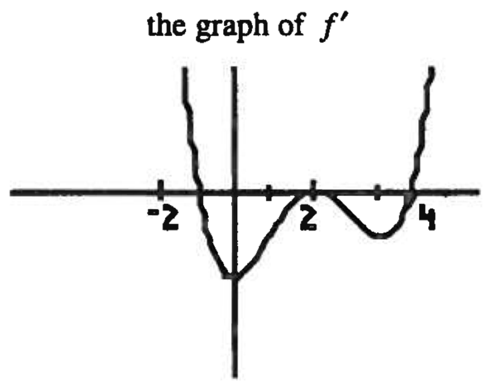
Round to 3 decimals places. Show all work.

**Multiple Choice** (3 pts. each)

4. What is the $x$-coordinate of the point of inflection on the graph of $y=\frac{1}{3}x^3+5x^2+24$?

(a) 5
(b) 0
(c) $-\frac{10}{3}$
(d) $-5$
(e) $-10$

5. Let $f$ be a function that has domain $[-2,\ 5]$. The graph of $f'$ is shown at the right. Which of the following statements are TRUE?

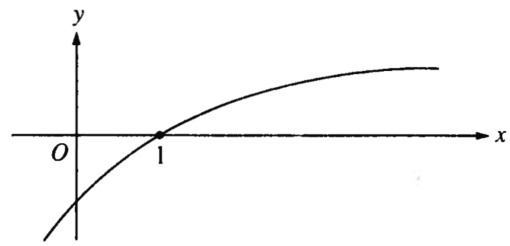



I. $f$ has a relative maximum at $x=-1$.
II. $f$ has an absolute minimum at $x=0$.
III. $f$ is concave down for $-2<x<0$.
IV. $f$ has inflection points at $x=0$, $x=2$, and $x=3$.

(a) I, II, IV
(b) I, III, IV
(c) II, III, IV
(d) I, II, III
(e) I, II, III, IV

6. The graph of the twice-differentiable function *f* is shown in the figure below. Which of the following is true?

(a) $f(1) < f'(1) < f''(1)$
(b) $f(1) < f''(1) < f'(1)$
(c) $f'(1) < f(1) < f''(1)$
(d) $f''(1) < f(1) < f'(1)$
(e) $f''(1) < f'(1) < f(1)$




**Free Response** (10 pts. each)

4. Using the traits found in FR #1-3, **sketch** $y = \ln\left(\frac{-3x}{x^2+9}\right)$.

Round at 3 decimal places. Show all work.

1. Find the extreme points of $y = 2x^2 e^{\frac{1}{2}x}$.

2. Find the points of inflection of $y = \ln\left(x^3 - 4x\right)$.

3. Make a key trait table for $y = \frac{4x}{x^2+2}$.

4. Sketch $y = x\sqrt{12+4x-x^2}$, labeling all the key traits.

Round at 3 decimal places. Show all work.

1. Find the extreme points of $y = x\sqrt{12 + 4x - x^2}$.

2. Find the points of inflection of $y = 2x^2 e^{\frac{1}{2}x}$.

3. Make a key trait table for $y=\ln\left(x^3-4x\right)$.

4. Sketch $y=\dfrac{4x}{x^2+2}$, labeling all the key traits.

Round at 3 decimal places. Show all work.

1. Find the extreme points of $y = \dfrac{4x}{x^2+2}$.

2. Find the points of inflection of $y = x\sqrt{12+4x-x^2}$.

3. Make a key trait table for $y = 2x^2 e^{\frac{1}{2}x}$.

4. Sketch $y = \ln\left(x^3 - 4x\right)$, labeling all the key traits.

Round at 3 decimal places. Show all work.

1. Find the extreme points of $y = \ln\left(x^3 - 4x\right)$.

2. Find the points of inflection of $y = \dfrac{4x}{x^2 + 2}$.

3. Make a key trait table for $y = x\sqrt{12 + 4x - x^2}$.

4. Sketch $y = 2x^2 e^{\frac{1}{2}x}$, labeling all the key traits

The Second Derivative Homework Answer Key

12-1 Free Response Homework

1. $60x^2+54x-8$

2. $168x^5-60x^3+18x$

3. $y''=-2\left(\sin x^2+2x^2\cos x^2\right)$

4. $y''=2\sec^2 x\left(2\tan^2 x+\sec^2 x\right)$

5. $\dfrac{d^2y}{dx^2}=9\sec 3x\left(\tan^2 3x+\sec^2 3x\right)$

6. $\dfrac{d^2y}{dx^2}=4e^{2x}(x+1)$

7. $f''(x)=\dfrac{-2\left(x^2-3\right)}{\left(x^2+3\right)^2}$

8. $g''(x)=\dfrac{-2}{(x-2)^2}$

9. $h''(x)=\dfrac{5}{\left(x^2+5\right)^{3/2}}$

10. $F''(x)=\dfrac{2}{\left(3x^2-2x+1\right)^{3/2}}$

11. $\dfrac{d^2y}{dx^2}=\dfrac{14\left(3x^2+10\right)}{\left(x^2-10\right)^3}$

12. $\dfrac{d^2y}{dx^2}=\dfrac{18x^2-18x}{\left(x^2-x+1\right)^3}$

13. $x\in(-\infty,\ -3)\cup(0,\ 3)$

14. $x\in\left(\dfrac{2}{3},\ 2\right)\cup(2,\ \infty)$

15a. Yes b. No

16a. $x\in(-\infty,\ -9)\cup(-3,\ -1)\cup(-1,\ 7)$

b. $x\in(-\infty,\ -7)\cup(-2,\ 0)\cup(5,\ \infty)$

c. $x\in(-7,\ -3)$

17.
$$\begin{array}{c|ccc} y'' & - & 0 & + \\ x & \longleftarrow & -\frac{1}{3} & \longrightarrow \end{array}$$

CD: $x\in\left(-\infty,\ -\dfrac{1}{3}\right)$; CU: $x\in\left(-\dfrac{1}{3},\ \infty\right)$ POI: $\left(-\dfrac{1}{3},\ -12.593\right)$

18.

$$\begin{array}{c|ccccc} y'' & + & 0 & - & 0 & + \\ \hline x & & 1 & & \frac{7}{3} & \end{array}$$

CD: $x \in \left(1,\ \frac{7}{3}\right)$; CU: $x \in (-\infty,\ 1) \cup \left(\frac{7}{3},\ \infty\right)$ POIs: $(1,\ 5)$, $\left(\frac{7}{3},\ -4.481\right)$

19.

$$\begin{array}{c|ccccccc} y'' & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -3.464 & & 0 & & 3.464 & \end{array}$$

CD: $x \in (-3.464,\ 0) \cup (3.464,\ \infty)$; CU: $x \in (-\infty,\ 3.464) \cup (0,\ 3.464)$

POIs: $(-3.464,\ 0.866)$, $(0,\ 0)$, $(3.464,\ -0.866)$

20.

$$\begin{array}{c|ccccc} y'' & + & 0 & - & 0 & + \\ \hline x & & -2 & & 2 & \end{array}$$

CD: $x \in (-2,\ 2)$; CU: $x \in (-\infty,\ -2) \cup (2,\ \infty)$; POI: none

21.

$$\begin{array}{c|ccccc} y'' & \text{DNE} & + & 0 & - & \text{DNE} \\ \hline x & -2\sqrt{2} & & 0 & & 2\sqrt{2} \end{array}$$

CD: $x \in \left(0,\ 2\sqrt{2}\right)$; CU: $x \in \left(-2\sqrt{2},\ 0\right)$ POI: $(0,\ 0)$

22.

$$\begin{array}{c|ccccc} y'' & \text{End} & - & 0 & + & \text{End} \\ \hline x & 0 & & \pi & & 2\pi \end{array}$$

CD: $x \in (0,\ \pi)$; CU: $x \in (\pi,\ 2\pi)$; $POI: (\pi,\ 0)$

23.

$$\begin{array}{c|ccc} y'' & - & 0 & + \\ \hline x & & 2 & \end{array}$$

CD: $x \in (-\infty,\ 2)$; CU: $x \in (2,\ \infty)$; POI: $\left(0,\ \frac{2}{e^2}\right)$

24.
$$\begin{array}{c|ccccc} y'' & + & 0 & - & 0 & + \\ x & & -\frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} & \end{array}$$

CD: $x \in \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$; CU: $x \in \left(-\infty,\ -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}},\ \infty\right)$

POIs: $\left(\frac{1}{\sqrt{2}},\ 0.607\right)$, $\left(-\frac{1}{\sqrt{2}},\ 0.607\right)$

25.
$$\begin{array}{c|ccccccc} y'' & - & \text{DNE} & + & 0 & - & \text{DNE} & + \\ x & & -3 & & 0 & & 3 & \end{array}$$

CD: $x \in (-\infty, -3) \cup (0,\ 3)$

CU: $x \in (-3,\ 0) \cup (3,\ \infty)$

POI: $(0,\ 0)$

26. Concave up for all domain

27. $a(t) = 6t - 12$

28. $a(t) = 60t^3 - 456t^2 + 846t - 360$

29. $a(t) = \dfrac{-9\pi^2}{16}\sin\left[\dfrac{\pi}{4}(t-5)\right]$

30. $a(t) = \dfrac{2t(t^2-3)}{(t^2+1)^2}$

31. $a(t) = \dfrac{e^t(2-e^t)}{2(1+e^t)^{3/2}}$

32. $a(6) = \dfrac{\pi^2}{2\sqrt{2}}$

33. $a(-1) = -2$

34. $a(2) = 0.736$

<u>12-1 Multiple Choice Homework</u>

1. D 2. D 3. A 4. B 5. E

6. A 7. A 8. C 9. D 10. E 11. E

<u>12-2 Free Response Homework</u>

1. $x=-7$ is @ a maximum point; $x=4$ is @ a minimum point

2. $x=3$ is @ a maximum point; $x=-3$ is @ a minimum point

3. $x=-0.863$ is @ a maximum point; $x=\pm\sqrt{3},\ \frac{9}{4}$ are @ minimum points

4. $x=0$ is @ a minimum point

5. $x=\pm\frac{\sqrt{27}}{2}$ are @ maximum points; $x=0$ is @ a minimum point

6. $x=0.618$ is @ a maximum point $x=-2.999$ is @ a minimum point

7. $x=-2$ is @ a maximum point; $x=2$ is @ a minimum point

8. $x=-2\sqrt{3}$ is @ a maximum point; $x=2\sqrt{3}$ is @ a minimum point

9. $x=-1$ is @ a maximum point; $x=4$ is @ a minimum point

10. $x=-3$ is @ a maximum point; $x=-\frac{1}{3}$ is @ a minimum point

11. $x=0$ is @ a maximum point; $x=\pm 1.581$ are @ a minimum points

12. $x=0$ is @ a maximum point; $x=\pm\sqrt{8.5}$ are @ a minimum points

13. $x=3$ is @ a maximum point; $x=-3$ is @ a minimum point

14. There are no extremes

15. $x=1.268$ is @ a maximum point; $x=4.732$ is @ a minimum point

16. $x=0.072$ is @ a maximum point; $x=13.928$ is @ a minimum point

17. $x=-3.236$ is @ a maximum point; $x=1.236$ is @ a minimum point

18. $x=2$ is @ a maximum point; $x=0$ is @ a minimum point

19. $x=\pm 3.464$ are @ a maximum points; $x=0$ is @ a minimum points. [NB. $x=\pm 4$ are also at minimums, but not by the 2nd Dx Test.]

20. $(1,2)$ is at a maximum

21. $(1,2)$ is at a maximum and $(-1,-2)$ is at a minimum.

22. There are no maximums or minimums. $\frac{dy}{dx}\neq 0$ anywhere.

23. $(-3,2)$ is at a minimum and $(3,-2)$ is at a maximum.

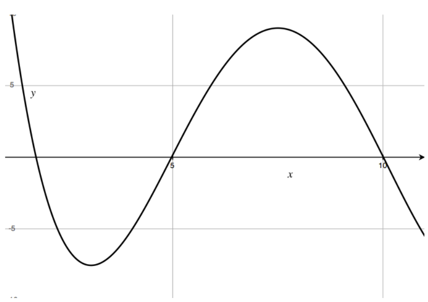
<u>12-2 Multiple Choice Homework</u>

1. B 2. D 3. C 4. D 5. D

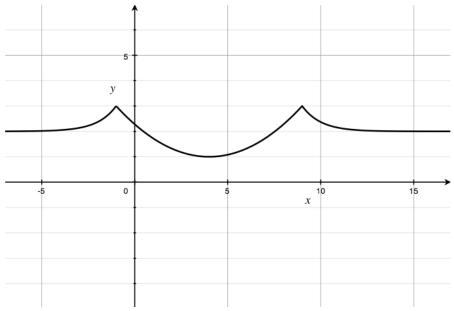
6. C 7. A

<u>12-3 Free Response Homework</u>

1.



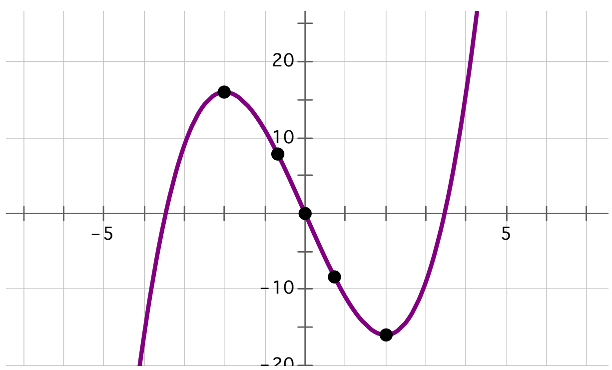

2.



y
5
-5
0
5
10
15
x

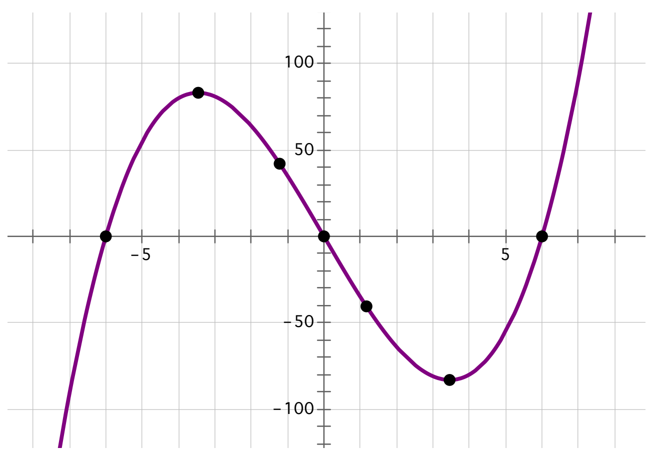
3. $y = x^3 - 12x$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $\left(-\infty,\ -2\sqrt{3}\right)$ | – | + | – | below, increasing, CD |
| $x = -2\sqrt{3}$ | 0 | + | – | zero |
| $\left(-2\sqrt{3},\ -2\right)$ | + | + | – | above, increasing, CD |
| $x = -2$ | 16 | 0 | – | maximum pt. |
| $\left(-2,\ 0\right)$ | + | – | – | above, decreasing, CD |
| $x = 0$ | 0 | – | 0 | POI |
| $\left(0,\ 2\right)$ | + | – | + | above, decreasing, CU |
| $x = 2$ | –16 | 0 | + | minimum pt. |
| $\left(2,\ 2\sqrt{3}\right)$ | – | + | + | below, increasing, CU |
| $x = 2\sqrt{3}$ | 0 | + | + | zero |
| $\left(2\sqrt{3},\ \infty\right)$ | + | + | + | above, increasing, CU |



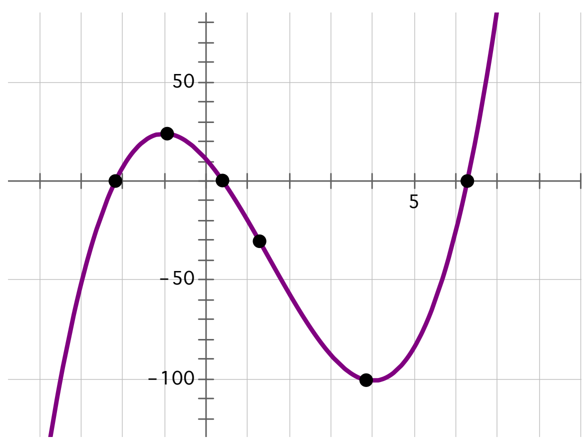
4. $y = x^3 - 36x$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -6)$ | – | + | – | below, increasing, CD |
| $x = -6$ | 0 | + | – | zero |
| $(-6,\ -2\sqrt{3})$ | + | + | – | above, increasing, CD |
| $x = -2\sqrt{3}$ | 16 | 0 | – | maximum pt. |
| $(-2\sqrt{3},\ 0)$ | + | – | – | above, decreasing, CD |
| $x = 0$ | 0 | – | 0 | POI |
| $(0,\ 2\sqrt{3})$ | + | – | + | above, decreasing, CU |
| $x = 2\sqrt{3}$ | –16 | 0 | + | minimum pt. |
| $(2\sqrt{3},\ 6)$ | – | + | + | below, increasing, CU |
| $x = 6$ | 0 | + | + | zero |
| $(6,\ \infty)$ | + | + | + | above, increasing, CU |



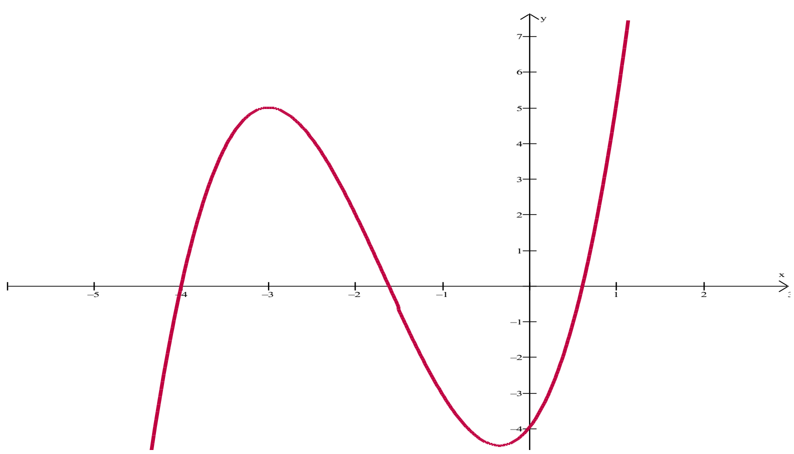
5. $y = 2x^3 - 9x^2 - 24x + 11$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -2.176)$ | – | + | – | below, increasing, CD |
| $x = -2.176$ | 0 | + | – | zero |
| $(-2.176,\ -1)$ | + | + | – | above, increasing, CD |
| $x = -1$ | 24 | 0 | – | maximum pt. |
| $(-1,\ 0.403)$ | + | – | – | above, decreasing, CD |
| $x = 0.403$ | 0 | – | 0 | zero |
| $(0.403,\ 1.5)$ | + | – | + | above, decreasing, CU |
| $x = 1.5$ | -38.5 | – | + | POI |
| $(1.5,\ 4)$ | – | – | + | below, decreasing, CU |
| $x = 4$ | –101 | 0 | + | minimum pt. |
| $(4,\ 6.273)$ | – | + | + | below, increasing, CU |
| $x = 6.273$ | 0 | + | + | zero |
| $(6.273,\ \infty)$ | + | + | + | above, increasing, CU |



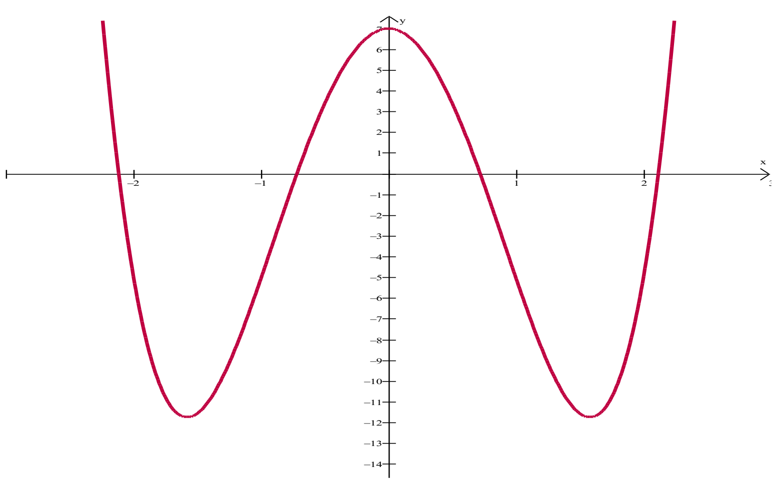
6. $y = x^3 + 5x^2 + 3x - 4$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty, -4)$ | – | + | – | below, increasing, CD |
| $x = -4$ | 0 | + | – | zero |
| $(-4, -3)$ | + | + | – | above, increasing, CD |
| $x = -3$ | 5 | 0 | – | maximum pt. |
| $\left(-3, -\frac{5}{3}\right)$ | + | – | – | above, decreasing, CD |
| $x = -\frac{5}{3}$ | 0.259 | – | 0 | POI |
| $\left(-\frac{5}{3}, -1.618\right)$ | + | – | + | above, decreasing, CU |
| $x = -1.618$ | 0 | – | + | zero |
| $\left(-1.618, -\frac{1}{3}\right)$ | – | – | + | below, decreasing, CU |
| $x = -\frac{1}{3}$ | –4.481 | 0 | + | minimum pt. |
| $\left(-\frac{1}{3}, 0.618\right)$ | – | + | + | below, increasing, CU |
| $x = 0.618$ | 0 | + | + | zero |
| $(0.618, \infty)$ | + | + | + | above, increasing, CU |



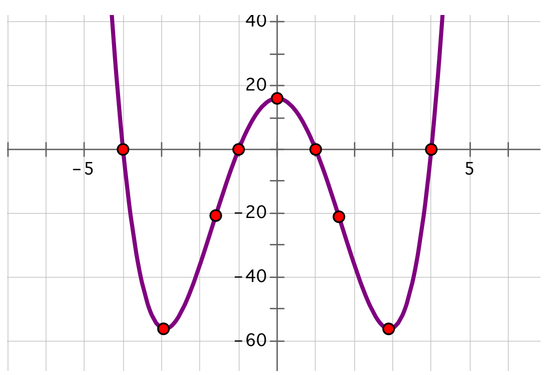
7. $y = 3x^4 - 15x^2 + 7$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty, -2.116)$ | + | – | + | above, decreasing, CU |
| $x = -2.116$ | 0 | – | + | zero |
| $(-2.116, -1.581)$ | – | – | + | below, decreasing, CU |
| $x = -1.581$ | –11.75 | 0 | + | minimum pt |
| $(-1.581, -0.913)$ | – | + | + | below, increasing, CU |
| $x = -0.913$ | –0.135 | + | 0 | POI |
| $(-0.913, -0.722)$ | – | + | – | below, increasing, CD |
| $x = -0.722$ | 0 | + | – | zero |
| $(-0.722, 0)$ | + | + | – | above, increasing, CD |
| $x = 0$ | 7 | 0 | – | maximum pt. |
| $(0, 0.722)$ | + | – | – | above, decreasing, CD |
| $x = 0.722$ | 0 | – | – | zero |
| $(0.722, 0.913)$ | – | – | – | below, decreasing, CD |
| $x = 0.913$ | –0.135 | – | 0 | POI |
| $(0.913, 1.581)$ | – | – | + | below, decreasing, CU |
| $x = 1.581$ | –11.75 | 0 | + | minimum pt. |
| $(1.581, 2.166)$ | – | + | + | below, increasing, CU |
| $x = 2.116$ | 0 | + | + | zero |
| $(2.116, \infty)$ | + | + | + | above, increasing, CU |



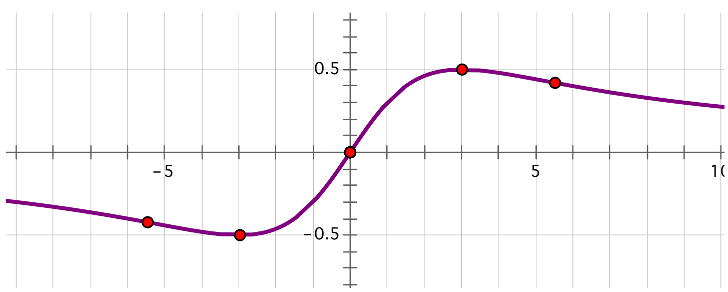
8. $y = x^4 - 17x^2 + 16$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -4)$ | $+$ | $-$ | $+$ | above, decreasing, CU |
| $x = -4$ | 0 | $-$ | $+$ | zero |
| $(-4,\ -2.915)$ | $-$ | $-$ | $+$ | below, decreasing, CU |
| $x = -2.915$ | $-56.25$ | 0 | $+$ | minimum pt |
| $(-2.915,\ -1.683)$ | $-$ | $+$ | $+$ | below, increasing, CU |
| $x = -1.683$ | $-24.138$ | $+$ | 0 | POI |
| $(-1.683,\ -1)$ | $-$ | $+$ | $-$ | below, increasing, CD |
| $x = -1$ | 0 | $+$ | $-$ | zero |
| $(-1,\ 0)$ | $+$ | $+$ | $-$ | above, increasing, CD |
| $x = 0$ | 36 | 0 | $-$ | maximum pt. |
| $(0,\ 1)$ | $+$ | $-$ | $-$ | above, decreasing, CD |
| $x = 1$ | 0 | $-$ | $-$ | zero |
| $(1,\ 1.683)$ | $-$ | $-$ | $-$ | below, decreasing, CD |
| $x = 1.683$ | $-24.138$ | $-$ | 0 | POI |
| $(0.913,\ 2.915)$ | $-$ | $-$ | $+$ | below, decreasing, CU |
| $x = 2.915$ | $-56.25$ | 0 | $+$ | minimum pt. |
| $(2.915,\ 4)$ | $-$ | $+$ | $+$ | below, increasing, CU |
| $x = 4$ | 0 | $+$ | $+$ | zero |
| $(2.116,\ \infty)$ | $+$ | $+$ | $+$ | above, increasing, CU |



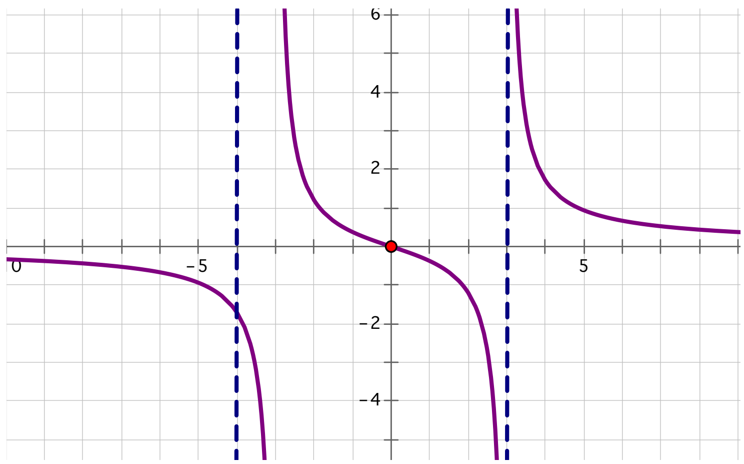
9. $y = \dfrac{3x}{x^2+9}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $\left(-\infty, -3\sqrt{3}\right)$ | – | – | – | below, decreasing, CD |
| $x=-3\sqrt{3}$ | –0.433 | – | 0 | POI |
| $\left(-3\sqrt{3}, -3\right)$ | – | – | + | below, decreasing, CU |
| $x=-3$ | –0.5 | 0 | + | minimum pt |
| $\left(-3, 0\right)$ | – | + | + | below, increasing, CU |
| $x=0$ | 0 | + | 0 | zero and POI |
| $\left(0, 3\right)$ | + | + | – | above, increasing, CD |
| $x=3$ | 0.5 | 0 | – | maximum pt. |
| $\left(3, 3\sqrt{3}\right)$ | + | – | – | above, decreasing, CD |
| $x=3\sqrt{3}$ | 0.433 | – | 0 | POI |
| $\left(3\sqrt{3}, \infty\right)$ | + | – | – | above, decreasing, CU |



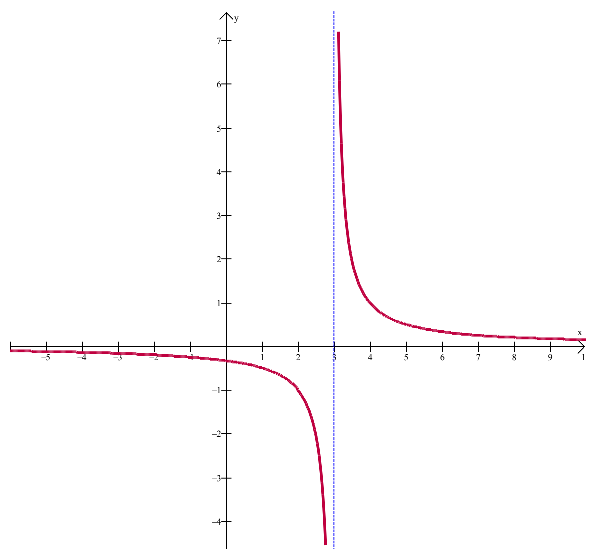
10. $y = \dfrac{3x}{x^2 - 9}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -3)$ | – | – | – | below, decreasing, CD |
| $x = -3$ | DNE | – | 0 | VA |
| $(-3,\ 0)$ | + | – | – | below, increasing, CU |
| $x = 0$ | 0 | + | 0 | zero and POI |
| $(0,\ 3)$ | - | – | – | above, increasing, CD |
| $x = 3$ | DNE | 0 | – | VA. |
| $(3,\ \infty)$ | + | – | – | above, decreasing, CU |



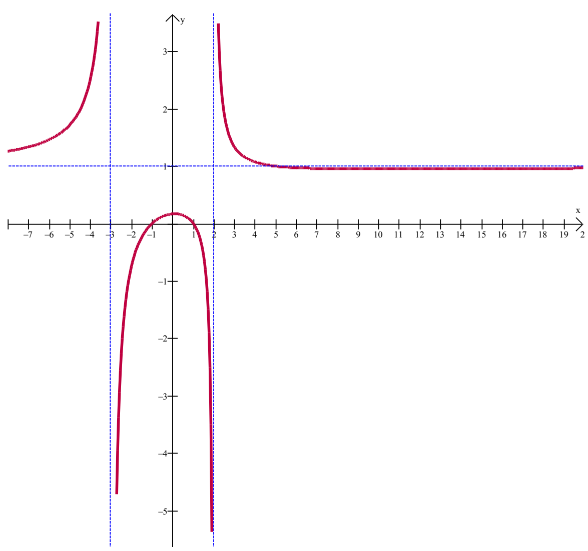
11. $y=\dfrac{x+1}{x^2-2x-3}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -1)$ | – | – | – | below, decreasing, CD |
| $x=-1$ | DNE | – | 0 | POE $\left(-1,\ -\frac{1}{4}\right)$ |
| $(-1,\ 3)$ | – | – | – | below, decreasing, CD |
| $x=-3$ | DNE | DNE | DNE | VA |
| $(3,\ \infty)$ | + | + | + | above, increasing, CU |



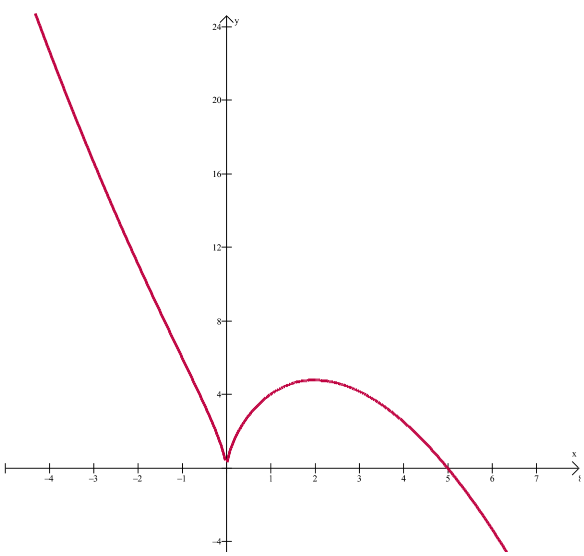
12. $y = \dfrac{x^2 - 1}{x^2 + x - 6}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty, -3)$ | + | + | + | above, increasing, CU |
| $x = -3$ | $\infty$ | $\infty$ | $\infty$ | VA |
| $(-3, -1)$ | – | + | – | below, increasing, CD |
| $x = -1$ | 0 | + | – | zero |
| $(-1, 0.101)$ | + | + | – | above, increasing, CD |
| $x = 0.101$ | 0.168 | 0 | – | maximum pt. |
| $(0.101, 1)$ | + | – | – | above, decreasing, CD |
| $x = 1$ | 0 | – | – | zero |
| $(1, 2)$ | – | – | – | below, decreasing, CD |
| $x = 2$ | $\infty$ | $\infty$ | $\infty$ | VA |
| $(2, 9.899)$ | + | – | + | above, decreasing, CU |
| $x = 9.899$ | 0.952 | 0 | + | minimum pt. |
| $(9.899, 14.929)$ | + | + | + | above, increasing, CU |
| $x = 14.929$ | 0.957 | + | 0 | POI |
| $(14.929, \infty)$ | + | + | – | above, increasing, CD |



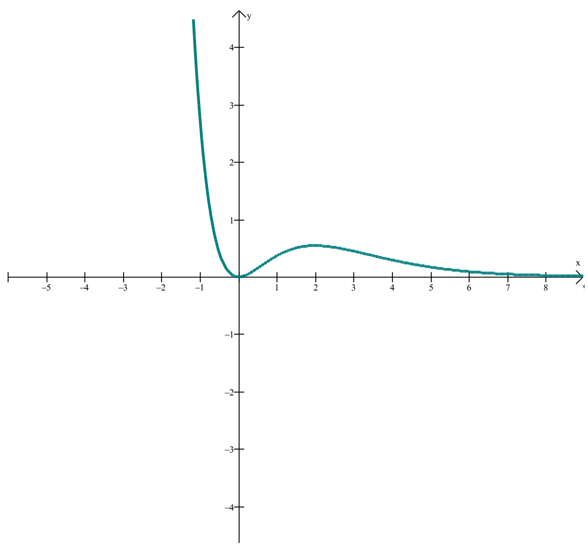
13. $y=5x^{2/3}-x^{5/3}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -1)$ | + | – | + | above, decreasing, CU |
| $x=-1$ | 6 | – | 0 | POI |
| $(-1,\ 0)$ | + | – | – | above, decreasing, CD |
| $x=0$ | 0 | $\infty$ | $\infty$ | zero, minimum pt. |
| $(0,\ 2)$ | + | + | – | above, increasing, CD |
| $x=2$ | 4.672 | 0 | – | maximum pt. |
| $(2,\ 5)$ | + | – | – | above, decreasing, CD |
| $x=5$ | 0 | – | – | zero |
| $(5,\ \infty)$ | – | – | – | below, decreasing, CD |



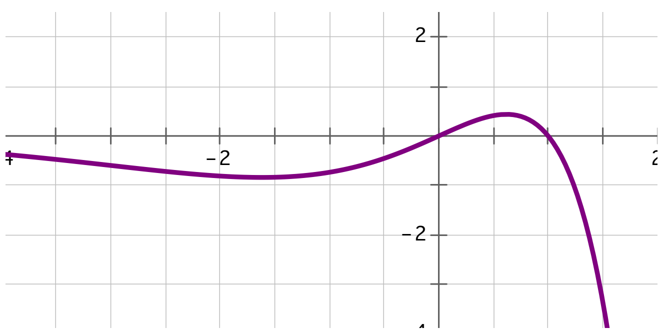
14. $y = x^2 e^{-x}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ 0)$ | + | – | + | above, decreasing, CU |
| $x = 0$ | 0 | 0 | + | zero, minimum pt. |
| $(-1,\ 0)$ | + | + | + | above, increasing, CU |
| $x = 0.586$ | 0.191 | + | 0 | POI |
| $(0.586,\ 2)$ | + | + | – | above, increasing, CD |
| $x = 2$ | 0.541 | 0 | – | maximum pt. |
| $(2,\ 3.41)$ | + | – | – | above, decreasing, CD |
| $x = 3.41$ | 0.384 | – | 0 | POI |
| $(3.41,\ \infty)$ | + | – | + | above, decreasing, CU |



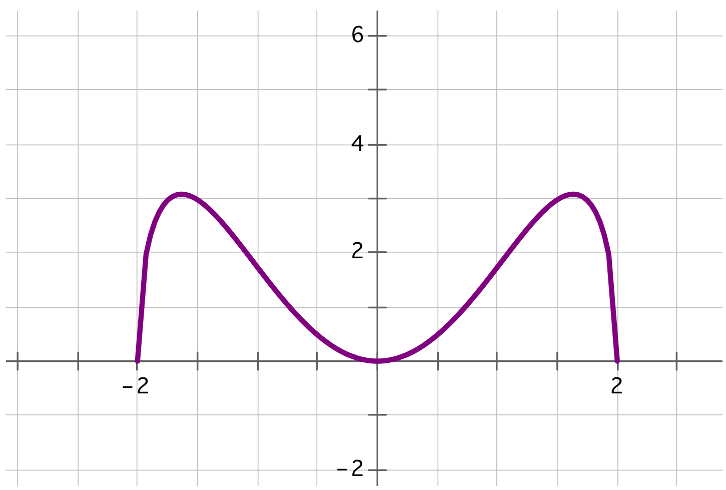
15. $y=\left(x-x^2\right)e^x$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $\left(-\infty,\ -3\right)$ | – | – | – | below, decreasing, CD |
| $x=-3$ | 16 | – | 0 | POI |
| $\left(-3,\ -1.618\right)$ | – | – | + | below, decreasing, CU |
| $x=-1.618$ | 3.542 | 0 | + | minimum pt. |
| $\left(-1.618, 0\right)$ | – | + | + | above, increasing, CU |
| $x=0$ | 0 | + | 0 | zero and POI |
| $\left(0,\ 0.618\right)$ | + | + | – | below, increasing, CD |
| $x=0.618$ | 16 | 0 | – | maximum pt. |
| $\left(0.618,\ 1\right)$ | + | – | – | above, decreasing, CD |
| $x=1$ | 0 | – | – | zero |
| $\left(1,\ \infty\right)$ | – | – | – | below, decreasing, CD |



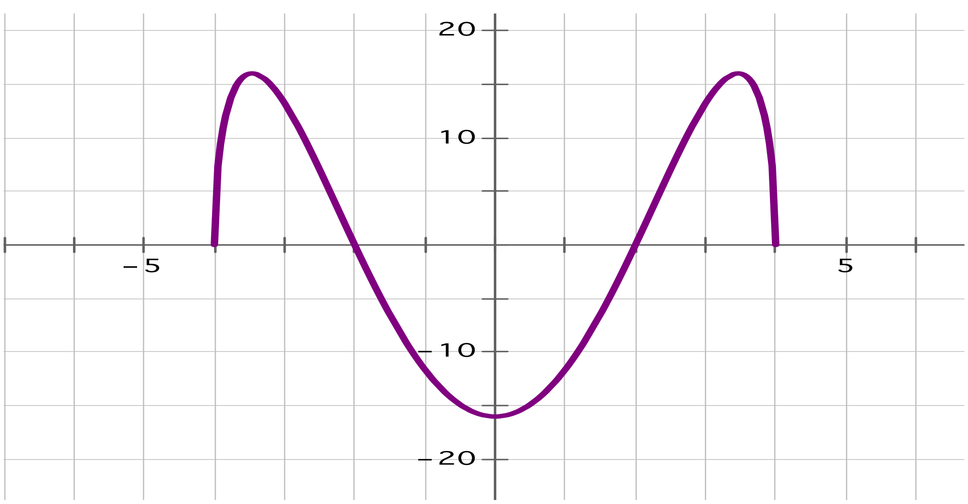
16. $y=\left(x^2\right)\sqrt{4-x}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ 0)$ | + | – | + | above, decreasing, CU |
| $x=0$ | 0 | 0 | + | zero, minimum pt. |
| $(0,\ 1.894)$ | + | + | + | above, increasing, CU |
| $x=1.894$ | 2.506 | + | 0 | POI |
| $\left(1.894,\ \frac{16}{5}\right)$ | + | + | – | above, increasing, CD |
| $x=\frac{16}{5}$ | 9.158 | 0 | – | maximum pt. |
| $\left(\frac{16}{5},\ 4\right)$ | + | + | + | above, decreasing, CD |
| $x=4$ | 0 | – | – | zero, endpoint, minimum pt. |



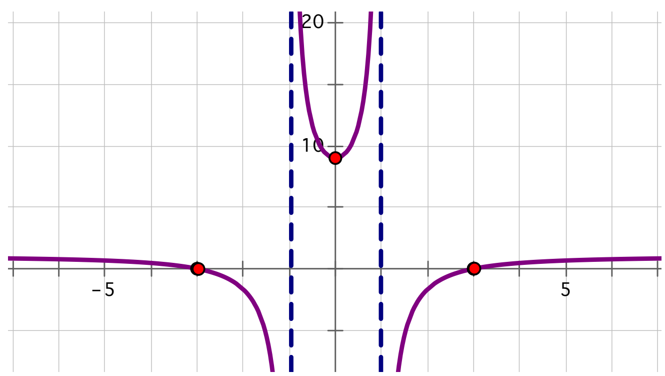
17. $y = (x^2 - 4)\sqrt{6 - x^2}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $x = 4$ | 0 | dne | – | zero, minimum pt. |
| $(-4, -2\sqrt{3})$ | + | + | – | above, increasing, CD |
| $x = -2\sqrt{3}$ | 16 | 0 | – | minimum pt. |
| $(-2\sqrt{3}, -2.252)$ | + | – | – | above, decreasing, CD |
| $x = -2.252$ | 3.542 | – | 0 | POI |
| $(-2.252, -2)$ | + | – | + | above, decreasing, CU |
| $x = -2$ | 0 | – | + | zero |
| $(-2, 0)$ | – | – | + | below, decreasing, CU |
| $x = 0$ | –16 | 0 | + | minimum pt. |
| $(0, 2)$ | – | + | + | below, increasing, CU |
| $x = 2$ | 0 | + | + | zero |
| $(2, 2.252)$ | + | + | + | above, increasing, CU |
| $x = 2.252$ | 3.542 | + | 0 | POI |
| $(2.252, -2\sqrt{3})$ | + | + | – | above, increasing, CD |
| $x = 2\sqrt{3}$ | 16 | 0 | – | maximum pt. |
| $(2\sqrt{3}, 4)$ | + | – | – | above, decreasing, CD |
| $x = 4$ | 0 | dne | – | zero, minimum pt. |



18. $y = \dfrac{x^2 - 9}{x^2 - 1}$

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty, -3)$ | – | – | – | below, decreasing, CD |
| $x = -3$ | 0 | – | 0 | zero |
| $(-3, -1)$ | – | – | + | below, decreasing, CU |
| $x = -1$ | dne | 0 | + | VA |
| $(-1, 0)$ | – | + | + | below, increasing, CU |
| $x = 0$ | 9 | + | 0 | maximum pt. |
| $(0, 1)$ | + | + | – | above, increasing, CD |
| $x = 1$ | dne | 0 | – | VA |
| $(1, 3)$ | + | – | – | above, decreasing, CD |
| $x = 3$ | 0 | – | – | zero |
| $(3\sqrt{3}, \infty)$ | + | – | – | above, decreasing, CU |



12-3 Multiple Choice Homework

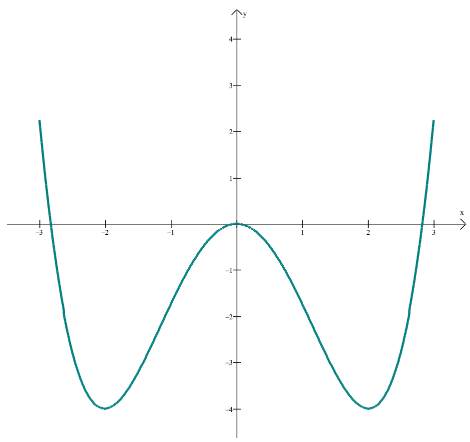
1. C 2. E 3. E 4. D 5. E 6. E

12-4 Free Response Homework

1a. min pts. @ $x = \pm 2$; max pt. @ $x = 0$ b. $x = \pm 1$

c. inc: $x \in (-2,\ 0) \cup (2, 3)$; dec: $x \in (-3,\ -2) \cup (0,\ 2)$

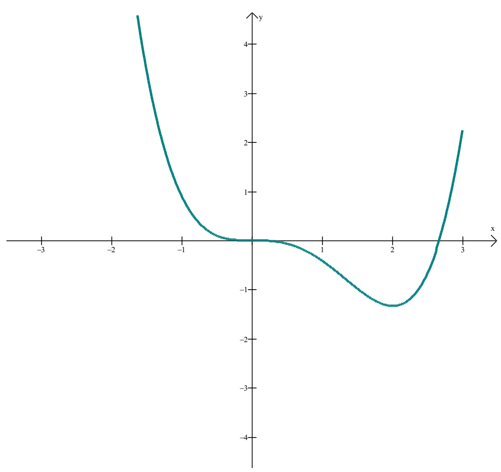
d. CU: $x \in (-3,\ -1) \cup (1,\ 3)$; CD: $x \in (-1,\ 1)$



2a. min pt. @ $x = 2$ b. $x = 0,\ 1.3$

c. inc: $x \in (2,\ 3)$; dec: $x \in (-3,\ 0) \cup (0,\ 2)$

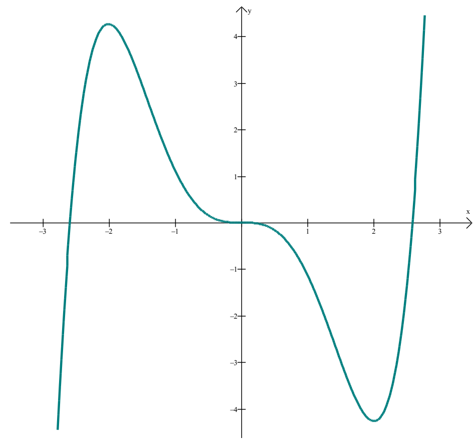
d. CU: $x \in (-3,\ 0) \cup (1.3,\ 3)$; CD: $x \in (0,\ 1.3)$



3a. min pt. @ $x=2$; max pt. @ $x=-2$ b. $x=\pm1.4,\ 0$

c. inc: $x\in(-3,\ -2)\cup(2,\ 3)$; dec: $x\in(-2,\ 0)\cup(0,\ 2)$

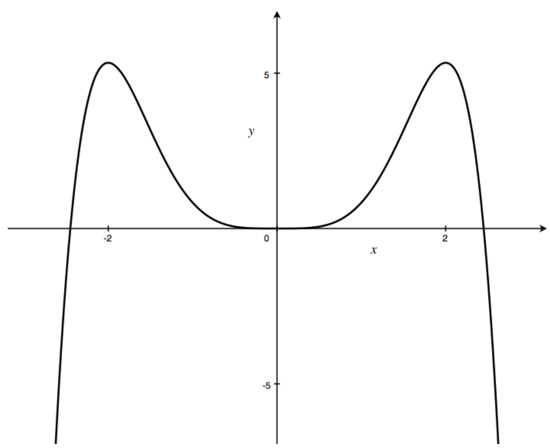
d. CU: $x\in(-1.4,\ 0)\cup(1.4,\ 3)$; CD: $x\in(-3,\ -1.4)\cup(0,\ 1.4)$



4a. max pts. @ $x=\pm2$; min pt. @ $x=0$ b. $x=\pm1.5$

c. inc: $x\in(-3,\ -2)\cup(0,\ 2)$; dec: $x\in(-2,\ 0)\cup(2,\ 3)$

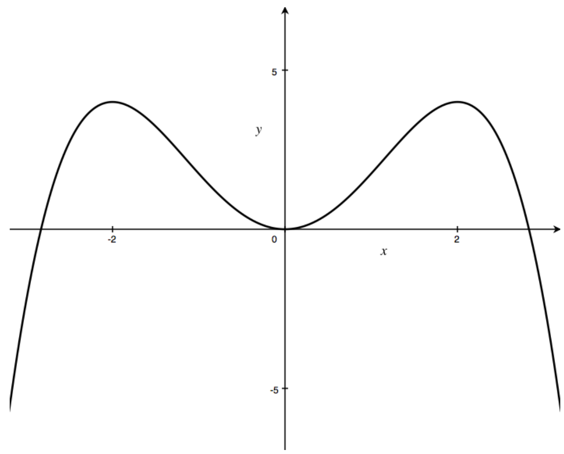
d. CU: $x\in(-1.5,\ 1.5)$; CD: $x\in(-3,\ -1.5)\cup(1.5,\ 3)$



5a. max pt. @ $x = \pm 2$; min pt. @ $x = 0$ b. $x = \pm 1.3$

c. inc: $x \in (-3,\ -2) \cup (0,\ 2)$; dec: $x \in (-2,\ 0) \cup (2,\ 3)$

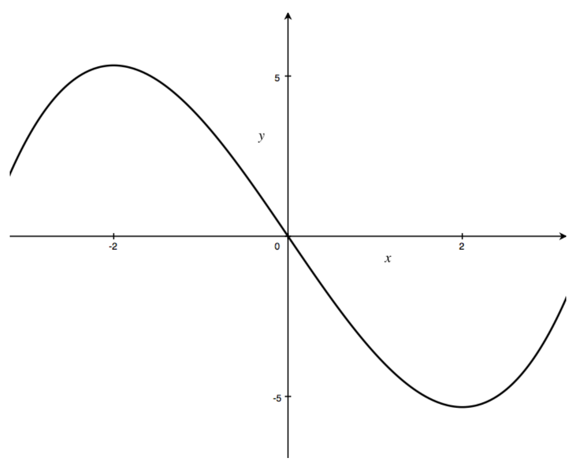
d. CU: $x \in (-1.3,\ 1.3)$; CD: $x \in (-3,\ -1.3) \cup (1.3,\ 3)$




6a. min pt. @ $x = 2$; max pt. @ $x = -2$ b. $x = 0$

c. inc: $x \in (-3,\ -2) \cup (2,\ 3)$; dec: $x \in (-2,\ 2)$

d. CU: $x \in (0,\ 3)$; CD: $x \in (-3,\ 0)$




7a. max pt. at $x = -4$ because $f'(x)$ switches from + to – and at $x = 7$ because $f'(x)$ is positive and the function stops; min pt. $x = 0$ because $f'(x)$ switches from – to + and at $x = -7$ because the function starts and $f'(x)$ is positive.

7b. $f(x)$ has points of inflection at $x = \pm 2$ and 4 where $f'(x)$ switches from increasing to decreasing or vice versa.

8a. max pt. at $x=0$ because $f'(x)$ switches from + to – , at $x=-7$ because the function starts and $f'(x)$is positive , and at $x=7$ because $f'(x)$is positive and the function stops; min pt. $x=-2$ and 3 because $f'(x)$ switches from – to + .

8b. $f(x)$ has points of inflection at $x=-4.5, -1,$ and 1.5 where $f'(x)$ switches from increasing to decreasing or vice versa.

9a. max pt. at $x=4$ because $f'(x)$ switches from + to –; min pt. at $x=-7$ because the function starts and $f'(x)$is positive , and at $x=7$ because $f'(x)$is negative and the function stops.

9b. $f(x)$ has points of inflection at $x=-4, 0,$ and 5.5 where $f'(x)$ switches from increasing to decreasing or vice versa.

12-4 Multiple Choice Homework

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | D | 2. | D | 3. | A | 4. | D | 5. | A | | |
| 6. | D | 7. | D | 8. | E | 9. | C | 10. | E | 11. | A |

## The Second Derivative Practice Test Answer Key

Multiple Choice

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | E | 2. | A | 3. | D | 4. | D | 5. | B | 6. | D |

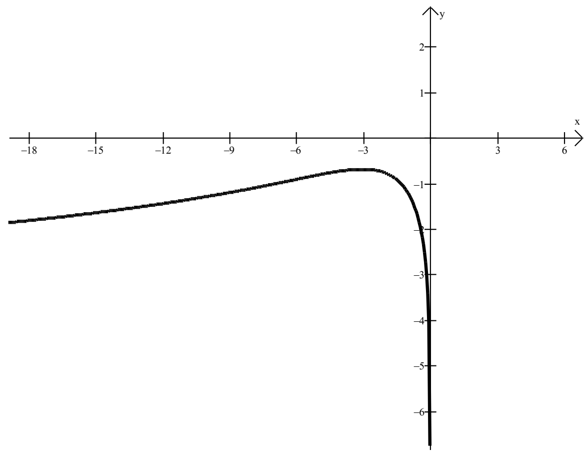
Free Response

1. Domain: $x \in (-\infty, 0)$; Zeros: None

2. Increasing: $x \in (-\infty, -3)$; Decreasing: $x \in (-3, 0)$
   Extreme Point: $(-3, -0.693)$

3. CD: $x \in (-\infty, -6.175)$; CU: $x \in (-6.175, 0)$; POI: $(-6.175, -0.934)$

4.




<u>Practice Group Test Answer Key—Page 1</u>

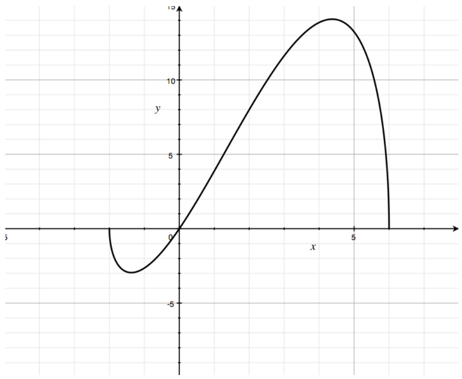
1. Extreme Points: $(0,\ 0)$, $(-4,\ 32e^{-2})$

2. No POIs

3.

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $(-\infty,\ -\sqrt{6})$ | + | – | – | below, decreasing, CD |
| $x=-\sqrt{6}$ | $-\frac{1}{2}\sqrt{6}$ | – | 0 | POI |
| $(-\sqrt{6},\ -\sqrt{2})$ | – | – | + | below, decreasing, CU |
| $x=-\sqrt{2}$ | $-\sqrt{2}$ | 0 | + | min pt. |
| $(-\sqrt{2},\ 0)$ | – | + | + | below, increasing, CU |
| $x=0$ | 0 | + | 0 | zero, POI |
| $(0,\ \sqrt{2})$ | + | + | – | above, increasing, CD |
| $x=\sqrt{2}$ | $\sqrt{2}$ | 0 | – | maximum pt. |
| $(\sqrt{2},\ \sqrt{6})$ | + | – | – | above, decreasing, CD |
| $x=\sqrt{6}$ | $\frac{1}{2}\sqrt{6}$ | – | 0 | POI |
| $(\sqrt{6},\ \infty)$ | + | – | + | above, decreasing, CU |

4.




Page 2

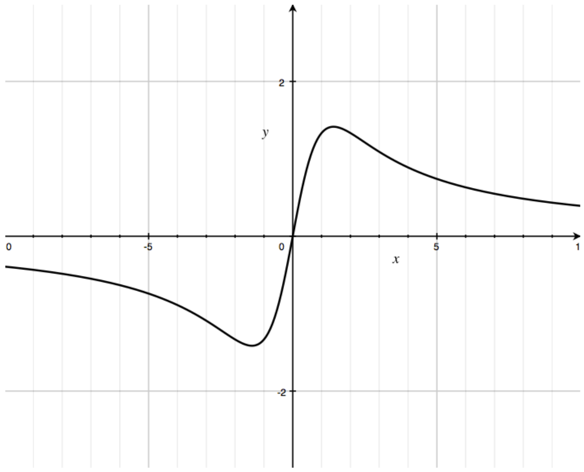
1. Extreme Points: $(-2,\ 0)$, $(-1.372,\ -2.952)$, $(4.372,\ 14.081)$, $(6,\ 0)$

2. POIs: $(-1.172,\ -1.845)$, $(-6.828,\ 0.635)$

3.

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $x=-2$ | DNE | DNE | DNE | VA |
| $(-2,\ -1.861)$ | – | + | – | below, increasing, CD |
| $x=-1.861$ | 0 | + | – | zero |
| $\left(-1.861,\ -\frac{2}{\sqrt{3}}\right)$ | + | + | – | above, increasing, CD |
| $x=-\frac{2}{\sqrt{3}}$ | 1.125 | 0 | – | max. pt. |
| $\left(-\frac{2}{\sqrt{3}},\ -0.254\right)$ | + | – | – | above, decreasing, CD |
| $x=-0.254$ | 0 | – | – | zero |
| $(-0.254,\ 0)$ | – | – | – | below, decreasing, CD |
| $x=0$ | DNE | DNE | DNE | VA |
| $x=2$ | DNE | DNE | DNE | VA |
| $(2,\ 2.115)$ | – | + | – | below, increasing, CD |
| $x=2.115$ | 0 | + | – | zero |
| $(2.115,\ \infty)$ | + | + | – | above, increasing, CD |

4.




Page 3

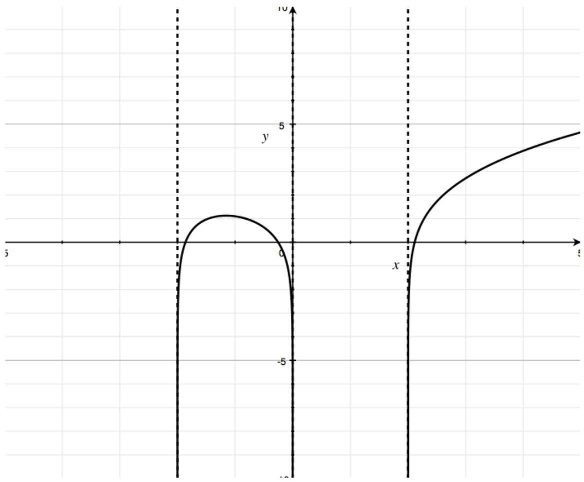
1. Extreme Points: $\left(\sqrt{2},\ \sqrt{2}\right),\ \left(-\sqrt{2},\ -\sqrt{2}\right)$

2. POIs: $\left(1.320,\ 5.203\right)$

3.

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $\left(-\infty,\ -6.828\right)$ | + | + | + | above, increasing, CU |
| $x=-6.828$ | 3.069 | + | 0 | POI |
| $\left(-6.828,\ -4\right)$ | + | + | – | above, increasing, CD |
| $x=-4$ | 4.331 | 0 | – | max pt. |
| $\left(-4,\ -1.472\right)$ | + | – | – | above, decreasing, CD |
| $x=-1.472$ | 1.529 | – | 0 | POI |
| $\left(-1.472,\ 0\right)$ | + | – | + | above, decreasing, CU |
| $x=0$ | 0 | 0 | + | zero, min pt. |
| $\left(0,\ \infty\right)$ | + | + | + | above, increasing, CU |

4.



Page 4

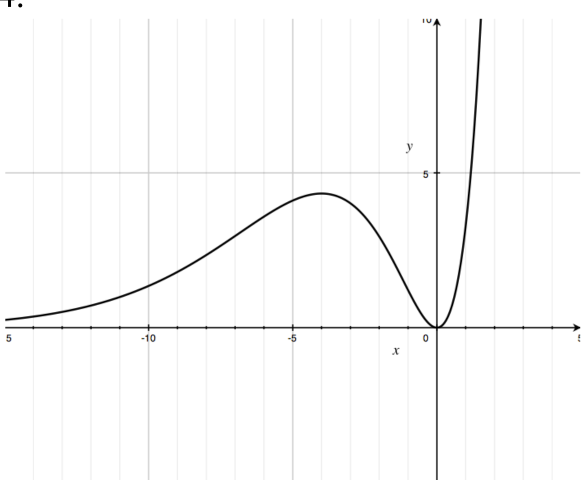
1. Extreme Points: $\left(\frac{-2}{\sqrt{3}},\ 1.125\right)$

2. POIs: $(0,\ 0)$, $\left(-\sqrt{6},\ -1.225\right)$, $\left(\sqrt{6},\ 1.225\right)$

3.

| $x$ | $y$ | $y'$ | $y''$ | Conclusion |
|---|---|---|---|---|
| $x=-2$ | 0 | DNE | DNE | zero |
| $(-2,\ -1.372)$ | – | – | + | below, decreasing, CU |
| $x=-1.372$ | –2.952 | 0 | + | min pt. |
| $(-1.372,\ 0)$ | – | + | + | below, increasing, CU |
| $x=0$ | 0 | + | + | zero |
| $(0,\ 1.320)$ | + | + | + | above, increasing, CU |
| $x=1.320$ | 5.203 | + | 0 | POI |
| $(1.320,\ 4.372)$ | + | + | – | above, increasing, CD |
| $x=4.372$ | 14.081 | 0 | – | max pt. |
| $(4.372,\ 6)$ | + | – | – | above, decreasing, CD |
| $x=6$ | 0 | DNE | DNE | zero |

4.



# Chapter 13:

# Integrals

Chapter 13 Overview: The Integral

Calculus is essentially comprised of two operations. Interspersed throughout the chapters of this book has been the first of these operations—the derivative. The surface of this field has barely been scratched and only those areas that are most related to Analytic Geometry were viewed. College Calculus will begin with the derivative and look much further into it, as well as looking at subjects other than Analytic Geometry where the derivative plays a major role. But some basics of the other operation—the **integral**—should be considered.

There are two kinds of integrals—the indefinite integral (or anti-derivative) and the definite integral. The indefinite integral is referred to as the anti-derivative, because, as an operation, it and the derivative are inverses (just as squares and square roots, or exponential and logarithmic functions). As it pertains to Analytic Geometry, the definite integral is an operation that gives the area between a curve and the $x$-axis. It has nothing to do with traits or sketching a graph. This area is purely Calculus and is an appropriate place to finish our introduction to the subject.

13-1: Anti-Derivatives: The Power Rule

As seen in a previous chapter, certain traits of a function can be deduced if its derivative is known. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called anti-differentiation, or integration.

| |
|---|
| Symbol: $\int \left(f(x)\right)\, dx =$ "the integral of *f* of *x dx*" |

The *dx* is called the differential. For now, just treat it as part of the integral symbol. It tells the independent variable of the function (usually, but not always, *x*) and, in a sense, is where the increase in the exponent comes from. It does have meaning on its own which will be explored in a different course.

Looking at the integral as an anti-derivative, that is, as an operation that reverses the derivative, a basic process for integration emerges.

Remember:

| |
|---|
| $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ and $D_x\left[\text{constant}\right] = 0$ |

(or, multiply the power in front and subtract one from the power). In reversing the process, the power must increase by one and divide by the new power. The derivative does not allude to what constant, if any, may have been attached to the original function.

**The Anti-Power Rule:**

$$\int \left(x^n\right)\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1$$

The "$+C$" is to account for any constant that might have been there before the derivative was taken. NB. This rule will not work if $n = -1$, because it would require division by zero. Recall from the derivative rules what yields $x^{-1}\left(\text{or } \frac{1}{x}\right)$ as the derivative— $\ln x$. The complete the Anti-Power Rule is:

**The Anti-Power Rule:**

$$\int \left(x^n\right)\,dx = \frac{x^{n+1}}{n+1} + C \text{ if } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Since $D_x\left[f(x)+g(x)\right] = D_x\left[f(x)\right] + D_x\left[g(x)\right]$ and $D_x\left[cx^n\right] = cD_x\left[x^n\right]$ then

$$\int \left(f(x)+g(x)\right)\,dx = \int f(x)\;dx + \int g(x)\;dx$$

$$\int c\left(f(x)\right)\,dx = c\int f(x)\;dx$$

These allow integratation of a polynomial by integrating each term separately.

LEARNING OUTCOMES

Find the indefinite integral of a polynomial.
Use integration to solve rectilinear motion problems.

EX 1 $\int\left(3x^2+4x+5\right)\,dx$

$$\int\left(3x^2+4x+5\right)\,dx = 3\frac{x^{2+1}}{2+1}+4\frac{x^{1+1}}{1+1}+5\frac{x^{0+1}}{0+1}+C$$
$$=\frac{3x^3}{3}+\frac{4x^2}{2}+\frac{5x^1}{1}+C$$
$$=x^3+2x^2+5x+C$$

EX 2 $\int\left(x^4+4x^2+5+\frac{1}{x}\right)\,dx$

$$\int\left(x^4+4x^2+5+\frac{1}{x}\right)\,dx=\frac{x^{4+1}}{4+1}+\frac{4x^{2+1}}{2+1}+\frac{5x^{0+1}}{0+1}+\ln x+C$$
$$=\frac{1}{5}x^5+\frac{4}{3}x^3+5x+\ln|x|+C$$

EX 3 $\int\left(x^2+\sqrt[3]{x}-\frac{4}{x}\right)\,dx$

$$\int\left(x^2+\sqrt[3]{x}-\frac{4}{x}\right)\,dx=\int\left(x^2+x^{1/3}-\frac{4}{x}\right)\,dx$$
$$=\frac{x^{2+1}}{2+1}+\frac{x^{1/3+1}}{\frac{1}{3}+1}-4\ln|x|+C$$
$$=\frac{1}{3}x^3+\frac{3}{4}x^{4/3}-4\ln|x|+C$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

EX 4 $\int\left(x^2+\sqrt[3]{x}\right)(2x+1)\,dx$

$$\int\left(x^2+\sqrt[3]{x}\right)(2x+1)\,dx=\int\left(2x^3+2x^{4/3}+x^2+x^{1/3}\right)dx$$
$$=\frac{2x^4}{4}+\frac{2x^{7/3}}{7/3}+\frac{x^3}{3}+\frac{x^{4/3}}{4/3}+C$$

$$=\frac{1}{2}x^4+\frac{6}{7}x^{7/3}+\frac{1}{3}x^3+\frac{3}{4}x^{4/3}+C$$

Example 5 is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for *C*.

EX 5 $f'(x)=4x^3-6x+3$. Find $f(x)$ if $f(0)=13$.

$$f(x)=\int\left(4x^3-6x+3\right)dx$$
$$=x^4-3x^2+3x+C$$

$$f(0)=0^4-3(0)^2+3(0)+C=13$$
$$\therefore\ C=13$$

$$f(x)=x^4-3x^2+3x+13$$

EX 6  The acceleration of a particle is described by $a(t)=3t^2+8t+1$. Find the distance equation for $x(t)$ if $v(0)=3$ and $x(0)=1$.

$$
\begin{aligned}
v(t) &= \int \big(a(t)\big)\,dt = \int \left(3t^2+8t+1\right)\,dt \\
&= t^3+4t^2+t+C_1 \\
3 &= (0)^3+4(0)^2+(0)+C_1 \\
3 &= C_1 \\
v(t) &= t^3+4t^2+t+3
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \int \big(v(t)\big)\,dt = \int \left(t^3+4t^2+t+3\right)\,dt \\
&= \frac{1}{4}t^4+\frac{4}{3}t^3+\frac{1}{2}t^2+3t+C_2 \\
1 &= \frac{1}{4}(0)^4+\frac{4}{3}(0)^3+\frac{1}{2}(0)^2+3(0)+C_2 \\
1 &= C_2
\end{aligned}
$$

$$x(t)=\frac{1}{4}t^4+\frac{4}{3}t^3+\frac{1}{2}t^2+3t+1$$

EX 7 The acceleration of a particle is described by $a(t)=12t^2-6t+4$. Find the distance equation for $x(t)$ if $v(1)=0$ and $x(1)=3$.

$$
\begin{aligned}
v(t)&=\int (a(t))\,dt=\int (12t^2-6t+4)\,dt\\
&=4t^3-3t^2+4t+C_1\\
0&=4(1)^3-3(1)^2+4(1)+C_1\\
-5&=C_1\\
v(t)&=4t^3-3t^2+4t-5
\end{aligned}
$$

$$
\begin{aligned}
x(t)&=\int (v(t))\,dt=\int (4t^3-3t^2+4t-5)\,dt\\
&=t^4-t^3+2t^2-5t+C_2\\
3&=(1)^4-(1)^3+2(1)^2-5(1)+C_2\\
6&=C_2
\end{aligned}
$$

$$x(t)=t^4-t^3+2t^2-5t+6$$

## 13-1 Free Response Homework

Perform the anti-differentiation.

1. $\int \left(6x^2 - 2x + 3\right) dx$

2. $\int \left(x^3 + 3x^2 - 2x + 4\right) dx$

3. $\int \frac{2}{\sqrt[3]{x}}\, dx$

4. $\int \left(8x^4 - 4x^3 + 9x^2 + 2x + 1\right) dx$

5. $\int x^3\left(4x^2 + 5\right) dx$

6. $\int (4x-1)(3x+8)\, dx$

7. $\int \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right) dx$

8. $\int \left(\frac{x^2 + \sqrt{x} + 3}{x}\right) dx$

9. $\int (x+1)^3\, dx$

10. $\int (4x-3)^2\, dx$

11. $\int \left(\sqrt{x} + 3\sqrt[2]{x^3} - \frac{6}{\sqrt{x}}\right) dx$

12. $\int \left(\frac{4x^3 + \sqrt{x} + 3}{x^2}\right) dx$

Solve the initial value problems.

13. $f'(x) = 3x^2 - 6x + 3$. Find $f(x)$ if $f(0) = 2$.

14. $f'(x) = x^3 + x^2 - x + 3$. Find $f(x)$ if $f(1) = 0$.

15. $f'(x) = \left(\sqrt{x} - 2\right)\left(3\sqrt{x} + 1\right)$. Find $f(x)$ if $f(4) = 1$.

16. The acceleration of a particle is described by $a(t) = 36t^2 - 12t + 8$. Find the distance equation for $x(t)$ if $v(1) = 1$ and $x(1) = 3$.

17. The acceleration of a particle is described by $a(t) = t^2 - 2t + 4$. Find the distance equation for $x(t)$ if $v(0) = 2$ and $x(0) = 4$.

13-1 Multiple Choice Homework

1. $\int \frac{1}{x^2}\ dx =$

a) $\ln x^2 + C$ b) $-\ln x^2 + C$ c) $x^{-1} + C$

d) $-x^{-1} + C$ e) $-2x^{-3} + C$

---

2. $\int x\left(10 + 8x^4\right)\ dx =$

a) $5x^2 + \frac{4}{3}x^6 + C$ b) $5x^2 + \frac{8}{5}x^5 + C$ c) $10x + \frac{4}{3}x^6 + C$

d) $5x^2 + 8x^6 + C$ e) $5x^2 + \frac{8}{7}x^6 + C$

---

3. $\int x\sqrt{3x}\ dx =$

a) $\frac{2\sqrt{3}}{5}x^{5/2} + C$ b) $\frac{5\sqrt{3}}{2}x^{5/2} + C$ c) $\frac{\sqrt{3}}{2}x^{1/2} + C$

d) $2\sqrt{3x} + C$ e) $\frac{5\sqrt{3}}{2}x^{3/2} + C$

---

## 13-2: Integration by Substitution: The Chain Rule

The other three derivative rules—the Product, Quotient and Chain Rules—are a little more complicated to reverse than the Power Rule. This is because they yield a more complicated function as a derivative, one which usually has several algebraic simplifications. The integral of a rational function is particularly difficult to unravel because, as seen previously, a rational derivative can be obtained by differentiating a composite function with a logarithm or a radical, or by differentiating another rational function. Reversing the Product Rule is as complicated, though for other reasons. Both these subjects can be left for a traditional Calculus class. The Chain Rule is another matter.

Composite functions are among the most pervasive situations in math. Though not as simple in reverse as the Power Rule, the overwhelming importance of this rule makes it imperative that it be addressed here.

Remember:

The Chain Rule: $\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)\bullet g'(x)$

The derivative of a composite function turns into a product of a composite and a non-composite. So if there is a product to integrate, it might be that the product came from the Chain Rule. The integration is not done by a formula so much as a process that might or might not work. One makes an educated guess and hopes it works out. There are other processes in Calculus for when it does not work.

Integration by Substitution:

1. Identify the inside function of the composite and call it *u*.
2. Find *du* from *u*.
3. If necessary, multiply a constant inside the integral to create *du*, and balance it by multiplying the reciprocal of that constant outside the integral. (See EX 2)
4. Substitute *u* and *du* into the equation.
5. Perform the integration using the Anti-Power Rule (or transcendental rules, in next section.)
6. Re-substitute for *u*.

This is one of those mathematical processes that makes little sense when first seen. But after seeing several examples, the meaning suddenly becomes clear.
**BE PATIENT!**

| LEARNING OUTCOME |
|---|
| Use the Unchain Rule to integrate composite, product expressions. |

EX 1 $\int\left(3x^2\left(x^3+5\right)^{10}\right)\,dx$

$\left(x^3+5\right)^{10}$ is the composite function. So $u = x^3+5$
$du = 3x^2\,dx$

$$\int\left(3x^2\left(x^3+5\right)^{10}\right)dx = \int\left(u^{10}\right)du$$
$$= \frac{u^{11}}{11} + C$$
$$= \frac{1}{11}\left(x^3+5\right)^{11} + C$$

EX 2 $\int \left( x\left(x^2+5\right)^3 \right) dx$

$\left(x^2+5\right)^3$ is the composite function. So $\begin{array}{l} u = x^2+5 \\ du = 2x\ dx \end{array}$

$$\begin{aligned} \int \left( x\left(x^2+5\right)^3 \right) dx &= \frac{1}{2}\int \left(x^2+5\right)^3 (2x\ dx) \\ &= \frac{1}{2}\int \left(u^3\right) du \\ &= \frac{1}{2} \bullet \frac{u^4}{4} + C \\ &= \frac{1}{8}\left(x^2+5\right)^4 + C \end{aligned}$$

EX 3 $\int \left( \left(x^3+x\right)\sqrt[5]{x^4+2x^2-5} \right) dx$

$\sqrt[4]{x^4+2x^2-5}$ is the composite function.

So $\begin{array}{l} u = x^4+2x^2-5 \\ du = \left(4x^3+4x\right)\ dx = 4\left(x^3+x\right)\ dx \end{array}$

$$\begin{aligned} \int \left( \left(x^3+x\right)\sqrt[5]{x^4+2x^2-5} \right) dx &= \frac{1}{4}\int \left(\sqrt[5]{x^4+2x^2-5}\right)\ 4\left(x^3+x\right) dx \\ &= \frac{1}{4}\int \left(\sqrt[5]{u}\right) du \\ &= \frac{1}{4}\int \left(u^{1/5}\right) du \\ &= \frac{1}{4}\frac{u^{6/5}}{\frac{6}{5}} + C \\ &= \frac{5}{24}\left(x^4+2x^2-5\right)^{6/5} + C \end{aligned}$$

EX 4 $\displaystyle\int\left(\frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\right)dx$

$$u = x^3+2x^2-5x+2$$
$$du = \left(3x^2+4x-5\right)\ dx$$

$$\begin{aligned}\int\left(\frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\right)dx &= \int\left(x^3+2x^2-5x+2\right)^{-3}\left(\left(3x^2+4x-5\right)\ dx\right)\\ &= \int\left(u^{-3}\right)du\\ &= \frac{u^{-2}}{-2}+C\\ &= \frac{-1}{2\left(x^3+2x^2-5x+2\right)^2}+C\end{aligned}$$

Sometimes, the other factor is not the *du*, or there is an extra *x* that must be replaced with some form of *u*.

EX 5 $\int (x+1)\sqrt{x-1}\,dx$

$$u = x-1$$
$$x = u+1$$
$$du = dx$$

$$\int (x+1)\sqrt{x-1}\,dx = \int \big((u+1)+1\big)\sqrt{u}\;du$$
$$= \int (u+2)u^{1/2}\,du$$
$$= \int \left(u^{3/2} + 2u^{1/2}\right) du$$
$$= \frac{u^{5/2}}{\frac{5}{2}} + \frac{2u^{3/2}}{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + C$$

EX 6 $\int \left( x^3 \left(x^2+4\right)^{3/2} \right) dx$

$$\begin{aligned} u &= x^2+4 \\ x^2 &= u-4 \\ du &= 2x\ dx \end{aligned}$$

$$\begin{aligned} \int \left( x^3 \left(x^2+4\right)^{3/2} \right) dx &= \frac{1}{2}\int \left( x^2 \left(x^2+4\right)^{3/2} \right)(2x\ dx) \\ &= \frac{1}{2}\int \left( (u-4)u^{3/2} \right) du \\ &= \frac{1}{2}\int \left( u^{5/2} - 4u^{3/2} \right) du \\ &= \frac{1}{2}\left( \frac{u^{7/2}}{\frac{7}{2}} - \frac{4u^{5/2}}{\frac{5}{2}} \right) + C \\ &= \frac{1}{7}\left(x^2+4\right)^{7/2} - \frac{4}{5}\left(x^2+4\right)^{5/2} + C \end{aligned}$$

13-2 Free Response Homework

Perform the anti-differentiation.

1. $\int (5x+3)^3\, dx$

2. $\int \left( x^3 \left( x^4+5 \right)^{24} \right) dx$

3. $\int \left(1+x^3\right)^2\, dx$

4. $\int (2-x)^{2/3}\, dx$

5. $\int \left( x\sqrt{2x^2+3} \right) dx$

6. $\int \dfrac{dx}{(5x+2)^3}$

7. $\int \dfrac{x^3}{\sqrt{1+x^4}}\, dx$

8. $\int \dfrac{x+1}{\sqrt[3]{x^2+2x+3}}\, dx$

9. $\int \left( x^5 \left( x^2+4 \right)^{12} \right) dx$

10. $\int \sqrt{x+3}\,(x+1)^2\, dx$

13-2 Multiple Choice Homework

1. $\displaystyle\int \frac{x}{x^2-4}\, dx =$

a) $\dfrac{-1}{4\left(x^2-4\right)^2}+C$ b) $\dfrac{1}{2\left(x^2-4\right)}+C$ c) $\dfrac{1}{2}\ln\left|x^2-4\right|+C$

d) $2\ln\left|x^2-4\right|+C$ e) $\dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)+C$

---

2. $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\ dx =$

a) $\ln\sqrt{x}+C$ b) $x+C$ c) $e^x+C$

d) $\frac{1}{2}e^{2\sqrt{x}}+C$ e) $e^{\sqrt{x}}+C$

---

3. When using the substitution $u=\sqrt{1+x}$, an anti-derivative of $\int 60x\sqrt{1+x}\ dx$ is

a) $20u^3-60u+C$ b) $15u^4-30u^2+C$ c) $30u^4-60u^2+C$

d) $24u^5-40u^3+C$ e) $12u^6-20u^4+C$

---

4. $\int \frac{4x}{1+x^2}\ dx =$

a) $4\arctan x+C$ b) $\frac{4}{x}\arctan x+C$ c) $\frac{1}{2}\ln\left(1+x^2\right)+C$

d) $2\ln\left(1+x^2\right)+C$ e) $2x^2+4\ln|x|+C$

---

5. $\int x\left(x^2-1\right)^4 dx =$

a) $\frac{1}{10}x^2\left(x^2-1\right)^5+C$ b) $\frac{1}{10}\left(x^2-1\right)^5+C$ c) $\frac{1}{5}\left(x^3-x\right)^5+C$

d) $\frac{1}{5}\left(x^2-1\right)^5+C$ e) $\frac{1}{5}\left(x^2-x\right)^5+C$

---

13-3: Anti-Derivatives: The Transcendental Rules

The proof of the transcendental integral rules can be left to a more formal Calculus course. But since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

**Derivative Rules:**

$$\frac{d}{dx}[\sin u]=(\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u]=(-\csc u\cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u]=(-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u]=(\sec u\tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u]=\left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u]=\left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right]=\left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u]=\left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right]=\left(a^u\bullet\ln a\right)\frac{du}{dx} \qquad \frac{d}{dx}[\log_a u]=\left(\frac{1}{u\bullet\ln a}\right)\frac{du}{dx}$$

**Integral Rules:**

$$\int(\cos u)\,du=\sin u+C \qquad \int(\csc u\cot u)\,du=-\csc u+C$$

$$\int(\sin u)\,du=-\cos u+C \qquad \int(\sec u\ \tan u)\,du=\sec u+C$$

$$\int\left(\sec^2 u\right)du=\tan u+C \qquad \int\left(\csc^2 u\right)du=-\cot u+C$$

$$\int\left(e^u\right)du=e^u+C \qquad \int\left(\frac{1}{u}\right)du=\ln|u|+C$$

$$\int\left(a^u\right)du=\frac{a^u}{\ln a}+C$$

LEARNING OUTCOME

Integrate functions involving transcendental operations.

EX 1 $\int(\sin x+3\cos x)\,dx$

$$\int(\sin x+3\cos x)\,dx=\int(\sin x)\,dx+3\int(\cos x)\,dx$$

$$=-\cos x+3\sin x+C$$

EX 2 $\int\left(e^x+4+3\csc^2 x\right)dx$

$$\int\left(e^x+4+3\csc^2 x\right)dx=\int\left(e^x\right)dx+4\int dx+3\int\left(\csc^2 x\right)dx$$

$$=e^x+4x-3\cot x+C$$

EX 3 $\int(\sec x(\sec x+\tan x))\,dx$

$$\int(\sec x(\sec x+\tan x))\,dx=\int\left(\sec^2 x\right)dx+\int(\sec x\tan x)\,dx$$

$$=\tan x+\sec x+C$$

The Unchain Rule will apply to the transcendental functions quite well.

EX 4 $\int(\sin 5x)\,dx$

$$u=5x$$
$$du=5\ dx$$

$$\int(\sin 5x)\,dx=\frac{1}{5}\int(\sin 5x)5\ dx$$
$$=\frac{1}{5}\int(\sin u)\,du$$
$$=\frac{1}{5}(-\cos u)+C$$

$$=-\frac{1}{5}\cos 5x+C$$

EX 5 $\int \left(\sin^6 x \cos x\right)\, dx$

$$u = \sin x$$
$$du = \cos x \;\, dx$$

$$\begin{aligned}\int \left(\sin^6 x \cos x\right)\, dx &= \int \left(u^6\right)\, du \\ &= \frac{1}{7}u^7 + C\end{aligned}$$

$$= \frac{1}{7}\sin^7 x \;+ C$$

EX 6 $\int \left(x^5 \sin x^6\right)\, dx$

$$u = x^6$$
$$du = 6x^5 \;\, dx$$

$$\begin{aligned}\int \left(x^5 \sin x^6\right)\, dx &= \frac{1}{6}\int \left(\sin x^6\right)\left(6x^5 \;\, dx\right) \\ &= \frac{1}{6}\int \left(\sin u\right)\, du \\ &= -\frac{1}{6}\cos u + C\end{aligned}$$

$$= -\frac{1}{6}\cos x^6 + C$$

EX 7 $\int\left(\cot^3 x\csc^2 x\right)\,dx$

$$u=\cot x$$
$$du=-\csc^2 x\;dx$$

$$\begin{aligned}\int\left(\cot^3 x\csc^2 x\right)\,dx&=\int\left(\cot^3 x\right)\left(-\csc^2 x\;dx\right)\\&=-\int\left(u^3\right)\,du\\&=-\frac{1}{4}u^4+C\end{aligned}$$

$$=-\frac{1}{4}\cot^4 x+C$$

EX 8 $\int\left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right)\,dx$

$$u=\sqrt{x}=x^{1/2}$$
$$du=\frac{1}{2}x^{-1/2}\;dx=\frac{1}{2x^{1/2}}\;dx$$

$$\begin{aligned}\int\left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right)\,dx&=2\int\left(\cos\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\;dx\right)\\&=2\int\left(\cos u\right)\,du\\&=2\sin u+C\end{aligned}$$

$$=2\sin\sqrt{x}+C$$

EX 9 $\int\left(xe^{x^2+1}\right)\,dx$

$$u = x^2 + 1$$
$$du = 2x\;dx$$

$$\int\left(xe^{x^2+1}\right)dx = \frac{1}{2}\int\left(e^{x^2+1}\right)(2x\;dx)$$
$$= \frac{1}{2}\int\left(e^u\right)du$$
$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2+1} + C$$

EX 10 $\int\left(\frac{\ln^3 x}{x}\right)dx$

$$u = \ln x$$
$$du = \frac{1}{x}\;dx$$

$$\int\left(\frac{\ln^3 x}{x}\right)dx = \int u^3\;du$$
$$= \frac{1}{4}u^4 + C$$
$$= \frac{1}{4}\ln^4 x + C$$

13-3 Free Response Homework

Perform the anti-differentiation.

1. $\int \left(x^4 \cos x^5\right) dx$

2. $\int \left(\sin\left(7x+1\right)\right) dx$

3. $\int \left(\sec^2\left(3x-1\right)\right) dx$

4. $\int \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right) dx$

5. $\int \left(\tan^4 x \sec^2 x\right) dx$

6. $\int \frac{\ln x}{x}\, dx$

7. $\int \left(e^{6x}\right) dx$

8. $\int \frac{\cos 2x}{\sin^3 2x}\, dx$

9. $\int \frac{x\ln\left(x^2+1\right)}{x^2+1}\, dx$

10. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

11. $\int \left(\sqrt{\cot x}\, \csc^2 x\right) dx$

12. $\int \frac{1}{x^2}\left(\sin\frac{1}{x}\right)\left(\cos\frac{1}{x}\right) dx$

13-3 Multiple Choice Homework

1. $\int \left(x^3+2+\frac{1}{x^2+1}\right) dx =$

a) $\frac{x^4}{4}+2x+\tan^{-1}x+C$

b) $x^4+2+\tan^{-1}x+C$

c) $\frac{x^4}{4}+2x+\frac{3}{x^3+3}+C$

d) $\frac{x^4}{4}+2x+\tan^{-1}2x^2+C$

e) $4+2x+\tan^{-1}x+C$

---

2. $\int \cos(3-2x)\ dx =$

a) $\sin(3-2x)+C$

b) $-\sin(3-2x)+C$

c) $\frac{1}{2}\sin(3-2x)+C$

d) $-\frac{1}{2}\sin(3-2x)+C$

e) $-\frac{1}{5}\sin(3-2x)+C$

---

3. $\int \frac{x-2}{x-1}\ dx =$

a) $-\ln|x-1|+C$

b) $x+\ln|x-1|+C$

c) $x-\ln|x-1|+C$

d) $x+\sqrt{x-1}+C$

e) $x-\sqrt{x-1}+C$

---

4. $\int \frac{e^{x^2}-2x}{e^{x^2}}\ dx =$

a) $x-e^{x^2}+C$

b) $x-e^{-x^2}+C$

c) $x+e^{-x^2}+C$

d) $-e^{x^2}+C$

e) $-e^{-x^2}+C$

---

5. $\int 6\sin x\cos^2 x \; dx =$

a) $2\sin^3 x + C$ b) $-2\sin^3 x + C$ c) $2\cos^3 x + C$

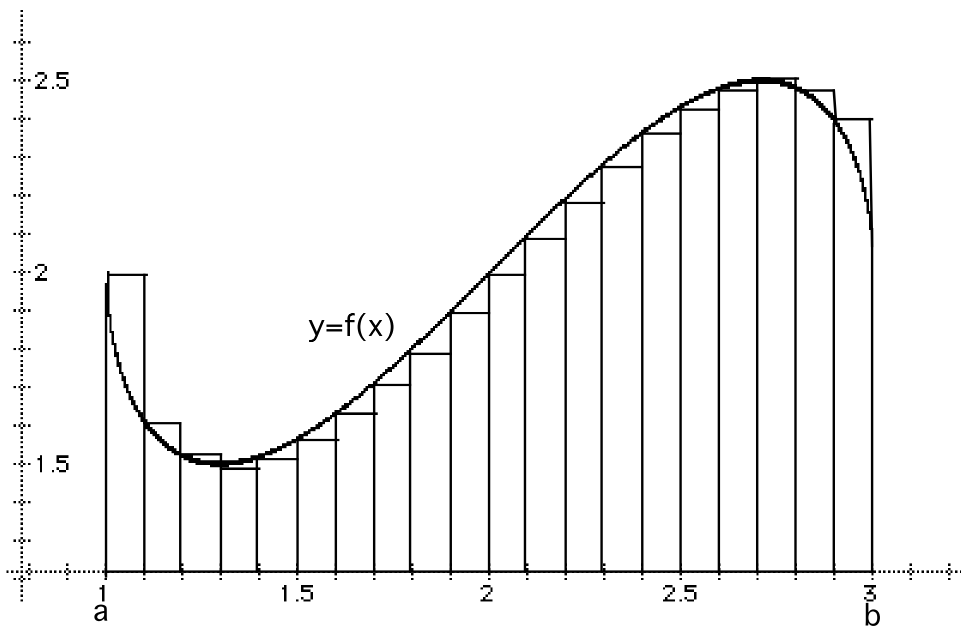
d) $-2\cos^3 x + C$ e) $3\sin^2 x\cos^2 x + C$

---

13-4: Definite Integrals

As noted in the overview, anti-derivatives are known as indefinite integrals because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an Analytic Geometry context of area. Though it is not necessary to know the theory behind this to be able to do it, the theory is a major subject of Integral Calculus, so it will be explored briefly here.

Geometry presented how to find the exact area of various polygons, but it never considered figures where one side is not made of a line segment. Here, consider a figure where one side is the curve $y = f(x)$ and the other sides are the $x$-axis and the lines $x = a$ and $x = b$.




As seen above, the area can be approximated by rectangles whose height is the $y$ value of the equation and whose width is $\Delta x$. The more rectangles made, the better the approximation. The area of each rectangle would be $f(x)\bullet\Delta x$ and the total area of $n$ rectangles would be $\sum_{i=1}^{n} f(x_i)\bullet\Delta x$. If an infinite number of rectangles (which would be infinitely thin) could be made, the exact area would be the resulting sum. The rectangles can be drawn several ways—with the left side at the height of the curve (as drawn above), with the right side at the curve, with the rectangle straddling the curve, or even with rectangles of different widths. But once they become infinitely thin, it will not matter how they were drawn—they will have no width and a height equal to the $y$-value of the curve.

An infinite number of rectangles can be made mathematically by taking the limit as $n$ approaches infinity, or

$$\boxed{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\bullet\Delta x.}$$

This limit is rewritten as the Definite Integral:

$$\boxed{\int_a^b f(x)\ dx}$$

$b$ is the "upper bound" and $a$ is the "lower bound," and would not mean much if it were not for the following rule:

**The Fundamental Theorem of Calculus:**

1. $\dfrac{d}{dx}\displaystyle\int_c^x f(t)\ dt = f(x)$

2. If $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\ dx = F(b) - F(a)$.

The First Fundamental Theorem of Calculus simply says what is already known—that an integral is an anti-derivative. The Second Fundamental Theorem says the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

This idea of the integral meaning the area may not make sense initially, mainly because Geometry was used, where area is always measured in square units. But that is only because the length and width are always measured in the same kind of units, so multiplying length and width must yield square units. Consider a graph where the $x$-axis is time in seconds and the $y$-axis is velocity in feet per second.

The area under the curve would be measured as seconds multiplied by feet/sec—that is, feet. So the area under the curve equals the distance traveled in feet. In other words, the integral of velocity is distance.

| LEARNING OUTCOMES |
|---|
| Evaluate definite integrals.<br>Interpret definite integrals as area under a curve. |

EX 1 Find $\int_1^4 x^2 \; dx$

$$\begin{aligned}\int_1^4 x^2 \; dx &= \left[\frac{x^3}{3}+C\right]_1^4 \\ &= \left(\frac{4^3}{3}+C\right)-\left(\frac{1^3}{3}+C\right) \\ &= 21\end{aligned}$$

Notice that the arbitrary constant $C$ does not effect the definite integral. It cancels itself out when we substitute $a$ and $b$ and subtract. So, with a definite integral, **<u>the + *C* can be ignored</u>**.

EX 2 Find $\int_{\pi/4}^{\pi/2} \sin x \; dx$

$$\begin{aligned}\int_{\pi/4}^{\pi/2} \sin x \; dx &= \left[-\cos x\right]_{\pi/4}^{\pi/2} \\ &= \left[-\cos\frac{\pi}{2}\right]-\left[-\cos\frac{\pi}{4}\right] \\ &= 0-\left(-\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}\end{aligned}$$

EX 3 $\int_0^3 \left(3x^2 + 4x + 5\right)\, dx$

$$\begin{aligned}\int_0^3 \left(3x^2 + 4x + 5\right)\, dx &= \left[x^3 + 2x^2 + 5x\right]_0^3 \\ &= \left[3^3 + 2(3)^2 + 5(3)\right] - \left[0^3 + 2(0)^2 + 5(0)\right] \\ &= 60 - 0 \\ &= 60\end{aligned}$$

EX 4 $\int_1^3 \left(\frac{3}{x}\right) dx$

$$\begin{aligned}\int_1^3 \left(\frac{3}{x}\right) dx &= 3\left[\ln x\right]_1^3 \\ &= 3\left[\ln 3\right] - 3\left[\ln 1\right] \\ &= 3\ln 3 = 3.295\end{aligned}$$

If a composite function requires integration by substitution, substitute for the boundaries as well. $a$ and $b$ are normally what $x$ would equal. If switching to $u$, the boundaries must equal what $\underline{u}$ would equal when $x = a$ and $x = b$.

EX 5 $\int_0^2 \left( x\left(x^2+5\right)^3 \right)\ dx$

$$u = x^2 + 5, \quad du = 2x, \quad u(0) = 5, \quad u(2) = 9$$

$$\begin{aligned} \int_0^2 \left( x\left(x^2+5\right)^3 \right)\ dx &= \frac{1}{2}\int_0^2 \left(x^2+5\right)^3 (2x\ dx) \\ &= \frac{1}{2}\int_5^9 u^3\, du \\ &= \frac{1}{2} \bullet \left[\frac{u^4}{4}\right]_5^9 \\ &= \frac{1}{2} \bullet \left[\frac{9^4}{4} - \frac{5^4}{4}\right] \\ &= 742 \end{aligned}$$

EX 6 $\int_0^{\pi/2} \left(\sin^6 x \cos x\right)\ dx$

$$u = \sin x, \ \ du = \cos x\ dx, \ \ u\left(\frac{\pi}{2}\right) = 1, \ \ u(0) = 0$$

$$\begin{aligned} \int_0^{\pi/2} \left(\sin^6 x \cos x\right)\ dx &= \int_0^1 \left(u^6\right)\ du \\ &= \left[\frac{1}{7}u^7\right]_0^1 \\ &= \frac{1}{7} \end{aligned}$$

13-4 Free Response Homework

Evaluate the definite integrals.

1. $\int_0^3 \left(x^2+5\right)\, dx$

2. $\int_2^6 \frac{1}{\sqrt{2x-3}}\, dx$

3. $\int_{-1}^1 \left(x\sqrt{1-x^2}\right)\, dx$

4. $\int_0^1 (x+2)^3\, dx$

5. $\int_{-2}^2 (x+5)\left(x^2-3\right)\, dx$

6. $\int_4^9 \frac{1-\sqrt{x}}{\sqrt{x}}\, dx$

7. $\int_{\pi/6}^{\pi/2} \cos^5 x \sin x\, dx$

8. $\int_1^3 \frac{x^2+1}{x^2}\, dx$

9. $\int_{-1}^2 \frac{dx}{2x+5}$

10. $\int_0^\pi \frac{\sin x}{2-\cos x}\, dx$

11. $\int_2^4 \frac{dx}{x \ln x}$

12. $\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\, dx$

13-4 Multiple Choice Homework

1. $\int_2^6 \left(\frac{1}{x}+2x\right)\, dx =$

a) $\ln 4 + 32$ b) $\ln 3 + 40$ c) $\ln 3 + 32$

d) $\ln 4 + 40$ e) $\ln 12 + 32$

---

2. $\int_0^1 \sin \pi x \, dx =$

a) $\dfrac{2}{\pi}$ b) $\dfrac{1}{\pi}$ c) $0$ d) $-\dfrac{2}{\pi}$ e) $-\dfrac{1}{\pi}$

---

3. $\int_{\pi/4}^{\pi/3} \dfrac{\sec^2 x}{\tan x} \, dx =$

a) $\ln\sqrt{3}$ b) $-\ln\sqrt{3}$ c) $\ln\sqrt{2}$

d) $\sqrt{3}-1$ e) $\ln\dfrac{\pi}{3}-\ln\dfrac{\pi}{4}$

---

4. $\int_0^2 \sqrt{x^2-4x+4} \, dx =$

a) 1 b) –1 c) –2 d) 2 e) None of these

---

5. If $\int_2^4 f(x) \, dx = 6$, then $\int_2^4 \left(f(x)+3\right) \, dx =$

a) 3 b) 6 c) 9 d) 12 e) 15

---

6. Let $f$ be the function defined by $f(x)=\begin{cases} x+1 & \text{for } x<0 \\ 1+\sin \pi x & \text{for } x \geq 0 \end{cases}$. Then

$\int_{-1}^{1} f(x)\,dx =$

a) $\frac{3}{2}$

b) $\frac{3}{2}-\frac{2}{\pi}$

c) $\frac{1}{2}-\frac{2}{\pi}$

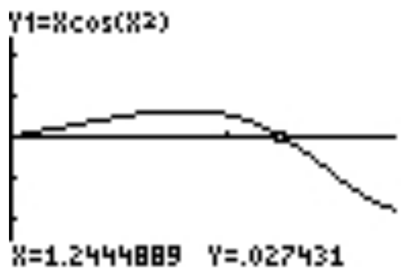
d) $\frac{3}{2}+\frac{2}{\pi}$

e) $\frac{1}{2}+\frac{2}{\pi}$

13-5: Definite Integrals and Area

Since the definite integrals were originally defined in terms of "area under a curve," what this context of "area" really means needs to be considered in relation to the definite integral.

Consider this function, $y=\left(x\cos\left(x^2\right)\right)$ on $x\in\left[0,\ \sqrt{\pi}\right]$. The graph looks like this:



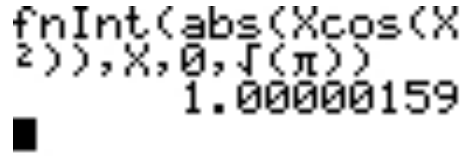

In the last section, $\int_0^{\sqrt{\pi}}\left(x\cos\left(x^2\right)\right)\,dx=0$. But it can be seen there is area under the curve, so how can the integral equal the area and equal 0? Remember that the integral was created from rectangles with width *dx* and height $f(x)$. So the area below the *x*-axis would be negative, because the $f(x)$ values are negative.

EX 1 What is the area under $y=\left(x\cos\left(x^2\right)\right)$ on $x\in\left[0,\ \sqrt{\pi}\right]$?

It is already known that $\int_0^{\sqrt{\pi}}\left(x\cos\left(x^2\right)\right)\,dx=0$, so this integral cannot represent the area. The question is to find the positive number that represents the area (total distance), not the difference between the positive and negative "areas" (displacement). The commonly accepted context for area is a positive value. So,

$$\begin{aligned}\text{Area}&=\int_0^{\sqrt{\pi}}\left|x\cos\left(x^2\right)\right|\,dx\\&=\int_0^{1.244}x\cos\left(x^2\right)\,dx-\int_{1.244}^{\sqrt{\pi}}x\cos\left(x^2\right)\,dx\\&=1\end{aligned}$$

The calculator could find this answer:




**When the phrase "area under the curve" is used, what is really meant is the area between the curve and the *x*-axis. CONTEXT IS EVERYTHING. The area under the curve is only equal to the definite integral when the curve is completely above the *x*-axis. When the curve goes below the *x*-axis, the definite integral is negative, but the area, by definition, is positive.**

LEARNING OUTCOMES

Relate definite integrals to area under a curve.
Understand the difference between displacement and total distance.
Extend that idea to understanding the difference between the two concepts in other contexts.

**Properties of Definite Integrals:**

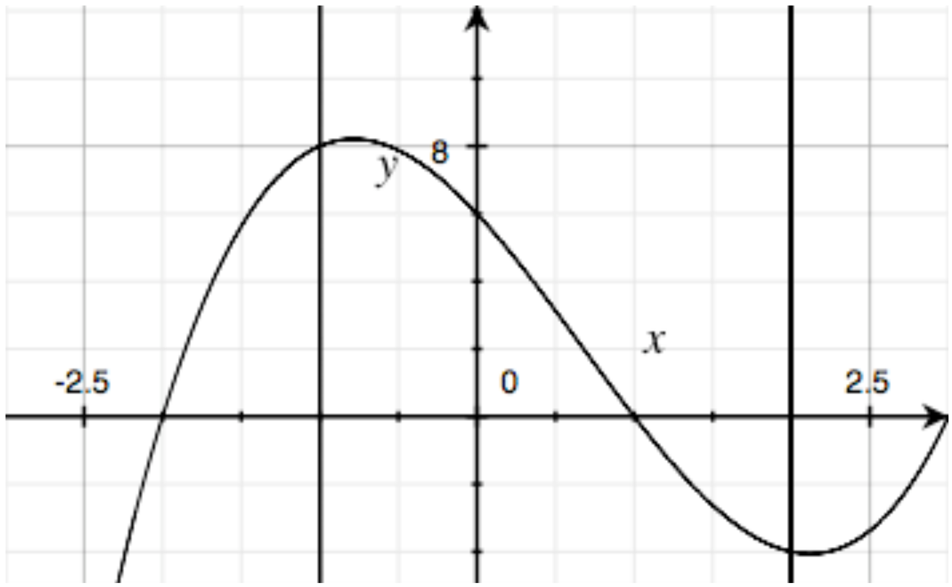
1. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

2. $\int_a^a f(x)\ dx = 0$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

EX 2 Find the area under $y = x^3 - 2x^2 - 5x - 6$ on $x \in [-1, 2]$.

A quick look at the graph reveals that the curve crosses the $x$-axis at $x = 2$.




Integrate $y$ on $x \in [-1, 2]$ to get the **difference** between the areas, not the sum. To get the total area, set up two integrals:

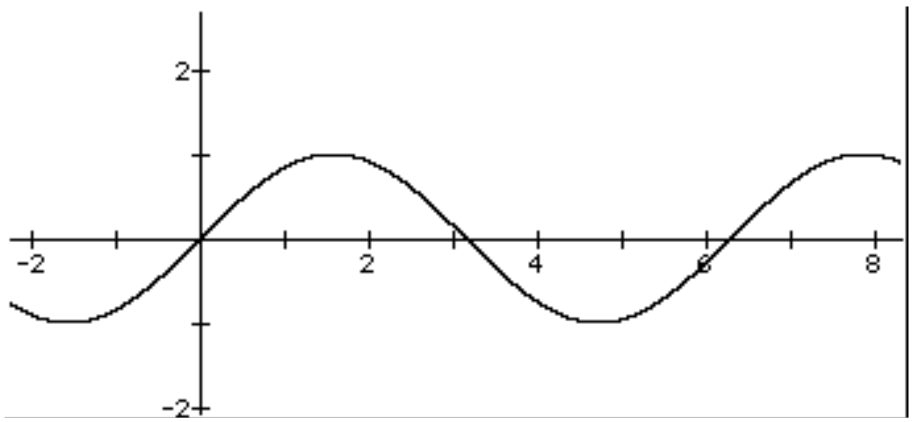
$$\begin{aligned}
\text{Area} &= \int_{-1}^{1} \left(x^3 - 2x^2 - 5x - 6\right)\, dx + \left(-\int_{1}^{2} \left(x^3 - 2x^2 - 5x - 6\right)\, dx\right) \\
&= \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\Bigg]_{-1}^{1} + \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{1}^{2} \\
&= \frac{32}{3} + \frac{29}{12} \\
&= \frac{157}{12}
\end{aligned}$$

EX 3 Find the area between $y=\sin x$ and the $x$-axis on $x\in[0,\ 2\pi]$.

Evaluating the integral $\int_0^{2\pi}\sin x\ dx$:

$$\begin{aligned}\int_0^{2\pi}\sin x\ dx &= [-\cos x]_0^{2\pi}\\ &= (-(-1))-1\\ &= 0\end{aligned}$$

Obviously, the area cannot be zero. Look at the graph:




What has happened with our integral here is that on $x\in[0,\ \pi]$ the curve is above the $x$-axis, so the "area" represented by the integral is positive. But on $x\in[\pi,\ 2\pi]$, the "area" is negative. Since these areas are the same size, the integrals added to zero.

There are several ways to account for this to find the area directly. The most common is to split the integral into two integrals where the boundaries are the zeros, and multiply the "negative area" by another negative to make it positive.

$$\begin{aligned}\text{Area} &= \int_0^{\pi}\sin x\ dx-\int_{\pi}^{2\pi}\sin x\ dx\\ &= [-\cos x]_0^{\pi}-[-\cos x]_{\pi}^{2\pi}\\ &= \big[(-(-1))-(-1)\big]-\big[(-1)-(-(-1))\big]\\ &= 4\end{aligned}$$

Here's a fun scenario—imagine leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your Calculus homework (gasp!). You head back home, pick up your assignment, and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? You are six miles from where you started but you have traveled 12 miles. These are the two different ideas behind displacement and total distance.

*Vocabulary*:
1. ***Displacement*** – how far apart the starting position and ending position of an object are (it can be positive or negative)
2. ***Total Distance*** – how far an object travels in total (this can only be positive)

$$\text{Displacement } = \int_a^b v\ dt$$

$$\text{Total Distance } = \int_a^b |v|\ dt$$

EX 4 A particle moves along a line so that its velocity at any time $t$ is $v(t) = t^2 - t - 6$ (measured in meters per second).

a) Find the displacement of the particle during the time period $1 \le t \le 4$.
b) Find the distance traveled during the time period $1 \le t \le 4$.

a) $\int_a^b v\ dt = \int_1^4 \left(t^2 - t - 6\right)\ dt$

$$= \left.\frac{t^3}{3} - \frac{t^2}{2} - 6t\right|_1^4$$

$$= -4.5$$

b) $\int_a^b |v| \; dt = \int_1^4 \left|t^2 - t - 6\right| \; dt$

$$= -\int_1^2 \left(t^2 - t - 6\right) \; dt + \int_2^4 \left(t^2 - t - 6\right) \; dt$$

$$= -\frac{t^3}{3} + \frac{t^2}{2} + 6t \Big|_1^2 + \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_2^4$$

$$= 10\frac{1}{6}$$

Note that the properties of integrals are used to split the integral into two integrals that represent the separate positive and negative distance traveled. Putting a "–" in front of the first integral turns the negative value from the integral into a positive value. Split the integral at $t = 2$ because that would be where $v(t) = 0$.

13-5 Free Response Homework

1. Find the area between $y = x^2 + 1$ and the $x$-axis on $x \in [-1,\ 1]$.

2. Find the area between $y = x\sqrt{9 - 4x^2}$ and the $x$-axis on $x \in \left[0,\ \frac{3}{2}\right]$.

3. Find the area between $y = \cos x$ and the $x$-axis on $x \in [0,\ 2\pi]$.

4. Find the area between $y = xe^{-x^2}$ and the $x$-axis on $x \in [-2,\ 2]$.

5. Find the area between $y = x\sqrt{9 - 4x^2}$ and the $x$-axis on $x \in \left[-\frac{3}{2},\ \frac{3}{2}\right]$.

6. Find the area between $y = x\sqrt{2x^2 - 18}$ and the $x$-axis on $x \in [-2,\ 1]$

7. Find the area between $y = 3\sin x\sqrt{1 - \cos x}$ and the $x$-axis on $x \in [-\pi,\ 0]$

8. Find the area between $y = x^2 e^{x^3}$ and the $x$-axis on $x \in [0,\ 1.5]$

9. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$ for $0 \le t \le 3$. Find a) the displacement and b) the distance traveled by the particle during the given time interval.

10. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$ for $1 \le t \le 6$. Find a) the displacement and b) the distance traveled by the particle during the given time interval.

11. Find the distance traveled by a particle in rectilinear motion whose velocity, in feet/sec, is described $v(t) = 4t + 1$ from $t = 1$ second to $t = 5$ seconds.

12. Find the distance traveled by a particle in rectilinear motion whose velocity, in feet/sec, is described $v(t) = 3t^2 - 4t + 1$ from $t = 0$ second to $t = 4$ seconds. [Be Careful!!!]

13-5 Multiple Choice Homework

1. The area under the graph of $y = 4x^3 + 6x - \frac{1}{x}$ on the interval $1 \le x \le 2$ is

a) $32 - \ln 2$ units$^2$ b) $30 - \ln 2$ units$^2$ c) $24 - \ln 2$ units$^2$

d) $\frac{99}{4}$ units$^2$ e) $21$ units$^2$

---

2. A particle starts at $(5,\ 0)$ when $t = 0$ and moves along the $x$-axis in such a way that at time $t > 0$ its velocity is given by $v(t) = \frac{1}{1+t}$. Determine the position of the particle at $t = 3$.

a) $\frac{97}{16}$ b) $\frac{95}{16}$ c) $\frac{79}{16}$ d) $5 + \ln 4$ e) $1 + \ln 4$

---

3. At $t = 0$, a particle starts at the origin with a velocity of 6 feet per second and moves along the $x$-axis in such a way that at time $t$ its acceleration is $12t^2$ feet per second per second. Through how many feet does the particle move during the first 2 seconds?

a) 16 ft b) 20 ft c) 24 ft d) 28 ft e) 32 ft

---

4. $\int_1^2 \frac{x^2 - x}{x^3}\ dx =$

a) $\ln 2 - \frac{1}{2}$ b) $\ln 2 + \frac{1}{2}$ c) $\frac{1}{2}$ d) $0$ e) $\frac{1}{4}$

---

5. $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\, dx =$

a) $\frac{1}{2}$ b) $1$ c) $2$ d) $\ln 2$ e) $\arctan 2 - \frac{\pi}{4}$

---

Integrals Practice Test
Part 1: CALCULATOR REQUIRED

Round to 3 decimal places. Show all work.

**Multiple Choice** (3 pts. each)

1. A particle moves along the *x*-axis with the velocity given by $v(t) = 3t^2 + 6t$ for time $t \geq 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at time $t = 1$?

(a) 4
(b) 6
(c) 9
(d) 11
(e) 12

2. If $\int_{-5}^{2} f(x)\, dx = -17$ and $\int_{5}^{2} f(x)\, dx = -4$, what is the value of $\int_{-5}^{5} f(x)\, dx$?

(a) –21
(b) –13
(c) 0
(d) 13
(e) 21

3. $\displaystyle\int \frac{x^2}{e^{x^3}}\, dx =$

(a) $-\dfrac{1}{3}\ln e^{x^3} + C$

(b) $-\dfrac{e^{x^3}}{3} + C$

(c) $-\dfrac{1}{3e^{x^3}} + C$

(d) $\dfrac{1}{3}\ln e^{x^3} + C$

(e) $\dfrac{x^3}{3e^{x^3}} + C$

4. $\int_0^x \sin t \; dt =$

(a) $\sin x$
(b) $-\cos x$
(c) $\cos x$
(d) $\cos x - 1$
(e) $1 - \cos x$

5. $\int_1^e \frac{x^2+1}{x} \; dx =$

(a) $\frac{e^2-1}{2}$
(b) $\frac{e^2+1}{2}$
(c) $\frac{e^2+2}{2}$
(d) $\frac{e^2-1}{e^2}$
(e) $\frac{2e^2-8e+6}{3e}$

6. $\int x\sqrt{4-x^2}\, dx =$

(a) $\frac{\left(4-x^2\right)^{3/2}}{3} + C$
(b) $-\left(4-x^2\right)^{3/2} + C$
(c) $\frac{x^2\left(4-x^2\right)^{3/2}}{3} + C$
(d) $-\frac{x^2\left(4-x^2\right)^{3/2}}{3} + C$

(e) $-\frac{\left(4-x^2\right)^{3/2}}{3}+C$

7. If $\int_{-2}^{2}\left(x^7+k\right)\ dx=16$, then $k=$

(a) –12
(b) 12
(c) –4
(d) 4
(e) 0

8. $\int \sin\left(2x+3\right)dx=$

(a) $\frac{1}{2}\cos\left(2x+3\right)+C$
(b) $\cos\left(2x+3\right)+C$
(c) $-\cos\left(2x+3\right)+C$
(d) $-\frac{1}{2}\cos\left(2x+3\right)+C$
(e) $-\frac{1}{5}\cos\left(2x+3\right)+C$

Integrals Practice Test
Part 2: NO CALCULATOR ALLOWED

Round to 3 decimal places. Show all work.

**Free Response** (10 pts. each)

1. Find the area between $y = \dfrac{1}{x^2}$ and the $x$-axis on the interval $x \in [1,\ 2]$.

2. Find the area between $y = \dfrac{x}{1+x^2}$ and the $x$-axis from $x = -2$ to $x = 2$.

3. A particle moves along a line with velocity function $v(t) = t^2 - t$, where $v$ is measured in meters per second. Set up, but do not solve, and integral expression for a) the displacement and b) the distance traveled by the particle during the time interval $[0,\ 5]$.

## Integrals Homework Answer Key

### 13-1 Free Response Homework

1. $2x^3 - x^2 + 3x + C$

2. $\frac{1}{4}x^4 + x^3 - x^2 + 4x + C$

3. $3x^{2/3} + C$

4. $\frac{8}{5}x^5 - x^4 + 3x^3 + x^2 + x + C$

5. $\frac{2}{3}x^6 + \frac{5}{4}x^4 + C$

6. $4x^3 + \frac{29}{2}x^2 - 8x + C$

7. $\frac{2}{3}x^{3/2} - 12x^{1/2} + C$

8. $\frac{1}{2}x^2 + 2x^{1/2} + 3\ln x + C$

9. $\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + C$

10. $\frac{16}{3}x^3 - 12x^2 + 9x + C$

11. $\frac{2}{3}x^{3/2} + \frac{6}{5}x^{5/2} - 12x^{1/2} + C$

12. $2x^2 - 2x^{-1/2} - \frac{1}{x} + C$

13. $f(x) = x^3 - 3x^2 + 3x + 2$

14. $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x - \frac{37}{12}$

15. $f(x) = \frac{3}{2}x^2 - \frac{10}{3}x^{3/2} - 2x + \frac{32}{3}$

16. $x(t) = 3t^4 - 2t^3 + 4t^2 - 13t + 11$

17. $x(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 2t^2 + 2t + 4$

### 13-1 Multiple Choice Homework

1. D
2. A
3. A

### 13-2 Free Response Homework

1. $\frac{1}{20}(5x+3)^4 + C$

2. $\frac{1}{100}(\mathrm{x}^4 + 5)^{25} + C$

3. $\frac{1}{7}x^7+\frac{1}{2}x^4+x+C$

4. $-\frac{3}{5}(2-x)^{5/3}+C$

5. $\frac{1}{6}\left(2x^2+3\right)^{3/2}+C$

6. $-\frac{1}{10}(5x+2)^{-2}+C$

7. $\frac{1}{2}\left(1+x^4\right)^{1/2}+C$

8. $\frac{3}{4}\left(x^2+2x+3\right)^{2/3}+C$

9. $\frac{1}{10}\left(x^2+4\right)^5-\left(x^2+4\right)^4+\frac{8}{3}\left(x^2+4\right)^3+C$

10. $\frac{2}{7}(x+3)^{7/2}-\frac{8}{5}(x+3)^{5/2}+\frac{8}{3}(x+3)^{3/2}+C$

13-2 Multiple Choice Homework

1. C 2. E 3. D 4. D 5. B

13-3 Free Response Homework

1. $\frac{1}{5}\sin x^5+C$

2. $-\frac{1}{7}\cos(7x+1)+C$

3. $\frac{1}{3}\tan(3x-1)+C$

4. $-2\cos\sqrt{x}+C$

5. $\frac{1}{5}\tan^5 x+C$

6. $\frac{1}{2}(\ln x)^2+C$

7. $\frac{1}{6}e^{6x}+C$

8. $-\frac{1}{4}\sin^{-2}2x+C=-\frac{1}{4}\csc^2 2x+C$

9. $\frac{1}{4}\left(\ln\left(x^2+1\right)\right)^2+C$

10. $2e^{\sqrt{x}}+C$

11. $-\frac{2}{3}\cot^{3/2}x+C$

12. $-\frac{1}{2}\left(\sin\frac{1}{x}\right)^2+C$, or $\frac{1}{2}\left(\cos\frac{1}{x}\right)^2+C$

13-3 Multiple Choice Homework

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | D | 3. | C | 4. | C | 5. | D |

13-4 Free Response Homework

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | 24 | 2. | 2 | 3. | 0 | 4. | 16.25 |
| 5. | $-33\frac{1}{3}$ | 6. | $-3$ | 7. | 0.070 | 8. | $\frac{8}{3}$ |
| 9. | 0.549 | 10. | 0.693 | 11. | 0.693 | 12. | 0.322 |

13-4 Multiple Choice Homework

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | C | 2. | A | 3. | A | 4. | D | 5. | D | 6. | D |

13-5 Free Response Homework

1. $\frac{8}{3}$ 2. 2.25 3. 4 4. 0.982

5. 4.5 6. $2\cos\sqrt{\pi}+2$ 7. $-\sqrt{32}$

8. $\frac{1}{3}\left(e^{3.375}-1\right)$ 9a. $-\frac{3}{2}$m b. $\frac{41}{6}$m

10a. $-\frac{10}{3}$m b. $\frac{98}{3}$m 11. 52 feet 12. 36.296 feet

13-5 Multiple Choice Homework

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | C | 2. | E | 3. | D | 4. | A | 5. | B |

# Integrals Practice Test Answer Key

## Multiple Choice

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | B | 2. | B | 3. | C | 4. | E |
| 5. | B | 6. | E | 7. | D | 8. | D |

## Free Response

1. Area $= \dfrac{1}{2}$

2. Area $= 2\ln 5$

3a. Displacement $= \int_0^5 \left(t^2 - t\right)\, dt$

b. Total Distance Traveled $= -\int_0^1 \left(t^2 - t\right)\, dt + \int_1^5 \left(t^2 - t\right)\, dt$